3

Bishop’s Mathematics: A Philosophical Perspective

Laura Crosilla

3.1 Introduction

The past 50 years have seen the flourishing of constructive approaches to mathematics and the growth of a variety of research groups working on constructive mathematics. This has given rise to a rich literature witnessing the depth and breadth of constructive mathematics, of which this volume is further proof. A principal drive for these developments has been the stimulus derived from the natural alliance between constructive mathematics and computer-aided computation, at first only an envisaged possibility and, more recently, a fact. The spark that started the present abundance of constructive mathematics was the publication in 1967 of Errett Bishop’s *Foundations of Constructive Analysis* [9]. Bishop’s book also prompted pivotal work in logic, with the emergence of new foundational frameworks, such as intuitionistic set and type theories, which, in turn, fostered fundamental work in computer-assisted theorem proving.[1]

Notwithstanding the extensive progress in constructive mathematics since 1967, there has been no corresponding advance in its philosophy. With very few exceptions, Bishop’s mathematics has at most received a quick mention, but no thorough consideration.[2] My overarching aim in this chapter is to enliven the philosophical debate about constructive mathematics Bishop-style.[3] There are a number of reasons to stimulate the philosophical reflection on constructive mathematics. First of

[1] See, for example, [14, 15, 22, 60]. See also [1, 3, 40, 43, 44, 59, 62, 63, 75] and [2, 25, 27, 65]. Note that Constable [64, p. 83] credits directly Bishop’s influence for the work that lead to the design of ‘a large computing system that would execute constructive proofs’.

[2] For example, Stewart Shapiro mentions Bishop very briefly in the well-known textbook [70] (e.g. at pages 184, in a footnote, 187, and 189) and in [71, 72]. The only exceptions I am aware of are [7, 45] and an exchange on the possibility of developing constructively the mathematical foundations of quantum physics [6, 8, 16, 47, 48, 49, 68, 69]. The lack of progress on the philosophy of Bishop’s mathematics is quite surprising, since there has been instead sustained analysis of other variants of constructivism inspired especially by the work of Brouwer, Gentzen, Dummett, Prawitz, and Martin-Löf.

[3] In the following, I write ‘constructive mathematics’ to denote the form of mathematics initiated by Bishop, and ‘intuitionistic mathematics’ to refer to other forms of mathematics that use intuitionistic logic.

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all, the significant advances of this form of mathematics in recent times and its natural bond with computation make constructive mathematics much more prominent within today's mathematical landscape than at its birth. An analysis of constructive mathematics is therefore important and pressing for a philosopher of mathematics who aims genuinely to engage with contemporary mathematics.[4] Second, Bishop's mathematics, as will be further argued, requires altogether different philosophical considerations compared with better-known approaches to intuitionistic mathematics, such as Brouwer's. Third, a sympathetic analysis of Bishop's philosophical remarks presents us with intriguing foundational ideas that deserve to be better understood and further developed.

The chapter is organised as follows. I begin by examining key elements of Bishop's philosophical remarks, especially focussing on Bishop's assessment of Brouwer, as it illuminates some of the most important aspects of Bishop's own foundational reflection. I then briefly sketch the most salient features of what I would like to call 'traditional' arguments for intuitionistic mathematics and consider whether these arguments could also support today's constructive mathematics. I argue that this is not the case, as traditional arguments are in tension with both Bishop's remarks and the constructive mathematical practice. This observation raises fundamental questions for the philosophy of constructive mathematics, and indicates that a thoroughly new approach is required. I conclude with the suggestion to look anew at Bishop's own remarks for inspiration.

3.2 Bishop on Brouwer

In this section, I review some prominent themes from Bishop's reflection on mathematics, especially focussing on the relation between Bishop and his main predecessor, Brouwer.[5] My aim is to understand Bishop's thought rather than defend his views. Bishop often combined severe criticism of Brouwer with recognition of his achievements. This complex relationship with the founder of intuitionism is aptly portrayed by Gabriel Stolzenberg in his review of [9], where Bishop's 'constructive framework' is presented as 'intimately related to Brouwer's intuitionism – though with important differences' [73].

An important and often emphasised difference between Bishop's and Brouwer's mathematics concerns the greater extent of the new form of mathematics. According to Bishop [9, p. ix], Brouwer and other constructivists were more successful in their criticism of classical mathematics than in replacing it with a better alternative. Soon after the publication of the fundamental [9], Bishop's work was celebrated for 'going substantially further mathematically' [41, p. 170]. For this reason, mathematicians

[4] A similar point is made in [33], in agreement with the spirit of the 'philosophy of the mathematical practice' (see, e.g., [24, 42, 58]).

[5] I especially draw from [9, 10, 11, 12].

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sympathetic to Bishop's project took it to refute the most prominent mathematical objection to Brouwerian intuitionism, famously emphasised by Hilbert and other critics of intuitionism, for which relinquishing the principle of excluded middle is tantamount to relinquishing the science of mathematics altogether.[6] Bishop himself writes that 'Hilbert's implied belief that there are a significant number of interesting theorems whose statements (standing alone) are constructive but whose proofs are not constructive (or cannot easily be made constructive) has not been justified. In fact we do not know of even one such theorem' [9, p. 354].

The greater extent of Bishop's mathematics is only one of the points of difference with Brouwer. Others relate to specific aspects of Brouwer's own approach, such as his treatment of the continuum and his philosophical ideas. It is plausible that when Bishop distanced himself from Brouwer and his followers, he aimed at separating aspects of the Brouwerian revolution he agreed with from others he found problematic. In so doing, he probably also hoped to attract to his new mathematics classical mathematicians who did care about constructivity but were sceptical of Brouwer's own approach. In the preface to his book, Bishop [9, p. ix] mentions previous attempts to constructivise mathematics, 'the most sustained' of which was made by L. E. J. Brouwer. He then notes that

> [t]he movement Brouwer founded has long been dead, killed partly by compromises of Brouwer's disciples with the viewpoint of idealism, partly by extraneous peculiarities of Brouwer's system which made it vague and even ridiculous to practising mathematicians, but chiefly by the failure of Brouwer and his followers to convince the mathematical public that abandonment of the idealistic viewpoint would not sterilize or cripple the development of mathematics. ([9], p. ix)

In this passage, Bishop criticises Brouwer not only for 'extraneous peculiarities' of his mathematics, but also for his perceived inability to communicate with and involve classical mathematicians.[7] In the following sections, I consider each point in turn.

3.3 Brouwer's Mathematics

In the 'Constructivist Manifesto' that opens [9], Bishop criticises Brouwer's mathematics especially for its use of free choice sequences in analysis. According to

[6] See [52, p. 476]. As noted by Myhill in [61], also constructivists such as Heyting thought that the "mutilation" of mathematics was an inevitable consequence of their standpoint [51, p. 74]. See [61, 73] for examples of favourable reception of Bishop's book. See Beeson's introduction to [13] for a more general discussion of the overall reception of Bishop's book.

[7] Note that in the quote above, Bishop claims that intuitionistic mathematics has long been dead. A similar image makes its way in Douglas Bridges's foreword to [34], where he writes that Bishop 'single-handedly showed that deep mathematics could be developed constructively, and thereby pulled the subject back from the edge of the grave'.

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Bishop, free choice sequences make the continuum 'not well enough defined' and the resulting mathematics 'so bizarre it becomes unpalatable to mathematicians' [9, p. 6]. As further mentioned in Section 3.4, a significant characteristic of Bishop's mathematics is its full *compatibility* with classical mathematics: every proof in Bishop's mathematics of a statement is also a classical proof of it.[8] In fact, one of the characteristics of Bishop's approach emphasised from the start is that it is not only compatible with classical mathematics, but its 'spirit' and 'execution' are 'much more like everyday modern mathematics than anything previously done in a systematic constructive way' [41, p. 171]. According to Bishop, Brouwer's treatment of the continuum, with the introduction of free choice sequences, marks instead a more drastic departure from the standard classical approach, and this, alone, makes Brouwer's mathematics less appealing, or even 'unpalatable' to mathematicians.[9]

Bishop also seems to think that Brouwer's free choice sequences represent an undesirable interference of 'metaphysical speculation' in mathematics, as they are dictated by Brouwer's philosophical view of the continuum rather than by the needs of the mathematical practice. Bishop further claims that Brouwer's mathematics (in general) shows 'a preoccupation with the philosophical aspects of constructivism at the expense of concrete mathematical activity'[10] [9, p. 6]. On the contrary, Bishop's book aimed to develop a large portion of abstract analysis within a constructive framework 'with an absolute minimum of philosophical prejudice concerning the nature of constructive mathematics' [9, p. ix].[11]

In subsequent texts, Bishop is more specific on what makes Brouwer's theory of the continuum problematic from his own point of view. For example, Bishop [10, p. 53] deplores the *lack of numerical interpretation* of free choice sequences: 'Brouwer's intuitionism at first glance contains elements that are extremely dubious; free choice sequences and allied concepts admit no ready numerical interpretation'. Similarly, in notes posthumously published as [12], Bishop writes that there seem to be at least two motivations for Brouwer's introduction of free choice sequences. First, 'it appears that Brouwer was troubled by a certain aura of the discrete clinging to the constructive real number system **R**'.[12] Second,

[8] See [21], especially Chapter 6.

[9] Feferman [41, p. 171] goes on writing: 'Indeed, a (philosophically unprepared) mathematician could pick up Bishop 1967 and read it as a straight piece of classical Cantorian mathematics. What would be puzzling to him is the more involved choice of certain notions and proofs, unless he also saw in what sense these were dictated by constructive requirements'. A similar point is made in Myhill's review [61, p. 744].

[10] Bishop [9, p. 6] writes: 'Brouwer became involved in metaphysical speculation by his desire to improve the theory of the continuum. A bugaboo of both Brouwer and the logicians has been compulsive speculation about the nature of the continuum. In the case of the logicians this leads to contortions in which various formal systems, all detached from reality, are interpreted within one another in the hope that the nature of the continuum will somehow emerge. In Brouwer's case there seems to have been a nagging suspicion that unless he personally intervened to prevent it the continuum would turn out to be discrete'.

[11] I discuss this point further in Section 3.6.

[12] See also the quote from [9, p. 6], in footnote [10].

> Brouwer had hopes of proving that every function from $\mathbf{R} \to \mathbf{R}$ is continuous, using arguments involving free choice sequences. [...] My objection to this is, that by introducing such a theorem as "all $f : \mathbf{R} \to \mathbf{R}$ are continuous" in the guise of axioms, we have lost contact with numerical meaning. Paradoxically this terrible price buys little or nothing of real mathematical value. The entire theory of free choice sequences seems to me to be made of very tenuous mathematical substance. [12, p. 26]

Summarising, Bishop seems to think that Brouwer's introduction of free choice sequences marks too drastic a departure from ordinary mathematics and is not sufficiently well motivated *mathematically* as it is not needed in practice. More importantly, free choice sequences cannot be directly explained in terms of *finitely performable computations with the integers*, therefore lacking clear *numerical meaning*. This is seen as a very substantial defect by Bishop, for whom the possibility of endowing mathematics with numerical meaning is a principal motive for developing his constructive mathematics.

One may wonder whether notwithstanding Bishop's disparaging remarks on free choice sequences, it would be beneficial for mathematics as a whole not to neglect forms of mathematics such as Brouwer's that countenance more abstract notions of construction.[13] As noted by Kreisel [56, pp. 146-147], since Bishop focusses on lawlike sequences, he does not offer a general explanation of why Brouwer's principles of continuity 'does not really affect mathematical practice'.[14] As further mentioned in Section 3.4, today's constructive mathematicians pledge to take a more encompassing approach compared with Bishop, as they hope to analyse from their general point of view a number of mathematical approaches, among which are the classical one, the Brouwerian one, and the recursion-theoretic approach of the Russian school of constructivism.[15]

Kreisel [56] further argues for the fruitfulness and the *naturalness* from a *mathematical* perspective of focusing on *abstract* (rather than more explicit) notions, and in particular on the most general notion of construction.[16] The interplay between

13 I would like to thank a referee for asking this question and for drawing my attention to the relevant passage of [56]. A related point is made by Veldman in [77, p. 61]: 'The principles proposed by Brouwer, even if one does not want to subscribe to the way he defends them, deserve to be discussed as possible starting points for our common mathematical discourse'.

14 It should be noted that Bishop [10, pp. 67–68] does consider the question of how one could accommodate Brouwer's theory of free choice sequences within a formal system. This, however, does not satisfy Kreisel who writes [56, pp. 146–147]: 'It is one thing to point out (correctly), as Bishop does, that Brouwer's assertion concerning the continuity of (extensional) functions does not really affect mathematical practice [...], if we simply take our functions as given *together* with a modulus of continuity. But it is a separate matter to explain this step by showing that any definition satisfying some abstract condition is bound to provide the additional information; in other words by analysing (when possible) the most general notion of construction, not merely definitions in some formal system such as Gödel's T'.

15 See [21] for a discussion of the most well-known varieties of constructive mathematics. Note that very recent work suggests that, contrary to what Bishop thought, there is the potential for practical results ensuing a Brouwerian approach to mathematics (see, e.g., [5]).

16 Note that the word 'abstract' is used here in its mathematical rather than philosophical sense. Kreisel [56, p

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more abstract and more explicit notions and their respective roles within different approaches to mathematics is a very significant point that ought to be central to the philosophy of mathematics, and especially to the philosophy of constructivism. Discussion of this point clearly exceeds the aims of this chapter. I wish, however, to point out the fact that Kreisel's comments clearly indicate that one of the most significant differences between a Brouwerian approach and Bishop's own, is the crucial foundational role the domain of the natural numbers has for Bishop. The role of the natural numbers for Bishop is the main focus of Section 3.6.

3.4 Persuasion and Dialogue

We saw Bishop's criticism of Brouwer's mathematics, but also of his inability to convince the mathematical public of the viability of his intuitionistic project. Bishop [9, p. 6] writes that Brouwer's programme 'failed to gain support' as Brouwer was

> an indifferent expositor and an inflexible advocate, contending against the great prestige of Hilbert and the undeniable fact that idealistic mathematics produced the most general results with the least effort.

Bishop was hopeful that constructive mathematics would eventually prevail over classical mathematics, to such an extent that in his preface [9, p. x], he writes that his ultimate goal is 'to hasten the inevitable day when constructive mathematics will be the accepted norm'. But he was aware that for this transformation to occur he needed to persuade his fellow mathematicians that the constructive program could succeed. This seems to motivate his criticism of Brouwer's 'inflexible' attitude towards classical mathematics.

Bishop was keen to reach out to the mathematical community, so much so that soon after the publication of [9], he embarked on a series of lectures on constructive mathematics across the USA. Although his lectures attracted large audiences, he had limited success in persuading classical mathematicians to join constructive mathematics.[17] In fact, Bishop thought that he had not been understood.

Notwithstanding his clear desire to communicate with classical mathematicians, some of Bishop's statements on classical mathematics may well have made the

122] claims that Bishop's 1967 book in some respects witnesses this attitude, too, as Bishop does not pin down his notion of algorithm to a specific notion like, for example, that of recursive function, working instead with a primitive notion of constructive function. The significance of Bishop's very abstract approach to the notion of computation is also emphasised, for example, in [17].

17 See [12, p. 1] where Bishop states that his general impression was that in those lectures he failed to communicate a real feeling for the philosophical issues involved. Nerode [64] recalls that after his tour of the eastern universities in the USA, Bishop told Nerode that he felt the trip might have been counterproductive, as the audiences did not take his work seriously. Bishop also thought that the difficulties experienced during the lecture tour contributed to the deterioration of his health, resulting in a heart attack [64, p. 80]. See also Beeson's foreword to [13].

communication problem with the classical mathematician worse.[18] For example, Bishop [9, p. ix] claims that his book is a 'piece of constructivist propaganda' and goes on to write:

> Our program is simple: to give numerical meaning to as much as possible of classical abstract analysis. Our motivation is the well-known scandal, exposed by Brouwer (and others) in great detail, that classical mathematics is deficient in numerical meaning.

More forceful are the remarks in [12], such as the claim that there is a 'philosophical deficit of major proportions' in contemporary mathematics, and that the latter manifests the 'debasement of meaning'.

A more conciliatory attitude characterises [11]. Here Bishop imagines an ideal dialogue between Hilbert and Brouwer in which the two mathematicians amicably discuss and compare their divergent foundational views.[19] Bishop [11, p. 510] claims that '[p]erhaps Brouwer should not have denounced the mathematics that Hilbert wished to do as meaningless'. In that text, Bishop strongly advocates a key role for constructive mathematics as *enhancing* or *deepening* the classical practice. The idea is that *within constructive mathematics* one can express distinctions in meaning which are not available to the classical mathematician, such as, for example, the distinction between statements that have a computational interpretation from those that lack one. Furthermore, constructive mathematics can be taken to be the basis over which one expresses and analyses a classically valid theorem T by means of implications of the form, for example, $\mathrm{LPO} \to \mathrm{T}^*$, with T^* a constructive substitute of the classical theorem T.[20] Incidentally, Bishop [11, p. 512] claims that implications such as $\mathrm{LPO} \to \mathrm{T}^*$ are 'ugly' and that we should try to obtain 'an implication which is natural and reflects the nature of the problem', namely one that is related to the structure of a particular theorem in some special way. If we do that, then working within a constructive context allows us to clearly single out any non-constructive assumption and identify and bring to the fore important aspects invisible from a classical perspective, especially the computational content of mathematics.

We see here the emergence of an idea that has been profitably refined in recent years giving rise to the *constructive reverse mathematics programme*. Constructive mathematics here is the ***core*** of a number of *varieties* of mathematics, among

[18] For example, [9] is portrayed as 'pure ideology' in [57] (p. 228) and the introduction to that book is termed 'embarrassing' (p. 239).

[19] Note that in this very text, Bishop [11, pp. 513–514] expresses, without argumentation, a very harsh opinion of non-standard analysis: 'It is difficult to believe that debasement of meaning could be carried so far'.

[20] Bishop calls 'Limited Principle of Omniscience' (LPO) the following statement: if $\{a_n\}$ is a binary sequence, then either there exists n such that $a_n = 1$, or else $a_n = 0$ for each n.

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which are **classical**, **Brouwerian**, and **Russian** constructive mathematics.[21] Indeed, each of the latter three forms of mathematics may be developed on the basis of some suitable extension of Bishop's mathematics by characteristic principles. For example, the principle of excluded middle and the axiom of choice can be added to Bishop's constructive mathematics, giving rise to a context for developing classical mathematics, while adding the principle of continuous choice to Bishop's mathematics, and the fan theorem allows one to develop a Brouwerian form of mathematics.[22] The constructive mathematician claims that due to its privileged position, constructive mathematics allows us to study from a 'neutral' perspective relations between mathematical notions belonging to these varieties, as well as comparing these varieties with each other.

3.5 Formalisation

As we saw on page 63, Bishop [9] also criticised Brouwer's disciples. His concern in that respect was especially Heyting's formalisation of intuitionistic logic and subsequent work in mathematical logic on intuitionistic formal systems. In this respect, Bishop's anti-formalist attitude appears particularly close to Brouwer's views. It is here that we find some of Bishop's strongest words of appreciation for Brouwer. Brouwer is often praised for his realisation of the defects of classical mathematics, especially its lack of numerical content, and his opposition to formalism. Bishop [9, p. 6] credits to Brouwer the 'disengagement of mathematics from logic':

> Brouwer fought the advance of formalism and undertook the disengagement of mathematics from logic. He wanted to strengthen mathematics by associating to every theorem and every proof a pragmatically meaningful interpretation.

As to the criticism of Brouwer's disciples, Bishop (ibid.) writes that Brouwer's precepts were

> formalized, giving rise to so-called intuitionistic number theory, and [...] the formal system so obtained turned out not to be of any constructive value. In fairness to Brouwer it should be said that he did not associate himself with these efforts to formalize reality [...].

Bishop's views on formalisation changed in some respects after the completion of *Foundations of Constructive Analysis*, as witnessed, for example, by [10]. Bishop did not give up criticising formalism and the dry study of formal systems as opposed to contentful mathematical practice (see especially [12]). However, by 1970 he seemed to have reached the conclusion that formalisation could be employed to

[21] See [21] for a comparison of these varieties of mathematics and see [35, 53, 54, 55, 76] for the constructive reverse mathematics programme.

[22] It should be noted that the constructive reverse mathematics programme is often developed informally, without fixing specific formal systems. This makes the above claims not completely precise and raises important questions. See the discussion in [35, p. 100], and especially footnote 1.

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the benefit of mathematics. For example, in [10], he employs Gödel's Dialectica interpretation to clarify the numerical content of mathematical statements. One of Bishop's main concerns in that text is the constructive interpretation of conditional statements. Bishop [10, p. 53] begins by giving a clear characterisation of constructive mathematics which makes more explicit ideas already hinted at in [9].[23] Constructive mathematics is here termed *predictive* since it

> describes or predicts the result of certain finitely performable, albeit hypothetical, computations within the set of integers.

This interpretation of *constructive* in terms of *finitely performable operations with the integers* is at the heart of Bishop's approach to constructive mathematics and his insistence on the *numerical content* of mathematical statements. After discussing some characteristic examples of mathematical problems, Bishop writes [10, p. 54]:

> The most urgent task of the constructivist is to give predictive embodiment to the ideas and techniques of classical mathematics. Classical mathematics is not totally divorced from reality. On the contrary, most of it has a strongly constructive cast.

A key step in the task of giving predictive embodiment to classical mathematics, is to endow conditional statements with suitable numerical meaning and it is here that Bishop employs Gödel's Dialectica interpretation. Bishop expresses his dissatisfaction with the standard Brouwer–Heyting–Kolmogorov (BHK) interpretation of implication as well as with a variant he proposed in [9]. He therefore suggests to use Gödel's Dialectica interpretation to offer a more satisfactory computational interpretation of conditional statements.[24] While Bishop's proposal in [10] deserves more careful analysis, I cannot go into more detail in the present context. The main point I wish to highlight is that [10] witnesses an apparent change of attitude, as formal systems are now taken to offer the means to tackle these urgent tasks and Bishop envisages the possibility of a fruitful cooperation between formalisation and mathematics. He writes [10, p. 60]:

> Another important foundational problem is to find a formal system that will efficiently express predictive mathematics. I think we should keep the formalism as primitive as possible, starting with a minimal system and enlarging it only if the enlargement serves a genuine mathematical need. In this way the formalism and the mathematics will hopefully interact to the advantage of both.

It is possible that the difficulties Bishop encountered in conveying his ideas to mainstream mathematicians and the more favourable reception of his mathematics

[23] See, for example, [9, p. viii].

[24] Bridges [18] reports a conversation with Bishop that suggests that Bishop's dissatisfaction with material implication was a major motive for his "conversion" to constructive mathematics.

among logicians had an impact on his apparent change in attitude.[25] It seems also likely that in the meantime Bishop had become more aware of the potential of applying constructive mathematics to computer programming. This was already prefigured in Appendix B of [9].[26] The remarkable point is that while in his 1967 book, formalisation was mainly seen as an artificial obstacle, distracting us from mathematics' genuine content, by 1970, Bishop appears more interested in formalisation, as long as it engages with questions of meaning. Even after 1967, formalization for the sake of formalization is strongly criticised. Now, however, Bishop thinks that when properly employed, formal systems *can* be a useful tool for clarifying issues of meaning and fostering possible applications to computers.

3.6 Philosophy

We have seen Bishop's concerns in [9] for Brouwer's introduction of free choice sequences and for his inflexible attitude. Furthermore, in his 1967 book, Bishop complained that Brouwer was preoccupied with philosophical aspects of constructivism at the expense of concrete mathematical activity and criticised his 'metaphysical speculation' over the nature of the continuum. This suggests a rather bleak view of philosophy and its relation with mathematics. Philosophy, however, has a more prominent and positive role in subsequent texts by Bishop. For example, in [11] Bishop clearly sees a role for philosophical thought in mathematics.[27] The article starts with the following very powerful statement.

> There is a crisis in contemporary mathematics and anybody who has not noticed it is being willfully blind. *The crisis is due to our neglect of philosophical issues*.[28]

Bishop [11, p. 507] complains that university courses in the foundations of mathematics focus on formal systems and their analysis 'at the expense of philosophical substance'. He writes that we need to change emphasis from proving theorems to knowing what they mean, 'from the mechanics of the assembly line which keeps grinding out the theorems, to an examination of what is being produced'. Philosophical reflection ought to contribute to this shift of focus and clarify the *meaning* of mathematical statements. Bishop writes that '[t]here is only one basic criterion to justify the philosophy of mathematics, and that is, does it contribute to making mathematics more meaningful'. [11, p. 508]

[25] See [64].
[26] There has been a recent discussion among constructive mathematicians on two unpublished manuscripts by Bishop, 'A general language' and 'How to compile mathematics into Algol'. These texts also witness Bishop's interest for formalisation as a tool for the application of constructive mathematics to computer programming. See, for example, http://www.cs.bham.ac.uk/∼mhe/Bishop/
[27] See also [10, p. 57], where Bishop claims that there must be a *philosophical* explanation of the empirical fact that intuitionistic implication admits a numerical interpretation.
[28] Italics in the original text.

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To explain why he takes issues of meaning as central to mathematics and philosophy, Bishop [11, p. 507] asks the question 'What do we mean by an integer?' He considers three possible answers:

(i) an integer that we can actually compute, for example, 3,
(ii) one that we can compute in principle only, for example, 9^{9^9},
(iii) one that is not computable by known techniques, even in principle, for example, the integer that is defined to be 1 if φ is true and 0 otherwise, where φ is an open problem such as the Riemann hypothesis.

A constructive approach to mathematics, so argues Bishop, is necessary if we want to bring to light important distinctions such as that between (i) and (ii) on the one side and (iii) on the other. Bishop [11, p. 507] adds:

> To my mind, it is a major defect of our profession that we refuse to distinguish, in a systematic way, between integers that are computable in principle and those that are not. We even refuse to do mathematics in such a way so as to permit one to make the distinction.[29]

Philosophy therefore can help clarify the computational meaning of mathematical statements and distinguish different statements depending on their meaning. In fact, Bishop's philosophy of mathematics rests on two crucial assumptions: the foundational role of the natural numbers within mathematics and the constructive interpretation of the logical constants. As to the *natural numbers*, Bishop [9, p. 2] asserts that 'the primary concern of mathematics is number, and this means the positive integers'. Bishop also mentions Kant, Kronecker, and echoes Brouwer, when he claims that

> the development of the theory of the positive integers from the primitive concept of the unit, the concept of adjoining a unit, and the process of mathematical induction carries complete conviction.

In later texts [10, 11, 12], it becomes even clearer that Bishop's insistence on the meaning (or lack of it) of mathematical statements is very much related to the availability (or not) of an interpretation of each statement in terms of some finitely performable operation with the natural numbers.[30] This fundamental role of the natural numbers within mathematics reminds us not only of Kronecker, but also of predicativism, especially as developed by Hermann Weyl [78, 79].[31]

With regard to the *constructive interpretation of the logical constants*, the philosophical import of this choice becomes apparent especially when Bishop considers

[29] One may wonder whether we should also pay attention to the distinction between (i) and (ii). Bishop [12, pp. 9–10] briefly discusses this question, noting the difficulties involved in demarcating (i) and (ii). See also [37, 45, 81].

[30] See Section 3.5.

[31] Bishop mentions Weyl, for example, in [9, p. 10].

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the interpretation of statements that quantify over infinite domains. Bishop often stresses the fact that we are *finite* beings, and claims that, for this reason, we should only be concerned with forms of mathematics that a finite being can carry out, at least *in principle*. Bishop's qualification 'in principle' is important, as it clarifies that his aim is not to ban infinite domains, rather to give prominence to the infinite domain of the natural numbers.

The thought that we should be concerned only with those forms of mathematics that a finite being can, in principle, carry out, brings Bishop to question the meaningfulness of *classical* quantification over infinite domains, which he implicitly assimilates to the doings of an *infinite mind*. Once more, this is already hinted at in [9, p. 2], where we read:

> We are not interested in properties of the positive integers that have no descriptive meaning for finite man. When a man proves a positive integer to exist, he should show how to find it. If God has mathematics of his own that needs to be done, let him do it himself.

These ideas are developed in more detail in [12], where Bishop explicitly frames the distinction between *classical* and *constructive* mathematics in terms of the opposition between agents with finite and infinite powers. For example, at page 12, he writes that while constructive mathematics describes mathematical operations that can be carried out by finite beings, 'classical mathematics concerns itself with operations that can be carried out by God'. Subsequently he considers the question of what powers should God (or a being with 'non-finite powers') have. A minimum requirement, according to Bishop, is a form of limited omniscience, that enables such an agent to search through a sequence of integers to determine whether they are all equal to 0 or not. In other terms, a minimum requirement is the principle LPO.[32]

To summarise, for Bishop, a classical interpretation of the truth of a universal statement whose quantifiers range over an infinite domain involves an *infinite search* through the domain to check each individual element. An aspect I find particularly fascinating is that this interpretation of classical quantification bears surprising similarities with how it is often framed by both predicativists and intuitionists. In this way the debate over classical versus constructive mathematics is brought back to the traditional theme of the opposition between finite and infinite domains, which was central to the thought of intuitionists and predicativists alike. For example, a predicativist would consider quantification (i.e., classical quantification) over an infinite domain justified only if some constraints are met (e.g., if a step-by-step specification of the domain is available). For an intuitionist, quantification over infinite domains has to be intuitionistic rather than classical.[33]

[32] See footnote 20 for the statement LPO.
[33] See, for example, [78, p. 23] and [39, p. 41] for a similar interpretation of classical quantification. See also

I will be returning to the role of Bishop's fundamental assumptions regarding the natural numbers and logic at the end of this chapter. Here I wish to get back to Bishop's views on philosophy. After Bishop's unfavourable comments on philosophy in [9], it is surprising to read Bishop's claims that mathematics is experiencing a crisis which is due to our *neglect of philosophical issues* and that philosophy can help clarify fundamental distinctions in meaning [11]. There is clearly a change in emphasis between the earlier and the later texts, and it is natural to ask if this signals also a deep change in Bishop's views on philosophy. I am inclined to think that there is no direct disagreement between [9] and subsequent texts. My impression is that Bishop may have thought he was focussing on different points. On the one hand, as already mentioned in Section 3.1, Bishop's most prominent criticism of Brouwer's philosophy is the charge of 'metaphysical speculation'. Though Bishop's remarks are not only sharp but also very brief, and therefore difficult to interpret, it is plausible that Bishop took certain philosophical questions, for example, whether there are mathematical entities and whether they are mind-dependent or not, as largely irrelevant to the mathematical practice, or 'superfluous'.[34] His criticism of Brouwer can therefore be explained by supposing that he thought Brouwer's mathematics was deeply bound up with Brouwer's views on these matters, while Bishop's own mathematics did not share these characteristics. On the other hand, Bishop's more positive comments on philosophy in [11] relate to its possible role in clarifying the *meaning* of mathematical statements, by distinguishing classical and constructive interpretations of the logical constants and highlighting the key foundational role of the natural numbers. It is possible that Bishop thought that his views on this matter did not require him to take a stance on the nature of mathematical entities, for example with regard to their existence and their mind-dependence (or independence). Issues of meaning, however, have for Bishop deep mathematical consequences, as they determine whether a piece of mathematics has computational content or not. To gain a computationally meaningful mathematics, Bishop thinks, we need to abandon non-constructive methods of proofs and reform mathematics constructively. These are the philosophical questions that deserve to be pursued and it is in pursuing them that philosophy can contribute to a fruitful development of mathematics.

3.7 Traditional Philosophical Arguments for Intuitionistic Logic

We have seen Bishop's thoughts on Brouwer, his criticism and, simultaneously, the appreciation for his predecessor's achievements. The philosophical literature presents us with a vast and important chapter in the philosophy of mathematics

[32] for a discussion of the key role of this interpretation of quantification within the predicativist literature. See [31] for a discussion of the relation between logic and infinite domains.

34 See especially [12, pp. 10–11].

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on arguments for intuitionistic logic and their critique. A central element of this debate is a number of arguments or argument schemas which are usually taken to be the most common defences of intuitionistic mathematics. Their key elements are inspired especially by the thought of Brouwer, Heyting, and Dummett.[35] Let us call them *traditional philosophical arguments for intuitionistic logic*. A natural question to ask is whether Bishop's views are compatible with traditional philosophical arguments for intuitionistic logic. In fact, I am interested in a more general question, as I would like to understand whether today's constructive mathematicians could employ traditional arguments for intuitionistic logic to support a shift from classical to intuitionistic mathematics, or if entirely different considerations are required. I take Bishop's views, as described above, as my starting point.

For our purposes, it is helpful to single out the most general characteristics of traditional philosophical arguments for intuitionistic logic. A typical feature of traditional arguments for intuitionistic logic is that they move from philosophical considerations and reach the conclusion that the general use of classical logic in mathematics is *illegitimate*. As a consequence, these arguments reject classical mathematics and propose its replacement with intuitionistic mathematics. The philosophical considerations may concern, for example, the nature of the mathematical entities, the nature of our mathematical activity, or important features of our mathematical language. Indeed, these traditional arguments are sometimes taken to entail not only that classical logic is illegitimate, but also that it is meaningless, incoherent, or even unintelligible.[36]

For example, one traditional 'Brouwerian' argument for intuitionistic logic starts from a view of mathematics as an essentially languageless activity of the mind and may also see mathematical entities as mental constructions.[37] This brings to the forefront the notion of *proof* of a mathematical statement (or construction), because to ascertain the truth of a mathematical statement, the mathematician needs to perform a certain mental construction by producing a proof of it. A purported proof of the principle of excluded middle is interpreted as a construction which either proves or reduces to absurdity any mathematical statement, the availability of which is highly implausible. Therefore the argument is seen to entail the rejection of the principle of excluded middle (and similar essentially classical laws). More precisely, the Brouwerian mathematician may accept the validity of the principle of excluded middle for finitary statements within a thoroughly finitary context, but objects to its assumption in infinitary contexts.[38]

[35] See, for example, [26].
[36] See, for example, [46].
[37] See, for example, [23, p. 141], [50, p. 53] and Dummett's discussion of traditional intuitionism in [36, 38].
[38] For example, Brouwer [23, p. 141] writes that 'every construction of a bounded finite character in a finite mathematical system can be attempted only in a finite number of ways, and each attempt can either be carried through to completion, or be continued until further progress is impossible. It follows that every assertion of

Another kind of traditional argument for intuitionistic logic is inspired by Dummett and proceeds from semantic considerations to a rejection of classical logic in favour of intuitionistic logic. A key element of this kind of argument is a view of language, and therefore meaning, as communicable and observable, with the related thought that use exhaustively determines meaning. This brings once more the focus on *proofs* as instruments of verification of mathematical statements. Classical logic is seen as embodying a verification-transcendent notion of truth, and for this reason rejected, while intuitionistic logic is seen as fully satisfying the requirement of meaning as communicable and observable.

These arguments' focus on proofs and constructions is clearly in agreement with the perspective of both Bishop and today's constructive mathematicians. However, in light of Bishop's criticism of Brouwer, it is important to see whether these traditional arguments would overall be acceptable to a constructive mathematician. In Sections 3.8 and 3.9, I consider three possible complaints that may be raised against traditional arguments for intuitionistic logic and argue that the third one highlights a conflict between these arguments and the very practice of constructive mathematics.

3.8 Philosophical Objections

A prominent reason for constructivism's lack of popularity among philosophers today is the fact that traditional arguments for intuitionistic logic are typically bound up with forms of *anti-realism* which are rather unpopular today. For example, the first kind of argument starts from a view of mathematics as free activity of the mind (and possibly of mathematical entities as mental constructions), and therefore is committed from the start to the mind-dependence of mathematical proofs (and possibly also of mathematical entities). The second kind of argument also gives rise to a form of anti-realism, as it focusses, once more, on proofs as instruments of verification and rejects a verification-transcendent notion of truth. Given that within today's philosophy of mathematics these forms of anti-realism are widely considered either unattractive or untenable, traditional arguments for intuitionistic logic do not enjoy widespread support among contemporary philosophers.

One may wonder whether the constructive mathematician would deem these arguments unfit for the same reason. Looking again at Bishop's criticism of Brouwer's philosophy, we see that it focusses especially on those parts of philosophy that Bishop seemed to consider 'superfluous' to the mathematical practice. Bishop, on the contrary, pledged to develop his mathematics with an absolute minimum of

possibility of a construction of a bounded finite character in a finite mathematical system can be judged. So, in this exceptional case, application of the principle of the excluded third is permissible'. Here 'judged' means 'either proved or reduced to absurdity'. Brouwer then goes on to use the example of 'fleeing properties' to argue that the principle of excluded middle is not permissible for 'infinite systems' such as the natural numbers.

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philosophical prejudice concerning the nature of constructive mathematics. Therefore, it is plausible that Bishop would have found traditional arguments for intuitionistic logic unpalatable in view of their alliance with anti-realism. I think that the same is probably true of many constructive mathematicians today who work in the tradition initiated by Bishop. It is, however, important to stress that Bishop and, plausibly, a constructive mathematician more generally, would object to traditional arguments for very different reasons compared with contemporary philosophers. While many contemporary philosophers find these arguments' anti-realism problematic, the constructive mathematician would not want to commit to a specific view on the nature of mathematics (mathematical entities, mathematical discourse) and for *that* reason would probably find the alliance with anti-realism unattractive.[39]

Philosophers of mathematics sometimes raise a different kind of objection against intuitionism: that it is a paradigmatic example of *philosophy-first*. For example, in Chapter 2 of [71], Shapiro discusses the relation between philosophy and mathematics and presents intuitionism, both in the Brouwerian and the Dummettian traditions, as paradigmatic examples of philosophy-first: 'the view that philosophical considerations should set the stage for and determine the proper practice of mathematics' [71, p. 21]. Traditional arguments for intuitionistic logic would seem to exemplify philosophy-first since they move from philosophical considerations, for example, specific thesis in the philosophy of mind or in the philosophy of language, and conclude with the rejection of classical mathematics. In the following, I review the key ideas of philosophy-first and argue that, although prima facie appealing, a rejection of traditional arguments for intuitionistic logic on the ground that they exemplify philosophy-first is problematic.

Shapiro claims that a philosophy-first approach to mathematics was once common, as exemplified, for example, by Plato's thought. Shapiro and other contemporary philosophers find philosophy-first approaches to mathematics problematic because purely philosophical considerations are taken to determine and fix the way mathematics is done. Many find this even more problematic when the philosophical conclusions, as in the case of intuitionism, impose a *revision* of standard mathematical practice. Shapiro [71, p. 30] writes: 'Many contemporary philosophers, including me, believe that scientists and mathematicians usually know what they are doing, and that what they are doing is worthwhile'. This has brought some philosophers to lean towards the opposite to philosophy-first, the thesis that philosophy is irrelevant to mathematics. Shapiro himself proposes a form of anti-revisionism, but does not go all the way to support what he calls a 'philosophy-last-if-at-all' approach. Furthermore, he objects to the exclusive use of philosophical considerations

[39] It is natural to ask whether the constructive mathematician's hope to maintain a neutral stance on crucial metaphysical issues can be sustained. While I cannot discuss this issue in this note, in my conclusion I suggest further work that could help clarify this point.

to restrict one's practice in general, thus even if, on their basis, one were to reject intuitionistic logic in favour of the more standard classical logic.

We have seen that Bishop, especially in [11, 12], did see a role for philosophical considerations in mathematics. He thought that disagreement over meaning has to be settled prior to disagreement over specific assumptions and techniques. He may therefore have found no fault with philosophy-first, as long as the philosophical considerations were prompted by issues of meaning, rather than based on what he considered 'speculation' regarding the nature of mathematics. What about today's constructive mathematicians? Would they find this objection to traditional arguments for intuitionistic logic compelling?

It is natural to expect that constructive mathematicians would be sympathetic to the thought that it should be the mathematician rather than the philosopher to decide which principles and techniques to employ in mathematics.[40] However, notwithstanding the appeal of this thought, I think that talk of philosophy-first may oversimplify the complex interaction between philosophy and mathematics. One may note, for example, that the debate on philosophy-first often artificially opposes mathematicians and philosophers, while historically many major mathematicians were also major philosophers (or philosophers of mathematics).[41] There is, of course, an obvious reply to this worry. One may observe that even if the same person, say Brouwer, engaged simultaneously in mathematical and philosophical inquiry, we may carefully distinguish between philosophical and mathematical components of his thought. For example, one may claim that Brouwer pursued philosophical rather than mathematical thoughts when he introduced his notion of free choice sequences. We have seen that Bishop probably thought of Brouwer along similar lines.

I find this attempt to rescue the philosophy-first objection to traditional arguments unconvincing, since more needs to be said on how to draw the line between mathematical and philosophical thinking. I tend to think that if we look more carefully at mathematicians' reasons for choosing a certain methodology or introducing some new concepts, it is likely that a number of different factors will have a role, some mathematical, some philosophical and some, furthermore, sociological in character. Typical discussions on mathematical methodology are a mixed bag, a blend of different issues that are difficult to categorise as exclusively mathematical or exclusively philosophical. This suggests that either the very notion of philosophy-first is hopelessly imprecise, or that one should offer a very careful formulation of it and, consequently, of this objection. One possible strategy would be to try and formulate

[40] This does not mean that they would also support a form of anti-revisionism which sanctions classical mathematics, as they would rather claim that there are good mathematical reasons to revise the standard classical practice.

[41] This point is acknowledged in [71, p. 31]. I would like to thank a referee for suggesting to further develop this point.

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the latter in terms of reasons that are *predominantly* philosophical or mathematical in character, rather than exclusively so. It is unclear to me if this could be done in a satisfactory way, but perhaps, if it can be done, it would suffice to express the worry that traditional arguments for intuitionistic logic are examples of philosophy-first.

Let us suppose, for the sake of argument, that we can meaningfully talk of philosophy-first and that traditional arguments for intuitionistic logic do constitute examples of it. Does this imply that the constructive mathematician should reject those arguments on this ground? I think the constructive mathematician should clearly say 'no'. Whatever the reasons for intuitionistic mathematics, the key question the mathematician will ask is whether the resulting mathematics is interesting and fruitful. Banning philosophy-first arguments a priori could then result in obstructing the development of fruitful and interesting mathematics, making mathematical progress more difficult. Brouwer's intuitionism is an emblematic example. One may claim that Brouwer's reasons for intuitionistic mathematics were predominantly philosophical in character, and find this unsatisfactory in some respect. However, it is clear that, this notwithstanding, the arguments Brouwer adduced for intuitionism had important and useful mathematical consequences, as they gave rise to the discovery of intuitionistic logic and opened up a whole new realm of mathematics. For these reasons, I think the constructive mathematician should not object to traditional arguments for intuitionistic logic on the sole basis that they are examples of philosophy-first.

On reflection, it seems that the philosophical discussion on philosophy-first highlights a different point. We have seen that those who object to traditional arguments for intuitionistic logic because they consider them paradigmatic examples of philosophy-first often express concerns for the revisionary spirit of these arguments. This suggests that they are concerned not only with the motives supporting the premises of these arguments, but also with these arguments' consequences, namely the fact that they demand a thorough change of the mathematical practice. In the next section, I focus, although from a different perspective, on crucial consequences of traditional arguments for intuitionistic logic and argue that they suggest the need for new arguments for constructive mathematics.

3.9 Too Strong

I believe that a better reason for objecting to traditional arguments for intuitionistic logic is that they are *too strong*. These arguments, as we have seen, entail the outright *rejection* of classical logic and, consequently, of classical mathematics. They are often taken to imply that classical mathematics is incoherent, and are sometimes also read as entailing the thorough unintelligibility of classical mathematics. For example, in his famous article 'The philosophical basis of intuitionist logic'

[36, p. 215], Dummett asks 'what plausible rationale can there be for repudiating, within mathematical reasoning, the canons of classical logic in favour of intuitionistic logic?'. Dummett clarifies that he is not concerned with 'justifications of intuitionistic mathematics from an eclectic point of view, that is, one which admits intuitionistic mathematics as a legitimate and interesting form of mathematics alongside classical mathematics'. Dummett's concern is rather the standpoint of the intuitionists themselves, who took classical mathematics to employ forms of reasoning which are *invalid on any legitimate way of construing mathematical statements*. Similarly to the view that Dummett examines in his article, also the traditional arguments for intuitionistic logic we considered above are usually taken to completely reject classical mathematics as illegitimate.

According to Bishop, classical mathematics is *deficient in meaning* so much so that he hopes that constructive mathematics will eventually replace it. Bishop does not, however, maintain that classical mathematics is outright illegitimate. The constructive mathematician may stress, like Bishop, that there are good, indeed, better reasons to work constructively, compared with working classically, and that classical mathematics as a whole cannot be given constructive meaning, as not every classical theorem can be given a computational interpretation. However, there are parts of classical mathematics that do have computational content, and we can make some sense of the rest, for example, in terms of conditional statements that prefix a suitable classical statement to a constructively meaningful one.[42] Furthermore, at least initially, classical mathematics is seen as a guide that helps the constructive mathematician develop new mathematics. For example, Bishop writes [9, p. x]:

> We are not contending that idealistic mathematics is worthless from the constructive point of view. This would be as silly as contending that unrigorous mathematics is worthless from the classical point of view. Every theorem proved with idealistic methods presents a challenge: to find a constructive version, and to give it a constructive proof.

For these reasons, I believe, Bishop should consider traditional arguments for intuitionistic logic not viable, as they imply the outright rejection of classical mathematics, if not its unintelligibility. I would think that many constructive mathematicians (Bishop-style) would also see things in essentially this way. One reason is that there is a tension between these arguments' conclusions and the contemporary constructive practice. First of all, claims of utter unintelligibility of classical mathematics are implausible given the above-mentioned use by the constructive mathematician of classical proofs as an initial guide. Second, constructive mathematicians believe that much of classical mathematics does not possess the same clear constructive meaning as a piece of constructive mathematics, and find classical

[42] See Section 3.4.

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mathematics unappealing for that reason. But they would certainly claim that they understand a classical theorem as clearly as any classical mathematician. In fact, as mentioned in Section 3.4, they would argue that they can offer a more precise analysis of a classical theorem, separating its constructive core from (possibly) an essentially classical component, such as, for example, LPO. More importantly, the constructive reverse mathematics programme, which was mentioned in Section 3.4, also requires a more tolerant approach to classical mathematics. One of its stated aims is to clarify the relation between concepts and theorems belonging to a number of mathematical practices, among which the classical one. A crucial claim of the constructive reverse mathematics programme is that constructive mathematics offers a 'neutral' perspective, on the basis of which to analyse classical mathematics.[43] Classical mathematics therefore is not to be rejected and, I would suggest, also not completely devoid of interest from the perspective of the constructive mathematician. Constructive mathematics is taken to be highly preferable, among other reasons for its computational content and because it offers an ideal ground from which to carry out a fine comparison between mathematical notions, theorems and proofs in different contexts. However, the constructive mathematician cannot on these sole grounds outlaw classical mathematics.

3.10 Concluding Remarks

If I am right to think that the constructive mathematician cannot accept traditional arguments for intuitionistic logic because their consequences are too strong, this raises a pressing question for the philosopher of mathematics: are there other arguments for constructive mathematics that can play a similar role as traditional arguments for intuitionistic mathematics? As a first step towards answering this question, we may consider what are the reasons for doing mathematics constructively. We have seen that Bishop's aim was to develop a 'meaningful' form of mathematics, one that 'predicts the results of certain finitely performable, albeit hypothetical, computations with the set of integers' [10, p. 53]. For Bishop, this meant that working constructively also deepens mathematics by making available important distinctions that a classical mathematician does not perceive. Furthermore, the constructive approach makes it possible to develop a computational form of mathematics which has systematic application to real-life computers. In fact, for today's constructive mathematician, the principal reason for working constructively is the *direct computational content* of constructive mathematics. The other reasons Bishop mentions are also important. Thirty-odd years ago, Fred Richman [66, 67] further developed some of Bishop's remarks, arguing that constructive mathematics has the advantage of being more *general* than classical mathematics. Since

[43] This is a strong claim. See, for example, [74] for criticism.

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constructive mathematics avoids the use of the excluded middle (and cognate principles such as LPO) but does not introduce principles that diverge from classical mathematics (contrary to Brouwerian mathematics), all of its theorems are also classically true. Therefore, constructive mathematics is more general than classical mathematics. Richman's notion of generality may be clarified by his comparison with algebra [66, p. 126]:

> Because intuitionistic mathematics is the weaker theory, its theorems have more models, so they are more general: for example, a theorem that holds for all groups is more general than one holding only for abelian groups.

Similarly to Bishop, Richman also stresses the importance of the possibility of distinguishing between mathematical concepts which are routinely identified in classical contexts.[44] I would like to call this feature of constructive mathematics *refinement*, as it allows for finer distinctions compared with classical mathematics. To summarise, the analysis of Bishop and Richman suggest that there are at least three main reasons for working constructively: (i) the possibility of giving direct *computational meaning* to mathematical statements, in particular one that can be readily applied to computers, (ii) the greater *generality* (in the sense above) of the resulting theorems, and (iii) *refinement*, namely the availability of significant distinctions that are unavailable within a classical context.

Can these reasons support a new argument for constructive mathematics? Billinge [7] seems to think so, but argues that it would be a mathematical rather than a philosophical argument. Billinge considers generality and refinement, and thinks of them primarily as *mathematical* rather than philosophical motives for constructive mathematics.[45] She also distinguishes between a *liberal* and a *radical* constructivist.[46] The first 'believes that constructive mathematics is preferable to classical mathematics, but that classical mathematics is at least coherent'. The second 'takes it that classical mathematics is absolutely illegitimate and cannot be rendered coherent under any interpretation' [7, p. 177]. She argues that Bishop did his mathematics in a constructive manner for explicit philosophical reasons and that he was a liberal rather than a radical constructivist. But she also argues that Bishop's philosophical comments cannot be fleshed out into an adequate philosophy of constructive mathematics. Billinge [7, p. 188] claims that the basic premises of Bishop's position are controversial, in particular so are Bishop's main assumptions – that all mathematical statements should have numerical content and that existence claims should be interpreted constructively. Her main complaint is that Bishop does not

44 See the example on page 76.

45 Surprisingly, in her concluding discussion in [7], Billinge does not mention the computational content as a key reason for doing mathematics constructively, although she discusses it in relation to the special status of the natural numbers within Bishop's philosophy.

46 See also [47, p. 222].

give good enough grounds for accepting these controversial assumptions, and that, as a consequence, Bishop's philosophical remarks cannot be taken to fully support liberal constructivism.

Billinge [7, p. 192] thinks that generality and refinement, as spelled out by Richman, are key *mathematical* advantages of working constructively and provide 'the most promising argument for liberal constructivism at the moment'. However, such an argument would not be a *philosophical* argument. In fact, Billinge [7, pp. 190–191] claims that she cannot see 'how one could give purely philosophical arguments for the superiority of constructive mathematics without overplaying one's hand and concluding that constructive mathematics is the only acceptable way of doing mathematics'. Succinctly: 'any adequate *philosophical* defence of constructive mathematics will justify *radical* constructivism'. [7, p. 192].

I agree with Billinge that overall Bishop's texts suggest a liberal rather than a radical constructivist position and also that Bishop does not offer a full philosophical defence of his claims. I think that this should not be surprising, as Bishop was a mathematician whose main focus was the concrete mathematical activity and whose philosophical views are briefly presented in remarks which appear primarily in introductions to technical work or in lecture notes. Furthermore, I also agree with Billinge that generality and refinement are important motives for constructive mathematics and that they are primarily motivated by mathematical needs, rather than explicit philosophical considerations. I am not persuaded, however, that it is utterly implausible that these reasons could play a key role in a philosophical argument for liberal constructivism. I cannot argue for this here due to space constraints. I will rather focus on a different point that is more relevant in the present context.

As already mentioned, for many contemporary constructive mathematicians the direct computational content of constructive mathematics is the primary motive for developing this form of mathematics. In view of Billinge's discussion, one may wonder whether this should count as primarily philosophical, mathematical or, perhaps, neither, for example, as an external practically motivated reason for doing mathematics constructively? I can think of at least two different ways of looking at the computational content of mathematical statements. First, there is the fact that if a mathematical statement can be given computational meaning then this can be employed in computer applications. In this sense, arguing for constructive mathematics on the basis of the availability of direct computational meaning seems to rely on pragmatic considerations, external to the mathematical practice. Second, the focus on the computational content may instead be determined by a preference for algorithmic proofs, independently from the possibility of applications.[47] Some

[47] See also [17, 20] for a discussion of the algorithmic nature of constructive mathematics. See [29] for a similar distinction.

mathematicians have a preference for proofs that are more algorithmic and explicit, proofs that construct their witnesses step-by-step. We have seen Bishop's focus on the natural numbers and on finite operations over the natural numbers. For Bishop the natural numbers have a key foundational role in mathematics, that he assimilates to the role they played for Kronecker. It is in this sense that the computational content can be taken as independent of practical considerations and not merely a mathematical but also as a philosophical reason for doing mathematics constructively.

Billinge takes the computational content (in this second sense) to be one of Bishop's main controversial assumptions, and argues that not only does Bishop inadequately support this assumption but, in fact, that it cannot be given adequate support in a way that coheres with Bishop's overall views. My impression is that Billinge reaches this conclusion because she takes Bishop to suggest an ontological reduction of mathematical entities to the natural numbers, that is, the thought that every mathematical entity can ultimately be reduced to some combination of natural numbers. She also suggests that for Bishop we have direct epistemic access to the natural numbers in a way that is analogous to our access to the physical world via sense perception. She then argues that this can only be supported if we take mathematical entities to be mental constructions, which would contradict Bishop's desire to remain neutral on metaphysical issues.

I do not think Bishop proposes such a reductive strategy, exactly because he argues against taking a specific stance on the nature of mathematical entities. This is also evidenced by the fact that Bishop frequently stresses the role in constructive mathematics of finitely performable *operations* with the natural numbers. This is suggestive that Bishop's focus is not the natural number themselves, but the possibility of dynamically developing mathematics via finite operations with the natural numbers, whatever the natural numbers may be. I take this to support the view that Bishop is not arguing, as Billinge [7, p. 188] claims, that every mathematical entity should be reduced to the natural numbers, that is, that the rational numbers, the real or the complex numbers are *really* just sets of natural numbers. I rather think that Bishop's remarks are better read in epistemological terms, leaving unanswered the question of the real nature of mathematical entities (including the natural numbers).

In fact, even if Bishop's own remarks are not to be read as I suggest, I believe that if we were to expand and build on his philosophical remarks more generally, the best strategy would be to focus on the epistemological claim that the natural numbers have a fundamental role in our understanding of mathematics in general. I take Bridges [19] to give a somewhat similar interpretation of contemporary constructivism, though without the commitment to the primality of the natural numbers that characterises Bishop's approach. Bridges [19] distinguishes between

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an ontological and an epistemological form of constructivism. He associates the first one with Brouwer and sees it as motivated by the belief that mathematical objects are mental creations. The second one focusses on methodological issues and takes them to motivate the shift to intuitionistic logic. Bridges takes today's constructivists of the Bishop school to be epistemological constructivists, rather than ontological constructivists.

I am tempted to think that there is the possibility of giving philosophical substance to a Bishop-inspired form of epistemological constructivism, i.e. one which focuses on the methodology of mathematics and reaches constructivism on the basis of a blend of mathematical and philosophical considerations. If understood in this way, Bishop's discussion of the distinction between finite and infinite domains gains a new prominence. A natural way to expand Bishop's remarks would be to look at the predicativist tradition, and especially Weyl's thought.[48] I am inclined to think that in this way Bishop's philosophical remarks may be enriched to give a new argument for constructive mathematics that takes the natural numbers as fundamental without rejecting *tout court* classical mathematics.[49]

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48 See [28, 30] for discussion and references.

49 This will be developed in future work. Note that there is a substantial difference between Bishop and Weyl, as Bishop's constructivism has a dynamic component, informed by the advent of the computer, that does nor figure in Weyl.

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