

BOUNDING THE BETTI NUMBERS OF REAL HYPERSURFACES NEAR THE TROPICAL LIMIT

BORNES SUR LES NOMBRES DE BETTI DES HYPERSURFACES ALGÈBRIQUES RÉELLES PROCHES DE LA LIMITE TROPICALE

ARTHUR RENAUDINEAU AND KRIS SHAW

ABSTRACT. We prove a bound conjectured by Itenberg on the Betti numbers of real algebraic hypersurfaces near non-singular tropical limits. These bounds are given in terms of the Hodge numbers of the complexification. To prove the conjecture we introduce a real variant of tropical homology and define a filtration on the corresponding chain complex inspired by Kalinin's filtration. The spectral sequence associated to this filtration converges to the homology groups of the real algebraic variety and we show that the terms of the first page are tropical homology groups with $\mathbb{Z}/2\mathbb{Z}$ -coefficients. The dimensions of these homology groups correspond to the Hodge numbers of complex projective hypersurfaces by combining results of Itenberg, Mikhalkin, Katzarkov, and Zharkov and the authors together with Arnal. The bounds on the Betti numbers of the real part follow, as well as a criterion to obtain a maximal variety. We also generalise Bertrand's formula relating the signature of the complex hypersurface and the Euler characteristic of the real algebraic hypersurface, as well as reprove Haas' combinatorial criterion for the maximality of plane curves near the tropical limit.

Résumé

On démontre une borne conjecturée par Itenberg sur les nombres de Betti des hypersurfaces algébriques réelles proches de la limite tropicale. Ces bornes sont exprimées en fonctions des nombres de Hodge de la complexification. Pour démontrer ces bornes nous introduisons une variante de l'homologie tropicale dans le cadre réel, et définissons une filtration sur le complexe de chaîne associé, inspirée par la filtration de Kalinin. La suite spectrale associée à cette filtration converge vers les groupes d'homologie de la variété algébrique réelle, et nous montrons que les termes de la première page sont les groupes d'homologie tropicale (à coefficients dans $\mathbb{Z}/2\mathbb{Z}$). Les dimensions de ces groupes d'homologie correspondent aux nombres de Hodge des hypersurfaces projectives complexes grâce aux résultats d'Itenberg, Mikhalkin, Katzarkov, Zharkov d'une part et des auteurs avec Arnal d'autre part. Les bornes sur les nombres de Betti s'ensuivent, ainsi qu'un critère pour obtenir une variété maximale. Nous généralisons également la formule due à Bertrand reliant la signature d'une hypersurface complexe et la caractéristique d'Euler d'une hypersurface réelle, et nous redémontrons le critère combinatoire de Haas sur la maximalité des courbes planes proches de la limite tropicale.

1. INTRODUCTION

A *real hypersurface* $V \subset \mathbb{P}^{n+1}$ of degree d is a hypersurface defined by a real homogeneous polynomial $f(z_0, \dots, z_{n+1}) \in \mathbb{R}[z_0, \dots, z_{n+1}]$ of degree d . We let $\mathbb{R}V$ denote the set of real points of V and $\mathbb{C}V$ denote the set of its complex points. The following fundamental question in real algebraic geometry can be traced back beyond Hilbert's sixteenth problem, see [Hil00], [Wil78], [DK00b] for a survey. In this paper we denote by \mathbb{Z}_2 the field with two elements.

Question 1.1. *For any $0 \leq q \leq n$, what is the maximal possible value of the q -th Betti number*

$$b_q(\mathbb{R}V) := \dim H_q(\mathbb{R}V; \mathbb{Z}_2)$$

among degree d non-singular real algebraic hypersurfaces V in \mathbb{P}^{n+1} ?

In 1876, Harnack [Har76] proved the optimal bound

$$b_0(\mathbb{R}V) \leq \frac{(d-1)(d-2)}{2} + 1,$$

for V a non-singular real algebraic plane curve of degree d . This bound was generalised to any compact non-singular real curve by Klein [Kle76] by replacing $\frac{(d-1)(d-2)}{2}$ with $g(\mathbb{C}V)$, where $g(\mathbb{C}V)$ denotes the genus of the complex curve. Beyond the case of plane curves, no optimal bounds are known in general on the individual Betti numbers of real algebraic varieties. For example, in the case of non-singular real algebraic surfaces in \mathbb{P}^3 , the maximal values of the individual Betti numbers are unknown beyond degree 5. It is known that the maximal number of connected components of a non-singular real algebraic quintic surface is either 23, 24, or 25 and the maximal value of the first Betti number is either 45 or 47, see [Ore01] and [IK96]. In general, we will denote by $h^{p,q}(\mathbb{C}V)$ the (p, q) -th Hodge number of $\mathbb{C}V$.

In relation to higher Betti numbers, in 1980 Viro formulated the following conjecture for all real projective surfaces.

Conjecture 1.2 (Viro). *If V is a non-singular real projective surface such that $\mathbb{C}V$ is simply connected, then*

$$b_1(\mathbb{R}V) \leq h^{1,1}(\mathbb{C}V),$$

where $h^{1,1}(\mathbb{C}V)$ denotes the $(1, 1)$ -th Hodge number of $\mathbb{C}V$.

When V is the double covering of \mathbb{P}^2 ramified along a curve of even degree, the above conjecture is equivalent to the slight weakening of Ragsdale's conjecture about plane curves [Rag04] made by Petrovsky [Pet38], see also [Vir80]. Counterexamples to Ragsdale's and Petrovsky's conjectured have been constructed by Viro [Vir80] and Itenberg [Ite93], respectively. Itenberg's counterexample paved the way to various counterexamples to Viro's conjecture and to constructions of real algebraic surfaces with many connected components, for example those in [Ite97], [Bih99], and [Bru06]. It is still not known whether Viro's conjecture is true for surfaces which are maximal in the sense of the Smith-Thom inequality (1.2).

There are two main directions in Question 1.1. The first is to obtain restrictions on the topologies of real algebraic varieties, as is the case for

Harnack's bound. The second direction is to provide constructions of real algebraic varieties with given topology. Viro's patchworking method provided a breakthrough in the second direction [Vir84]. This technique continues to be the most powerful tool to construct real algebraic varieties in toric varieties with effectively computable topology. Here we will restrict our attention to Viro's primitive combinatorial patchworking, see Remark 3.8. The following was conjectured by Itenberg around 2005, and later appeared in [Ite17].

Conjecture 1.3. [Ite17, Conjecture 2.5] *Let V be a real hypersurface in \mathbb{P}^{n+1} obtained by a primitive patchworking. Then for any integer $q = 0, \dots, n$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) & \text{for } q = n/2, \\ h^{q,n-q}(\mathbb{C}V) + 1 & \text{otherwise.} \end{cases}$$

In the case of real algebraic surfaces in \mathbb{P}^3 arising from primitive patchworkings the above bounds were already proven by Itenberg [Ite97], and are explicitly,

$$b_0(\mathbb{R}V) \leq \binom{d-1}{3} + 1 \quad \text{and} \quad b_1(\mathbb{R}V) \leq \frac{2d^3 - 6d^2 + 7d}{3}.$$

For example, real algebraic surfaces of degree 5 arising from a primitive patchworking satisfy $b_0(\mathbb{R}V) \leq 5$ and $b_1(\mathbb{R}V) \leq 45$. Furthermore, asymptotic analogues of the bounds in Conjecture 1.3 were proved by Itenberg and Viro in [IV07].

Viro's method for patchworking applies not only to real hypersurfaces in projective space but also to hypersurfaces in more general toric varieties, see for example [Ris93]. Real algebraic hypersurfaces arising from primitive patchworking were later interpreted by Viro [Vir01] as real algebraic hypersurfaces near non-singular tropical limits, see Definition 3.9 and [BIMS15, Section 5.3]. Here we will use this contemporary point of view on Viro's method and relate it to Viro's original formulation in Remark 3.8. A hypersurface near the non-singular tropical limit will always be (partially) compactified in a toric variety whose fan is a subfan of the dual fan of the Newton polytope of the hypersurface. We say a Newton polytope Δ is non-singular if the associated toric variety Y_Δ is non-singular. In this paper we establish the following theorem for real algebraic hypersurfaces in compact non-singular toric varieties near a non-singular tropical limit.

Theorem 1.4. *Let V be a compact real algebraic hypersurface with non-singular Newton polytope and near a non-singular tropical limit. Then for any integer $q = 0, \dots, n$,*

$$b_q(\mathbb{R}V) \leq \begin{cases} h^{q,q}(\mathbb{C}V) & \text{for } q = n/2, \\ h^{q,n-q}(\mathbb{C}V) + h^{q,q}(\mathbb{C}V) & \text{otherwise.} \end{cases}$$

When the toric variety is projective space, Conjecture 1.3 follows directly from the following theorem and the fact that $h^{q,q}(\mathbb{C}V) = 1$ for $2q \neq n$ by the Lefschetz hyperplane section theorem.

1.1. A guide to the proof of Theorem 1.4. To prove Theorem 1.4, we use a tropical description of primitive patchworking in terms of real phase structures, which we present in Section 3. We recall the relation to the standard version of primitive patchworking in Remark 3.8. For an n -dimensional non-singular real tropical hypersurface X in a tropical toric variety Y the notion of a real phase structure \mathcal{E} is described in Definition 3.1. In Section 3.2 we describe a cellular cosheaf on X called the sign cosheaf $\mathcal{S}_{\mathcal{E}}$. A cellular cosheaf \mathcal{G} on a tropical hypersurface X consists of a vector space $\mathcal{G}(\sigma)$ for each face σ of X together with linear maps $i_{\sigma\tau}: \mathcal{G}(\sigma) \rightarrow \mathcal{G}(\tau)$ for each inclusion of faces $\tau \subset \sigma$. These linear maps must satisfy commutativity conditions for all face relations $\rho \subset \tau_1, \tau_2 \subset \sigma$. The cellular chain complex $C_{\bullet}(X; \mathcal{S}_{\mathcal{E}})$ with coefficients in $\mathcal{S}_{\mathcal{E}}$ together with its homology groups $H_{\bullet}(X; \mathcal{S}_{\mathcal{E}})$ are defined in Definition 3.16. Note that all the cosheaves in this paper are considered over \mathbb{Z}_2 , unless it is otherwise clearly stated. In Proposition 3.17, we prove that the homology groups of the sign cosheaf are isomorphic to the homology groups of the real part of a real algebraic hypersurface near the tropical limit. The next step of the proof is to construct a filtration of the chain complex with coefficients in $\mathcal{S}_{\mathcal{E}}$,

$$0 \subset C_{\bullet}(X; \mathcal{K}_n) \subset \cdots \subset C_{\bullet}(X; \mathcal{K}_p) \subset \cdots \subset C_{\bullet}(X; \mathcal{S}_{\mathcal{E}}), \quad (1.1)$$

where the \mathcal{K}_p 's are the collection of cellular cosheaves on X from Definition 4.5.

The tropical homology groups, as introduced by Itenberg, Katzarkov, Mikhakin and Zharkov [IKMZ19], are also homology groups of cosheaves on tropical varieties but with rational coefficients. Here we use a \mathbb{Z}_2 -variant of this homology theory and denote the cosheaves by \mathcal{F}_p . For every p and each face τ of X , we define linear maps $bv_p: \mathcal{K}_p(\tau) \rightarrow \mathcal{F}_p(\tau)$ in Definition 4.9, which come from the augmented filtration of a group algebra which was highlighted by Quillen [Qui68]. It follows from Lemma 4.8, that these linear maps are surjective and satisfy $\text{Ker}(bv_p: \mathcal{K}_p(\tau) \rightarrow \mathcal{F}_p(\tau)) = \mathcal{K}_{p+1}(\tau)$. Proposition 4.10 shows that these linear maps commute with the cosheaf maps and thus induce morphisms of chain complexes and produce the filtration in (1.1). Then we consider the spectral sequence associated to this filtration which we denote by $(E_{\bullet, \bullet}^{\bullet}, \partial_{\bullet})$. This spectral sequence degenerates since it arises from a filtration of a chain complex consisting of finite dimensional chain groups. Therefore we obtain,

$$\dim H_q(X; \mathcal{S}_{\mathcal{E}}) = \sum_{p=0}^n \dim E_{q,p}^{\infty}.$$

By Corollary 4.11 we get

$$E_{q,p}^1 = H_q(C_{\bullet}(X, \mathcal{K}_p)/C_{\bullet}(X, \mathcal{K}_{p+1})) \cong H_q(X; \mathcal{F}_p).$$

This establishes the following theorem which bounds the Betti numbers of real algebraic hypersurfaces near the tropical limit. The theorem holds not just for a compact hypersurface in the toric variety defined by its Newton polytope, but also for a non-compact hypersurface obtained by removing the intersection with any of the torus orbits of that toric variety. This includes for example, hypersurfaces near the tropical limit contained in the torus or affine space. In the statement below, the notation b_q^{BM} denotes the q -th

Betti number of the Borel-More homology group [BM60] and $H_q^{BM}(X; \mathcal{F}_p)$ denotes the Borel-Moore variant of tropical homology [JRS18]. For Δ a lattice polytope, denote by Y_Δ^o a partial compactification of the torus corresponding to a subfan of the dual fan of Δ .

Theorem 1.5. *Let V be a real algebraic hypersurface with Newton polytope Δ in the non-singular toric variety Y_Δ^o and near the non-singular tropical limit X . Then for all q we have,*

$$b_q(\mathbb{R}V) \leq \sum_{p=0}^n \dim H_q(X; \mathcal{F}_p),$$

and

$$b_q^{BM}(\mathbb{R}V) \leq \sum_{p=0}^n \dim H_q^{BM}(X; \mathcal{F}_p).$$

The main ingredient for proving Theorem 1.4 from Theorem 1.5 is to relate the dimensions of tropical homology groups with \mathbb{Z}_2 -coefficients and the Hodge numbers of complex hypersurfaces.

Theorem 1.6. *Let X be a non-singular compact tropical hypersurface with non-singular Newton polytope Δ . Let V be a non-singular complex hypersurface in the non-singular complex toric variety Y_Δ also with Newton polytope Δ . Then for all p and q we have*

$$\dim H^{p,q}(\mathbb{C}V) = \text{rank } H_q(X; \mathcal{F}_p).$$

Proof. By [ARS, Corollary 1.4], the integral tropical homology groups of the hypersurface X are torsion free. Therefore, we have

$$\text{rank } H_q(X; \mathcal{F}_p^{\mathbb{Z}}) = \dim H_q(X; \mathcal{F}_p) = \dim H_q(X; \mathcal{F}_p^{\mathbb{Q}}),$$

where $H_q(X; \mathcal{F}_p^{\mathbb{Z}})$ and $H_q(X; \mathcal{F}_p^{\mathbb{Q}})$ are the tropical homology group with \mathbb{Z} and \mathbb{Q} -coefficients, respectively. By [IKMZ19, Corollary 2] and [ARS, Corollary 1.9] we have $\dim H_q(X; \mathcal{F}_p^{\mathbb{Q}}) = h^{p,q}(\mathbb{C}V)$ and this completes the proof. \square

Finally, the expression of the bounds in Theorem 1.4 is obtained by the Lefschetz Hyperplane Theorem which implies that if $p+q \neq n$ or $p \neq q$ then then $h^{p,q}(\mathbb{C}V) = 0$.

Remark 1.7. *We expect that the above theorem may also hold beyond the case of hypersurfaces. For example, Viro's patchworking construction has been generalised to complete intersections [Stu94], [Bih02]. In order to obtain a version of Theorem 1.4 for complete intersections it remains to relate the dimensions of the tropical homology groups with \mathbb{Z}_2 -coefficients to the Hodge numbers of complete intersections over the complex numbers. One possible route to making this connection is to establish tropical Lefschetz section theorems and torsion freeness of the integral tropical homology groups for tropical complete intersections as in the case for hypersurfaces in [ARS].*

Even more generally, a tropical manifold is a polyhedral space locally modelled on matroid fans [MR, Section 7.4]. A real phase structure on a tropical manifold would consist of specifying orientations of the local matroids, subject to compatibility conditions. It would be interesting to study real phase structures in this context and to generalise Theorem 1.5.

1.2. Further consequences of the spectral sequence. In Section 6, we describe how to go further in the spectral sequence to give criteria in Theorem 6.1 for a real algebraic hypersurface arising from a primitive patchworking to attain the bounds in Theorem 1.4. For a real algebraic variety, the Smith-Thom inequality bounds the sum of the Betti numbers of the real part by the sum of the complexification,

$$\sum_{q=0}^n b_q(\mathbb{R}V) \leq \sum_{q=0}^{2n} b_q(\mathbb{C}V). \quad (1.2)$$

A real algebraic variety is called an *M-variety*, or a *maximal variety*, if it satisfies equality in (1.2). The spectral sequence gives a necessary and sufficient condition for a compact real algebraic hypersurface near the tropical limit to be maximal in the sense of the Smith-Thom inequality. Moreover, Theorem 6.1 gives a criterion for individual Betti numbers to attain the bounds of Theorem 1.4.

Theorem 1.8. *A compact real hypersurface with non-singular Newton polytope and near a non-singular tropical limit is maximal in the sense of the Smith-Thom inequality (1.2) if and only if the associated spectral sequence $(E_{\bullet, \bullet}^{\bullet}, \partial_{\bullet})$ degenerates at the first page.*

Proof. A compact real hypersurface near the tropical limit is maximal in the sense of the Smith-Thom inequality if and only if all the inequalities in Theorem 1.4 are equalities. This happens exactly when the spectral sequence $(E_{\bullet, \bullet}^{\bullet}, \partial_{\bullet})$ degenerates at the first page. \square

Viro proved the existence of non-singular maximal surfaces of any degree in \mathbb{P}^3 [Vir79]. Later, Itenberg and Viro proved that there exist non-singular projective hypersurfaces of any dimension that are asymptotically maximal [IV07]. This was generalised by Bertrand [Ber06] to hypersurfaces and complete intersections in arbitrary toric varieties. Bertrand also proved in [Ber06] that there exist toric varieties in any dimension that do not admit torically non-degenerate maximal hypersurfaces. However, all known examples are singular.

Question 1.9. *For every non-singular Newton polytope Δ , does there exist a maximal real hypersurface in the toric variety Y_{Δ} with Newton polytope Δ ?*

Theorem 6.1 describes the possible non-zero differentials on further pages of the spectral sequence $(E_{\bullet, \bullet}^{\bullet}, \partial_{\bullet})$. If the following conjecture is true, then it would simplify the statement of Theorem 6.1, since we would only need to consider the differentials on the first page.

Conjecture 1.10. *For a compact non-singular real tropical hypersurface (X, \mathcal{E}) the spectral sequence associated to the filtration*

$$0 \subset C_{\bullet}(X; \mathcal{K}_n) \subset \cdots \subset C_{\bullet}(X; \mathcal{K}_1) \subset C_{\bullet}(X; \mathcal{S}_{\mathcal{E}})$$

degenerates at the second page.

In Section 7, we restrict our attention to real plane curves near the tropical limit. In this case, the only possible non-zero differential of the spectral

sequence is on the first page. Using the isomorphism in Corollary 4.11, this differential is $\partial_1: H_1(C; \mathcal{F}_0) \rightarrow H_0(C; \mathcal{F}_1)$. In Theorem 7.2, we explicitly describe this linear map using the twist description of patchworking for curves [BIMS15, Section 3]. Using this description we also recover Haas' criterion for the maximality of curves in toric surfaces near non-singular tropical limits in Theorem 7.5.

A real algebraic hypersurface near the non-singular tropical limit has the same signature as the Euler characteristic of its real part. This relation was first proved in the case of surfaces by Itenberg [Ite97], and then later generalised to arbitrary dimensions by Bertrand [Ber10]. By comparing Euler characteristics of different pages of the above spectral sequence, we recover this result and a generalisation to the non-compact case. A different proof of this signature formula in the compact case was given by Arnal in his master's thesis [Arn17] also using tropical homology. We recall that as in Theorem 1.5, we let Y_Δ^o denote a partial compactification of the torus corresponding to a subfan of the dual fan of Δ .

Corollary 1.11. *Let V be a real algebraic hypersurface in the non-singular toric variety Y_Δ^o with Newton polytope Δ and near the non-singular tropical limit X , then*

$$\chi^{BM}(\mathbb{R}V) = \chi_y(\mathbb{C}V)|_{y=-1},$$

where $\chi_y(\mathbb{C}V)$ denotes the χ_y genus of $\mathbb{C}V$ and $\chi(\mathbb{R}V)$ is the topological Euler characteristic of $\mathbb{R}V$. In particular, if $Y_{\Delta^o} = Y_\Delta$ is compact, then

$$\chi(\mathbb{R}V) = \sigma(\mathbb{C}V),$$

where $\sigma(\mathbb{C}V)$ denotes the signature of $\mathbb{C}V$.

Proof. The Euler characteristic of the r -th page of a spectral sequence is

$$\chi(E^r) = \sum_{p,q} (-1)^q \dim E_{q,p}^r.$$

Since any page is by definition the homology of the preceding page, one has

$$\chi(E^\infty) = \dots = \chi(E^0) = \sum_p \chi(C_\bullet^{BM}(X; \mathcal{F}_p)),$$

where $\chi(C_\bullet^{BM}(X; \mathcal{F}_p))$ denotes the Euler characteristic of the chain complex for the Borel-Moore homology of X with coefficients in \mathcal{F}_p .

A complex hypersurface $\mathbb{C}V$ which is near a non-singular tropical limit is torically non-degenerate in the sense of [DK86]. Therefore, by [ARS, Theorem 1.8], the Euler characteristics of the chain complexes for tropical homology give the coefficients of the χ_y genus of $\mathbb{C}V$,

$$\chi_y(\mathbb{C}V) = \sum_p (-1)^p \chi(C_\bullet^{BM}(X; \mathcal{F}_p)) y^p.$$

Therefore,

$$\chi_y(\mathbb{C}V)|_{y=-1} = \chi(E^0)$$

Moreover, the Euler characteristic of the infinity page is equal to the Borel-Moore Euler characteristic of $\mathbb{R}V$,

$$\chi(E^\infty) = \chi^{BM}(\mathbb{R}V),$$

and this proves the first claim.

Finally, when the ambient toric variety is compact, then so is the hypersurface. Then by the definition of the χ_y genus, we have $\sigma(CV) = \chi_{-1}(CV)$ and the corollary is proved. \square

To end the introduction we would like to make a few remarks about the geometric inspiration behind the proofs of Theorem 1.4. The construction of the cosheaves $\mathcal{S}_{\mathcal{E}}$, \mathcal{K}_p , and \mathcal{F}_p , together with the linear maps bv_p , are all presented using linear algebra, but their definitions are geometrically motivated. In the case of the sign cosheaf, the \mathbb{Z}_2 -vector space $\mathcal{S}_{\mathcal{E}}(\sigma)$ associated to a face σ of X is isomorphic to $H_0(\mathbb{R}P^n \setminus \mathbb{R}\mathcal{A}_{\sigma}; \mathbb{Z}_2)$, where \mathcal{A}_{σ} is a real hyperplane arrangement determined by the face σ and n is the dimension of X . The cosheaves \mathcal{F}_p from tropical homology satisfy $\mathcal{F}_p(\sigma) = H_p(\mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}_{\sigma}; \mathbb{Z}_2)$ [Zha13]. This is described in the proof of Lemma 2.6.

For real varieties, the Viro homomorphism is a partially defined multivalued homomorphism

$$bv_*: H_*(\mathbb{R}V; \mathbb{Z}_2) \dashrightarrow H_*(CV; \mathbb{Z}_2),$$

where $H_*(\mathbb{R}V; \mathbb{Z}_2)$ and $H_*(CV; \mathbb{Z}_2)$ denote the total homology of the real and complex parts respectively. A description of these homomorphisms is given in [DK00b, Appendix A2]. The complement of a real hyperplane arrangement \mathcal{A} in $\mathbb{R}P^n$ is a disjoint union of convex regions and therefore satisfies $H_q(\mathbb{R}P^n \setminus \mathbb{R}\mathcal{A}; \mathbb{Z}_2) = 0$ for all $q \neq 0$. Moreover, the complement of a real hyperplane arrangement is a maximal variety in the sense of the Smith-Thom inequality (1.2) [OT92, Introduction p.6]. Therefore, in this special case the Viro homomorphism gives a collection of well defined graded maps

$$bv_p: \text{Ker}(bv_{p-1}) \rightarrow H_p(\mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}; \mathbb{Z}_2).$$

The map $bv_0: H_0(\mathbb{R}P^n \setminus \mathbb{R}\mathcal{A}; \mathbb{Z}_2) \rightarrow H_0(\mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}; \mathbb{Z}_2)$ is induced by the inclusion $i: \mathbb{R}P^n \setminus \mathbb{R}\mathcal{A} \rightarrow \mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}$. To define the map bv_p , given $\alpha \in \text{Ker}(bv_{p-1})$, consider a p -chain β in $\mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}$ such that $\partial\beta = bv_{p-1}(\alpha)$. Then $bv_p(\alpha)$ is the homology class of the cycle $\beta + \text{conj}(\beta)$. It follows from the maximality of $\mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}$ that the complex conjugation acts as the identity on homology groups, see [Wil78, Corollary A.2]. Therefore, the maps bv_p are well defined as they do not depend on the choice of the chain β . Kalinin's spectral sequence [Kal05] induces a filtration on the real homology of a variety, which in the case of a real hyperplane arrangement is given by

$$0 = \text{Ker}(bv_n) \subset \text{Ker}(bv_{n-1}) \subset \cdots \subset \cdots \subset \text{Ker}(bv_0) \subset H_0(\mathbb{R}P^n \setminus \mathbb{R}\mathcal{A}; \mathbb{Z}_2).$$

Although we do not use this geometry in the presentation of our arguments, we borrow the notation for the Viro homomorphisms for our maps $bv_p: \mathcal{K}_p \rightarrow \mathcal{F}_p$ and use the letter \mathcal{K} to denote the pieces of the filtration of $\mathcal{S}_{\mathcal{E}}$ in reference to Kalinin's filtration.

1.3. Related works. In the case of real complete toric varieties, Hower [How08] used a spectral sequence to relate the Betti numbers of a real toric variety to a \mathbb{Z}_2 variant of Brion's Hodge spaces for fans [Bri97]. The \mathbb{Z}_2 -Hodge spaces for complete regular fans vanishes outside of the line $p = q$ [BFMvH06, Proof of Theorem 1.2], and nonsingular complete real toric varieties are all maximal in the sense of the Smith-Thom inequality. Hower

proved that in the case of real toric varieties coming from reflexive polytopes, the spectral sequence also degenerates at the first page, and those toric varieties are again all maximal. Hower also exhibits an example of a six-dimensional projective toric variety which is not maximal, disproving a conjecture in [BFMvH06].

The Hodge spaces for fans coincide with the tropical cohomology groups of the corresponding tropical toric variety, since their defining chain complexes are isomorphic by definition. Finding necessary and sufficient conditions for fans to satisfy a version of Poincaré duality for Brion’s Hodge spaces with integer coefficients would lead to a better understanding of fans defining maximal toric varieties.

For Lagrangian toric fibrations equipped with an anti-symplectic involution, Castaño-Bernard and Matessi study the cohomology of the fixed point locus using a long exact sequence which relates it to the cohomology of the Calabi-Yau manifold [CnBM10]. This is inspired by the Leray spectral sequence of Gross that relates the cohomologies of the Calabi-Yau manifold and the base space [Gro01], see also [GS10]. Very recently Argüz and Prince, computed the connecting maps of this long exact sequence and the cohomology groups with \mathbb{Z}_2 -coefficients of real Lagrangians in the quintic 3-fold and its mirror [AP20].

1.4. Outline of the paper. In Section 2, we review the definitions of tropical hypersurfaces in toric varieties and their tropical homology groups. Section 3 uses real phase structures on non-singular tropical hypersurfaces to describe Viro’s primitive patchworking. In Subsection 3.2, we introduce the sign cosheaf and the real tropical homology groups. Section 4 describes the augmentation filtration in general and adapts it to filter the sign cosheaves and the chain complexes. Section 5 contains the proof of Theorem 1.4. Section 6 investigates going further in the spectral sequence and lists all possible non-zero maps at further pages. Lastly in Section 7, we illustrate the situation in the case of plane curves and describe the only possibly non-zero differential map in the spectral sequence in this case. This allows us to recover Haas’ condition to obtain maximal curves.

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2. PROJECTIVE TROPICAL HYPERSURFACES

The tropical numbers are the set $\mathbb{T} = [-\infty, \infty)$. We equip \mathbb{T} with the topology of a half open interval and \mathbb{T}^N with the product topology. Tropical toric varieties are tropical manifolds in the sense of [MR, Chapter 7] with charts to \mathbb{T}^N . The space \mathbb{T}^N itself is tropical affine space. Just like toric varieties over a field, they are constructed from rational polyhedral fans see [MS15, Section 6.2], [MR, Section 3.2]. We recall that a rational polyhedral fan Σ is *simplicial* if each of its cones is the cone over a simplex. A simplicial rational polyhedral fan in \mathbb{R}^N is unimodular if the primitive integer directions of the rays of each cone can be completed to a basis of \mathbb{Z}^N . A tropical toric variety is *non-singular* if it is built from a simplicial unimodular rational polyhedral fan. A tropical toric variety is compact if and only if the corresponding fan is complete.

A tropical toric variety $\mathbb{T}Y$ has a stratification and the combinatorics of the stratification is governed by its fan Σ . For a cone ρ of Σ , let $T(\rho)$ denote its tangent space, meaning the linear span of the cone. For a face ρ the corresponding stratum of $\mathbb{T}Y$ is defined to be $\mathbb{T}Y_\rho := \mathbb{R}^N/T(\rho)$. We assume the vertex of the fan to be 0, so the corresponding open stratum of $\mathbb{T}Y$ is denoted by $\mathbb{T}Y_0$.

Example 2.1. If the fan Σ in \mathbb{R}^N consists of a single point, then the associated tropical toric variety is called an N -dimensional tropical torus. A tropical torus can be identified with \mathbb{R}^N as a vector space and there is an integer lattice \mathbb{Z}^N contained in it. Furthermore, each stratum of an N -dimensional tropical toric variety $\mathbb{T}Y$ is a tropical torus of dimension $N - \dim \rho$. The stratum $\mathbb{T}Y_\rho$ can be non-canonically identified with $\mathbb{R}^{N-\dim \rho}$, and contains a lattice which maps to \mathbb{Z}^N under this identification.

Notice that $\mathbb{T}Y_\rho$ is not necessarily closed in $\mathbb{T}Y$, and we let $\overline{\mathbb{T}Y}_\rho$ denote its closure. For two cones ρ and ρ' of the fan Σ we have $\mathbb{T}Y_{\rho'} \subset \overline{\mathbb{T}Y}_\rho$ if and only if ρ is a face of ρ' in Σ . Moreover, if ρ is a face of ρ' in Σ , we have a projection map denoted by $\pi_{\rho, \rho'}: \mathbb{T}Y_\rho \rightarrow \mathbb{T}Y_{\rho'}$ which is dual to the inclusion of $T(\rho')$ into $T(\rho)$.

Example 2.2. Tropical projective space $\mathbb{T}P^N$ is constructed from the complete fan Σ in \mathbb{R}^N whose rays are in directions $-e_1, \dots, -e_N, e_0 = \sum_{i=1}^N e_i$, where e_i denote the standard basis vectors. For every proper subset $I \subsetneq \{0, \dots, N\}$ there is a cone of the fan defining $\mathbb{T}P^N$ of dimension equal to $|I|$. Moreover, analogous to projective space over a field, tropical projective space can also be defined as the quotient

$$\mathbb{T}P^N = \frac{\mathbb{T}^N \setminus (-\infty, \dots, -\infty)}{(x_0, \dots, x_N) \sim (a + x_0, \dots, a + x_N)},$$

where $a \in \mathbb{T} \setminus -\infty$.

The cones of Σ correspond to subsets $I \subsetneq \{0, \dots, N\}$. The I -th open stratum of the stratification of $\mathbb{T}P^N$ is then

$$\mathbb{T}P_I^N = \{x \in \mathbb{T}P^N \mid x_i = -\infty \text{ iff } i \in I\}.$$

Using the above quotient description, the open stratum $\mathbb{T}P_0^N$ can be identified with the vector space $\mathbb{R}^{N+1}/\langle(1, \dots, 1)\rangle$. Hence it is a tropical torus of dimension N and has the structure of a real vector space. Similarly, an open stratum $\mathbb{T}P_I^N$ can be identified with a quotient of a real vector space determined by the cone corresponding to I . It is a tropical torus of dimension $N - |I|$. \triangle

2.1. Tropical hypersurfaces. A tropical polynomial in $n + 1$ variables is a function $F_{\text{trop}}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ of the form

$$F_{\text{trop}}(x) = \max_{i \in A} (a_i + \langle i, x \rangle), \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^{n+1} , the set A is a finite subset of \mathbb{Z}^{n+1} , and $a_i \in \mathbb{T}$ for all $i \in A$.

A tropical polynomial of the form (2.1) induces a regular subdivision of the Newton polytope of its defining polynomial. A *tropical hypersurface* X in \mathbb{R}^{n+1} is the locus of non-linearity of the function defined by a tropical polynomial together with weights naturally assigned to its top dimensional faces, also known as *facets*. The tropical hypersurface of a polynomial is dual to the regular subdivision of its Newton polytope induced by the convex-hull of the graph of $i \rightarrow a_i$, hence this subdivision is called the *dual subdivision* of X . The weight of a facet is the integer length of the segment of the dual subdivision dual to the facet. We refer the reader to [MS15, Section 3.1], [BIMS15, Section 5.1], and [MR, Section 2.3] for further details and examples.

A tropical hypersurface in \mathbb{R}^{n+1} is *non-singular* if its dual subdivision is primitive, meaning that each $n + 1$ dimensional polytope of the subdivision has normalised lattice volume equal to 1. In particular, the weights on all facets of a non-singular tropical hypersurface are equal to one. We define a tropical hypersurface in a tropical toric variety TY to be the closure of a tropical hypersurface in $\text{TY}_0 \subset \text{TY}$ where TY_0 is the open stratum corresponding to the vertex 0 of the fan defining TY .

Given a non-singular tropical hypersurface X in \mathbb{R}^{n+1} with Newton polytope Δ , we will consider its (partial) compactification in a tropical toric variety TY which is defined by a subfan of the dual fan of Δ . In this case, the compactification of X in TY_Δ is said to be a non-singular tropical variety if TY_Δ is non-singular, which is guaranteed if the Newton polytope Δ is non-singular.

Example 2.3. The fan Σ defining $\mathbb{T}P^{n+1}$ from Example 2.2 is a complete fan. Therefore, the non-singular tropical hypersurfaces that we consider in $\mathbb{T}P^{n+1}$ must satisfy that the dual fan of their Newton polytopes are equal to Σ . Therefore, the Newton polytope of such a hypersurface is equal up to translation to $d\Delta_{n+1}$ for some d , where

$$d\Delta_{n+1} = \text{ConvHull}\{0, de_1, \dots, de_{n+1}\}.$$

If X is non-singular in \mathbb{R}^{n+1} with the above Newton polytope, then it is dual to a unimodular subdivision of $d\Delta_{n+1}$. The intersection $X_I \subset \mathbb{T}P_I^{n+1}$ is also a non-singular tropical hypersurface dual to the subdivision of the corresponding face of $d\Delta_{n+1}$.

We follow the convention that all faces of a hypersurface X in a tropical toric variety are closed, see [JRS18, Section 2.A]. We let $X_\rho := X \cap \mathbb{T}Y_\rho$. The faces of X_ρ are also closed in the tropical torus $\mathbb{T}Y_\rho$, when considered with its own topology. However, since $\mathbb{T}Y_\rho$ is not closed in $\mathbb{T}Y$, the closure in $\mathbb{T}Y$ of a face σ in $\mathbb{T}Y_\rho$ may not be equal to σ . If σ is a face in $\mathbb{T}Y_\rho$, upon taking the closure $\bar{\sigma}$ in $\mathbb{T}Y$ only faces of X contained in strata $\mathbb{T}Y_\eta$ satisfying $\rho \subset \eta$ can be contained in $\bar{\sigma}$.

For a face σ of a tropical hypersurface, we let $\text{int}(\sigma)$ denote its relative interior. The *sedentarity* of a point y in $\mathbb{T}Y$ is ρ if y is contained in the stratum $\mathbb{T}Y_\rho$. The *sedentarity* of a face σ is denoted by $\text{sed}(\sigma)$ and is equal to ρ if $\text{int}(\sigma) \subset \mathbb{T}Y_\rho$. The *parent face* of a face τ of X of dimension k and sedentarity ρ is the unique face σ of X of zero sedentarity and dimension $k + \dim \rho$ such that τ is in the boundary of σ . The *star* of a face σ is the star of any point in its relative interior. More precisely,

$$\text{Star}(\sigma) = \{v \in \mathbb{T}Y_{\text{sed}(\sigma)} \mid \exists \delta > 0 \text{ s.t. } \forall \epsilon < \delta, x_\sigma + \epsilon v \in X_{\text{sed}(\sigma)}\}.$$

where x_σ is any point in $\text{int}(\sigma)$.

2.2. Tropical homology. The cosheaves that we use throughout the text will always be vector spaces over \mathbb{Z}_2 . Let X be a non-singular tropical hypersurface in a tropical toric variety $\mathbb{T}Y$. Let the defining fan of $\mathbb{T}Y$ be the simplicial unimodular fan Σ in \mathbb{R}^{n+1} , and let \mathbb{Z}^{n+1} denote the standard lattice in \mathbb{R}^{n+1} . Let ρ be a cone of Σ of dimension s with primitive integer generators r_1, \dots, r_s and define the integral tangent space to $\mathbb{T}Y_\rho$ as

$$T_{\mathbb{Z}}(\mathbb{T}Y_\rho) := \frac{\mathbb{Z}^{n+1}}{\langle r_1, \dots, r_s \rangle}.$$

For a face σ of X of sedentarity $\rho \in \Sigma$, denote the tangent space of $\text{int} \sigma$ by $T(\sigma)$. Notice that $T(\sigma) \subset T(\mathbb{T}Y_\rho)$. The integral tangent space of $\text{int}(\sigma)$ is $T_{\mathbb{Z}}(\sigma) := T(\sigma) \cap T_{\mathbb{Z}}(\mathbb{T}Y_\rho)$. Since σ is a rational polyhedron, we can consider the lattice points of $\mathbb{T}Y_\rho$ contained in the tangent space. Since X is a non-singular hypersurface the rank of $T_{\mathbb{Z}}(\sigma)$ is the same as the dimension of σ and also the reduction modulo 2 of the free \mathbb{Z} -module $T_{\mathbb{Z}}(\sigma)$ is a vector space of the same dimension as σ . In fact, at any vertex v adjacent to σ , one can complete a basis of the free \mathbb{Z} -module $T_{\mathbb{Z}}(\sigma)$ into a basis of $T_{\mathbb{Z}}(\mathbb{T}Y_\rho)$ with vectors in $T_{\mathbb{Z}}(\sigma_i)$ for σ_i faces adjacent to v . We denote this vector space over \mathbb{Z}_2 by $\mathcal{F}_1(\sigma)$.

If $\mathbb{T}Y_\rho$ and $\mathbb{T}Y_\eta$ are a pair of strata corresponding to cones $\rho \subset \eta$ of Σ then $\mathbb{T}Y_\eta \subset \bar{\mathbb{T}Y}_\rho$ then the generators of the cone η contain the generators of the cone ρ . Therefore there is a projection map:

$$\pi_{\rho\eta}: T_{\mathbb{Z}}(\mathbb{T}Y_\rho) \rightarrow T_{\mathbb{Z}}(\mathbb{T}Y_\eta). \quad (2.2)$$

Upon taking the reduction modulo 2 we get a map $\pi_{\sigma\tau}: \mathcal{F}_1(\mathbb{T}Y_\rho) \rightarrow \mathcal{F}_1(\mathbb{T}Y_\eta)$. For faces σ and τ in $\mathbb{T}Y_\rho$ and $\mathbb{T}Y_\eta$, respectively, such that $\tau \subset \bar{\sigma}$, Equation (2.2) restricts to a map

$$\pi_{\sigma\tau}: \mathcal{F}_1(\sigma) \rightarrow \mathcal{F}_1(\tau). \quad (2.3)$$

Definition 2.4. *Let X be a non-singular tropical hypersurface in a tropical toric variety $\mathbb{T}Y$ with defining fan Σ . The p -multi-tangent spaces of X are*

cellular cosheaves \mathcal{F}_p on X . For $\rho \in \Sigma$ and τ a face of $X_\rho := X \cap \mathbb{T}Y_\rho$ we have

$$\mathcal{F}_p(\tau) = \sum_{\substack{\tau \subset \sigma \subset X_\rho \\ \dim(\sigma) = n - \dim \rho}} \bigwedge^p \mathcal{F}_1(\sigma). \quad (2.4)$$

When $\tau \subset \sigma$, the maps of the cellular cosheaf $i_{\sigma\tau}: \mathcal{F}_p(\sigma) \rightarrow \mathcal{F}_p(\tau)$ are induced by the inclusions $\mathcal{F}_1(\sigma) \rightarrow \mathcal{F}_1(\tau)$ when σ and τ have the same sedentarity and otherwise are induced by the quotient map $\pi_{\sigma\tau}$ from (2.3).

Example 2.5. The tropical plane $X \subset \mathbb{T}P^3$ is the closure of a two dimensional fan X_0 in \mathbb{R}^3 . The fan X_0 has rays τ_1, τ_2, τ_3 , and τ_0 in respective directions $-e_1, -e_2, -e_3$, and $e_0 = e_1 + e_2 + e_3$, where e_i 's are the standard basis vectors. Every pair of rays generates a two dimensional face of $X \cap \mathbb{R}^3$, see the right hand side of Figure 2. Denote by ε_i the reduction of $e_i \bmod 2$.

Let σ_{ij} denote the two dimensional face spanned by rays τ_i and τ_j . Then $\mathcal{F}_1(\sigma_{ij}) = \mathbb{Z}_2\langle \varepsilon_i, \varepsilon_j \rangle$ and $\mathcal{F}_2(\sigma_{ij}) = \mathbb{Z}_2\langle \varepsilon_i \wedge \varepsilon_j \rangle$. For the ray τ_i , we obtain $\mathcal{F}_1(\tau_i) = \mathbb{Z}_2^3$ and $\mathcal{F}_0(\tau_i) = \mathbb{Z}_2\langle \varepsilon_i \wedge \varepsilon_j, \varepsilon_i \wedge \varepsilon_{j'} \rangle$ where $\varepsilon_i, \varepsilon_j, \varepsilon_{j'}$ form a basis of \mathbb{Z}_2^3 . For any face τ of X , we have $\mathcal{F}_0(\tau) = \mathbb{Z}_2$.

An integral version of the next lemma can also be found in [ARS, Corollary 2.16]. We include its proof here for completeness.

Lemma 2.6. *Let X be an n -dimensional non-singular tropical hypersurface of a tropical toric variety $\mathbb{T}Y$. For a face τ of X of dimension k and sedentarity ρ the polynomial defined as*

$$\chi_\tau(\lambda) := \sum_{p=0}^n (-1)^p \dim \mathcal{F}_p(\tau) \lambda^p,$$

is equal to

$$\chi_\tau(\lambda) = (1 - \lambda)^k \left[(1 - \lambda)^{n-k+1-\dim \rho} - (-\lambda)^{n-k+1-\dim \rho} \right].$$

Proof. By [Zha13, Theorem 4], the \mathbb{Z} -multi-tangent spaces $\mathcal{F}_p^{\mathbb{Z}}(\sigma)$ are isomorphic to the dual of the p -th graded piece of the Orlik-Solomon algebra of the matroid of an associated projective hyperplane arrangement \mathcal{A}_σ defined over the complex numbers. The Orlik-Solomon algebra of this arrangement is isomorphic to the cohomology ring of the complement of the arrangement in projective space so that $\text{Hom}(\mathcal{F}_p^{\mathbb{Z}}(\sigma), \mathbb{Z}) \cong H^p(\mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}_\sigma; \mathbb{Z})$. The homology groups of the complement of a complex hyperplane arrangement are torsion free so $\mathcal{F}_p(\sigma) \cong H_p(\mathbb{C}P^n \setminus \mathbb{C}\mathcal{A}_\sigma; \mathbb{Z}_2)$.

Let \mathcal{P}_{n-k} denote the $(n-k)$ -dimensional pair of pants; that is the complement of $n-k+2$ hyperplanes in general position in $\mathbb{C}P^{n-k}$. For a face τ of X of dimension k and sedentarity ρ , the complement of the associated arrangement is $\mathcal{P}_{n-k-\dim \rho} \times (\mathbb{C}^*)^k$. Therefore, we have the isomorphism $\mathcal{F}_p(\tau) \cong H_p(\mathcal{P}_{n-k-\dim \rho} \times (\mathbb{C}^*)^k; \mathbb{Z}_2)$, and by the Künneth formula for the homology groups we have

$$\chi_\tau(\lambda) = \chi_{\mathcal{P}_{n-k-\dim \rho}}(\lambda) \chi_{(\mathbb{C}^*)^k}(\lambda),$$

where $\chi_{\mathcal{P}_{n-k-\dim \rho}}(\lambda)$ and $\chi_{(\mathbb{C}^*)^k}(\lambda)$ are the Euler-Poincaré polynomials of $\mathcal{P}_{n-k-\dim \rho}$ and $(\mathbb{C}^*)^k$ respectively. Calculating the homology of these spaces shows that

$$\begin{aligned} \chi_{\mathcal{P}_{n-k}}(\lambda) &= \sum_{r=0}^{n-k-\dim \rho} (-1)^r \binom{n-k-\dim \rho+1}{r} \lambda^r \\ &= (1-\lambda)^{n-k+1-\dim \rho} - (-\lambda)^{n-k+1-\dim \rho}, \end{aligned}$$

and

$$\chi_{(\mathbb{C}^*)^k}(\lambda) = \sum_{s=0}^k (-1)^s \binom{k}{s} \lambda^s = (1-\lambda)^k.$$

The product of these two polynomials is precisely the description of $\chi_\tau(\lambda)$ in the lemma. \square

Definition 2.7. *Let X be a non-singular tropical hypersurface of a tropical toric variety $\mathbb{T}Y$. The groups of cellular q -chains with coefficients in \mathcal{F}_p are*

$$C_q(X; \mathcal{F}_p) = \bigoplus_{\dim \sigma = q} \mathcal{F}_p(\sigma).$$

The boundary maps $\partial: C_q(X; \mathcal{F}_p) \rightarrow C_{q-1}(X; \mathcal{F}_p)$ are given by the direct sums of the cosheaf maps $i_{\sigma\tau}$ for $\tau \subset \sigma$. The (p, q) -th tropical homology group is

$$H_q(X; \mathcal{F}_p) = H_q(C_\bullet(X; \mathcal{F}_p)).$$

3. REAL TROPICAL HYPERSURFACES

In this section we describe a tropical approach to Viro's primitive patchworking construction via real phase structures. For an explanation of how it relates to Viro's original construction see Remark 3.8. Section 3.1 defines real phase structures on tropical hypersurfaces and describes how to obtain the real part of the tropical hypersurface. In Subsection 3.2, we introduce the sign cosheaf on a tropical hypersurface and prove that its homology groups are isomorphic to the homology groups of the real part of a tropical variety equipped with a real phase structure.

3.1. Real phase structures and patchworking.

Definition 3.1. *A real phase structure on an n -dimensional non-singular tropical hypersurface X in a tropical toric variety $\mathbb{T}Y$ is a collection $\mathcal{E} = \{\mathcal{E}_\sigma\}_{\sigma \in \text{Facet}(X_0)}$ where $\mathcal{E}_\sigma \subset \mathbb{Z}_2^{n+1}$ is an n -dimensional affine subspace whose direction is $\mathcal{F}_1(\sigma)$. The collection \mathcal{E} must satisfy the following property:*

If τ is a face of X_0 of codimension 1, then for any facet σ adjacent to τ and any element $\varepsilon \in \mathcal{E}_\sigma$, there exists a unique facet $\sigma' \neq \sigma$ adjacent to τ such that $\varepsilon \in \mathcal{E}_{\sigma'}$.

A non-singular tropical hypersurface equipped with a real phase structure is called a non-singular real tropical hypersurface.

Example 3.2. Figure 1 depicts a real tropical line X in the tropical projective plane $\mathbb{T}P^2$. On each edge $\sigma_0, \sigma_1, \sigma_2$ of the line there is a set of vectors in \mathbb{Z}_2^2 . These vectors indicate all the points in the affine subspace \mathcal{E}_{σ_i} for a real phase structure \mathcal{E} .

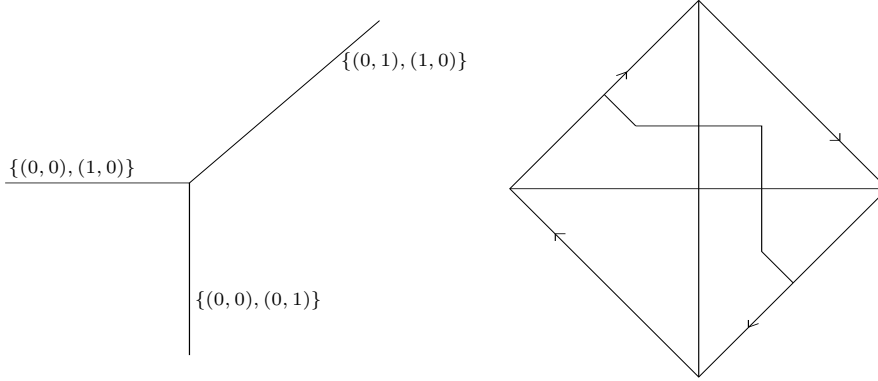


FIGURE 1. On the left is the real tropical line $X \subset \mathbb{TP}^2$ with a real phase structure \mathcal{E} from Example 3.2. On the right hand side is its real part $\mathbb{R}X_{\mathcal{E}}$ in \mathbb{RP}^2 .

The vertex of the tropical line is the only codimension one face. For $(0, 0) \in \mathcal{E}_{\sigma_1}$, we have that $(0, 0) \in \mathcal{E}_{\sigma_2}$ and $(0, 0) \notin \mathcal{E}_{\sigma_0}$. This is the condition in Definition 3.1 for the face σ_1 and the element $(0, 0)$.

Example 3.3. Recall the tropical hyperplane described in Example 2.5 and depicted on the right of Figure 2. The following collection of affine spaces forms a real phase structure on X ,

$$\mathcal{E}_{\sigma_{12}} = \langle \varepsilon_1, \varepsilon_2 \rangle, \quad \mathcal{E}_{\sigma_{13}} = \langle \varepsilon_1, \varepsilon_3 \rangle, \quad \mathcal{E}_{\sigma_{23}} = \langle \varepsilon_2, \varepsilon_3 \rangle,$$

$$\mathcal{E}_{\sigma_{01}} = \langle \varepsilon_0, \varepsilon_1 \rangle + \varepsilon_3, \quad \mathcal{E}_{\sigma_{02}} = \langle \varepsilon_0, \varepsilon_2 \rangle + \varepsilon_1, \quad \text{and} \quad \mathcal{E}_{\sigma_{03}} = \langle \varepsilon_0, \varepsilon_3 \rangle + \varepsilon_2.$$

Given a plane $P \subset \mathbb{RP}^3$, the intersection of L with the coordinate hyperplanes of \mathbb{RP}^3 defines an arrangement of real hyperplanes on $\mathbb{RP} \cong \mathbb{RP}^2$. Such is the picture on the left hand side of Figure 2. Each region of the complement of this hyperplane arrangement on $\mathbb{RP} \subset \mathbb{RP}^3$ lives in a single orthant of $\mathbb{R}^3 = \mathbb{RP}^3 \setminus \{x_0 = 0\}$. In Figure 2, each connected component of the complement of the line arrangement is labelled with the vector in \mathbb{Z}_2^3 corresponding to this orthant. Let $L_i = \{x_i = 0\} \cap P \subset \mathbb{RP}^3$ and set $p_{ij} = L_i \cap L_j$. Notice that the points contained in the affine space $\mathcal{E}_{\sigma_{ij}}$ of the real phase structure on X coincide with the collection of signs of the regions of the complement of the line arrangement which are adjacent to the point p_{ij} .

Following [GKZ08, Chapter 11], we now describe how to obtain a space homeomorphic to the real part of a toric variety by glueing together multiple copies of a tropical toric variety. Let Σ be a rational polyhedral fan in \mathbb{R}^{n+1} defining a tropical toric variety \mathbb{TY} . For every $\varepsilon \in \mathbb{Z}_2^{n+1}$, let $\mathbb{TY}(\varepsilon)$ denote a copy of \mathbb{TY} indexed by ε . Define

$$\tilde{\mathbb{TY}} := \bigsqcup_{\varepsilon \in \mathbb{Z}_2^{n+1}} \mathbb{TY}(\varepsilon) / \sim, \quad (3.1)$$

where \sim identifies strata $\mathbb{TY}_{\rho}(\varepsilon)$ and $\mathbb{TY}_{\rho}(\varepsilon')$ if and only if $\varepsilon + \varepsilon'$ is in the reduction modulo 2 of the linear space spanned by the cone ρ .

The following theorem is a direct translation of [GKZ08, Theorem 11.5.4].

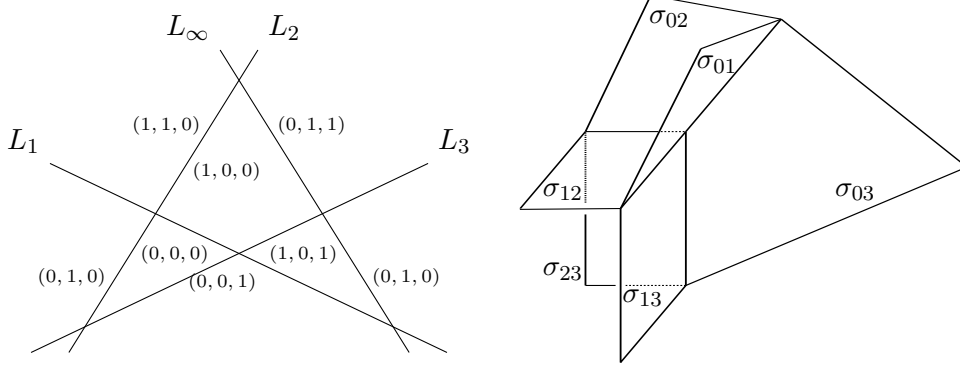


FIGURE 2. The left hand side depicts a real line arrangement in $\mathbb{R}P^2$ arising from a linear embedding $\mathbb{R}P^2 \rightarrow \mathbb{R}P^3$. On the right is a tropical plane X in $\mathbb{T}P^3$. A real phase structure on X is described in Example 3.3.

Theorem 3.4. *The topological space $\tilde{\mathbb{T}}Y$ is homeomorphic to the real point set of the toric variety $\mathbb{R}Y$.*

Example 3.5. We explicitly describe how to obtain $\mathbb{R}P^{n+1}$ by glueing together multiple copies of $\mathbb{T}P^{n+1}$. For every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n+1}) \in \mathbb{Z}_2^{n+1}$, let $\mathbb{T}P^{n+1}(\varepsilon)$ denote a copy of $\mathbb{T}P^{n+1}$ indexed by ε . Then

$$\mathbb{R}P^{n+1} \cong \bigsqcup_{\varepsilon \in \mathbb{Z}_2^{n+1}} \mathbb{T}P^{n+1}(\varepsilon) / \sim,$$

where \sim is the equivalence relation generated by identifying $x \in \mathbb{T}P^{n+1}(\varepsilon)$ and $x' \in \mathbb{T}P^{n+1}(\varepsilon')$ for $\varepsilon \neq \varepsilon'$, such that $[x_0, \dots, x_{n+1}] = [x'_0, \dots, x'_{n+1}]$ and

- if $x_0 \neq -\infty$, then there exist a unique $1 \leq j \leq n+1$ such that $\varepsilon_j \neq \varepsilon'_j$. Moreover, we must have $x_j = x'_j = -\infty$.
- if $x_0 = -\infty$, then we must have $\varepsilon_i \neq \varepsilon'_i$ for all $1 \leq i \leq n+1$.

Given a polyhedron σ of sedentarity 0 contained in $\mathbb{T}Y$ and $\varepsilon \in \mathbb{Z}_2^{n+1}$ we let σ^ε denote its copy in $\mathbb{T}Y(\varepsilon)$.

Definition 3.6. *Let X be a non-singular tropical hypersurface in a tropical toric variety $\mathbb{T}Y$ together with a real phase structure \mathcal{E} .*

The real part of X with respect to the real phase structure \mathcal{E} is denoted $\mathbb{R}X_{\mathcal{E}}$ and is the image in $\mathbb{R}Y$ of

$$\bigcup_{\substack{\text{facets of } \sigma \subset X_0 \\ \varepsilon \in \mathcal{E}_\sigma}} \bar{\sigma}^\varepsilon.$$

where $\bar{\sigma}^\varepsilon$ denotes the closure of $\sigma^\varepsilon \subset \mathbb{T}Y_0(\varepsilon)$ in $\mathbb{T}Y(\varepsilon)$.

The following theorem is the tropical reformulation of a particular case of the combinatorial version of Viro's patchworking theorem from [Vir84].

Theorem 3.7 (Viro's patchworking [Vir84]). *Let (X, \mathcal{E}) be a non-singular real tropical hypersurface with Newton polytope Δ in a non-singular tropical*

toric variety TY_Δ^o corresponding to a subfan of the dual fan of Δ . Then there exists a non-singular real algebraic hypersurface V of Y_Δ^o also with Newton polytope Δ such that

$$(\tilde{\mathrm{TY}}_\Delta^o, \mathbb{R}X_\mathcal{E}) \cong (\mathbb{R}Y_\Delta^o, \mathbb{R}V).$$

Remark 3.8. For the reader's convenience, we explain the connection between the tropical version of primitive patchworking and Viro's original formulation as described in [Vir84, Ite93, Ris93].

The input of Viro's original formulation of primitive patchworking is a regular subdivision of a lattice polytope $\Delta \subset \mathbb{R}^{n+1}$, whose normal fan is unimodular, together with a choice of sign $\delta_i \in \{+, -\}$ for each lattice point $i \in \Delta \cap \mathbb{Z}^{n+1}$.

Given a tropical hypersurface $X \in \mathbb{R}^{n+1}$, its dual subdivision is a regular subdivision of Δ which, by definition, is primitive if X is non-singular. Every edge e of the dual subdivision of Δ is dual to a facet σ_e of X . From a real phase structure \mathcal{E} on X we produce a collection of signs δ_i for all $i \in \Delta \cap \mathbb{Z}^{n+1}$ as follows. Two vertices of an edge e of the subdivision of Δ are assigned different signs if and only if \mathcal{E}_{σ_e} contains the origin $(0, \dots, 0)$ in \mathbb{Z}_2^{n+1} . For more details we refer to [Ren17, Lemma 1]. Upon choosing the sign of one lattice point in Δ arbitrarily, this rule determines a collection of signs for each integer point in Δ .

From the subdivision of Δ and the assignment of signs to all lattice points in Δ , Viro's construction builds a polyhedral complex in the following way. For $\varepsilon \in \mathbb{Z}_2^{n+1}$, let $\Delta(\varepsilon)$ denote the symmetric copy of Δ in the orthant of \mathbb{R}^{n+1} corresponding to ε . Then define

$$\tilde{\Delta} := \bigcup_{\varepsilon \in \mathbb{Z}_2^{n+1}} \Delta(\varepsilon) / \sim, \quad (3.2)$$

where the equivalence relation \sim is the same as described for the tropical toric variety coming from the fan Σ which is a subfan of the dual fan of Δ .

The triangulation of Δ induced by X induces a symmetric triangulation of $\Delta(\varepsilon)$. Moreover, the sign choices $\delta_i \in \{+, -\}$ for $i \in \Delta \cap \mathbb{Z}^{n+1}$ induce choices of signs for $\Delta(\varepsilon)$ for all $\varepsilon \in \mathbb{Z}_2^{n+1}$ by way of the following rule: For $i_1, \dots, i_n \in \Delta(\varepsilon)$

$$\delta_{(i_1, \dots, i_n)} = \left(\prod_{j=1}^{n+1} (-1)^{\varepsilon_j i_j} \right) \delta_{(|i_1|, \dots, |i_{n+1}|)}.$$

In other words, when passing from a lattice point to its reflection in a coordinate hyperplane, the sign is preserved if the distance from the lattice point to the hyperplane is even, and the sign is changed if the distance is odd.

For a simplex T in the subdivision of $\Delta(\varepsilon)$ let S_T denote the convex hull of the midpoints of the edges of T having endpoints of opposite signs. Denote by S the union of all such S_T considered in the quotient to $\tilde{\Delta}$ as in (3.2). Then S is an n -dimensional piecewise-linear manifold contained in $\tilde{\Delta}$. It turns out that pairs $(\tilde{\mathrm{TY}}, \mathbb{R}X_\mathcal{E})$ and $(\tilde{\Delta}, S)$ are combinatorially isomorphic and homeomorphic. Thus the two formulations of patchworking are equivalent. For more details see [Ren17, Lemma 1].

From here a polynomial defining the hypersurface V from Theorem 3.7 can be written down explicitly. The defining polynomial of V is

$$\mathbf{F}_t(x) = \sum_{(i_1, \dots, i_{n+1}) \in \mathbb{Z}^{n+1} \cap \Delta} \delta_{i_1, \dots, i_n} \left(x_1^{i_1} \cdots x_{n+1}^{i_{n+1}} x_0^{d - \sum i_j} \right) t^{-a_{(i_1, \dots, i_n)}}, \quad (3.3)$$

where the $a_{(i_1, \dots, i_n)}$'s are the coefficients from the tropical polynomial in (2.1) and $t > 0$ is a sufficiently large real number.

Definition 3.9. A real algebraic hypersurface V in a toric variety Y is called near a non-singular tropical limit if it is defined by a polynomial $\mathbf{F}_t(x)$ of the form (3.3) for t sufficiently large coming from a non-singular tropical hypersurface X with a real phase structure \mathcal{E} and the fan defining Y is a subfan of the fan dual to the Newton polytope of \mathbf{F}_t

In particular, a hypersurface near a non-singular tropical limit X with real phase structure \mathcal{E} will satisfy the homeomorphism of pairs from Theorem 3.7

$$(\tilde{\mathbb{T}}Y_\Delta^o, \mathbb{R}X_\mathcal{E}) \cong (\mathbb{R}Y_\Delta^o, \mathbb{R}V).$$

3.2. The sign cosheaf. Let X be a non-singular real tropical hypersurface equipped with a real phase structure \mathcal{E} . By definition, for any facet σ of X of sedentarity 0, the real phase structure \mathcal{E} gives an affine space \mathcal{E}_σ of direction $\mathcal{F}_1(\sigma)$. Let us extend the real phase structure to facets of higher sedentarity as follows.

Recall the definition of the map $\pi_{\sigma\tau}: \mathcal{F}_1(\sigma) \rightarrow \mathcal{F}_1(\tau)$ from (2.3) via the projection maps $\pi_{\rho\eta}: \mathcal{F}_1(\mathbb{T}Y_\rho) \rightarrow \mathcal{F}_1(\mathbb{T}Y_\eta)$ when σ and τ are in strata $\mathbb{T}Y_\rho$ and Y_η , respectively. Also recall that the parent face of a face τ of X of dimension k and sedentarity ρ is the unique face σ of X of zero sedentarity and dimension $k + \dim \rho$ such that τ is in the boundary of σ . Let τ_0 denote the parent face of a facet τ in X_ρ so that $\pi_{\tau_0\tau}(\tau_0) = \tau$. Define $\mathcal{E}_\tau = \pi_{\tau_0\tau}(\mathcal{E}_{\tau_0})$. Notice that \mathcal{E}_τ is an affine space of $\mathcal{F}_1(\mathbb{T}Y_\rho)$ which is parallel to $\mathcal{F}_1(\tau)$.

Example 3.10. The tropical line in $\mathbb{T}P^2$ from Example 3.2 contains three points of non-zero sedentarity. The projection of the affine vector spaces $\mathcal{E}_{\sigma_1}, \mathcal{E}_{\sigma_2}$ for the horizontal and vertical edges are simply $0 \in \mathcal{F}_1(\mathbb{T}P_{\{1\}}^{n+1})$ and $0 \in \mathcal{F}_1(\mathbb{T}P_{\{2\}}^{n+1})$, respectively. For the diagonal edge we obtain $\mathcal{E}_{\sigma_0} = 1 \in \mathcal{F}_1(\mathbb{T}P_{\{0\}}^{n+1})$. Here both $\mathcal{F}_1(\mathbb{T}P_{\{1\}}^{n+1})$ and $\mathcal{F}_1(\mathbb{T}P_{\{2\}}^{n+1})$ are one dimensional vector spaces over \mathbb{Z}_2 , so we identify them with \mathbb{Z}_2 in the unique way.

Example 3.11. The real phase structure on the tropical plane $X \subset \mathbb{T}P^3$ from Example 3.3 can be extended to the facets of all strata X_I for I a proper subset of $\{0, \dots, n+1\}$. If $|I| = 1$ then X_I is a tropical line in as in Example 3.2. Consider for example $I = \{1\}$ and the facet $\rho_2 = \overline{\sigma_{12}} \cap X_{\{1\}}$. The projection $\pi_{\sigma_{12}\rho_2}$ has kernel the first coordinate direction and therefore $\mathcal{E}_{\rho_2} = \langle e_1, e_2 \rangle / \langle e_1 \rangle \subset \mathcal{F}_1(\mathbb{T}P_{\{1\}}^3)$. Furthermore, $X_{\{12\}}$ is a point and $\mathcal{E}_{X_{\{12\}}} = 0$.

For any facet σ of any sedentarity, we define the abstract vector space $\mathcal{S}_\mathcal{E}(\sigma)$ with basis in bijection with the elements of \mathcal{E}_σ ,

$$\mathcal{S}_\mathcal{E}(\sigma) = \mathbb{Z}_2 \langle w_\varepsilon \mid \varepsilon \in \mathcal{E}_\sigma \rangle.$$

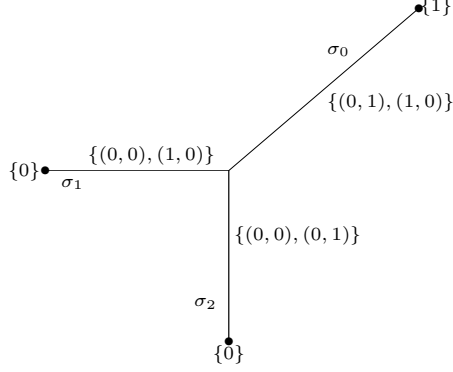


FIGURE 3. The extension of the real phase structure on the tropical line from Example 3.2 to faces of non-zero sedentarity.

The vector space $\mathcal{S}_{\mathcal{E}}(\sigma)$ is a linear subspace of the abstract vector space $\mathbb{Z}_2 \langle w_{\varepsilon} \mid \varepsilon \in \mathcal{F}_1(\mathbb{T}Y_{\text{sed}(\sigma)}) \rangle$.

Definition 3.12. *Let X be a non-singular real tropical hypersurface equipped with a real phase structure \mathcal{E} . The sign cosheaf $\mathcal{S}_{\mathcal{E}}$ on X is defined by*

$$\mathcal{S}_{\mathcal{E}}(\tau) = \sum_{\substack{\sigma \mid \tau \subset \sigma \\ \text{sed}(\sigma) = \text{sed}(\tau)}} \mathcal{S}_{\mathcal{E}}(\sigma), \quad (3.4)$$

where the sum is over the facets σ of $X_{\text{sed}(\tau)}$ containing τ . The maps of the cellular cosheaf

$$i_{\sigma\tau} : \mathcal{S}_{\mathcal{E}}(\sigma) \rightarrow \mathcal{S}_{\mathcal{E}}(\tau)$$

are induced by natural inclusions when σ and τ are of the same sedentarity and otherwise are induced by the quotients $\pi_{\sigma\tau}$ from (2.3) composed with inclusions.

Example 3.13. We describe some of the vector spaces $\mathcal{S}_{\mathcal{E}}(\tau)$ and maps between them for the real phase structure on the tropical plane $X \subset \mathbb{T}P^3$ from Example 3.3.

For the facets σ_{01} , σ_{12} , and σ_{13} of sedentarity 0 from Example 3.3 we have,

$$\mathcal{S}_{\mathcal{E}}(\sigma_{01}) = \langle w_{\varepsilon_3}, w_{\varepsilon_1+\varepsilon_3}, w_{\varepsilon_1+\varepsilon_2}, w_{\varepsilon_2} \rangle,$$

$$\mathcal{S}_{\mathcal{E}}(\sigma_{12}) = \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_2}, w_{\varepsilon_1+\varepsilon_2} \rangle, \text{ and}$$

$$\mathcal{S}_{\mathcal{E}}(\sigma_{13}) = \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_3}, w_{\varepsilon_1+\varepsilon_3} \rangle.$$

Consider the one dimensional face τ_1 of sedentarity 0 and in direction e_1 . Then we have

$$\mathcal{S}_{\mathcal{E}}(\tau_1) = \langle w_{\varepsilon} \mid \varepsilon \in \mathcal{E}_{\sigma_{12}} \cup \mathcal{E}_{\sigma_{13}} \cup \mathcal{E}_{\sigma_{01}} \rangle = \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_2}, w_{\varepsilon_1+\varepsilon_2}, w_{\varepsilon_3}, w_{\varepsilon_1+\varepsilon_3} \rangle,$$

and there is an injection $i_{\sigma_{12}\tau_1} : \mathcal{S}_{\mathcal{E}}(\sigma_{12}) \rightarrow \mathcal{S}_{\mathcal{E}}(\tau_1)$.

For the face ρ_2 from Example 3.11 we have

$$\mathcal{S}_{\mathcal{E}}(\rho_2) = \langle w_0, w_{\varepsilon_2} \rangle \subset \mathbb{Z}_2 \langle w_{\varepsilon} \mid \varepsilon \in \mathcal{F}_1(\mathbb{T}P_{\{1\}}^{n+1}) \rangle.$$

The map $i_{\sigma_{12}\rho_2} : \mathcal{S}_{\mathcal{E}}(\sigma_{12}) \rightarrow \mathcal{S}_{\mathcal{E}}(\rho_2)$ has kernel generated by w_{ε_1} and $w_{\varepsilon_1+\varepsilon_2}$. These bases elements correspond to the points in the kernel of the map $\pi_{\sigma_{12}\rho_2} : \mathcal{F}_1(\sigma_{12}) \rightarrow \mathcal{F}_1(\rho_2)$.

Lemma 3.14. *Let X be a non-singular real tropical hypersurface equipped with a real phase structure \mathcal{E} . If τ is a face of dimension k of the stratum X_ρ , the dimension of $\mathcal{S}_\mathcal{E}(\tau)$ is*

$$\dim \mathcal{S}_\mathcal{E}(\tau) = 2^{\dim X_\rho + 1} - 2^k.$$

Proof. It follows from the definition that $\mathcal{S}_\mathcal{E}(\tau)$ is the \mathbb{Z}_2 -vector space generated by w_ε for elements ε in the set

$$\mathcal{E}(\tau) := \bigcup_{\substack{\sigma \mid \tau \subset \sigma \\ \text{sed}(\sigma) = \text{sed}(\tau)}} \mathcal{E}(\sigma),$$

where the union is taken over the facets σ of $X_{\text{sed}(\tau)}$ containing τ . It follows from [Ite97, Proposition 3.1], which is formulated in the original description of Viro's patchworking described in Remark 3.8, that $\#(\mathcal{E}(\tau)) = 2^{\dim X_\rho + 1} - 2^k$. \square

Corollary 3.15. *For any face τ of X , we have*

$$\sum_{0 \leq p \leq \dim \tau} \dim \mathcal{F}_p(\tau) = \dim \mathcal{S}_\mathcal{E}(\tau).$$

Proof. Let τ be a face of dimension k of the stratum X_ρ . Since $\dim X_\rho = n - k - \dim \rho$, it follows from Lemma 2.6 that

$$\chi_\tau(\lambda) = (1 - \lambda)^k \left[(1 - \lambda)^{\dim X_\rho + 1} - (-\lambda)^{\dim X_\rho + 1} \right],$$

and then

$$\sum_{0 \leq p \leq \dim \tau} \dim \mathcal{F}_p = \chi_\tau(-1) = 2^{\dim X_\rho + 1} - 2^k = \dim \mathcal{S}_\mathcal{E}(\tau).$$

\square

Definition 3.16. *Let X be a non-singular tropical hypersurface equipped with a real phase structure \mathcal{E} . The groups of cellular q -chains with coefficients in $\mathcal{S}_\mathcal{E}$ are*

$$C_q(X; \mathcal{S}_\mathcal{E}) = \bigoplus_{\dim \sigma = q} \mathcal{S}_\mathcal{E}(\sigma).$$

The boundary maps $\partial: C_q(X; \mathcal{S}_\mathcal{E}) \rightarrow C_{q-1}(X; \mathcal{S}_\mathcal{E})$ are given by the direct sums of the cosheaf maps $i_{\sigma\tau}$ for $\tau \subset \sigma$. The real tropical homology groups are

$$H_q(X; \mathcal{S}_\mathcal{E}) := H_q(C_\bullet(X; \mathcal{S}_\mathcal{E})).$$

For a non-singular tropical hypersurface X equipped with a real phase structure \mathcal{E} , we relate the homology of the cellular cosheaf $\mathcal{S}_\mathcal{E}$ to the homology of the real part $\mathbb{R}X_\mathcal{E}$.

Proposition 3.17. *Let X be a non-singular tropical hypersurface equipped with a real phase structure \mathcal{E} . There is an isomorphism of chain complexes*

$$C_\bullet(\mathbb{R}X_\mathcal{E}; \mathbb{Z}_2) \cong C_\bullet(X; \mathcal{S}_\mathcal{E}).$$

It follows that $H_q(\mathbb{R}X_\mathcal{E}; \mathbb{Z}_2) \cong H_q(X; \mathcal{S}_\mathcal{E})$ for all q .

Proof. Firstly, we have

$$C_q(\mathbb{R}X_{\mathcal{E}}; \mathbb{Z}_2) = \bigoplus_{\substack{\tilde{\sigma} \in \mathbb{R}X \\ \dim \tilde{\sigma} = q}} \mathbb{Z}_2 \langle \tilde{\sigma} \rangle$$

and the differential

$$\partial: C_q(\mathbb{R}X_{\mathcal{E}}; \mathbb{Z}_2) \rightarrow C_{q-1}(\mathbb{R}X_{\mathcal{E}}; \mathbb{Z}_2) \quad (3.5)$$

is given componentwise by maps $\tilde{\sigma} \rightarrow \sum_{\tilde{\tau} \in \partial \tilde{\sigma}} \tilde{\tau}$. We can rewrite these chain groups by summing instead over the faces of X

$$C_q(\mathbb{R}X_{\mathcal{E}}; \mathbb{Z}_2) = \bigoplus_{\substack{\sigma \in X \\ \dim \sigma = q}} \left(\bigoplus_{\substack{\tilde{\sigma} \in \mathbb{R}X_{\mathcal{E}} \\ \exists \varepsilon | \sigma^\varepsilon = \tilde{\sigma}}} \mathbb{Z}_2 \langle \tilde{\sigma} \rangle \right) = \bigoplus_{\substack{\sigma \in X \\ \dim \sigma = q}} \mathcal{S}_{\mathcal{E}}(\sigma).$$

By Definition 3.16 we have $C_q(\mathbb{R}X_{\mathcal{E}}; \mathbb{Z}_2) \cong C_q(X; \mathcal{S}_{\mathcal{E}})$ for all q . Also by the definition of the maps $i_{\sigma\tau}: \mathcal{S}_{\mathcal{E}}(\sigma) \rightarrow \mathcal{S}_{\mathcal{E}}(\tau)$ for σ and τ of dimensions q and $q-1$ respectively, we see that the differentials of the chain complex $C_{\bullet}(X; \mathcal{S}_{\mathcal{E}})$ coincide with the differentials in (3.5) above. Therefore the chain complexes are isomorphic and the isomorphism of homology groups follows. \square

4. A FILTRATION OF THE CHAIN COMPLEX

We begin by describing the augmentation filtration highlighted by Quillen [Qui68] on the abstract vector space $\mathbb{Z}_2 \langle w_{\varepsilon} \mid \varepsilon \in V \rangle$, where V is a vector space defined over \mathbb{Z}_2 . This same filtration was used in [BFMvH06] and [How08] to give criteria for a toric variety to be maximal in the sense of the Smith-Thom inequality. We then adapt the filtration to the situation when V is an affine subspace and not only a vector space, and apply this to filter first the vector spaces $\mathcal{S}(\sigma)$ where σ is a top dimensional face of a tropical hypersurface. This produces filtrations of the spaces $\mathcal{S}(\tau)$ for any face τ . Finally, we show that this produces a filtration of the chain complex $C_{\bullet}(X; \mathcal{S})$. Throughout this section X will denote an n -dimensional tropical hypersurface with Newton polytope Δ contained in the tropical toric variety $\mathbb{T}Y_{\Delta}$.

4.1. The augmentation filtration. Let V be a vector space defined over \mathbb{Z}_2 . The vector space $\mathbb{Z}_2 \langle w_{\varepsilon} \mid \varepsilon \in V \rangle$ can also be considered as the group algebra $\mathbb{Z}_2[V]$, where the algebra structure is given by $w_{\varepsilon}w_{\eta} = w_{\varepsilon+\eta}$. Any element of $\mathbb{Z}_2[V]$ can be written as $\sum a_i w_{\varepsilon_i}$, where $a_i \in \mathbb{Z}_2$ and $\varepsilon_i \in V$. For a subset $G \subset V$ we define

$$w_G := \sum_{\varepsilon \in G} w_{\varepsilon}.$$

The *augmentation morphism* is a morphism of algebras given by

$$\begin{aligned} \varphi : \mathbb{Z}_2[V] &\rightarrow \mathbb{Z}_2 \\ \sum a_i w_{\varepsilon_i} &\mapsto \sum a_i. \end{aligned}$$

The *augmentation ideal* of V , denoted \mathcal{I}_V is the kernel of this morphism. For all $p \geq 1$, define

$$\mathcal{I}_V^p = \{w_1 \cdots w_p \mid w_1, \dots, w_p \in \mathcal{I}_V\}.$$

Notice that $\mathcal{I}_V = \mathcal{I}_V^1$. Since \mathcal{I}_V is the kernel of a homomorphism it is an ideal, and for all $p \geq 1$ we have $\mathcal{I}_V^{p+1} \subset \mathcal{I}_V^p$. We obtain a filtration of $\mathbb{Z}_2[V]$:

$$\cdots \subset \mathcal{I}_V^p \subset \cdots \subset \mathcal{I}_V^2 \subset \mathcal{I}_V \subset \mathbb{Z}_2[V]. \quad (4.1)$$

The following lemma and proposition are generalisations of [BFMvH06, Lemma 6.1 and Proposition 6.1], but we recall their proofs for convenience. The Grassmannian of p dimensional vector subspaces of V is denoted by $\text{Gr}_p(V)$.

Lemma 4.1. *For $p \geq 1$, the power \mathcal{I}_V^p is additively generated by $\{w_F \mid F \in \text{Gr}_p(V)\}$.*

Proof. The proof is by induction on p . If $p = 1$, an element $w \in \mathcal{I}_V$ is a sum of an even number of elements $w_{\varepsilon_1}, \dots, w_{\varepsilon_{2l}} \in \mathbb{Z}_2[V]$. Then we can write

$$w = \sum_{i \mid w_{\varepsilon_i} \neq w_0} w_{\{0, \varepsilon_i\}},$$

and $\{0, \varepsilon_i\}$ is a 1-dimensional subspace of V .

Now assume that the claim is true for p . Then \mathcal{I}_V^{p+1} is additively generated by products $w_{F'}w_{\{0, \varepsilon\}}$, where F' is a vector subspace of V of dimension p and $\varepsilon \in V$. If $\varepsilon \in F'$ then $w_{F'}w_{\{0, \varepsilon\}} = 0$ and otherwise $w_{F'}w_{\{0, \varepsilon\}} = w_F$, where $F = \langle F', \varepsilon \rangle$ is a vector subspace of V of dimension $p + 1$. This completes the proof. \square

Corollary 4.2. *If $p > \dim V$, then $\mathcal{I}_V^p = 0$, and the filtration in (4.1) is a filtration of length $\dim V$.*

Proposition 4.3. *For all p there is an isomorphism of vector spaces $\mathcal{I}_V^p / \mathcal{I}_V^{p+1} \cong \bigwedge^p V$.*

Proof. Consider the map

$$\begin{aligned} f &: V \rightarrow \mathbb{Z}_2[V] \\ \varepsilon &\mapsto w_0 + w_\varepsilon \end{aligned}$$

This map is not a homomorphism but $f(V) \subset \mathcal{I}_V$. One can consider the composition of f with the quotient map $\mathcal{I}_V \rightarrow \mathcal{I}_V / \mathcal{I}_V^2$, and we denote this map by f_1 . Then $f: V \rightarrow \mathcal{I}_V / \mathcal{I}_V^2$ is a homomorphism since

$$f(\varepsilon) + f(\eta) + f(\varepsilon + \eta) = (w_0 + w_\varepsilon)(w_0 + w_\eta) \in \mathcal{I}_V^2.$$

For all p , define the map

$$\begin{aligned} f_p &: V^p \rightarrow \mathcal{I}_V^p / \mathcal{I}_V^{p+1} \\ (\varepsilon_1, \dots, \varepsilon_p) &\mapsto f(\varepsilon_1) \cdots f(\varepsilon_p), \end{aligned}$$

where $(\mathcal{I}_V / \mathcal{I}_V^2)^p$ is naturally identified with $\mathcal{I}_V^p / \mathcal{I}_V^{p+1}$, and $V^1 = V$. These maps are p -linear and alternating since

$$(f(\varepsilon))^2 = [(w_0 + w_\varepsilon)^2] = w_0^2 + w_\varepsilon^2 = w_0 + w_0 = 0,$$

and it descends to a linear map

$$\begin{aligned} \hat{f}_p &: \bigwedge^p V \rightarrow \mathcal{I}_V^p / \mathcal{I}_V^{p+1} \\ \varepsilon_1 \wedge \cdots \wedge \varepsilon_p &\mapsto f(\varepsilon_1) \cdots f(\varepsilon_p). \end{aligned}$$

By Lemma 4.1, the power \mathcal{I}_V^p is generated by w_F for all vector subspaces F of V of dimension p , so the maps \hat{f}_p are surjective. Since $\mathcal{I}_V = \ker \varphi$, its dimension is $2^{\dim V} - 1$, and the set of generators $\{w_F \mid F \in \text{Gr}_1(V)\}$ is a basis of \mathcal{I}_V . Therefore,

$$\sum_{p=1}^{\dim V} \dim \mathcal{I}_V^p / \mathcal{I}_V^{p+1} = \dim \mathcal{I}_V = 2^{\dim V} - 1$$

and

$$\sum_{p=1}^{\dim V} \dim \bigwedge^p V = 2^{\dim V} - 1,$$

so all the map \hat{f}_p are isomorphisms. \square

4.2. The filtration of the sign cosheaf. Here we adapt the augmentation filtration from the last subsection to filter the vector spaces $\mathcal{S}(\tau)$ for τ a face of a tropical hypersurface X equipped with a real phase structure.

Let σ be a facet of X_ρ . By choosing a vector θ in the affine hyperplane \mathcal{E}_σ , we obtain an identification

$$\psi_\theta: \mathcal{S}_\mathcal{E}(\sigma) \cong \mathbb{Z}_2[\vec{\mathcal{E}}_\sigma],$$

where $\vec{\mathcal{E}}_\sigma$ denotes the vector subspace of $\mathcal{F}_1(\text{TY}_\rho)$ parallel to \mathcal{E}_σ of dimension $n + 1 - \dim \rho$. Transporting the augmentation filtration of $\mathbb{Z}_2[\vec{\mathcal{E}}_\sigma]$ by the isomorphism ψ_θ , one obtains a filtration of $\mathcal{S}_\mathcal{E}(\sigma)$

$$0 = \mathcal{K}_{n+1-\dim \rho}(\sigma) \subset \mathcal{K}_n(\sigma) \subset \cdots \subset \mathcal{K}_0(\sigma) = \mathcal{S}_\mathcal{E}(\sigma).$$

In the following lemma, we show that this filtration does not depend on the choice of an element θ in \mathcal{E}_σ we choose. Let $\text{Aff}_p(\mathcal{E}_\sigma)$ denote the space of all p -dimensional affine subspaces of \mathcal{E}_σ .

Lemma 4.4. *For any facet σ of X_ρ one has*

$$\mathcal{K}_p(\sigma) = \langle w_G \mid G \in \text{Aff}_p(\mathcal{E}_\sigma) \rangle$$

Proof. Recall that we choose an element θ of the affine hyperplane \mathcal{E}_σ . By Lemma 4.1, the vector space $\mathcal{K}_p(\sigma)$ is generated by the w_G for all affine subspaces G of \mathcal{E}_σ of dimension p passing through θ . Let G be an affine subspace of \mathcal{E}_σ of dimension p not passing through θ , and let H be any affine hyperplane H of G . Since we are over \mathbb{Z}_2 , one has $G = H \cup H'$, where H' is the affine hyperplane of G parallel to and disjoint from H . Denote by H_1 the affine subspace of \mathcal{E}_σ parallel to H and passing through θ . Then one has

$$w_G = w_{H \cup H_1} + w_{H' \cup H_1},$$

and the lemma is proved. \square

Definition 4.5. *Let X be a real tropical hypersurface with real phase structure \mathcal{E} . For all p , we define a collection of cosheaves \mathcal{K}_p on X . For τ a face of X of sedentarity ρ , let*

$$\mathcal{K}_p(\tau) = \sum_{\sigma \supset \tau} \mathcal{K}_p(\sigma) \subset \mathcal{S}_\mathcal{E}(\tau)$$

where the sum is over facets σ of X_ρ . The cosheaf maps $\mathcal{K}_p(\tau_1) \rightarrow \mathcal{K}_p(\tau_2)$ for $\tau_2 \subset \tau_1$ are the restrictions of the maps $i_{\tau_1\tau_2}: \mathcal{S}_\mathcal{E}(\tau_1) \rightarrow \mathcal{S}_\mathcal{E}(\tau_2)$.

For each face τ of X we obtain a filtration of $\mathcal{S}_\mathcal{E}(\tau)$ given by

$$\mathcal{K}_n(\tau) \subset \cdots \subset \mathcal{K}_2(\tau) \subset \mathcal{K}_1(\tau) \subset \mathcal{K}_0(\tau) = \mathcal{S}_\mathcal{E}(\tau). \quad (4.2)$$

If $\tau_2 \subset \tau_1$, then the facets adjacent to τ_1 are a subset of the facets adjacent to τ_2 . It follows that $i_{\tau_1\tau_2}(\mathcal{K}_p(\tau_1)) \subset \mathcal{K}_p(\tau_2)$ so that the cosheaf maps for \mathcal{K}_p are well-defined.

Example 4.6. Any two vectors $w_{\varepsilon_1}, w_{\varepsilon_2}$ for $\varepsilon_1, \varepsilon_2 \in \mathcal{F}_1(\text{TY}_\rho)$ are on an affine line. Moreover, these are the only points over \mathbb{Z}_2 contained on the line. Therefore, every facet σ of sedentarity ρ of a real tropical hypersurface X , the vector subspace $\mathcal{K}_1(\sigma) \subset \mathcal{S}_\mathcal{E}(\sigma)$ is generated by $w_{\varepsilon_1} + w_{\varepsilon_2}$, for any vectors $\varepsilon_1, \varepsilon_2 \in \mathcal{E}(\sigma)$. This implies that $\mathcal{K}_1(\sigma)$ is the hyperplane inside $\mathcal{S}_\mathcal{E}(\sigma)$ defined by the linear form $\sum_{\varepsilon \in \mathcal{E}_\sigma} x_\varepsilon = 0$ where the x_ε 's form a dual basis to the w_ε 's.

For a face τ of sedentarity ρ and higher codimension, we will prove that the space $\mathcal{K}_1(\tau)$ is also an hyperplane inside $\mathcal{S}_\mathcal{E}(\tau)$ defined by the linear form

$$\sum_{\substack{\varepsilon \in \mathcal{E}_\sigma \\ \sigma \supset \tau}} x_\varepsilon = 0,$$

where the σ considered in the summation are facets of $X_{\text{sed}(\tau)}$.

By the description of $\mathcal{K}_1(\tau)$ in terms of generators we see that it is contained in the claimed hyperplane. To prove the reverse inclusion, it is enough to show that $w_\varepsilon + w_{\varepsilon'} \in \mathcal{K}_1(\tau)$, for any $w_\varepsilon, w_{\varepsilon'} \in \mathcal{S}_\mathcal{E}(\tau)$. Let σ and σ' be two facets of X_ρ containing τ such that $\varepsilon \in \mathcal{E}_\sigma$ and $\varepsilon' \in \mathcal{E}_{\sigma'}$. If $\sigma = \sigma'$, then $w_\varepsilon + w_{\varepsilon'} \in \mathcal{K}_1(\sigma)$ and the claim is true.

Suppose the facets σ, σ' are distinct. Since they intersect in τ , they necessarily intersect in a vertex v of the same sedentarity as τ . There are $n - \dim \rho + 2$ rays of X_ρ containing the vertex v , and the facets of X_ρ containing v correspond to the cones generated by subsets of size $n - \dim \rho$ of these rays. Therefore, any two distinct cones σ, σ' intersecting in τ must have at least $n - \dim \rho - 2$ rays in common. It follows that σ and σ' necessarily intersect in a face of codimension either one or two.

If it is a face of codimension one, then by the condition on a real phase structure in Definition 3.1, there exists $\varepsilon_1 \in \mathcal{E}_\sigma \cap \mathcal{E}_{\sigma'}$. But then $w_\varepsilon + w_{\varepsilon'} = (w_\varepsilon + w_{\varepsilon_1}) + (w_{\varepsilon_1} + w_{\varepsilon_2}) \in \mathcal{K}_1(\tau)$. If $\sigma \cap \sigma'$ is a face of codimension two, then there exists a facet σ'' such that $\tau \subset \sigma''$, the intersections $\sigma \cap \sigma''$ and $\sigma' \cap \sigma''$ are of codimension 1 and $\sigma \cap \sigma' \cap \sigma'' = \sigma \cap \sigma'$. This follows from the above description of facets adjacent to a vertex v in terms of the span of subsets of the rays at v . Then, there exist $\varepsilon_1 \in \mathcal{E}_\sigma \cap \mathcal{E}_{\sigma''}$ and $\varepsilon_2 \in \mathcal{E}_{\sigma''} \cap \mathcal{E}_{\sigma'}$ such that

$$w_\varepsilon + w_{\varepsilon'} = (w_\varepsilon + w_{\varepsilon_1}) + (w_{\varepsilon_1} + w_{\varepsilon_2}) + (w_{\varepsilon_2} + w_{\varepsilon'}) \in \mathcal{K}_1(\tau).$$

This shows that $\mathcal{K}_1(\tau)$ is also a hyperplane inside $\mathcal{S}_\mathcal{E}(\tau)$ for all faces τ .

Example 4.7. For the real tropical plane from Example 3.3 we describe the filtration in (4.2) for some faces. Following Example 4.6, for every facet σ_{ij} of X , the vector space $\mathcal{K}_1(\sigma_{ij})$ is of codimension one in $\mathcal{S}_\mathcal{E}(\sigma_{ij})$. Moreover, the vector space $\mathcal{S}_\mathcal{E}(\sigma_{ij})$ is two dimensional. Therefore, the only

element in $\text{Aff}_2(\mathcal{S}_\mathcal{E}(\sigma))$ is the whole vector space itself. This implies that $\mathcal{K}_2(\sigma_{ij}) = \langle w_{\mathcal{S}_\mathcal{E}(\sigma_{ij})} \rangle$, in particular it is one dimensional. For instance for σ_{12} we have,

$$\begin{aligned}\mathcal{S}_\mathcal{E}(\sigma_{12}) &= \langle w_0, w_{\varepsilon_1}, w_{\varepsilon_2}, w_{\varepsilon_1+\varepsilon_2} \rangle, \\ \mathcal{K}_1(\sigma_{12}) &= \langle w_0 + w_{\varepsilon_1}, w_0 + w_{\varepsilon_2}, w_0 + w_{\varepsilon_1+\varepsilon_2} \rangle, \text{ and} \\ \mathcal{K}_2(\sigma_{12}) &= \langle w_0 + w_{\varepsilon_1} + w_{\varepsilon_2} + w_{\varepsilon_1+\varepsilon_2} \rangle.\end{aligned}$$

For the face τ_1 from Example 3.13, since $\mathcal{K}_1(\tau_1)$ is generated by $\mathcal{K}_1(\sigma_{01})$, $\mathcal{K}_1(\sigma_{12})$, and $\mathcal{K}_1(\sigma_{13})$, we have

$$\mathcal{K}_1(\tau_1) = \langle w_0 + w_{\varepsilon_1}, w_0 + w_{\varepsilon_2}, w_0 + w_{\varepsilon_1+\varepsilon_2}, w_0 + w_{\varepsilon_3}, w_0 + w_{\varepsilon_1+\varepsilon_3} \rangle.$$

For $p = 2$ we have

$$\mathcal{K}_2(\tau_1) = \langle w_{\mathcal{S}_\mathcal{E}(\sigma_{01})}, w_{\mathcal{S}_\mathcal{E}(\sigma_{12})} \rangle,$$

since $w_{\mathcal{S}_\mathcal{E}(\sigma_{01})} + w_{\mathcal{S}_\mathcal{E}(\sigma_{12})} + w_{\mathcal{S}_\mathcal{E}(\sigma_{13})} = 0$.

Lemma 4.8. *For any face τ of X , there is an isomorphism*

$$\mathcal{F}_p(\tau) \cong \mathcal{K}_p(\tau) / \mathcal{K}_{p+1}(\tau).$$

Proof. Extend the map \widehat{f}_p from the proof of Proposition 4.3 to a map defined on any face τ of X :

$$\widehat{f}_p(\tau) : \mathcal{F}_p(\tau) \longrightarrow \mathcal{K}_p(\tau) / \mathcal{K}_{p+1}(\tau).$$

This map is again surjective and it follows from Corollary 3.15 that it is a isomorphism. \square

Using the isomorphism from Lemma 4.8, we can define homomorphisms $\mathcal{K}_p(\tau) \rightarrow \mathcal{F}_p(\tau)$, which we call the Viro homomorphisms following [DK00a].

Definition 4.9. *For any face τ of X , define the Viro homomorphisms $bv_p : \mathcal{K}_p(\tau) \rightarrow \mathcal{F}_p(\tau)$ as the composition of the quotient map*

$$\mathcal{K}_p(\tau) \rightarrow \mathcal{K}_p(\tau) / \mathcal{K}_{p+1}(\tau)$$

with the inverse of the isomorphism $\widehat{f}_p(\tau)$.

If τ is a face of X , then $\mathcal{K}_p(\tau)$ is also generated by vectors of the form w_G , where G is an element of $\text{Aff}_p(\mathcal{E}_\sigma)$ for some top dimensional face σ containing τ . The Viro map on the generators is

$$bv_p(w_G) = v_1 \wedge \cdots \wedge v_p,$$

where v_1, \dots, v_p is a basis of the vector space parallel to the affine space G .

Proposition 4.10. *For all faces $\tau \subset \sigma$ of X , the following diagram commutes and the rows are exact*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_{p+1}(\sigma) & \xrightarrow{i} & \mathcal{K}_p(\sigma) & \xrightarrow{bv_p} & \mathcal{F}_p(\sigma) \longrightarrow 0 \\ & & \downarrow i_{\sigma\tau} & & \downarrow i_{\sigma\tau} & & \downarrow i_{\sigma\tau} \\ 0 & \longrightarrow & \mathcal{K}_{p+1}(\tau) & \xrightarrow{i} & \mathcal{K}_p(\tau) & \xrightarrow{bv_p} & \mathcal{F}_p(\tau) \longrightarrow 0. \end{array} \quad (4.3)$$

Proof. The exactness of the rows follows from Lemma 4.8. Since the augmentation morphism commutes with linear projections and inclusions, the left-hand square is commutative. The commutativity of the square on the right follows from the description of bv_p on the generators. \square

The cellular q -chains with coefficients in \mathcal{K}_p are defined by

$$C_q(X; \mathcal{K}_p) = \bigoplus_{\dim \sigma = q} \mathcal{K}_p(\sigma).$$

Thanks to the commutativity of the left hand square of the diagram in Proposition 4.10, there is the complex of relative chains

$$C_\bullet(X; \mathcal{K}_p, \mathcal{K}_{p+1}) := C_\bullet(X; \mathcal{K}_p) / C_\bullet(X; \mathcal{K}_{p+1}).$$

We let $H_q(X; \mathcal{K}_p, \mathcal{K}_{p+1})$ denote the q -th homology group of this complex.

Corollary 4.11. *For all p and q we have isomorphisms*

$$H_q(X; \mathcal{K}_p, \mathcal{K}_{p+1}) \cong H_q(X; \mathcal{F}_p).$$

Proof. There is an isomorphism $bv_p: C_q(X; \mathcal{K}_p, \mathcal{K}_{p+1}) \rightarrow C_q(X; \mathcal{F}_p)$ for each q . The commutativity on the right hand side of Proposition 4.10 implies that bv_p induces an isomorphism of complexes $C_\bullet(X; \mathcal{K}_p, \mathcal{K}_{p+1}) \rightarrow C_\bullet(X; \mathcal{F}_p)$. Since the complexes are isomorphic, so are their homology groups and this proves the statement of the corollary. \square

Proposition 4.12. *The first page of the spectral sequence associated to the filtration of the chain complex $C_\bullet(X; \mathcal{S}_\mathcal{E})$ by the chain complexes $C_\bullet(X; \mathcal{K}_p)$ has terms*

$$E_{q,p}^1 \cong H_q(X; \mathcal{F}_p).$$

Proof. Proposition 4.10 implies that the chain complexes $C_\bullet(X; \mathcal{K}_p)$ filter the chain complex $C_\bullet(X; \mathcal{S}_\mathcal{E})$ from Definition 3.16

$$0 \subset C_\bullet(X; \mathcal{K}_n) \subset \cdots \subset C_\bullet(X; \mathcal{K}_1) \subset C_\bullet(X; \mathcal{S}_\mathcal{E}).$$

This is a finite filtration of a complex of finite dimensional vector spaces, therefore the spectral sequence associated to this filtration converges [McC01, Theorem 2.6]. By definition, the first page of the spectral sequence of the filtered complex consists of the relative chain groups,

$$E_{q,p}^1 \cong H_q(X; \mathcal{K}_p, \mathcal{K}_{p+1}).$$

Then the proposition follows from Corollary 4.11. \square

Proof of Theorem 1.5. The pages of a spectral sequence satisfy $\dim E_{q,p}^\infty \leq \dim E_{q,p}^r$ for all r . By Propositions 3.17, 4.12, and the convergence of the spectral sequence associated to the filtration we obtain

$$\dim H_q(\mathbb{R}V) = \dim H_q(X; \mathcal{S}_\mathcal{E}) = \sum_{p=0}^{\dim X} E_{q,p}^\infty \leq \sum_{p=0}^{\dim X} E_{q,p}^1 = \sum_{p=0}^{\dim X} H_q(X; \mathcal{F}_p).$$

When the tropical hypersurface X is contained in a partial compactification of the torus $\mathbb{T}Y_\Delta^o$ corresponding to a subfan of the dual fan of the Newton polytope Δ the filtration of the chain complex $C_\bullet(\bar{X}; \mathcal{S}_\mathcal{E})$ can be restricted to the cells contained in $X = \bar{X} \cap Y_\Delta^o$ to give a filtration of $C_\bullet^{BM}(X; \mathcal{S}_\mathcal{E})$. Variants of Propositions 3.17 and 4.12 also hold in the non-compact case and the argument given above completes the proof. \square

5. PROOF OF THEOREM 1.4

Lemma 5.1. *Let X be a compact tropical non-singular hypersurface of dimension n in the non-singular tropical toric variety $\mathbb{T}Y_\Delta$, where Δ is the Newton polytope of X . Then*

$$\dim H_q(X; \mathcal{F}_p) = 0$$

unless $p + q = n$ or $p = q$.

Proof. Let V be a complex non-singular hypersurface of the same dimension and Newton polytope as X considered in the complex toric variety Y_Δ . The Lefschetz Hyperplane Section Theorem together with Poincaré duality for $\mathbb{C}V$ implies that $h^{p,q}(\mathbb{C}V) = 0$ unless $p + q = n$ or $p = q$. The statement of the lemma now follows by applying Theorem 1.6. \square

Proof of Theorem 1.4. It follows from the statement of Theorem 1.5 that

$$b_q(\mathbb{R}V) \leq \sum_{p=0}^d \dim H_q(X; \mathcal{F}_p).$$

By Lemma 5.1, the sum on the right hand side is equal to $\sum_{p=0} h^{p,q}(\mathbb{C}V)$ which completes the proof. \square

6. GOING FURTHER IN THE SPECTRAL SEQUENCE

In addition to bounding the Betti numbers of real hypersurfaces close to a non-singular tropical limit, the spectral sequence provides immediate criteria for the optimality of the bounds on individual Betti numbers from Theorem 1.4, in addition to the criterion for maximality in the sense of the Smith-Thom inequality from Theorem 1.8.

Theorem 6.1. *Let V be a compact hypersurface with non-singular Newton polytope near a non-singular tropical limit X , then the q -th Betti number of $\mathbb{R}V$ attains the bound in Theorem 1.4 if and only if all of the following maps from the spectral sequence for X are zero*

(1) when $q = n/2$,

$$\partial_1: E_{q,q}^1 \rightarrow E_{q-1,q+1}^1 \quad \text{and} \quad \partial_1: E_{q+1,q-1}^1 \rightarrow E_{q,q}^1,$$

(2) when $q < n/2$,

$$\partial_1: E_{q,n-q}^1 \rightarrow E_{q-1,n-q+1}^1, \quad \partial_1: E_{q+1,n-q-1}^1 \rightarrow E_{q,n-q}^1, \quad \text{and}$$

$$\partial_{2q-n+1}: E_{q+1,q-r}^{2q-n+1} \rightarrow E_{q,q}^{2q-n+1}.$$

Remark 6.2. *If V is a compact hypersurface in a non-singular toric variety near a non-singular tropical limit, then the real point set $\mathbb{R}V$ is a smooth n -dimensional manifold and its Betti numbers over \mathbb{Z}_2 satisfy Poincaré duality. This ensures that*

$$b_i(\mathbb{R}V) = b_{n-i}(\mathbb{R}V).$$

Therefore in order to determine all of the Betti numbers of $\mathbb{R}V$ we only need to determine the Betti numbers $b_q(\mathbb{R}V)$ for $q \leq n/2$.

Lemma 6.3. *Let V be a compact hypersurface with non-singular Newton polytope near a non-singular tropical limit X . The only possible non-zero differentials of the spectral sequence $(E_{\bullet,\bullet}^r, \partial^\bullet)$ for X are*

$$\partial_1: E_{q,p}^1 \rightarrow E_{q-1,p+1}^1 \quad \text{for } p+q=n \quad (6.1)$$

$$\partial_r: E_{q+1,q-r}^r \rightarrow E_{q,q}^r \quad \text{for } r=2q-n+1 \quad (6.2)$$

$$\partial_r: E_{q,q}^r \rightarrow E_{q-1,q+r}^r \quad \text{for } r=n-2q+1. \quad (6.3)$$

Proof. If a boundary map $\partial_r: E_{q,p}^r \rightarrow E_{q-1,p+r}^r$ is non-zero, then necessarily both $E_{q,p}^r$ and $E_{q-1,p+r}^r$ must be non-zero. This implies that both $E_{q,p}^1 \cong H_q(X; \mathcal{F}_p)$ and $E_{q-1,p+r}^1 \cong H_{q-1}(X; \mathcal{F}_{p+r})$ must be non-zero. But Lemma 5.1 implies that $E_{q,p}^1 \cong H_q(X; \mathcal{F}_p) = 0$ unless $p+q=n$ or $p=q$.

Case 1: Suppose $p+q=n$, then for $E_{q-1,p+r}^r$ to be non-zero we must have either $q-1+p+r=n$ or $q-1=p+r$. In the first case $r=1$. In the second case $r=2q-n-1$. These are the maps listed in (6.1) and (6.2) in the statement of the lemma.

Case 2: If $p=q$, then we have the non-zero map in (6.1) when $r=1$. If $r>1$, then $q-1+q+r=n$ so that we find the condition in (6.3) above. This completes the proof of the lemma. \square

Proof of Theorem 6.1. The q -th Betti number attains the bounds in Theorem 1.4 if and only if for all p the maps

$$\partial_r: E_{q,p}^r \rightarrow E_{q-1,p+r}^r \quad \text{and} \quad \partial_r: E_{q+1,p-r}^r \rightarrow E_{q,p}^r,$$

are zero for all r . The theorem follows from the list of possible non-zero differential maps in Lemma 6.3. \square

Example 6.4. Applying Theorem 6.1 to the case $n=2$ implies that for all r the only possibly non-zero differentials of the spectral sequence are

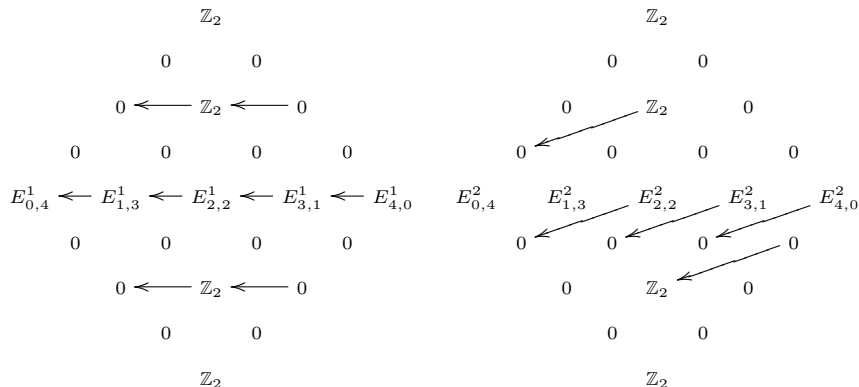
$$\partial_1: H_2(X; \mathcal{K}_0, \mathcal{K}_1) \rightarrow H_1(X; \mathcal{K}_1, \mathcal{K}_2) \quad \text{and} \quad \partial_1: H_1(X; \mathcal{K}_1, \mathcal{K}_2) \rightarrow H_0(X; \mathcal{K}_2). \quad (6.4)$$

Recall that Corollary 1.11 relates the signature of $\mathbb{C}V$ to the Euler characteristic to $\mathbb{R}V$ for a non-singular real hypersurface obtained from a primitive patchworking. Combining this with Poincaré duality for $\mathbb{R}V$, Serre duality for $\mathbb{C}V$, and also the Lefschetz Hyperplane Theorem for $\mathbb{C}V$ and $\mathbb{C}Y_\Delta$ we obtain the following equality,

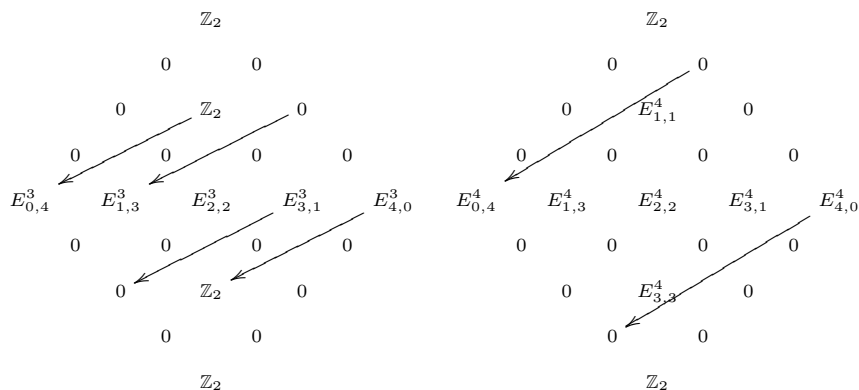
$$2b_0(\mathbb{R}V) - b_1(\mathbb{R}V) = 2 + 2h^{2,0}(\mathbb{C}V) - h^{1,1}(\mathbb{C}V).$$

Therefore a compact surface in a three dimensional toric variety obtained by primitive patchworking is maximal if and only if one of the maps in (6.4) is zero.

Example 6.5. For $n = 4$ we show the first pages of the spectral sequence. The first page on the left below has terms $E_{q,p}^1 \cong H_q(X; \mathcal{F}_p)$.



Notice that all differentials on the second page are trivial since the conditions (6.2) and (6.3) in Lemma 6.3 cannot be satisfied for $n = 4$ and $r = 2$. Therefore $E_{q,p}^3 = E_{q,p}^2$ and the arrows of the third page are depicted on the left.



On the right hand side above is the fourth page of the spectral sequence. Here all differentials are zero, moreover for $r \geq 4$ all differentials are zero by Lemma 6.3.

7. CASE OF PLANE CURVES

In this section we explicitly describe the only possibly non-zero differential map in the spectral sequence in the case of curves. In this case, Viro's primitive patchworking construction, equivalently, the real phase structures on tropical curves from Section 3.1, can be reformulated in terms of *admissible twists*. For definitions and examples of tropical curves, tropical toric surfaces, the twist description of Viro's patchworking, and the tropical homology of tropical curves we refer to [BIMS15].

Given a compact non-singular tropical curve C in a tropical toric surface there is another equivalent way of describing a real phase structure on C in terms choosing a subset of *twisted edges* of the bounded edges of $C \cap \mathbb{R}^2$ satisfying an admissibility condition. A collection T of bounded edges of a

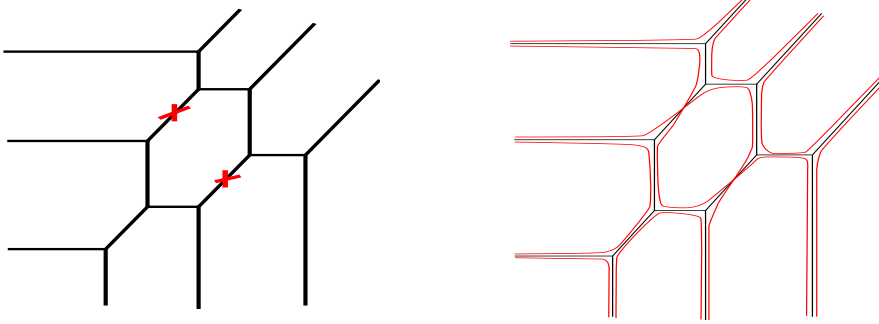


FIGURE 4. On the left is a non-singular cubic with a twist-admissible set of edges. On the right hand side is the image by the coordinatewise logarithm map of $\mathbb{R}V$.

tropical curve $C \cap \mathbb{R}^2$ is *admissible* if for all $\gamma \in H_1(C; \mathcal{F}_0)$ we have

$$\sum_{e \in T \cap \text{Supp}(\gamma)} v_e = 0 \in \mathbb{Z}_2^2,$$

where v_e is the primitive integer direction of the edge e . The edges in T are called twisted edges because of how a real algebraic curve near a tropical limit C with a real phase structure given by the twisted edges behaves under the logarithm map. See the right hand-side of Figure 4.

Let C be a non-singular compact tropical curve with a real phase structure \mathcal{E} . For a bounded edge e of C_0 , its symmetric copy, e^ε in $\mathbb{R}C_\varepsilon$, is adjacent to two other edges $e_1^\varepsilon, e_2^\varepsilon$ of $\mathbb{R}C_\varepsilon$ which are also contained in the quadrant corresponding to ε . The twisted edges for a real phase structure \mathcal{E} correspond to those edges e of C for which e_1, e_2 are not contained in a closed half space of $\mathbb{R}^2(\varepsilon)$ whose boundary contains e^ε . A detailed description of this approach can be found in [BIMS15, Section 3.2].

Using the twist formulation we describe explicitly the map

$$\partial_1 : H_1(C; \mathcal{F}_0) \rightarrow H_0(C; \mathcal{F}_1)$$

arising from the spectral sequence on the chain level when the curve C is compact. In this case, both of the above homology groups are isomorphic to \mathbb{Z}_2^g , where g is the first Betti number of C . For explicit calculations of the tropical (co)homology of tropical curves see [BIMS15, Section 7.8].

Example 7.1. Figure 4, shows a non-singular plane tropical cubic with a twist-admissible set of edges, and the image by coordinatewise logarithm map Log of the real part $\mathbb{R}V$ of the curve V which is defined by the polynomial \mathbf{P}_t from (3.3) for t sufficiently large. Figure 5 depicts $\mathbb{R}C_T$. Notice that this curve is maximal in the sense of Harnack's inequality, namely $b_0(\mathbb{R}V) = g(\mathbb{C}V) + 1$.

Let \tilde{C} denote the first barycentric subdivision of C , which results in adding a vertex in the middle of each edge. Then the vertices of \tilde{C} are the vertices of C together with additional vertices v_e for each edge of C . For every edge e of C there are now two edges e' and e'' of \tilde{C} , moreover v_e is in the boundary of each of these edges.

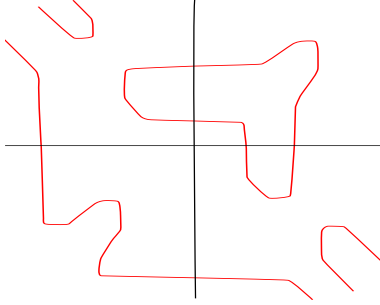


FIGURE 5. The real part $\mathbb{R}V$ for the real cubic from Figure 4 and Example 7.1.

We can extend any cellular cosheaf \mathcal{G} , in particular, $\mathcal{F}_0, \mathcal{F}_1$, or $\mathcal{S}_{\mathcal{E}}$, to a cellular cosheaf on \tilde{C} in the following way. Set $\mathcal{G}(\tilde{e}') = \mathcal{G}(\tilde{e}'') = \mathcal{G}(e)$. If v_e is the midpoint of an edge e then define $\mathcal{G}(v_e) = \mathcal{G}(e)$. The cosheaf morphisms $\mathcal{G}(\tilde{e}') \rightarrow \mathcal{G}(v_e)$ are the identity maps. Changing the cellular structure does not change the homology groups of the cosheaves $\mathcal{F}_0, \mathcal{F}_1$, and $\mathcal{S}_{\mathcal{E}}$. Namely, $H_i(\tilde{C}; \mathcal{F}_0) \cong H_i(C; \mathcal{F}_0)$, $H_i(\tilde{C}; \mathcal{F}_1) \cong H_i(C; \mathcal{F}_1)$, and $H_i(\tilde{C}; \mathcal{S}_{\mathcal{E}}) \cong H_i(C; \mathcal{S}_{\mathcal{E}})$.

For a cellular homology class $\gamma \in H_1(C; \mathcal{F}_0)$, we denote by $\text{Supp}(\gamma)$ the collection of edges of C appearing in some chain representing γ . This is well defined since we are working with \mathbb{Z}_2 -coefficients.

Theorem 7.2. *Let C be a non-singular compact tropical curve in a tropical toric surface. Suppose C is equipped with a real phase structure corresponding to a collection of twists T of edges of C . Then the boundary map of the spectral sequence $\partial_1: H_1(\tilde{C}; \mathcal{F}_0) \rightarrow H_0(\tilde{C}; \mathcal{F}_1)$ is given by*

$$\partial_1(\gamma) = \sum_{e \in T \cap \text{Supp}(\gamma)} v_e \otimes s_e,$$

where s_e is the generator of $\mathcal{F}_1(v_e)$. In particular, the number of connected components of $\mathbb{R}C$ is equal to $\dim \text{Ker}(\partial_1) + 1$

Proof. It is enough to prove the statement for cycles in C which are boundaries of bounded connected components of the complement $\mathbb{R}^2 \setminus C$ since they form a basis of $H_1(C; \mathcal{F}_0)$. Given such a cycle $\gamma \in C_1(\tilde{C}; \mathcal{F}_0)$, we first choose a lift $\tilde{\gamma} \in C_1(C; \mathcal{S}_{\mathcal{E}})$ as follows. Let v be a trivalent vertex of C and suppose that v is in the cycle γ . Let \tilde{e}_1 and \tilde{e}_2 be the two edges of \tilde{C} (or half edges in C) which share the endpoint v and are contained in γ , see Figure 6. Let $\varepsilon(v)$ denote the unique element in $\mathcal{E}(\tilde{e}_1) \cap \mathcal{E}(\tilde{e}_2)$ by Definition 3.1.

We set

$$\tilde{\gamma} = \sum_{\tilde{e} \in \gamma \cap \tilde{C}} \tilde{e} \otimes w_{\varepsilon(v)} \in C_1(C; \mathcal{S}_{\mathcal{E}}),$$

where in the sum above v , is the unique trivalent vertex of \tilde{C} adjacent to the edge \tilde{e} .

If $e \in \text{Edge}(C \cap \gamma) \cap T$ and v, v' are the two adjacent vertices of e , then $w_{\varepsilon(v)}$ and $w_{\varepsilon(v')}$ are different and

$$bv_1(w_{\varepsilon(v)} + w_{\varepsilon(v')}) = w_e \in \mathcal{F}_1(e).$$

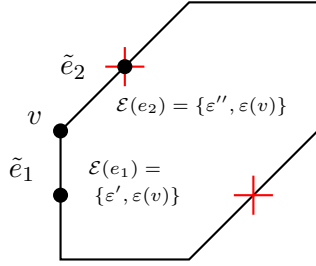


FIGURE 6. The cycle γ of the cubic from Figure 4 and the lift around a vertex.

If e is not twisted, then $w_{\varepsilon(v)} = w_{\varepsilon(v')}$. This proves that $\partial\tilde{\gamma} \in C_0(C; \mathcal{K}_1)$ is supported by the midpoints of twisted edges and that the image by bv_1 of the coefficient over e is exactly the generator of $\mathcal{F}_1(e)$. This proves the lemma. \square

Example 7.3. Consider the tropical curve C in both sides of Figure 7. The red markings on the edges denote collections of twisted edges T_1 on the left and T_2 and the right. It can be verified that both collections of twists are admissible.

Consider the basis $\gamma_1, \gamma_2, \gamma_3$ of $H_1(C; \mathcal{F}_0)$ where γ_i 's are the boundaries of the three bounded connected components of $\mathbb{R}^2 \setminus C$. Let $\gamma_1^*, \gamma_2^*, \gamma_3^*$ denote the dual basis of $H_0(C; \mathcal{F}_1)$. We can represent the map from ∂_1 from Theorem 7.2 by a matrix using these two ordered bases and we obtain the matrices

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

for the twists T_1 and T_2 , respectively. The matrix on the left has a 1-dimensional null space, and therefore a real algebraic curve produced from the collection of twists on the left of Figure 7 has two connected components. On the right the matrix has a 2-dimensional null space and the curve from the twists on the right of Figure 7 has 3 connected components.

7.1. M-curves and Haas theorem. Haas in his thesis [Haa97] studied maximal curves obtained by primitive patchworking. In particular, he found a necessary and sufficient criterion for maximality (see also [BIMS15, Section 3.3] and [BBR17]). Here as an example we reformulate and reprove Haas' criterion for maximality using the techniques of the last section.

Definition 7.4. An edge e of a tropical curve C in a 2-dimensional non-singular tropical toric variety $\mathbb{T}Y$ is called exposed if e is in the closure of an unbounded connected component of $\mathbb{T}Y_0 \setminus C_0$. The set of exposed edges is denoted by $\text{Ex}(C)$. Denote by $\text{Ex}^c(C)$ the complement of $\text{Ex}(C)$ in the set of bounded edges of $C_0 \cap \mathbb{T}Y_0$.

The following theorem is a reformulation of Haas' maximality condition reproved using the description of the map in the spectral sequence from Theorem 7.2.

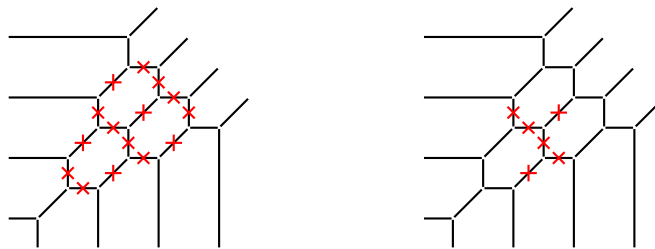


FIGURE 7. The curve C with two collections of twists T_1 and T_2 from Example 7.3.

Theorem 7.5 (Haas' maximality condition [Haa97]). *Let C be a non-singular compact tropical curve in a tropical toric variety equipped with a real phase structure corresponding to a collection of twisted edges $T \subset \text{Edges}(C)$. Then $\mathbb{R}C$ is a maximal curve if and only if $T \cap \text{Ex}^c(C) = \emptyset$ and for every cycle $\gamma \in H_1(C; \mathbb{Z}_2)$ the intersection $\gamma \cap T$ consists of an even number of edges.*

Proof. By Theorem 1.8, the curve $\mathbb{R}C$ is maximal if and only if $\partial_1 = 0$. Cycles in $C_1(C; \mathcal{F}_0)$ which are boundaries of connected components of the complement $\mathbb{R}^2 \setminus C$ form a basis of $H_1(C; \mathcal{F}_0)$. There are $g := b_1(C)$ such cycles and we denote them by $\gamma_1, \dots, \gamma_g$. Therefore, it suffices to show that $\partial_1(\gamma_i) = 0$ for all i .

For C a non-singular tropical curve there is a non-degenerate pairing:

$$\langle \cdot, \cdot \rangle: H_0(C; \mathcal{F}_1) \times H_1(C; \mathcal{F}_0) \rightarrow \mathbb{Z}_2$$

induced from the pairing on integral homology groups for non-singular tropical curves in [Sha11]. A similar non-degenerate pairing defined between tropical homology and cohomology groups is also defined in [BIMS15, Section 7.8] and [MZ14, Section 3.2]. On the chain level this pairing is:

$$\langle \beta, \gamma \rangle = |\text{EdgeSupp}(\beta') \cap \gamma| \pmod{2},$$

where $\beta' \sim \beta$ and $\beta' \in C_0(\tilde{C}; \mathcal{F}_1)$ is supported on the midpoints of edges of C . The set $\text{EdgeSupp}(\beta')$ consists of the edges of C whose midpoint is in the support of β' . Therefore, it suffices to show that for all pairs of such cycles γ_i and γ_j , the non-degenerate pairing $\langle \partial_1(\gamma_i), \gamma_j \rangle$ is zero.

The intersection $\gamma_i \cap T$ is even if and only if $\langle \partial_1(\gamma_i), \gamma_i \rangle = 0$. Secondly, the pairing $\langle \partial_1(\gamma_i), \gamma_j \rangle = 0$ if and only if $\gamma_i \cap \gamma_j \cap T$ is a set of even cardinality. Since γ_i and γ_j are boundaries of convex regions in \mathbb{R}^2 they can only intersect in at most one edge of C . Therefore, the intersection $\gamma_i \cap \gamma_j \cap T$ must be empty and the statement is proved. \square

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ARTHUR RENAUDINEAU, UNIV. LILLE, CNRS, UMR 8524 - LABORATOIRE PAUL PAINLEVÉ, F-59000 LILLE, FRANCE.

Email address: `arthur.renaudineau@univ-lille.fr`

KRIS SHAW, UNIVERSITY OF OSLO, OSLO, NORWAY.

Email address: `krisshaw@math.uio.no`