THE BIGGEST FIVE OF REVERSE MATHEMATICS

DAG NORMANN AND SAM SANDERS

Abstract. The aim of *Reverse Mathematics* (RM for short) is to find the minimal axioms needed to prove a given theorem of ordinary mathematics. These minimal axioms are almost always *equivalent* to the theorem, working over the *base theory* of RM, a weak system of computable mathematics. The *Big Five phenomenon* of RM is the observation that a large number of theorems from ordinary mathematics are either provable in the base theory or equivalent to one of only four systems; these five systems together are called the 'Big Five'. The aim of this paper is to greatly extend the Big Five phenomenon as follows: there are two supposedly *fundamentally different* approaches to RM where the main difference is whether the language is restricted to *second-order* objects or if one allows *third-order* objects. In this paper, we unite these two strands of RM by establishing numerous equivalences involving the **second-order** Big Five systems on one hand, and well-known **third-order** theorems from analysis about (possibly) discontinuous functions on the other hand. We both study relatively tame notions, like cadlag or Baire 1, and potentially wild ones, like quasi-continuity. We also show that *slight* generalisations and variations of the aforementioned third-order theorems fall *far* outside of the Big Five.

1. Introduction and preliminaries

1.1. **Short summary.** The aim of the program *Reverse Mathematics* (RM for short) is to find the minimal axioms needed to prove a given theorem of ordinary mathematics. In a nutshell, the aim of this paper is to greatly extend the so-called Big Five phenomenon, a central topic in RM according to Montalbán, as follows.

> [...] we would still claim that the great majority of the theorems from classical mathematics are equivalent to one of the big five. This phenomenon is still quite striking. Though we have some sense of why this phenomenon occurs, we really do not have a clear explanation for it, let alone a strictly logical or mathematical reason for it. The way I view it, gaining a greater understanding of this phenomenon is currently one of the driving questions behind reverse mathematics. (see [67, p. 432])

In more detail, there are at least two supposedly *fundamentally different*[1] approaches to RM where the main difference is whether the language is restricted to *second-order* objects or if one allows *third-order* objects. In this paper, we unite these two strands of RM by establishing numerous equivalences involving the

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Department of Mathematics, The University of Oslo, P.O. Box 1053, Blindern N-0316 Oslo, Norway

Department of Philosophy II, RUB Bochum, Germany

E-mail addresses: `dnormann@math.uio.no, sasander@me.com`.

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[1] This opinion is for instance expressed in the latest textbook on RM, namely in [23, §12.4].

second-order Big Five systems on one hand, and well-known **third-order** theorems from analysis about (possibly) discontinuous functions on the other hand. We both study relatively 'tame' notions, like cadlag and Baire 1, and potentially 'wild' ones, like quasi-continuity. We also show that *slight* generalisations and variations of the aforementioned third-order theorems fall *far* outside of the Big Five and much stronger (second- and higher-order) systems. The reader will agree that while our results are comprehensive, they only scratch the surface of what is possible and lead the way to a whole new research area. In evidence, we sketch analogous results for the RM of the second-order *weak weak König's lemma* and the third-order Vitali covering theorem for uncountable coverings in Section 2.3.3.

Finally, we discuss the detailed aim and motivation of this paper within RM in Section 1.2 and introduce essential definitions in Section 1.3.

1.2. **Aim and motivation.** Reverse Mathematics (RM for short) is a program in the foundations of mathematics initiated by Friedman ([28, 29]) and developed extensively by Simpson and others ([23, 93, 94]); an introduction to RM for the 'mathematician in the street' may be found in [95]. We assume basic familiarity with RM, including Kohlenbach's *higher-order* RM introduced in [55], while a brief sketch may be found in Section 1.3.1. Recent developments in higher-order RM, including our own, are in [70–76]. All equivalences are proved over Kohlenbach's base theory RCA_0^ω (or slight extensions), as defined in the appendix (Section A.1).

The biggest difference between 'classical' RM and higher-order RM is that the former makes use of the language of *second-order* arithmetic, while the latter uses the language of *higher-order* arithmetic (see Section 1.3.1 for details). Thus, higher-order objects are only indirectly available via so-called codes or representations in classical RM. It is then a natural question -in the very spirit of RM- what the connection is between third-order objects and their second-order codes. Now, continuous functions constitute perhaps the most basic case study and Kohlenbach in [53, §4] studies the connection between:

- third-order functions on Baire or Cantor space that satisfy the standard 'epsilon-delta' definition of continuity,
- second-order codes for continuous functions on Baire or Cantor space, following the definition from [94, II.6].

Kohlenbach shows that *weak König's lemma* (WKL for short) suffices to show that a (third-order) continuous function on Cantor space can be represented by a code. In Section 2.2, we adapt some of Kohlenbach's results to the unit interval, which turns out to be surprisingly hard. The representation of the reals in (both second- and higher-order) RM may be found in Section A.2.

With these 'coding results' on $[0, 1]$ in place, we establish in Section 2.3 equivalences between the second Big Five system WKL and the following third-order theorems; all definitions may be found in Section 1.3.2.

- A cadlag function on the unit interval is bounded (or: Riemann integrable).
- A cadlag function on the unit interval has a supremum.
- A regulated function on the unit interval is bounded.
- A bounded upper semi-continuous[2] function on $[0, 1]$ has a supremum.
- A bounded Baire 1 function $F : [0, 1] \to \mathbb{R}$ has a supremum.

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[2] A 'famous' recent reference for the study of semi-continuity is Villani's work [99].

- A bounded upper semi-continuous function on the unit interval that has a supremum, attains it.
- Cousin's lemma for cadlag (or: lower semi-continuous) functions.
- Cousin's lemma for regulated $F : [0,1] \to \mathbb{R}$ such that $F(x) = \frac{F(x-)+F(x+)}{2}$ for all $x \in [0,1]$.
- Cousin's lemma for quasi-continuous functions.
- Cousin's lemma for Baire 1 functions.
- ...

While cadlag -or even Baire 1- functions can be said to be 'close to continuous', quasi-continuous functions can be quite exotic, as discussed in Remark 2.13.

We obtain similar equivalences for the other Big Five systems, namely ACA_0 (Section 2.4), ATR_0 (Section 2.6), and Π^1_1-CA_0 (Section 2.5), involving the Jordan decomposition theorem, Cousin's lemma, and supremum principles. We suggest many other possible equivalences involving third-order theorems, i.e. this paper may be lengthy but only scratches the surface of what is possible. In evidence, we sketch similar results for WWKL_0 and the Vitali covering theorem in Section 2.3.3. Thus, the distinction between second- and third-order statements does not seem that crucial to RM as there are **many** interesting equivalences across this distinction.

Now, many of the aforementioned results are based on the higher-order RM of the following central axiom from [55]:

$$(\exists E : \mathbb{N}^{\mathbb{N}} \to \{0,1\})(\forall f \in \mathbb{N}^{\mathbb{N}})\big((\exists n \in \mathbb{N})(f(n) = 0) \leftrightarrow E(f) = 0\big). \tag{$\exists^2$}$$

The functional $E : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is *dis*continuous at $f = 11\dots$ and is usually called 'Kleene's quantifier $\exists^2$'. Kohlenbach shows the equivalence between the existence of a discontinuous function on $\mathbb{R}$ and $(\exists^2)$ in [55, §3]. We establish a number of interesting equivalences for $(\exists^2)$ in Section 2.7, including the well-known fact that the Riemann integrable functions are not closed under composition.

Finally, we show in Section 2.8 that *slight* variations or generalisations of all the aforementioned third-order statements cannot be proved from the Big Five or $(\exists^2)$, and *much* stronger systems. This is done by deriving from these statements the following version of the uncountability of the reals:

$$\mathsf{NIN}_{[0,1]} : \text{there is no injection from the unit interval } [0,1] \text{ to } \mathbb{N}.$$

Basic mathematical fact as $\mathsf{NIN}_{[0,1]}$ may be, it cannot be proved in Z_2^ω from Section A.3, which is a conservative extension of second-order arithmetic Z_2. As a side-result, many well-known inclusions among function spaces, like the statement *all regulated functions are Baire 1*, also imply $\mathsf{NIN}_{[0,1]}$; these inclusions can therefore not be proved in the Big Five and much stronger systems.

In conclusion, many third-order statements fall into the Big Five classification, while *slight* variations or generalisations of the former fall *far outside* this classification. We have no explanation for this phenomenon at this point.

1.3. **Preliminaries.**

We briefly discuss Reverse Mathematics (Section 1.3.1) and introduce some mainstream definitions (Section 1.3.2).

1.3.1. *Introducing Reverse Mathematics.*

We refer to [95] for a basic introduction to RM and to [23, 93, 94] for an overview of RM. We expect familiarity with RM, including Kohlenbach's *higher-order* RM from [55]. A more detailed description of the latter, including the definition of the base theory RCA_0^ω, can be found in a technical appendix (Section A). We do introduce the language of higher-order RM, namely as follows.

First of all, in contrast to 'classical' RM based on L_2, the language of *second-order arithmetic* Z_2, higher-order RM uses L_ω, the richer language of *higher-order arithmetic*. Indeed, while L_2 is restricted to natural numbers and sets of natural numbers, L_ω can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of *all finite types* $\mathbf{T}$, defined by the two clauses:

$$\text{(i) } 0 \in \mathbf{T} \text{ and (ii) if } \sigma, \tau \in \mathbf{T} \text{ then } (\sigma \to \tau) \in \mathbf{T},$$

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type σ to objects of type τ. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, we note that Z_2 only deals with objects of type 0 and 1.

Secondly, the language L_ω includes variables $x^\rho, y^\rho, z^\rho, \ldots$ of any finite type $\rho \in \mathbf{T}$. Types may be omitted when they can be inferred from context. The constants of L_ω include the type 0 objects $0, 1$ and $<_0, +_0, \times_0, =_0$ which are intended to have their usual meaning as operations on $\mathbb{N}$. Equality at higher types is defined in terms of '$=_0$' as follows: for any objects x^τ, y^τ, we have

$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \ldots z_k^{\tau_k})[x z_1 \ldots z_k =_0 y z_1 \ldots z_k], \tag{1.1}$$

if the type τ is composed[3] as $\tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0)$. Furthermore, L_ω also includes the *recursor constant* $\mathbf{R}_\sigma$ for any $\sigma \in \mathbf{T}$, which allows for iteration on type σ-objects. Formulas and terms are defined as usual.

Thirdly, while not strictly speaking necessary, it is often convenient to explicitly include types for *finite sequences* of objects. For a given type ρ, the associated type ρ^* is the type of finite sequences of type ρ objects. We discuss the latter and related notations in detail in Notation A.5.

Finally, sets of objects of any finite type can be represented via characteristic functions in L_ω, an approach well-known from measure and probability theory and adopted in this paper as in Definition 1.5.

1.3.2. *Some definitions.* We introduce some standard definitions from analysis, all rather mainstream and taking place in RCA_0^ω.

First of all, we use the standard definition of (uniform) continuity as follows, where $I \equiv [0, 1]$ is the unit interval.

Definition 1.1. [Continuity]

- A function $F : [0, 1] \to \mathbb{R}$ is *continuous at* $x \in [0, 1]$ if

$$(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall y \in [0, 1])(|x - y| < \tfrac{1}{2^N} \to |F(x) - F(y)| < \tfrac{1}{2^k}). \tag{1.2}$$

  A function $F : [0, 1] \to \mathbb{R}$ is *continuous* if (1.2) holds for all $x \in [0, 1]$
- A *modulus* of continuity is any $G : (\mathbb{N} \times \mathbb{R}) \to \mathbb{N}$ such that $G(k, x) = N$ as in (1.2), for $k \in \mathbb{N}, x \in [0, 1]$.
- A function $F : [0, 1] \to \mathbb{R}$ is *uniformly* continuous if:

$$(\forall k \in \mathbb{N})(\exists N \in \mathbb{N})(\forall x, y \in [0, 1])(|x - y| < \tfrac{1}{2^N} \to |F(x) - F(y)| < \tfrac{1}{2^k}). \tag{1.3}$$

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[3] We recall the convention of right associativity of the type arrow, i.e. the type $\tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0)$ stands for $\tau_1 \to (\tau_2 \to (\ldots \to (\tau_k \to 0) \ldots))$.

- A *modulus of uniform continuity* is any $h : \mathbb{N} \to \mathbb{N}$ such that $h(k) = N$ as in (1.3) for any $k \in \mathbb{N}$.

Secondly, we shall study the following weaker notions, many of which are well-known and hark back to the days of Baire, Darboux, Hankel, and Volterra ([4, 5, 20, 37, 38, 101]). We will use 'sup' and related operators in the same 'virtual' or 'comparative' way as in second-order RM (see e.g. [94, X.1]). In this way, a formula of the form '$\sup A > a$' makes sense as shorthand for a formula in the language of all finite types, even when $\sup A$ need not exist in RCA_0^ω. As in [6, 7], the definition of Baire n-function proceeds via (external) induction over standard n. Sets are defined in Definition 1.5 below, namely via characteristic functions.

Definition 1.2. For $f : [0, 1] \to \mathbb{R}$, we have the following definitions:

- f is *upper semi-continuous* at $x_0 \in [0, 1]$ if $f(x_0) \geq_{\mathbb{R}} \limsup_{x \to x_0} f(x)$,
- f is *lower semi-continuous* at $x_0 \in [0, 1]$ if $f(x_0) \leq_{\mathbb{R}} \liminf_{x \to x_0} f(x)$,
- f is *quasi-continuous* at $x_0 \in [0, 1]$ if for $\epsilon > 0$ and an open neighbourhood U of x_0, there is a non-empty open $G \subset U$ with $(\forall x \in G)(|f(x_0) - f(x)| < \varepsilon)$.
- f is *cliquish* at $x_0 \in [0, 1]$ if for $\epsilon > 0$ and an open neighbourhood U of x_0, there is a non-empty open $G \subset U$ with $(\forall y, z \in G)(|f(y) - f(z)| < \varepsilon)$.
- f is *regulated* if for every x_0 in the domain, the 'left' and 'right' limit $f(x_0-) = \lim_{x \to x_0-} f(x)$ and $f(x_0+) = \lim_{x \to x_0+} f(x)$ exist.
- f is *càdlàg* if it is regulated and $f(x) = f(x+)$ for $x \in [0, 1)$.
- f is *Darboux* if it has the intermediate value property, i.e. if $a, b \in [0, 1], c \in \mathbb{R}$ are such that $a \leq b$ and either $f(a) \leq c \leq f(b)$ or $f(b) \leq c \leq f(a)$, then there is $d \in [a, b]$ with $f(d) = c$.
- f is *Baire 0* if it is a continuous function.
- f is *Baire $n+1$* if it is the pointwise limit of a sequence of Baire n functions.
- f is *effectively Baire n* ($n \geq 2$) if there is a sequence $(f_{m_1,\dots,m_n})_{m_1,\dots,m_n \in \mathbb{N}}$ of continuous functions such that for all $x \in [0, 1]$, we have
$$f(x) = \lim_{m_1 \to \infty} \lim_{m_2 \to \infty} \dots \lim_{m_n \to \infty} f_{m_1,\dots,m_n}(x).$$
- f is *Baire 1** if[4] there is a sequence of closed sets $(C_n)_{n \in \mathbb{N}}$ such $[0, 1] = \cup_{n \in \mathbb{N}} C_n$ and $f_{\upharpoonright C_m}$ is continuous for all $m \in \mathbb{N}$.
- f is *continuous almost everywhere* if it is continuous at all $x \in [0, 1] \setminus E$, where E is a measure zero[5] set.
- f is *pointwise discontinuous* if for any $x \in [0, 1]$ and $\varepsilon > 0$, there is $y \in [0, 1]$ such that f is continuous at y and $|x - y| < \varepsilon$ (Hankel, 1870, [37]).

As to notations, a common abbreviation is 'usco' and 'lsco' for the first two items, while one often just writes 'cadlag', i.e. without the accents. Moreover, if a function has a certain weak continuity property at all reals in $[0, 1]$ (or its intended domain), we say that the function has that property.

Regarding the notion of 'effectively Baire n' in Definition 1.2, the latter is used, using codes for continuous functions, in second-order RM (see [6, 7]). Baire himself notes in [4, p. 69] that Baire 2 functions can be *represented* by effectively

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[4] The notion of Baire 1* goes back to [24] and equivalent definitions may be found in [50]. In particular, Baire 1* is equivalent to the Jayne-Rogers notion of *piecewise continuity* from [46].

[5] A set $A \subset \mathbb{R}$ is *measure zero* if for any $\varepsilon > 0$ there is a sequence of basic open intervals $(I_n)_{n \in \mathbb{N}}$ such that $\cup_{n \in \mathbb{N}} I_n$ covers A and has total length below ε. Note that this notion does not depend on (the existence of) the Lebesgue measure.

Baire 2 functions. By Theorem 2.34, there is a significant difference between the latter two notions. Similarly, cliquish functions are exactly those functions that can be expressed as the sum of two quasi-continuous functions ([10, 64]). Nonetheless, comparing Theorems 2.14 and 2.34, these notions behave fundamentally different in RM. Analogously, functions continuous almost everywhere are exactly those functions that can be expressed as the sum of two 'strong' quasi-continuous functions (see [35] for the latter notion).

Thirdly, the notion of *bounded variation* (abbreviated BV) was first explicitly[6] introduced by Jordan around 1881 ([47]) yielding a generalisation of Dirichlet's convergence theorems for Fourier series. Indeed, Dirichlet's convergence results are restricted to functions that are continuous except at a finite number of points, while functions of bounded variation can have (at most) countable many points of discontinuity, as already studied by Jordan, namely in [47, p. 230]. Nowadays, the *total variation* of $f : [a, b] \to \mathbb{R}$ is defined as follows:

$$V_a^b(f) := \sup_{a \leq x_0 < \cdots < x_n \leq b} \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|. \tag{1.4}$$

If this quantity exists and is finite, one says that f has bounded variation on $[a, b]$. Now, the notion of bounded variation is defined in [69] *without* mentioning the supremum in (1.4); see also [11, 12, 57]. Hence, we shall distinguish between the following notions. Jordan seems to use item (a) of Definition 1.3 in [47, p. 228-229].

Definition 1.3. [Variations on variation]

(a) The function $f : [a, b] \to \mathbb{R}$ *has bounded variation* on $[a, b]$ if there is $k_0 \in \mathbb{N}$ such that $k_0 \geq \sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|$ for any partition $x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b$.

(b) The function $f : [a, b] \to \mathbb{R}$ *has a variation* on $[a, b]$ if the supremum in (1.4) exists and is finite.

The fundamental theorem about BV-functions (see e.g. [47, p. 229]) is as follows.

Theorem 1.4 (Jordan decomposition theorem)**.** *A function $f : [0, 1] \to \mathbb{R}$ of bounded variation is the difference of two non-decreasing functions $g, h : [0, 1] \to \mathbb{R}$.*

Theorem 1.4 has been studied extensively via second-order representations in e.g. [36, 57, 69, 104]. The same holds for constructive analysis by [11, 12, 39, 85], involving different (but related) constructive enrichments. Now, arithmetical comprehension suffices to derive Theorem 1.4 for various kinds of second-order *representations* of BV-functions in [57, 69]. By contrast, the results in [75–78] show that the Jordan decomposition theorem is even 'explosive': combining with the Suslin functional from $\Pi_1^1\text{-}\mathsf{CA}_0^\omega$ (see Section A.3), one derives $\Pi_2^1\text{-}\mathsf{CA}_0$.

Fourth, we shall make use of the following notion of (open and closed) set, which was studied in e.g. [75–78, 89].

Definition 1.5. [Sets in RCA_0^ω] We let $Y : \mathbb{R} \to \mathbb{R}$ represent subsets of $\mathbb{R}$ as follows: we write '$x \in Y$' for '$Y(x) >_{\mathbb{R}} 0$' and call a set $Y \subseteq \mathbb{R}$ 'open' if for every $x \in Y$, there is an open ball $B(x, \frac{1}{2^N}) \subset Y$ with $N \in \mathbb{N}$. A set Y is called 'closed' if the complement is open.

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[6] Lakatos in [59, p. 148] claims that Jordan did not invent or introduce the notion of bounded variation in [47], but rather discovered it in Dirichlet's 1829 paper [21].

For open Y as in the previous definition, the formula '$x \in Y$' has the same complexity (modulo higher types) as in second-order RM (see [94, II.5.6]), while given $(\exists^2)$ from Section 1, the former becomes a 'proper' characteristic function, only taking values '0' and '1'. Hereafter, an 'open set' refers to Definition 1.5, while 'RM-open set' refers to the second-order definition from RM. For simplicity, we sometimes assume $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ and work with characteristic functions of open sets directly. Nonetheless, combining Theorem 2.2 and [94, II.7.1], an RM-open set is indeed an open set as in Definition 1.5, working over RCA_0^ω.

Next, the notion of 'countable set' can be formalised in various ways, namely via Definitions 1.6 and 1.7.

Definition 1.6. [Enumerable sets of reals] A set $A \subset \mathbb{R}$ is *enumerable* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(\forall x \in \mathbb{R})(x \in A \rightarrow (\exists n \in \mathbb{N})(x =_{\mathbb{R}} x_n))$.

This definition reflects the RM-notion of 'countable set' from [94, V.4.2]. We note that given μ^2 from Section A.3, we may replace the final implication in Definition 1.6 by an equivalence. Our definition of 'countable set' is now as follows in RCA_0^ω.

Definition 1.7. [Countable subset of $\mathbb{R}$] A set $A \subset \mathbb{R}$ is *countable* if there exists $Y : \mathbb{R} \rightarrow \mathbb{N}$ such that $(\forall x, y \in A)(Y(x) =_0 Y(y) \rightarrow x =_{\mathbb{R}} y)$. If $Y : \mathbb{R} \rightarrow \mathbb{N}$ is also *surjective*, i.e. $(\forall n \in \mathbb{N})(\exists x \in A)(Y(x) = n)$, we call A *strongly countable*.

The first part of Definition 1.7 is from Kunen's set theory textbook ([58, p. 63]) and the second part is taken from Hrbacek-Jech's set theory textbook [44] (where the term 'countable' is used instead of 'strongly countable'). For the rest of this paper, 'strongly countable' and 'countable' shall exclusively refer to Definition 1.7, *except when explicitly stated otherwise.*

2. MAIN RESULTS

2.1. Introduction.

We obtain the following results in Sections 2.2-2.8.

- We study the connection between continuous functions on the reals and their codes in Section 2.2, mostly working over RCA_0^ω or assuming WKL.
- We obtain numerous equivalences involving the Big Five and third-order theorems about (possibly) discontinuous functions (Sections 2.3-2.6).
- We obtain equivalences for $(\exists^2)$ in Section 2.7 where the associated principles also stem from mainstream mathematics.
- In Section 2.8, we show that slight variations or generalisations from the third-order statements in the previous three items cannot be proved from the Big Five, $(\exists^2)$, and much stronger systems, like Z_2^ω from Section A.3.

As discussed in Remark 2.13, some of our results deal with functions 'close to continuous', like the cadlag ones, while other results deal with functions that can be 'far from continuous', like the quasi-continuous ones.

Finally, we discuss some known results due to Kohlenbach regarding continuous and discontinuous functions in the following remark.

Remark 2.1. First of all, Kohlenbach establishes a number of interesting 'coding results' for functions on $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ in [53, §4], as follows.

- By [53, Theorem 4.4], RCA_0^ω proves the equivalence between the following for a functional $Y : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$ continuous on Baire space $\mathbb{N}^{\mathbb{N}}$:
  - the functional Y has a *continuous* modulus of continuity,

  - there is a total *RM-code* ([94, II.6.1]) that equals Y on $\mathbb{N}^{\mathbb{N}}$,
  - there is a total *Kleene associate* ([53, Def. 4.3]) that equals Y on $\mathbb{N}^{\mathbb{N}}$.
- Using a construction due to Dag Normann, $\mathsf{RCA}_0^\omega + \mathsf{WKL}$ proves that a continuous $Y : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ has a modulus of (uniform) continuity ([53, Prop. 4.10]). By the previous items, there is also an RM-code that equals Y on $2^{\mathbb{N}}$. In this way, the usual second-order RM-results apply to such Y, namely via the aforementioned code. For instance, over $\mathsf{RCA}_0^\omega + \mathsf{WKL}$, such Y is bounded on $2^{\mathbb{N}}$ by [94, IV.2.2], and similar results apply immediately.

Secondly, working over RCA_0^ω, Kohlenbach establishes a number of interesting equivalences involving discontinuous functions in [55, §3], as follows.

- The axiom $(\exists^2)$ from Section 1 is equivalent to the existence of a discontinuous function on $\mathbb{R}$, like e.g. Heaviside's function.
- The axiom $(\exists^2)$ from Section 1 is equivalent to (μ^2), i.e. the existence of Feferman's mu-operator from Section A.3.

Using classical logic, the first item yields that $\neg(\exists^2)$ is equivalent to *Brouwer's theorem*, i.e. the statement that all functions on $\mathbb{R}$ are continuous. In the below, we will make use of the above facts, often without very detailed references.

2.2. **From codes to continuous functions and back again.** We establish the following connections between continuous functions on the reals and their codes.

- A code for a continuous function on $\mathbb{R}$ represents a third-order continuous function, working over RCA_0^ω (Theorem 2.2).
- A third-order continuous function on $[0, 1]$ can be represented by an RM-code (Theorem 2.3), working over $\mathsf{RCA}_0^\omega + \mathsf{WKL}$.
- Over RCA_0^ω, WKL is equivalent to basic properties of (third-order) continuous functions on the unit interval (Theorem 2.8).

The proof of Theorem 2.3 is rather involved, while similar results like the boundedness of continuous functions, have (more) basic proofs by Theorems 2.6 and 2.7.

First of all, RCA_0^ω is a conservative extension of RCA_0 (see e.g. Remark A.3). In this light, it is desirable that theorems of RCA_0 also yield theorems of RCA_0^ω. Given the coding practise of RM, this is not always straightforward and we therefore establish Theorem 2.2, which expresses that (second-order) codes for continuous functions give rise to third-order continuous functions, working in the base theory. Our definition of 'RM-code for continuous function' is the standard one ([94, II.6.1]) and as in the latter, we often identify a code and the function it represents. The following proof is also evidence for the necessity of $\mathsf{QF\text{-}AC}^{1,0}$ in RCA_0^ω.

Theorem 2.2 (RCA_0^ω). *Let Φ be an RM-code for an $\mathbb{R} \to \mathbb{R}$-function. There is a third-order $F : \mathbb{R} \to \mathbb{R}$ such that $F(x)$ equals the value of Φ at x for any $x \in \mathbb{R}$.*

Proof. For total RM-codes of functionals $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, one applies $\mathsf{QF\text{-}AC}^{1,0}$ to:

$$\text{'the RM-code is defined at each point of } \mathbb{N}^{\mathbb{N}}\text{'}$$

to obtain a third-order functional $\Psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ equal to the (value of the) code everywhere; this argument may be found in e.g. the proof of 1) → 3) in [53, Prop. 4.4]. We now show that the same procedure works for RM-codes of $[0, 1] \to \mathbb{R}$-functions. Indeed, a code for an $\mathbb{R} \to \mathbb{R}$-function is a set $\Phi \subset [\mathbb{N} \times \mathbb{Q} \times \mathbb{Q}^+ \times \mathbb{Q} \times \mathbb{Q}^+]$ satisfying certain properties. The formula 'Φ is total on $\mathbb{R}$' has the following form (which is suitable for $\mathsf{QF\text{-}AC}^{1,0}$):

$$(\forall x \in \mathbb{R}, k \in \mathbb{N})(\exists (n,a,r,b,s) \in \Phi)(d(x,a) <_{\mathbb{R}} r \wedge s <_{\mathbb{Q}} \tfrac{1}{2^k}). \tag{2.1}$$

Intuitively, the fourth component $b \in \mathbb{Q}$ of Φ contains rational approximations to the value of the code Φ at $x \in \mathbb{R}$, while $s \in \mathbb{Q}$ is an upper bound on the difference between b and the value of Φ at $x \in \mathbb{R}$. Hence, apply $\mathsf{QF\text{-}AC}^{1,0}$ to (2.1) to obtain G such that $G(x,k)$ is the quintuple as in (2.1). Note that $G(x,k)(4)$ may not be extensional on the reals as in item (e) in Definition A.4. Now define $F : \mathbb{R} \to \mathbb{R}$ by $[F(x)](k) := G(x,k+1)(4)$ and note that F is indeed extensional on the reals. Clearly, $F(x)$ equals the value of Φ at every $x \in \mathbb{R}$. □

Unfortunately, the theorem does not generalise to codes for Baire 1 functions (in the sense of [6, 7]). Indeed, by Theorem 2.28, $(\exists^2)$ is equivalent to the statement that a code for a Baire 1 function represents a third-order function.

Secondly, by Theorem 2.2, we can make the leap from 'second-order codes for continuous functions' to 'third-order continuous functions' without problems. Theorem 2.3 expresses that the other direction is possible too, additionally assuming *weak König's lemma* WKL in the base theory. As will become clear, the associated proof is based on that of [53, Prop. 4.10], which is in turn based on a construction due to Dag Normann, as noted in [53, p. 94].

Theorem 2.3 ($\mathsf{RCA}_0^\omega + \mathsf{WKL}$)**.** *Any $F : \mathbb{R} \to \mathbb{R}$ continuous on $[0,1]$ has a modulus of uniform continuity $h : \mathbb{N} \to \mathbb{N}$ on $[0,1]$.*

Proof. First of all, [53, Prop. 4.10] establishes that, working over $\mathsf{RCA}_0^\omega + \mathsf{WKL}$, any $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ continuous on $2^\mathbb{N}$ has a modulus of *uniform* continuity. In the proof of [53, Prop. 4.4], there is an explicit formula for an RM-code defined in terms of such a modulus. For completeness, we now sketch the proof of [53, Prop. 4.10], which consists of two steps. As a first step, the formula $A(k,\sigma)$ in (2.2) is a slight modification of the innermost universal formula in the definition of continuity for F on $2^\mathbb{N}$, where $\sigma^{0^*} \leq_{0^*} 1$ is a finite binary sequence:

$$(\forall g, h \leq_1 1)(\overline{g}|\sigma| =_{0^*} \sigma =_{0^*} \overline{h}|\sigma| \to F(g)(k) = F(h)(k))\big]. \tag{2.2}$$

By definition, we have $(\forall f \leq_1 1, k^0)(\exists N^0)A(k, \overline{f}N)$. Despite the quantifiers in (2.2), WKL suffices to define its characteristic function $\chi_A^{(0\times 0^*)\to 0}$, i.e. we have

$$(\forall \sigma^{0^*} \leq_{0^*} 1, k^0)(\chi_A(k,\sigma) = 0 \leftrightarrow A(k,\sigma)). \tag{2.3}$$

The existence of χ_A is proved in the next paragraph of this proof. Now, $\sigma \in T_k \leftrightarrow \neg A(k,\sigma)$ defines a 0/1-tree T_k, which has no path by the above. By WKL, the tree T_k is finite for any k^0, implying $(\forall k^0)(\exists N^0)(\forall f \leq_1 1)A(k, \overline{f}N)$. The latter yields

$$(\forall k^0)(\exists N^0)(\forall \sigma^{0^*} \leq_{0^*} 1)[|\sigma| = N \to A(k, \overline{\sigma}N)],$$

and applying $\mathsf{QF\text{-}AC}^{0,0}$ readily yields the required modulus of uniform continuity.

As a second step, we now establish the existence of χ_A as in (2.3). Due to the continuity of F, it suffices to prove the existence of χ such that:

$$(\forall \sigma^{0^*} \leq 1, k^0)\big(\chi(\sigma,k) = 0 \leftrightarrow (\forall \tau^{0^*} \leq 1)(F(\sigma * \tau * 00\dots)(k) = F(\sigma * 00\dots)(k))\big).$$

Now define a sequence of 0/1 trees as follows: $\tau \in T_{k,\sigma}$ in case either of the following:

- $(\forall \gamma^{0^*} \leq 1)\big(|\gamma| \leq |\tau| \to F(\sigma * \tau * 00\dots)(k) = F(\sigma * \gamma * 00\dots)(k)\big)$,

- $(\exists \tilde{\gamma}^{0^*} \leq 1)(\exists l \leq |\tau|)\big(\tau = \tilde{\gamma} * \overline{00\dots}l$ with $|\tilde{\gamma}|$ minimal such that:
$$F(\sigma * \tilde{\gamma} * 00\dots)(k) \neq F(\sigma * 00\dots)(k)\big).$$

Now, each tree $T_{k,\sigma}$ is infinite and by the sequential version of WKL (equivalent to WKL by [54, Prop. 3.1]), there is a sequence of paths $f_{k,\sigma}$ in $T_{k,\sigma}$ for $k \in \mathbb{N}$ and $\sigma^{0^*} \leq 1$. Using the continuity of F, one readily verifies that for $\sigma^{0^*} \leq 1, k^0$:

$$(\forall \tau^{0^*} \leq 1)(F(\sigma*\tau*00\dots)(k) = F(\sigma*00\dots)(k)) \leftrightarrow F(\sigma*f)(k) = F(\sigma*00\dots)(k).$$

which is as required to obtain (2.3). For the next paragraph, we point out the following, assuming a fixed enumeration of all finite sequences: if we require that in the second item defining $T_{k,\sigma}$, the sequence $\tilde{\gamma}$ is the minimal sequence with the stated property, measured by sequence number, then $\neg A(k,\sigma)$ implies that $T_{k,\sigma}$ has a single branch witnessing $\neg A(k,\sigma)$, while if $A(k,\sigma)$ holds, any branch in $T_{k,\sigma}$ will witness $A(k,\sigma)$.

Finally, we modify the previous paragraph to accommodate functions continuous on the unit interval. For convenience, we work with *ternary* trees where a tree element $\sigma \in \{-1,0,1\}^{<\mathbb{N}}$ is a finite sequence in the alphabet $\{-1,0,1\}$. Similarly, $f \in \{-1,0,1\}^{\mathbb{N}}$ means that $f(k) \in \{-1,0,1\}$ for all $k \in \mathbb{N}$. Clearly, each $f \in \{-1,0,1\}^{\mathbb{N}}$ codes a real number $\rho(f) = 1/2 + \sum_{n=0}^{\infty} f(n)2^{-(n+2)}$, where the partial sums form a fast converging Cauchy-sequence as in Definition A.4. Now, it is well-known in computer science that any k-ary tree admits a representation as a binary tree (see [51, 63]), and the associated (effective) conversion is sometimes called the *Knuth transform* ([81, p. 146]). As expected, the latter is readily formalised in RCA_0 and hence WKL is equivalent to the existence of a path for infinite ternary trees, and the same for the associated sequential versions from [54, Prop. 3.1].

Next, fix $F : \mathbb{R} \to \mathbb{R}$ continuous on $[0,1]$ and consider the formula:

$$A(\sigma, n) \equiv (\forall g \in \{-1,0,1\}^{\mathbb{N}})[|F(\rho(\sigma * 00\dots)) - F(\rho(\sigma * g))| \leq \tfrac{1}{2^n}].$$

We now use WKL to prove the existence of a function $B^{(0^*\times 0)\to 0}$ such that for all $\sigma \in \{-1,0,1\}^{<\mathbb{N}}$ and $n \in \mathbb{N}$:

$$A(\sigma, n+1) \to (B(\sigma,n) = 0) \to A(\sigma, n). \tag{2.4}$$

Using (2.4), one readily finds a modulus of uniform continuity for F as in the first part of the proof. In order to define B satisfying (2.4), we define a sequence $S_{\sigma,n}$ of infinite ternary trees. By WKL, these have a sequence of infinite branches, and the actual B depends on which sequence of branches we select. We use the convention that the elements of $\{-1,0,1\}^{<\mathbb{N}}$ are enumerated first by length, and then by the lexicographical ordering.

We now define $S_{\sigma,n}$ as follows: for the (finite) set of sequences $\gamma \in \{-1,0,1\}^{<\mathbb{N}}$ of length $k \in \mathbb{N}$, there are two cases to be considered, namely items (1) and (2).

(1) If for all $\gamma \in \{-1,0,1\}^{<\mathbb{N}}$ of length k, we have that for all $l \leq k$,
$$\big|[F(\rho(\sigma * 00\dots))](l) - [F(\rho(\sigma * \gamma * 00\dots))](l)\big| \leq_{\mathbb{Q}} 2^{-n} + 2^{1-l}$$
then all $\gamma \in \{-1,0,1\}^{<\mathbb{N}}$ of length k are in $S_{\sigma,n}$.

(2) If the previous item is false, there is a least γ' of length $\leq k$ such that for some $l \leq k$ we have that
$$\big|[F(\rho(\sigma * 00\dots))](l) - [F(\rho(\sigma * \gamma' * 00\dots))](l)\big| >_{\mathbb{Q}} 2^{-n} + 2^{1-l}$$
We then let the extension $\gamma' * 0 \cdots 0$ to a sequence of length k be in $S_{\sigma,n}$.

We now make two important observations about the trees $S_{\sigma,k}$. Firstly, if for a fixed $k \in \mathbb{N}$, there is a sequence of length k in $S_{\sigma,n}$ following item (2), then the same sequence, only extended with zeros, will be the single sequence of length k' for any $k' > k$. In this case, the only branch in $S_{\sigma,n}$ is a ternary g^1 such that $|F(\rho(\sigma * 0^*)) - F(\rho(\sigma * g))| > 2^{-n}$. Secondly, if $|F(\rho(\sigma * 00\dots)) - F(\rho(\sigma * g))| \leq 2^{-(n+1)}$ holds for all ternary g^1, then this formula holds for all branches g in $S_{\sigma,n}$.

Now, let $g_{\sigma,n}$ be a branch in $S_{\sigma,n}$ provided by sequential WKL. Then at least one of the following two items is the case:

- $|F(\rho(\sigma * 00\dots)) - F(\rho(\sigma * g_{\sigma,n}))| < 2^{-n}$,
- $|F(\rho(\sigma * 00\dots)) - F(\rho(\sigma * g_{\sigma,n}))] > 2^{-(n+1)}$.

The $n+4$-th rational approximation of $|F(\rho(\sigma * 00\dots)) - F(\rho(\sigma * g_{\sigma,n}))|$ tells us which item holds. In case the first item holds, we put $B(\sigma, n) = 0$, and 1 otherwise. This function B satisfies (2.4) and we are done. □

The following remark discusses the representation used in the previous proof.

Remark 2.4 (Representations)**.** Regarding the proof of Theorem 2.3, the use of sequences based on $\{-1, 0, 1\}$ and the map ρ is known as the *negative binary representation*. The set of such representations is a computable retract of the set of representations as given in Section A.2; this representation is useful for representing $[0, 1]$ over a compact space, or $\mathbb{R}$ over a σ-compact space, as in e.g. the proof of item (xxiii) of Theorem 2.9.

As an exercise, the reader can verify that a continuous *increasing* function on $[0, 1]$ has a modulus of continuity in RCA_0^ω. The following corollary is useful.

Corollary 2.5 ($\mathsf{RCA}_0^\omega + \mathsf{WKL}$)**.** *For a sequence $(F_n)_{n\in\mathbb{N}}$ of continuous $[0, 1] \to \mathbb{R}$-functions, there is a sequence of RM-codes $(\Phi_n)_{n\in\mathbb{N}}$ such that $F_n(x)$ equals $\Phi_n(x)$ for all $x \in [0, 1]$ and $n \in \mathbb{N}$.*

Proof. One readily defines an RM-code from a modulus of uniform continuity for a $[0, 1] \to \mathbb{R}$-function. The principle WKL is equivalent to the 'sequential' version of WKL, i.e. that for a sequence of infinite 0/1-trees, there is a sequence of paths through the respective trees ([54, Prop. 3.1]). The latter readily yields the required sequence of RM-codes, via a sequence of moduli of uniform continuity. □

Thirdly, a continuous function on $[0, 1]$ has a modulus of uniform continuity by Theorem 2.3 but the proof is rather involved. As it happens, the proof that continuous functions are bounded is easier, and (mostly) suffices for the development of higher-order RM.

Theorem 2.6 ($\mathsf{RCA}_0^\omega + \mathsf{WKL}$)**.** *A continuous $F : [0, 1] \to \mathbb{R}$ is bounded.*

Proof. For F as in the theorem, define $G : 2^{\mathbb{N}} \to \mathbb{N}$ by

$$G(f) := \lceil [F(\mathbb{r}(f))](2) \rceil + 1, \tag{2.5}$$

where $\mathbb{r}(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}$ can also be found in Definition A.4. We now split the proof in two cases. First of all, if G as in (2.5) is discontinuous on $2^{\mathbb{N}}$, we obtain $(\exists^2)$ as the latter is equivalent to the existence of a discontinuous function on $\mathbb{N}^{\mathbb{N}}$ by [55, Prop. 3.7]. Suppose F is unbounded on $[0, 1]$; by the continuity of the former, we have $(\forall n \in \mathbb{N})(\exists q \in \mathbb{Q} \cap [0, 1])(|F(q)| >_{\mathbb{R}} n)$. Applying $\mathsf{QF\text{-}AC}^{0,0}$, we obtain a sequence $(q_n)_{n\in\mathbb{N}}$ such that $(\forall n \in \mathbb{N})(q_n \in [0, 1] \wedge |F(q_n)| > n)$. Since $(\exists^2) \to \mathsf{ACA}_0$, we may use the well-known (second-order) convergence theorems by [94, III.2.7]. Thus, $(q_n)_{n\in\mathbb{N}}$ has a sub-sequence with limit $y \in [0,1]$. Clearly, F is discontinuous at y, a contradiction. Hence, F is bounded on $[0,1]$.

Secondly, if G as in (2.5) is continuous on $2^{\mathbb{N}}$, then it has a modulus of uniform continuity by [53, Prop. 4.11]. Hence, G is bounded on $2^{\mathbb{N}}$, implying that F is also bounded on $[0,1]$; the latter follows by contradiction and the fact that individual real numbers have binary representations in RCA_0 (see [41]). □

Fourth, we recall that WKL is equivalent to the statement *for a code of a uniformly continuous function, there is a modulus of uniform continuity* ([94, IV.2.9]).

Theorem 2.7 (RCA_0^ω+WKL)**.** *A uniformly continuous $F : [0,1] \to \mathbb{R}$ has a modulus of uniform continuity.*

Proof. Let $F : [0,1] \to \mathbb{R}$ be uniformly continuous. In particular, we have

$$(\forall k \in \mathbb{N})(\exists g \in 2^{\mathbb{N}})\underline{(\forall x, y \in [0,1] \cap \mathbb{Q})(|x-y| < \mathbb{r}(g) \to |F(x) - F(y)| \leq \tfrac{1}{2^k})}, \quad (2.6)$$

where $\mathbb{r}(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}$ is a real number in $[0,1]$. As noted in [94, Table 4, Notes], WKL is equivalent to Π_1^0-AC_0, where the latter is:

$$(\forall n \in \mathbb{N})(\exists X \subset \mathbb{N})\varphi(n, X) \to (\exists (Z_n)_{n\in\mathbb{N}})(\forall n \in \mathbb{N})[\varphi(n, Z_n) \wedge Z_n \subseteq \mathbb{N}],$$

for any $\varphi \in \Pi_1^0$. The underlined formula in (2.6) is Π_1^0, as $\lambda q.F((q, q, \dots))$ is merely a sequence of reals if q is a variable over $\mathbb{Q}$. Hence, apply Π_1^0-AC_0 to (2.6) and note that the resulting function yields a modulus of uniform continuity. □

Fifth, we obtain the following equivalences.

Theorem 2.8 (RCA_0^ω)**.** *The following are equivalent to* WKL.

- *A continuous $F : [0,1] \to \mathbb{R}$ is bounded.*
- *A uniformly continuous $F : [0,1] \to \mathbb{R}$ has a modulus of uniform continuity.*
- *A continuous $F : [0,1] \to \mathbb{R}$ is Riemann integrable* ([94, IV.2.7]).

Proof. That WKL implies the first two items from the theorem, follows from Theorems 2.6 and 2.7. To obtain the third item from WKL, use Corollary 2.5, combined with the second-order results for Riemann integration ([94, IV.2.5]). To show that the first item implies WKL, note that an RM-code Φ for a continuous function on $[0,1]$ yields a (third-order) continuous function $F : [0,1] \to \mathbb{R}$ by Theorem 2.2. Then F is bounded and so is the function represented by Φ. We now obtain WKL via [94, IV.2.3]. An analogous proof goes through for the second and third items. Indeed, the latter for codes are equivalent to WKL by [94, IV.2.7 and IV.2.9]. □

In conclusion, we have adapted some of Kohlenbach's 'coding results' from [53, §4], namely from $2^{\mathbb{N}}$ to $[0,1]$. We have presented a fairly constructive but lengthy proof (Theorem 2.3). We have also obtained shorter but less constructive proofs of similar results (Theorems 2.6 and 2.7). Along the way, we have shown that WKL is equivalent to *third-order* statements (see Theorem 2.8). The consensus view here seems to be that (third-order) continuous functions are 'really' second-order, as evidenced by Corollary 2.5. In this way, equivalences like Theorem 2.8 do not really connect second- and third-order arithmetic. The aim of the next section is to exhibit 'more real' connections, i.e. equivalences between WKL and third-order theorems that do not have an obvious second-order counterpart.

2.3. **Equivalences for weak König's lemma.** We obtain equivalences between WKL and certain third-order statements in higher-order RM (Sections 2.3.1 and 2.3.2). In Section 2.3.3, we sketch similar results for the RM of *weak weak König's lemma* (WWKL for short) from [94, X.1].

2.3.1. *Boundedness and supremum principles.* We establish our first series of third-order statements equivalent to WKL (Theorem 2.9), the former being boundedness and supremum principles from analysis. We note in passing that the textbook proof that BV-functions are bounded on $[0,1]$ (see e.g. [2]) goes through in RCA_0^ω.

While some of the theorems under study are basic, others like item (xx) seem advanced as the class of *quasi-continuous* functions goes *far* beyond even the Borel or measurable functions, as discussed in Remark 2.13. Quasi-continuity goes back to Baire ([4]) and is used in domain theory ([17, 30, 31, 60]).

Regarding item (xvi), the assumption $F(x) = \frac{F(x-)+F(x+)}{2}$ and variations is found in e.g. [2, 33, 34, 102]. Regarding item (xxi), cadlag functions are an important class in stochastics and econometrics while Remark 2.13 explains why items (ii)-(iv), (vi)-(vii), (ix), (x), (xviii), and (xix) are non-trivial. Regarding item (xxiv), Darboux sub-classes are topics of study in their own right (see e.g. [1, 64, 65, 79]). The fragment of countable choice $\mathsf{QF\text{-}AC}^{0,1}$ is defined in Section A.1 while the exact role of the Axiom of Choice is discussed in Remark 2.10.

Theorem 2.9 ($\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$)**.** *The following are equivalent to* WKL.

(i) *A regulated* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(ii) *A regulated and continuous almost everywhere* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(iii) *A regulated and pointwise discontinuous* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(iv) *A regulated and not everywhere discontinuous* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(v) *Any usco* $F : [0,1] \to \mathbb{R}$ *is bounded above.*
(vi) *Any usco and not everywhere discontinuous* $F : [0,1] \to \mathbb{R}$ *is bounded above.*
(vii) *Any usco and pointwise discontinuous* $F : [0,1] \to \mathbb{R}$ *is bounded above.*
(viii) *Any lsco* $F : [0,1] \to \mathbb{R}$ *is bounded below.*
(ix) *Any usco and Baire 1 function* $F : [0,1] \to \mathbb{R}$ *is bounded above.*
(x) *Any usco and effectively Baire* n $F : [0,1] \to \mathbb{R}$ *is bounded above* ($n \geq 2$).
(xi) *A regulated and usco* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(xii) *A regulated and quasi-continuous* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(xiii) *A bounded usco function on* $[0,1]$ *that has a supremum, attains it.*
(xiv) *A bounded regulated usco function on* $[0,1]$ *with a supremum, attains it.*
(xv) *A bounded lsco function on* $[0,1]$ *that has an infimum, attains it.*
(xvi) *A regulated* $F : [0,1] \to \mathbb{R}$ *such that* $F(x) = \frac{F(x-)+F(x+)}{2}$ *for all* $x \in (0,1)$, *is bounded.*
(xvii) *A regulated and lsco* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(xviii) *A regulated and Baire 1 function* $F : [0,1] \to \mathbb{R}$ *is bounded.*
(xix) *A regulated effectively Baire* n *function* $F : [0,1] \to \mathbb{R}$ *is bounded* ($n \geq 2$).
(xx) *A bounded and quasi-continuous* $F : [0,1] \to \mathbb{R}$ *has a sup* (*and inf*).
(xxi) *A cadlag function* $F : [0,1] \to \mathbb{R}$ *is bounded* ([83, Problem IV.3]).
(xxii) *A cadlag function* $F : [0,1] \to \mathbb{R}$ *has a sup* (*and inf*).
(xxiii) *A bounded Baire 1 function* $F : [0,1] \to \mathbb{R}$ *has a supremum.*
(xxiv) *A bounded Darboux Baire 1 function* $F : [0,1] \to \mathbb{R}$ *has a supremum.*

We do not use $\mathsf{QF\text{-}AC}^{0,1}$ *in relation to items* (xii), (xvi), (xvii), *and* (xx)-(xxiv).

Proof. First of all, item (i) readily implies WKL as $F(x+) = F(x) = F(x-)$ for all $x \in (0,1)$ in case F is continuous; Theorem 2.8 now yields WKL. For the reversal, assume WKL and fix some regulated $F : [0,1] \to \mathbb{R}$. In case the latter is continuous, it is also bounded by Theorem 2.8. In case F is discontinuous, we have access to $(\exists^2)$ by [55, §3]. Now suppose $(\forall n \in \mathbb{N})(\exists x \in [0,1])(|F(x)| > n)$ and apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain $(x_n)_{n\in\mathbb{N}}$ such that $|F(x_n)| > n$ for all $n \in \mathbb{N}$. Use μ^2 to guarantee $F(x_{n+1}) > \max(n+1, F(x_n))$ for all $n \in \mathbb{N}$, if necessary. Since $(\exists^2) \to \mathsf{ACA}_0$, we have access to the well-known second-order convergence theorems (see [94, III.2]). Thus, there is a convergent sub-sequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, say with limit $y \in [0,1]$. Then either there are infinitely many $n \in \mathbb{N}$ such that $y_n < y$ or infinitely many $m \in \mathbb{N}$ such that $y_m > y$; note that this case distinction is decidable using $\exists^2$. In the former case (the latter being symmetric), $F(y_n)$ becomes arbitrarily large as $n \to \infty$. In particular, $F(y-)$ does not exist, a contradiction, and F must be bounded on $[0,1]$, and item (i) follows.

Secondly, the equivalence for item (v) (and items (ii)-(iv), (vi)-(vii), (viii)-(xi), and (xviii)-(xix)) follows in the same way as for item (i). Indeed, item (v) for instance implies WKL since a continuous function is trivially usco (and lsco, Baire 1, cadlag, or effectively Baire n), while WKL already follows in [94, IV.2.3] from the existence of an upper bound. For the reversal, one proceeds as in the previous paragraph, noting that F cannot be usco at the limit point $y \in [0,1]$. The equivalence involving items (viii)-(xi) and (xviii)-(xix) is now immediate.

Thirdly, item (xiii) readily implies WKL as a continuous function is trivially usco, i.e. combining Theorem 2.2 and [94, IV.2.3] yields WKL. For the reversal, assume WKL and fix an usco function $f : [0,1] \to [0,1]$ that has a supremum $y_0 \in [0,1]$. In case f is continuous, WKL yields an RM-code (Corollary 2.5). Hence, the well-known second-order result in [94, IV.2.3] yields the required maximum. In case F is discontinuous, we have access to $(\exists^2)$ by [55, §3]. By definition, we have $(\forall n \in \mathbb{N})(\exists x \in [0,1])(f(x) \geq y_0 - \frac{1}{2^n})$. Apply $\mathsf{QF\text{-}AC}^{0,1}$ to obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(\forall n \in \mathbb{N})(f(x_n) \geq y_0 - \frac{1}{2^n})$. Since $(\exists^2) \to \mathsf{ACA}_0$, we have access to the well-known second-order convergence theorems (see [94, III.2]). Let $(z_n)_{n\in\mathbb{N}}$ be a convergent sub-sequence of $(x_n)_{n\in\mathbb{N}}$, say with limit w_0. By assumption (f being usco and y_0 its supremum), we have

$$y_0 \geq f(w_0) \geq \limsup_{x\to w_0} f(x) \geq \limsup_{n\to\infty} f(z_n) \geq \lim_{n\to\infty} y_0 - \tfrac{1}{2^n} = y_0,$$

which implies $f(w_0) = y_0$ as required for item (xiii). The equivalence involving items (xiv) and (xv) is now immediate.

Fourth, for items (xii), (xvi), (xvii), and (xxi), the equivalence is proved as for items (i) and (v) with the only modification that $(\forall n \in \mathbb{N})(\exists x \in [0,1])(|F(x)| > n)$ implies $(\forall n \in \mathbb{N})(\exists q \in [0,1] \cap \mathbb{Q})(|F(q)| > n)$ due to the extra conditions in these items. Hence, we can apply $\mathsf{QF\text{-}AC}^{0,0}$ (rather than $\mathsf{QF\text{-}AC}^{0,1}$), included in RCA_0^ω.

Fifth, for item (xx), the latter yields WKL by [94, IV.2.3]; indeed, Theorem 2.2 converts an RM-code for a continuous function into a third-order continuous function, which is trivially quasi-continuous. Now assume WKL and let $F : [0,1] \to \mathbb{R}$ be quasi-continuous and bounded. In case the latter is also continuous, Theorem 2.3 provided a modulus of uniform continuity and [94, IV.2.3] provides the required supremum. In case F is discontinuous, we obtain $(\exists^2)$ by [55, §3]. The usual interval-halving technique (using $\exists^2$) then readily yields the required supremum as

$$(\exists x \in [0,1])(F(x) > r) \leftrightarrow (\exists q \in [0,1] \cap \mathbb{Q})(F(q) > r), \tag{2.7}$$

for any $r \in \mathbb{R}$, as cadlag implies quasi-continuity. An analogous proof goes through for item (xxii), as cadlag functions are quasi-continuous.

For item (xxiii), let $F : [0,1] \to \mathbb{R}$ be the pointwise limit of $(F_n)_{n\in\mathbb{N}}$, where each $F_n : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$. Let Φ_n be an RM-code for F_n and M_n be the modulus of uniform continuity for F_n, all provided by Corollary 2.5 (and Remark 2.1). We may assume that F is not continuous, whence we have access to $\exists^2$ by [55, §3]. We now show that for $r \in \mathbb{Q}$, $\sup_{x\in[0,1]} F(x) > r$ is definable in $\exists^2$; the proof is based on the equivalence between items (A) and (B) below.

Now, $\exists^2$ can (uniformly) convert between various representations of real numbers (see [41] for the latter). Thus, we may assume that any $x \in [0,1]$, which actually is a fast converging sequence of rational numbers (see Definition A.4), is obtained from a negative binary representation f_x as in Remark 2.4. Note that there is a bijective correspondence between the negative binary representations and the fast converging sequences obtained from them.

We let $I(x,k)$ be the interval of reals $y \in [0,1]$ represented by a negative binary representation extending that of $\overline{f_x}k$. We assume Φ_n to be given as a set of pairs of intervals $\langle [a,b],[c,d]\rangle$ with rational endpoints such that, in addition to an approximation requirement, if $y \in [a,b]$ then $F_n(y) \in [c,d]$; this is the most frequently used domain representation. Using $\exists^2$, the latter representation is equivalent to any other (RM-)representation.

We now show that the following are equivalent:

- (A) $\sup_{x\in[0,1]} F(x) > r$,
- (B) There exists $x \in [0,1]$, $n,k \in \mathbb{N}$ such that for $m \geq n$ and $\langle [a,b],[c,d]\rangle \in \Phi_m$, if $j = M_m(k+1)$ and $I(x,j) \cap [a,b] \neq \emptyset$, then $[c,d]$ contains an element $\geq r + 2^{-(k+1)}$.

To show that (A) → (B), assume for some $x \in [0,1]$ that $F(x) > r$ and let $n,k \in \mathbb{N}$ be such that $F_m(x) > r - 2^{-k}$ for all $m \geq n$. We now verify (B) for this choice of n, k and x. Let $m \geq n$ and $\langle [a,b],[c,d]\rangle \in \Phi_m$ with $j = M_m(k+1)$ be such that $I(x,j) \cap [a,b] \neq \emptyset$. Let $y \in [0,1]$ be in this intersection, implying $|x-y| < \frac{1}{2^j}$ and hence $|F_m(x) - F_m(y)| < 2^{-(k+1)}$. Since $F_m(y) \in [c,d]$ and $F_m(y) > r + 2^{-(k+1)}$ by the triangle inequality, (B) follows for the aforementioned choice of n,k,x.

To show that (B) → (A), let x, n and k be as stated in (B). We show that $F(x) > r$ by showing that $F_m(x) \geq r + 2^{-(k+2)}$ for all $m \geq n$. Let $\langle [a,b],[c,d]\rangle \in \Phi_m$ be such that $x \in [a,b]$ and $|d-c| < 2^{-(k+2)}$. Clearly $[a,b] \cap I(x,k) \neq \emptyset$, since e.g. x is in both sets. Since $F_m(x) \in [c,d]$ and $[c,d]$ contains an element $\geq r + 2^{-(k+1)}$, we must have that $F_m(x) \geq r + 2^{-(k+2)}$.

In (B), we first have existential quantifiers for x, n and k, then universal quantifiers over $\mathbb{N}$ and $\mathbb{Q}^4$ and the remaining matrix is decidable in the parameters. As '$(\exists x \in [0,1])$' actually is a quantifier over $\{-1,0,1\}^{\mathbb{N}}$ and the latter is computably identifiable with $2^{\mathbb{N}}$, the equivalence (A) ↔ (B) shows that $\sup_{x\in[0,1]} F(x) > r$ can be expressed as a second-order formula of the form

$$(\exists n \in \mathbb{N})(\exists f \in 2^{\mathbb{N}})(\forall m \in \mathbb{N})R(r,n,f,m),$$

where R is Turing computable in the second-order objects $(\Phi_k)_{k\in\mathbb{N}}$ and $(M_k)_{k\in\mathbb{N}}$. By WKL, the formula $(\exists f \in 2^{\mathbb{N}})(\forall m \in \mathbb{N})R(r, n, f, m)$ is equivalent to:

$$(\forall k \in \mathbb{N})(\exists \sigma \in 2^{<\mathbb{N}})\big[|\sigma| = k \wedge (\forall m \in \mathbb{N})R(r, n, \sigma * 00\ldots, m)\big],$$

which is arithmetical. This shows the existence of $\sup_{x\in[0,1]} F(x)$ for $F : [0, 1] \rightarrow \mathbb{R}$ in Baire 1, given as the limit of a sequence of continuous functions, assuming $\exists^2$. The previous goes through for item (xxiv) since continuous functions on $[0, 1]$ are Darboux, which follows by imitating the second-order intermediate value theorem as can be found in [94, II.6.6]. $\square$

In light of Theorem 2.9, a single second-order equivalence from analysis can give rise to *many different* equivalences in higher-order RM. There is however a limit: while the supremum principle for effectively Baire 2 functions is equivalent to the Big Five system Π^1_1-CA_0 (see Theorem 2.22), the former principle for Baire 1* or Baire 2 functions is not provable in Z_2^ω by Theorem 2.32. Nonetheless, the third-order RM of WKL can only be called *extremely robust* following Section 1.1.

In the next remark, we discuss the role of the Axiom of Choice in Theorem 2.9.

Remark 2.10 (On the Axiom of Choice)**.** The Axiom of Choice (AC for short) plays an interesting role in Theorem 2.9, namely related to the results in [72, 73]. As in the latter, we say that a statement T in the language of finite types *exhibits the Pincherle phenomenon* if the following two items are satisfied.

- The statement T is provable *without* AC but only in relatively strong systems, namely T is provable in Z_2^Ω, but not in Z_2^ω (see Section A.3).
- The statement T is provable in weak systems *assuming* (fragments of) AC, namely the system $\mathsf{RCA}_0^\omega + \mathsf{WKL} + \mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ proves T.

In short, the Pincherle phenomenon is the observation that AC makes certain theorems 'easier to prove' even though we do not strictly *need* AC. We first observed this phenomenon in [73] for a theorem by Salvatore Pincherle from [82, p. 67], while *many* examples may be found in [72] and elsewhere.

One readily verifies that the Pincherle phenomenon is exhibited by items (i), (v), (viii), (xi), (xiii), and (xiv) from Theorem 2.9. Indeed, the model $\mathbf{Q}^*$ of Z_2^ω from [76] is such that there is an unbounded regulated function on $[0, 1]$, i.e. item (i) is not provable in Z_2^ω and the same for the other items. More interestingly, items (ix), (x), (xviii), and (xix) exhibit a kind of *weak* Pincherle phenomenon as these items are already[7] provable in $\mathsf{RCA}_0^\omega + \Sigma^1_1$-$\mathsf{AC}_0$, which is conservative over ACA_0. Thus, the extra 'Baire 1' condition in these items makes them 'easier to prove', where we note that the addition of this condition is non-trivial by Remark 2.13.

Next, we have an important corollary to Theorem 2.9, where a *set* is 'Baire 1' if the characteristic function is Baire 1. The general notion of Baire set may be found in [52, p. 21, Def. 4] under a different name; we refer to [62] for an introduction and to [22, §7] for equivalent definitions, including that of Borel set in Euclidean space.

Theorem 2.11 ($\mathsf{RCA}_0^\omega + \mathsf{WKL}$)**.** *For any open Baire 1 set $O \subset [0, 1]$, there exist $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ such that $x \in O \leftrightarrow (\exists n \in \mathbb{N})(x \in (a_n, b_n))$ for all $x \in [0, 1]$.*

---

[7] The use of $\mathsf{QF}\text{-}\mathsf{AC}^{0,1}$ can be replaced by Σ^1_1-AC_0 in light of the equivalence (A)$\leftrightarrow$(B) from the proof of Theorem 2.9.

Proof. We make use of item (xxiii) of Theorem 2.9. In particular, the proof of this item immediately generalises to infima involving rational parameters, i.e. we have:

for a bounded Baire 1 function $f : [0,1] \to \mathbb{R}$*, there is* $F : \mathbb{Q}^2 \to \mathbb{R}$ *such that for all* $p, q \in \mathbb{Q} \cap [0,1]$*, the real* $F(p,q)$ *equals* $\inf_{x\in[p,q]} f(x)$*.*

To see this, we observe that when $p < q$ for $p, q \in \mathbb{Q}$, the formula $x \in [p,q]$ can be expressed by a Π^0_1-formula in the negative binary representation f_x from Remark 2.4. Now let O be an open Baire 1 set and note that the representation is trivial in case $O = \emptyset$. Hence, we may assume there is $x_0 \in O$ and m_0 such that $B(x_0, \frac{1}{2^{m_0}}) \subset O$. Let $\big((p_n, q_n)\big)_{n\in\mathbb{N}}$ be an enumeration of all intervals in $[0,1]$ with rational end-points. Now define the following sequence of intervals:

$$(a_n, b_n) := \begin{cases} B(x_0, \frac{1}{2^{m_0}}) & \text{in case } \inf_{x\in[p_n,q_n]} \mathbb{1}_O(x) < \frac{1}{2} \\ (p_n, q_n) & \text{in case } \inf_{x\in[p_n,q_n]} \mathbb{1}_O(x) > 0 \end{cases}. \tag{2.8}$$

Note that the case distinction in (2.8) is decidable (in RCA_0^ω) and that in each case $(a_n, b_n) \subset O$. The theorem is now immediate. □

Theorem 2.11 essentially expresses that a Baire 1 open set can be represented by a code for an open set (see [94, II.5.6]). The general case for arbitrary open sets is not provable in Z_2^ω from Section A.3 (see [72]). Nonetheless, assuming WKL, any applicable second-order theorem generalises from 'codes for open sets' to 'third-order open sets that are Baire 1'. Examples include the Heine-Borel theorem for countable coverings of closed sets ([13, Lemma 3.13] and [94, IV.1.5]), the Tietze extension theorem ([32]), the Urysohn lemma ([94, II.7.3]), and the Baire category theorem ([94, II.4.10]). The same holds *mutatis mutandis* for open sets with quasi-continuous characteristic functions or open sets as in Definition 1.5 where $Y : \mathbb{R} \to \mathbb{R}$ is Baire 1. The following theorem establishes a similar theorem for countable sets; enumerating general (strongly) countable sets cannot be done in Z_2^ω (see [75, 77]).

Theorem 2.12 (ACA_0^ω)**.** *For any Baire 1 set* $A \subset [0,1]$ *and Baire 1 function* $Y : [0,1] \to \mathbb{N}$ *injective on* A*, there is* $(x_n)_{n\in\mathbb{N}}$ *which includes all elements of* A*.*

Proof. Let $A \subset [0,1]$ and $Y : [0,1] \to \mathbb{N}$ be as in the theorem. The standard proof shows that the product of Baire 1 functions is Baire 1. Thus, Theorem 2.9 guarantees that $\inf_{[p,q]}(\mathbb{1}_A(x)Y(x))$ makes sense for rational $p, q \in [0,1]$. Assuming $A \neq \emptyset$, we have $\inf_{[0,1]}(\mathbb{1}_A(x)Y(x)) = n_0 > 0$. Now replace $[0,1]$ by $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ to check in which of the latter sub-intervals the unique $x_0 \in A$ with $Y(x_0) = n_0$ is to be found. The usual interval-halving technique now provides this real and repeat for $\inf_{[0,1]}(\mathbb{1}_{A\setminus\{x_0\}}(x)Y(x)) = n_1$ to enumerate A. □

Finally, we finish this section with some conceptual remarks.

Remark 2.13. First of all, it is a basic fact that BV, usco, lsco, and regulated functions are Baire 1, but this **cannot** be proved from the Big Five or $(\exists^2)$, and much stronger systems by Theorem 2.34. To be absolutely clear, $\mathsf{ACA}_0^\omega + \Pi^1_1\text{-}\mathsf{CA}_0$ and much stronger systems are consistent with the existence of BV, usco, lsco, and regulated functions that are not Baire 1, explaining e.g. items (ix) and (xviii) in Theorem 2.9. Similar results hold for 'effectively Baire n', explaining for instance item (xix) in Theorem 2.9.

Secondly, the cadlag functions are arguably 'close to continuous', but one should be careful with such claims: by Theorem 2.38, it is consistent with the Big Five and much stronger systems that there is a regulated (or usco) function that is discontinuous everywhere (see also [92]), explaining items (ii)-(iv) and (vi)-(vii) in Theorem 2.9. Furthermore, quasi-continuous functions can be quite 'wild': if $\mathfrak{c}$ is the cardinality of $\mathbb{R}$, there are $2^{\mathfrak{c}}$ non-measurable quasi-continuous $[0,1] \to \mathbb{R}$-functions and $2^{\mathfrak{c}}$ measurable quasi-continuous $[0,1] \to [0,1]$-functions (see [43]). Also, the class of quasi-continuous functions is closed under taking *transfinite* limits ([68]).

Thirdly, the regulated functions boast *many* sub-spaces (see [2]) and the same for the Baire 1 functions (see [48]); one can presumably formulate a version of items (i) or (xviii) for many of those. In general, many variations of Theorem 2.9 are possible, based on any function space containing the continuous functions. For instance, one can replace 'quasi-continuity' by the weaker notion 'lower quasi-continuity' (see [26]) in most equivalences in this paper. Moreover, derivatives that are continuous almost everywhere, are quasi-continuous ([66]), suggesting many possible variations. The notion of *strong* quasi-continuity also seems very promising, especially in light of its intimate connection to continuity almost everywhere ([35]), as well as the notion of *countably continuous* and related concepts ([97]).

Fourth, we have studied the (countable) Axiom of Choice in higher-order RM in [76]. We believe that the choice principle NCC from the latter, which is provable in ZF, can replace $\mathsf{QF\text{-}AC}^{0,1}$ in Theorem 2.9 and the below.

2.3.2. *Covering lemmas.* We show that WKL is equivalent to a number of third-order covering lemmas (Theorem 2.14). We have shown in [70] that the general case, called *Cousin's lemma*, is not provable from WKL and much stronger systems.

First of all, WKL is equivalent to compactness results like the Heine-Borel theorem for countable coverings ([94, IV.1]) and Cousin's lemma for (codes of) continuous functions ([6,7]). In general, Cousin's lemma ([19]) is formulated as follows.

> *For* $\Psi : [0,1] \to \mathbb{R}^+$, *the covering* $\cup_{x\in[0,1]} B(x, \Psi(x))$ *of* $[0,1]$ *has a finite sub-covering, i.e. there are* $x_0, \dots, x_k \in [0,1]$ *where* $\cup_{i\leq k} B(x_i, \Psi(x_i))$ *covers* $[0,1]$.

Secondly, we establish the following theorem to be contrasted with items (i)-(vi) in Theorem 2.34. We stress that Cousin's lemma deals with uncountable coverings. Recall Remark 2.13 which explains why items (iv)-(viii) are non-trivial.

Theorem 2.14 (RCA_0^ω). *The following are equivalent to* WKL.

(i) *Cousin's lemma for RM-codes of continuous functions.*
(ii) *Cousin's lemma for continuous functions.*
(iii) *Cousin's lemma for lsco functions.*
(iv) *Cousin's lemma for lsco Baire 1 functions.*
(v) *Cousin's lemma for lsco effectively Baire* $n+2$ *functions.*
(vi) *Cousin's lemma for lsco functions that are continuous almost everywhere.*
(vii) *Cousin's lemma for lsco functions that are pointwise discontinuous.*
(viii) *Cousin's lemma for lsco functions that are not everywhere discontinuous.*
(ix) *Cousin's lemma for quasi-continuous functions.*
(x) *Cousin's lemma for cadlag functions.*
(xi) *Cousin's lemma for regulated* $F : [0,1] \to \mathbb{R}$ *such that* $F(x) = \frac{F(x-)+F(x+)}{2}$ *for all* $x \in [0,1]$.
(xii) *Cousin's lemma for regulated* $F : [0,1] \to \mathbb{R}$ *such that for all* $x \in [0,1]$*:*

$$\min(F(x-), F(x+)) \leq F(x) \leq \max(F(x-), F(x+)).$$

(xiii) *Cousin's lemma for Baire 1 functions.*

Proof. Zeroth of all, the equivalence between WKL and item (i) has been proved in [6, 7]. The combination of Theorems 2.2 and 2.3 then establishes the equivalence between WKL and item (ii).

First of all, assume item (iii) and note that by Theorem 2.2, an RM-code for a continuous function equals a (third-order) continuous function. The latter is trivially lsco and item (iii) establishes Cousin's lemma for RM-codes of continuous functions, and hence WKL. Now assume WKL and fix lsco $\Psi : [0,1] \rightarrow \mathbb{R}^+$. If the latter is continuous, item (ii) provides a finite sub-covering. If the latter is discontinuous, we have access to $(\exists^2)$ by [55, §3]. In case $(\exists N \in \mathbb{N})(\forall q \in [0,1] \cap \mathbb{Q})(\Psi(q) \geq \frac{1}{2^N})$, item (iii) is immediate as enough rational numbers form a finite sub-covering. Finally, in case $(\forall N \in \mathbb{N})(\exists q \in [0,1] \cap \mathbb{Q})(\Psi(q) < \frac{1}{2^N})$, apply $\mathsf{QF\text{-}AC}^{0,0}$ included in the base theory to obtain a sequence of rationals $(q_n)_{n\in\mathbb{N}}$ such that $\Psi(q_n) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Since $(\exists^2) \rightarrow \mathsf{ACA}_0$, we have access to the usual second-order convergence theorems (see [94, III.2]), i.e. $(q_n)_{n\in\mathbb{N}}$ has a convergent sub-sequence $(r_n)_{n\in\mathbb{N}}$, say with limit $y \in [0,1]$. Then $\Psi(y) = 0$ as Ψ is lsco and Ψ comes arbitrarily close to 0 close to y, a contradiction, and we are done.

Secondly, item (ix) implies WKL in the same way as for item (iii). Now assume WKL and fix quasi-continuous $\Psi : [0,1] \rightarrow \mathbb{R}^+$. In case the latter is also continuous, Theorem 2.3 provides a modulus of uniform continuity and [94, IV.2.3] implies that $\inf_{x\in[0,1]} \Psi(x) > 0$, which readily yields the finite sub-covering. In case F is discontinuous, we obtain $(\exists^2)$ by [55, §3]. Now consider the following for any $r \in \mathbb{R}$:

$$(\exists x \in [0,1])(\Psi(x) > r) \leftrightarrow (\exists q \in [0,1] \cap \mathbb{Q})(\Psi(q) > r), \tag{2.9}$$

which holds due to the definition of quasi-continuity. In this light, use μ^2 to define $G(x)$ as the least $N \in \mathbb{N}$ such that

$$(\exists q \in \mathbb{Q} \cap [0,1])(\tfrac{1}{2^N} < |x - (q + \Psi(q))| \wedge x \in B(q, \Psi(q))). \tag{2.10}$$

Now define an increasing sequence $(r_n)_{n\in\mathbb{N}}$ of rationals via $r_0 := \frac{1}{2^{G(0)}}$ and $r_{n+1} := r_n + \frac{1}{2^{G(r_n)}}$. In case this sequence stays in $[0,1]$, it converges to some $y \in [0,1]$, which leads to a contradiction. Hence, $r_{n_0} > 1$ for some $n_0 \in \mathbb{N}$, readily yielding a finite sub-covering. Since cadlag functions are (trivially) quasi-continuous, the equivalence for item (x) also follows. Similarly, one readily observes that (2.9) also holds for functions as in items (xi) and (xii), i.e. the above proof for item (ix) yields the equivalences involving items (xi) and (xii). An alternative proof of items (iii) and (ix) proceeds by noting that one can restrict $\cup_{x\in[0,1]}B(x, \Psi(x))$ to rationals for quasi-continuous or lsco $\Psi : [0,1] \rightarrow \mathbb{R}^+$, yielding a countable sub-covering $\cup_{q\in[0,1]\cap\mathbb{Q}}B(q, \Psi(q))$ of $[0,1]$ to which the second-order Heine-Borel theorem from [94, IV.1] applies; this is relevant to the proof of Corollary 2.35.

Thirdly, for item (xiii), it suffices to prove the latter from WKL as continuous functions are trivially Baire 1. To establish item (xiii), in case $\Psi : [0,1] \rightarrow \mathbb{R}^+$ is continuous, use item (ii). In case $\Psi : [0,1] \rightarrow \mathbb{R}^+$ is discontinuous, we obtain $(\exists^2)$ by [55, §3], and hence ACA_0. Let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions with pointwise limit Ψ. Corollary 2.5 converts this sequence into a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of codes for continuous functions. However, the latter is a code for a Baire 1 function in the sense of [6, 7]. By the latter, Cousin's lemma for codes for Baire 1 functions, is also equivalent to ACA_0, i.e. Cousin's lemma for Ψ now follows via the second-order lemma for $(\Phi_n)_{n\in\mathbb{N}}$ as $(\exists^2) \rightarrow \mathsf{ACA}_0$. □

Now, Cousin's lemma for *codes for* Baire 1 functions, is equivalent to ACA_0 ([6, 7]). Moreover, by Theorem 2.28, $(\exists^2)$ is equivalent to the statement: *a code for a Baire 1 function denotes a third-order function*. Hence, following item (xiii) of Theorem 2.14, the use of second-order codes changes the logical strength of Cousin's lemma. We obtain sharper results on Cousin's lemma in Section 2.8.3.

Finally, Theorem 2.11 expresses that open Baire 1 sets have RM-codes, assuming WKL. Now, consider the following version of the (countable) Heine-Borel theorem:

let $(O_n)_{n\in\mathbb{N}}$ be a sequence of open Baire 1 sets, the union of which covers $[0,1]$. Then there is $m\in\mathbb{N}$ such that $\cup_{n\leq m}O_n$ covers $[0,1]$.

This is a direct generalisation of [94, V.1.5] and can be included in Theorem 2.14. The general case, i.e. with 'Baire 1' omitted, exhibits the Pincherle phenomenon from Remark 2.10, as shown in [72].

2.3.3. *More on covering lemmas.* We show that WWKL is equivalent to a number of third-order covering theorems (Theorem 2.15), where the former is *weak weak König's lemma* as in [94, X.1.7]. We conjecture that $\mathsf{RCA}_0^\omega+\mathsf{WWKL}$ cannot prove Theorem 2.3, i.e. we cannot use the associated coding results in this section.

First of all, as suggested by its name, WWKL is a certain restriction of WKL, namely to trees of positive measure. Montalbán states in [67] that WWKL is robust, i.e. equivalent to small perturbations of itself, in the same way as the Big Five are; WWKL_0 is even called the 'sixth Big system' in [87]. Now, WWKL is equivalent to the *Vitali covering theorem* for countable coverings, and to numerous variations (see [94, X.1] and [87, Lemma 8]), including the following.

Let $\big((a_n,b_n)\big)_{n\in\mathbb{N}}$ be a sequence of open intervals that covers $[0,1]$. Then for any $\varepsilon>0$, there is $m\in\mathbb{N}$ such that $\cup_{n\leq m}(a_n,b_n)$ has measure $>1-\varepsilon$.

The following generalisation to uncountable coverings, in the spirit of Cousin's lemma, is not provable from the Big Five and much stronger systems (see [72]).

For $\Psi:[0,1]\to\mathbb{R}^+$ and $\varepsilon>0$, there are $x_0,\dots,x_k\in[0,1]$ such that $\cup_{i\leq k}B(x_i,\Psi(x_i))$ has measure $1-\varepsilon$.

Vitali indeed considers uncountable coverings in [100], going as far as expressing his surprise regarding the uncountable case. We shall refer to the second centred statement as *Vitali's principle* as it constitutes the 'combinatorial essence' of the Vitali covering theorem, in our opinion. The first centred statement will be called *Vitali's principle for countable coverings.*

Secondly, we establish the following theorem to be contrasted with Corollary 2.35. Recall Remark 2.13 which explains why items (iv)-(viii) are non-trivial.

Theorem 2.15 (RCA_0^ω). *The following are equivalent to* WWKL.

(i) *Vitali's principle for RM-codes of continuous functions.*
(ii) *Vitali's principle for continuous functions.*
(iii) *Vitali's principle for lsco functions.*
(iv) *Vitali's principle for lsco Baire 1 functions.*
(v) *Vitali's principle for lsco effectively Baire $n+2$ functions.*
(vi) *Vitali's principle for lsco functions that are continuous almost everywhere.*
(vii) *Cousin's lemma for lsco functions that are pointwise discontinuous.*
(viii) *Vitali's principle for lsco functions that are not everywhere discontinuous.*
(ix) *Vitali's principle for quasi-continuous functions.*
(x) *Vitali's principle for cadlag functions.*

(xi) *Vitali's principle for regulated* $F : [0,1] \to \mathbb{R}$ *such that* $F(x) = \frac{F(x-)+F(x+)}{2}$ *for all* $x \in [0,1]$.

(xii) *Vitali's principle for regulated* $F : [0,1] \to \mathbb{R}$ *such that for all* $x \in [0,1]$*:*

$$\min(F(x-), F(x+)) \leq F(x) \leq \max(F(x-), F(x+)).$$

(xiii) *Vitali's principle for Baire 1 functions.*

Proof. We establish the equivalences involving WWKL and items (i) and (ii). Invoking the law of excluded middle as in $(\exists^2) \vee \neg(\exists^2)$ then finishes the proof. Indeed, in the former case, $(\exists^2) \to \mathsf{ACA}_0$, which makes WWKL and all items outright provable in light of Theorem 2.14. In case $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous ([55, §3]) and items (iii)-(xiii) reduce to item (ii).

Assume WWKL and fix continuous $\Psi : [0,1] \to \mathbb{R}^+$. To show that the countable union $\cup_{q\in[0,1]\cap\mathbb{Q}} B(q, \Psi(q))$ covers $[0,1]$, consider $x \in [0,1]$ and apply the definition of continuity of Ψ for k such that $\frac{1}{2^k} \leq \Psi(x)/2$, i.e. we obtain $N \in \mathbb{N}$ such that for $y \in B(x, \frac{1}{2^N})$, we have $|\Psi(x) - \Psi(y)| < \Psi(x)/2$. Then for any $q \in B(x, \frac{1}{2^N}) \cap \mathbb{Q}$ close enough to x, we have $x \in B(q, \Psi(q))$, as required. As noted above, WWKL is equivalent to Vitali's principle for countable coverings ([94, X.1]), i.e. we may apply the latter to $\cup_{q\in\mathbb{Q}\cap[0,1]} B(q, \Psi(q))$ to obtain item (ii). For item (i), apply Theorem 2.2 and use item (ii).

For the reversals, these essentially follow from the proof of [7, Theorem 4.2], which takes place in RCA_0 and establishes that Cousin's lemma for (codes for) continuous functions, implies WKL. In more detail, in the aforementioned proof, one fixes a countable covering $\cup_{n\in\mathbb{N}}(a_n, b_n)$ of $[0,1]$ and defines a continuous function $\delta : [0,1] \to \mathbb{R}^+$. This function is then shown to have an RM-code and to satisfy:

$$(\forall x \in [0,1])\big[\delta(x) > \tfrac{1}{2^k} \to B(x, \delta(x)) \subseteq \cup_{m\leq k}(a_m, b_m)\big]. \tag{2.11}$$

In light of (2.11), a finite sub-covering for $\cup_{x\in[0,1]} B(x, \delta(x))$ immediately yields a finite sub-covering for $\cup_{n\in\mathbb{N}}(a_n, b_n)$, i.e. Cousin's lemma for (codes for) continuous functions implies the Heine-Borel theorem for countable coverings, and hence WKL via [94, IV.1.1]. Now, the definition of the RM-code of δ and the proof of (2.11) take place in RCA_0, i.e. we may simply apply item (i) to $\cup_{x\in[0,1]} B(x, \delta(x))$ and obtain Vitali's principle for countable coverings, and hence WWKL via [94, X.1]. In light of Theorem 2.2, item (ii) also yield WWKL. □

Finally, certain equivalences from Theorem 2.15 can be proved (or expanded) using [88, Cor. 2.6], where it is shown that WWKL is equivalent to Vitali's principle restricted to $\Psi : [0,1] \to \mathbb{R}^+$ that are continuous *almost everywhere.*

2.4. **Equivalences for arithmetical comprehension.** In this section, we establish some equivalences between arithmetical comprehension ACA_0 and third-order theorems from analysis, including the *Jordan decomposition theorem* as in Theorem 1.4. We have shown in [75] that the general case of the latter cannot be proved from the Big Five and much stronger systems like Z_2^ω. Regarding definitions, the system ACA_0^ω is defined as $\mathsf{RCA}_0^\omega + (\exists^2)$ and basic properties are in Section A.3.

First of all, we need the following theorem, where a *jump discontinuity* of a function $f : \mathbb{R} \to \mathbb{R}$ is a real $x \in \mathbb{R}$ such that the left and right limits $f(x-)$ and $f(x+)$ exist, but are not equal. By [75, §3.3], listing *all* points of discontinuity of BV-functions cannot be done in in the Big Five and much stronger systems.

Theorem 2.16 (ACA_0^ω). *If $f : [0,1] \to \mathbb{R}$ is regulated, there is a sequence of reals containing all jump discontinuities of f.*

Proof. Let $f : [0,1] \to \mathbb{R}$ be regulated. We say that $x \in (0,1)$ is a *jump* if $\mathsf{Jump}(f,x) := |f(x+) - f(x-)|$ is > 0, which is equivalent to the following:

$$(\exists k \in \mathbb{N})(\forall m \in \mathbb{N})(\exists q, r \in [0,1] \cap \mathbb{Q})(q < x < r \wedge |q - r| < \tfrac{1}{2^m} \wedge |f(q) - f(r)| > \tfrac{1}{2^k}),$$

where we note that f only occurs with rational inputs. Hence, the set of jumps is arithmetically definable from f.

In the below, p, q, a, b and $\delta > 0$ etc. are assumed to be variables over the rationals, while x is a variable over the reals. First of all, we prove that items (1) and (2) as follows are equivalent.

(1) There is exactly one jump $x \in (a,b)$ with $\mathsf{Jump}(f,x) \geq \delta$.

(2) (a) For all $n \in \mathbb{N}$ there are p, q such that $a < p < q < b$, $q - p < 2^{-n}$, and $|f(p) - f(q)| > \delta - 2^{-n}$.
  (b) There is an $n \in \mathbb{N}$ such that for all pairs (p_1, q_1) and (p_2, q_2) satisfying (a) with respect to n, δ, a, b and f, we have that $(p_1, q_1) \cap (p_2, q_2) \neq \emptyset$.

Assume item (1) and note that (2).(a) follows from the fact that there is at least one point in (a,b) with a jump $\geq \delta$. To prove (2).(b), we use the fact there is only one $x \in (a,b)$ with a jump $\geq \delta$. To prove this uniqueness, we show that:

there is $n \in \mathbb{N}$ such that (p,q) satisfying (2).(a) contains an x with jump $\geq \delta$.

Suppose the centred claim is false. Then for each k, there will be $p_k < q_k$ such that $|f(p_k) - f(q_k)| > \delta - 2^{-k}$, $q_k - p_k < 2^{-k}$, and $x \notin (p_k, q_k)$. Since we may pick a convergent sub-sequence (provable in ACA_0^ω by [94, III.2]), we can, without loss of generality, assume that $(p_k)_{k \in \mathbb{N}}$ has a limit y. If y is one of the objects a or b , both p_k and q_q will approximate y from the same side, violating the assumption of one-sided limits. If $y = x$, then for each k both p_k and q_k will be on the same side of x, and then infinitely many will approach x from the same side, again violating the assumption of one-sided limits. For any other value of y we can either argue as above, or obtain that there is also a jump at y of a size $\geq \delta$, contradicting the assumption of item (1).

Now assume (2) and note that by (2).(a) there cannot be more than one jump x in (a,b) with $\mathsf{Jump}(f,x) \geq \delta$. Now, let n be as in (2).(b) and note that there is a pair $p_k < q_p$ of rational points for each $k \geq n$ satisfying (2).(b). We can find such p_k, q_k via an effective search, and by (2).(b) we must have that both $(p_k)_{k \geq n}$ and $(q_k)_{k \geq n}$ converge, and to the same limit x. Clearly, using arguments as in the previous paragraph, x is a jump in (a,b) with $\mathsf{Jump}(f,x) \geq \delta$.

Having established the equivalence between (1) and (2), we see that the set of triples (a, b, δ) such that (a,b) contains exactly one jump x with $\mathsf{Jump}(f,x) \geq \delta$ is arithmetically definable from the restriction of f to the rationals, and using the characterisation we see that the unique x then is definable from the same restriction using $\exists^2$. In this way, we can enumerate the set of jumps. ☐

We can now generalise Corollary 2.5 as follows.

Corollary 2.17 ($\mathsf{RCA}_0^\omega + \mathsf{WKL}$)**.** *A cadlag function $f : [0,1] \to \mathbb{R}$ has a modulus of cadlag, i.e. there is $G : (\mathbb{R} \times \mathbb{N}) \to \mathbb{N}$ such that*

$$(\forall k \in \mathbb{N}, x, y, z \in [0,1]) \left[\begin{array}{c} y, z \in (x - \frac{1}{2^{G(x,k)}}, x) \to |f(y) - f(z)| < \frac{1}{2^k} \\ \wedge \\ y \in (x, x + \frac{1}{2^{G(x,k)}}) \to |f(x) - f(y)| < \frac{1}{2^k} \end{array}\right]. \tag{2.12}$$

Proof. In case $f : [0,1] \to \mathbb{R}$ is continuous, use Corollary 2.5 to obtain a modulus of continuity, which readily yields a modulus of cadlag. In case $f : [0,1]$ is discontinuous, we obtain $(\exists^2)$ by [55, §3] and we may use the theorem to obtain a sequence $(x_n)_{n\in\mathbb{N}}$ that lists all points of discontinuity of f. Now define a modulus of cadlag $G : (\mathbb{R} \times \mathbb{N}) \to \mathbb{N}$ based on the following case distinction.

- In case $x \neq x_n$ for all $n \in \mathbb{N}$, then $G(x,k)$ is the least $N \in \mathbb{N}$ such that for all $y \in (x - \frac{1}{2^N}, x + \frac{1}{2^N}) \cap \mathbb{Q}$, we have $|f(x) - f(y)| < \frac{1}{2^{k+1}}$.
- In case $x = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then $G(x, k-1)$ is the least $N \in \mathbb{N}$ such that the formula in big square brackets in (2.12) holds for all $y, z \in \mathbb{Q} \cap [0,1]$.

Then G is as required by the corollary and we are done. □

One can also use the previous theorem and corollary to show that cadlag functions are Baire 1 in a relatively weak system, but the technical details are somewhat tedious. This should be contrasted with Theorem 2.34 as by the latter the Big Five cannot prove that e.g. regulated functions are Baire 1. One similarly establishes (part of) the *Lebesgue decomposition theorem* (see e.g. [61]).

Secondly, Theorem 2.16 has interesting consequences, e.g. Theorem 2.19, which should be contrasted with Theorem 2.37. We shall make (seemingly essential) use of the following fragment of the induction axiom, which also follows from $\mathsf{QF\text{-}AC}^{0,1}$.

Definition 2.18. [IND_2] Let Y^2, k^0 satisfy $(\forall n \leq k)(\exists f \in 2^{\mathbb{N}})(Y(f,n) = 0)$. There is w^{1^*} such that $(\forall n \leq k)(\exists i < |w|)(Y(w(i), n) = 0)$.

We note that the class NBV from [2, Def. 1.2], [84, §1.1], or [27, p. 103], is essentially the intersection between BV and the cadlag functions. As discussed in [2], the classical Riemann-Stieltjes integral provides a natural one-to-one correspondence between the dual of the space $C([a,b])$ of continuous functions and the space $NBV([a,b])$ of (normalized) BV-functions.

Theorem 2.19 ($\mathsf{RCA}_0^\omega + \mathsf{IND}_2$)**.** *The following are equivalent to* ACA_0.

- *The Jordan decomposition theorem for cadlag BV-functions.*
- *The Jordan decomposition theorem for BV-functions satisfying the equality $f(x) = \frac{f(x+)+f(x-)}{2}$ for $x \in (0,1)$.*
- *The Jordan decomposition theorem for quasi-continuous BV-functions.*

We do not need IND_2 *for the first item.*

Proof. We establish the equivalence between ACA_0 and the first item based on Theorem 2.16 and the observation that cadlag functions do not have removable discontinuities. One proceeds analogously for the second and third item, as the functions therein also do not have removable discontinuities. As shown in [75, Theorem 3.33], a BV-function is regulated assuming IND_2.

First of all, assume ACA_0 and fix a cadlag BV-function $f : [0,1] \to \mathbb{R}$. In case the latter is continuous, Corollary 2.5 provides an RM-code. We can now apply the second-order RM results from [69, §3] to obtain codes for (continuous) increasing functions $g, h : [0,1] \to \mathbb{R}$ such that $f = g - h$ on $[0,1]$. Theorem 2.2 thus yields the first item from the theorem, in this case. In case $f : [0,1] \to \mathbb{R}$ is discontinuous, we have access to $(\exists^2)$ by [55, §3]. By Theorem 2.16, there is a sequence $(x_n)_{n\in\mathbb{N}}$ of all reals in $[0,1]$ where f is discontinuous; indeed, since f is cadlag, it only has jump discontinuities by definition. Given $(x_n)_{n\in\mathbb{N}}$, the supremum in (1.4) can be replaced by a supremum over $\mathbb{Q}$ and $\mathbb{N}$. As a result, we can define $V_a^b(f)$ as in (1.4) using $\exists^2$, where $a, b \in [0,1]$ are parameters. Clearly, $g(x) := \lambda x.V_0^x(f)$ is an increasing function, and the same for $h(x) := g(x) - f(x)$ via an elementary argument. Hence, $f = g - h$ in this case as well, and the first item follows.

Secondly, assume the first item of the theorem. We now invoke the law of excluded middle as in $(\exists^2) \vee \neg(\exists^2)$. In the former case we are done, as $(\exists^2) \to \mathsf{ACA}_0$ is trivial. In the latter case, i.e. we have $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous by [55, §3]. Hence, the first item now expresses:

a continuous $f : [0,1] \to \mathbb{R}$ *of bounded variation is the difference of two continuous non-decreasing* $g, h : [0,1] \to \mathbb{R}$.

Now fix some code Φ for a continuous BV-function and use Theorem 2.2 to obtain third-order $f : [0,1] \to \mathbb{R}$ that equals the value of Φ everywhere. By the centred statement, there are two continuous non-decreasing $g, h : [0,1] \to \mathbb{R}$ such that $f = g - h$ on $[0,1]$. Now consider:

$$(\forall x \in [0,1], k \in \mathbb{N})(\exists N \in \mathbb{N})(|g(x) - g(x + \tfrac{1}{2^N})| < \tfrac{1}{2^k} \wedge |g(x) - g(x - \tfrac{1}{2^N})| < \tfrac{1}{2^k}).$$

Applying $\mathsf{QF\text{-}AC}^{1,0}$, one obtains a (continuous) modulus of continuity for g, as g is non-decreasing. Following Remark 2.1, this readily yields an RM-code for g (and h), i.e. we have also established the second-order version of the centred statement. The latter implies ACA_0 by [69, §3], and we are done. □

One possible addition to the previous theorem is as follows: a real function is usco if and only if it is *sequentially usco*, i.e. the pointwise limit of a *descreasing* sequence of continuous functions (see e.g. [25, p. 62]). This sequential notion goes back to Baire's equivalent definition of usco (see [5]) and the associated restriction of Jordan decomposition theorem is readily[8] seen to be equivalent to ACA_0. A similar result can be obtained for Baire 1^* formulated using RM-codes for closed sets, in light of [78, Lemma 4.11].

Next, Theorem 2.16 has the following consequence. We refer to [94, p. 136] for the details on Riemann integration in RCA_0.

Theorem 2.20 (RCA_0^ω)**.** *The axiom* WKL *is equivalent to:*

- *a code for a continuous function on the unit interval is Riemann integrable.*
- *a continuous function on the unit interval is Riemann integrable.*
- *a cadlag function on the unit interval is Riemann integrable.*

---

[8] Fix $f \in BV$ and let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence of continuous functions with pointwise limit f. By Theorem 2.16, we only need to enumerate the removable discontinuities of f. Using $\exists^2$, one readily enumerates the *strict local maxima* of a continuous $g : [0,1] \to \mathbb{R}$ ([91, p. 272]), i.e. those $x \in [0,1]$ such that $(\exists N \in \mathbb{N})(\forall y \in B(x, \frac{1}{2^N}))(x \neq y \to g(y) < g(x))$. Now let $(x_m)_{n\in\mathbb{N}}$ be an enumeration of all strict local maxima of all f_n. For any $m \in \mathbb{N}$, x_m is a removable discontinuity of f if and only if there is $n_0 \in \mathbb{N}$ such that x_m is a strict local maximum of f_n for $n \geq n_0$.

Proof. The equivalence for the first item is immediate by [94, IV.2.7]. For the second item, one additionally uses Theorem 2.2 and Corollary 2.5.

Now assume the third item and fix some code Φ for a continuous function on $[0,1]$. Use Theorem 2.2 to convert the latter into a continuous third-order function, which is trivially cadlag. By the first item, this function is Riemann integrable, and hence so is the function represented by Φ. We obtain WKL by the first item.

Now assume WKL and let $f:[0,1]\to\mathbb{R}$ be cadlag. If the latter is also continuous, we may use the second item to obtain the third one. In case f is discontinuous, we obtain $(\exists^2)$ by [55, §3]. Use Theorem 2.16 to obtain a sequence $(x_n)_{n\in\mathbb{N}}$ which enumerates all the points where f is discontinuous (as cadlag functions do not have removable discontinuities). The usual 'epsilon-delta' proof now goes through assuming a modulus as provided by Theorem 2.17. $\square$

Finally, we mention some related results from the RM of ACA_0.

Remark 2.21. The RM of ACA_0 involves some theorems from analysis, like e.g. [94, IV.2.11 and III.2.2]. In the same way as above, one shows that the following are also equivalent to ACA_0 over RCA_0^ω. We use 'RM-closed' to refer to the second-order definition of codes for closed set in RM ([94, II.5.6]).

- Let $F:C\to\mathbb{R}$ be cadlag where $C\subset[0,1]$ is an RM-closed set. Then $\sup_{x\in C}F(x)$ exists.
- Let $F:C\to\mathbb{R}$ be cadlag and usco where $C\subset[0,1]$ is an RM-closed set. Then F attains a maximum value on C.
- Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence (relative to the sup norm) of continuous functions. Then the limit function exists and is continuous.
- Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence (relative to the sup norm) of cadlag functions. Then the limit function exists and is cadlag.

Another promising theorem is the compactness theorem ([8, Theorem 14.3]) for the Skorohod space (of cadlag functions), which is presented as a generalisation of the Arzelà-Ascoli theorem. The latter is part of the RM of ACA_0 by [94, III.2.9]. Similarly, a version of the Arzelà-Ascoli theorem for quasi-continuous functions exists, namely [42, Prop. 2.22] and related theorems.

2.5. Equivalences for Π_1^1-comprehension.

We establish some equivalences for Π_1^1-CA_0 involving third-order theorems from analysis.

First of all, we establish Theorem 2.22 to be contrasted with Theorem 2.32. Here, Σ_1^1-IND is the induction axiom for Σ_1^1-formulas and IND_2 is as in Definition 2.18. As to notation, fix $(r_n)_{n\in\mathbb{N}}$, a standard injective enumeration of the non-negative rational numbers. For $B\subset\mathbb{Q}^+$, we say that 'B is Σ_1^1 with parameter $x\in\mathbb{N}^{\mathbb{N}}$', if $A=\{a:r_a\in B\}$ is Σ_1^1 with parameter x. Since we do not always have access to Σ_1^1-comprehension, we refer to both A and B as (defined) classes.

Theorem 2.22 (ACA_0^ω). *The following are equivalent.*

(i) *For any $x\in\mathbb{N}^{\mathbb{N}}$, any bounded $\Sigma_1^{1,x}$-class in $\mathbb{Q}^+$ has a supremum.*
(ii) *A bounded effectively Baire 2 $f:[0,1]\to\mathbb{R}$ has a supremum.*
(iii) *For $n\geq 2$, a bounded and effectively Baire n $f:[0,1]\to\mathbb{R}$ has a supremum.*

Assuming $\mathsf{IND}_2+\Sigma_1^1$-$\mathsf{IND}$, these items are equivalent to Π_1^1-CA_0.

Proof. We first prove that item (i) implies items (ii) and (iii). Let $f:[0,1]\to[0,1]$ be effectively Baire 2, i.e. there is a double sequence $(f_{n,m})_{n,m\in\mathbb{N}}$ of continuous functions such that $f(x) = \lim_{n\to\infty}\lim_{m\to\infty} f_{n,m}(x)$ for $x \in [0,1]$. Now consider the following for $r \in \mathbb{Q}$:

$$\begin{aligned}&(\exists y \in [0,1])(f(y) > r)\\ &\leftrightarrow (\exists x \in [0,1])(\exists n \in \mathbb{N})(\forall i \geq n)(\exists m \in \mathbb{N})(\forall j \geq m)(f_{i,j}(x) > r).\end{aligned} \tag{2.13}$$

By Corollary 2.5, we can replace $f_{i,j}$ by a sequence of RM-codes, rendering (2.13) part of the language of second-order arithmetic. Using $\exists^2$, (2.13) is equivalent to a Σ^1_1-formula, i.e. we may form the set $\{r \in \mathbb{Q} : (\exists y \in [0,1])(f(y) > r)\}$ using item (i), from which the supremum of f is readily defined using $\exists^2$, i.e. item (ii) follows. Item (iii) is proved in the same way, where (2.13) becomes more complicated due to the presence of more arithmetical quantifiers originating from the definition of 'effectively Baire n'.

Secondly, we prove that item (ii) implies item (i). Let $B = \{r_a : a \in A\}$ be bounded, where A is Σ^1_1 and given by:

$$a \in A \leftrightarrow (\exists x \in 2^{\mathbb{N}})(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})R(a, x, m, n),$$

where R is primitive recursive. We now construct continuous functions $F_{n,m} : 2^{\mathbb{N}} \to \mathbb{R}$ such that the double limit $F = \lim_{m\to\infty}\lim_{n\to\infty} F_{n,m}$ is well defined and such that $\sup F = \sup B$. Identifying $2^{\mathbb{N}}$ with the Cantor set, we extend each $F_{n,m}$ to a continuous function $f_{n,m} : [0,1] \to \mathbb{R}$ by extending the graph with straight lines. Note that all limits commute with this extension and that the corresponding extension f of F is Baire 2 with the same supremum. For each $a \in \mathbb{N}$, let $G_a(x) = \lim_{m\to\infty}\lim_{n\to\infty} G_{a,n,m}(x)$ be the characteristic function of the set $\{x \in 2^N : (\forall m \in \mathbb{N})(\exists n \in \mathbb{N})R(a, x, m, n)\}$, where

$$G_{a,n,m}(x) := \begin{cases} 1 & \text{if } (\forall i \leq m)(\exists j \leq n)R(a, x, i, j) \\ 0 & \text{otherwise.}\end{cases}.$$

We now define $F_{n,m} : [0,1] \to \mathbb{R}$ by cases as follows.

- If x is of the form $\underbrace{1 * \cdots * 1}_{m+1 \text{ times}} * y$, we define $F_{n,m}(x) := 0$.
- If for $a \leq m$, x is of the form $\underbrace{1 * \cdots * 1}_{a \text{ times}} * 0 * y$, define $F_{n,m}(x) := r_a G_{a,n,m}(y)$.

If $x = 11\ldots$ then $F_{n,m}(x) = 0$ for all $n, m \in \mathbb{N}$, so in the double limit we have that $F(x) = 0$. If not, x is of the form $\underbrace{1 * \cdots * 1}_{a \text{ times}} * 0 * y$ for some $a \geq 0$. For all $m \geq a$ and all $n \in \mathbb{N}$ we have that $F_{n,m}(x) = r_a G_{a,n,m}(y)$. Then $F(x) = r_a$ if $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})R(a, y, m, n)$, and 0 otherwise. Then $\sup F = \sup B$, so the latter exists by the assumption that the former exists.

Thirdly, item (i) clearly follows from $\Pi^1_1\text{-}\mathsf{CA}_0$ and it is a tedious but straightforward verification that the reversal goes through assuming $\mathsf{IND}_2 + \Sigma^1_1\text{-}\mathsf{IND}$. □

We note that the use of $(\exists^2)$ as part of the base theory in Theorem 2.22 is necessary: in isolation, items (ii) and (iii) do not exceed WKL_0 in terms of second-order consequences. This follows via the ECF-interpretation from Remark A.3.

Secondly, we have the following corollary to Theorem 2.22, to be contrasted with Theorem 2.33. We say that a *set* is 'effectively Baire n' if the characteristic function has this property. The notion of Baire set may be found in [52, p. 21] under a different name; we refer to [62] for an introduction and to [22, §7] for equivalent definitions, including that of Borel set in Euclidean space.

Theorem 2.23 ($\mathsf{ACA}_0^\omega + \Pi_1^1$-$\mathsf{CA}_0$)**.** *For any open effectively Baire n set $O \subset [0,1]$, there exists an RM-code ($n \geq 2$).*

Proof. We make use of the items in Theorem 2.22. In particular, the proof of these immediately generalises to infima involving rational parameters, i.e. we have

> *For a bounded effectively Baire n function $f : [0,1] \to \mathbb{R}$, there is $F : \mathbb{Q}^2 \to \mathbb{R}$ such that for all $p, q \in \mathbb{Q} \cap [0,1]$, the real $F(p,q)$ equals $\inf_{x\in[p,q]} f(x)$.*

Now consider the sequence in (2.8) as in the proof of Theorem 2.11. □

The previous proof essentially establishes that an effectively Baire n open set can be represented by a code for an open set (see [94, II.5.6]). Hence, any theorem from the RM of Π_1^1-CA_0 immediately generalises from 'codes for open sets' to 'third-order open sets that are effectively Baire n'. The RM of Π_1^1-CA_0 contains considerable results on codes for open and closed sets sets (see [13–15, 94]), including the Cantor-Bendixson theorem. The same holds *mutatis mutandis* for open sets with quasi-continuous characteristic functions. One can similarly generalise Theorem 2.12 to effectively Baire n functions.

2.6. **Equivalences for arithmetical transfinite recursion.**

We establish equivalences for ATR_0 involving third-order theorems from analysis. We also establish Theorem 2.26 which shows that adding the extra condition 'Baire 1' converts theorems about BV-functions from 'not provable in Z_2^ω' to 'provable from ATR_0 plus induction'. Remark 2.13 again explains why there is no contradiction here.

First of all, we have a corollary to [7, Theorem 6.5], to be contrasted with item (vi) from Theorem 2.34. Here, Δ_2^1-IND is the induction axiom for Δ_2^1-formulas.

Theorem 2.24 (ACA_0^ω + Δ_2^1-IND)**.** *The following are equivalent to* ATR_0*.*

- *Cousin's lemma for codes for Baire 2 functions.*
- *Cousin's lemma for effectively Baire 2 $\Psi : [0,1] \to \mathbb{R}^+$.*
- *Cousin's lemma for effectively Baire n $\Psi : [0,1] \to \mathbb{R}^+$ ($n \geq 2$).*

Proof. It is known that ATR_0 is equivalent to Cousin's lemma for codes for Baire 2 (or: any $n \geq 2$) functions, working over RCA_0^ω plus Δ_2^1-induction (see [6, 7]). Now, a code for a Baire n function is essentially an effectively Baire n function where the continuous functions are given by codes. As noted below [7, Def. 6.1], ACA_0 suffices to show that a code for a Baire n function has a (unique) value. Hence, $\exists^2$ readily defines a third-order function taking these values everywhere on $[0,1]$. Similarly, an effective Baire n function readily becomes a code for a Baire n function by replacing the continuous functions by codes for continuous functions (see Corollary 2.5). In this way, the base theory connects the items from the theorem and we are done. □

By the previous, (full) Cousin's lemma plus $(\exists^2)$ implies ATR_0 assuming some induction. The use of $(\exists^2)$ is again essential as Cousin's lemma in isolation does not exceed WKL_0 in terms of second-order consequences. By Theorem 2.28, $(\exists^2)$ is equivalent to the statement that a code for a Baire 1 (or Baire n) function denotes a third-order function. Hence, the strength of Cousin's lemma for codes for Baire 2 functions is actually due to the coding of Baire 2 functions.

Secondly, we obtain equivalences involving ATR_0 and the Jordan decomposition theorem, to be contrasted with Theorem 2.37. Moreover, Theorem 2.25 also shows that the RM of ATR_0 is a special case of the higher-order RM of the (full) Jordan decomposition theorem, where the latter is developed in [75, §3.3]. In the below, notions like *arithmetical*, Σ_1^1, etc. are based on the 'standard' definition, i.e. with the understanding that we (only) allow parameters of type 0 and of type 1. We say that a function $f : [0,1] \to \mathbb{R}$ is Σ_1^1 if its graph is Σ_1^1, which is equivalent to the graph being Δ_1^1, and to being Borel measurable.

Theorem 2.25 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2 + \Sigma_2^1$-$\mathsf{IND}$)**.** *The following are equivalent to* ATR_0.

(i) *For arithmetical formulas φ such that*

$$(\forall n \in \mathbb{N})(\exists \text{ at most one } X \subset \mathbb{N})\varphi(X,n), \tag{2.14}$$

the set $\{n \in \mathbb{N} : (\exists X \subset \mathbb{N})\varphi(X,n)\}$ exists.

(ii) *For arithmetical $f : [0,1] \to \mathbb{R}$ in BV, there is a sequence $(x_n)_{n\in\mathbb{N}}$ enumerating all points where f is discontinuous.*

(iii) *For a Σ_1^1-function $f : [0,1] \to \mathbb{R}$ in BV, there is a sequence $(x_n)_{n\in\mathbb{N}}$ enumerating all points where f is discontinuous.*

(iv) *The Jordan decomposition theorem* (*Theorem 1.4*) *restricted to arithmetical* (*or: Σ_1^1*) *functions in BV.*

(v) *A non-enumerable arithmetical set in $\mathbb{R}$ has a limit point.*

Proof. The equivalence between ATR_0 and item (i) is found in [94, V.5.2]. Now assume item (ii) and fix arithmetical φ such that (2.14). Note that we can use μ^2 to find those $n \in \mathbb{N}$ such that $(\exists X \subset \mathbb{N})\varphi(X,n)$ and there is m_0 such that $X(m) = 1$ for $m \geq m_0 \in \mathbb{N}$, so without loss of generality we may assume that there are no such n. Define the function $f : [0,1] \to \mathbb{R}$ as follows:

$$f(x) := \begin{cases} \frac{1}{2^{n+3}} & \text{the least } n \in \mathbb{N} \text{ such that } \varphi(\mathfrak{b}(x), n) \\ 0 & \text{otherwise} \end{cases}, \tag{2.15}$$

where $\mathfrak{b} : [0,1] \to 2^{\mathbb{N}}$ converts real numbers to a binary representation, choosing a tail of zeros if applicable. In light of (2.14), for every $n \in \mathbb{N}$, there is at most one $x \in [0,1]$ such that $f(x) = \frac{1}{2^n}$. Hence, the sum $\sum_{i=0}^{k-1} |f(x_i) - f(x_{i+1})|$ as in (1.4) is at most $\sum_{n=1}^{k} \frac{1}{2^n}$, which is at most 1. By definition, f is arithmetical and item (ii) provides a sequence $(x_m)_{m\in\mathbb{N}}$ with all points where f is discontinuous. Hence, we have for all $n \in \mathbb{N}$ that

$$(\exists X \subset \mathbb{N})\varphi(X,n) \leftrightarrow (\exists m \in \mathbb{N})\varphi(\mathfrak{b}(x_m), n),$$

where, as we assumed, X in the left-hand side will not have a tail of 1's.

Item (iv) implies item (iii) as $\exists^2$ allows us to enumerate the points of discontinuity of increasing functions (see [77, Lemma 7]). Of course item (iii) implies (ii). We also have that item (iii) implies item (iv) as follows: the sequence in item (iii) allows us to replace the supremum in (1.4) by one over $\mathbb{N}$ and $\mathbb{Q}$. Hence, $\exists^2$ can define the increasing function $g(x) := \lambda x.V_0^x(f)$. By noting that $h := f - g$ is also increasing, item (iv) follows.

Finally, assume item (i) and fix a Σ_1^1-function $f \in BV$ with bound $k_0 = 1$ as in Definition 1.3. Now consider the set

$$D_k := \{x \in [0,1] : |f(x+) - f(x-)| > \tfrac{1}{2^k}\}, \tag{2.16}$$

where we note that IND_2 suffices to show that $f \in BV$ is regulated ([75, Theorem 3.33]). The set D_k is Σ^1_1 because the graph of f is. Moreover, since each element $x \in D_k$ contributes at least $\frac{1}{2^k}$ to the variation of f, D_k can have at most 2^k many elements. Using Σ^1_2-induction, one obtains[9] an enumeration of D_k for fixed $k \in \mathbb{N}$. By [94, V.4.10], which is provable in ATR_0, $\cup_{k\in\mathbb{N}} D_k$ can now be enumerated, and item (iii) follows.

For item (v), fix φ as in (2.14) and define the set $A \subset \mathbb{R}$ by putting $x \in A$ in case $x \in [n+1, n+2)$ and $\varphi(\mathfrak{b}(x-(n+1)), n)$. Since $A \cap [0, n]$ contains at most n elements, A has no limit points, i.e. item (v) readily yields item (i). For the reversal, let $A \subset \mathbb{R}$ be a set without limit points. By contraposition, $A \cap [-n, n]$ is finite for each fixed $n \in \mathbb{N}$ (for which we use Σ^1_1-AC_0). As in the previous paragraphs of the proof, we can enumerate $A = \cup_{n\in\mathbb{N}}\big(A \cap [-n, n]\big)$. □

As above, $(\exists^2)$ is essential for the equivalence in Theorem 2.25 as the Jordan decomposition theorem in isolation cannot go beyond ACA_0 in terms of second-order consequences, a fact observed again using ECF from Remark A.3.

Thirdly, we obtain a version of Theorem 2.16 for BV-functions that are also in Baire 1. As discussed in Remark 2.13, while BV-functions are Baire 1, this basic fact is not provable in ACA^ω_0 and much stronger systems like Z^ω_2. By [75, §3.3], listing *all* points of discontinuity of BV-functions similarly cannot be done in Z^ω_2.

Theorem 2.26 ($\mathsf{ACA}^\omega_0 + \mathsf{IND}_2 + \Sigma^1_2$-$\mathsf{IND} + \mathsf{ATR}_0$). *For Baire 1 $f : [0,1] \to \mathbb{R}$ in BV, there is a sequence $(x_n)_{n\in\mathbb{N}}$ enumerating all points where f is discontinuous.*

Proof. Let $f : [0,1] \to \mathbb{R}$ be Baire 1 and in BV, say with variation bounded by 1. In light of Theorem 2.16, we only need to enumerate the 'removable' discontinuities if f, i.e. those $x \in (0,1)$ for which $f(x) \neq f(x+)$ and $f(x+) = f(x-)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions with pointwise limit f on $[0,1]$. Now consider the following formula

$$(\exists n_0 \in \mathbb{N})(\forall n, m \geq n_0)(\forall q \in B(x, \tfrac{1}{2^m}) \cap \mathbb{Q})(|f_n(x) - f(q)| > \tfrac{1}{2^k}), \tag{2.19}$$

which holds in case f has a removable discontinuity at $x \in (0,1)$ such that $|f(x) - f(x+)| > \frac{1}{2^k}$. There can only be 2^k many pairwise distinct $x \in [0,1]$ such that (2.19), as each such real contributes at least $\frac{1}{2^k}$ to the total variation. Clearly, the formula (2.19) is equivalent to (second-order) arithmetical as f only occurs with rational input and f_n can be replaced uniformly by a sequence of codes Φ_n. Using Σ^1_2-induction, one can enumerate all reals satisfying (2.19) for fixed $k \in \mathbb{N}$, as in the proof of Theorem 2.25 and Footnote 9. Again using [94, V.4.10], we can enumerate all reals satisfying (2.19), and we are done. □

With some effort, one generalises Theorem 2.26 from 'Baire 1' to 'effectively Baire n'; it goes without saying that (2.19) becomes more complicated. The same goes

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[9] For $X \subset \mathbb{R}$, $N \in \mathbb{N}$, define the notation '$|X| \leq N$', i.e. X has at most N elements, as:

$$(\forall w^{1^*})\big([|w| > N \wedge (\forall i, j < |w|)(i \neq j \to w(i) \neq w(j))] \to (\exists k < |w|)((w(k) \notin X))\big). \tag{2.17}$$

Using (2.17), let $\varphi(n, X)$ be the following formula:

$$|X| \leq n \to (\exists v^{1^*})(\forall x \in \mathbb{R})\big([x \in X \to (\exists i < |v|)(v(i) = x)] \wedge |v| \leq n\big), \tag{2.18}$$

expressing that a set with at most n elements can be enumerated by a finite sequence of length n. Then (2.17) is Π^1_1 if X is Σ^1_1 while (2.18) is then Σ^1_2. Hence, for X in Σ^1_1, Σ^1_2-induction on $\varphi(n, X)$ establishes the desired enumeration.

for the generalisation from 'BV' to 'regulated', which seems to require $\mathsf{QF\text{-}AC}^{0,1}$. Unfortunately, Theorem 2.26 cannot be pushed down to ACA_0 as the union of enumerable arithmetical sets does not necessarily[10] have an arithmetical enumeration.

Finally, we now establish the following, to be contrasted with the final item of Theorem 2.32, Theorem 2.37, and Theorem 2.38. We again stress Remark 2.13 which explains why there is no contradiction here: rather strong systems are unable to prove that BV or usco functions are in fact Baire 1.

Theorem 2.27 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_2 + \Sigma^1_2\text{-}\mathsf{IND} + \mathsf{ATR}_0$). *The following are provable.*

- *The Jordan decomposition theorem (Theorem 1.4) for BV-functions in Baire 1.*
- *A bounded Baire 1 BV-function $F : [0,1] \to \mathbb{R}$ has a supremum.*
- *For a Riemann integrable BV-function $f : [0,1] \to [0,1]$ in Baire 1 with $\int_0^1 f(x)dx = 0$, there is $x \in [0,1]$ such that $f(x) = 0$.*

Proof. For the first item, the sequence provided by Theorem 2.26 allows one to replace 'supremum over $\mathbb{R}$' by 'supremum over $\mathbb{N}$' in (1.4). Hence, we can define $g(x) := \lambda x.V_0^x(f)$ using $\exists^2$. Now, g is (trivially) increasing and one readily verifies the same for $h = f - g$, i.e. a Jordan decomposition is immediate. The second item follows in the same way. The third item follows by using [94, II.4.10] to obtain a real $y \in [0,1]$ not in the sequence provided by Theorem 2.26; by definition, f must be continuous at y, and $f(y) = 0$ readily follows. □

With some effort, one generalises Theorem 2.27 from 'Baire 1' to 'effectively Baire n'. It goes without saying that the proof becomes more complicated.

2.7. **Equivalences for Kleene's arithmetical quantifier.** We establish interesting equivalences for $(\exists^2)$. To this end, Thomae's function as follows is useful:

$$f(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ and } p, q \text{ are co-prime} \end{cases}. \tag{2.20}$$

Thomae introduces this function around 1875 in [96, p. 14, §20]) to show that Riemann integrable functions can have a dense set of discontinuity points. As in the previous, the coding of Baire n functions is taken from [6,7].

Theorem 2.28 ($\mathsf{RCA}_0^\omega + \mathsf{WKL}$). *The following are equivalent to $(\exists^2)$.*

(i) *There exists Riemann integrable $f : [0,1] \to [0,1], g : [0,1] \to \mathbb{R}$ such that $g \circ f$ is not Riemann integrable.*
(ii) *There exists a function that is not Riemann integrable.*
(iii) *There exists regulated $f : [0,1] \to [0,1], g : [0,1] \to \mathbb{R}$ such that $g \circ f$ is not regulated.*
(iv) *There exists a function that is not regulated.*
(v) *There exists $f : [0,1] \to [0,1], g : [0,1] \to \mathbb{R}$ in Baire 1 such that $g \circ f$ is not in Baire 1.*
(vi) *There exists a function $f : [0,1] \to \mathbb{R}$ that is not Baire 1.*
(vii) *There exists usco $f : [0,1] \to [0,1], g : [0,1] \to \mathbb{R}$ such that $g \circ f$ is not usco.*

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[10] To see, let $X = \langle X_1, ..., X_n\rangle$ be in A_n if and only if $X_1 = \emptyset$ and for all $i < n$, X_{i+1} is the Turing jump of X_i. This constitutes the first n elements in the jump hierarchy, coded as one object, and A_n is arithmetical of a complexity independent of n. Now, each A_n is a singleton with an arithmetical element, but the union does not have any arithmetical enumeration.

(viii) *There exists a function that is not usco.*
(ix) *There exists a function that is not quasi-continuous.*
(x) *There exists a function that is not cliquish.*
(xi) *There exists a function* $f : [0, 1] \to \mathbb{R}$ *that is unbounded.*
(xii) *There exists a Baire 1 function* $f : [0, 1] \to \mathbb{R}$ *that is unbounded.*
(xiii) *There exists a function* $f : [0, 1] \to \mathbb{R}$ *that is not locally bounded*[11].
(xiv) *There exists Darboux functions* $f : [0, 1] \to [0, 1], g : [0, 1] \to \mathbb{R}$ *such that* $g + f$ *is not Darboux.*
(xv) *There is a bounded Darboux* $f : [0, 1] \to \mathbb{R}$ *which does not attains its sup.*
(xvi) *For a code for a Baire 1 function on* $[0, 1]$*, there exists a third-order function that equals the value of the code on* $[0, 1]$.
(xvii) *For a code for a Baire n function on* $[0, 1]$*, there exists a third-order function that equals the value of the code on* $[0, 1]$ $(n \geq 2)$.

We only need WKL *for the items* (i)*,* (ii)*,* (xi)*,* (xii)*,* (xv)*.*

Proof. First of all, assume $(\exists^2)$ and consider Thomae's function f as in (2.20); one readily verifies that f is Riemann integrable (with integral equal to zero) and regulated (with zero as left and right limits) on any interval. Now define $g : [0, 1] \to \mathbb{R}$ as 0 in case $x = 0$, and 1 otherwise; this function is trivially Riemann integrable and regulated. However, $g \circ f$ is Dirichlet's function $\mathbb{1}_{\mathbb{Q}}$, i.e. the characteristic function of the rationals, which is readily shown to be *not* Riemann integrable and *not* regulated. Thus, $(\exists^2)$ implies items (i)-(iv).

Secondly, assume item (iii) (similar for item (iv)) and note that $g \circ f$ must be discontinuous, as continuous functions are trivially regulated. However, the existence of a discontinuous function on $\mathbb{R}$ yields $(\exists^2)$ by [55, §3]. Similarly, for items (i) and (ii), WKL suffices to obtain an RM-code for a continuous function on Cantor space (see [53, §4]); the same goes through *mutatis mutandis* for functions on $[0, 1]$ by Corollary 2.5. Hence, WKL suffices to show that a continuous function on $[0, 1]$ is Riemann integrable by [94, IV.2.6]. Thus, $g \circ f$ must be discontinuous, which yields $(\exists^2)$ by [55, §3]. Similarly, for items (v) and (vi), a function not in Baire 1 must be discontinuous, as continuous functions are trivially Baire 1; in this way, we obtain a discontinuous function and hence $(\exists^2)$ by [55, §3]. The first six items now each imply $(\exists^2)$, the first two using WKL as noted above.

Thirdly, assume $(\exists^2)$ and note that Thomae's function is Baire 1. In particular, finding a sequence of continuous functions converging to f as in (2.20) is straightforward (using $\exists^2$). The same holds for $g : [0, 1] \to \mathbb{R}$ defined as 0 in case $x = 0$, and 1 otherwise. We now show that $\mathbb{1}_{\mathbb{Q}} = g \circ f$ is not Baire 1, establishing items (v) and (vi). To this end, suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions with pointwise limit $\mathbb{1}_{\mathbb{Q}}$. We first prove the following statement in the next paragraph.

For any non-empty $[a, b] \subset [0, 1]$*, there is an arbitrarily large* $N \in \mathbb{N}$ *and a non-empty* $[c, d] \subset [a, b]$ *such that* $f_N([c, d]) = [\frac{1}{4}, \frac{3}{4}]$.

To establish the previous centred statement, fix a non-trivial interval $[a, b] \subset [0, 1]$ and fix $x < y$ such that $x \in \mathbb{Q} \cap [a, b]$ and $y \in [a, b] \setminus \mathbb{Q}$. Since $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $\mathbb{1}_{\mathbb{Q}}$, there exists arbitrarily large $N \in \mathbb{N}$ such that $f_N(x) \geq \frac{3}{4}$ and $f_N(y) \leq \frac{1}{4}$. By the intermediate value theorem (provable in RCA_0 for RM-codes by

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[11] A function $f : [0, 1] \to \mathbb{R}$ is *locally bounded* if for all $x \in [0, 1]$, there is $N \in \mathbb{N}$ such that $(\forall y \in B(x, \frac{1}{2^N}) \cap [0, 1])(|f(y)| \leq N)$.

[94, II.6.6], and hence in ACA_0^ω for continuous functions), there exists an interval $[c,d] \subseteq [x,y] \subset [a,b]$ such that $f_N([c,d]) = [\frac{1}{4},\frac{3}{4}]$. The previous centred statement has been proved, working in ACA_0^ω.

By [56, §3], $(\exists^2)$ is equivalent to the existence of a functional witnessing the intermediate value theorem. Hence, following the previous paragraph, $\exists^2$ readily yields a functional that returns the numbers $N \in \mathbb{N}$ and $c,d \in [0,1]$ as in the centred statement on input $[a,b]$ and $m \in \mathbb{N}$, where $N \geq m$. Using the latter functional, one readily obtains sequences $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$, and $g \in \mathbb{N}^\mathbb{N}$ such that $g(n) \geq n$, $f_{g(n)}([c_n,d_n]) = [\frac{1}{4},\frac{3}{4}]$, and $|c_n - d_n| < \frac{1}{2^n}$ for all $n \in \mathbb{N}$. However, if $c = \lim_{n\to\infty} c_n$, then $\mathbb{1}_{\mathbb{Q}}(c) = \lim_{n\to\infty} f_{g(n)}(c) \in [\frac{1}{4},\frac{3}{4}]$, a contradiction. Hence, we have proved item (v) and (vi). Since $\mathbb{1}_{\mathbb{Q}}$ is not usco (or quasi-continuous or cliquish), the equivalence between $(\exists^2)$ and items (vii)-(x) follows in the same way.

To prove item (xii) (and item (xi)) from $(\exists^2)$, let $f_n(x)$ be 2^{2n} in case $x \in [0,\frac{1}{2^n}]$ and $1/x$ if $x \in (\frac{1}{2^n},1]$. Then each $f_n : [0,1] \to \mathbb{R}$ is continuous and the sequence converges pointwise to the function which is $1/x$ for $x > 0$ and 0 otherwise. The latter is unbounded and Baire 1. To prove $(\exists^2)$ from item (xii) (or item (xi)), note that the function provided by the latter must be discontinuous by Theorem 2.6. However, a discontinuous function yields $(\exists^2)$ by [55, §3]. The equivalence for items (xiii) follows in the same way, but without using WKL as continuous functions are trivially locally bounded.

For item (xiv), use $(\exists^2)$ to define the following functions $f,g : [0,1] \to \mathbb{R}$

$$f(x) := \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 1 & x = 0 \end{cases} \qquad g(x) := \begin{cases} -\sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Clearly, $f(x) + g(x) = \mathbb{1}_{\{0\}}$, which is not Darboux. For the reversal, a continuous function has the intermediate value property, which is provable in $\mathsf{RCA}_0^\omega + \mathsf{WKL}$ by combining Corollary 2.5 with the second-order intermediate value theorem ([94, II.6.6]). Hence, a function that is not Darboux, is discontinuous, yielding $(\exists^2)$ by [55, §3]. Note that we can avoid the use of WKL by imitating the proof of [94, II.6.6] in RCA_0^ω for (third-order) continuous functions.

For item (xv), consider $f : [0,1] \to [0,1]$ defined by $f(0) = 0$ and $f(x) := e^{-x}\cos(\frac{1}{x})$ for $x > 0$ using $(\exists^2)$. For the reversal, a continuous function is Darboux as in the previous paragraph, and attains is supremum by combining Cor. 2.5 and the second-order results in [94, IV.2.3]. Hence, item (xv) expresses the existence of a discontinuous function, and $(\exists^2)$ follows by [55, §3].

For items (xvi) and (xvii), a code for a Baire n function is essentially an effectively Baire n function where the continuous functions are given by codes. As noted below [7, Def. 6.1], ACA_0 suffices to show that a code for a Baire n function has a (unique) value. Hence, $\exists^2$ readily defines a third-order function taking these values everywhere on $[0,1]$. For the reversal, define a 'Baire 1 code for the Heaviside function' in RCA_0^ω and use items (xvi) or (xvii) to obtain the (discontinuous) Heaviside function, yielding $(\exists^2)$ by [55, §3]. □

The previous theorem yields the following strange result by contraposition:

if all functions on $\mathbb{R}$ are Baire 1, then all functions on $\mathbb{R}$ are continuous.

In this light, Brouwer's theorem is not an isolated event, but rather the limit of a certain restriction process.

Finally, one cannot generalise item (vi) of Theorem 2.28 to Baire 2 by Theorem 2.30. One *can* generalise the latter item to 'effectively Baire 2' by considering a well-known effectively Baire 3 function: the characteristic function of Borel's normal numbers ([9]). The technical details are however tedious and the same holds for 'effectively Baire n', where examples of such functions are given in [49].

2.8. **Beyond the Big Five.**

2.8.1. *Introduction.* In the above, we have obtained equivalences between well-known second-order principles like the Big Five on one hand, and a number of third-order theorems on the other hand. This was based on the higher-order RM of $(\exists^2)$, which we have also developed. In this section, we show that similarly basic third-order statements or slight generalisations, go **far** beyond the RM of the Big Five and $(\exists^2)$. We do so by deriving from the former the following:

$$(\forall Y:[0,1]\to\mathbb{N})(\exists x,y\in[0,1])(x\neq y\wedge Y(x)=Y(y)), \tag{$\mathsf{NIN}_{[0,1]}$}$$

which expresses that there is no injection from $[0,1]$ to $\mathbb{N}$. By [74, §4], $\mathsf{NIN}_{[0,1]}$ is not provable in relatively strong systems like Z_2^ω, which is a conservative extension of Z_2 (see Section A.3). The following list provides some interesting examples.

- The existence of a function not in Baire 1 is equivalent to $(\exists^2)$ (Theorem 2.28), while the existence of a function not in Baire 2 (or Baire 1^*) implies $\mathsf{NIN}_{[0,1]}$ (Theorem 2.30).
- Cousin's lemma for lsco (or: quasi-continuous) functions is equivalent to WKL (Theorem 2.14), while this lemma for usco (or: cliquish) functions implies $\mathsf{NIN}_{[0,1]}$ (Theorem 2.34).
- Cousin's lemma for effectively (or: codes for) Baire 2 functions is part of the RM of ATR_0 (Theorem 2.24) while the generalisation to Baire 2 (or Baire 1^*) functions implies $\mathsf{NIN}_{[0,1]}$ (Theorem 2.34).
- The supremum principle for effectively (or: codes for) Baire 2 functions is part of the RM of Π^1_1-CA_0 (Theorem 2.22) while the generalisation to Baire 2 (or Baire 1^*) functions implies $\mathsf{NIN}_{[0,1]}$ (Theorem 2.32).
- Jordan's decomposition theorem for cadlag BV-functions is equivalent to ACA_0 (Theorem 2.19), while this theorem for usco BV-functions implies $\mathsf{NIN}_{[0,1]}$ (Theorem 2.37).

In our opinion, these examples show that one should not put too much emphasis on the distinction 'second- versus third-order', as there are plenty equivalences between second- and third-order theorems. The real fundamental 'divide' is whether a given theorem follows from conventional comprehension alone (say up to Z_2^ω), or whether it implies $\mathsf{NIN}_{[0,1]}$ or similar principles not provable in Z_2^ω.

An important side-result of this section (see Theorem 2.34) is that many well-known inclusions among function spaces, like the statements *BV-functions are Baire 1* and *Baire 1^* functions are Baire 1*, also imply $\mathsf{NIN}_{[0,1]}$. In this way, such inclusions cannot be established in the Big Five and much stronger systems.

Finally, we mention in passing that the results in this section also identify certain problems with the representation or coding of (slightly) discontinuous functions in the language of second-order arithmetic.

2.8.2. *Beyond Baire 1 functions.* In this section, we show that the equivalences in Theorems 2.9 and 2.28 cannot be generalised to Baire 2 or Baire 1^*.

First of all, we shall need the following fragment of the induction axiom, also studied in [75, §2.2.2] with some non-trivial equivalences.

Definition 2.29. [IND_0] Let Y^2 satisfy $(\forall n \in \mathbb{N})(\exists$ at most one $f \in 2^{\mathbb{N}})(Y(f,n) = 0)$. For $k \in \mathbb{N}$, there is w^{1^*} with $|w| = k$ such that for $m \leq k$, we have:

$$(w(m) \in 2^{\mathbb{N}} \wedge Y(w(m), m) = 0) \leftrightarrow (\exists f \in 2^{\mathbb{N}})(Y(f,m) = 0).$$

We now have the following result, to be contrasted with item (vi) in Theorem 2.28. There is no contradiction here: Baire 1^* functions are of course Baire 1, but by Theorem 2.34, this is not provable from the Big Five and much stronger systems.

Theorem 2.30 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_0$). *The principle* $\mathsf{NIN}_{[0,1]}$ *follows from either:*

- *there is a* $[0,1] \to \mathbb{R}$ *function that is not Baire 2,*
- *there is a* $[0,1] \to \mathbb{R}$ *function that is not Baire* 1^*.

Proof. Fix an arbitrary function $f : [0,1] \to \mathbb{R}$ and let $Y : [0,1] \to \mathbb{N}$ be injective. Now define $f_n(x)$ as $f(x)$ in case $Y(x) \leq n$, and 0 otherwise. Clearly, f is the pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}}$. Now fix some $n_0 \in \mathbb{N}$ and use IND_0 to enumerate all $x \in [0,1]$ such that $Y(x) \leq n_0$. With this finite sequence, one readily defines a sequence of continuous functions converging to f_{n_0}, which shows that the latter is Baire 1. This shows that any function $f : [0,1] \to \mathbb{R}$ is Baire 2 assuming $\neg\mathsf{NIN}_{[0,1]}$. Now define the closed set $C_n := \{x \in [0,1] : Y(x) = n\}$ and note that $f_{\restriction C_n}$ is indeed continuous for all $n \in \mathbb{N}$. Hence, f is also Baire 1^* assuming $\neg\mathsf{NIN}_{[0,1]}$, and we are done. □

We believe the first item in Theorem 2.30 is related to the Vitali-Carathéodory theorem as in [105, Cor. 4], but we can only conjecture a connection.

Secondly, we study the following supremum principle which Theorem 2.22 establishes for effectively Baire n functions assuming Π_1^1-CA_0. Theorem 2.32 shows that slight generalisations are not provable in Z_2^ω.

Principle 2.31 (Supremum principle for Γ). *For bounded* $f : [0,1] \to \mathbb{R}$ *in* Γ, *there is* $F : \mathbb{Q}^2 \to \mathbb{R}$ *such that for* $p, q \in \mathbb{Q} \cap [0,1]$, *the real* $F(p,q)$ *equals* $\inf_{x\in[p,q]} f(x)$.

The following theorem is to be contrasted with items (v), (xx), and (xxiii) of Theorem 2.9 and with Theorem 2.22. There is no contradiction here as $\mathsf{NIN}_{[0,1]}$ follows from the fact that regulated or usco functions are Baire 1 by Theorem 2.34.

Theorem 2.32 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_0$). *The principle* $\mathsf{NIN}_{[0,1]}$ *follows from either:*

- *The supremum principle* (*Princ. 2.31*) *for Baire* 1^* *or Baire 2 functions.*
- *The supremum principle* (*Princ. 2.31*) *for cliquish functions.*
- *The supremum principle* (*Princ. 2.31*) *for regulated functions.*
- *The supremum principle* (*Princ. 2.31*) *for usco functions.*
- *The supremum principle* (*Princ. 2.31*) *for* BV*-functions.*

The theorem still goes through if we limit the items to functions that are pointwise discontinuous or continuous almost everywhere.

Proof. First of all, let $Y : [0,1] \to \mathbb{N}$ be an injection and define $f(x) := \frac{1}{2^{Y(x)+5}}$, which is Baire 2, Baire 1^*, usco, regulated, cliquish, and BV, as we show next. Indeed, for fixed $x_0 \in [0,1]$, IND_0 can enumerate the finitely many reals that are mapped below $n_0 := Y(x_0)$ by Y. Thus, f is arbitrary close to 0 in a small enough punctured neighbourhood of x_0, readily implying that it is usco, regulated, and cliquish. Moreover, since Y is an injection, the sums $\sum_{i=0}^{n-1} |f(x_i) - f(x_{i+1})|$ as in (1.4) are bounded by $\sum_{i=0}^{n} \frac{1}{2^{i+5}}$, which is at most 2, i.e. f is in BV. For the Baire 2 property, define $f_n(x)$ as $f(x)$ in case $Y(x) \leq n$, and define $f_n(x)$ as 0 if $Y(x) > n$. Clearly, for fixed $n_0 \in \mathbb{N}$, the function f_{n_0} has got at most finitely many points of discontinuity, which can be enumerated using IND_0. Using this finite list, one readily defines a sequence of continuous functions converging to f_{n_0}, i.e. the latter is Baire 1 and f is Baire 2. That f is Baire 1^* follows as for Theorem 2.30.

Secondly, $\sup_{x\in[0,1]} f(x)$ is $\frac{1}{2^{n_1+5}}$ where $n_1 \in \mathbb{N}$ is the least number such that $Y(x_1) = n_1$ for some $x_1 \in [0,1]$. Comparing $\sup_{x\in[0,\frac{1}{2}]} f(x)$ and $\sup_{x\in[\frac{1}{2},1]} f(x)$, we obtain the first bit of the binary expansion of x_1. Using the usual interval-halving technique, we then obtain x_1 itself. Now repeat this process for f replaced by $f_1 : [0,1] \to \mathbb{R}$ which is f for $x \neq x_1$ and 0 otherwise. Thus, we obtain an enumeration of $[0,1]$ and by [94, II.4.9], there is $y \in [0,1]$ not in the latter sequence, a contradiction, and the items from the theorem follow.

Thirdly, for the final sentence in the theorem, define $g : [0,1] \to \mathbb{R}$ as $g(x) := 0$ if $x \in \mathbb{Q} \cap [0,1]$, and $g(x) := f(x)$ otherwise. Then g is continuous at each rational number and thus pointwise discontinuous, which one readily establishes using IND_0. In the same way as for f in the previous paragraph, the function g is BV, usco, regulated, Baire 2, and cliquish. Repeating the 'sup construction' from the previous paragraph for f replaced by g, one obtains an enumeration of $[0,1] \setminus \mathbb{Q}$, which yields the required contradiction.

Finally, for the restriction to functions continuous almost everywhere, let $\mathcal{C}$ be the Cantor (middle-third) set, which has an RM-code and is recursively homomorphic to Cantor space, all in RCA_0 by [94, I.8.6]. Let $Y : \mathcal{C} \to \mathbb{N}$ be an injection and define $h : [0,1] \to \mathbb{R}$ as 0 in case $x \notin \mathcal{C}$ and $\frac{1}{2^{Y(x)+1}}$ otherwise. In the same way as for f and g, the function h is BV, usco, regulated, Baire 2, and cliquish. Since $\mathcal{C}$ is closed and has measure zero, h is continuous almost everywhere. Repeating the 'sup construction' from the previous paragraph for f replaced by h, one obtains an enumeration of $\mathcal{C}$, which yields a contradiction by [94, II.5.9]. Moreover, in case $\mathcal{C}$ is not countable, neither is $2^{\mathbb{N}}$ and $[0,1]$ by the results in [90], and we are done. □

One can derive NIN from the fact that a Baire 2 function has a supremum, i.e. not involving parameters, but the technical details are somewhat messy. Now, Theorem 2.32 identifies a problem with the coding of Baire 2 functions in the language of second-order arithmetic. Indeed, comparing Theorems 2.22 and 2.32, we observe that the logical properties of the supremum principle for Baire 2 functions changes dramatically upon restriction to *effectively* Baire 2 functions; the latter using codes for continuous functions is essentially the second-order representation used in [6, 7].

Thirdly, we have the following theorem to be contrasted with Theorem 2.23. Note that $\mathsf{Z}_2^\omega + \mathsf{IND}_0$ cannot prove $\mathsf{NIN}_{[0,1]}$ by [76, §3].

Theorem 2.33 ($\mathsf{ACA}_0^\omega + \Pi_1^1\text{-}\mathsf{CA}_0 + \mathsf{IND}_0$). *The principle* $\mathsf{NIN}_{[0,1]}$ *follows from:*

$$\textit{for any open Baire 2 set in } \mathbb{R}\textit{, there is an RM-code.} \tag{2.21}$$

Proof. Fix $A \subset [0,1]$ and $Y : [0,1] \to \mathbb{N}$ injective on A. Note that we can use μ^2 from Section A.3 to remove any rationals from A. Note that we can also guarantee that Y maps to $\mathbb{N} \setminus \{0, 1\}$. Now define the closed set $C \subset \mathbb{R}$ as follows:

$$y \in C \leftrightarrow (\exists n \in \mathbb{N})(n < y < n + 1 \wedge (y - n) \in A \wedge Y(y - n) = n).$$

Since Y is an injection on A, each interval $(n, n+1)$ for $n \geq 2$ contains at most one $y \in C$. Clearly, C is closed and $O := \mathbb{R} \setminus C$ is open. For fixed $n_0 \in \mathbb{N}$, we can enumerate the (at most n_0) reals in $C \cap (0, n_0 + 1)$ using IND_0. With the finite list, one readily defines a sequence of continuous functions that converges to $\mathbb{1}_{O\cap(0,n_0+1)}$, i.e. the latter is Baire 1. Then $\mathbb{1}_O$ is Baire 2 and an RM-code for O (and hence C) is provided by the centred item. Suppose $O = \cup_{n\in\mathbb{N}}(a_n, b_n)$ where the latter intervals have rational end-points. Use Π^1_1-CA_0 to form the following set $B := \{m \in \mathbb{N} : (\exists x \in \mathbb{R})\varphi(x, m)\}$, where

$$\varphi(x, m) \equiv (m < x < m + 1 \wedge x \in \mathbb{R} \setminus \cup_{n\in\mathbb{N}}(a_n, b_n))$$

is (equivalent to) an L_2-formula. Now apply Σ^1_1-AC_0, provable in ATR_0 by [94, V.8.3], to $(\forall m \in B)(\exists x \in \mathbb{R})\varphi(x, m)$. The resulting sequence readily yields an enumeration of A. By [94, II.4.9], there is $y \in [0, 1]$ not in this enumeration, and hence we can find $z \in [0, 1] \setminus A$. In this way, for any countable set $A \subseteq [0, 1]$ (Def. 1.7), there is $y \in [0, 1] \setminus A$. Thus, there is injection from $[0, 1]$ to $\mathbb{N}$. $\square$

One can replace $\mathbb{R}$ in (2.21) by $[0, 1]$ and still obtain $\mathsf{NIN}_{[0,1]}$; we leave this as an exercise to the reader.

In conclusion, we have shown that certain *slight* generalisations or variations of the third-order theorems from Sections 2.3-2.7 go far beyond the RM of the Big Five or $(\exists^2)$. It is however hard to draw a 'borderline': Theorem 2.30 involves a super-class, namely Baire 2, *and* a sub-class, namely Baire 1^*, of Baire 1. Now, Baire 1^* functions are of course Baire 1, but only in strong enough systems by Theorem 2.34. A similar result can be obtained for Baire 1^{**} functions ([80]).

2.8.3. *Variations on a theme.* In this section, we establish the results sketched in Section 2.8.1, i.e. we show that slight generalisations or variations of the third-order theorems from Sections 2.3-2.7 are not provable from any Big Five system, the axiom $(\exists^2)$, and much stronger systems.

First of all, we have the following theorem, where the first five items are to be contrasted with Theorem 2.14. Regarding items (xi)-(xv), we recall that Z^ω_2 cannot prove $\mathsf{NIN}_{[0,1]}$, i.e. ATR_0 is relatively weak in comparison.

Theorem 2.34 ($\mathsf{ACA}^\omega_0 + \mathsf{IND}_0$)**.** *The following statements imply* $\mathsf{NIN}_{[0,1]}$.

- (i) *Cousin's lemma for BV-functions* $\Psi : [0, 1] \to \mathbb{R}^+$.
- (ii) *Cousin's lemma for regulated functions* $\Psi : [0, 1] \to \mathbb{R}^+$.
- (iii) *Cousin's lemma for usco* $\Psi : [0, 1] \to \mathbb{R}^+$.
- (iv) *Cousin's lemma for cliquish* $\Psi : [0, 1] \to \mathbb{R}^+$.
- (v) *Cousin's lemma for Baire* 1^* $\Psi : [0, 1] \to \mathbb{R}^+$.
- (vi) *Cousin's lemma for Baire 2* $\Psi : [0, 1] \to \mathbb{R}^+$.
- (vii) *All usco* (*or: lsco*) *functions* $f : [0, 1] \to \mathbb{R}$ *are Baire 1.*
- (viii) *All BV-functions* $f : [0, 1] \to \mathbb{R}$ *are Baire 1.*
- (ix) *All regulated functions* $f : [0, 1] \to \mathbb{R}$ *are Baire 1.*
- (x) *All Baire* 1^* *functions* $f : [0, 1] \to \mathbb{R}$ *are Baire 1.*
- (xi) *All usco cliquish* $f : [0, 1] \to \mathbb{R}$ *are Baire 1.*

Given $\mathsf{ATR}_0 + \Delta^1_2$-$\mathsf{IND}$*, the principle* $\mathsf{NIN}_{[0,1]}$ *also follows from the following.*

- (xi) *All BV* (*or: regulated, usco, or lsco*) $f : [0, 1] \to \mathbb{R}$ *are effectively Baire 2.*

(xii) *All* BV *(or: regulated, usco, or lsco)* $f : [0,1] \to \mathbb{R}$ *are eff. Baire* $n+2$.
(xiii) *All* BV *(or: regulated, usco, or lsco)* $f : [0,1] \to \mathbb{R}$ *have a Borel code.*
(xiv) *All Baire 2* $f : [0,1] \to \mathbb{R}$ *are effectively Baire 2 (Baire,* [4, p. 69]*).*
(xv) *All Baire 2* $f : [0,1] \to \mathbb{R}$ *are effectively Baire* n $(n \geq 3)$.

The theorem still goes through if we limit items (i)-(xi) *to functions that are pointwise discontinuous or continuous almost everywhere.*

Proof. Let $Y : [0,1] \to \mathbb{N}$ be an injection and define $\Psi(x) := \frac{1}{2^{Y(x)+5}}$. Recall that Ψ is usco (and cliquish, regulated, and BV) as shown in the proof of Theorem 2.32. Now, for distinct reals $x_0, \dots, x_k \in [0,1]$ we must have that $\cup_{i \leq k} B(x_i, \Psi(x_i))$ has measure at most $1/2$, since $\Psi(x_0), \dots, \Psi(x_k)$ are all distinct due to Y being an injection. However, item (iii) provides a finite sub-covering of $\cup_{x \in [0,1]} B(x, \Psi(x))$, which must have measure at least 1, a contradiction. Thus, there is no injection from $[0,1]$ to $\mathbb{N}$, i.e. $\mathsf{NIN}_{[0,1]}$ follows from item (iii). The same proof goes through for items (i), (ii), and (iv). That f is Baire 1^* follows as for Theorem 2.30, i.e. item (v) also follows in the same way.

For item (vi), one shows that $\Psi(x) := \frac{1}{2^{Y(x)+5}}$ as above, is Baire 2 in the same way as in the proof of Theorem 2.30. Indeed, define $\Psi_n(x)$ as $\Psi(x)$ if $Y(x) \leq n$, and 0 otherwise. Clearly, Ψ is the pointwise limit of Ψ_n while the latter has at most $n+5$ points of discontinuity. For fixed $n \in \mathbb{N}$, the discontinuity points of Ψ_n can be enumerated using IND_0, which readily yields a sequence of continuous functions that converges to Ψ_n. In this way, Ψ as above is Baire 2, and $\mathsf{NIN}_{[0,1]}$ follows.

For item (vii), we derive item (iii) from the latter. To this end, fix usco $\Psi : [0,1] \to \mathbb{R}^+$ and let $(\Psi_n)_{n \in \mathbb{N}}$ be a sequence of continuous functions with pointwise limit Ψ. Use Corollary 2.5 to obtain a sequence $(\Phi_n)_{n \in \mathbb{N}}$ of codes for continuous functions. The latter sequence is a code for a Baire 1 function in the sense of [6, 7]. Since ACA_0 proves Cousin's lemma for codes for Baire 1 functions (see [6, 7]), we obtain Cousin's lemma for usco functions and hence $\mathsf{NIN}_{[0,1]}$ by item (vii). By definition, for usco $f : [0,1] \to \mathbb{R}$ with upper bound $N \in \mathbb{N}$ (provided by Theorem 2.9), $g(x) := N - f(x)$ is lsco, i.e. item (vii) can be formulated with either lsco or usco.

For items (viii)-(xi), the same proof as for item (vii) goes through for regulated or BV-functions. In particular, $\Psi(x) := \frac{1}{2^{Y(x)+5}}$ is BV (and regulated, cliquish, and Baire 1^*) in case $Y : [0,1] \to \mathbb{N}$ is an injection by the proof of Theorem 2.32.

For item (xi), recall that $\mathsf{ATR}_0 + \Delta^1_2$-induction proves (second-order) Cousin's lemma for codes of Baire 2 (or even Borel) functions ([6,7]). Moreover, we note that $\exists^2$ can convert an effectively Baire 2 function into a code for a Baire 2 function (by uniformly replacing the continuous functions by codes). Hence, assuming item (xi), we obtain the Cousin lemma for BV (or: regulated, or: usco) functions, and hence $\mathsf{NIN}_{[0,1]}$ by the results for item (i) (and items (ii) and (iii)). The case for lsco functions implies the case for usco functions as for item (vii). Items (xii) and (xiii) follow in the same way. Items (xiv) and (xv) follow in the same way, in combination with the fact that item (vi) proves $\mathsf{NIN}_{[0,1]}$.

For the final sentence, the supremum principle for Baire 1 functions is provable in $\mathsf{RCA}_0^\omega + \mathsf{WKL}$ by Theorem 2.9. Hence, items (vii)-(xi) yield the supremum principle for the associated classes. We now obtain $\mathsf{NIN}_{[0,1]}$ via Theorem 2.32. For items (ii)-(vi), let $Y : 2^{\mathbb{N}} \to \mathbb{N}$ be an injection and recall the Cantor set $\mathcal{C}$ from the proof of Theorem 2.32, with associated recursive homomorphism $H : 2^{\mathbb{N}} \to [0,1]$ defined as $H(f) := \sum_{n=0}^{\infty} \frac{2f(n)}{3^{n+1}}$. Now define $\Psi : [0,1] \to \mathbb{R}^+$ using $(\exists^2)$ as:

$$\Psi(x) := \begin{cases} d(x, \mathcal{C}) & x \notin \mathcal{C} \\ \frac{1}{2^{Y(I(x))+5}} & \text{otherwise} \end{cases}, \tag{2.22}$$

where $I(x)$ is the unique $f \in 2^{\mathbb{N}}$ such that $H(f) = x$ in case $x \in \mathcal{C}$, and $00\dots$ otherwise. As in the proof of Theorem 2.32, Ψ is continuous almost everywhere, pointwise discontinuous, and has the properties required for items (ii)-(vi). To show that Ψ is in BV, note that $\lambda x.d(x, \mathcal{C})$ is Lipschitz (with constant 1) and hence BV as the textbook proof goes through in RCA_0^ω ([2, p. 74]). Since Y is an injection, the function $\lambda x.\big(\Psi(x) - d(x, \mathcal{C})\big)$ is also in BV. The sum of two BV-functions is in BV, as the textbook proof goes through in RCA_0^ω ([2, Prop. 1.3]).

Finally, by the definition of Ψ in (2.22), if $x \in [0,1] \setminus \mathcal{C}$, then $\mathcal{C} \cap I_x^\Psi = \emptyset$. Hence, let $z_0, \dots, z_m$ be those $y_i \in \mathcal{C}$ for $i \leq k$ and note that $\cup_{j \leq m} I_{z_j}^\Psi$ covers $\mathcal{C}$. Then $I(z_0), \dots, I(z_m)$ yields a finite sub-cover of $\cup_{f \in 2^{\mathbb{N}}} [\overline{f}(Y(f)+5)]$. However, the measure of the latter is below $1/2$, a contradiction. Moreover, in case $2^{\mathbb{N}}$ is not countable, neither is $[0,1]$ by the results in [90], i.e. $\mathsf{NIN}_{[0,1]}$ follows. □

We could restrict item (i) in Theorem 2.34 to functions continuous on a set of positive measure; this would mean replacing the Cantor set $\mathcal{C}$ in (2.22) by a 'fat' Cantor set, i.e. having positive measure. Item (vii)-(xi) are also robust in that they still imply $\mathsf{NIN}_{[0,1]}$ upon replacing 'Baire 1' by the equivalent definition (involving perfect sets) provided by the Baire characterisation theorem ([5]).

Recall the Vitali principle from Section 2.3.3, which yields the following immediate corollary, to be contrasted with Theorem 2.15.

Corollary 2.35 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_0$). *The following statements imply* $\mathsf{NIN}_{[0,1]}$.

- (i) *Vitali's principle for BV-functions* $\Psi : [0,1] \to \mathbb{R}^+$.
- (ii) *Vitali's principle for regulated functions* $\Psi : [0,1] \to \mathbb{R}^+$.
- (iii) *Vitali's principle for usco* $\Psi : [0,1] \to \mathbb{R}^+$.
- (iv) *Vitali's principle for cliquish* $\Psi : [0,1] \to \mathbb{R}^+$.
- (v) *Vitali's principle for Baire* 1^* $\Psi : [0,1] \to \mathbb{R}^+$.
- (vi) *Vitali's principle for Baire 2* $\Psi : [0,1] \to \mathbb{R}^+$.

Proof. The proof of Theorem 2.34 goes through: the single use of Cousin's lemma can be replaced by the associated version of Vitali's principle for $\varepsilon > \frac{1}{2}$. □

We note that the Theorem 2.34 identifies a significant problem with the coding of Baire 2 functions in the language of second-order arithmetic. Indeed, Cousin's lemma for *codes for* Baire 2 functions is equivalent to ATR_0 ([6,7]). In light of item (vi) of Theorem 2.34, this coding seriously changes the logical strength of Cousin's lemma for Baire 2 functions. We do have the following nice corollary to the theorem and Theorem 2.24, to be contrasted with the fact that Z_2^ω cannot prove $\mathsf{NIN}_{[0,1]}$.

Corollary 2.36. *For $n \geq 2$, the system* $\mathsf{ACA}_0^\omega + \mathsf{ATR}_0 + \Delta_2^1\text{-}\mathsf{IND}$ *proves that there is no effectively Baire n function $Y : [0,1] \to \mathbb{Q}$ that is injective on* $[0,1]$.

Proof. By Theorem 2.24, the system at hand proves Cousin's lemma for effectively Baire n functions. As in the (first paragraph of the) proof of the theorem, one obtains the relevant restriction of $\mathsf{NIN}_{[0,1]}$, and we are done. □

Secondly, we have the following theorem, to be contrasted with Theorem 2.19.

Theorem 2.37 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_0$)**.** *The principle* $\mathsf{NIN}_{[0,1]}$ *follows from the following:*

Jordan decomposition theorem (Theorem 1.4) restricted to usco BV-functions.

Proof. Let $Y : [0,1] \to \mathbb{N}$ be an injection and define $f(x) := \frac{1}{2^{Y(x)+5}}$. In the same way as in the proof of Theorem 2.32, $f : [0,1] \to \mathbb{R}$ is usco and BV. The centred statement from Theorem 2.37 now provides non-decreasing $g, h : [0,1] \to \mathbb{R}$ such that $f = g - h$ on $[0,1]$. By [75, Lemma 3], $\exists^2$ can enumerate the points of discontinuity of non-decreasing functions. As in [94, II.4.9], this enumeration is not all of $[0,1]$, i.e. there is $y \in [0,1]$ such that g and h are continuous at y. Hence f is continuous at y, but this is a contradiction as there are points z arbitrarily close to y such that $f(z)$ is arbitrarily small (use IND_0 to establish this claim). □

There are a number of similar 'decomposition theorems', e.g. implying that cliquish and usco functions can be expressed as the sum of two quasi-continuous functions ([10, 64]). One readily shows that the latter decompositions also yield $\mathsf{NIN}_{[0,1]}$, even when restricted to BV-functions.

Thirdly, by Theorems 2.19 and 2.20, $\mathsf{RCA}_0^\omega + \mathsf{ACA}_0$ proves certain basic properties of the Riemann integral and BV-functions. By contrast, Theorem 2.38 shows that other basic properties do not follow from the former and much stronger systems. We note that the negation of the first item implies a very strong 'non-uniqueness' of the Riemann integral. Similarly, the negation of the second item states that BV-functions need not be differentiable *anywhere*. A version of the second item, called Lebesgue's theorem, involving codes is provable in WKL_0 by [69, §6].

Theorem 2.38 ($\mathsf{ACA}_0^\omega + \mathsf{IND}_0$)**.** *The following principles imply* $\mathsf{NIN}_{[0,1]}$.

- For a Riemann integrable BV-function $f : [0,1] \to [0,1]$ with $\int_0^1 f(x)dx = 0$, there is $x \in [0,1]$ such that $f(x) = 0$.
- For $f : [0,1] \to [0,1]$ in BV, there is $x \in [0,1]$ where f is differentiable.
- For regulated $f : [0,1] \to [0,1]$, there is $x \in [0,1]$ where f is continuous.
- For usco $f : [0,1] \to \mathbb{R}$, there is $x \in [0,1]$ where f is continuous.

Proof. Let $Y : [0,1] \to \mathbb{N}$ be an injection and define $f(x) := \frac{1}{2^{Y(x)+5}}$. As in the proof of Theorems 2.34 and 2.37, this function is in BV and regulated and usco.

To show that f is Riemann integrable with integral equal to zero, use the 'epsilon-delta' definition of Riemann integrability and (essentially) the same argument why f is in BV. However, $f(x) > 0$ for all $x \in [0,1]$ by definition, i.e. we obtain a contradiction from the first item of Theorem 2.38.

Similarly, the second and third item imply there is $y \in [0,1]$ where f is continuous. This is a contradiction as there are points z arbitrarily close to y such that $f(z)$ is arbitrarily small (use IND_0 to establish this claim). □

It is an interesting exercise to show that in the final two items, *continuity* can be replaced by much weaker properties, including *feeble continuity* as in [97, §24, p. 53]. As noted in the latter, *every* $\mathbb{R} \to \mathbb{R}$-function is feebly continuous at *all but countably* many reals, i.e. we are dealing with a *very* weak continuity notion. These results go back to Young ([103], 1907) with a (historical) overview in [18].

We note that Theorems 2.37 and 2.38 identify a problem with the coding of functions in the language of second-order arithmetic. Indeed, we observe that the logical properties of Cousin's lemma for Baire 2 functions, the Jordan decomposition theorem for BV-functions, and Lebesgue's (differentiability) theorem for BV-functions change dramatically upon introducing second-order codes.

Finally, while the main focus of [76] is the study of $\mathsf{NIN}_{[0,1}$, some results are obtained for $\mathsf{NBI}_{[0,1]}$, i.e. the statement *there is no* ***bijection*** *from* $[0,1]$ *to* $\mathbb{N}$. By [76, §4], $\mathsf{RCA}_0^\omega + \mathsf{QF\text{-}AC}^{0,1}$ proves $\mathsf{NBI}_{[0,1]}$ but Z_2^ω cannot. In this way, $\mathsf{NBI}_{[0,1]}$ exhibits the Pincherle phenomenon from Remark 2.10. Now, we have used item (a) in Definition 1.3 as our definition of BV-functions. One could define 'strong BV' as item (b) in Definition 1.3, i.e. the supremum (1.4) must additionally exist. One readily verifies that e.g. Cousin's lemma for strong BV-functions (see item (i) in Theorem 2.34) implies $\mathsf{NBI}_{[0,1]}$. Similarly, we can derive $\mathsf{NBI}_{[0,1]}$ from any of the above theorems implying $\mathsf{NIN}_{[0,1]}$, even after replacing 'BV' by 'strong BV'.

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Appendix A. Higher-order Reverse Mathematics

We introduce the base theory of higher-order RM (Section A.1), some essential notations (Section A.2), and some axioms (Section A.3).

A.1. **The base theory of higher-order Reverse Mathematics.**

We introduce Kohlenbach's base theory RCA_0^ω, first introduced in [55, §2].

Definition A.1. The base theory RCA_0^ω consists of the following axioms.

(a) Basic axioms expressing that $0, 1, <_0, +_0, \times_0$ form an ordered semi-ring with equality $=_0$.

(b) Basic axioms defining the well-known Π and Σ combinators (aka K and S in [3]), which allow for the definition of λ*-abstraction.*

(c) The defining axiom of the recursor constant $\mathbf{R}_0$: for m^0 and f^1:

$$\mathbf{R}_0(f,m,0) := m \text{ and } \mathbf{R}_0(f,m,n+1) := f(n,\mathbf{R}_0(f,m,n)). \tag{A.1}$$

(d) The *axiom of extensionality*: for all $\rho,\tau \in \mathbf{T}$, we have:

$$(\forall x^\rho, y^\rho, \varphi^{\rho\to\tau})\big[x =_\rho y \to \varphi(x) =_\tau \varphi(y)\big]. \tag{$\mathsf{E}_{\rho,\tau}$}$$

(e) The induction axiom for quantifier-free formulas of L_ω.

(f) $\mathsf{QF\text{-}AC}^{1,0}$: the quantifier-free Axiom of Choice as in Definition A.2.

Note that variables (of any finite type) are allowed in quantifier-free formulas of the language L_ω: only quantifiers are banned. Recursion as in (A.1) is called *primitive recursion*; the class of functionals obtained from $\mathbf{R}_\rho$ for all $\rho \in \mathbf{T}$ is called *Gödel's system* T of all (higher-order) primitive recursive functionals.

Definition A.2. The axiom $\mathsf{QF\text{-}AC}$ consists of the following for all $\sigma,\tau \in \mathbf{T}$:

$$(\forall x^\sigma)(\exists y^\tau)A(x,y) \to (\exists Y^{\sigma\to\tau})(\forall x^\sigma)A(x,Y(x)), \tag{$\mathsf{QF\text{-}AC}^{\sigma,\tau}$}$$

for any quantifier-free formula A in the language of L_ω.

As discussed in [55, §2], RCA_0^ω and RCA_0 prove the same sentences 'up to language' as the latter is set-based and the former function-based. This conservation result is obtained via the so-called ECF-interpretation discussed in Remark A.3.

Remark A.3 (The ECF-interpretation)**.** The (rather) technical definition of ECF may be found in [98, p. 138, §2.6]. Intuitively, the ECF-interpretation $[A]_{\mathsf{ECF}}$ of a formula $A \in \mathsf{L}_\omega$ is just A with all variables of type two and higher replaced by type one variables ranging over so-called 'associates' or 'RM-codes' (see [53, §4]); the latter are countable representations of continuous functionals. Thus, the formula $[A]_{\mathsf{ECF}}$ is just A in case $A \in \mathsf{L}_2$. The ECF-interpretation connects RCA_0^ω and RCA_0 (see [55, Prop. 3.1]) in that if RCA_0^ω proves A, then RCA_0 proves $[A]_{\mathsf{ECF}}$, again 'up to language', as RCA_0 is formulated using sets, and $[A]_{\mathsf{ECF}}$ is formulated using types, i.e. using type zero and one objects. In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the *canonical* embedding of higher-order into second-order arithmetic.

A.2. **Notations and the like.** We introduce the usual notations for common mathematical notions, like real numbers, as also introduced in [55].

Definition A.4 (Real numbers and related notions in RCA_0^ω)**.**

(a) Natural numbers correspond to type zero objects, and we use 'n^0' and '$n \in \mathbb{N}$' interchangeably. Rational numbers are defined as signed quotients of natural numbers, and '$q \in \mathbb{Q}$' and '$<_\mathbb{Q}$' have their usual meaning.

(b) Real numbers are coded by fast-converging Cauchy sequences $q_{(\cdot)} : \mathbb{N} \to \mathbb{Q}$, i.e. such that $(\forall n^0, i^0)(|q_n - q_{n+i}| <_\mathbb{Q} \frac{1}{2^n})$. We use Kohlenbach's 'hat function' from [55, p. 289] to guarantee that every q^1 defines a real number.

(c) We write '$x \in \mathbb{R}$' to express that $x^1 := (q^1_{(\cdot)})$ represents a real as in the previous item and write $[x](k) := q_k$ for the k-th approximation of x.

(d) Two reals x, y represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are *equal*, denoted $x =_\mathbb{R} y$, if $(\forall n^0)(|q_n - r_n| \leq 2^{-n+1})$. Inequality '$<_\mathbb{R}$' is defined similarly. We sometimes omit the subscript '$\mathbb{R}$' if it is clear from context.

(e) Functions $F : \mathbb{R} \to \mathbb{R}$ are represented by $\Phi^{1\to1}$ mapping equal reals to equal reals, i.e. extensionality as in $(\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \to \Phi(x) =_\mathbb{R} \Phi(y))$.

(f) The relation '$x \leq_\tau y$' is defined as in (1.1) but with '$\leq_0$' instead of '$=_0$'. Binary sequences are denoted '$f^1, g^1 \leq_1 1$', but also '$f, g \in C$' or '$f, g \in 2^\mathbb{N}$'. Elements of Baire space are given by f^1, g^1, but also denoted '$f, g \in \mathbb{N}^\mathbb{N}$'.

(g) For a binary sequence f^1, the associated real in $[0, 1]$ is $\mathbb{r}(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}$.

(h) Sets of type ρ objects $X^{\rho\to0}, Y^{\rho\to0}, \ldots$ are given by their characteristic functions $F_X^{\rho\to0} \leq_{\rho\to0} 1$, i.e. we write '$x \in X$' for $F_X(x) =_0 1$.

For completeness, we list the following notational convention for finite sequences.

Notation A.5 (Finite sequences)**.** The type for 'finite sequences of objects of type ρ' is denoted ρ^*, which we shall only use for $\rho = 0, 1$. Since the usual coding of pairs of numbers goes through in RCA_0^ω, we shall not always distinguish between 0 and 0^*. Similarly, we assume a fixed coding for finite sequences of type 1 and shall make use of the type '1^*'. In general, we do not always distinguish between 's^ρ' and '$\langle s^\rho \rangle$', where the former is 'the object s of type ρ', and the latter is 'the sequence of type ρ^* with only element s^ρ'. The empty sequence for the type ρ^* is denoted by '$\langle\rangle_\rho$', usually with the typing omitted.

Furthermore, we denote by '$|s| = n$' the length of the finite sequence $s^{\rho^*} = \langle s_0^\rho, s_1^\rho, \dots, s_{n-1}^\rho\rangle$, where $|\langle\rangle| = 0$, i.e. the empty sequence has length zero. For sequences s^{ρ^*}, t^{ρ^*}, we denote by '$s{*}t$' the concatenation of s and t, i.e. $(s{*}t)(i) = s(i)$ for $i < |s|$ and $(s{*}t)(j) = t(|s|-j)$ for $|s| \leq j < |s|+|t|$. For a sequence s^{ρ^*}, we define $\overline{s}N := \langle s(0), s(1), \dots, s(N-1)\rangle$ for $N^0 < |s|$. For a sequence $\alpha^{0\to\rho}$, we also write $\overline{\alpha}N = \langle \alpha(0), \alpha(1), \dots, \alpha(N-1)\rangle$ for *any* N^0. By way of shorthand, $(\forall q^\rho \in Q^{\rho^*})A(q)$ abbreviates $(\forall i^0 < |Q|)A(Q(i))$, which is (equivalent to) quantifier-free if A is. For sequences f^1, g^1, the sequence $f \oplus g$ is $f(0) * g(0) * f(1) * g(1) * \dots$.

A.3. **Some comprehension functionals.** In second-order RM, the logical hardness of a theorem is measured via what fragment of the comprehension axiom is needed for a proof. For this reason, we introduce some axioms and functionals related to *higher-order comprehension* in this section. We are mostly dealing with *conventional* comprehension here, i.e. only parameters over $\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ are allowed in formula classes like Π_k^1 and Σ_k^1.

First of all, the following functional is clearly discontinuous at $f = 11\dots$; in fact, $(\exists^2)$ is equivalent to the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ if $x >_\mathbb{R} 0$, and 0 otherwise ([55, §3]). This fact shall be repeated often.

$$(\exists\varphi^2 \leq_2 1)(\forall f^1)\big[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0\big]. \tag{$\exists^2$}$$

Related to $(\exists^2)$, the functional μ^2 in (μ^2) is also called *Feferman's* μ ([3]) and can be found in Hilbert-Bernays' *Grundlagen* ([40, Supplement V]).

$$\begin{aligned}(\exists\mu^2)(\forall f^1)\big[(\exists n)(f(n) = 0) \to [f(\mu(f)) = 0 \wedge (\forall i < \mu(f))(f(i) \neq 0)]& \\ \wedge [(\forall n)(f(n) \neq 0) \to \mu(f) = 0]\big],&\end{aligned} \tag{μ^2}$$

We have $(\exists^2) \leftrightarrow (\mu^2)$ over RCA_0^ω and $\mathsf{ACA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\exists^2)$ proves the same sentences as ACA_0 by [45, Theorem 2.5].

Secondly, the functional S^2 in (S^2) is called *the Suslin functional* ([55]).

$$(\exists\mathsf{S}^2 \leq_2 1)(\forall f^1)\big[(\exists g^1)(\forall n^0)(f(\overline{g}n) = 0) \leftrightarrow \mathsf{S}(f) = 0\big], \tag{S^2}$$

The system $\Pi_1^1\text{-}\mathsf{CA}_0^\omega \equiv \mathsf{RCA}_0^\omega + (\mathsf{S}^2)$ proves the same Π_3^1-sentences as $\Pi_1^1\text{-}\mathsf{CA}_0$ by [86, Theorem 2.2]. By definition, the Suslin functional S^2 can decide whether a Σ_1^1-formula as in the left-hand side of (S^2) is true or false. We similarly define the functional S_k^2 which decides the truth or falsity of Σ_k^1-formulas from L_2; we also define the system $\Pi_k^1\text{-}\mathsf{CA}_0^\omega$ as $\mathsf{RCA}_0^\omega + (\mathsf{S}_k^2)$, where (S_k^2) expresses that S_k^2 exists. We note that the operators ν_n from [16, p. 129] are essentially S_n^2 strengthened to return a witness (if existant) to the Σ_n^1-formula at hand. The operator ν_n is essentially Hilbert-Bernays' operator ν (see [40, Supplement V]) restricted to Σ_n^1-formulas.

Thirdly, full second-order arithmetic Z_2 is readily derived from $\cup_k\Pi_k^1\text{-}\mathsf{CA}_0^\omega$, or from:

$$(\exists E^3 \leq_3 1)(\forall Y^2)\big[(\exists f^1)(Y(f) = 0) \leftrightarrow E(Y) = 0\big], \tag{$\exists^3$}$$

and we therefore define $\mathsf{Z}_2^\Omega \equiv \mathsf{RCA}_0^\omega + (\exists^3)$ and $\mathsf{Z}_2^\omega \equiv \cup_k\Pi_k^1\text{-}\mathsf{CA}_0^\omega$, which are conservative over Z_2 by [45, Cor. 2.6]. Despite this close connection, Z_2^ω and Z_2^Ω can behave quite differently, as discussed in e.g. [70, §2.2]. The functional from $(\exists^3)$ is also called '$\exists^3$', and we use the same convention for other functionals. Hilbert-Bernays' operator ν (see [40, Supplement V]) is essentially Kleene's $\exists^2$, modulo a non-trivial fragment of the Axiom of (quantifier-free) Choice.

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