APPROXIMATION AND ACCUMULATION RESULTS OF HOLOMORPHIC MAPPINGS WITH DENSE IMAGE

GIOVANNI D. DI SALVO

ABSTRACT. We present four approximation theorems for manifold–valued mappings. The first one approximates holomorphic embeddings on pseudoconvex domains in $\mathbb{C}^n$ with holomorphic embeddings with dense images. The second theorem approximates holomorphic mappings on complex manifolds with bounded images with holomorphic mappings with dense images. The last two theorems work the other way around, constructing (in different settings) sequences of holomorphic mappings (embeddings in the first one) converging to a mapping with dense image defined on a given compact minus certain points (thus in general not holomorphic).

1. INTRODUCTION

Calling Y a connected complex manifold and $\triangle$ the unit open disc in $\mathbb{C}$, F. Forstnerič and J. Winkelmann proved in [5] that the set $\mathcal{D}$ of all holomorphic maps $f\colon \triangle \to Y$ with dense image, is dense in $\mathcal{O}(\triangle, Y)$.

We are going to present four results which generalize in different directions the above–mentioned one. The first one is the most straightforward generalization in higher dimension, which is stated for holomorphic embeddings. The second one still provides approximating holomorphic mappings with dense images but in a more general setting. Moreover, it is no more a "density result" as the approximating mappings do not belong to the set of mappings we approximate. The last two theorems work the other way around, describing mappings with dense images approximated by holomorphic mappings.

Another result accounting this topic can be found in [1]: given a closed complex submanifold $X \subsetneq \mathbb{C}^n$, for $n > 1$, there exists a *complete* (the image of every divergent path in X has infinite length in $\mathbb{C}^n$) holomorphic embedding $f\colon X \hookrightarrow \mathbb{C}^n$ with everywhere dense image; for $n = 1$ the same result holds for complete holomorphic embeddings $f\colon \mathbb{C} \hookrightarrow \mathbb{C}^2$. If moreover $X \cap \mathbb{B}^n \neq 0$ and $K \subset X \cap \mathbb{B}^n$ is compact, there exists a Runge domain $\Omega \subset X$ containing K which admits a complete holomorphic embedding $f\colon \Omega \hookrightarrow \mathbb{B}^n$ with dense image.

Finally in [3] it is proved the existence of a holomorphic injective mapping with dense image from the open unit polydisc in $\mathbb{C}^m$ to an m–dimensional paracompact connected complex manifold M.

The paper is organized as follows: we start by presenting the main results, then a technical section follows. The subsequent section shows the inductive procedure on which the proofs of the main results rely, which in turn are provided in the last section.

I would like to warmly thank E. F. Wold for directing me to write this paper.

2. Main Results

All the manifolds in this paper are connected. We will denote with $\mathcal{O}(X, Y)$ the set of all holomorphic mappings $h \colon X \to Y$, with $\mathcal{O}_{\mathrm{emb}}(X, Y)$ and $\mathcal{O}_{\mathrm{inj}}(X, Y)$ the set of all holomorphic embeddings and injections respectively.

Theorem 2.1. Let $\Omega \subset \mathbb{C}^n$ open, bounded, star–shaped (with respect to 0) and pseudoconvex. Let Y be a complex manifold. Then

$$\{h \in \mathcal{O}_{\mathrm{emb}}(\Omega, Y) \ : \ \overline{h(\Omega)} = Y\}$$

is dense inside $\mathcal{O}_{\mathrm{emb}}(\Omega, Y)$ (with respect the compact convergence topology).

Observe that the density is considered with respect to the topology of Y. The distance d on Y exploited throughout the paper to make computations (namely: it allows to define the distance between two mappings $f, g \colon X \to Y$ on a given $M \subset X$ by $\|f - g\|_M := \sup\{d(f(x), g(x)) \ : \ x \in M\}$) is induced by a Riemannian metric. Such d induces the same topology Y is endowed with.

Remark 2.1. We will implicitly assume $n \leq \dim Y$, otherwise $\mathcal{O}_{\mathrm{emb}}(\Omega, Y) = \emptyset$.

Theorem 2.2. Let X be a complex manifold, Y Stein manifold. Then

$$\mathscr{H}(X, Y) := \{h \in \mathcal{O}(X, Y) \ : \ \overline{h(X)} = Y\}$$

compactly approximates

$$\mathscr{G}(X, Y) := \{g \in \mathcal{O}(X, Y) \ : \ g \text{ non constant}, \ g(X) \subset\subset Y\}$$

meaning that, given $g \in \mathscr{G}(X, Y)$, for every $M \subset X$ compact and for every $\epsilon > 0$ there exists $h \in \mathscr{H}(X, Y)$ such that $\|g - h\|_M < \epsilon$.

Observe that $\mathscr{G}(X, Y) = \emptyset$ whenever X is either compact or euclidean: in the former case the holomorphic mappings are constant, in the latter Liouville holds, obtaining again g constant, as its image is bounded. An example of a domain X such that $\mathscr{G}(X, Y) \neq \emptyset$ is given by any open relatively compact domain $X \subset\subset Y$ in a Stein manifold; just consider any non–constant holomorphic bounded mapping $g \colon X \to Y$, e.g., the inclusion $\iota \colon X \hookrightarrow Y$.

To present the last two theorems, we need the following definition:

Definition 2.1. Let K be a compact in Y complex manifold. A point $\zeta \in K$ is *locally exposable* if there exists a $\mathscr{C}^2$–smooth strictly plurisubharmonic function ρ on some open neighborhood U of ζ such that

(1) $\rho(\zeta) = 0$ and $d\rho(\zeta) \neq 0$, and
(2) $\rho < 0$ on $(K \cap U) \setminus \{\zeta\}$.

Definition 2.1 was originally presented in [2]. It is a generalization of the more standard definition of *local peak point* (see e.g., [7] pag. 354).

Theorem 2.3. Let Y be a complex manifold, $K \subset Y$ connected not finite Stein compact. Then the set of locally exposable points is non–empty and for any such point $x_0 \in K$, there exist $U_k \subset Y$ open neighborhood of K and $F_k \in \mathcal{O}_{\mathrm{inj}}(U_k, Y)$ compactly convergent to $F \colon K \setminus \{x_0\} \to Y$ such that $\overline{F(K \setminus \{x_0\})} = Y$.

Theorem 2.4. Let Y be a Stein manifold, $K \subset Y$ connected not finite compact. Denote with Γ the closure in Y of the set of locally exposable points for K, which is non–empty. Then there exists $U_k \subset Y$ open neighborhood of K and $F_k \in \mathcal{O}(U_k, Y)$ compactly convergent to $F \colon K \setminus \Gamma \to Y$ such that $\overline{F(K \setminus \Gamma)} = Y$.

In Theorems 2.3 and 2.4 the limit mapping F is not in general holomorphic, as the domain is not in general an open set. But being F obtained as uniform limit of holomorphic mappings, the realization of F could be a consistent definition for a holomorphic map on $K \setminus \{x_0\}$ or $K \setminus \Gamma$.

3. Technical Tools

The proofs will extensively exploit Theorem 1.1 in [2] and a slightly different version of it (whose proof follows automatically from the original one) which is as follows:

Theorem 3.1. Let Y be a complex manifold and $Y_0 \subset Y$ Stein compact. Let $\zeta \in Y_0$ be locally exposable and $\gamma \colon [0,1] \to Y$ smoothly embedded curve such that $\gamma(0) = \zeta$ and $\gamma((0,\delta]) \subset Y \setminus Y_0$ for some $\delta > 0$. Then, for every V open neighborhood of γ and for every $\epsilon > 0$, there exist the following:

(1) $U \subset Y$ neighborhood of Y_0 ,
(2) an arbitrarily small $V' \subset V$ neighborhood of ζ , and
(3) $f \colon U \to Y$ holomorphic mapping such that
   - $f(\zeta) = \gamma(1)$;
   - $f(V') \subset V$;
   - $\|f - \mathrm{Id}\|_{Y_0 \setminus V'} < \epsilon$.

If the whole curve $\gamma((0,1])$ lies in $Y \setminus Y_0$ then f can be chosen to be injective on U.

In [2] the theorem is formulated in a slightly more general setting and considering only the case $\gamma((0,1]) \subset Y \setminus Y_0$.

Lemma 3.1. Let X be a metrizable topological space, $W \subset\subset X$ connected not finite. Let $f_k \colon U_k \to X$ be a sequence of continuous mappings, where $U_k \subset X$ is some open neighborhood of K_k and $K_1 := \overline{W}$, $K_{k+1} := f_k(K_k)$. Define $F_k := f_k \circ \cdots \circ f_1 \colon \overline{W} \to X$ and consider

   - $\{\epsilon_k\}_k$ positive real numbers ,
   - $\{C_k\}_k$ *compact exhaustion* of W, that is $C_k \subset X$ compact, $C_k \subsetneq C_{k+1}$ and $\bigcup_k C_k = W$, and
   - $\{V'_k\}_k$ open sets in X ,

such that, setting $r_k := \max\{d(x, F_k(C_k)) \, : \, x \in K_{k+1}\}$, the following hold:

(i)$_k$ $V'_k \subset U_k$ and $V'_k \cap K_k \neq \emptyset$,
(ii)$_k$ $\max\{r_k, \|f_k - \mathrm{Id}\|_{K_k \setminus V'_k}\} \leq \epsilon_k$, and
(iii)$_k$ $V'_{k+1} \cap F_k(C_k) = \emptyset$.

If F_k converges uniformly on compacts of W to $F \colon W \to X$, then for every $x \in K_{k+1}$

$$d(x, F(W)) \leq \sum_{n \geq k} \epsilon_n . \tag{3.1}$$

Proof. Let $x \in K_{k+1}$. Then by definition and from (ii)$_k$ one gets

$$d(x, F_k(C_k)) \leq r_k \leq \epsilon_k .$$

Then (iii)$_k$ implies $F_k(C_k) \subset K_{k+1} \setminus V'_{k+1}$ hence (ii)$_{k+1}$ says that f_{k+1} moves $F_k(C_k)$ less than ϵ_{k+1}, therefore

$$d(x, f_{k+1}(F_k(C_k))) \leq \epsilon_k + \epsilon_{k+1} .$$

Since $f_{k+1}(F_k(C_k)) = F_{k+1}(C_k) \subset F_{k+1}(C_{k+1}) \subset K_{k+2} \setminus V'_{k+2}$ we can repeat the argument getting

$$d(x, \underbrace{f_{k+2}f_{k+1}(F_k(C_k))}_{F_{k+2}(C_k)}) \leq \epsilon_k + \epsilon_{k+1} + \epsilon_{k+2} .$$

Inductively, for every $m \geq 0$ we get

$$d(x, F_{k+m}(C_k)) \leq \sum_{j=0}^{m} \epsilon_{k+j}$$

and passing to $\lim_{m\to\infty}$ (which is well defined in left hand side, since the distance is continuous and $\{F_n\}_n$ uniformly converges to F on C_k) we get $d(x, F(C_k)) \leq \sum_{j\geq 0} \epsilon_{k+j}$. Since $F(C_k) \subset F(W)$, we have $d(x, F(W)) \leq d(x, F(C_k))$ and we conclude. $\qquad \square$

Recall now a useful property of Stein manifolds ([7], §2, Proposition 2.21 and Theorem 3.24):

Theorem 3.2. Let Y be a Stein manifold. Then there exists ρ, a $\mathscr{C}^2$–smooth strictly plurisubharmonic exhausting function for Y, such that the set of critical points $C := \{z \in Y : d\rho(z) = 0\}$ is discrete in Y. In particular, for every $c \in \mathbb{R}$, $\{\rho < c\} \subset\subset Y$ and $Y_c := \{\rho \leq c\}$ is a Stein compact.

Remark 3.1. With the notation of Theorem 3.2, we see that every regular boundary point of a strictly pseudoconvex domain $\{\rho < c\}$ is locally exposable: take $\zeta \in \{\rho = c\} \setminus C$; we assume it is the origin in suitable local coordinates. Then considering $\tilde{\rho}(z) := \rho(z) - c - \epsilon|z|^2$ for ϵ small enough, we conclude.

4. Inductive procedure

Consider $Q = \{q_n\}_n \subset Y$ such that $\overline{Q} = Y$. Fix $\epsilon > 0$ and define $\epsilon_k := \frac{\epsilon}{2^{k+1}}$. For Theorems 2.1 and 2.2 we fix a compact subset M of the domain, Ω and X respectively. In what follows, we exploit Theorem 3.1 to get a suitable sequence of holomorphic functions f_k allowing to reach all the points of Q which have not been already reached, so that the image of the composition tends to invade the whole codomain. Along with the f_k, we construct both a compact exhaustion $\{C_k\}_k$ of the case domain Ω, X, $K \setminus \{x_0\}$ or $K \setminus \Gamma$ so that it fulfills the hypothesis of Lemma 3.1, and all the other sequences needed. For the construction of the f_k, we will focus on Theorems 2.2 and 2.1; the procedure for the remaining two theorems is alike and it is afterwards explained. Similarly for the proofs: we worked out the details of the proof of Theorem 2.2, which is displayed by proving convergence of the composition of the f_k, approximation of the given function, and density of the image. The argument for the remaining three results is analogous and it is subsequently illustrated.

4.1. **Construction of f_k for Theorem 2.2.**

4.1.1. *Existence of locally exposable points and base of the induction.* Consider $g \colon X \to Y$ holomorphic non–constant such that $g(X) \subset\subset Y$. Set $K_1 := \overline{g(X)}$ and take $c_1 \in \mathbb{R}$ such that $K_1 \subseteq Y_{c_1}$ and $K_1 \cap \partial Y_{c_1} \neq \emptyset$ (see notation of Theorem 3.2) . Consider $\zeta_1 \in K_1 \cap \partial Y_{c_1}$; if $\zeta_1 \notin C$ then it is locally exposable by Remark 3.1, otherwise we slightly move K_1 by composing g with a suitable holomorphic small perturbation defined as follows. Assume ζ_1 is the only point in $K_1 \cap \partial Y_{c_1}$. Observe that in suitable local coordinates on $\mathbb{C}^n$, which we split as $z = (z', z'') = x + iy = (x' + iy', x'' + iy'') \in \mathbb{C}^k \oplus \mathbb{C}^{n-k}$, we can express ρ near the origin ([5], Lemma 3.10.1) as

$$\rho(z) = \rho(0) + E(x', x'', y', y'') + o(|z|^2)$$

where

$$E(x', x'', y', y'') := \sum_{j=1}^{k} (\lambda_j y_j^2 - x_j^2) + \sum_{j=k+1}^{n} (\lambda_j y_j^2 + x_j^2)$$

with $\lambda_j > 1$ for $j = 1, \ldots, k$ and $\lambda_j \geq 1$ for $j = k+1, \ldots, n$, for some $k \in \{0, 1, \ldots, n\}$. Assume ζ_1 to be the origin $0 \in \mathbb{C}^n$ in these coordinates, so $\rho(0) = \rho(\zeta_1) = c_1$. The boundary of any Y_c is defined by ρ, so we want to move a small neighborhood of ζ_1 in a suitable direction allowing it to go across the level set ∂Y_{c_1}. Take then any nonzero vector $v = (\xi' + i\eta', \xi'' + i\eta'')$ with $\xi' = 0 \in \mathbb{R}^k$. By standard results on Stein Manifolds ([6], Corollary 5.6.3), given $w \in T_{\zeta_1} Y$ there exists a holomorphic vector field $V \colon Y \to TY$ such that $V(\zeta_1) = w$. In our case we take w to correspond to v. The flow of V on some neighborhood W of K_1 is a holomorphic mapping $\phi_t \colon W \to Y$ defined for complex times sufficiently small in modulus, say $|t| < T$, $t \in \mathbb{C}$. In local coordinates around ζ_1 it is $\phi_t(0) = tv + o(|t|^2)$ and up to shrinking T, we have that

$$\rho(\phi_t(0)) = \rho(0) + t^2 E(0, \xi'', \eta', \eta'') + o(|t|^3) > \rho(0) = c_1$$

for any $|t| < T$, considering now $t \in \mathbb{R}$. By continuity of $(z, t) \mapsto \phi_t(z)$ and of ρ, the flow will drag a whole small neighborhood of ζ_1 beyond ∂Y_{c_1} and for sufficiently small times it will be the only piece of K_1 crossing the boundary (as we are assuming that $K_1 \cap \partial Y_{c_1}$ is just one point), that is: there exist $T' < T$ and U neighborhood of ζ_1 sufficiently small, so that

- $\rho(\phi_t(z)) > \rho(0)$, $\forall z \in U \cap K_1$, $T'/2 < t < T'$;
- $\rho(\phi_t(z)) \leq \rho(0)$, $\forall z \in K_1 \setminus U$, $0 \leq t < T'$;
- $\phi_t(U \cap K_1) \cap C = \emptyset$, $\forall T'/2 < t < T'$.

So, considering now $\widetilde{K}_1 := \phi_t(K_1)$ for any $T'/2 < t < T'$, there exists some $c > c_1$ such that $\widetilde{K}_1 \subseteq Y_c$ and $\widetilde{K}_1 \cap \partial Y_c \setminus C \neq \emptyset$.

If $K_1 \cap \partial Y_{c_1}$ contains more than one point, either it is not discrete or it is a finite set. In the former case $K_1 \cap \partial Y_{c_1} \setminus C \neq \emptyset$, in the latter we may assume $K_1 \cap \partial Y_{c_1} = \{\zeta_1, \zeta_1'\}$; then, applying the previous argument to one of these points, the piece of K_1 that is dragged across ∂Y_{c_1}, could be not only $U \cap K_1$, but also $U' \cap K_1$, for any small time (where $U, U' \subset Y$ are suitably small neighborhoods of ζ_1 and ζ_1' respectively), leading to a similar $\widetilde{K}_1$ and achieving the same conclusion. Therefore, up to consider $\phi_t \circ g$ instead of g, we may assume that $\exists \zeta_1 \in K_1 \cap \partial Y_{c_1} \setminus C$, which is then locally exposable by Remark 3.1 and could be sent to any point of $Y \setminus K_1$ by the holomorphic mapping f_1 provided by Theorem 3.1 (see next section).

Finally define $f_0 := \mathrm{Id}_Y$, $C_0 := \emptyset$, $F_0 := f_0$ and $n_0 := 0$.

4.1.2. *Inductive step.* Assume we have the following: $K_k \subset Y$ compact, $c_k \in \mathbb{R}$ such that $K_k \subseteq Y_{c_k}$ and $\exists \zeta_k \in K_k \cap \partial Y_{c_k} \setminus C$, F_{k-1} holomorphic on some neighborhood of K_1, with $F_{k-1}(K_1) = K_k$, $C_{k-1} \subset X$ compact and $n_{k-1} \in \mathbb{N}$. Consider then a smoothly embedded curve $\gamma_k \colon [0,1] \to Y$ such that

(i)$_k$ $\gamma_k(0) = \zeta_k$;
(ii)$_k$ $\gamma_k(1) = q_{n_k}$ where $n_k := \min\{n > n_{k-1} \ : \ q_n \notin K_k\}$;
(iii)$_k$ $\gamma_k((0, \delta_k]) \subset Y \setminus Y_{c_k}$ for some $\delta_k > 0$.

Then for every V_k open neighborhood of γ_k, Theorem 3.1 guarantees there exist

(1)$_k$ $U_k \subset Y$ neighborhood of Y_{c_k} ;
(2)$_k$ $V'_k \subset (V_k \cap B(\zeta_k, \epsilon_k)) \setminus F_{k-1}(g(M \cup C_{k-1}))$;
(3)$_k$ $f_k \colon U_k \to Y$ holomorphic such that the following hold:
    (a)$_k$ $f_k(\zeta_k) = q_{n_k}$;
    (b)$_k$ $f_k(V'_k) \subset V_k$;
    (c)$_k$ $\|f_k - \mathrm{Id}\|_{Y_{c_k} \setminus V'_k} < \epsilon_k$.

Observe that to apply Theorem 3.1, V'_k needs to be a neighborhood of the locally exposable point ζ_k; moreover to have convergence (see next section) it has to avoid the image of (the fixed compact M and) the compact C_{k-1}. This cannot happen if g is constant, as $F_{k-1}(g(M \cup C_{k-1})) = K_k = \{\zeta_k\}$ so V'_k would be a punctured neighborhood of ζ_k; moreover $\zeta_k \in Y_{c_k} \setminus V'_k$ and (c)$_k$ would fail for ϵ_k small enough. Set $K_{k+1} := f_k(K_k)$; up to perturbing f_k as we did with g, there exists $c_{k+1} \in \mathbb{R}$ such that $K_{k+1} \subseteq Y_{c_{k+1}}$ and $\exists \zeta_{k+1} \in K_{k+1} \cap \partial Y_{c_{k+1}} \setminus C$; then $F_k := f_k \circ F_{k-1}$ is holomorphic on some neighborhood of K_1 and $F_k(K_1) = K_{k+1}$; we finally choose a compact $C_k \subset X$ large so that $\max\{d(x, F_k(g(C_k))) : x \in K_{k+1}\} \leq \epsilon_k$ and $C_{k-1} \subsetneq C_k$. The induction may proceed.

4.2. Construction of f_k for Theorem 2.1.

4.2.1. *Existence of a locally exposable point.* Let $g \in \mathcal{O}_{\mathrm{emb}}(\Omega, Y)$. Exploiting sharshapedness and up to considering $g_\delta(z) = g((1 - \delta)z)$ for $0 < \delta < 1$, we can suppose without loss of generality g to be holomorphic and injective on U a Stein neighborhood of $\overline{\Omega}$. Call $R = \max_{z \in \overline{\Omega}} |z|$, let $\zeta_0 \in \partial \Omega$ be such that $|\zeta_0| = R$ and define $\rho(z) := |z|^2 - R^2$ which is strictly plurisubharmonic, hence ζ_0 is a locally exposable point by Remark 3.1. Define $\zeta_1 := g(\zeta_0)$ and $K_1 := g(\overline{\Omega})$.

If $\dim Y = n$, then ζ_1 is locally exposable with respect to $\rho_1 := \rho \circ g^{-1}$ and K_1 is a Stein compact, in fact $W_\alpha = g_\alpha(\Omega)$, $0 < \alpha < \delta$ is a neighbourhood basis of Stein domains since each W_α is biholomorphic to Ω which is holomorphically convex. In this case, ζ_1 is locally exposable and K_1 is a Stein compact asking g for mere injectivity.

Let us now prove that ζ_1 and K_1 are still a locally exposable point and a Stein compact respectively even in the case $\dim Y = m > n$. Working with local charts we can assume to work on open subsets of $\mathbb{C}^m$. Since $dg(\zeta)$ has full rank n at each point $\zeta \in U$, it is $\mathrm{Im}\, dg(\zeta) \simeq \mathbb{C}^n$ and up to a linear change of coordinates, we can assume $\mathrm{Im}\, dg(\zeta_0) = \mathbb{C}^n \times \{0\}^{m-n}$ and obviously $\mathrm{Im}\, dg(\zeta_0) \bigoplus \mathrm{Span}_{\mathbb{C}}(e_{n+1}, \dots, e_m) = \mathbb{C}^m$. Define then, for $(z_1, \dots, z_m) \in U \times \mathbb{C}^{m-n}$

$$\tilde{g}(z_1, \dots, z_m) := g(z_1, \dots, z_n) + z_{n+1} e_{n+1} + \cdots + z_m e_m .$$

Clearly $\tilde{g}(\zeta_0, 0) = \zeta_1$ and it is locally invertible near $(\zeta_0, 0)$. Call $h \colon A \to B$ the inverse, where $A, B \subset \mathbb{C}^m$ are open neighborhoods of ζ_1 and $(\zeta_0, 0)$ respectively and since

$$\pi_j \circ h(z) = z_j \ \text{ for } \ j = n+1, \ldots, m \,,$$

we have that

$$g(U) \cap A = \{z \in A \ : \ z_{n+1} = \cdots = z_m = 0\} \,;$$

we worked around ζ_1, but the same argument holds for any other point (regardless of whether it is regular or singular) so $g(U)$ is a complex subvariety; in particular, it is *locally closed* (every point in $g(U)$ has an open neighborhood W such that $g(U) \cap W$ is closed in W), thus it admits a Stein neighborhood basis ([5], Theorem 3.1.1). The same holds for any $g(U')$, with $\overline{\Omega} \subset U' \subset U$, therefore K_1 is a Stein compact. Define now

$$\rho_1(z) := \rho \circ \alpha(z) + \sum_{j=n+1}^{m} |z_j|^2 \,,$$

where $\alpha := \pi_{|\mathbb{C}^n} \circ h \colon A \to U$ and $\pi_{|\mathbb{C}^n} \colon \mathbb{C}^m \to \mathbb{C}^n$ is the projection on the first n coordinates. The term $\rho \circ \alpha$ is plurisubharmonic as ρ is such and α is holomorphic; moreover

$$L_w(\rho \circ \alpha; t) = L_{\alpha(w)}(\rho; \alpha'(w)t) > 0$$

for every $t \in \mathbb{C}^m$, $\pi_{|\mathbb{C}^n}(t) \neq 0$, $w \in A$, in fact $\ker \alpha'(w) = \{0\}^n \times \mathbb{C}^{m-n}$ for any $w \in A$ and ρ is strictly plurisubharmonic. Therefore we add the plurisubharmonic term $\beta(z) := \sum_{j=n+1}^{m} |z_j|^2$. Clearly

$$L_w(\beta; t) = \beta(t) > 0$$

for every $t \in \mathbb{C}^m$, $\pi_{|\mathbb{C}^{m-n}}(t) \neq 0$, $w \in A$. Therefore ρ_1 is strictly plurisubharmonic on ζ_1 (actually on the whole A). It is clear that $\rho_1(\zeta_1) = 0$ and $\rho_1 < 0$ on $A \cap K_1 \setminus \{\zeta_1\}$; then a computation shows that $d\rho_1(\zeta_1) \neq 0$, hence ζ_1 is locally exposable in K_1 with respect to ρ_1.

4.2.2. *Inductive procedure.* Let us observe that $Y \setminus K_1$ is connected, in fact if $n = 1$, Ω is simply connected (being starshaped) thus its boundary is connected and so is its image.

Let $n \geq 2$ and $\dim Y = n$; then $g(\Omega)$ is a Stein domain, which has connected boundary for $\dim Y \geq 2$ (see [8], pag. 22).

The remaining case, by Remark 2.1, is $n \geq 2$ and $m = \dim Y > n$. In this case just observe that $g(\Omega)$ is a complex submanifold of complex codimension $m - n \geq 1$, so its complement is connected.

Assume we have $K_k \subset Y$ Stein compact, $\zeta_k \in K_k$ locally exposable with respect to some strictly plurisubharmonic ρ_k, $C_{k-1} \subset \Omega$ compact, F_{k-1} holomorphic injective on some neighborhood of K_1 such that $F_{k-1}(K_1) = K_k$. The construction of $f_k \colon U_k \to Y$ is as in Section 4.1.2, with $Y_{c_k} = K_k$ (Stein compact) and $\gamma_k((0, 1]) \subset Y \setminus Y_{c_k}$ (which can be achieved since $Y \setminus Y_{c_k}$ is connected, as above), allowing f_k to be injective on U_k and setting $\zeta_{k+1} := f_k(\zeta_k) = q_{n_k}$ (which is locally exposable with respect to $\rho_{k+1} := \rho_k \circ f_k^{-1}$). Finally set $K_{k+1} := f_k(K_k)$ Stein compact, $F_k := f_k \circ F_{k-1}$ holomorphic injective on some open neighborhood of K_1 and $C_k \subset \Omega$ as in Section 4.1.2. In particular $F_k(K_1) = K_{k+1}$ and $\{C_k\}_k$ is a compact exhaustion of Ω.

4.3. **Construction of f_k for Theorem 2.3.** Since K is a Stein compact, there exists $U \subset Y$ Stein neighborhood of K and consequently a plurisubharmonic exhausting function $\rho \colon U \to \mathbb{R}$ as recalled in Theorem 3.2. Then, the existence of at least one locally exposable point $x_0 \in K$ is guaranteed by the argument of Section 4.1.1, with x_0, K, U playing the role of ζ_1, K_1 and Y respectively and being $\{C_k\}_k$ compact exhaustion for $K \setminus \{x_0\}$. So, given any $x_0 \in K$ locally exposable, the rest of the inductive procedure is as in Section 4.2.2, except for M and g which here just do not play any role, with x_0, K, ρ playing the role of ζ_1, K_1 and ρ_1 respectively.

4.4. **Construction of f_k for Theorem 2.4.** The construction is as in Section 4.1, with no M, no g, with K playing the role of K_1 and $\{C_k\}_k$ compact exhaustion for $K \setminus \Gamma$. At each step we get a locally exposable point $\zeta_k \in K_k \cap \partial Y_{c_k} \setminus C$, sent to q_{n_k} by f_k and corresponding to some $x_k \in K = K_1$, which is locally exposable as well (thus $\{x_k\}_k \subset \Gamma$).

5. PROOFS

5.1. **Proof of Theorem 2.2.**

Proof. $F_k \colon K_1 = \overline{g(X)} \to Y$ is holomorphic. Then from $(2)_{k+1}$ it follows that for every fixed j

$$F_k(g(C_j)) \subset K_{k+1} \setminus V'_{k+1} \tag{5.1}$$

holds true for every $k \geq j$. Hence we get that, for every fixed j

$$\|F_{k+1} - F_k\|_{g(C_j)} = \|f_{k+1} - \mathrm{Id}\,\|_{F_k(g(C_j))} \leq \|f_{k+1} - \mathrm{Id}\,\|_{K_{k+1} \setminus V'_{k+1}} < \epsilon_k \tag{5.2}$$

is true for every $k \geq j$, so $\{F_k\}_k$ converges on compact subsets of $g(X)$ to $F \colon g(X) \to Y$ holomorphic. As above, $(2)_{k+1}$ implies $F_k(g(M)) \subset K_{k+1} \setminus V'_{k+1}$ for every k, hence inequality (5.2) holds true for all k, thus

$$\|F - \mathrm{Id}\,\|_{g(M)} \leq \sum_{k \geq 0} \|F_{k+1} - F_k\|_{g(M)} < \epsilon \,,$$

allowing us to conclude that $\|h - g\|_M < \epsilon$, where $h = F \circ g \colon X \to Y$ is the approximating mapping. We now check it actually has dense image. If $h(X)$ is not dense in Y, then there exists an open ball $B \subset Y$ such that

$$\beta := d(B, h(X)) > 0 \,.$$

The construction of the sets K_k and the sequence $\{n_k\}_{k \geq 1}$ allows to consider a partition of Q as

$$q_{n_{k-1}}, \ldots, q_{n_k - 1} \in K_k, \quad k \geq 1 \,.$$

In this way we can define the sequence

$$k(n) := j \quad \text{for} \quad n = n_{j-1}, \ldots, n_j - 1, \quad \text{for} \quad j \geq 1$$

and we have $q_n \in K_{k(n)}$ holds true for all n; the sequence $n \mapsto k(n)$ is increasing and such that $k(n) \to +\infty$ as $n \to +\infty$ (otherwise $\exists \widetilde{k}, N$ such that $q_n \in K_{\widetilde{k}}$ for all $n \geq N$, so Q would not be dense). Since $g(X) \subset\subset Y$, it follows by Lemma 3.1 that

$$d(q_n, h(X)) = d(q_n, F(g(X))) \leq \sum_{j \geq k(n) - 1} \epsilon_j \,.$$

This last sum is less than β for any $n \geq n_\beta$, for a suitably large n_β. Therefore $\{q_n\}_{n > n_\beta}$, which is still dense in Y, does not meet an open ball, contradiction. $\qquad\square$

5.2. **Proof of Theorem 2.1.**

Proof. It is the same as the previous proof. Just observe that now F_k is defined on $K_1 = g(\overline{\Omega})$, it is holomorphic injective and so is F. Since g is injective by assumption, then the approximating mapping $h = F \circ g$ is holomorphic injective as well. $\qquad\square$

5.3. **Proof of Theorem 2.3.**

Proof. The mappings F_k are defined, holomorphic and injective on some open neighborhood of K and converge to $F \colon K \setminus \{x_0\} \to Y$ uniformly on compacts of $K \setminus \{x_0\}$. The construction of mappings f_k ensures, as for Theorem 2.2, to achieve $\overline{F(K \setminus \{x_0\})} = Y$. $\qquad\square$

5.4. **Proof of Theorem 2.4.**

Proof. As for Theorem 2.3 (except for injectivity of F_k), with $K \setminus \Gamma$ instead of $K \setminus \{x_0\}$. $\qquad\square$

References

[1] Alarcón, A. and Forstnerič, F.; Complete densely embedded complex lines in $\mathbb{C}^2$. *Proc. Amer. Math. Soc.*, 146(3):1059–1067, 2018.

[2] Deng, F., Fornæss, J. E. and Wold, E. F.; *Exposing Boundary Points of Strongly Pseudoconvex Subvarieties in Complex Spaces.* Proceedings Of The AMS, Volume 146, Number 6, June 2018, Pages 2473–2487, http://dx.doi.org/10.1090/proc/13693, Article electronically published on March 9, 2018.

[3] Fornæss, J. E. and Stout, E. L.; *Polydiscs in complex manifolds.. Math. Ann.* 227 (1977), no. 2, 145–153.

[4] Forstnerič, F.; Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 56. Springer, Cham, 2017.

[5] Forstnerič, F. and Winkelmann, J.; *Holomorphic Discs with Dense Images.* Mathematical Research Letters 12, 265–268 (2005)

[6] Hörmander, L.; *An introduction to Complex Analysis in Several Variables.* North–Holland, 3rd edition (1990)

[7] Range, R. M.; *Holomorphic Functions and Integral Representations in Several Complex Variables.* Springer, Graduate Text in Mathematics 108, (1986)

[8] Serre, J. P.; Un théorème de dualité. *Comment. Math. Helv.* **29** (1955), 9–26.