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Forward backward stochastic differential equations with delayed generators

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Abstract

In this paper, we prove a result of existence and uniqueness of solutions to coupled forward backward stochastic differential equations with delayed generators under a Lipschitz condition.

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1 Introduction

Backward stochastic differential equations (BSDEs, in short) were developed in the early 1990s, by Pardoux and Peng [23]. They established results on the existence and uniqueness of the adapted solutions under Lipschitz condition. Since then, BSDEs have been intensively developed both theoretically and in various applications. In [24, 25], authors gave a probabilistic representation for the solutions of some quasilinear parabolic partial

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differential equations in terms of solutions of BSDEs which is a generalisation of the well-known Feynman-Kac formula. Furthermore, BSDEs are encountered in many fields of applied mathematics such as finance, stochastic games and optimal control, homogenisation, etc. For more details, the reader can see [12, 13, 14, 15].

Inspired with an application, i.e. with the model for stochastic differential utility in finance, Antonelli and later Pardoux and Tang introduced respectively in [1, 26] a notion of forward backward differential equations (FBSDEs, for short). Using the "*Method of Contraction Mapping*" authors gave an existence and uniqueness result when the time duration T is sufficiently small. Using a PDE approach called "*The Four Step Scheme*, Ma et al. in [20] gave the existence and uniqueness to a class of FBSDEs in which the forward SDE is non-degenerate (i.e the diffusion coefficient of forward SDE, denoted with σ, is non-degenerate). Application of this method provided possibility to remove the restriction on the time duration in the Markovian framework.

Initiated by Hu and Peng [17] and Peng and Wu [28], and later developed by Yong [34], [35], the "*Method of Continuation*" solved non-Markovian FBSDEs with arbitrary duration without the non degeneracy condition of σ. The main assumption for this method is the so-called "*monotonicity conditions*" on the coefficients, which is restrictive in a different way. This method has been used widely in applications (see, e.g., [31, 32, 36]) because of its pure probabilistic nature. Recently, Ma et al. [21], study the well-posedness of the FBSDEs in a general non-Markovian framework. They derive a unified scheme which combines all above methodology and address some fundamental long-standing problems for non Markovian context. The study of such equations is very interesting. They are encountered when one applies the stochastic maximum principle to optimal stochastic control problems (see [16] for a linear version in an optimal stochastic control problem, see [27] for the probabilistic interpretation of a general type of systems of quasilinear PDEs, for the application in finance see [8, 9] etc.).

In all those previously mentioned models via FBSDEs, it is assumed that coefficients have a Markovian structure with respect to the triple (X,Y,Z), i.e at time t, coefficients depend only on the values of $X(t)$, $Y(t)$ and $Z(t)$. For example let consider the following simple FBSDE

$$X(t) = x+\sigma\int_0^t Z(s)dW(s), \tag{1.1}$$

$$Y(t) = X(T)-\int_t^T Z(s)dW(s). \tag{1.2}$$

This equation appears in many stochastic control problems when diffusion contains control, which is often the case in the optimal investment problems in finance.

However, in view of many papers (e.g [2, 10, 18, 19, 29]), it is well-know that a

Markovian structure becomes too restrictive in some models in finance. Indeed, it requires choices which are quantified by the values of all coefficients at time t , which are only based on the current information without taking in to account anything from the past. While in reality the investor compare previous and current opportunities, take into account experienced trends in the prices or satisfaction from the past consumptions, form a priori expectations about the projects, compare their past expectations with the current pay-offs, study the risk factors and the realized gains, and finally make decisions. Therefore, taking in to account the memory of the past values mentioned in [18] by Loewenstein (as one of three factors which could help in understanding preferences and inter-temporal choices of agents) is clearly very important in the applications, and it was a motivation for us to consider non-Markovian coefficients and study associated FBSDEs. More precisely, several works (e.g [2, 10, 19, 29]) explained that delay on the value function comes from the notion of disappointment effect on non-monotone preferences modelling due to aversion against volatility. Thus, according to the work of Delong in [3], on can derive a delayed version of FBSDE (1.1)-(1.2) as follows: for all $t \in [0,T]$

$$X(t) = x + \int_0^t \frac{\sigma}{s}\left(\int_0^s Z(u)du\right) dW(s), \qquad \text{a.s} \tag{1.3}$$

$$Y(t) = \frac{1}{T}\int_0^T X(s)ds - \int_t^T Z(s)dW(s), \qquad \text{a.s.} \tag{1.4}$$

Even though there exist results in the literature about the study of the FBSDEs (1.1)-(1.2), there exist no results for FBSDEs (1.3)-(1.4) which can be viewed as a simple version of delayed FBSDEs. Motivated by the above, this paper is devoted to study the fully FBSDEs with delayed generator in the form: for all $t \in [0,T]$,

$$X(t) = x + \int_0^t b(s,X_s,Y_s,Z_s)ds + \int_0^t \sigma(s,X_s,Y_s,Z_s)dW(s), \qquad \text{a.s.} \tag{1.5}$$

$$Y(t) = \xi + \int_t^T f(s,X_s,Y_s,Z_s)ds - \int_t^T (g(s,X_s,Y_s) + Z(s))dW(s), \qquad \text{a.s.} \tag{1.6}$$

where $(X_t,Y_t,Z_t) = (X(s+u),Y(s+u),Z(s+u))_{u\in[-T,0]}$ design the past of the triple processes (X,Y,Z) until moment t.

In the special case that the function f does not depend to X and g identically null, BSDEs (1.6) have been already studied in [4]. They established an existence and uniqueness result under sufficiently small Lipschitz constant or sufficiently small terminal times T. Authors illustrated in two examples that the previously developed conditions are necessary and sufficient. However, they also showed that, for some special class of generators, existence and uniqueness may still hold for an arbitrary time horizon and for arbitrary Lipschitz constant. Furthermore, in [5] authors extended previous result to BSDEs with time delayed generators driven by Brownian motions and Poisson random measures.

When coefficients b and σ do not depend of Y and Z, equation (1.5) reduces to the well-know delayed SDEs which have been studied in many papers under an arbitrary Lipschitz condition constant and/or arbitrary time horizon, (see [30, 33]).

Paper is organized as follows in the sequel: In Section 2 we recall some preliminary notations and results. Section 3 is dedicated to derive our main result which is an existence and uniqueness results for the solution of FBSDEs (1.5)-(1.6).

2 Preliminaries

For a strict positive real number T, let us consider $(\Omega, \mathbf{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq T})$ a a filtered probability space, where the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is assumed to be complete, right continuous and generated by a $(W(t))_{0\leq t\leq T}$, a d-dimensional Brownian motion, for $d \in \mathbb{N}^*$.

On the other hand, since we deal with delayed FBSDEs, let us set the following space on which all generators will be defined:

- Let $L^2_{-T}(\mathbb{R}^{n\times d})$ denote the space of measurable functions $z : [-T;0] \to \mathbb{R}^{n\times d}$ satisfying

$$\int_{-T}^{0} \mid z(t) \mid^2 dt < \infty;$$

- Let $L^\infty_{-T}(\mathbb{R}^n)$ denote the space of bounded, measurable functions $y : [-T,0] \to \mathbb{R}^n$ satisfying

$$\sup_{-T\leq t\leq 0} \mid y(t) \mid^2 < +\infty.$$

Finally, in order to give what we mean by solution of Eqs. (1.5)-(1.6), let us set the following spaces. For any $\beta > 0$,

- Let $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ be the space of $\mathcal{F}_T$-measurable random variables $\xi : \Omega \to \mathbb{R}^n$ endowed with the norm

$$\|\xi\|^2_{L^2} := \mathbb{E}(|\xi|^2);$$

- Let $\mathcal{S}^2(\mathbb{R}^n)$ denote the space of all predictable process $\eta = (\eta(t))_{-T\leq t\leq T}$ with values in $\mathbb{R}^n$ such that

$$\mathbb{E}\left(\sup_{-T\leq s\leq T} e^{\beta s}|\eta(s)|^2\right) < +\infty;$$

- Let $\mathcal{H}^2(\mathbb{R}^{n\times d})$ denote the space of all predictable process $\eta = (\eta(t))_{-T\leq t\leq T}$ with values in $\mathbb{R}^{n\times d}$ such that

$$\mathbb{E}\left(\int_{-T}^{T} e^{\beta s}|\eta(s)|^2 ds\right) < +\infty.$$

In all this paper, we will use the following notations; $|.|$ denotes the usual norm in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ with it associated Euclidian norm. For $u=(x,y,z)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d}$,

$$\|u\|^2 := |x|^2+|y|^2+|z|^2$$

and

$$h(t,u) := (f(t,u),b(t,u),\sigma(t,u)).$$

We are now able to introduce definition of the solution to the Eqs. (1.5)-(1.6).

Definition 2.1. *A triple of adapted processes (X,Y,Z) is called solution of Eqs.* (1.5)-(1.6) *for $t\in[0,T]$ if it satisfies* (1.5)-(1.6) $\mathbb{P}$*-almost surely (a.s.).*

Solution $U=(X,Y,Z)$ is unique, if for any other solution $U'=(X',Y',Z')$ and for $t\in[0,T]$ following holds:

$$X=X',\ Y=Y',\ Z=Z'\ a.s.$$

3 Mains results

In this section we derive an existence and uniqueness result for FBSDE with delayed. Our method differ to one applied by Hu and Peng in [17] because in our context, we applied Itô's formula to $|X|^2$ and $|Y|^2$ rather than to $\langle X,Y\rangle$ which permit us to relax some of their assumptions. Indeed, we work under the following classical assumptions:

(A1) $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$,

(A2) $\Phi:\Omega\times[-T,T]\times L^{\infty}_{-T}(\mathbb{R}^n)\times L^{\infty}_{-T}(\mathbb{R}^n)\times L^{2}_{-T}(\mathbb{R}^{n\times d})\to\mathbb{R}^{\ell}$ is a product measurable and $\mathbf{F}$-adapted function such that for some probability measure α defined on $([-T,0],\mathcal{B}([-T,0]))$ and any $u_t=(x_t,y_t,z_t), u'_t=(x'_t,y'_t,z'_t)\in L^{\infty}_{-T}(\mathbb{R}^n)\times L^{\infty}_{-T}(\mathbb{R}^n)\times L^{2}_{-T}(\mathbb{R}^{n\times d})$, there exist two positive constant K such that following holds:

(i) $|\,\Phi(t,u_t)-\Phi(t,u'_t)\,|^2\leq K\int_{-T}^{0}\|u(t+v)-u'(t+v)\|^2\alpha(dv)$ a.s., $\forall\ t\in[0,T]$.

(ii) For $t<0$, $\Phi(t,u_t)=0$ a.s.,

(iii) $\mathbb{E}\left[\int_0^T |\Phi(t,0)|^2 dt\right] < +\infty$,

where Φ is respectively b,σ,f and $\ell=n$ if $\Phi=b,f$ and $\ell=n\times d$ if $\Phi=\sigma$.

(A3) $g:\Omega\times[-T,T]\times L^\infty_{-T}(\mathbb{R}^n)\times L^\infty_{-T}(\mathbb{R}^n)\to\mathbb{R}^{n\times d}$ is a product measurable and **F**-adapted function such that for some probability measure α defined on $([-T,0],\mathcal{B}([-T,0]))$ and any $(x_t,y_t),(x'_t,y'_t)\in L^\infty_{-T}(\mathbb{R}^n)\times L^\infty_{-T}(\mathbb{R}^n)$, there exist positive constant K such that following holds:

(i) $|\, g(t,x_t,y_t)-g(t,x'_t,y'_t)\,|^2\leq$
$K\int_{-T}^0(|x(t+u)-x'(t+u)|^2+|y(t+u)-y'(t+u)|^2)\alpha(du)$ a.s. $,\forall\ t\in[0,T]$,

(ii) For $t<0$, $g(t,x_t,y_t)=0$, a.s,

(iii) $\mathbb{E}\left[\int_0^T |g(t,0,0)|^2 dt\right] < +\infty$.

Remark 3.1. *(a) Assumption* (**A2**)-*(ii) allows us to take* $(X(t),Y(t),Z(t))=(X(0),Y(0),0)$ *for* $t<0$*, as a solution of eqs.* (1.5)-(1.6).

(b) The quantity $\Phi(t,0)$ *in* (**A2**)-*(iii) and* $g(t,0,0)$ *in* (**A3**)-*(iii) should be understood respectively as a value of the generator* ϕ *at* $u_t=(0,0,0)$ *and* g *at* $(x_t,y_t)=(0,0)$.

Remark 3.2. *In view of* (a) *of Remark 3.1, if the process* $(X(t),Y(t),Z(t))_{-T\leq t\leq T}$ *belongs to* $\mathcal{S}^2(\mathbb{R}^n)\times\mathcal{S}^2(\mathbb{R}^n)\times\mathcal{H}^2(\mathbb{R}^{n\times d})$ *and satisfies FBSDE* (1.5)-(1.6)*, then for all* $t<0$, $X(t)=X(0)$, $Y(t)=Y(0)$ *and* $Z(t)=0$*. Therefore we have*

$$\mathbb{E}\left(\sup_{-T\leq s\leq T} e^{\beta s}|\eta(s)|^2\right)=\mathbb{E}\left(\sup_{0\leq s\leq T} e^{\beta s}|\eta(s)|^2\right),$$

for $\eta=X,Y$ *and*

$$\mathbb{E}\left(\int_{-T}^T e^{\beta s}|Z(s)|^2 ds\right)=\mathbb{E}\left(\int_0^T e^{\beta s}|Z(s)|^2 ds\right).$$

(c) *Conditions* (**A2**) (i) *and* (**A3**) (i) *are versions of Lipschitz condition with a constant* K.

First we deal with a special form of FBSDEs (1.5)-(1.6) which arise when the function g is identically null. More precisely let us consider the following FBSDEs

$$X(t) = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dW(s), \quad \text{a.s.} \tag{3.1}$$

$$Y(t) = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z(s)dW(s), \quad \text{a.s.} \tag{3.2}$$

Theorem 3.1. *Let assumptions* (**A1**)-(**A2**) *hold, and let horizon time* $T > 0$ *and Lipschitz constant* K *satisfy*

$$21Ke\max(1, T^2) < 1.$$

Then delayed FBSDEs (3.1)-(3.2) *has an unique solution* (X, Y, Z) *in* $\mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n\times d})$

Proof. This proof is subdivided in two parts.

Step 1: *Existence.*

Let set $U^0 = (X^0, Y^0, Z^0) = (0, 0, 0)$ and consider $U^n = (X^n, Y^n, Z^n)$ defined recursively as follows: For all $t \in [0, T]$,

$$X^n(t) = x + \int_0^t b(s, U_s^{n-1})ds + \int_0^t \sigma(s, U_s^{n-1})dW(s),$$

$$Y^n(t) = \xi + \int_t^T f(s, U_s^{n-1})ds - \int_t^T Z^n(s)dW(s).$$

Setting

$$\bar{\phi}^{n+1} = \phi^{n+1} - \phi^n,$$

for ϕ equal to X, Y, Z and

$$\bar{\Phi}(s) = \Phi(s, X_s^n, Y_s^n, Z_s^n) - \Phi(s, X_s^{n-1}, Y_s^{n-1}, Z_s^{n-1}),$$

for Φ equal to b, σ, f, it follows that

$$\begin{aligned} \bar{X}^{n+1}(t) &= \int_0^t \bar{b}(s)ds + \int_0^t \bar{\sigma}(s)dW(s), \\ \bar{Y}^{n+1}(t) &= \int_t^T \bar{f}(s)ds - \int_t^T \bar{Z}^{n+1}(s)dW(s). \end{aligned} \tag{3.3}$$

Applying Itô's formula to $e^{\frac{\beta}{2}t}\bar{X}^{n+1}(t)$, taking the modulus and applied the triangle inequality, one obtains

$$e^{\frac{\beta}{2}t}|\bar{X}^{n+1}(t)| \leq \frac{\beta}{2}\int_0^t e^{\frac{\beta}{2}s}|\bar{X}^{n+1}(s)|ds + \int_0^t e^{\frac{\beta}{2}s}|\bar{b}(s)|ds + \left|\int_0^t e^{\frac{\beta}{2}s}\bar{\sigma}(s)dW(s)\right|.$$

Next, by Young's inequality and isometry principle,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq t\leqslant T} e^{\beta t} \mid \bar{X}^{n+1}(t) \mid^2\right)\\
\leq\ & \frac{\beta^2}{2}T\int_0^T \mathbb{E}\left(e^{\beta s} \mid \bar{X}^{n+1}(s) \mid^2\right) ds + 4\mathbb{E}\left(T\int_0^T e^{\beta s} \mid \bar{b}(s) \mid^2 ds + \int_0^T e^{\beta s}|\bar{\sigma}(s)|^2 ds\right)\\
\leq\ & \frac{\beta^2}{2}T^2\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\beta t} \mid \bar{X}^{n+1}(t) \mid^2\right) + 4\mathbb{E}\left(T\int_0^T e^{\beta s} \mid \bar{b}(s) \mid^2 ds + \int_0^T e^{\beta s}|\bar{\sigma}(s)|^2 ds\right).
\end{aligned}$$

Hence,

$$\mathbb{E}\left(\sup_{0\leq t\leqslant T} e^{\beta t} \mid \bar{X}^{n+1}(t) \mid^2\right) \leq 4\left(1-\frac{\beta^2}{2}T^2\right)^{-1} \max(1,T)\mathbb{E}\left(\int_0^T e^{\beta s} \mid \bar{b}(s) \mid^2 ds + \int_0^T e^{\beta s}|\bar{\sigma}(s)|^2 ds\right). \tag{3.4}$$

Let us now treat the backward SDE. Applying again Itô's formula to $e^{\beta t} \mid \bar{Y}^{n+1}(t) \mid^2$, we obtain

$$\begin{aligned}
&\mathbb{E}\left(e^{\beta t} \mid \bar{Y}^{n+1}(t) \mid^2 + \beta\int_t^T e^{\beta s} \mid \bar{Y}^{n+1}(s) \mid^2 ds + \int_t^T e^{\beta s} \mid \bar{Z}^{n+1}(s) \mid^2 ds\right)\\
&= 2\mathbb{E}\int_t^T e^{\beta s}\bar{Y}^{n+1}(s)\bar{f}(s)ds.
\end{aligned}$$

From Young's inequality it follows that there exists a positive constant β such that

$$2\bar{Y}^{n+1}(s)\bar{f}(s) \leq \beta|\bar{Y}^{n+1}(s)|^2 + \frac{1}{\beta}|\bar{f}(s)|^2,$$

so we obtain

$$\mathbb{E}\left(e^{\beta t} \mid \bar{Y}^{n+1}(t) \mid^2 + \int_t^T e^{\beta s} \mid \bar{Z}^{n+1}(s) \mid^2 ds\right) \leq \frac{1}{\beta}\mathbb{E}\int_t^T |\bar{f}(s)|^2 ds. \tag{3.5}$$

On the other hand, according to the backward component of (3.3) and applying conditional expectation which respect $\mathcal{F}_t$, we have

$$e^{\frac{\beta}{2}t}\bar{Y}^{n+1}(t) \leq \mathbb{E}\left(\int_t^T e^{\frac{\beta}{2}s}\bar{f}(s)ds \mid \mathcal{F}_t\right). \tag{3.6}$$

Squaring each member of (3.6), we apply Doob's martingale and Hôlder inequality to obtain

$$\mathbb{E}\left(\sup_{0\leq t\leqslant T} e^{\beta t} \mid \bar{Y}^{n+1}(t)\mid^2\right) \leq 4T\mathbb{E}\left(\int_0^T e^{\beta s}\mid \bar{f}(s)\mid^2 ds\right). \tag{3.7}$$

Next, it follows from (3.7),(3.5) and (3.4) that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leqslant T} e^{\beta t}\mid \bar{X}^{n+1}(t)\mid^2 + \sup_{0\leq t\leqslant T} e^{\beta t}\mid \bar{Y}^{n+1}(t)\mid^2 + \int_0^T e^{\beta s}\mid \bar{Z}^{n+1}(s)\mid^2 ds\right]\\
\leq\ & \mathbb{E}\left[4\left(1-\frac{\beta^2}{2}T^2\right)^{-1}\max(1,T)\left(\int_0^T e^{\beta s}|\bar{b}(s)|^2ds + \int_0^T e^{\beta s}|\bar{\sigma}(s)|^2ds\right)\right.\\
&\left.+\left(4T+\frac{1}{\beta}\right)\int_0^T e^{\beta s}|\bar{f}(s)|^2ds\right].
\end{aligned} \tag{3.8}$$

Further, from the Lipschitz condition $(\mathbf{A2})$ on b, σ and f that, it follows that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta s}|\bar{\Phi}(s)|^2ds\right]\\
\leq\ & K\mathbb{E}\left[\int_0^T\int_{-T}^0 e^{\beta s}\left(\mid \bar{X}^n(s+u)\mid^2 + \mid \bar{Y}^n(s+u)\mid^2 + \mid \bar{Z}^n(s+u)\mid^2\right)\alpha(du)ds\right].
\end{aligned}$$

Since for each $n \geq 1$, $X^n(s) = x$, $Y^n(s) = Y^n(0)$ and $Z^n(s) = 0$ for all $s < 0$, we obtain respectively by Fubini's theorem and the change of variable

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{\beta t}|\bar{\Phi}(s)|^2ds\right]\\
&\leq K\mathbb{E}\left[\int_{-T}^0 e^{-\beta u}\left(\int_u^{T+u} e^{\beta v}\left(\mid \bar{X}^n(v)\mid^2 + \mid \bar{Y}^n(v)\mid^2 + \mid \bar{Z}^n(v)\mid^2\right)dv\right)\alpha(du)\right]\\
&\leq Ke\max(1,T)\mathbb{E}\left[\sup_{0\leq t\leqslant T} e^{\beta t}\mid \bar{X}^n(t)\mid^2 + \sup_{0\leq t\leqslant T} e^{\beta t}\mid \bar{Y}^n(t)\mid^2 + \int_0^T e^{\beta s}\mid \bar{Z}^n(s)\mid^2 ds\right]
\end{aligned} \tag{3.9}$$

Putting (3.9) in (3.8) we get

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{X}^{n+1}(t)\mid^2 + \sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{Y}^{n+1}(t)\mid^2 + \int_0^T e^{\beta s} \mid \bar{Z}^{n+1}(s)\mid^2 ds\right] \\
\le & \left[8\left(1-\frac{\beta^2}{2}T^2\right)^{-1}\max(1,T) + \left(4T+\frac{1}{\beta}\right)\right] \\
\times & \mathbb{E}\left[\sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{X}^{n}(t)\mid^2 + \sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{Y}^{n}(t)\mid^2 + \int_0^T e^{\beta s} \mid \bar{Z}^{n}(s)\mid^2 ds\right].
\end{aligned}
$$

Therefore, setting $\beta = \frac{1}{T}$, we obtain

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{X}^{n+1}(t)\mid^2 + \sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{Y}^{n+1}(t)\mid^2 + \int_0^T e^{\beta s} \mid \bar{Z}^{n+1}(s)\mid^2 ds\right] \\
\le \; & 21Ke\max(1,T^2)\mathbb{E}\left[\sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{X}^{n}(t)\mid^2 + \sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{Y}^{n}(t)\mid^2 + \int_0^T e^{\beta s} \mid \bar{Z}^{n}(s)\mid^2 ds\right].
\end{aligned}
$$

Finally, by iterative argument, it not difficult to derive

$$
\begin{aligned}
& \mathbb{E}\left[\sup_{0\le t\leqslant T} \mid e^{\beta t}\bar{X}^{n+1}(t)\mid^2 + \sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{Y}^{n+1}(t)\mid^2 + \int_0^T e^{\beta s} \mid \bar{Z}^{n+1}(s)\mid^2 ds\right] \\
& \le \left[21Ke\max(1,T^2)\right]^n \mathbb{E}\left[\sup_{0\le t\leqslant T} \mid e^{\beta t}\bar{X}^{1}(t)\mid^2 + \sup_{0\le t\leqslant T} e^{\beta t} \mid \bar{Y}^{1}(t)\mid^2 + \int_0^T e^{\beta s} \mid \bar{Z}^{1}(s)\mid^2 ds\right].
\end{aligned}
$$

Since $21Ke\max(1,T^2) < 1$, $(X^n, Y^n, Z^n)_{n\ge 1}$ is a Cauchy sequence in the Banach space $\mathcal{S}^2(\mathbb{R}^n)\times\mathcal{S}^2(\mathbb{R}^n)\times\mathcal{H}^2(\mathbb{R}^{n\times d})$. It is then easy to conclude that $(X,Y,Z) = \lim_{n\to+\infty}(X^n,Y^n,Z^n)$ solves delayed FBSDEs (3.1)-(3.2).

Step 2: *Uniqueness.*

Let define $U = (X,Y,Z)$ and $U' = (X',Y',Z')$ be two solutions of Eqs. (3.1)-(3.2). Let us set $\delta\phi = \phi - \phi'$, for $\phi = X,Y,Z$. Then $(\delta X, \delta Y, \delta Z)$ satisfies

$$
\begin{aligned}
\delta X(t) &= \int_0^t \bar{b}(s)ds + \int_0^t \bar{\sigma}(s)dW(s) \\
\delta Y(t) &= \int_t^T \bar{f}(s)ds - \int_t^T \delta Z(s)dW(s),
\end{aligned}
$$

where $\bar{\Phi}(s) = \Phi(s, X_s, Y_s, Z_s) - \Phi(s, X'_s, Y'_s, Z'_s)$ for $\Phi = b, \sigma$ and f. With the similar steps used in existence part, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leqslant T} e^{\beta t} \mid \delta X(t) \mid^2 + \sup_{0\leq t\leqslant T} e^{\beta t} \mid \delta Y(t) \mid^2 + \int_0^T e^{\beta s} \mid \delta Z(s) \mid^2 ds\right] \\
\leq\ & 21Ke \max(1, T^2) \mathbb{E}\left[\sup_{0\leq t\leqslant T} e^{\beta t} \mid \delta X(t) \mid^2 + \sup_{0\leq t\leqslant T} e^{\beta t} \mid \delta Y(t) \mid^2 + \int_0^T e^{\beta s} \mid \delta Z(s) \mid^2 ds\right].
\end{aligned}$$

In the fact that $21Ke \max(1, T^2) < 1$ we have

$$\mathbb{E}\left[\sup_{0\leq t\leqslant T} e^{\beta t} \mid \delta X(t) \mid^2 + \sup_{0\leq t\leqslant T} e^{\beta t} \mid \delta Y(t) \mid^2 + \int_0^T e^{\beta s} \mid \delta Z(s) \mid^2 ds\right] \leq 0,$$

which implies that $X = X'$, $Y = Y'$ and $Z = Z'$ a.s. □

Now let us study a more general FBSDE delayed FBSDEs (1.5)-(1.6).

Theorem 3.2. *Let assumptions* (**A1**)-(**A3**) *hold, and let horizon time $T > 0$ and Lipschitz constant K satisfy*

$$21Ke \max(1, T^2) < 1.$$

Then delayed FBSDEs (1.5)-(1.6) *admit an solution (X, Y, Z) in $\mathcal{S}^2(\mathbb{R}^n) \times \mathcal{S}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n\times d})$.*

Proof. Our method is an adaptation of idea used by Janković, Đorđević and Jovanović [6, 7, 11]. In this fact, let us consider this following

$$\begin{aligned}
X(t) &= x + \int_0^t \widetilde{b}(r, X_r, Y_r, Z_r) dr + \int_0^t \widetilde{\sigma}(r, X_r, Y_r, Z_r) dW(r), \\
Y(t) &= \xi + \int_t^T \widetilde{f}(r, X_r, Y_r, Z_r) dr - \int_t^T Z(r) dW(r),
\end{aligned} \tag{3.10}$$

where $\widetilde{\Phi}(t, x_t, y_t, z_t) = \Phi(t, x_t, y_t, z_t - g(t, x_t, y_t))$, with $\Phi = b, \sigma, f$.

According to Assumption (**A2**) and (**A3**), the function $\tilde{\Phi}$ satisfies the following assumption

(A4) $\widetilde{\Phi} : \Omega \times [-T, T] \times L^\infty_{-T}(\mathbb{R}^n) \times L^\infty_{-T}(\mathbb{R}^n) \times L^2_{-T}(\mathbb{R}^{n\times d}) \to \mathbb{R}^\ell$ is a product measurable and $\mathbf{F}$-adapted function such that for some probability measure α on $([-T, 0], \mathcal{B}([-T, 0]))$ and for any $u(t) = (x_t, y_t, z_t)$, $u'(t) = (x'_t, y'_t, z'_t) \in L^\infty_{-T}(\mathbb{R}^n) \times L^\infty_{-T}(\mathbb{R}^n) \times L^2_{-T}(\mathbb{R}^{n\times d})$ there exists a positive constant K such that following holds:

(i) $\widetilde{\Phi}(t,y_t,z_t)-\widetilde{\Phi}(t,y'_t,z'_t)|^2 \leq K\int_{-T}^{0}\|u(t+v)-u'(t+v)\|^2\alpha(dv)$ a.s.,

(ii) $\mathbb{E}\left[\int_0^T |\widetilde{\Phi}(t,0,0,0)|^2 dt\right] < +\infty$

(iii) $\widetilde{\Phi}(t,\cdot,\cdot,\cdot)=0$ for $t<0$.

Further, in view of Theorem 3.1, FBSDDE (3.10) admit an unique solution $(\overline{X},\overline{Y},\overline{Z})$ in $\mathcal{S}^2(\mathrm{IR})\times\mathcal{S}^2(\mathrm{IR})\times\mathcal{H}^2(\mathrm{IR})$. Setting $X=\overline{X},\ Y=\overline{Y},\ Z=\overline{Z}-g(.,\overline{X},\overline{Y})$, it not difficult to prove that (X,Y,Z) is the solution of FBSDDE (1.5)-(1.6). It remain to show that such solution is unique.

Let us suppose that it exists (X',Y',Z'), an other solution of FBSDDE (1.5)-(1.6). Therefore setting $\Delta X=\overline{X}-X'$, $\Delta Y=\overline{Y}-Y'$, we have

$$\begin{aligned}
\Delta X(t) &= \int_0^t \left[b(s,Y'_s,Z'_s)-b(s,\overline{Y}_s,\overline{Z}_s-g(s,\overline{X}_s,\overline{Y}_s))\right]ds\\
&\quad \int_0^t \left[\sigma(s,Y'_s,Z'_s)-\sigma(s,\overline{Y}_s,\overline{Z}_s-g(s,\overline{Y}_s))\right]dW(s),\\
\Delta Y(t) &= \int_t^T \left[f(s,Y'_s,Z'_s)-f(s,\overline{Y}_s,\overline{Z}_s-g(s,\overline{X}_s,\overline{Y}_s))\right]ds\\
&\quad -\int_t^T \left[Z'(s)+g(s,Y'_s)-\overline{Z}(s))\right]dW(s).
\end{aligned}$$

With the same computations used in the uniqueness part of the proof of Theorem 3.2, we obtain that

$$\begin{aligned}
&\mathrm{I\!E}\left[\sup_{0\leq t\leqslant T} e^{\beta t}\mid \Delta X(t)\mid^2+\sup_{0\leq t\leqslant T} e^{\beta t}\mid \Delta Y(t)\mid^2+\int_0^T e^{\beta s}\mid Z'(s)+g(s,X'_s,Y'_s)-\overline{Z}(s))\mid^2 ds\right]\\
\leq\ & 21Ke\max(1,T^2)\mathrm{I\!E}\left[\sup_{0\leq t\leqslant T} e^{\beta t}\mid \Delta X(t)\mid^2+\sup_{0\leq t\leqslant T} e^{\beta t}\mid \Delta Y(t)\mid^2\right.\\
&\left.+\int_0^T e^{\beta s}\mid Z'(s)+g(s,X'_s,Y'_s)-\overline{Z}(s))\mid^2 ds\right].
\end{aligned}$$

Suppose again that $21Ke\max(1,T^2)<1$, than

$$\mathrm{I\!E}\left[\sup_{0\leq t\leqslant T} e^{\beta t}\mid \Delta X(t)\mid^2+\sup_{0\leq t\leqslant T} e^{\beta t}\mid \Delta Y(t)\mid^2+\int_0^T e^{\beta s}\mid Z'(s)+g(s,X'_s,Y'_s)-\overline{Z}(s))\mid^2 ds\right]=0.$$

Finally we get $X'=\overline{X}$, $Y'=\overline{Y}$ and $Z'=\overline{Z}-g(.,\overline{X},\overline{Y})$, which completes the proof. □

As a corollary of Theorem 3.2, let us state this existence and uniqueness result for the BSDE (1.6) in the special case that function f and g are defined respectively in $L^{\infty}_{-T}(\mathbb{R}) \times L^{2}_{-T}(\mathbb{R})$ and $L^{\infty}_{-T}(\mathbb{R})$. (Proof is the same as in the more general case given with Theorem 3.2.

Corollary 3.1. *Let assume f depends only to y_t and z_t and g depend only to y_t for all $t \in [0,T]$ and satisfy* (**A1**)-(**A3**) *such that*

$$5KTe\max(1,T) < 1.$$

Then delayed BSDEs (3.2) *admit an unique solution (Y,Z) in $\mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(\mathbb{R})$.*

Remark 3.3. *In view of our work, we note that the condition on T and K introduced in [4] has been weakened. On the other hand, condition appear in Theorem 3.1 and 3.2 is equivalent to $21TKe\max(1,T)$ and is justified by the addition of SDEs.*

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