

# Simplified Approaches to Buckling of Composite Plates

by

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# Preface

This thesis has been written to fulfill the degree of Master of Science at the University of Oslo, Department of Mathematics, Mechanics Division.

I would like to thank Professor Brian Hayman at the University of Oslo and Det Norske Veritas (DNV), who has been my supervisor during this project. His knowledge and support is highly appreciated. Then I want to thank Dr. Scient. student Henrik Mathias Eiding at the University of Oslo for his patient assistance related to ANSYS programming.



# Abstract

Composite structures consisting of plates or plate-like elements are used widely in wind turbine blades and in certain types of ships, particularly naval ships. These structural elements are often subjected to significant forces. Buckling analyses are often conducted by FE analyses. But sometimes these analyses are really complex and make heavy demands on both computer resources and the analyst's expertise. There is a need for simplified but reliable analysis methods.

Both CLPT and FSDT have been applied to the estimation of elastic critical loads for plates. Thus, the method includes out-of-plane shear deformation.

Further, the method is developed for plates subjected to uniaxial compression load, both simply supported and clamped edges have been studied. The analysis method will also cover cases with in-plane biaxial compression, in-plane shear loading and combined loadings. These are confined to plates with simply supported edges. The case of a plate having an initial geometric imperfection will also be investigated and it is been tried to establish the onset of first ply failure.

To validate the methods, FE analyses is performed using ANSYS.

The methods based on FSDT give a better estimation than CLPT. It is best suited for thin and moderately thick plates. Higher order deformation theories should be considered for really thick plates. In addition, the methods are limited to linear cases.



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# 1 Introduction

## 1.1 Background and Motivation



Figure 1.1: Windturbines and the norwegian naval ship, Fridtjof Nansen.

Composite structures consisting of plates or plate-like elements are used widely in windturbine blades and in certain types of ships, particularly naval ships. These structural elements are often subjected to significant forces such as in-plane compressive or shear loading. So understanding and proper application of composite materials have helped to influence the lifetime and stability of these constructions. Thus in the design context buckling analysis plays a crucial role.

Buckling analysis or parametric studies are often conducted by FE analyses. But sometimes these FE analyses are quite complex and make heavy demands on both computer resources and the analyst's expertise. There is a need for simplified but reliable analysis methods that can readily be used for parametric studies.

## 1.2 Problem Definition

The general aim of the thesis will be the investigation of simplified approaches to the estimation of failure loads for composite plates under in-plane loadings. The specific objectives will be as follows:

- For a selected number of laminate lay-ups and plate aspect ratios, uniaxial critical loads having different thicknesses shall be estimated using the Matlab routines developed in the pre-project and also, in some cases, the ANSYS FE software. The analysis includes simply supported and clamped edges. Both Classical Laminated Plate Theory (CLPT) and First-order Shear Deformation Theory (FSDT) shall be applied. The results presented in a graphical (preferably non-dimensional) format and in the form of tables.
- To extend the analysis method to cover cases with in-plane biaxial compression and in-plane shear loading. Only simply supported edges using FSDT shall be applied. A limited parametric study shall be performed for each of these loading cases and the results presented graphically and in form of tables.
- The method shall be extended to analyse cases with combined compression and shear loadings. This will be confined to shear combined with uniaxial compression loading and to the case of simply supported plates. A limited parametric study shall be performed and the results shall be presented in one or more interaction diagrams.
- To extend the approach to analyse the plates with initial out-of-flatness. For uniaxial compression loading, the case of a plate having an initial geometric imperfection in the form of the first buckling mode shall be analysed such that the deflections and stresses can be estimated for increasing values of applied loading. This will be confined to small-deflection (linearised) buckling theory and plates with simply supported edges, but will include out-of-plane shear deformation (FSDT). Corresponding FE analyses shall be performed using ANSYS for some selected cases to validate the method.
- Using the imperfection analysis, a suitable material failure criterion shall be applied to find the value of applied load at which material failure first occurs (first ply failure). A limited parametric study shall be performed for square, simply supported plates with a selected lay-up type and the results compared with those from the ongoing studies in the MARSTRUCT Network of Excellence.

### 1.3 Contents of The Thesis

Section 2 gives a presentation of theories related to buckling analysis of composite plates. Both classical laminated plate theory (CLPT) and first-order shear deformation laminated plate theory would be briefly reviewed.

Section 3 deals with the analysis of specially orthotropic laminates using the CLPT. Both simply supported and clamped plates subjected to uniaxial compressive load will be investigated.

Analysis of specially orthotropic laminates based on the FSDT is devoted to section 4. Here analytical solutions are developed for simply supported and clamped plates with uniaxial compression. The analysis method will also cover the cases with in-plane biaxial compression, in-plane shear loading and combined loadings related to simply supported plates. Further, the investigation is concerned about plates with an initial geometric imperfection.

Section 5 deals with finite element analysis to validate the present method.

Section 6 contains the results from the analysis. The critical buckling loads are estimated using Matlab. Corresponding FE analyses have been performed using ANSYS. The results will be presented both graphically and in tables.

Section 7 contains conclusion and suggestions for further work.

Material properties for a selected number of laminate lay-ups are listed in Appendix A, while Appendix B gives a presentation of some useful expressions and deduction of buckling equations. A part of Matlab, Maple and ANSYS codes are listed in Appendix C.

## 2 Classical and First-Order Theories of Composite Plates

### 2.1 Introduction

According to Reddy [1], analysis of composite plates is based on the following approaches:

1. Equivalent single-layer theories (2-D)
  - (a) Classical laminated plate theory
  - (b) Shear deformation laminated plate theories
2. Three-dimensional elasticity theory (3-D)
  - (a) Traditional 3-D elasticity formulations
  - (b) Layerwise theories

This section gives a presentation, and explains the main differences between classical laminated plate theory (CLPT) and first-order shear deformation laminated plate theory (FSDT). The other theories have not been investigated further and are not a part of this project.

The simplest theory is the CLPT and requires that the Kirchhoff hypothesis holds, which assumes that plane cross sections remain plane and normal to the middle-plane during deformations. This implies that the transverse shear strains vanish. The FSDT is a bit more complicated and is build on the Reissner-Mindlin hypothesis, where plane cross sections remain plane after deformation, but not necessarily normal to the refrence plane. This results inclusion of out-of-plane shear deformation.

### 2.2 The Classical Laminated Plate Theory, CLPT

#### 2.2.1 Kinematics

The in-plane displacements are related to the normal displacements as follows [2]:

$$u(x, y, z) = u_0 - z \frac{\partial w}{\partial x}, \quad v(x, y, z) = v_0 - z \frac{\partial w}{\partial y}, \quad w(x, y, z) = w_0(x, y)$$

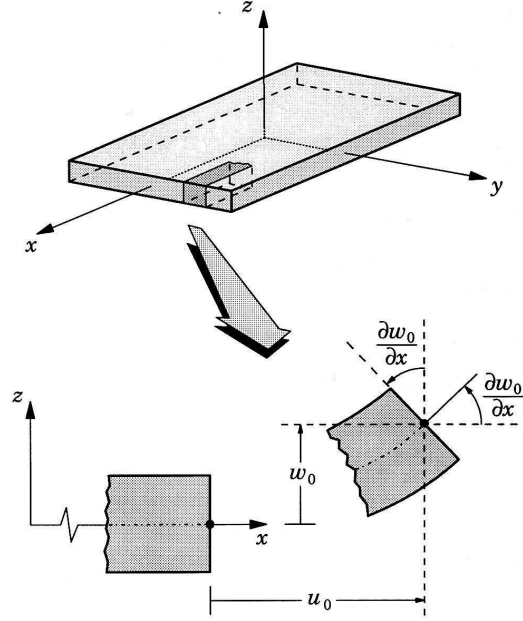


Figure 2.1: Undeformed and deformed geometries of an edge of a plate under the Kirchhoff assumptions [1].

So  $u$  is displacement in  $x$  direction,  $v$  is displacement in  $y$  direction and  $w$  is displacement in  $z$  direction, while  $u_0$ ,  $v_0$  and  $w_0$  are displacements of the midplane in  $x$ ,  $y$  and  $z$  directions, respectively. Based on the displacement field above, we can find the strains as follows:

$$\begin{aligned}
 \varepsilon &= \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \\
 &= \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \\
 &= \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \tag{2.1}
 \end{aligned}$$

### 2.2.2 The Material Law

Definition of tensor strains [2]:

$$\begin{Bmatrix} \varepsilon_L \\ \varepsilon_T \\ \frac{1}{2}\gamma_{LT} \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

Using this definition, the stress strain relations are given by [2]:

$$\begin{aligned} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} &= [T]^{-1} \begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = [T]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_L \\ \varepsilon_T \\ \frac{1}{2}\gamma_{LT} \end{Bmatrix} \\ &= [T]^{-1} [Q^*] [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix} = [\bar{Q}^*] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix} = [\bar{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \\ &= \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \end{aligned}$$

Inserting of equation (2.1) gives:

$$\begin{aligned} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} &= \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \\ &+ z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \end{aligned} \quad (2.2)$$

### 2.2.3 Resultant Forces and Moments

The stresses in a laminate vary layer to layer. Hence it is convenient to deal with a simpler but equivalent system of forces and moments acting on a laminate cross section. Resultant force is obtained by integrating the corresponding stress through the laminate thickness  $h$  [2]:



$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = \sum_{i=1}^n \int_{h_{i-1}}^{h_i} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_i dz \quad (2.3)$$

Similarly, the resultant moment is obtained by integration through the thickness of the corresponding stress times the moment arm with respect to the midplane [2]:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z dz = \sum_{i=1}^n \int_{h_{i-1}}^{h_i} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_i z dz \quad (2.4)$$

Substitution of equation (2.2) in equations (2.3) and (2.4) gives:

$$\begin{aligned} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} &= \sum_{i=1}^n \int_{h_{i-1}}^{h_i} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_i dz \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \\ &+ \sum_{i=1}^n \int_{h_{i-1}}^{h_i} z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_i dz \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \\ &+ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \end{aligned} \quad (2.5)$$

$$\begin{aligned} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= \sum_{i=1}^n \int_{h_{i-1}}^{h_i} z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_i dz \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \\ &+ \sum_{i=1}^n \int_{h_{i-1}}^{h_i} z^2 \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_i dz \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \\ &= \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \\ &+ \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w_0}{\partial x^2} \\ -\frac{\partial^2 w_0}{\partial y^2} \\ -2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} \end{aligned} \quad (2.6)$$

## 2.2.4 Equilibrium Equations in Terms of Displacements

From Appendix B.2, the equations (B.1), (B.2) and (B.7) are the equilibrium equations for a laminated thin plate:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p^* = 0$$

where  $p^* = p + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} - \rho^* \frac{\partial^2 w}{\partial t^2}$ .

The last equation solves a buckling problem. Insert equation (2.6), we obtain:

$$\begin{aligned} & -D_{11} \frac{\partial^4 w}{\partial x^4} - 4D_{16} \frac{\partial^4 w}{\partial x^3 \partial y} - (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} - 4D_{26} \frac{\partial^4 w}{\partial x \partial y^3} - D_{22} \frac{\partial^4 w}{\partial y^4} \\ & + B_{11} \frac{\partial^3 u_0}{\partial x^3} + 3B_{16} \frac{\partial^3 u_0}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \frac{\partial^3 u_0}{\partial x \partial y^2} + B_{26} \frac{\partial^3 u_0}{\partial y^3} \\ & + B_{16} \frac{\partial^3 v_0}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 v_0}{\partial x^2 \partial y} + 3B_{26} \frac{\partial^3 v_0}{\partial x \partial y^2} + B_{22} \frac{\partial^3 v_0}{\partial y^3} \\ & + p^* = 0 \end{aligned} \tag{2.7}$$

For *specially orthotropic laminates*, their constitutive equations satisfy the following conditions [2]:

$$A_{16} = A_{26} = 0$$

$$B_{ij} = 0$$

$$D_{16} = D_{26} = 0$$

Incorporation of conditions above into equation (2.7) simplifies the equilibrium equation for specially orthotropic laminates as follow:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = p^* \tag{2.8}$$

## 2.3 The First-Order Shear Deformation Theory, FSDT

### 2.3.1 Kinematics

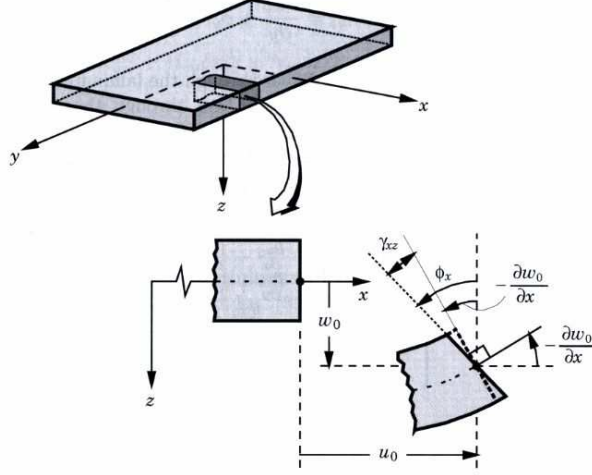


Figure 2.2: Undeformed and deformed geometries of an edge of a plate under the assumptions of the first-order plate theory [3].

The displacement field for the FSDT based on the assumption from chapter 2.1 and the figure (2.1) can be expressed as [3]:

$$u(x, y, z) = u_0 + z\phi_x, \quad v(x, y, z) = v_0 + z\phi_y, \quad w(x, y, z) = w_0(x, y)$$

where:

$$\frac{\partial u}{\partial z} = \phi_x, \quad \frac{\partial v}{\partial z} = \phi_y$$

which indicate that  $\phi_x$  and  $\phi_y$  are the rotations of a transverse normal about the  $y$  and  $x$  axes, respectively.

It is convenient to split the strain vector into two parts, where  $\varepsilon_b$  is the bending part and  $\varepsilon_s$  is the shear part [4]:

$$\varepsilon_{\mathbf{b}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \quad (2.9)$$

$$\varepsilon_s = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial v}{\partial z} + \frac{\partial w_0}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w_0}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix} \quad (2.10)$$

### 2.3.2 The Material Law

Definition of tensor strains [2]:

$$\begin{Bmatrix} \varepsilon_L \\ \varepsilon_T \\ \frac{1}{2}\gamma_{LT} \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix}$$

The relations between stresses and strains are from the relation for linearized elasticity. For FSDT it is convenient to split it into two parts, bending and shear. Similar to section 2.2.2, by using the tensor strains, the bending part can be expressed as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} + z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \quad (2.11)$$

Then the shear part [2]:

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = k \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = k \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix} \quad (2.12)$$

where  $k$  is shear correction coefficient.

### 2.3.3 Resultant Forces and Moments

The resultant force and resultant moment are obtained in the same way as the CLPT:

$$\begin{aligned}
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \\
&+ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \quad (2.13)
\end{aligned}$$

$$\begin{aligned}
\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \\
&+ \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \quad (2.14)
\end{aligned}$$

Equations relating the shear-force resultants  $R_{xz}$  and  $R_{yz}$  to the shear strains  $\gamma_{xz}$  and  $\gamma_{yz}$  can be written as [2]:

$$\begin{aligned}
\begin{Bmatrix} R_{yz} \\ R_{xz} \end{Bmatrix} &= k \sum_{i=1}^n \int_{h_{i-1}}^{h_i} \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}_i dz \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz} \end{Bmatrix} \\
&= k \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \frac{\partial w_0}{\partial y} \\ \phi_x + \frac{\partial w_0}{\partial x} \end{Bmatrix} \quad (2.15)
\end{aligned}$$

### 2.3.4 Equilibrium Equations in Terms of Displacements

To solve a buckling problem, we need equations (B.4), (B.5) and (B.6) from Appendix B.2:

$$\begin{aligned}
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - R_{xz} &= 0 \\
\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - R_{yz} &= 0 \\
\frac{\partial R_{xz}}{\partial x} + \frac{\partial R_{yz}}{\partial y} + p^* &= 0
\end{aligned}$$

where  $p^* = p + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} - \rho^* \frac{\partial^2 w}{\partial t^2}$

Constitutive equations for a specially orthotropic plate with the new displacement field still satisfy the conditions stated in chapter 2.2.4:  $A_{16} = A_{26} = 0$ ,  $B_{ij} = 0$  and  $D_{16} = D_{26} = 0$ . In addition,  $A_{45} = A_{54} = 0$ . In view of these conditions, equilibrium equations above can be written in terms of the displacement field as follows:

$$D_{11} \frac{\partial^2 \phi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_y}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_x}{\partial y^2} - A_{55} k \left( \phi_x + \frac{\partial w}{\partial x} \right) = 0 \quad (2.16)$$

$$D_{22} \frac{\partial^2 \phi_y}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_y}{\partial x^2} - A_{44} k \left( \phi_y + \frac{\partial w}{\partial y} \right) = 0 \quad (2.17)$$

$$A_{55} k \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{44} k \left( \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + p^* = 0 \quad (2.18)$$

Equations (2.16)-(2.18) are three coupled second-order differential equations with  $w$ ,  $\phi_x$  and  $\phi_y$  as the three unknowns.

### 3 Analysis of Specially Orthotropic Plates Using CLPT

#### 3.1 Buckling of Simply Supported Plates under Uniaxial Compressive Load

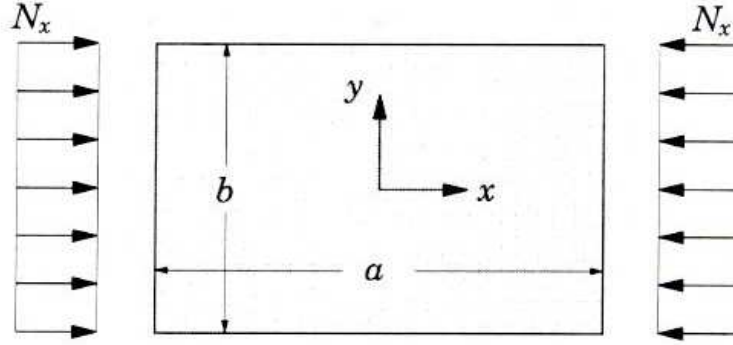


Figure 3.1: Plate with uniaxial compression load [1].

For the buckling analysis, we assume that the only applied load is the in-plane force in  $x$  direction. All other loads are zero. From equation (2.8) we put  $p^* = N_x \frac{\partial^2 w}{\partial x^2} = -N \frac{\partial^2 w}{\partial x^2}$ . Now the equation that solves the buckling problem is given by:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + N \frac{\partial^2 w}{\partial x^2} = 0 \quad (3.1)$$

The plate edges are simply supported so that the transverse displacements at the edges and resultant moments about each edge are zero. These edge conditions are the boundary conditions, and mathematically expressed as follows [2]:

$$\begin{aligned} x = 0 : \quad & w(0, y) = 0 & M_x(0, y) = 0 \\ x = a : \quad & w(a, y) = 0 & M_x(a, y) = 0 \\ y = 0 : \quad & w(x, 0) = 0 & M_y(x, 0) = 0 \\ y = b : \quad & w(x, b) = 0 & M_y(x, b) = 0 \end{aligned}$$

A Navier solution of equation (3.1) that also satisfies the preceding boundary conditions is given by [2]:

$$w(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

where  $w_{mn}$  are the displacement coefficients,  $m$  and  $n$  are positive integers.

We now assume that

$$w(x, y) = w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (3.2)$$

Substituting equation (3.2) in equation (3.1) gives:

$$N = D_{11} \left(\frac{m\pi}{a}\right)^2 + (2D_{12} + 4D_{66}) \left(\frac{n\pi}{b}\right)^2 + D_{22} \left(\frac{a\pi}{m}\right)^2 \left(\frac{n}{b}\right)^4 \quad (3.3)$$

Thus, for each choice of  $m$  and  $n$  there corresponds a unique value of  $N$ . The *critical buckling load* is the smallest of  $N$ , which can be obtained by  $n = 1$  and  $m$  varying.

### 3.2 Buckling of Clamped Plates under Uniaxial Compressive Load

Still, we assume that the only applied load is the in-plane force in  $x$  direction. All other loads are zero. For plates with all edges clamped we have chosen Rayleigh-Ritz method to solve the buckling problem. The method is based on the plate's potential energy. We now split the total potential energy in two parts, bending and external forces [5]:

$$\Pi = U_b + U_p \quad (3.4)$$

where

$$\begin{aligned} U_b &= \frac{1}{2} \int_V \varepsilon^T \sigma \, dV = \frac{1}{2} \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon^T \bar{\mathbf{Q}} \varepsilon \, dz \, dA = \frac{1}{2} \int_A \kappa^T \mathbf{D} \kappa \, dA \\ &= \frac{1}{2} \int_0^b \int_0^a D_{11} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12} \left(\frac{\partial^2 w}{\partial x^2}\right) \left(\frac{\partial^2 w}{\partial y^2}\right) \\ &\quad + D_{22} \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \, dx \, dy \end{aligned} \quad (3.5)$$



$$U_p = \frac{1}{2} \int_0^b \int_0^a -N \left( \frac{\partial w}{\partial x} \right)^2 dx dy \quad (3.6)$$

The boundary conditions associated with the clamped edges are [1]:

$$\begin{aligned} x = 0 : \quad w(0, y) = 0 \quad \frac{\partial w(0, y)}{\partial x} = 0 \\ x = a : \quad w(a, y) = 0 \quad \frac{\partial w(a, y)}{\partial x} = 0 \\ y = 0 : \quad w(x, 0) = 0 \quad \frac{\partial w(x, 0)}{\partial y} = 0 \\ y = b : \quad w(x, b) = 0 \quad \frac{\partial w(x, b)}{\partial y} = 0 \end{aligned}$$

A solution that satisfies the preceding boundary conditions is given by [6]:

$$w(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} w_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right)$$

where  $w_{mn}$  are the displacement coefficients,  $m$  and  $n$  are positive integers.

The equation above with only one term,  $m$  and  $n$  varying, is usually enough to solve the buckling problem. So we assume that:

$$w(x, y) = w_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) \quad (3.7)$$

Substitution of equation (3.7) in equation (3.4) gives:

$$\Pi = \begin{cases} \frac{1}{2} \pi^4 w_{mn}^2 \left[ D_{11} \frac{3}{4} \frac{b}{a^3} m^4 + D_{22} \frac{3}{4} \frac{a}{b^3} + \left( \frac{1}{2} D_{12} + D_{66} \right) \frac{m^2}{ab} \right] \\ - \frac{3}{4} \frac{w_{mn}^2 \pi^2 b m^2}{a^3} N, & m = n \\ \frac{1}{4} \pi^4 w_{mn}^2 \left[ D_{11} \frac{3}{16} \frac{b}{a^3} (n^4 + 6m^2 n^2 + m^4) + D_{22} \frac{a}{b^3} \right. \\ \left. + \left( \frac{1}{2} D_{12} + D_{66} \right) \frac{n^2 + m^2}{ab} \right] - \frac{3}{64} \frac{w_{mn}^2 \pi^2 b}{a} (n^2 + m^2) N, & m \neq n \end{cases} \quad (3.8)$$

Equilibrium requires that  $\delta\Pi = 0$ , thus

$$\frac{\partial \Pi}{\partial w_{mn}} \delta w_{mn} = 0 \quad \Leftrightarrow \quad \frac{\partial \Pi}{\partial w_{mn}} = 0$$

$$\frac{\partial \Pi}{\partial w_{mn}} = \begin{cases} \pi^2 w_{mn} \left[ D_{11} \pi^2 \frac{3}{4} \frac{b}{a^3} m^4 + D_{22} \pi^2 \frac{3}{4} \frac{a}{b^3} + (\frac{1}{2} D_{12} + D_{66}) \pi^2 \frac{m^2}{ab} \right. \\ \left. - \frac{3}{16} \frac{bm^2}{a} N \right] = 0, & m = n \\ \frac{1}{2} \pi^2 w_{mn} \left[ D_{11} \frac{3}{16} \frac{b}{a^3} \pi^2 (n^4 + 6m^2 n^2 + m^4) + D_{22} \frac{a}{b^3} \pi^2 \right. \\ \left. + (\frac{1}{2} D_{12} + D_{66}) \frac{\pi^2}{ab} (n^2 + m^2) - \frac{3}{16} \frac{b}{a} (n^2 + m^2) N \right] = 0, & m \neq n \end{cases} \quad (3.9)$$

Solving equation (3.9) for  $N$ , we obtain:

$$N = \begin{cases} \frac{4\pi^2 D_{11} m^2}{a^2} + \frac{4\pi^2 D_{22} a^2}{b^4 m^2} + \frac{16\pi^2}{3b^2} (\frac{1}{2} D_{12} + D_{66}), & m = n \\ \frac{D_{11} \frac{\pi^2}{a^2} (n^4 + 6n^2 m^2 + m^4) + D_{22} \pi^2 \frac{16}{3} \frac{a^2}{b^4} + (\frac{1}{2} D_{12} + D_{66}) \frac{16}{3} \frac{\pi^2}{b^2} (n^2 + m^2)}{n^2 + m^2}, & m \neq n \end{cases} \quad (3.10)$$

Thus, combination of  $m$  and  $n$  that gives the smallest value of  $N$  is the *critical buckling load* for a clamped plate.

## 4 Analysis of Specially Orthotropic Plates Using FSDT

### 4.1 Buckling of Simply Supported Plates under Uniaxial Compressive Load

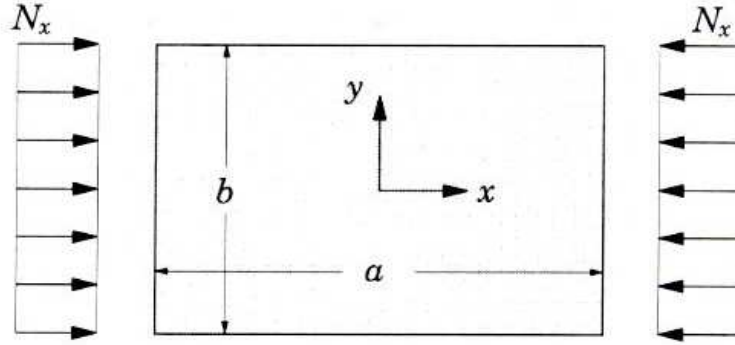


Figure 4.1: Plate with uniaxial compression load [1].

Since the only applied load is the force in  $x$  direction, from equation (2.18),  $p^* = N_x \frac{\partial^2 w}{\partial x^2} = -N \frac{\partial^2 w}{\partial x^2}$ . Based on equations (2.16)-(2.18), the equation set that solves the buckling problem is given by:

$$\begin{aligned}
 D_{11} \frac{\partial^2 \phi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_y}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_x}{\partial y^2} - A_{55} k \left( \phi_x + \frac{\partial w}{\partial x} \right) &= 0 \\
 D_{22} \frac{\partial^2 \phi_y}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_y}{\partial x^2} - A_{44} k \left( \phi_y + \frac{\partial w}{\partial y} \right) &= 0 \quad (4.1) \\
 A_{55} k \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{44} k \left( \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) - N \frac{\partial^2 w}{\partial x^2} &= 0
 \end{aligned}$$

Boundary conditions for this plate are the same as those for CLPT:

$$\begin{aligned}
 x = 0 : \quad w(0, y) = 0 \quad M_x(0, y) = 0 \\
 x = a : \quad w(a, y) = 0 \quad M_x(a, y) = 0 \\
 y = 0 : \quad w(x, 0) = 0 \quad M_y(x, 0) = 0 \\
 y = b : \quad w(x, b) = 0 \quad M_y(x, b) = 0
 \end{aligned}$$

The following double Fourier series are assumed to represent  $w$ ,  $\phi_x$  and  $\phi_y$  [2]:

$$w(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (4.2)$$

$$\phi_x(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (4.3)$$

$$\phi_y(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} y_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \quad (4.4)$$

where  $w_{mn}$ ,  $x_{mn}$  and  $y_{mn}$  are the series coefficients,  $m$  and  $n$  are positive integers.

For simply supported plates, it is enough to consider one term with  $m$  and  $n$  varying from each equation. Substitution of equations (4.2)-(4.4) into equation set (4.1) gives the following matrix equation:

$$\begin{bmatrix} -D_{11}\alpha^2 - D_{66}\beta^2 - A_{55}k & -D_{12}\alpha\beta - D_{66}\alpha\beta & -A_{55}k\alpha \\ -D_{12}\alpha\beta - D_{66}\alpha\beta & -D_{22}\beta^2 - D_{66}\alpha^2 - A_{44}k & -A_{44}k\beta \\ -A_{55}k\alpha & -A_{44}k\beta & N\alpha^2 - A_{55}k\alpha^2 - A_{44}k\beta^2 \end{bmatrix} \cdot \begin{Bmatrix} x_{mn} \\ y_{mn} \\ w_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.5)$$

where  $\alpha = \frac{m\pi}{a}$  og  $\beta = \frac{n\pi}{b}$ .

By defining

$$C_1 = -D_{11}\alpha^2 - D_{66}\beta^2 - A_{55}k$$

$$C_2 = -D_{12}\alpha\beta - D_{66}\alpha\beta$$

$$C_3 = -A_{55}k\alpha$$

$$C_4 = -D_{22}\beta^2 - D_{66}\alpha^2 - A_{44}k$$

$$C_5 = -A_{44}k\beta$$

simplify the equation (4.5):

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & N\alpha^2 + \alpha C_3 + \beta C_5 \end{bmatrix} \begin{Bmatrix} x_{mn} \\ y_{mn} \\ w_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.6)$$

We are seeking non-trivial solutions, thus

$$\begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & N\alpha^2 + \alpha C_3 + \beta C_5 \end{vmatrix} = 0 \quad (4.7)$$

Solving equation (4.7) for  $N$ , we obtain:

$$N = \frac{C_1 C_5^2 + \alpha C_3 C_2^2 + \beta C_2^2 C_5 + C_3^2 C_4 - \alpha C_1 C_3 C_4 - \beta C_1 C_4 C_5 - 2C_2 C_3 C_5}{\alpha^2(C_1 C_4 - C_2^2)} \quad (4.8)$$

The *critical buckling load* occurs at  $n = 1$ , while  $m$  can vary.

## 4.2 Buckling of Clamped Plates under Uniaxial Compressive Load

As the CLPT, the Rayleigh-Ritz method has been used to solve the buckling problem for a clamped plate. It is convenient to split the total potential energy in three parts, bending, shear and external forces:

$$\Pi = U_b + U_s + U_p \quad (4.9)$$

where

$$\begin{aligned} U_b &= \frac{1}{2} \int_V \varepsilon_b^T \sigma_b dV = \frac{1}{2} \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon_b^T \bar{\mathbf{Q}} \varepsilon_b dz dA = \frac{1}{2} \int_A \kappa^T \mathbf{D} \kappa dA \\ &= \frac{1}{2} \int_0^b \int_0^a D_{11} \left( \frac{\partial^2 w_b}{\partial x^2} \right)^2 + 2D_{12} \left( \frac{\partial^2 w_b}{\partial x^2} \right) \left( \frac{\partial^2 w_b}{\partial y^2} \right) \\ &\quad + D_{22} \left( \frac{\partial^2 w_b}{\partial y^2} \right)^2 + 4D_{66} \left( \frac{\partial^2 w_b}{\partial x \partial y} \right)^2 dx dy \end{aligned} \quad (4.10)$$

$$\begin{aligned} U_s &= \frac{1}{2} \int_V \varepsilon_s^T \sigma_s dV = \frac{1}{2} \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon_s^T \bar{\mathbf{Q}}_{skj} \varepsilon_s dz dA = \frac{1}{2} \int_A \varepsilon_s^T \mathbf{A}_{skj} \varepsilon_s dA \\ &= \frac{1}{2} k \int_0^b \int_0^a A_{44} \left( \frac{\partial w_s}{\partial y} \right)^2 + A_{55} \left( \frac{\partial w_s}{\partial x} \right)^2 dx dy \end{aligned} \quad (4.11)$$

$$U_p = \frac{1}{2} \int_0^b \int_0^a -N \left( \frac{\partial w}{\partial x} \right)^2 dx dy \quad (4.12)$$

The boundary conditions associated with the clamped edges are still:

$$\begin{aligned} x = 0 : \quad w(0, y) = 0 \quad \frac{\partial w(0, y)}{\partial x} = 0 \\ x = a : \quad w(a, y) = 0 \quad \frac{\partial w(a, y)}{\partial x} = 0 \\ y = 0 : \quad w(x, 0) = 0 \quad \frac{\partial w(x, 0)}{\partial y} = 0 \\ y = b : \quad w(x, b) = 0 \quad \frac{\partial w(x, b)}{\partial y} = 0 \end{aligned}$$

A solution that satisfies the preceding boundary conditions is given by [6]:

$$\begin{aligned} w(x, y) &= w_b + w_s \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \bar{w}_b \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) \\ &\quad + \bar{w}_s \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \end{aligned} \quad (4.13)$$

where  $\bar{w}_b$  and  $\bar{w}_s$  are the displacement coefficients for bending and shear,  $m$  and  $n$  are positive integers.

We now assume that:

$$w(x, y) = \bar{w}_b \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{\pi x}{a} \right) \sin^2 \left( \frac{\pi y}{b} \right) + \bar{w}_s \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) \quad (4.14)$$

Equilibrium requires that  $\delta\Pi = 0$ , thus:

$$\frac{\partial\Pi}{\partial\bar{w}_b} \delta\bar{w}_b + \frac{\partial\Pi}{\partial\bar{w}_s} \delta\bar{w}_s = 0$$

This implies

$$\left\{ \begin{array}{l} \frac{\partial\Pi}{\partial\bar{w}_b} = 0 \\ \frac{\partial\Pi}{\partial\bar{w}_s} = 0 \end{array} \right\} \quad (4.15)$$

Substitution of equations (4.9) and (4.14) in equation (4.15) gives two solutions. One for  $m \neq 1$ , and another  $m = 1$ . For  $m \neq 1$ , equation (4.15) gives following matrix equation:

$$\begin{bmatrix} H_1 + NH_2 & NH_3 \\ NH_3 & H_4 + NH_5 \end{bmatrix} \cdot \begin{Bmatrix} \bar{w}_b \\ \bar{w}_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4.16)$$

where

$$H_1 = \frac{\pi^4}{32a^3b^3} (D_{11}b^4(18m^2 + 3 + 3m^4) + D_{12}a^2b^2(8 + 8m^2) + 16D_{22}a^4 + D_{66}a^2b^2(16 + 16m^2))$$

$$H_2 = -\frac{\pi^2b(3m^6 + 3 + 3n^4 - 3m^4 - 3m^2 + 3m^2n^4 - 12m^2n^2 - 6n^2 - 6m^4n^2)}{32a(1 - 2n^2 + m^4 - 2m^2n^2 - 2m^2 + n^4)}$$

$$H_3 = -\frac{8b(-mn^3 + (-1)^{m+n+1}mn^3)}{3a(1 - 2n^2 + m^4 - 2m^2n^2 - 2m^2 + n^4)}$$

$$H_4 = \frac{k\pi^2}{4ab}(A_{44}a^2 + A_{55}n^2b^2)$$

$$H_5 = -\frac{\pi^2b(n^2 + n^6 - 2m^2n^4 + m^4n^2 - 2n^4 - 2m^2n^2)}{4a(1 - 2n^2 + m^4 - 2m^2n^2 - 2m^2 + n^4)}$$

Non-trivial solutions if the determinant of the matrix expressed in equation (4.16) is zero. This leads us to a second-order equation:

$$(H_2H_5 - H_3^2)N^2 + (H_1H_5 + H_2H_4)N + H_1H_4 = 0$$

The smallest value of  $N$  is given by:

$$N = \frac{-(H_1H_5 + H_2H_4) - \sqrt{(H_1H_5 + H_2H_4)^2 - 4(H_2H_5 - H_3^2)H_1H_4}}{2(H_2H_5 - H_3^2)} \quad (4.17)$$

Combination of positive integers  $m$  and  $n$  gives the *critical buckling load*.

For  $m = 1$ , equation (4.15) gives following matrix equation:

$$\begin{bmatrix} G_1 + NG_2 & NG_3 \\ NG_3 & G_4 + NG_5 \end{bmatrix} \cdot \begin{Bmatrix} \bar{w}_b \\ \bar{w}_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4.18)$$

where

$$G_1 = \frac{\pi^4}{4a^3b^3} (3D_{11}b^4 + 2D_{12}a^2b^2 + 3D_{22}a^4 + 4D_{66}a^2b^2)$$

$$G_2 = -\frac{\pi^2b(3n^4 - 12n^2)}{16a(n^4 - 4n^2)}$$

$$G_3 = -\frac{8b(-n^3 + (-1)^{n+2}n^3)}{3a(n^4 - 4n^2)}$$

$$G_4 = \frac{k\pi^2}{4ab}(A_{44}a^2 + A_{55}n^2b^2)$$

$$G_5 = -\frac{\pi^2b(n^6 - 4n^4)}{4a(n^4 - 4n^2)}$$

Non-trivial solution if the determinant of the matrix in equation (4.18) equals zero. A result of this, is a second-order equation:

$$(G_2G_5 - G_3^2)N^2 + (G_1G_5 + G_2G_4)N + G_1G_4 = 0$$

Solving this, we obtain the smallest value of  $N$ :

$$N = \frac{-(G_1G_5 + G_2G_4) - \sqrt{(G_1G_5 + G_2G_4)^2 - 4(G_2G_5 - G_3^2)G_1G_4}}{2(G_2G_5 - G_3^2)} \quad (4.19)$$

The *critical buckling load* depends on the positive integer  $n$ .



### 4.3 Buckling of Simply Supported Plates under Biaxial Compressive Load

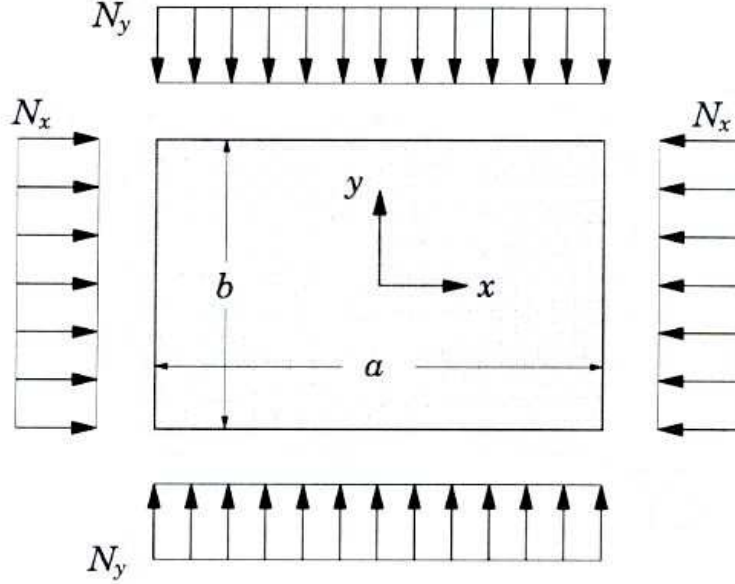


Figure 4.2: Plate with biaxial compressive load [1].

For the buckling analysis, we assume that the only applied loads are the in-plane compression edge forces. We now define [1]:

$$N_x = -N, \quad N_y = -\zeta N, \quad \zeta = \frac{N_y}{N_x}$$

That implies  $p^* = N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} = -N \frac{\partial^2 w}{\partial x^2} - \zeta N \frac{\partial^2 w}{\partial y^2}$  from equation (2.18). Based on equations (2.16)-(2.18), the equation set that solves the buckling problem takes the form:

$$\begin{aligned} D_{11} \frac{\partial^2 \phi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_y}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_x}{\partial y^2} - A_{55} k \left( \phi_x + \frac{\partial w}{\partial x} \right) &= 0 \\ D_{22} \frac{\partial^2 \phi_y}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_y}{\partial x^2} - A_{44} k \left( \phi_y + \frac{\partial w}{\partial y} \right) &= 0 \quad (4.20) \\ A_{55} k \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{44} k \left( \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) - N \frac{\partial^2 w}{\partial x^2} - \zeta N \frac{\partial^2 w}{\partial y^2} &= 0 \end{aligned}$$

Boundary conditions for simply supported plates are still:

$$x = 0 : \quad w(0, y) = 0 \quad M_x(0, y) = 0$$

$$x = a : \quad w(a, y) = 0 \quad M_x(a, y) = 0$$

$$y = 0 : \quad w(x, 0) = 0 \quad M_y(x, 0) = 0$$

$$y = b : \quad w(x, b) = 0 \quad M_y(x, b) = 0$$

The following double Fourier series are assumed to represent  $w$ ,  $\phi_x$  and  $\phi_y$ :

$$w(x, y) = w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (4.21)$$

$$\phi_x(x, y) = x_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad (4.22)$$

$$\phi_y(x, y) = y_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \quad (4.23)$$

where  $w_{mn}$ ,  $x_{mn}$  and  $y_{mn}$  are the series coefficients,  $m$  and  $n$  are positive integers.

Substitution of equations (4.21)-(4.23) into equation set (4.20) gives the following matrix equation:

$$\begin{bmatrix} -D_{11}\alpha^2 - D_{66}\beta^2 - A_{55}k & -D_{12}\alpha\beta - D_{66}\alpha\beta & -A_{55}k\alpha \\ -D_{12}\alpha\beta - D_{66}\alpha\beta & -D_{22}\beta^2 - D_{66}\alpha^2 - A_{44}k & -A_{44}k\beta \\ -A_{55}k\alpha & -A_{44}k\beta & N\alpha^2 + \zeta N\beta^2 - A_{55}k\alpha^2 - A_{44}k\beta^2 \end{bmatrix} \cdot \begin{Bmatrix} x_{mn} \\ y_{mn} \\ w_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.24)$$

where  $\alpha = \frac{m\pi}{a}$  og  $\beta = \frac{n\pi}{b}$ .

By defining

$$C_1 = -D_{11}\alpha^2 - D_{66}\beta^2 - A_{55}k$$

$$C_2 = -D_{12}\alpha\beta - D_{66}\alpha\beta$$

$$C_3 = -A_{55}k\alpha$$

$$C_4 = -D_{22}\beta^2 - D_{66}\alpha^2 - A_{44}k$$

$$C_5 = -A_{44}k\beta$$

simplify the equation (4.24):

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & N(\alpha^2 + \zeta\beta^2) + \alpha C_3 + \beta C_5 \end{bmatrix} \begin{Bmatrix} x_{mn} \\ y_{mn} \\ w_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.25)$$

We are seeking non-trivial solutions, thus

$$\begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & N(\alpha^2 + \zeta\beta^2) + \alpha C_3 + \beta C_5 \end{vmatrix} = 0 \quad (4.26)$$

Solving equation (4.26) for  $N$ , we obtain:

$$N = \frac{C_1 C_5^2 + \alpha C_2^2 C_3 + \beta C_2^2 C_5 + C_3^2 C_4 - \alpha C_1 C_3 C_4 - \beta C_1 C_4 C_5 - 2C_2 C_3 C_5}{\alpha^2 (C_1 C_4 - C_2^2) + \zeta \beta^2 (C_1 C_4 - C_2^2)} \quad (4.27)$$

Right combination of  $m$  and  $n$  gives the *critical buckling load*. Various values of  $\zeta$  will also be investigated.

#### 4.4 Buckling of Simply Supported Plates under In-plane Shear Load

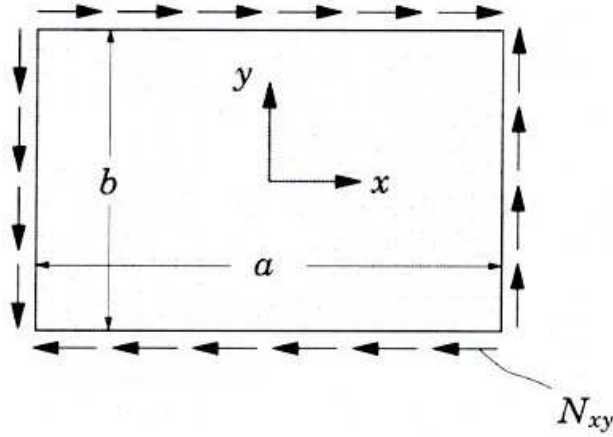


Figure 4.3: Plate with in-plane shear load [1].

In this section we consider buckling of specially orthotropic plates under in-plane shear load,  $N_{xy}$ . The problem does not permit the Navier solution, so as for clamped plates, we use Rayleigh-Ritz method to solve the problem. When everything else but in-plane shear load is zero,  $p^* = 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = -2\hat{N}_{xy} \frac{\partial^2 w}{\partial x \partial y}$ .

For simply supported plates subjected to in-plane shear load, the same boundary conditions are valid with corresponding expressions:

$$\begin{aligned} w(x, y) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\ \phi_x(x, y) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\ \phi_y(x, y) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} y_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \end{aligned}$$

where  $w_{mn}$ ,  $x_{mn}$  and  $y_{mn}$  are the series coefficients,  $m$  and  $n$  are positive integers.

We now split the total potential energy functional for the Rayleigh-Ritz method in three parts (bending, shear and external forces):

$$\Pi = U_b + U_s + U_p \quad (4.28)$$

where

$$\begin{aligned} U_b &= \frac{1}{2} \int_V \varepsilon_b^T \sigma_b dV = \frac{1}{2} \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon_b^T \bar{\mathbf{Q}} \varepsilon_b dz dA = \frac{1}{2} \int_A \kappa^T \mathbf{D} \kappa dA \\ &= \frac{1}{2} \int_0^b \int_0^a D_{11} \left(\frac{\partial \phi_x}{\partial x}\right)^2 + 2D_{12} \left(\frac{\partial \phi_x}{\partial x}\right) \left(\frac{\partial \phi_y}{\partial y}\right) \\ &\quad + D_{22} \left(\frac{\partial \phi_y}{\partial y}\right)^2 + 4D_{66} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2 dx dy \end{aligned} \quad (4.29)$$

$$\begin{aligned} U_s &= \frac{1}{2} \int_V \varepsilon_s^T \sigma_s dV = \frac{1}{2} \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon_s^T \bar{\mathbf{Q}}_{skj} \varepsilon_s dz dA = \frac{1}{2} \int_A \varepsilon_s^T \mathbf{A}_{skj} \varepsilon_s dA \\ &= \frac{1}{2} k \int_0^b \int_0^a A_{44} \left(\phi_y + \frac{\partial w}{\partial y}\right)^2 + A_{55} \left(\phi_x + \frac{\partial w}{\partial x}\right)^2 dx dy \end{aligned} \quad (4.30)$$

$$U_p = \frac{1}{2} \int_0^b \int_0^a -2\hat{N}_{xy} \left(\frac{\partial w}{\partial x}\right) \left(\frac{\partial w}{\partial y}\right) dx dy \quad (4.31)$$

Substituting the Fourier approximations for  $w$ ,  $\phi_x$  and  $\phi_y$  gives:

$$\begin{aligned} U_b &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\pi^2}{8ab} [(D_{11}m^2b^2 + D_{66}n^2a^2)x_{mn}^2 \\ &\quad + (2D_{12}mnab + 2D_{66}mnab)x_{mn}y_{mn} \\ &\quad + (D_{22}n^2a^2 + D_{66}m^2b^2)y_{mn}^2] \end{aligned} \quad (4.32)$$

$$\begin{aligned}
U_s = & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{k}{8ab} [A_{55}a^2b^2x_{mn}^2 + A_{44}a^2b^2y_{mn}^2 + 2A_{55}m\pi ab^2x_{mn}w_{mn} \\
& + 2A_{44}n\pi a^2by_{mn}w_{mn} + (A_{44}n^2\pi^2a^2 + A_{55}m^2\pi^2b^2)w_{mn}^2] \quad (4.33)
\end{aligned}$$

Using the following identities on equation (4.31) [7]:

$$\begin{aligned}
\int_0^a \sin \frac{m\pi x}{a} \cos \frac{p\pi x}{a} dx &= \begin{cases} 0, & \text{if } m \pm p \text{ is an even number} \\ \frac{2a}{\pi} \frac{m}{m^2-p^2}, & \text{if } m \pm p \text{ is an odd number} \end{cases} \\
\int_0^b \sin \frac{n\pi y}{b} \cos \frac{q\pi y}{b} dy &= \begin{cases} 0, & \text{if } n \pm q \text{ is an even number} \\ \frac{2b}{\pi} \frac{n}{n^2-q^2}, & \text{if } n \pm q \text{ is an odd number} \end{cases}
\end{aligned}$$

We arrive at:

$$U_p = -4\hat{N}_{xy} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} \frac{mnpq}{(m^2-p^2)(n^2-q^2)} w_{mn} w_{pq} \quad (4.34)$$

where  $m \pm p$  and  $n \pm q$  are odd numbers.

Equilibrium requires that  $\delta\Pi = 0$ , thus:

$$\begin{aligned}
\frac{\partial\Pi}{\partial x_{mn}} &= 0, \quad m = 1, \dots, \infty, \quad n = 1, \dots, \infty \\
\frac{\partial\Pi}{\partial y_{mn}} &= 0, \quad m = 1, \dots, \infty, \quad n = 1, \dots, \infty \\
\frac{\partial\Pi}{\partial w_{mn}} &= 0, \quad m = 1, \dots, \infty, \quad n = 1, \dots, \infty
\end{aligned} \quad (4.35)$$

Inserting of equations (4.32)-(4.34), equation set (4.35) becomes:

$$\begin{aligned}
\frac{\partial\Pi}{\partial x_{mn}} &= \left( \frac{1}{4} \frac{\pi^2 m^2 b}{a} D_{11} + \frac{1}{4} \frac{\pi^2 n^2 a}{b} D_{66} + \frac{1}{4} kab A_{55} \right) x_{mn} \\
&+ \left( \frac{1}{4} \pi^2 mn D_{12} + \frac{1}{4} \pi^2 mn D_{66} \right) y_{mn} \\
&+ \left( \frac{1}{4} \pi k mb A_{55} \right) w_{mn} \\
&= I_1 x_{mn} + I_2 y_{mn} + I_3 w_{mn} = 0 \quad (4.36)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \Pi}{\partial y_{mn}} &= \left( \frac{1}{4} \pi^2 mn D_{12} + \frac{1}{4} \pi^2 mn D_{66} \right) x_{mn} \\
&+ \left( \frac{1}{4} \frac{\pi^2 n^2 a}{b} D_{22} + \frac{1}{4} \frac{\pi^2 m^2 b}{a} D_{66} + \frac{1}{4} kab A_{44} \right) y_{mn} \\
&+ \left( \frac{1}{4} \pi kna A_{44} \right) w_{mn} \\
&= I_4 x_{mn} + I_5 y_{mn} + I_6 w_{mn} = 0
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
\frac{\partial \Pi}{\partial w_{mn}} &= \left( \frac{1}{4} \pi kmb A_{55} \right) x_{mn} + \left( \frac{1}{4} \pi kna A_{44} \right) y_{mn} \\
&+ \left( \frac{1}{4} \frac{\pi^2 kn^2 a A_{44}}{b} + \frac{1}{4} \frac{\pi^2 km^2 b A_{55}}{a} \right) w_{mn} \\
&+ \left( -8 \hat{N}_{xy} \frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} \right) w_{pq} \\
&= I_7 x_{mn} + I_8 y_{mn} + I_9 w_{mn} + I_{10} w_{pq} = 0
\end{aligned} \tag{4.38}$$

$m, n, p$  and  $q$  in equations (4.36)-(4.38) are positive integers, and run from 1 to  $\infty$ .  $I_{10}$  is valid for  $m \pm p, n \pm q$  odd numbers, otherwise zero.

Equation set (4.35) can be expressed in matrix form:

$$\left\{ \begin{array}{c} \frac{\partial \Pi}{\partial x_{11}} \\ \frac{\partial \Pi}{\partial x_{12}} \\ \vdots \\ \frac{\partial \Pi}{\partial x_{1n}} \\ \frac{\partial \Pi}{\partial x_{21}} \\ \vdots \\ \frac{\partial \Pi}{\partial x_{m1}} \\ \vdots \\ \frac{\partial \Pi}{\partial x_{mn}} \\ \frac{\partial \Pi}{\partial y_{11}} \\ \vdots \\ \frac{\partial \Pi}{\partial y_{mn}} \\ \frac{\partial \Pi}{\partial w_{11}} \\ \vdots \\ \frac{\partial \Pi}{\partial w_{mn}} \end{array} \right\} = [MXI] \left\{ \begin{array}{c} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \\ x_{21} \\ \vdots \\ x_{m1} \\ \vdots \\ x_{mn} \\ y_{11} \\ \vdots \\ y_{mn} \\ w_{11} \\ \vdots \\ w_{mn} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right\} \tag{4.39}$$

A simplified version of equation (4.39) takes the form:

$$\begin{bmatrix} \text{matrix1} & \text{matrix2} & \text{matrix3} \\ \text{matrix4} & \text{matrix5} & \text{matrix6} \\ \text{matrix7} & \text{matrix8} & \text{matrix9} \end{bmatrix} \begin{Bmatrix} x \\ y \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.40)$$

If we now assume that  $m = 1, \dots, M$  and  $n = 1, \dots, N$ , then this  $[MXI]$  is a  $3MN \times 3MN$  matrix. Entries of  $[MXI]$  are based on equations (4.36)-(4.38). To get a idea how  $[MXI]$  looks like, it is convenient to split this huge matrix  $[MXI]$  into 9 small matrixes.  $[\text{matrix1}]$  has row number 1 to  $MN$ , column number 1 to  $MN$  in  $[MXI]$ , and is based on  $I_1$  in equation (4.36). This results a diagonal matrix:

$$[\text{matrix1}] = \begin{bmatrix} I_1(m=1, n=1) & 0 & \dots & 0 \\ 0 & I_1(1, 2) & 0 & \dots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & I_1(m=M, n=N) \end{bmatrix}$$

$[\text{matrix2}]$  has row number 1 to  $MN$ , column number  $MN + 1$  to  $2MN$ , and is based on  $I_2$  in equation (4.36). Also a diagonal matrix:

$$[\text{matrix2}] = \begin{bmatrix} I_2(m=1, n=1) & 0 & \dots & 0 \\ 0 & I_2(1, 2) & 0 & \dots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & I_2(m=M, n=N) \end{bmatrix}$$

Diagonal matrix  $[\text{matrix3}]$  has row number 1 to  $MN$ , column number  $2MN + 1$  to  $3MN$ , and is based on  $I_3$  in equation (4.36):

$$[\text{matrix3}] = \begin{bmatrix} I_3(m=1, n=1) & 0 & \dots & 0 \\ 0 & I_3(1, 2) & 0 & \dots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & I_3(m=M, n=N) \end{bmatrix}$$

$[\text{matrix4}]$  has row number  $MN + 1$  to  $2MN$ , column number 1 to  $MN$ , and is based on  $I_4$  in equation (4.37).

$[\text{matrix5}]$  has row number  $MN + 1$  to  $2MN$ , column number  $MN + 1$  to  $2MN$ , and is based on  $I_5$  in equation (4.37).

$[\text{matrix6}]$  has row number  $MN + 1$  to  $2MN$ , column number  $2MN + 1$  to  $3MN$ , and is based on  $I_6$  in equation (4.37).

[*matrix7*] has row number  $2MN + 1$  to  $3MN$ , column number 1 to  $MN$ , and is based on  $I_7$  in equation (4.38).

[*matrix8*] has row number  $2MN + 1$  to  $3MN$ , column number  $MN + 1$  to  $2MN$ , and is based on  $I_8$  in equation (4.38).

[*matrix9*] has row number  $2MN + 1$  to  $3MN$ , column number  $2MN + 1$  to  $3MN$ , and is based on  $I_9$  and  $I_{10}$  in equation (4.38).

All the matrixes are diagonal matrixes except [*matrix9*] because of the term  $w_{pq}$ .  $I_9$  denotes the diagonal entries, while  $I_{10}$  denotes others.

To do the the buckling analysis, we need to find the non-trivial solution of equation (4.40). By solving  $\det[MXI] = 0$ , we will have the *critical buckling load*  $(\hat{N}_{xy})_{cr}$ . The accuracy of the result depends on the number of  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  terms. Timoshenko [7] suggests that we divide this system into two groups, one containing constants  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  for which  $m + n$  are odd numbers and the other for which  $m + n$  are even numbers. For shorter plates ( $a/b < 2$ ), it is enough to consider the second group. For longer plates both groups of equations should be considered.

## 4.5 Buckling of Simply Supported Plates under Combined Loads

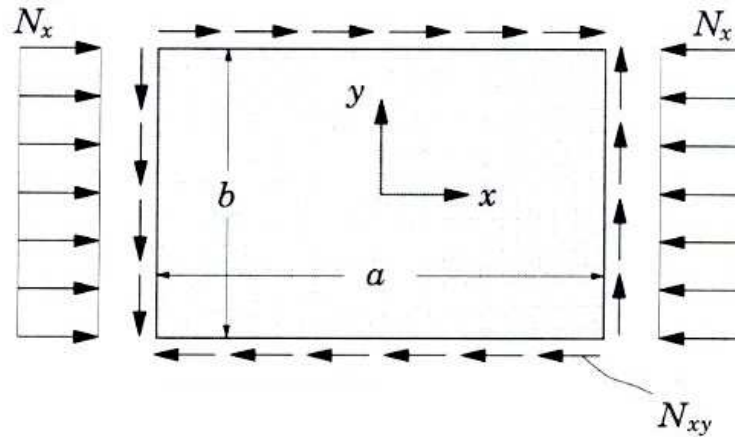


Figure 4.4: Plate with uniaxial compressive load and in-plane shear load [1].

For the plate subjected to combined loads, we assume that the only applied loads are uniaxial compression load in  $x$  direction and in-plane shear load. Now



introducing a new constant,  $\mu$ :

$$\mu = \frac{N_{xy}}{N_x} \quad \Rightarrow \quad N_x = -N, \quad N_{xy} = -\mu N$$

The boundary conditions and the corresponding double Fourier series for  $w_{mn}$ ,  $\phi_x$  and  $\phi_y$  are still valid here. Further, the calculation procedure is the same as the section 4.4. Again, we use Rayleigh-Ritz method to solve the problem. Potential energy due to bending and shear are given by equations (4.29) and (4.30), and the results are presented in equations (4.32) and (4.33).  $U_p$  is given by:

$$\begin{aligned} U_p &= \frac{1}{2} \left[ \int_0^b \int_0^a N_x \left( \frac{\partial w}{\partial x} \right)^2 + 2N_{xy} \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) dx dy \right] \\ &= -\frac{1}{2} \left[ \int_0^b \int_0^a N \left( \frac{\partial w}{\partial x} \right)^2 + 2\mu N \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) dx dy \right] \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ -\frac{1}{8} N \frac{\pi^2 m^2 b}{a} w_{mn}^2 \right. \\ &\quad \left. - 4\mu N \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} \frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} w_{mn} w_{pq} \right] \end{aligned} \quad (4.41)$$

where  $m \pm p$  and  $n \pm q$  are odd numbers.

As we know, the total potential energy is given by  $\Pi = U_b + U_s + U_p$ . Equilibrium requires that  $\delta\Pi = 0$ , thus:

$$\begin{aligned} \frac{\partial\Pi}{\partial x_{mn}} &= \left( \frac{1}{4} \frac{\pi^2 m^2 b}{a} D_{11} + \frac{1}{4} \frac{\pi^2 n^2 a}{b} D_{66} + \frac{1}{4} kab A_{55} \right) x_{mn} \\ &+ \left( \frac{1}{4} \pi^2 mn D_{12} + \frac{1}{4} \pi^2 mn D_{66} \right) y_{mn} \\ &+ \left( \frac{1}{4} \pi kmb A_{55} \right) w_{mn} \\ &= J_1 x_{mn} + J_2 y_{mn} + J_3 w_{mn} = 0 \end{aligned} \quad (4.42)$$

$$\begin{aligned}
\frac{\partial \Pi}{\partial y_{mn}} &= \left( \frac{1}{4} \pi^2 mn D_{12} + \frac{1}{4} \pi^2 mn D_{66} \right) x_{mn} \\
&+ \left( \frac{1}{4} \frac{\pi^2 n^2 a}{b} D_{22} + \frac{1}{4} \frac{\pi^2 m^2 b}{a} D_{66} + \frac{1}{4} kab A_{44} \right) y_{mn} \\
&+ \left( \frac{1}{4} \pi kna A_{44} \right) w_{mn} \\
&= J_4 x_{mn} + J_5 y_{mn} + J_6 w_{mn} = 0
\end{aligned} \tag{4.43}$$

$$\begin{aligned}
\frac{\partial \Pi}{\partial w_{mn}} &= \left( \frac{1}{4} \pi kmb A_{55} \right) x_{mn} + \left( \frac{1}{4} \pi kna A_{44} \right) y_{mn} \\
&+ \left( \frac{1}{4} \frac{\pi^2 kn^2 a A_{44}}{b} + \frac{1}{4} \frac{\pi^2 km^2 b A_{55}}{a} - \frac{1}{4} \frac{\pi^2 m^2 b}{a} N \right) w_{mn} \\
&+ \left( -8\mu N \frac{mnpq}{(m^2 - p^2)(n^2 - q^2)} \right) w_{pq} \\
&= J_7 x_{mn} + J_8 y_{mn} + J_9 w_{mn} + J_{10} w_{pq} = 0
\end{aligned} \tag{4.44}$$

$m, n, p$  and  $q$  in equations (4.42)-(4.44) are positive integers, and run from 1 to  $\infty$ .  $I_{10}$  is valid for  $m \pm p, n \pm q$  odd numbers, otherwise zero.

Equations (4.42)-(4.44) can be expressed in matrix form:

$$\left\{ \begin{array}{c} \frac{\partial \Pi}{\partial x_{11}} \\ \frac{\partial \Pi}{\partial x_{12}} \\ \vdots \\ \frac{\partial \Pi}{\partial x_{1n}} \\ \frac{\partial \Pi}{\partial x_{21}} \\ \vdots \\ \frac{\partial \Pi}{\partial x_{m1}} \\ \vdots \\ \frac{\partial \Pi}{\partial x_{mn}} \\ \frac{\partial \Pi}{\partial y_{11}} \\ \vdots \\ \frac{\partial \Pi}{\partial y_{mn}} \\ \frac{\partial \Pi}{\partial w_{11}} \\ \vdots \\ \frac{\partial \Pi}{\partial w_{mn}} \end{array} \right\} = [MXJ] \left\{ \begin{array}{c} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \\ x_{21} \\ \vdots \\ x_{m1} \\ \vdots \\ x_{mn} \\ y_{11} \\ \vdots \\ y_{mn} \\ w_{11} \\ \vdots \\ w_{mn} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right\} \tag{4.45}$$

A simplified version of equation (4.45) is given by:

$$\begin{bmatrix} \text{mat1} & \text{mat2} & \text{mat3} \\ \text{mat4} & \text{mat5} & \text{mat6} \\ \text{mat7} & \text{mat8} & \text{mat9} \end{bmatrix} \begin{Bmatrix} x \\ y \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.46)$$

Now this  $[MXJ]$  is based on equations (4.42)-(4.44). If  $m = 1, \dots, M$  and  $n = 1, \dots, N$ , then  $[MXJ]$  is a  $3MN \times 3MN$  matrix. Just like section 4.4, we can now split  $[MXJ]$  in 9 smaller matixes, where  $[\text{mat1}], \dots, [\text{mat8}]$  are diagonal matrixes based on expressions  $J1, \dots, J8$  in equations (4.42)-(4.44).  $J9$  denotes the diagonal entries in  $[\text{mat9}]$ , while  $J10$  denotes others.

To find the the *critical buckling load*  $N$ , we need to solve  $\det[MXJ] = 0$ . The accuracy of the result depends on the number of  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  terms.

## 4.6 Plates with Initial Geometric Imperfection under Uniaxial Compressive Load

### 4.6.1 Relationship Between Displacement and Applied Load

It is normally unrealistic to assume that a plate is perfect. We consider now a simply supported plate with an initial deformation,  $w_{int}$ . Usually such deformation occurs by production faults or welding faults in assembly of the plate. When the plate subjected to in-plane stresses, it will receive an additional deformation,  $w$ . Thus, the total deformation is  $w_{tot} = w_{int} + w$ . In this section we assume that the only applied load is the in-plane compressive force in  $x$  direction. This means  $p^* = N_x \frac{\partial^2 w_{tot}}{\partial x^2} = -N \frac{\partial^2 w_{tot}}{\partial x^2}$ . We are now interested in analysing the response under increasing load. In other words, finding the relationship between applied load and the displacement. Based on equations (2.16)-(2.18) from section 2.3.4, the equation set that solves the problem is given by:

$$\begin{aligned} D_{11} \frac{\partial^2 \phi_x}{\partial x^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_y}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_x}{\partial y^2} - A_{55} k \left( \phi_x + \frac{\partial w}{\partial x} \right) &= 0 \\ D_{22} \frac{\partial^2 \phi_y}{\partial y^2} + (D_{12} + D_{66}) \frac{\partial^2 \phi_x}{\partial x \partial y} + D_{66} \frac{\partial^2 \phi_y}{\partial x^2} - A_{44} k \left( \phi_y + \frac{\partial w}{\partial y} \right) &= 0 \\ A_{55} k \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{44} k \left( \frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) - N \frac{\partial^2 w_{tot}}{\partial x^2} &= 0 \end{aligned} \quad (4.47)$$

For simply supported plate, we have the usual boundary conditions with corresponding double Fourier series:

$$\phi_x(x, y) = x_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$\phi_y(x, y) = y_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

and

$$\begin{aligned} w_{tot}(x, y) &= w(x, y) + w_{int}(x, y) \\ &= w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) + w_i \sin\left(\frac{m_i\pi x}{a}\right) \sin\left(\frac{n_i\pi y}{b}\right) \end{aligned} \quad (4.48)$$

where  $w_{mn}$ ,  $x_{mn}$  and  $y_{mn}$  are the unknown series coefficients.  $w_i$  is a given imperfection amplitude at centre.

If we now assume that  $m = m_i$  and  $n = n_i$ , substitution of  $\phi_x$ ,  $\phi_y$  and  $w_{tot}$  into equation set (4.47) gives the following matrix equation:

$$\begin{bmatrix} -D_{11}\alpha^2 - D_{66}\beta^2 - A_{55}k & -D_{12}\alpha\beta - D_{66}\alpha\beta & -A_{55}k\alpha \\ -D_{12}\alpha\beta - D_{66}\alpha\beta & -D_{22}\beta^2 - D_{66}\alpha^2 - A_{44}k & -A_{44}k\beta \\ -A_{55}k\alpha & -A_{44}k\beta & N\alpha^2 - A_{55}k\alpha^2 - A_{44}k\beta^2 \end{bmatrix} \cdot \begin{Bmatrix} x_{mn} \\ y_{mn} \\ w_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -N\alpha^2 w_i \end{Bmatrix} \quad (4.49)$$

where  $\alpha = \frac{m\pi}{a}$  og  $\beta = \frac{n\pi}{b}$ .

By defining

$$C_1 = -D_{11}\alpha^2 - D_{66}\beta^2 - A_{55}k$$

$$C_2 = -D_{12}\alpha\beta - D_{66}\alpha\beta$$

$$C_3 = -A_{55}k\alpha$$

$$C_4 = -D_{22}\beta^2 - D_{66}\alpha^2 - A_{44}k$$

$$C_5 = -A_{44}k\beta$$

simplify the equation (4.49):

$$\begin{bmatrix} C_1 & C_2 & C_3 \\ C_2 & C_4 & C_5 \\ C_3 & C_5 & N\alpha^2 + \alpha C_3 + \beta C_5 \end{bmatrix} \begin{Bmatrix} x_{mn} \\ y_{mn} \\ w_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -N\alpha^2 w_i \end{Bmatrix} \quad (4.50)$$

Now solving equation (4.50) for  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$ :

$$w_{mn} = \frac{-N\alpha^2 w_i}{\frac{(C_5 - \frac{C_2 C_3}{C_1})(\frac{C_2 C_3}{C_1} - C_5)}{(C_4 - \frac{C_2^2}{C_1})} - \frac{C_3^2}{C_1} + \alpha C_3 + \beta C_5 + N\alpha^2} \quad (4.51)$$

$$y_{mn} = \frac{(\frac{C_2 C_3}{C_1} - C_5)}{(C_4 - \frac{C_2^2}{C_1})} w_{mn} \quad (4.52)$$

$$x_{mn} = \frac{-C_2 y_{mn} - C_3 w_{mn}}{C_1} \quad (4.53)$$

Relationship between displacement and applied load for a point  $(x, y)$  is given by equation (4.48). Inserting of  $w_{mn}$  obtained in equation (4.51), we arrive at:

$$w_{tot}(x, y) = \frac{-N\alpha^2 w_i}{\frac{(C_5 - \frac{C_2 C_3}{C_1})(\frac{C_2 C_3}{C_1} - C_5)}{(C_4 - \frac{C_2^2}{C_1})} - \frac{C_3^2}{C_1} + \alpha C_3 + \beta C_5 + N\alpha^2} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) + w_i \sin\left(\frac{m_i \pi x}{a}\right) \sin\left(\frac{n_i \pi y}{b}\right) \quad (4.54)$$

for given  $N$ ,  $w_{tot}$  is the only unknown.

#### 4.6.2 First Ply Failure

For composite materials, strengths in different directions can vary widely. For example, a unidirectional lamina could withstand a lot more of tension along the fibres, compared to tension perpendicular to the fibres. So it is interesting to calculate the allowable strength for composites. The failure criteria discussed here are limited to first ply failure, which gives a conservative estimate of the strength of the laminate.

There are a several models which can be used to calculate the Failure Index (FI). The one we have chosen is a widely used criterion, *Tsai-Wu*. In its most general form, it can be written as [5]:

$$FI = F_{ij} \sigma_i \sigma_j + F_i \sigma_i = 1 \quad (4.55)$$

where  $i, j = 1, \dots, 6$ .

For an orthotropic material in the 2-D plane stress state, the Tsai-Wu failure criterion becoming:

$$FI = F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 + F_1\sigma_1 + F_2\sigma_2 = 1 \quad (4.56)$$

This has four quadratic strength parameters, analogous to the modulus components, and two linear strength parameters which account for the differences in the tensile and compressive strengths. Five of the six strength parameters are obtained from simple mechanical tests, and they are given by:

$$F_{11} = \frac{1}{X_t X_c} \quad , \quad F_{22} = \frac{1}{Y_t Y_c} \quad , \quad F_1 = \frac{1}{X_t} - \frac{1}{X_c}$$

$$F_2 = \frac{1}{Y_t} - \frac{1}{Y_c} \quad , \quad F_{66} = \frac{1}{S_{12}^2}$$

where

$X_t$  = Tension strength along the  $x$ -axis

$X_c$  = Compression strength along the  $x$ -axis

$Y_t$  = Tension strength along the  $y$ -axis

$Y_c$  = Compression strength along the  $y$ -axis

$S_{12}$  = Shear strength in the  $xy$ -plane

The sixth parameter,  $F_{12}$ , represents the interaction of two stress components,  $\sigma_1$  and  $\sigma_2$ , in a combined strength test. A biaxial test must be conducted to determine the  $F_{12}$ . This is a much more difficult test to perform experimentally. In practice, we usually set the normalised interaction term ( $F_{12}^*$ ) to a fixed value, then  $F_{12}$  can be found by:

$$F_{12}^* = \frac{F_{12}}{\sqrt{F_{11}F_{22}}}$$

These are just default values. Often  $F_{12}^* = -0.5$  or  $F_{12}^* = 0$ .

In practice, instead of equation (4.56), there is another way to define Tsai-Wu criterion. By introducing a factor  $R$  [5], equation (4.56) becoming:

$$R^2 (F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2) + R (F_1\sigma_1 + F_2\sigma_2) - 1 = 0 \quad (4.57)$$

$$\Rightarrow R^2\xi_1 + R\xi_2 - 1 = 0$$

Solving this, we will have positive  $R$  given by:

$$R = \frac{-\xi_2 + \sqrt{\xi_2^2 + 4\xi_1}}{2\xi_1} \quad (4.58)$$

$1/R$  equals 1 gives first ply failure.

Now we are going to analyse at which material failure first occurs for a simply supported plate with an initial geometric imperfection. Stresses in each ply or ply  $i$  is given by [5]:

$$\begin{aligned} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_i &= \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_i \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \\ &+ z_i \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_i \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \end{aligned} \quad (4.59)$$

First, we need to find the midplane strains  $\varepsilon^0$  for the entire laminate by solving:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \quad (4.60)$$

For a applied load  $N_x = -N$ , equation (4.60) becoming:

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} -N \\ 0 \\ 0 \end{Bmatrix} \quad (4.61)$$

Then from section 2.3.2, using FSDT, we know that the midplane curvature for entire laminate is given by:

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \quad (4.62)$$

where:

$$\begin{aligned} \phi_x(x, y) &= x_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\ \phi_y(x, y) &= y_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \end{aligned}$$

For a given point  $(x, y)$  equation (4.62) becoming:

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} -x_{mn} \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\ -y_{mn} \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \\ \left(x_{mn} \left(\frac{n\pi}{b}\right) + y_{mn} \left(\frac{m\pi}{a}\right)\right) \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \end{Bmatrix} \quad (4.63)$$

where  $x_{mn}$  and  $y_{mn}$  are given by equations (4.52) and (4.53) in section 4.6.1.

Now determine  $[\bar{Q}]$  and coordinate  $z$  for each ply. Thereafter calculate lamina stresses  $\sigma_x$ ,  $\sigma_y$  and  $\tau_{xy}$  along the  $x$  and  $y$  axes in each ply by substituting equations (4.61) and (4.63) into (4.59).

To produce a first ply failure, we need to scale up (or down) the applied load  $N$  until we find the Failure Index  $FI = 1$  in equation (4.56), or find the inverse of the "strength ratio"  $1/R = 1$  in equation (4.57).



## 5 Finite Element Model

For verification of the present methods, a variety of plate dimensions and three cases with different lay-ups have been considered. For more details, see Appendix A. Computed results by the present methods have been compared with finite element (FE) analyses using ANSYS. Case A and B are modelled using SHELL281 elements, while case C SHELL91.

FE analysis is performed in several steps:

- Choose the correct element type (SHELL91 or SHELL281) and apply the lay-ups.
- Build up the model with geometry and material properties. Then set the element size (25 for all cases).
- Define the boundary conditions and apply the load.
- Static analysis followed by a buckling analysis. List the eigenvalues and the corresponding buckling modes.

For analyses with imperfection:

- Perform the steps above.
- Do A non-linear analysis with initial imperfection (= 1. buckling mode). To show this non-linear behavior, we have to apply a new load which is much larger than buckling load. It is been chosen twice the critical buckling load for all cases.
- Having the results plotted (load-displacement).
- For first ply failure, we have to add the failure criteria before the non-linear analysis.
- Then investigate the plies and try to find the load which gives first ply failure. "Inverse of Tsai-Wu Strength Ratio Index" has been used.

To do the analyses, it is very important to define the correct boundary conditions and applied loads for each case. We explain the conditions using figures.

1. For **simply supported, uniaxial loads**, the boundary conditions and the applied load are defined below.

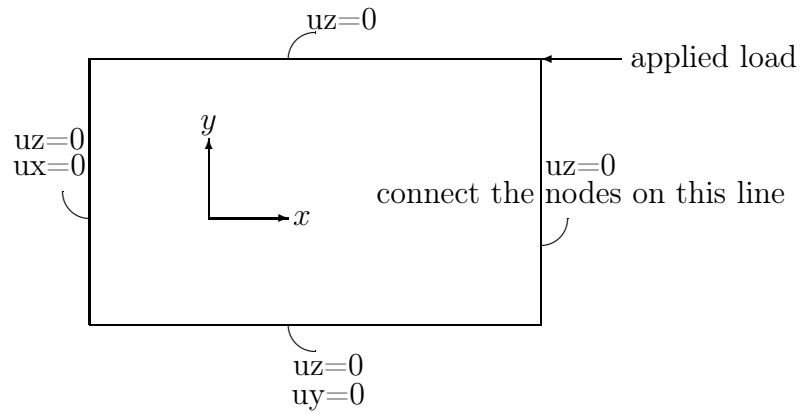


Figure 5.1: Applied boundary conditions and load for simply supported plate.

2. For **clamped edges, uniaxial loads**, the boundary conditions and the applied load are listed below.

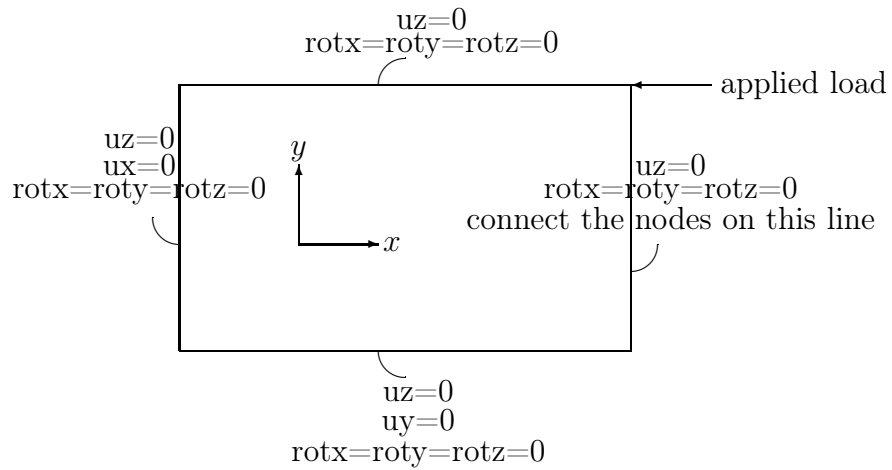


Figure 5.2: Applied boundary conditions and load for clamped plate.

3. Now, the boundary conditions and the applied loads for **simply supported, biaxial load**.

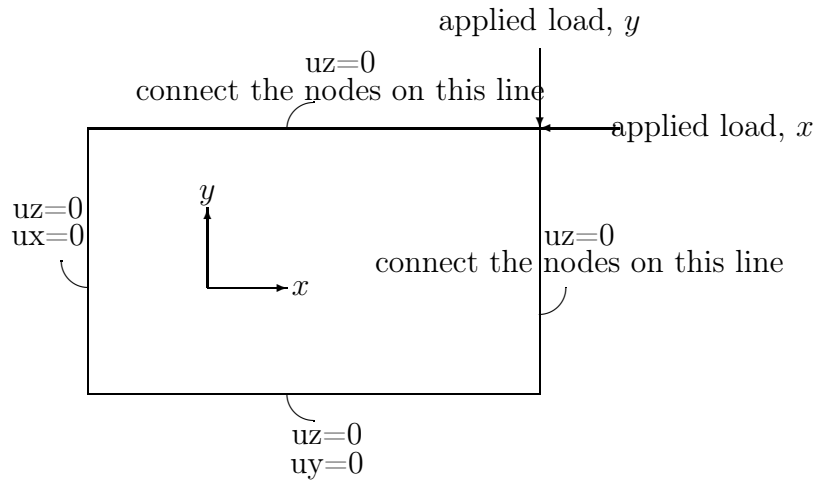


Figure 5.3: Applied boundary conditions and loads for simply supported plate subjected to biaxial load.

4. For **simply supported, in-plane shear**.

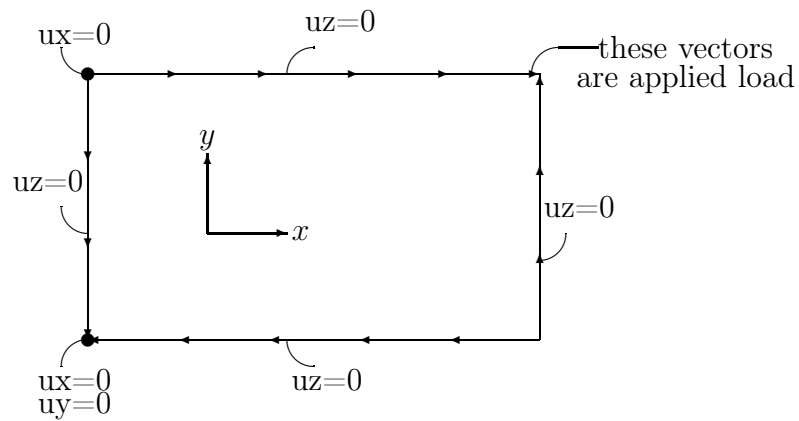


Figure 5.4: Applied boundary conditions and load for simply supported plate subjected to in-plane shear load.

The main idea is to apply each node a point load. The value is given by:

$$\frac{\text{total load}}{\text{number of element} \times \text{number of nodes per element}}$$

(Using "nlist" in ANSYS will return a list of nodes.)

5. For **simply supported, combined loads (uniaxial and in-plane)**, the boundary conditions and the applied loads are defined below.

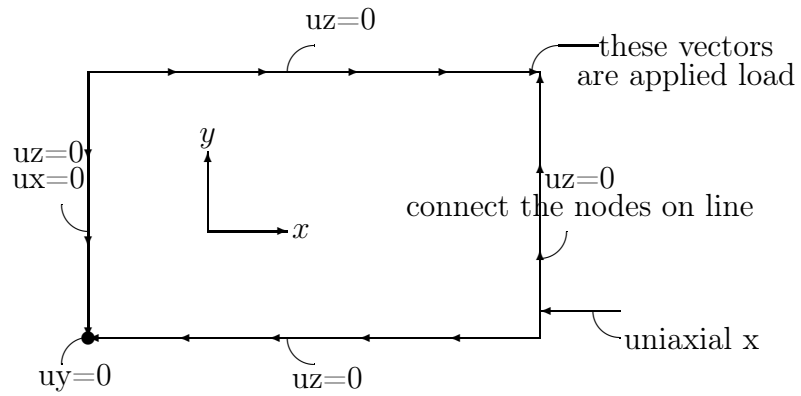


Figure 5.5: Applied boundary conditions and load for simply supported plate subjected to combined loads.

## 6 Presentation of Results

### 6.1 Uniaxial Compressive Load, CLPT and FSDT

This section contains results from both simply supported and clamped edges using CLPT and FSDT. Corresponding FE analyses have been performed using ANSYS. Results are presented in the form of tables and graphs. Three cases with different lay-ups and thickness have been investigated. Their material properties are listed in Appendix A.

#### 6.1.1 Simply Supported Plates

From section 3.1, using CLPT, the critical buckling load is given by equation (3.3). From section 4.1, using FSDT, the critical buckling load is given by equation (4.8). The results are estimated using the routines developed in Matlab.

#### CASE A - 500×500

	<i>Case A - Simply supported, 500 × 500mm</i>		
Plate thickness	CLPT [ <i>N/mm</i> ]	FSDT [ <i>N/mm</i> ]	ANSYS [ <i>N/mm</i> ]
<i>t = 8.0mm</i>	175.0 (n=1,m=1)	174.5 (n=1,m=1)	172.2
<i>t = 16.0mm</i>	1372.0 (n=1,m=1)	1356.4 (n=1,m=1)	1323.7
<i>t = 24.0mm</i>	4600.9 (n=1,m=1)	4484.7 (n=1,m=1)	4332.4
<i>t = 32.0mm</i>	10871.0 (n=1,m=1)	10394.0 (n=1,m=1)	9947.4
<i>t = 48.0mm</i>	36578.0 (n=1,m=1)	33159.0 (n=1,m=1)	31236.0

Table 6.1: Case A, simply supported - 500×500.

	<i>Failure in percent - Simply supported, A 500 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
<i>t = 8.0mm</i>	0.3%	1.6%	1.3%
<i>t = 16.0mm</i>	1.2%	3.6%	2.5%
<i>t = 24.0mm</i>	2.6%	6.2%	3.5%
<i>t = 32.0mm</i>	4.6%	9.3%	4.5%
<i>t = 48.0mm</i>	10.3%	17.1%	6.2%

Table 6.2: Case A, simply supported - 500×500. Failure in percent.

The first buckling mode for case A-1 ( $500 \times 500$ ) is plotted in figure (6.1).

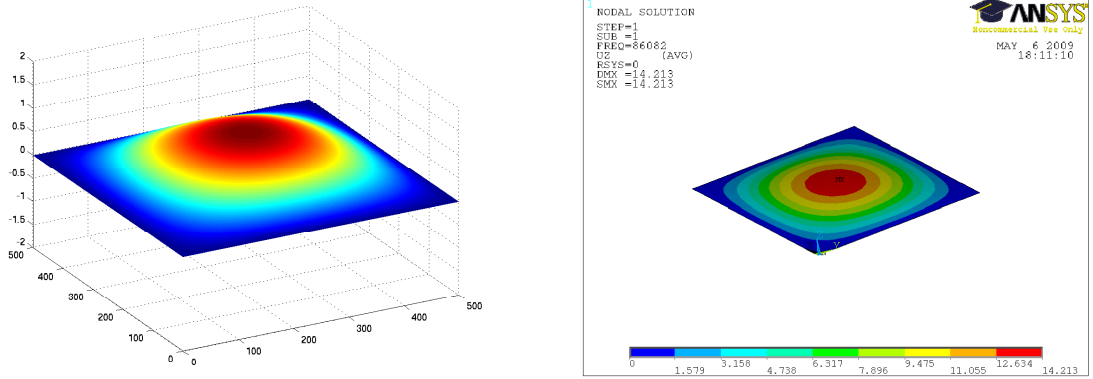


Figure 6.1: Left one shows first buckling mode for case A1, simply supported plate,  $500 \times 500 \text{ mm}$ , based on FSDT. Right one is from ANSYS.

Now, to show the results graphically, it is been chosen to introduce two new parameters, *affine plate buckling coefficient*,  $k_0$ , and *generalized rigidity ratio*,  $D^*$  [8]:

$$k_0 = \frac{N_{cr} b^2}{\pi^2 \sqrt{D_{11} D_{22}}}$$

$$D^* = \frac{D_{12} + 2D_{66}}{\sqrt{D_{11} D_{22}}}$$

The modified buckling coefficient is given by:

$$k_0 - 2D^* \tag{6.1}$$

Further, equation (6.1) is shown in figure below, plotted against  $b/t$ .

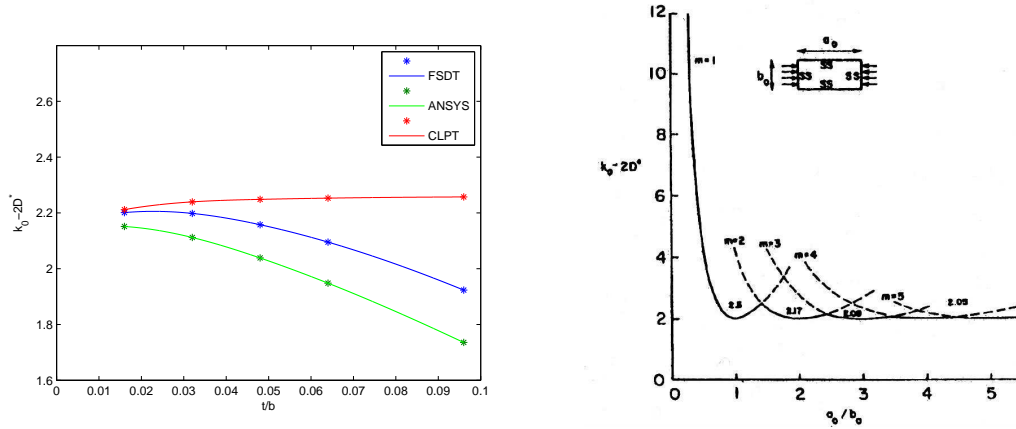


Figure 6.2: Left one shows simply supported plate, case A,  $500 \times 500 \text{ mm}$ . Uniaxial buckling coefficient (modified)  $k_0 - 2D^*$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8]. Here modified buckling coefficient is plotted against plate affine aspect ratio  $a_0/b_0$  using CLPT.

According to Brunelle and Oyibo, we should have got a straight horizontal line with modified buckling coefficient equals to 2 on CLPT. But for this lay-up, we have got a  $D_{11}$  and  $D_{22}$  which result a plate affine ratio smaller than 1. Maybe this is because the lay-up contains many nonzero degree plies. For example, for case A1  $500 \times 500$ :

$$D_{11} = 1.8820 \times 10^6 \quad , \quad D_{22} = 0.7557 \times 10^6$$

$$\Rightarrow \frac{a_0}{b_0} = \frac{a}{(D_{11})^{1/4}} \frac{(D_{22})^{1/4}}{b} \approx 0.8$$

This gives us a modified buckling coefficient greater than 2. The results from FSDT and ANSYS are under CLPT, which is logical.

## CASE A - 2000×500

<i>Case A - Simply supported, 2000 × 500mm</i>			
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 8.0mm$	165.7 (n=1,m=3)	165.4 (n=1,m=3)	163.9
$t = 16.0mm$	1284.5 (n=1,m=3)	1274.8 (n=1,m=3)	1253.7
$t = 24.0mm$	4291.7 (n=1,m=3)	4219.9 (n=1,m=3)	4119.8
$t = 32.0mm$	10123.0 (n=1,m=3)	9826.7 (n=1,m=3)	9527.2
$t = 48.0mm$	33998.0 (n=1,m=3)	31850.0 (n=1,m=3)	30494.0

Table 6.3: Case A, simply supported - 2000×500.

<i>Failure in percent - Simply supported, A 2000 × 500mm</i>			
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 8.0mm$	0.2%	1.1%	0.9%
$t = 16.0mm$	0.8%	2.5%	1.7%
$t = 24.0mm$	1.7%	4.2%	2.4%
$t = 32.0mm$	3.0%	6.3%	3.1%
$t = 48.0mm$	6.7%	11.5%	4.4%

Table 6.4: Case A, simply supported - 2000×500. Failure in percent.

For 2000 × 500, it is chosen to show the first buckling mode for case A-4.

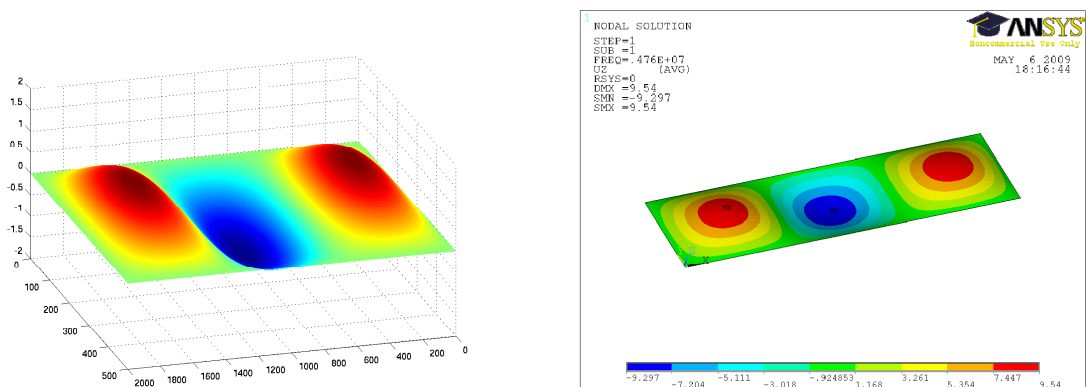


Figure 6.3: Left one shows first buckling mode for case A4, simply supported plate, 2000×500mm, based on FSDT. Right one is from ANSYS.

Again, equation (6.1) has been used to show the results graphically.



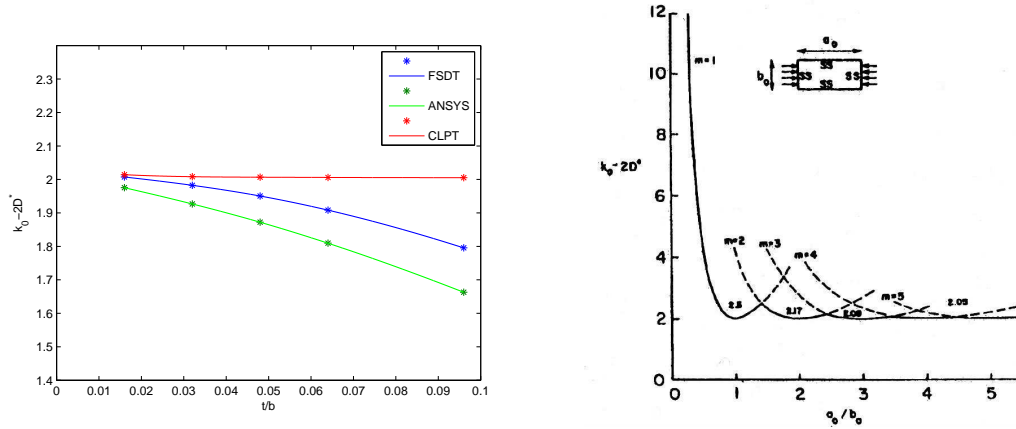


Figure 6.4: Left one shows simply supported plate, case A,  $2000 \times 500 \text{mm}$ . Uniaxial buckling coefficient (modified)  $k_0 - 2D^*$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

We see that the results for CLPT  $2000 \times 500$  give a better match. (For case A1,  $a_0/b_0 \approx 3.2$ . According to the article,  $k_0 - 2D^*$  should be a little higher than 2.)

### CASE B - $500 \times 500$

	<i>Case B - Simply supported, <math>500 \times 500 \text{mm}</math></i>		
Plate thickness	CLPT [ $N/\text{mm}$ ]	FSDT [ $N/\text{mm}$ ]	ANSYS [ $N/\text{mm}$ ]
$t = 8.0 \text{mm}$	179.4 (n=1,m=1)	178.9 (n=1,m=1)	173.8
$t = 16.0 \text{mm}$	1484.2 (n=1,m=1)	1467.1 (n=1,m=1)	1412.4
$t = 24.0 \text{mm}$	5064.3 (n=1,m=1)	4933.9 (n=1,m=1)	4678.8
$t = 32.0 \text{mm}$	12070.0 (n=1,m=1)	11526.0 (n=1,m=1)	10766.0
$t = 48.0 \text{mm}$	40955.0 (n=1,m=1)	37007.0 (n=1,m=1)	33688.0

Table 6.5: Case B, simply supported -  $500 \times 500$ .

	<i>Failure in percent - Simply supported, B <math>500 \times 500 \text{mm}</math></i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 8.0 \text{mm}$	0.3%	3.2%	2.9%
$t = 16.0 \text{mm}$	1.2%	5.1%	3.9%
$t = 24.0 \text{mm}$	2.6%	8.2%	5.5%
$t = 32.0 \text{mm}$	4.7%	12.1%	7.1%
$t = 48.0 \text{mm}$	10.7%	21.6%	9.9%

Table 6.6: Case B, simply supported -  $500 \times 500$ . Failure in percent.

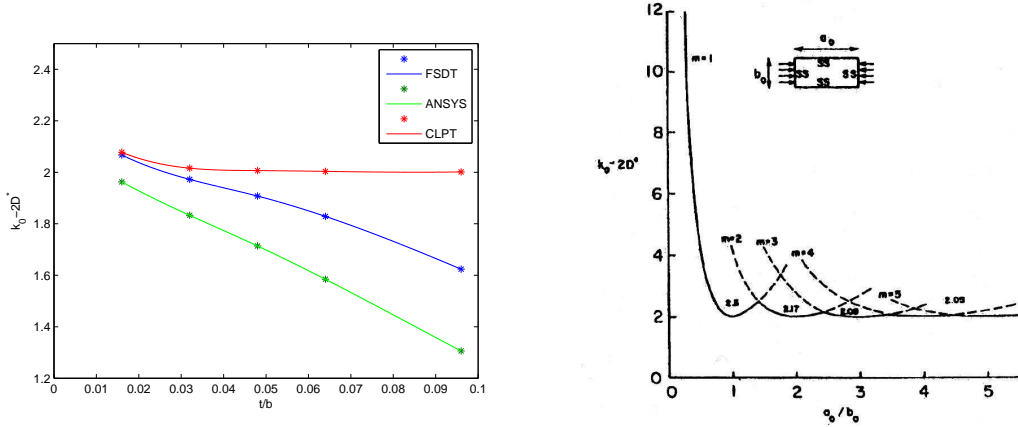


Figure 6.5: Left one shows simply supported plate, case B,  $500 \times 500 \text{ mm}$ . Uniaxial buckling coefficient (modified)  $k_0 - 2D^*$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

For case B1,  $500 \times 500$ :

$$D_{11} = 1.6413 \times 10^6 \quad , \quad D_{22} = 0.9412 \times 10^6$$

$$\Rightarrow \frac{a_0}{b_0} = \frac{a}{(D_{11})^{1/4}} \frac{(D_{22})^{1/4}}{b} \approx 0.87$$

which results  $k_0 - 2D^*$  greater than 2 according to Brunelle and Oyibo. For CLPT plotted in Matlab, we see that case B1 lies little higher than 2.

### CASE B - $2000 \times 500$

	Case B - Simply supported, $2000 \times 500 \text{ mm}$		
Plate thickness	CLPT [ $N/\text{mm}$ ]	FSDT [ $N/\text{mm}$ ]	ANSYS [ $N/\text{mm}$ ]
$t = 8.0 \text{ mm}$	179.4 (n=1,m=4)	178.9 (n=1,m=4)	173.9
$t = 16.0 \text{ mm}$	1484.2 (n=1,m=4)	1467.1 (n=1,m=4)	1428.7
$t = 24.0 \text{ mm}$	5064.3 (n=1,m=4)	4933.9 (n=1,m=4)	4763.2
$t = 32.0 \text{ mm}$	12070.0 (n=1,m=4)	11526.0 (n=1,m=4)	11018.0
$t = 48.0 \text{ mm}$	40955.0 (n=1,m=4)	37007 (n=1,m=4)	34736.0

Table 6.7: Case B, simply supported -  $2000 \times 500$ .

	<i>Failure in percent - Simply supported, B 2000 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 8.0mm$	0.3%	3.2%	2.9%
$t = 16.0mm$	1.2%	3.9%	2.7%
$t = 24.0mm$	2.6%	6.3%	3.6%
$t = 32.0mm$	4.7%	9.5%	4.6%
$t = 48.0mm$	10.7%	17.9%	6.5%

Table 6.8: Case B, simply supported - 2000×500. Failure in percent.

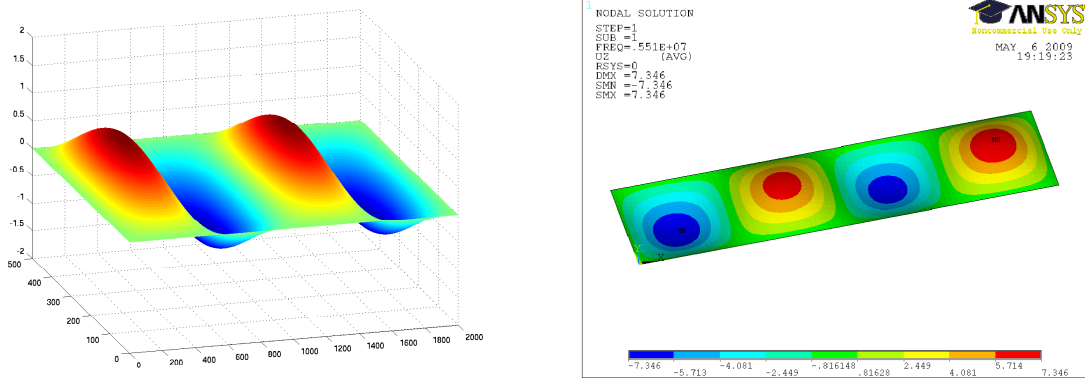


Figure 6.6: Left one shows first buckling mode for case B4, simply supported plate, 2000×500mm, based on FSDT. Right one is from ANSYS.

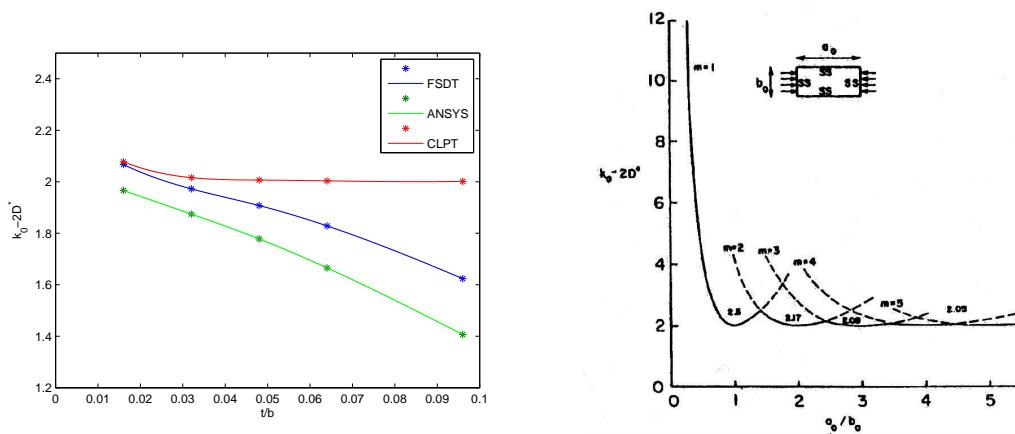


Figure 6.7: Left one shows simply supported plate, case B, 2000×500mm. Uniaxial buckling coefficient (modified)  $k_0 - 2D^*$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

For case B1  $2000 \times 500$ ,  $a_0/b_0 \approx 3.5$ . This value can result a modified buckling coefficient greater than 2 according to Brunelle and Oyibo.

### CASE C - $500 \times 500$

<i>Case C - Simply supported, <math>500 \times 500mm</math></i>			
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 49.98mm$	22787.0 (n=1,m=1)	21638.0 (n=1,m=1)	20634.0
$t = 42.14mm$	13658.0 (n=1,m=1)	13161.0 (n=1,m=1)	12630.2
$t = 28.42mm$	4189.6 (n=1,m=1)	4118.8 (n=1,m=1)	4001.6
$t = 21.56mm$	1829.1 (n=1,m=1)	1811.2 (n=1,m=1)	1771.4
$t = 16.66mm$	844.0 (n=1,m=1)	839 (n=1,m=1)	824.6
$t = 9.80mm$	171.8 (n=1,m=1)	171.4 (n=1,m=1)	169.7

Table 6.9: Case C, simply supported -  $500 \times 500$ .

<i>Failure in percent - Simply supported, C <math>500 \times 500mm</math></i>			
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 49.98mm$	5.3%	10.4%	4.9%
$t = 42.14mm$	3.8%	8.1%	4.2%
$t = 28.42mm$	1.7%	4.7%	2.9%
$t = 21.56mm$	1.0%	3.3%	2.2%
$t = 16.66mm$	0.6%	2.4%	1.7%
$t = 9.80mm$	0.2%	1.2%	1.0%

Table 6.10: Case C, simply supported -  $500 \times 500$ . Failure in percent.

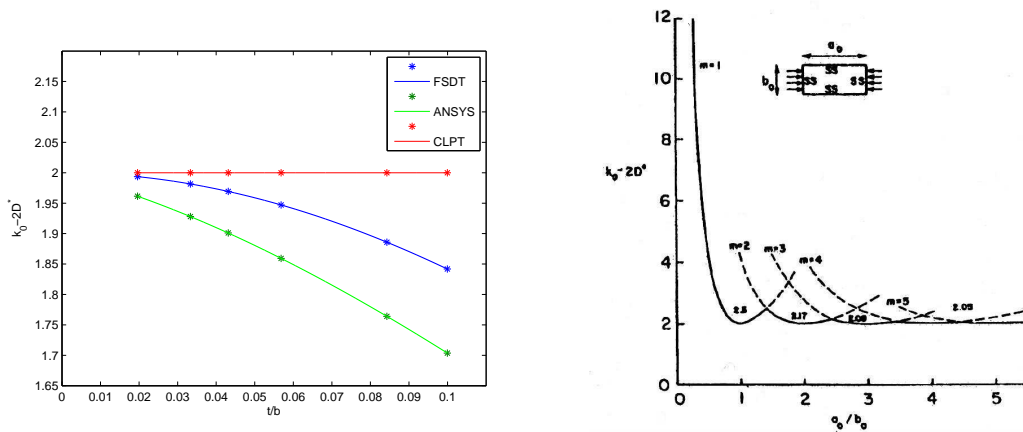


Figure 6.8: Left one shows simply supported plate, case C,  $500 \times 500mm$ . Uniaxial buckling coefficient (modified)  $k_0 - 2D^*$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

Since  $a_0/b_0$  equals to 1 for all thicknesses, we will now have a straight line with value 2 for CLPT. The result is in accordance with Brunelle and Oyibo.

### CASE C - 2000×500

	<i>Case C - Simply supported, 2000 × 500mm</i>		
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 49.98mm$	22787.0 (n=1,m=4)	21638.0 (n=1,m=4)	20974.0
$t = 42.14mm$	13658.0 (n=1,m=4)	13161.0 (n=1,m=4)	12813.2
$t = 28.42mm$	4189.6 (n=1,m=4)	4118.8 (n=1,m=4)	4043.2
$t = 21.56mm$	1829.1 (n=1,m=4)	1811.2 (n=1,m=4)	1785.7
$t = 16.66mm$	844.0 (n=1,m=4)	839 (n=1,m=4)	829.8
$t = 9.80mm$	171.8 (n=1,m=4)	171.4 (n=1,m=4)	170.3

Table 6.11: Case C, simply supported - 2000×500.

	<i>Failure in percent - Simply supported, C 2000 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 49.98mm$	5.3%	8.6%	3.2%
$t = 42.14mm$	3.8%	6.6%	2.7%
$t = 28.42mm$	1.7%	3.6%	1.9%
$t = 21.56mm$	1.0%	2.4%	1.4%
$t = 16.66mm$	0.6%	1.7%	1.1%
$t = 9.80mm$	0.2%	0.9%	0.6%

Table 6.12: Case C, simply supported - 2000×500. Failure in percent.

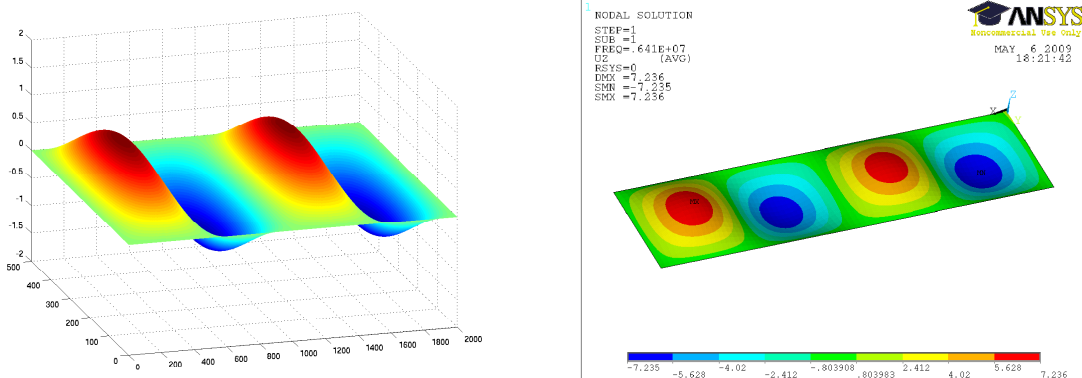


Figure 6.9: Left one shows first buckling mode for case C2, simply supported plate, 2000×500mm, based on FSDT. Right one is from ANSYS.

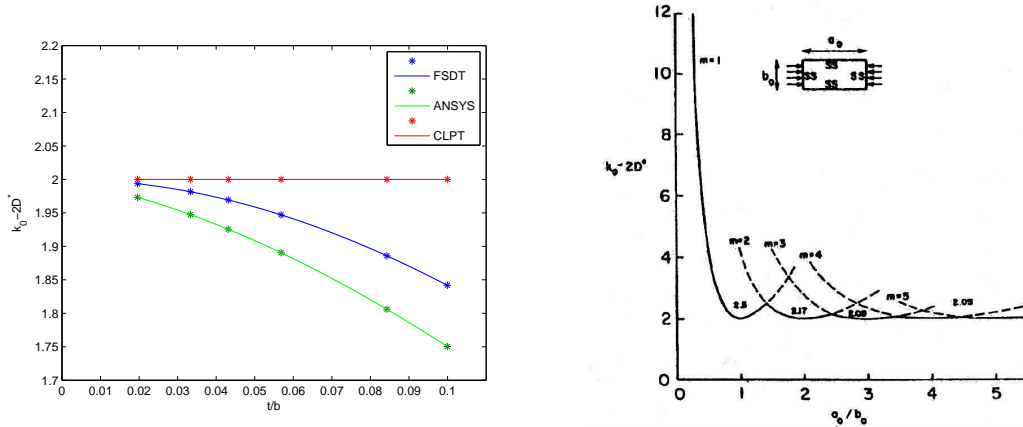


Figure 6.10: Left one shows simply supported plate, case C,  $2000 \times 500mm$ . Uniaxial buckling coefficient (modified)  $k_0 - 2D^*$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

$a_0/b_0$  equals to 4 for all thicknesses, we will now have a straight line with value 2 for CLPT. The result is in accordance with Brunelle and Oyibo.

### 6.1.2 Clamped Plates

For clamped edges, it is been chosen to show the results in tables and their buckling modes.

#### CASE A - $500 \times 500$

	<i>Case A - Clamped edges, <math>500 \times 500mm</math></i>		
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 8.0mm$	511.1 (n=1,m=1)	508.3 (n=1,m=1)	491.1
$t = 16.0mm$	4088.5 (n=1,m=1)	3999.3 (n=1,m=1)	3817.6
$t = 24.0mm$	13799.0 (n=1,m=1)	13132.0 (n=1,m=1)	12276.2
$t = 32.0mm$	32708.0 (n=1,m=1)	29960.0 (n=1,m=1)	27282.0
$t = 48.0mm$	110390.0 (n=1,m=1)	90885.0 (n=1,m=1)	77936.0

Table 6.13: Case A, clamped edges -  $500 \times 500$ .

	<i>Failure in percent - Clamped edges, A 500 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 8.0mm$	0.6%	4.1%	3.5%
$t = 16.0mm$	2.2%	7.1%	4.8%
$t = 24.0mm$	5.1%	12.4%	7.0%
$t = 32.0mm$	9.2%	19.9%	9.8%
$t = 48.0mm$	21.5%	41.6%	16.6%

Table 6.14: Case A, clamped edges - 500×500. Failure in percent.

Buckling mode for case A3 is shown below.

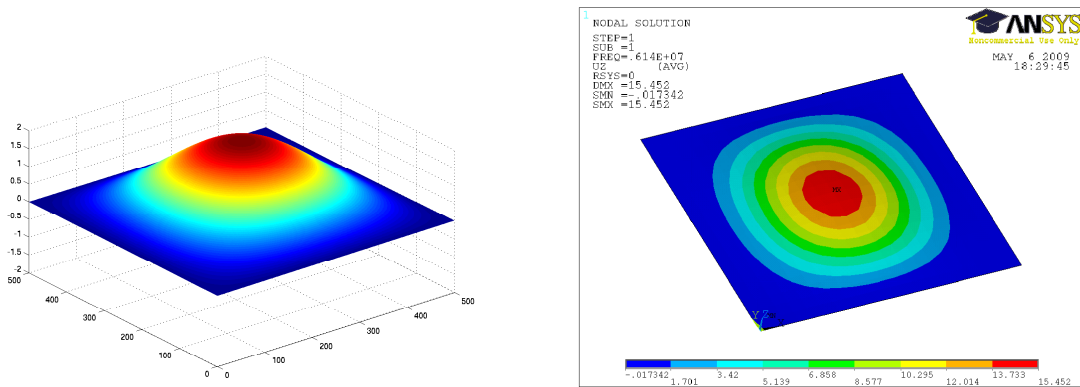


Figure 6.11: Left one shows first buckling mode for case A3, clamped plate, 500×500mm, based on FSDT. Right one is from ANSYS.

### CASE A - 2000×500

	<i>Case A - Clamped edges, 2000 × 500mm</i>		
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 8.0mm$	331.1 (n=1,m=5)	329.5 (n=5,m=5)	315.9
$t = 16.0mm$	2610.2 (n=1,m=5)	2561.2 (n=5,m=5)	2438.8
$t = 24.0mm$	8769.1 (n=1,m=5)	8406.8 (n=5,m=5)	7945.6
$t = 32.0mm$	20740.0 (n=1,m=5)	19254.0 (n=5,m=5)	18045.4
$t = 48.0mm$	69843.0 (n=1,m=5)	59323.0 (n=5,m=5)	54548.0

Table 6.15: Case A, clamped edges - 2000×500.

	<i>Failure in percent - Clamped edges, A 2000 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 8.0mm$	0.5%	4.8%	4.3%
$t = 16.0mm$	1.9%	7.0%	5.0%
$t = 24.0mm$	4.3%	10.4%	5.8%
$t = 32.0mm$	7.7%	14.9%	6.7%
$t = 48.0mm$	17.7%	28.0%	8.8%

Table 6.16: Case A, clamped edges - 2000×500. Failure in percent.

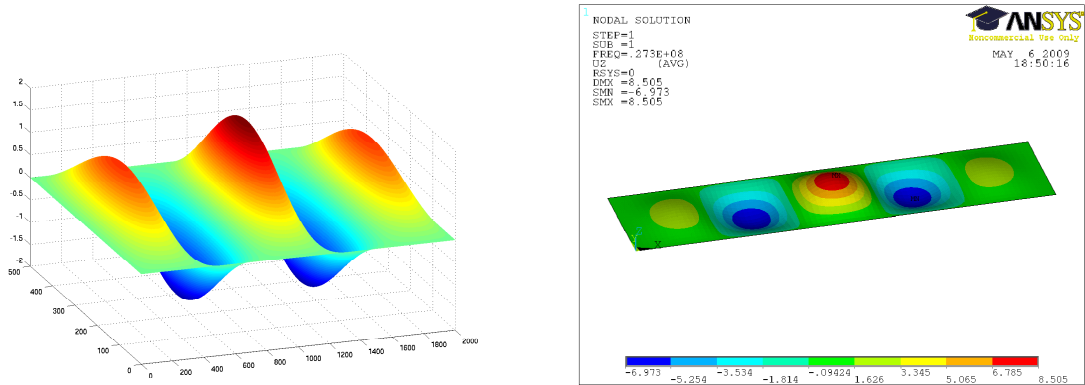


Figure 6.12: Left one shows first buckling mode for case A5, clamped plate, 2000×500mm, based on FSDT. Right one is from ANSYS.

### CASE B - 500×500

	<i>Case B - Clamped edges, 500 × 500mm</i>		
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 8.0mm$	511.1 (n=1,m=1)	508.3 (n=1,m=1)	482.4
$t = 16.0mm$	4088.5 (n=1,m=1)	3999.3 (n=1,m=1)	3729.6
$t = 24.0mm$	13799.0 (n=1,m=1)	13132.0 (n=1,m=1)	11935.8
$t = 32.0mm$	32708.0 (n=1,m=1)	29960.0 (n=1,m=1)	26388.0
$t = 48.0mm$	110390.0 (n=1,m=1)	89199.0 (n=2,m=2)	74482.0

Table 6.17: Case B, clamped edges - 500×500.



	<i>Failure in percent - Clamped edges, B 500 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 8.0mm$	0.6%	5.9%	5.4%
$t = 16.0mm$	2.2%	9.6%	7.2%
$t = 24.0mm$	5.1%	15.6%	10.0%
$t = 32.0mm$	9.2%	24.0%	13.5%
$t = 48.0mm$	23.8%	48.2%	19.8%

Table 6.18: Case B, clamped edges - 500×500. Failure in percent.

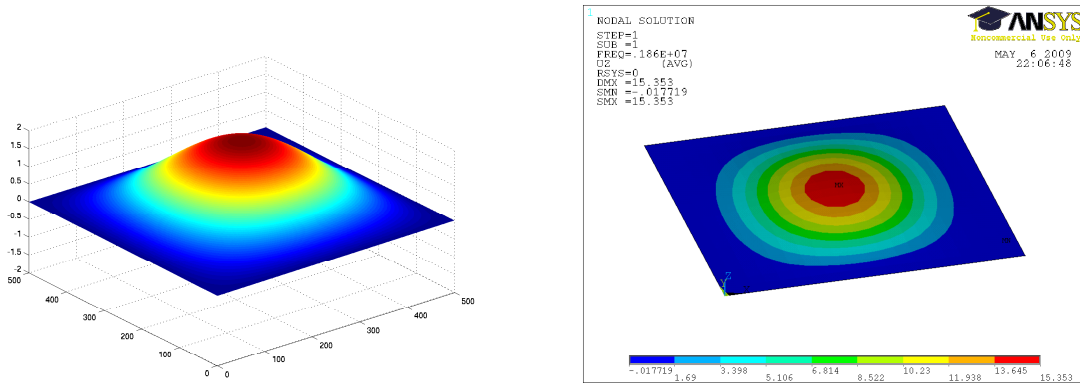


Figure 6.13: Left one shows first buckling mode for case B2, clamped plate, 500×500mm, based on FSDT. Right one is from ANSYS.

### CASE B - 2000×500

	<i>Case B - Clamped edges, 2000 × 500mm</i>		
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 8.0mm$	346.1 (n=1,m=5)	344.3 (n=5,m=5)	324.0
$t = 16.0mm$	2859.7 (n=1,m=6)	2792.9 (n=6,m=6)	2621.2
$t = 24.0mm$	9679.8 (n=1,m=6)	9179.8 (n=6,m=6)	8542.2
$t = 32.0mm$	22979.0 (n=1,m=6)	20927.0 (n=6,m=6)	19155.0
$t = 48.0mm$	77673.0 (n=1,m=6)	63294.0 (n=6,m=6)	55898.0

Table 6.19: Case B, clamped edges - 2000×500.

	<i>Failure in percent - Clamped edges, B 2000 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 8.0mm$	0.5%	6.8%	6.3%
$t = 16.0mm$	2.4%	9.1%	6.6%
$t = 24.0mm$	5.4%	13.3%	7.5%
$t = 32.0mm$	9.8%	20.0%	9.3%
$t = 48.0mm$	22.7%	39.0%	13.2%

Table 6.20: Case B, clamped edges - 2000×500. Failure in percent.

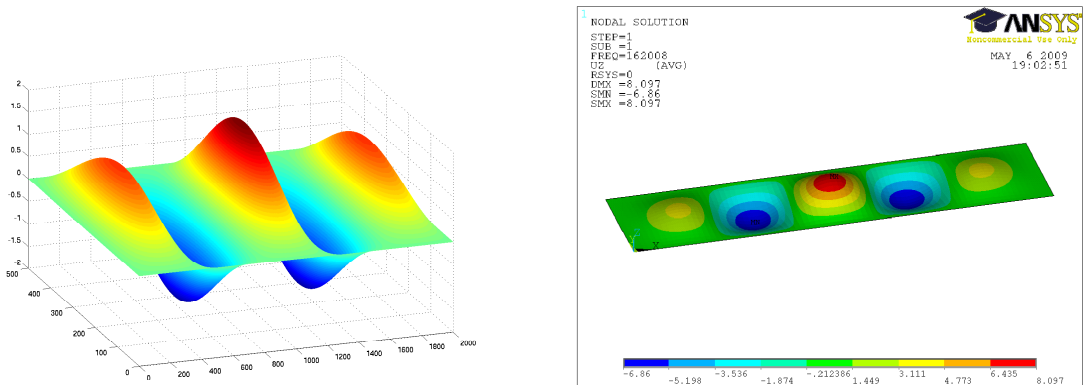


Figure 6.14: Left one shows first buckling mode for case B1, clamped plate, 2000×500mm, based on FSDT. Right one is from ANSYS.

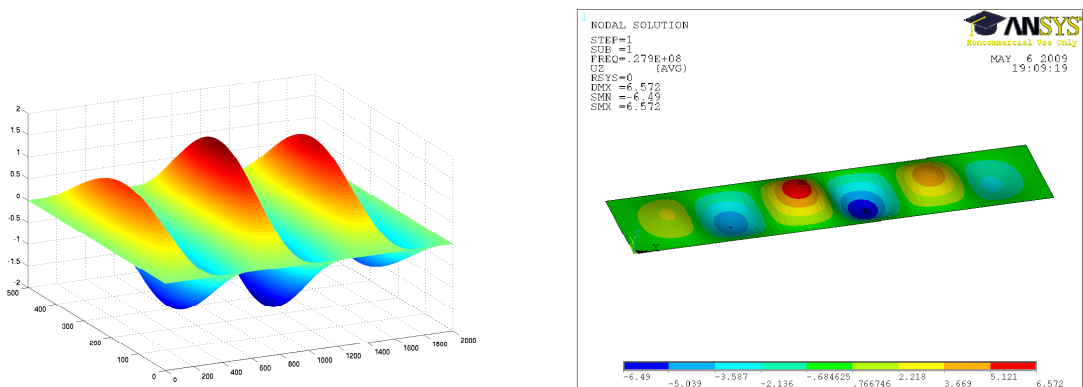


Figure 6.15: Left one shows first buckling mode for case B5, clamped plate, 2000×500mm, based on FSDT. Right one is from ANSYS.

## CASE C - 500×500

	<i>Case C - Clamped edges, 500 × 500mm</i>		
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 49.98mm$	69137.0 (n=1,m=1)	61910.0 (n=1,m=1)	53718.0
$t = 42.14mm$	41439.0 (n=1,m=1)	38299.0 (n=1,m=1)	34012.0
$t = 28.42mm$	12711.0 (n=1,m=1)	12262.0 (n=1,m=1)	11272.2
$t = 21.56mm$	5549.7 (n=1,m=1)	5435.8 (n=1,m=1)	5064.2
$t = 16.66mm$	2560.6 (n=1,m=1)	2529.1 (n=1,m=1)	2374.2
$t = 9.80mm$	521.2 (n=1,m=1)	519.0 (n=1,m=1)	490.9

Table 6.21: Case C, clamped edges - 500×500.

	<i>Failure in percent - Clamped edges, C 500 × 500mm</i>		
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 49.98mm$	11.7%	28.7%	15.3%
$t = 42.14mm$	8.2%	21.8%	12.6%
$t = 28.42mm$	3.7%	12.8%	8.8%
$t = 21.56mm$	2.1%	9.6%	7.3%
$t = 16.66mm$	1.2%	7.9%	6.5%
$t = 9.80mm$	0.4%	6.2%	5.7%

Table 6.22: Case C, clamped edges - 500×500. Failure in percent.

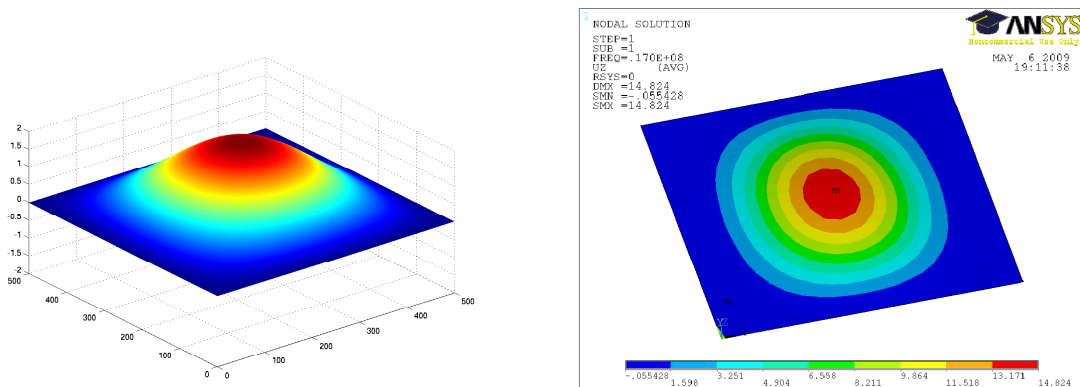


Figure 6.16: Left one shows first buckling mode for case C2, clamped plate, 500×500mm, based on FSDT. Right one is from ANSYS.

### CASE C - 2000×500

<i>Case C - Clamped edges, 2000 × 500mm</i>			
Plate thickness	CLPT [ $N/mm$ ]	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]
$t = 49.98mm$	46335.0 (n=1,m=6)	41424.0 (n=6,m=6)	38064.0
$t = 42.14mm$	27772.0 (n=1,m=6)	25630.0 (n=6,m=6)	23836.0
$t = 28.42mm$	8519.1 (n=1,m=6)	8210.8 (n=6,m=6)	7787.0
$t = 21.56mm$	3719.4 (n=1,m=6)	3641.0 (n=6,m=6)	3481.0
$t = 16.66mm$	1716.1 (n=1,m=6)	1694.4 (n=6,m=6)	1627.5
$t = 9.80mm$	349.3 (n=1,m=6)	347.8 (n=6,m=6)	335.6

Table 6.23: Case C, clamped edges - 2000×500.

<i>Failure in percent - Clamped edges, C 2000 × 500mm</i>			
Plate thickness	CLPT - FSDT	CLPT - ANSYS	FSDT - ANSYS
$t = 49.98mm$	11.9%	21.7%	8.8%
$t = 42.14mm$	8.4%	16.5%	7.5%
$t = 28.42mm$	3.8%	9.4%	5.4%
$t = 21.56mm$	2.2%	6.8%	4.6%
$t = 16.66mm$	1.3%	5.4%	4.1%
$t = 9.80mm$	0.4%	4.1%	3.6%

Table 6.24: Case C, clamped edges - 2000×500. Failure in percent.

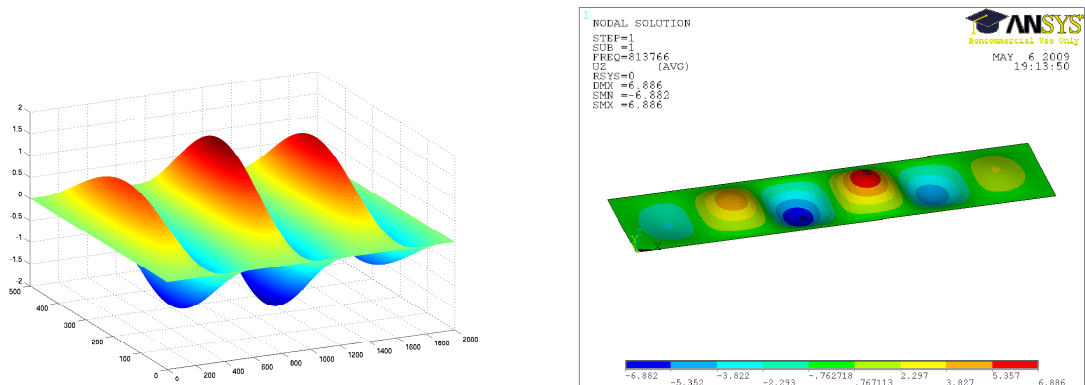


Figure 6.17: Left one shows first buckling mode for case C5, clamped plate, 2000×500mm, based on FSDT. Right one is from ANSYS.

## 6.2 Biaxial Compressive Load, FSDT

From section 4.3, equation (4.27) gives critical buckling load for biaxial compressive load. The results are estimated using the routines developed in Matlab.  $\zeta = 0.5$ ,  $\zeta = 1$  and  $\zeta = 2$  have been investigated.

### CASE A - 500×500

<i>Case A - Simply supported, 500 × 500mm, <math>\zeta = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	87.3 (n=1,m=1)	86.1	1.4%
$t = 16.0mm$	678.2 (n=1,m=1)	661.8	2.5%
$t = 24.0mm$	2242.4 (n=1,m=1)	2166.2	3.5%
$t = 32.0mm$	5196.9 (n=1,m=1)	4973.4	4.5%
$t = 48.0mm$	16579.0 (n=1,m=1)	15617.0	6.2%

Table 6.25: Case A, simply supported - 500×500.

<i>Case A - Simply supported, 500 × 500mm, <math>\zeta = 0.5</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	116.4 (n=1,m=1)	114.8	1.4%
$t = 16.0mm$	904.2 (n=1,m=1)	882.4	2.5%
$t = 24.0mm$	2989.8 (n=1,m=1)	2888.2	3.5%
$t = 32.0mm$	6929.2 (n=1,m=1)	6631.4	4.5%
$t = 48.0mm$	22106.0 (n=1,m=1)	20824.0	6.2%

Table 6.26: Case A, simply supported - 500×500.

<i>Case A - Simply supported, 500 × 500mm, <math>\zeta = 2</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	58.2 (n=1,m=1)	57.4	1.4%
$t = 16.0mm$	452.1 (n=1,m=1)	441.2	2.5%
$t = 24.0mm$	1494.9 (n=1,m=1)	1444.1	3.5%
$t = 32.0mm$	3464.6 (n=1,m=1)	3315.4	4.5%
$t = 48.0mm$	11053.0 (n=1,m=1)	10410.8	6.2%

Table 6.27: Case A, simply supported - 500×500.

To show the results graphically, we have chosen interaction diagram. To do that, we need critical buckling load for  $N_y$ . Performing the same procedure as section

4.1, we arrive:

$$N_{y0cr} = \frac{C_1 C_5^2 + \alpha C_3 C_2^2 + \beta C_2^2 C_5 + C_3^2 C_4 - \alpha C_1 C_3 C_4 - \beta C_1 C_4 C_5 - 2C_2 C_3 C_5}{\beta^2 (C_1 C_4 - C_2^2)} \quad (6.2)$$

where

$$\alpha = \frac{m\pi}{a}, \quad \beta = \frac{n\pi}{b}, \quad C_1 = -D_{11}\alpha^2 - D_{66}\beta^2 - A_{55}k, \quad C_2 = -D_{12}\alpha\beta - D_{66}\alpha\beta$$

$$C_3 = -A_{55}k\alpha, \quad C_4 = -D_{22}\beta^2 - D_{66}\alpha^2 - A_{44}k, \quad C_5 = -A_{44}k\beta$$

Now, for interaction diagram,  $x$  axis denoted by  $N_{xcr}/N_{x0cr}$  and  $y$  axis denoted by  $N_{ycr}/N_{y0cr}$ .

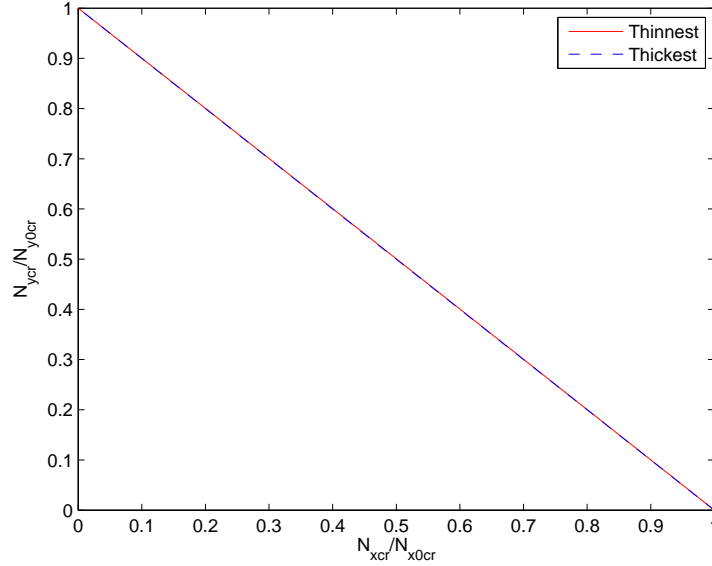


Figure 6.18: Interaction diagram for case A, Simply supported plate,  $500 \times 500 \text{mm}$ .

**CASE A - 2000×500**

<i>Case A - Simply supported, 2000 × 500mm, <math>\zeta = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	32.5 (n=1,m=1)	32.4	0.3%
$t = 16.0mm$	251.4 (n=1,m=1)	250.2	0.5%
$t = 24.0mm$	836.3 (n=1,m=1)	830.4	0.7%
$t = 32.0mm$	1959.9 (n=1,m=1)	1941.9	0.9%
$t = 48.0mm$	6461.0 (n=1,m=1)	6376.6	1.3%

Table 6.28: Case A, simply supported - 2000×500.

<i>Case A - Simply supported, 2000 × 500mm, <math>\zeta = 0.5</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	61.4 (n=1,m=1)	61.2	0.3%
$t = 16.0mm$	474.9 (n=1,m=1)	472.5	0.5%
$t = 24.0mm$	1579.7 (n=1,m=1)	1568.5	0.7%
$t = 32.0mm$	3702.0 (n=1,m=1)	3668.0	0.9%
$t = 48.0mm$	12204.0 (n=1,m=1)	12044.4	1.3%

Table 6.29: Case A, simply supported - 2000×500.

<i>Case A - Simply supported, 2000 × 500mm, <math>\zeta = 2</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	16.7 (n=1,m=1)	16.7	0%
$t = 16.0mm$	129.5 (n=1,m=1)	128.9	0.5%
$t = 24.0mm$	430.8 (n=1,m=1)	427.8	0.7%
$t = 32.0mm$	1009.6 (n=1,m=1)	1000.4	0.9%
$t = 48.0mm$	3328.4 (n=1,m=1)	3285.0	1.3%

Table 6.30: Case A, simply supported - 2000×500.

For case with 2000×500, the interaction diagram is not a straight line. We need to calculate  $N_{xcr}$  and  $N_{ycr}$  for more  $\zeta$  values to show the right curve.

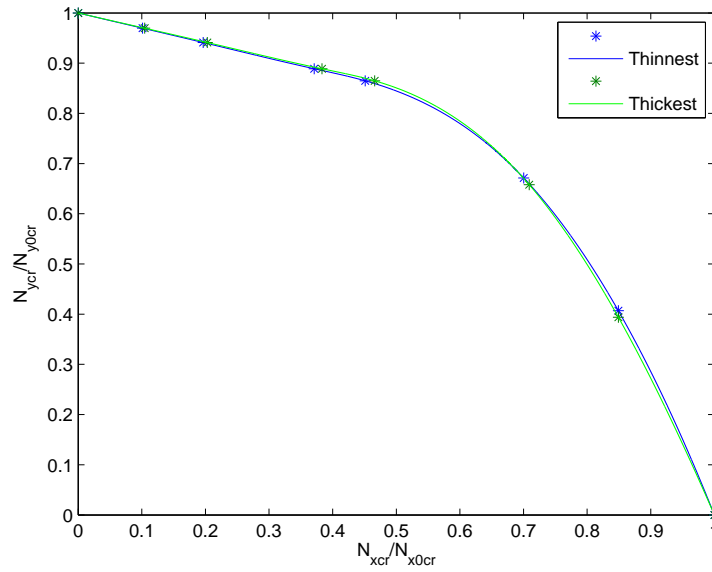


Figure 6.19: Interaction diagram for case A, Simply supported plate,  $2000 \times 500 \text{mm}$ .

### CASE B - $500 \times 500$

<i>Case B - Simply supported, <math>500 \times 500 \text{mm}</math>, <math>\zeta = 1</math></i>			
Plate thickness	FSDT [ $N/\text{mm}$ ]	ANSYS [ $N/\text{mm}$ ]	Failure in percent
$t = 8.0 \text{mm}$	89.4 (n=1,m=1)	86.9	2.9%
$t = 16.0 \text{mm}$	733.6 (n=1,m=1)	706.2	3.9%
$t = 24.0 \text{mm}$	2467.0 (n=1,m=1)	2339.4	5.5%
$t = 32.0 \text{mm}$	5762.8 (n=1,m=1)	5383.0	7.1%
$t = 48.0 \text{mm}$	18503.0 (n=1,m=1)	16845.2	9.8%

Table 6.31: Case B, simply supported -  $500 \times 500$ .



<i>Case B - Simply supported, 500 × 500mm, ζ = 0.5</i>			
Plate thickness	FSDT [ <i>N/mm</i> ]	ANSYS [ <i>N/mm</i> ]	Failure in percent
<i>t = 8.0mm</i>	119.3 (n=1,m=1)	115.8	3.0%
<i>t = 16.0mm</i>	978.1 (n=1,m=1)	941.6	3.9%
<i>t = 24.0mm</i>	3289.3 (n=1,m=1)	3119.2	5.5%
<i>t = 32.0mm</i>	7683.8 (n=1,m=1)	7177.4	7.1%
<i>t = 48.0mm</i>	24671.0 (n=1,m=1)	22460.0	9.8%

Table 6.32: Case B, simply supported - 500×500.

<i>Case B - Simply supported, 500 × 500mm, ζ = 2</i>			
Plate thickness	FSDT [ <i>N/mm</i> ]	ANSYS [ <i>N/mm</i> ]	Failure in percent
<i>t = 8.0mm</i>	59.6 (n=1,m=1)	57.9	2.9%
<i>t = 16.0mm</i>	489.0 (n=1,m=1)	470.8	3.9%
<i>t = 24.0mm</i>	1644.6 (n=1,m=1)	1559.6	5.5%
<i>t = 32.0mm</i>	3841.9 (n=1,m=1)	3588.6	7.1%
<i>t = 48.0mm</i>	12336.0 (n=1,m=1)	11230.0	9.8%

Table 6.33: Case B, simply supported - 500×500.

Interaction diagram has only been made for the thinnest and thickest plates.

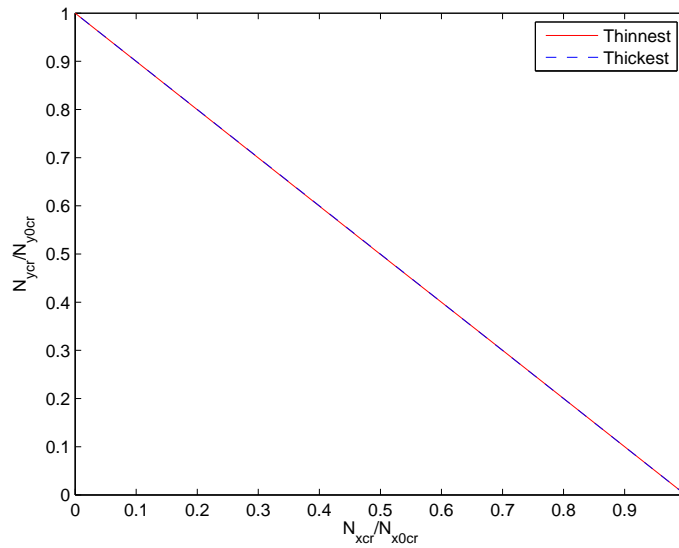


Figure 6.20: Interaction diagram for case B, Simply supported plate, 500×500mm.

**CASE B - 2000×500**

<i>Case B - Simply supported, 2000 × 500mm, <math>\zeta = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	39.7 (n=1,m=1)	39.4	0.8%
$t = 16.0mm$	365.8 (n=1,m=1)	363.4	0.7%
$t = 24.0mm$	1272.7 (n=1,m=1)	1262.1	0.8%
$t = 32.0mm$	3037.1 (n=1,m=1)	3004.8	1.1%
$t = 48.0mm$	10111.0 (n=1,m=1)	9958.4	1.5%

Table 6.34: Case B, simply supported - 2000×500.

<i>Case B - Simply supported, 2000 × 500mm, <math>\zeta = 0.5</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	75.0 (n=1,m=1)	74.5	0.7%
$t = 16.0mm$	690.9 (n=1,m=1)	686.5	0.6%
$t = 24.0mm$	2403.9 (n=1,m=1)	2383.8	0.8%
$t = 32.0mm$	5736.7 (n=1,m=1)	5675.2	1.1%
$t = 48.0mm$	19098 (n=1,m=1)	18804.6	1.6%

Table 6.35: Case B, simply supported - 2000×500.

<i>Case B - Simply supported, 2000 × 500mm, <math>\zeta = 2</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	20.5 (n=1,m=1)	20.3	1.0%
$t = 16.0mm$	188.4 (n=1,m=1)	187.2	0.6%
$t = 24.0mm$	655.6 (n=1,m=1)	650.2	0.8%
$t = 32.0mm$	1564.6 (n=1,m=1)	1548.0	1.1%
$t = 48.0mm$	5208.5 (n=1,m=1)	5130.4	1.5%

Table 6.36: Case B, simply supported - 2000×500.

For case B, 2000×500, more values of  $\zeta$  have been calculated and three thicknesses have been plotted.

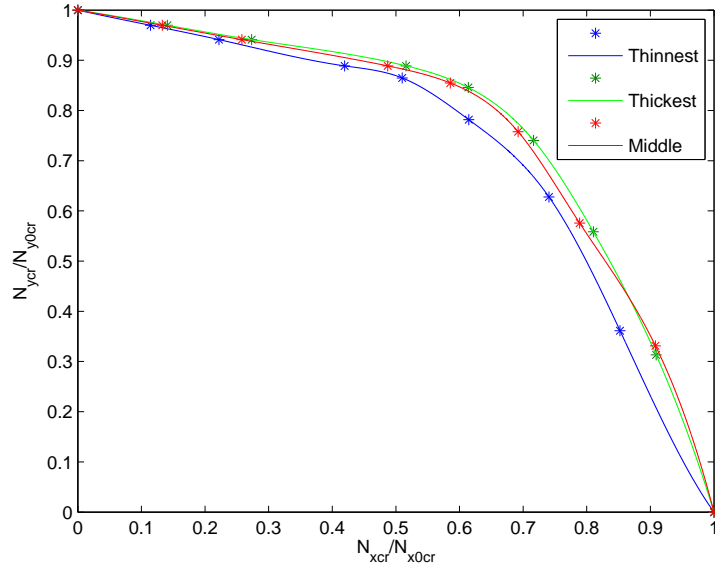


Figure 6.21: Interaction diagram for case B, Simply supported plate,  $2000 \times 500 \text{mm}$ .

Due to numerical approaches, the curves differ slightly from each other. But the difference is negligible.

### CASE C - $500 \times 500$

<i>Case C - Simply supported, <math>500 \times 500 \text{mm}</math>, <math>\zeta = 1</math></i>			
Plate thickness	FSDT [ $N/\text{mm}$ ]	ANSYS [ $N/\text{mm}$ ]	Failure in percent
$t = 49.98 \text{mm}$	10819.0 (n=1,m=1)	10316.8	4.9%
$t = 42.14 \text{mm}$	6580.4 (n=1,m=1)	6315.2	4.2%
$t = 28.42 \text{mm}$	2059.4 (n=1,m=1)	2000.8	2.9%
$t = 21.56 \text{mm}$	905.6 (n=1,m=1)	885.7	2.2%
$t = 16.66 \text{mm}$	419.5 (n=1,m=1)	412.3	1.7%
$t = 9.80 \text{mm}$	85.7 (n=1,m=1)	84.8	1.1%

Table 6.37: Case C, simply supported -  $500 \times 500$ .

	<i>Case C - Simply supported, 500 × 500mm, <math>\zeta = 0.5</math></i>		
Plate thickness	FSDT [ <i>N/mm</i> ]	ANSYS [ <i>N/mm</i> ]	Failure in percent
<i>t = 49.98mm</i>	14425 (n=1,m=1)	13755.8	4.9%
<i>t = 42.14mm</i>	8773.9 (n=1,m=1)	8420.2	4.2%
<i>t = 28.42mm</i>	2745.9 (n=1,m=1)	2667.8	2.9%
<i>t = 21.56mm</i>	1207.5 (n=1,m=1)	1180.9	2.3%
<i>t = 16.66mm</i>	559.3 (n=1,m=1)	549.7	1.7%
<i>t = 9.80mm</i>	114.3 (n=1,m=1)	113.1	1.1%

Table 6.38: Case C, simply supported - 500×500.

	<i>Case C - Simply supported, 500 × 500mm, <math>\zeta = 2</math></i>		
Plate thickness	FSDT [ <i>N/mm</i> ]	ANSYS [ <i>N/mm</i> ]	Failure in percent
<i>t = 49.98mm</i>	7212.5 (n=1,m=1)	6878.0	4.9%
<i>t = 42.14mm</i>	4386.9 (n=1,m=1)	4210.0	4.2%
<i>t = 28.42mm</i>	1372.9 (n=1,m=1)	1333.9	2.9%
<i>t = 21.56mm</i>	603.7 (n=1,m=1)	590.5	2.2%
<i>t = 16.66mm</i>	279.7 (n=1,m=1)	274.9	1.7%
<i>t = 9.80mm</i>	57.1 (n=1,m=1)	56.6	0.9%

Table 6.39: Case C, simply supported - 500×500.

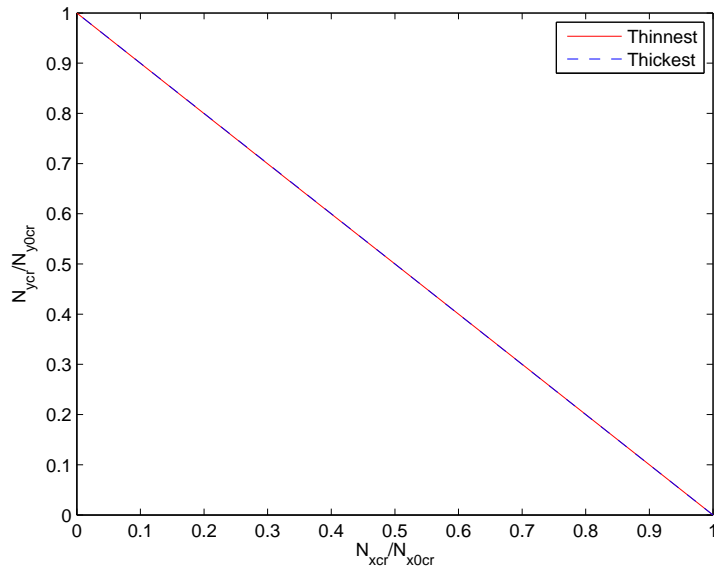


Figure 6.22: Interaction diagram for case C, Simply supported plate,  $500 \times 500 \text{mm}$ .

### CASE C - $2000 \times 500$

	<i>Case C - Simply supported, <math>2000 \times 500 \text{mm}</math>, <math>\zeta = 1</math></i>		
Plate thickness	FSDT [ $N/\text{mm}$ ]	ANSYS [ $N/\text{mm}$ ]	Failure in percent
$t = 49.98 \text{mm}$	7105.6 (n=1,m=1)	7064.6	0.6%
$t = 42.14 \text{mm}$	4300.5 (n=1,m=1)	4279.2	0.5%
$t = 28.42 \text{mm}$	1336.7 (n=1,m=1)	1332.1	0.3%
$t = 21.56 \text{mm}$	586.3 (n=1,m=1)	584.8	0.3%
$t = 16.66 \text{mm}$	271.2 (n=1,m=1)	270.7	0.2%
$t = 9.80 \text{mm}$	55.3 (n=1,m=1)	55.3	0%

Table 6.40: Case C, simply supported -  $2000 \times 500$ .

<i>Case C - Simply supported, 2000 × 500mm, <math>\zeta = 0.5</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 49.98mm$	12580.0 (n=1,m=2)	12348.8	1.9%
$t = 42.14mm$	7618.6 (n=1,m=2)	7499.2	1.6%
$t = 28.42mm$	2370.2 (n=1,m=2)	2344.6	1.1%
$t = 21.56mm$	1040.0 (n=1,m=2)	1031.5	0.8%
$t = 16.66mm$	481.2 (n=1,m=2)	478.1	0.6%
$t = 9.80mm$	98.2 (n=1,m=2)	97.8	0.4%

Table 6.41: Case C, simply supported - 2000×500.

<i>Case C - Simply supported, 2000 × 500mm, <math>\zeta = 2</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 49.98mm$	3660.5 (n=1,m=1)	3639.4	0.6%
$t = 42.14mm$	2215.4 (n=1,m=1)	2204.6	0.5%
$t = 28.42mm$	688.6 (n=1,m=1)	686.3	0.3%
$t = 21.56mm$	302.0 (n=1,m=1)	301.3	0.2%
$t = 16.66mm$	139.7 (n=1,m=1)	139.4	0.2%
$t = 9.80mm$	28.5 (n=1,m=1)	28.5	0%

Table 6.42: Case C, simply supported - 2000×500.

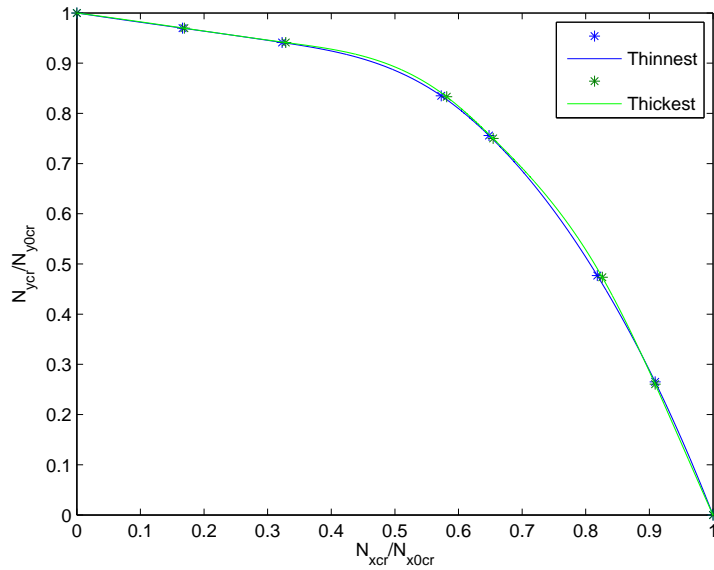


Figure 6.23: Interaction diagram for case C, Simply supported plate,  $2000 \times 500mm$ .

### 6.3 In-plane Shear Load, FSDT

For simply supported plates subjected to in-plane shear loads, the critical buckling load are found by equation (4.40) from section 4.4. Both groups of equations ( $m + n$  odd and  $m + n$  even, suggested by Timoshenko) will be considered, even for a shorter plate ( $a/b < 2$ ). The results are estimated using Matlab, and listed below for  $M = 5$  and  $N = 5$ , which means the total number of  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  terms are 75. For some cases it is been tested for 147 terms ( $M = N = 7$ ) and others 243 terms ( $M = N = 9$ ), but it makes heavy demands on Matlab. It could run hours without any results.

## CASE A - 500×500

Case A - Simply supported, 500 × 500mm			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	$\pm 404.0$ (N=5,M=5)	399.6	1.1%
$t = 16.0mm$	$\pm 3081.0$ (N=5,M=5)	3019.0	2.1%
$t = 24.0mm$	$\pm 9885.0$ (N=5,M=5)	9616.6	2.8%
$t = 32.0mm$	$\pm 22030.0$ (N=5,M=5)	21312.0	3.7%
$t = 48.0mm$	$\pm 63800.0$ (N=5,M=5)	61330.0	4.0%

Table 6.43: Case A, simply supported - 500×500.

To show the results graphically, we plot the buckling coefficient,  $k_0$ , against  $t/b$ . We know from section 6.1.1,  $k_0$  is given by:

$$k_0 = \frac{N_{cr} b^2}{\pi^2 \sqrt{D_{11} D_{22}}}$$

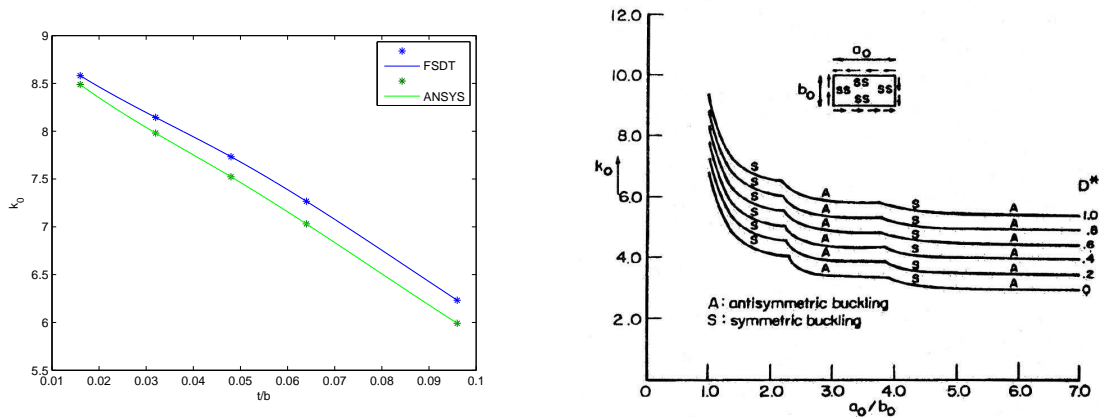


Figure 6.24: Left one shows simply supported plate, case A, 500×500mm. Shear buckling coefficient  $k_0$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8]. Here shear buckling coefficient is plotted against plate affine aspect ratio  $a_0/b_0$  using CLPT.

Generalized rigidity ratio,  $D^*$ , for case A1-5 are given by:

$$A1 \Rightarrow D^* = 0.7531$$

$$A2 \Rightarrow D^* = 0.6937$$

$$A3 \Rightarrow D^* = 0.6754$$

$$A4 \Rightarrow D^* = 0.6666$$



$$A5 \Rightarrow D^* = 0.6579$$

According to Brunelle and Oyibo, for those  $D^*$  values listed above, we should have got  $k_0$  around 8.5-9.0 with  $a_0/b_0$  around 1 using CLPT. We see that all the results, both FSDT and ANSYS, are beneath this value. Besides using FSDT gave us a really good approximation to ANSYS. We have got some useful results.

### CASE A - 2000×500

<i>Case A - Simply supported, 2000 × 500mm</i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	$\pm 198.9$ (N=5,M=5)	197.6	0.7%
$t = 16.0mm$	$\pm 1524.0$ (N=5,M=5)	1505.2	1.2%
$t = 24.0mm$	$\pm 5005.0$ (N=5,M=5)	4916.0	1.8%
$t = 32.0mm$	$\pm 11520.0$ (N=5,M=5)	11264.6	2.3%
$t = 48.0mm$	$\pm 36140.0$ (N=5,M=5)	35106.0	2.9%

Table 6.44: Case A, simply supported - 2000×500.

The plate with thickness  $t = 48.0mm$  has been tested for  $M = N = 7$ . We arrive at  $\pm 35970 N/mm$ , which is 0.47% better than the result with  $M = N = 5$ . But it took twice the running time in Matlab.

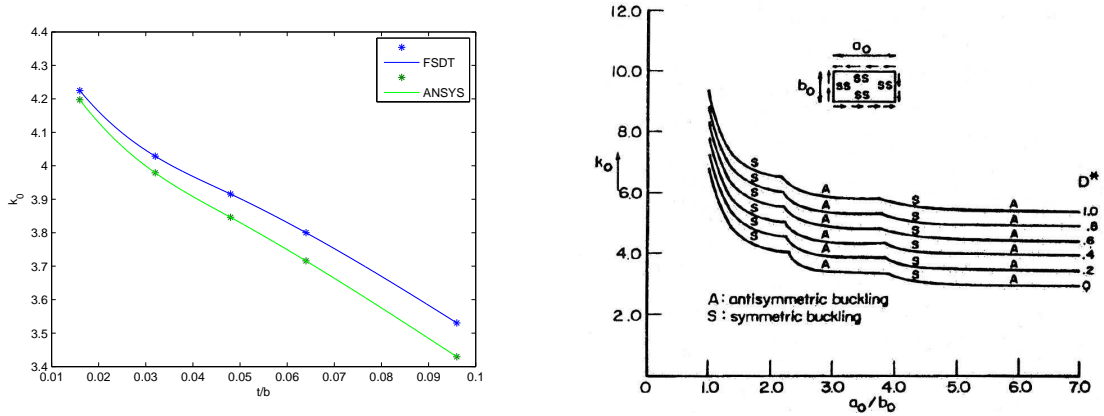


Figure 6.25: Left one shows simply supported plate, case A, 2000×500mm. Shear buckling coefficient  $k_0$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

According to Brunelle and Oyibo, we should have got  $k_0 \approx 5$  for  $a_0/b_0 \approx 3.2$  using CLPT. Both FSDT and ANSYS gave us a acceptable result.

## CASE B - 500×500

Case B - Simply supported, 500 × 500mm			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	$\pm 416.3$ (N=5,M=5)	354.6	17.4%
$t = 16.0mm$	$\pm 3347.0$ (N=5,M=5)	3013.2	11.1%
$t = 24.0mm$	$\pm 10898.0$ (N=5,M=5)	9964.8	9.4%
$t = 32.0mm$	$\pm 24416.0$ (N=5,M=5)	22454.0	8.7%
$t = 48.0mm$	$\pm 70700.0$ (N=5,M=5)	65436.0	8.0%

Table 6.45: Case B, simply supported - 500×500.

Case B1 ( $t = 8.0mm$ ) with  $M = N = 7$  gives  $\pm 415.6 N/mm$ . The result is still not good enough. We notice that the failure percentages for case B are really high compared with case A. This could be caused by  $D_{16}$  and  $D_{26}$ , which we assumed to be zero during the calculation. We can make a test to confirm the suspicion. Assume that we have a plate just like case B5, but the plate thickness and the ply thickness are reduced to 1/6 of the original values (like case B1). In this way the effects from  $D_{16}$  and  $D_{26}$  are reduced. From ANSYS we obtained  $420.2N/mm$  and from Matlab with  $M = N = 5$ , we arrived at  $\pm 439.1N/mm$ . The failure in percent is 4.5%. So the inaccuracy partially caused by neglecting of  $D_{16}$  and  $D_{26}$ .

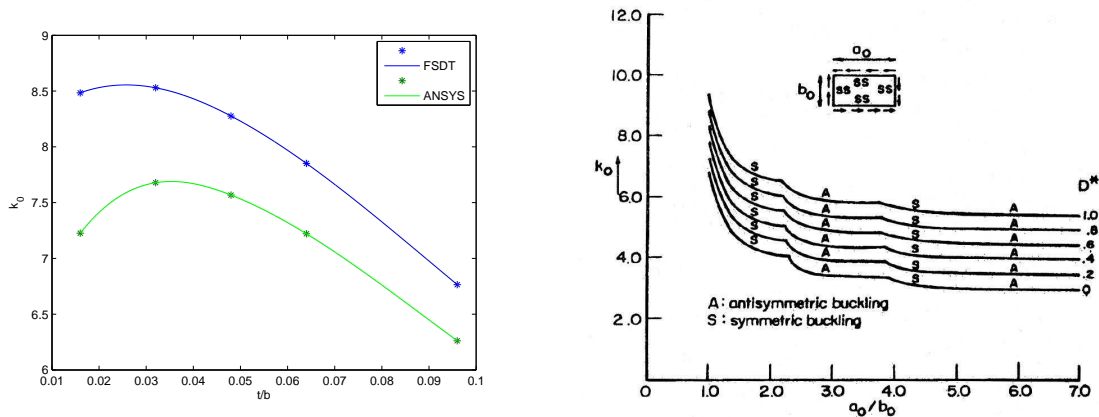


Figure 6.26: Left one shows simply supported plate, case B, 500×500mm. Shear buckling coefficient  $k_0$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

Generalized rigidity ratio,  $D^*$ , for case B1-5:

$$B1 \Rightarrow D^* = 0.7892$$

$$B2 \Rightarrow D^* = 0.8831$$

$$B3 \Rightarrow D^* = 0.9196$$

$$B4 \Rightarrow D^* = 0.9388$$

$$B5 \Rightarrow D^* = 0.9586$$

Using CLPT, we should have got shear buckling coefficients around 9.0-9.5 for  $a_0/b_0 \approx 1$  according to Brunelle and Oyibo.

### CASE B - 2000×500

	Case B - Simply supported, 2000 × 500mm		
Plate thickness	FSDT [N/mm]	ANSYS [N/mm]	Failure in percent
$t = 8.0mm$	$\pm 226.1$ (N=5,M=5)	192.5	17.5%
$t = 16.0mm$	$\pm 1971.0$ (N=5,M=5)	1796.8	9.7%
$t = 24.0mm$	$\pm 6667.0$ (N=5,M=5)	6197.2	7.6%
$t = 32.0mm$	$\pm 15450.0$ (N=5,M=5)	14473.4	6.7%
$t = 48.0mm$	$\pm 48030.0$ (N=5,M=5)	45144.0	6.4%

Table 6.46: Case B, simply supported - 2000×500.

Here, the inaccuracy partially caused by neglecting of  $D_{16}$  and  $D_{26}$ .

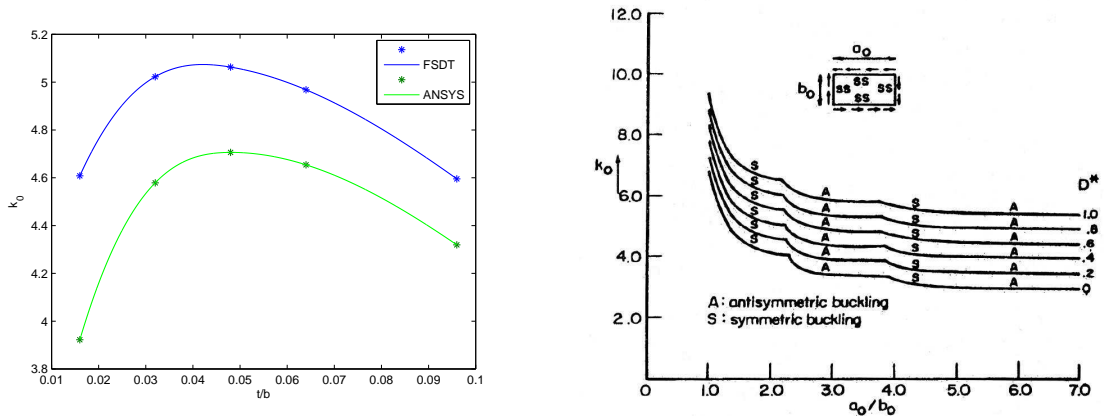


Figure 6.27: Left one shows simply supported plate, case B, 2000×500mm. Shear buckling coefficient  $k_0$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

According to Brunelle and Oyibo, we should have got  $k_0$  around 5.5-6 for  $a_0/b_0$  around 3.5-4 using CLPT.

### CASE C - 500×500

Case C - Simply supported, 500 × 500mm			
Plate thickness	FSDT [N/mm]	ANSYS [N/mm]	Failure in percent
$t = 49.98mm$	$\pm 45810.0$ (N=5,M=5)	44190.0	3.7%
$t = 42.14mm$	$\pm 28710.0$ (N=5,M=5)	27790.0	3.3%
$t = 28.42mm$	$\pm 9383.0$ (N=5,M=5)	9153.8	2.5%
$t = 21.56mm$	$\pm 4194.0$ (N=5,M=5)	4111.4	2.0%
$t = 16.66mm$	$\pm 1960.0$ (N=5,M=5)	1929.2	1.6%
$t = 9.80mm$	$\pm 404.2$ (N=5,M=5)	400.0	1.1%

Table 6.47: Case C, simply supported - 500×500.

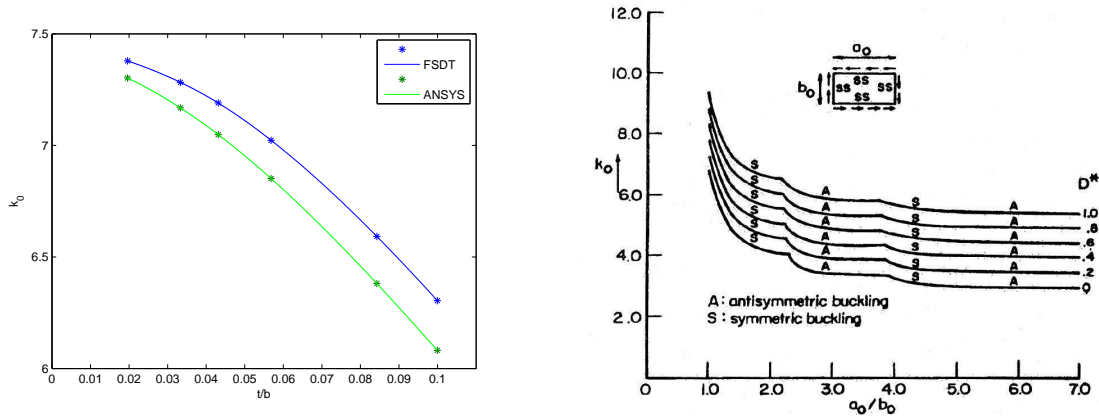


Figure 6.28: Left one shows simply supported plate, case C, 500×500mm. Shear buckling coefficient  $k_0$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

For case C1-6,  $D^* = 0.5679$  for all thicknesses. Thereby according to fig (6.28) by Brunelle and Oyibo,  $k_0 \approx 8$  for  $a_0/b_0 = 1$ . We see that both curves from FSDT and ANSYS are beneath this value.

## CASE C - 2000×500

Case C - Simply supported, 2000 × 500mm			
Plate thickness	FSDT [N/mm]	ANSYS [N/mm]	Failure in percent
$t = 49.98mm$	$\pm 31400.0$ (N=5,M=5)	30578.0	2.7%
$t = 42.14mm$	$\pm 19360.0$ (N=5,M=5)	18911.6	2.4%
$t = 28.42mm$	$\pm 6181.0$ (N=5,M=5)	6066.4	1.9%
$t = 21.56mm$	$\pm 2738.0$ (N=5,M=5)	2694.0	1.6%
$t = 16.66mm$	$\pm 1274.0$ (N=5,M=5)	1255.2	1.5%
$t = 9.80mm$	$\pm 261.3$ (N=5,M=5)	258.1	1.2%

Table 6.48: Case C, simply supported - 2000×500.

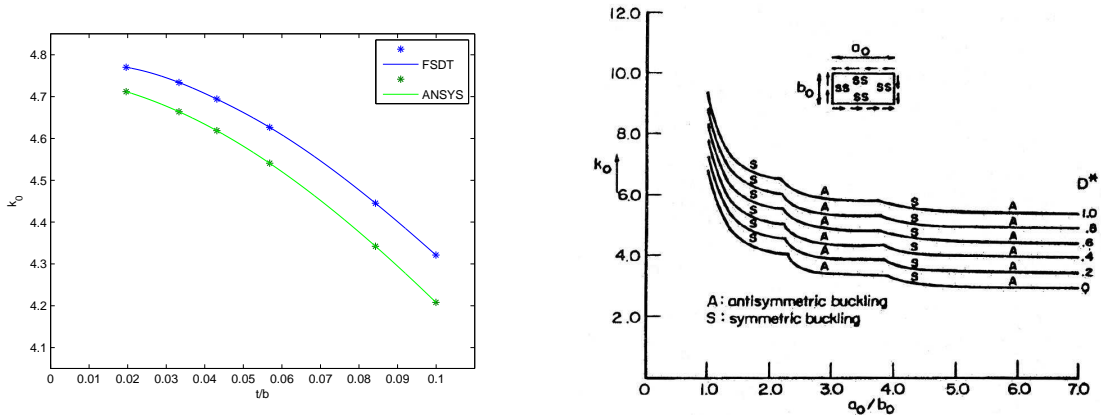


Figure 6.29: Left one shows simply supported plate, case C, 2000×500mm. Shear buckling coefficient  $k_0$  vs  $t/b$ . Right one is from the article "Generic Buckling Curves for Specially Orthotropic Rectangular Plates" written by Brunelle and Oyibo [8], using CLPT.

Using CLPT, we should arrive at  $k_0 \approx 5$  for  $a_0/b_0 = 4$  according to Brunelle and Oyibo. Thus, FSDT gives a better approximation to ANSYS than CLPT.

## 6.4 Combined Load , FSDT

From section 4.5, equation (4.46) gives critical buckling load for uniaxial  $x$  direction load combined with in-plane shear. The results are estimated using the routines developed in Matlab, and computed for  $M = 5$  and  $N = 5$ . The tables below show the results for  $\mu = 1$  only.

## CASE A - 500×500

<i>Case A - Simply supported, 500 × 500mm, <math>\mu = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	$\pm 151.0$ (N=5,M=5)	148.7	1.5%
$t = 16.0mm$	$\pm 1170.0$ (N=5,M=5)	1137.7	2.8%
$t = 24.0mm$	$\pm 3830.0$ (N=5,M=5)	3698.6	3.6%
$t = 32.0mm$	$\pm 8780.0$ (N=5,M=5)	8416.0	4.3%
$t = 48.0mm$	$\pm 27300.0$ (N=5,M=5)	25806.0	5.8%

Table 6.49: Case A, simply supported - 500×500.

We show the results graphically in interaction diagram.  $x$  axis denotes uniaxial loads in  $x$  direction, while  $y$  axis denotes in-plane shear. Since the results will not give us a straight line, we have to calculate more values of  $\mu$  to obtain the correct curve.

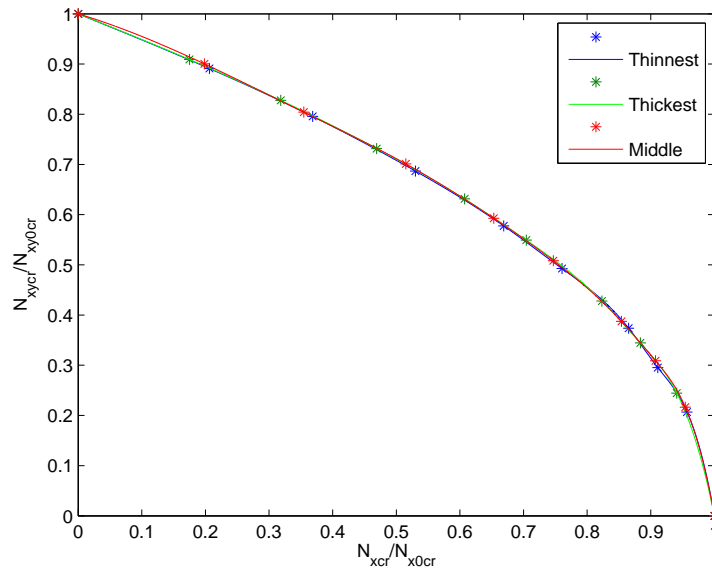


Figure 6.30: Interaction diagram for case A, Simply supported plate, 500×500mm.

### CASE A - 2000×500

<i>Case A - Simply supported, 2000 × 500mm, <math>\mu = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	$\pm 113.0$ (N=5,M=5)	111.7	1.2%
$t = 16.0mm$	$\pm 870.0$ (N=5,M=5)	852.4	2.1%
$t = 24.0mm$	$\pm 2900.0$ (N=5,M=5)	2792.4	3.9%
$t = 32.0mm$	$\pm 6600.0$ (N=5,M=5)	6428.0	2.8%
$t = 48.0mm$	$\pm 21100.0$ (N=5,M=5)	20300.0	3.9%

Table 6.50: Case A, simply supported - 2000×500.

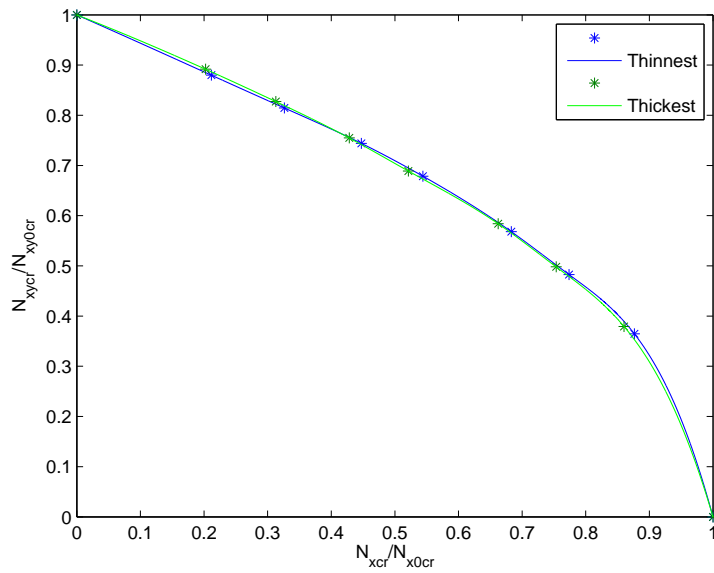


Figure 6.31: Interaction diagram for case A, Simply supported plate, 2000×500mm.

## CASE B - 500×500

<i>Case B - Simply supported, 500 × 500mm, <math>\mu = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	$\pm 154.0$ (N=5,M=5)	139.5	10.4%
$t = 16.0mm$	$\pm 1260.0$ (N=5,M=5)	1165.8	8.1%
$t = 24.0mm$	$\pm 4210.0$ (N=5,M=5)	3887.0	8.3%
$t = 32.0mm$	$\pm 9720.0$ (N=5,M=5)	8930.2	8.8%
$t = 48.0mm$	$\pm 30300.0$ (N=5,M=5)	27518.0	10.1%

Table 6.51: Case B, simply supported - 500×500.

For case B1 and B3,  $M = N = 3$  gives  $155.1N/mm$  and  $4227.0N/mm$ . We see that reducing terms from 25 to 9 for  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  will not effect the results that much. But this is only the cese of square plates.

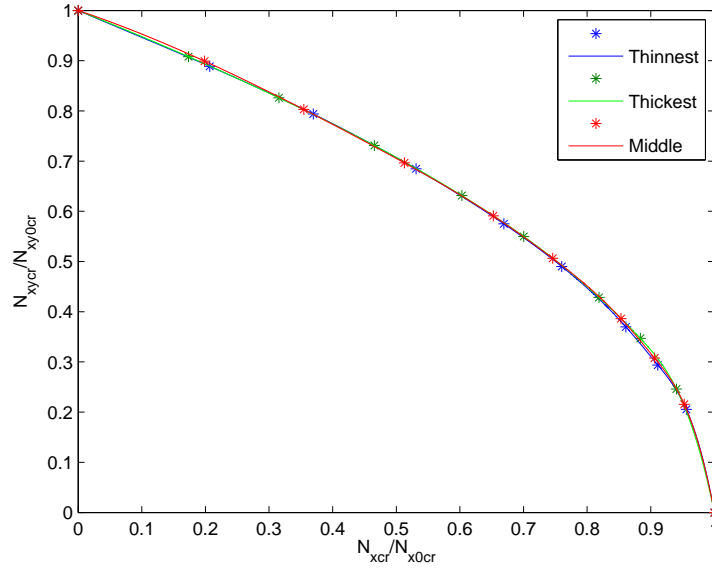


Figure 6.32: Interaction diagram for case B, Simply supported plate, 500×500mm.



## CASE B - 2000×500

<i>Case B - Simply supported, 2000 × 500mm, <math>\mu = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 8.0mm$	$\pm 124.7$ (N=5,M=5)	109.1	14.3%
$t = 16.0mm$	$\pm 1070.0$ (N=5,M=5)	984.8	8.7%
$t = 24.0mm$	$\pm 3600.0$ (N=5,M=5)	3353.4	7.4%
$t = 32.0mm$	$\pm 8500.0$ (N=5,M=5)	7811.4	8.8%
$t = 48.0mm$	$\pm 26711.1$ (N=5,M=5)	24548.0	8.8%

Table 6.52: Case B, simply supported - 2000×500.

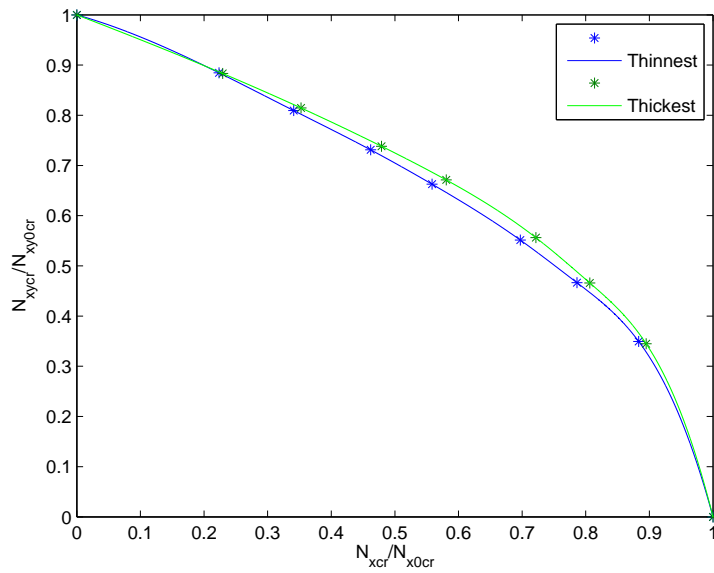


Figure 6.33: Interaction diagram for case B, Simply supported plate, 2000×500mm.

### CASE C - 500×500

<i>Case C - Simply supported, 500 × 500mm, <math>\mu = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 49.98mm$	$\pm 18200.0$ (N=5,M=5)	17411.4	4.5%
$t = 42.14mm$	$\pm 11200.0$ (N=5,M=5)	10730.6	4.4%
$t = 28.42mm$	$\pm 3530.0$ (N=5,M=5)	3432.0	2.9%
$t = 21.56mm$	$\pm 1560.0$ (N=5,M=5)	1524.5	2.3%
$t = 16.66mm$	$\pm 720.0$ (N=5,M=5)	711.0	1.3%
$t = 9.80mm$	$\pm 148.0$ (N=5,M=5)	146.6	1.0%

Table 6.53: Case C, simply supported - 500×500.

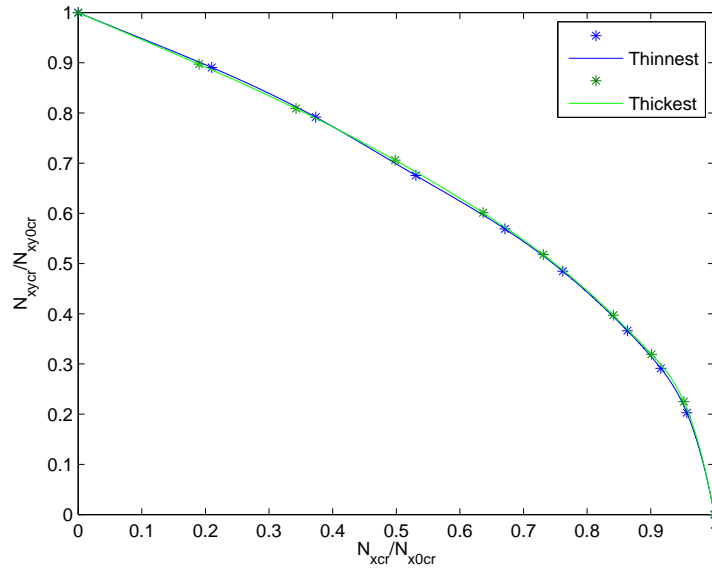


Figure 6.34: Interaction diagram for case C, Simply supported plate, 500×500mm.

### CASE C - 2000×500

<i>Case C - Simply supported, 2000 × 500mm, <math>\mu = 1</math></i>			
Plate thickness	FSDT [ $N/mm$ ]	ANSYS [ $N/mm$ ]	Failure in percent
$t = 49.98mm$	$\pm 16306.6$ (N=5,M=5)	15597.0	4.5%
$t = 42.14mm$	$\pm 10000.0$ (N=5,M=5)	9581.2	4.4%
$t = 28.42mm$	$\pm 3100.0$ (N=5,M=5)	3045.6	1.8%
$t = 21.56mm$	$\pm 1400.0$ (N=5,M=5)	1348.4	3.8%
$t = 16.66mm$	$\pm 640.0$ (N=5,M=5)	627.4	2.0%
$t = 9.80mm$	$\pm 130.0$ (N=5,M=5)	128.9	0.9%

Table 6.54: Case C, simply supported - 2000×500.

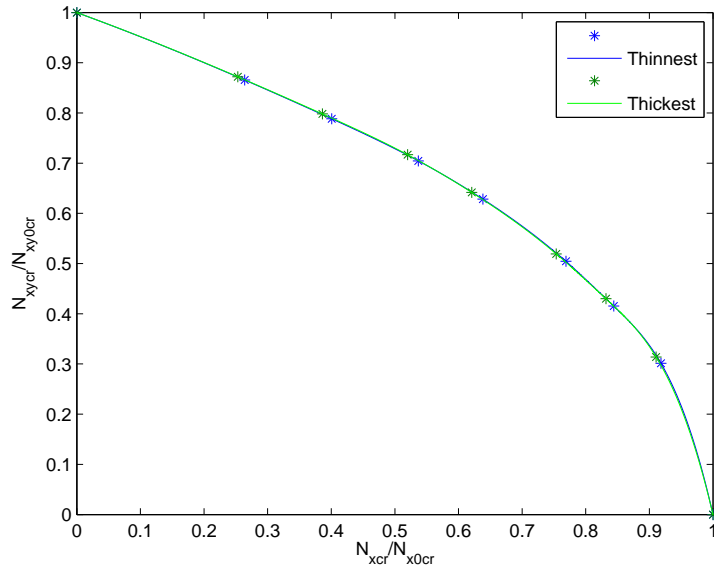


Figure 6.35: Interaction diagram for case C, Simply supported plate, 2000×500mm.

## 6.5 Plates with Initial Geometric Imperfection, FSDT

### 6.5.1 Displacement - Applied Load

Using equation (4.54) in section 4.6.1, we are now able to plot the load-displacement curve. This equation is valid only when  $m = m_i$  and  $n = n_i$ . Now introducing a magnification factor:

$$\lambda = \frac{1}{1 - \frac{N}{N_{cr}}}$$

where  $N$  is applied load and  $N_{cr}$  is the critical buckling load.

If maximum imperfection is  $w_i$ , then displacement causing by load  $N$  is given by:

$$w_{tot} = \lambda w_{int}(x, y) = \frac{w_i \sin\left(\frac{m_i \pi x}{a}\right) \sin\left(\frac{n_i \pi y}{b}\right)}{1 - \frac{N}{N_{cr}}} \quad (6.3)$$

Equations (4.54) and (6.3) are plotted below for comparison with ANSYS. Note that ANSYS includes non-linear effect. Further, we have chosen the imperfection magnitude at centre (or maximum imperfection) to be 0.1%, 1%, 2% and 3% of  $b \Rightarrow w_i = 0.5mm, w_i = 5mm, w_i = 10mm$  and  $w_i = 15mm$ . Note that in ANSYS, the applied loads are twice the current critical loads.

#### CASE A - $500 \times 500$

For case A,  $500 \times 500$ , load-displacement curves are plotted for  $(x, y) = (250, 250)$ .

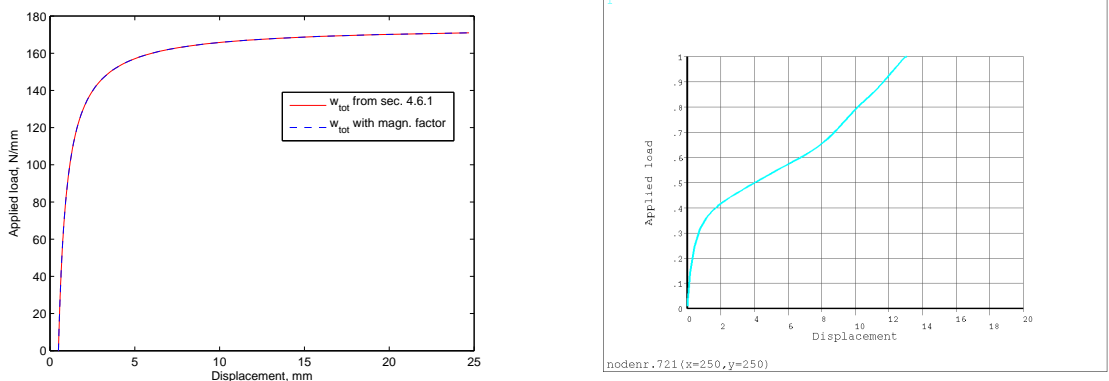


Figure 6.36: Case A1,  $500 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $0.5mm$ . Right one is from ANSYS.

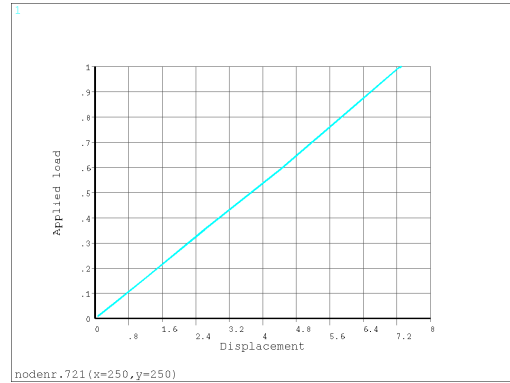
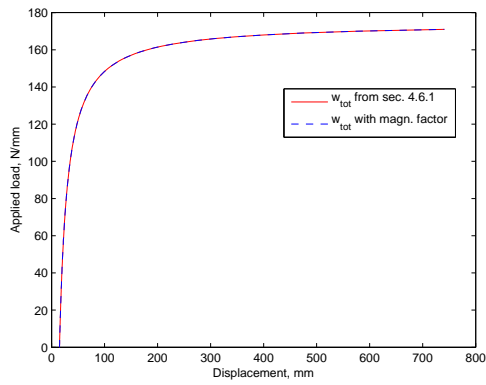


Figure 6.37: Case A1,  $500 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $15\text{mm}$ . Right one is from ANSYS.

### CASE A - $2000 \times 500$

For case A,  $2000 \times 500$ , we now select an arbitrary point. The load-displacement curves are plotted for  $(x, y) = (325, 250)$ .

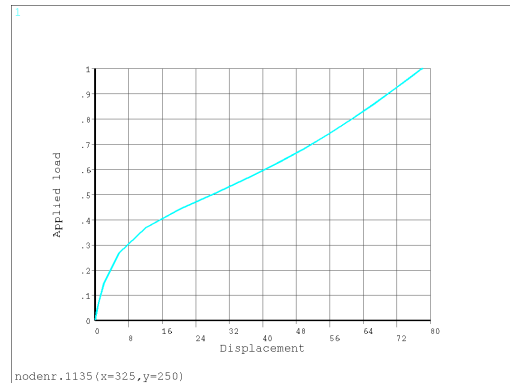
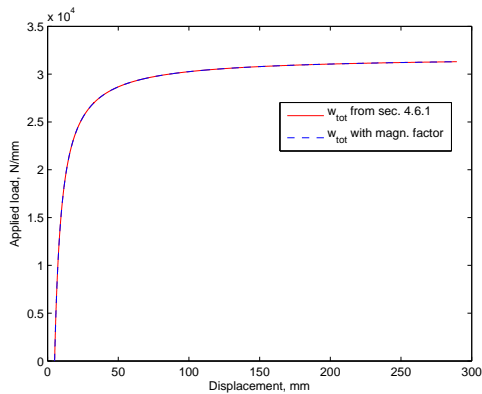


Figure 6.38: Case A5,  $2000 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $5\text{mm}$ . Right one is from ANSYS.

### CASE B - $500 \times 500$

For case B,  $500 \times 500$ , we plot for  $(x, y) = (250, 250)$ .

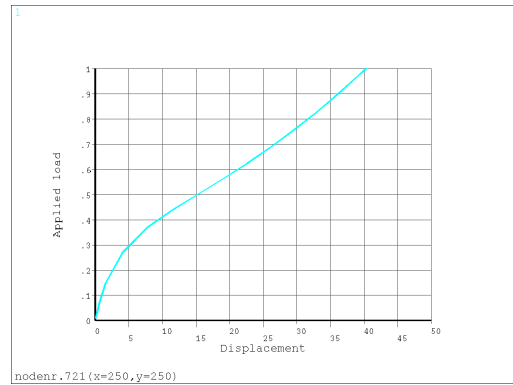
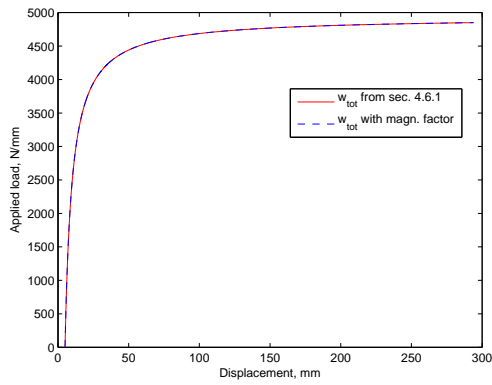


Figure 6.39: Case B3,  $500 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $5mm$ . Right one is from ANSYS.

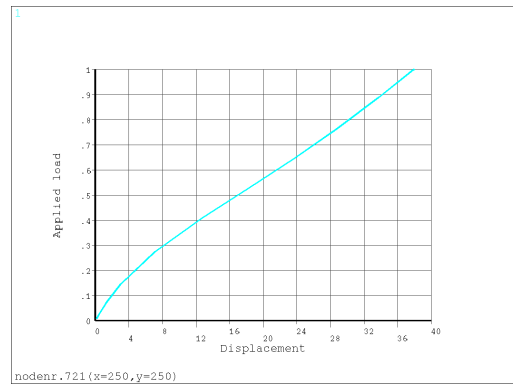
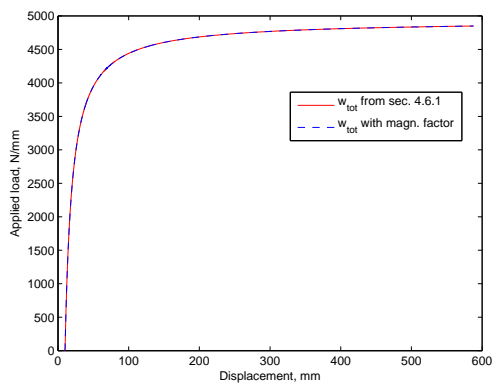


Figure 6.40: Case B3,  $500 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $10mm$ . Right one is from ANSYS.

### CASE B - $2000 \times 500$

For case B,  $2000 \times 500$ , we have selected point  $(x, y) = (250, 250)$ .

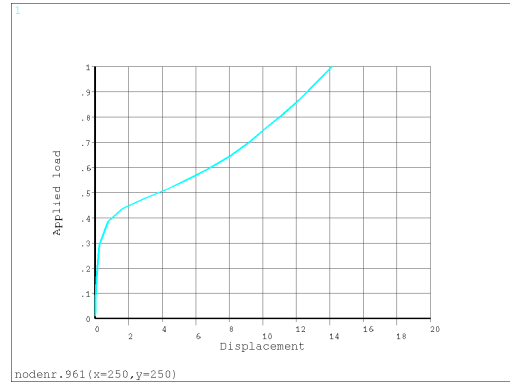
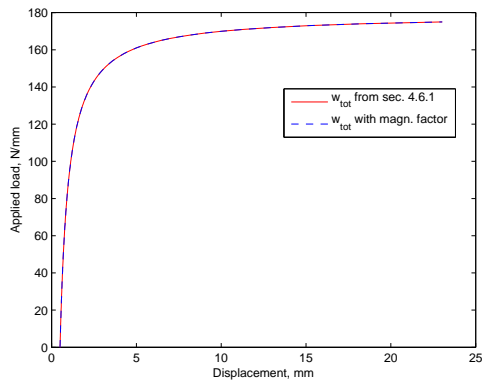


Figure 6.41: Case B1,  $2000 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $0.5mm$ . Right one is from ANSYS.

### CASE C - $500 \times 500$

For case C,  $500 \times 500$ , the load-displacement curves are plotted for  $(x, y) = (250, 250)$ .

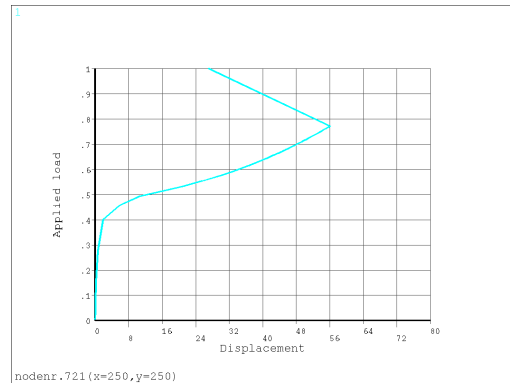
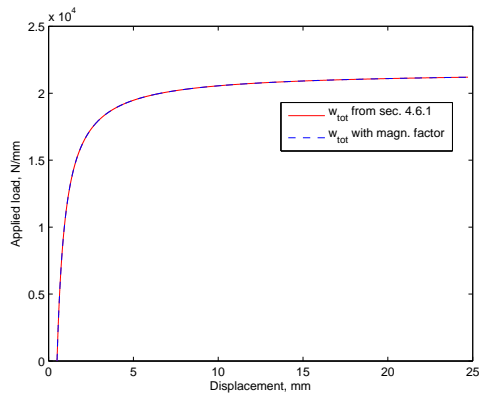


Figure 6.42: Case C1,  $500 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $0.5mm$ . Right one is from ANSYS.

### CASE C - $2000 \times 500$

For case C,  $2000 \times 500$ , we have chosen  $(x, y) = (250, 250)$ .

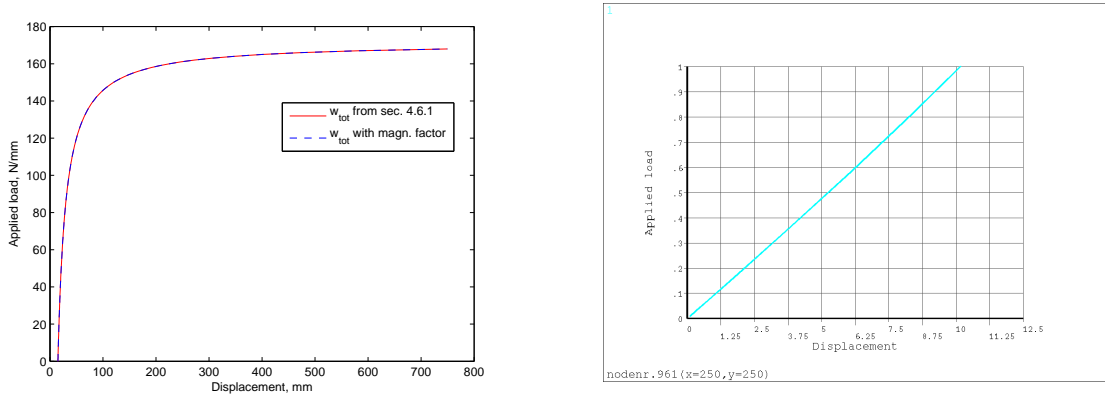


Figure 6.43: Case C6,  $2000 \times 500$ . Left one shows load-displacement curve based on equation (4.54) and (6.3) with imperfection  $15\text{mm}$ . Right one is from ANSYS.

We notice that equation (4.54) matches (6.3) perfectly. Since the models are confined to small-deflection buckling theory, the graphs from Matlab will never exceed the current critical buckling loads. ANSYS includes nonlinear effects, thus its load-displacement curves show us postbuckling behavior. The curve will continue growing although the critical buckling load is reached. We also see that load-displacement curve from ANSYS becomes a straight line for larger value of imperfection. For case C1, the load-displacement curve from ANSYS is "unstable" (see figure (6.42)). There is a bending of the graph. It can be interpreted as a change of the buckling modes.

### 6.5.2 First Ply Failure - Tsai-Wu

Now using equations (4.57) and (4.58) to find the inverse of the "strength ratio". The analysis is confined to square plates. Only case A and B, and their 3-4 outer layers will be investigated for first ply failure. The results are estimated using Matlab. Corresponding analysis is performed in ANSYS. The main differences between these two methods are 3D Tsai-Wu modeling and including of non-linear effect in ANSYS. It is been suggested  $S_{13} = S_{12} = 65$  in appendix A. But  $S_{13} = S_{12} = 1000$  have also been tested, and it seems that it does not give any noticeable differences. ANSYS has obtained almost identically results on those two values. Note that the results are calculated for middle of the layers.



## CASE A

	<i>Case A - Simply supported, 500 × 500mm, Using FSDT</i>				
Imp.=0.1%	<i>t = 8mm</i>	<i>t = 16mm</i>	<i>t = 24mm</i>	<i>t = 32mm</i>	<i>t = 48mm</i>
Stress [ <i>N/mm<sup>2</sup></i> ]	19.7	72.3	153	237.5	364.2
Ply no. (degree)	12 (-45)	24 (-45)	34 (0)	46 (0)	70 (0)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

	<i>Case A - Simply supported, 500 × 500mm, ANSYS</i>				
Imp.=0.1%	<i>t = 8mm</i>	<i>t = 16mm</i>	<i>t = 24mm</i>	<i>t = 32mm</i>	<i>t = 48mm</i>
Stress [ <i>N/mm<sup>2</sup></i> ]	30.8	72.1	147.1	228.1	357.9
Ply no. (degree)	12 (-45)	22 (0)	34 (0)	46 (0)	70 (0)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

	<i>Case A - Simply supported, 500 × 500mm, Using FSDT</i>				
Imp.=1%	<i>t = 8mm</i>	<i>t = 16mm</i>	<i>t = 24mm</i>	<i>t = 32mm</i>	<i>t = 48mm</i>
Stress [ <i>N/mm<sup>2</sup></i> ]	10.1	27.2	48.5	75.0	135.4
Ply no. (degree)	12 (-45)	24 (-45)	36 (-45)	48 (-45)	70 (0)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

	<i>Case A - Simply supported, 500 × 500mm, ANSYS</i>				
Imp.=1%	<i>t = 8mm</i>	<i>t = 16mm</i>	<i>t = 24mm</i>	<i>t = 32mm</i>	<i>t = 48mm</i>
Stress [ <i>N/mm<sup>2</sup></i> ]	27.9	47.1	63.1	90.3	125.0
Ply no. (degree)	10 (0)	22 (0)	34 (0)	46 (0)	70 (0)
Coordinate	(250,125)	(250,250)	(250,250)	(250,250)	(250,250)

	<i>Case A - Simply supported, 500 × 500mm, Using FSDT</i>				
Imp.=2%	<i>t = 8mm</i>	<i>t = 16mm</i>	<i>t = 24mm</i>	<i>t = 32mm</i>	<i>t = 48mm</i>
Stress [ <i>N/mm<sup>2</sup></i> ]	6.5	15.4	25.4	36.6	62.1
Ply no. (degree)	12 (-45)	24 (-45)	36 (-45)	48 (-45)	72 (-45)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

	<i>Case A - Simply supported, 500 × 500mm, ANSYS</i>				
Imp.=2%	<i>t = 8mm</i>	<i>t = 16mm</i>	<i>t = 24mm</i>	<i>t = 32mm</i>	<i>t = 48mm</i>
Stress [ <i>N/mm<sup>2</sup></i> ]	25	34.4	45.0	56.3	76.5
Ply no. (degree)	10 (0)	22 (0)	34 (0)	46 (0)	70 (0)
Coordinate	(250,62.5)	(250,125)	(250,250)	(250,250)	(250,250)

<i>Case A - Simply supported, 500 × 500mm, Using FSDT</i>					
Imp.=3%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	4.8	10.7	17.0	23.8	38.6
Ply no. (degree)	12 (-45)	24 (-45)	36 (-45)	48 (-45)	72 (-45)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

<i>Case A - Simply supported, 500 × 500mm, ANSYS</i>					
Imp.=3%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	22.5	29.4	37.1	43.8	57.5
Ply no. (degree)	10 (0)	22 (0)	34 (0)	46 (0)	70 (0)
Coordinate	(250,62.5)	(250,125)	(250,125)	(250,250)	(250,250)

We see that the results are not what we have expected. Almost everything from Matlab differs from ANSYS: Stresses, ply numbers, coordinates. What is most surprising is that for some cases, the first ply failure stresses from ANSYS are lower than stresses obtained by Matlab. Since ANSYS includes this non-linear effect, it should withstand more. The results from the tables are plotted below.

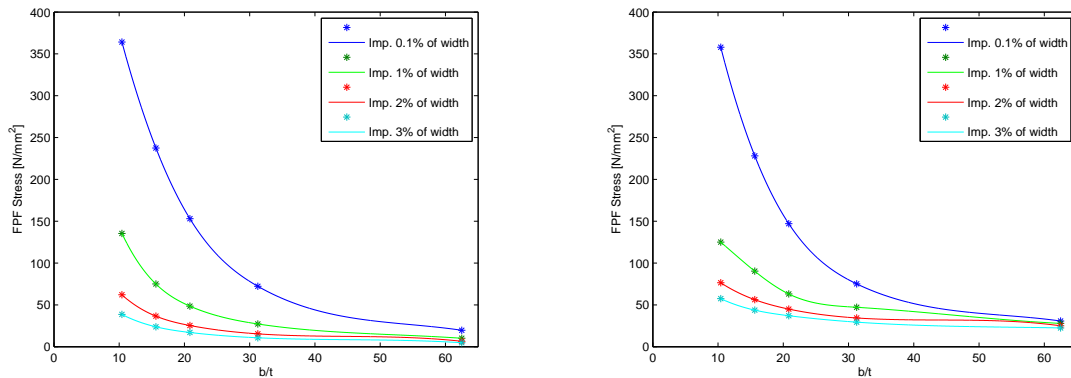


Figure 6.44: Case A, 500×500. Left one shows first ply stresses, using FSDT, plotted against  $b/t$ . Right one is from ANSYS.

The graphs have the same shape. Both models give almost identically results for thick plates with small imperfections. Below, it is been included a plot from ANSYS that show us the Tsai-wu stress distribution.

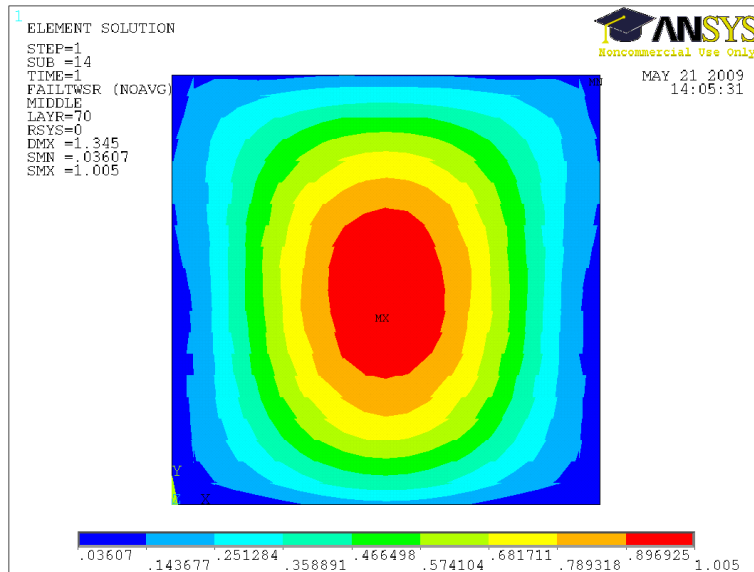


Figure 6.45: Tsai-Wu stress distribution for case A5 with imperfection 3% of the width. Plotted in ANSYS.

Different stress values and ply numbers from ANSYS and Matlab can be explained that computation in ANSYS is based on large-deflection theory and 3D Tsai-Wu modeling, while Matlab is based on small-deflection theory and 2D Tsai-Wu model. When the plate is almost perfect (large thickness and small imperfection), this effect decreases. We will then obtain almost similar results from Matlab and ANSYS. But it is just a hypothesis and need more investigation.

### CASE B

Now, we look at case B with the same model.

	<i>Case B - Simply supported, 500 × 500mm, Using FSDT</i>				
Imp.=0.1%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	20.1	40.5	44.6	45.9	47.3
Ply no.	6 (90)	14 (90)	22 (90)	30 (90)	46 (90)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

<i>Case B - Simply supported, 500 × 500mm, FSDT</i>					
Imp.=0.1%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	25.0	70.9	134.2	187.5	191.7
Ply no. (degree)	1 (0)	1 (0)	1 (0)	1 (0)	46 (90)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(500,0)

<i>Case B - Simply supported, 500 × 500mm, Using FSDT</i>					
Imp.=1%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	11.9	18.8	23.3	26.9	31.5
Ply no. (degree)	6 (90)	14 (90)	22 (90)	30 (90)	46 (90)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

<i>Case B - Simply supported, 500 × 500mm, FSDT</i>					
Imp.=1%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	27.5	46.1	75.0	101.9	142.1
Ply no. (degree)	8 (0)	16 (0)	24 (0)	32 (0)	48 (0)
Coordinate	(250,375)	(250,250)	(250,250)	(250,250)	(500,0)

<i>Case B - Simply supported, 500 × 500mm, Using FSDT</i>					
Imp.=2%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	8.6	12.3	15.7	18.5	23.0
Ply no. (degree)	6 (90)	14 (90)	22 (90)	30 (90)	46 (90)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

<i>Case B - Simply supported, 500 × 500mm, FSDT</i>					
Imp.=2%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	25	34.4	49.2	61.3	85.3
Ply no. (degree)	8 (0)	16 (0)	24 (0)	32 (0)	48 (0)
Coordinate	(300,437.5)	(250,250)	(250,250)	(250,250)	(500,0)

<i>Case B - Simply supported, 500 × 500mm, Using FSDT</i>					
Imp.=3%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	6.7	9.2	11.8	14.2	18.2
Ply no. (degree)	6 (90)	14 (90)	22 (90)	30 (90)	46 (90)
Coordinate	(250,250)	(250,250)	(250,250)	(250,250)	(250,250)

<i>Case B - Simply supported, 500 × 500mm, FSDT</i>					
Imp.=3%	$t = 8mm$	$t = 16mm$	$t = 24mm$	$t = 32mm$	$t = 48mm$
Stress [ $N/mm^2$ ]	23.5	31.3	38.8	45.3	63.0
Ply no. (degree)	8 (0)	16 (0)	24 (0)	32 (0)	48 (0)
Coordinate	(300,437.5)	(250,312.5)	(250,250)	(250,250)	(500,0)

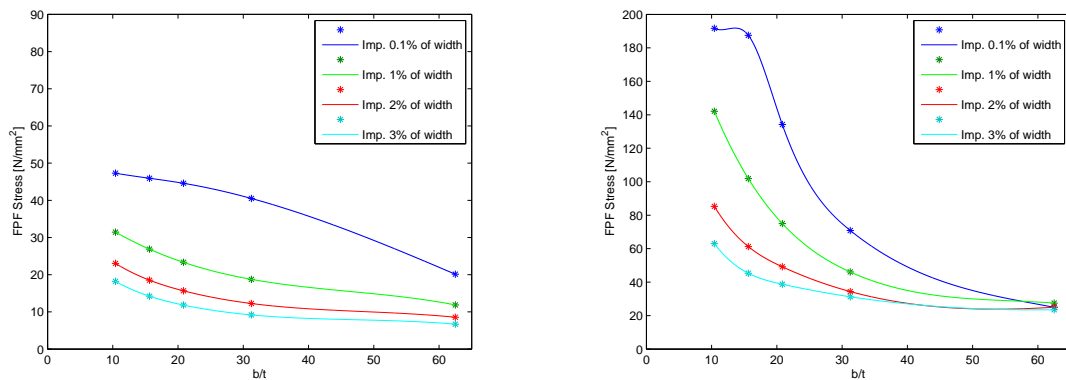


Figure 6.46: Case B, 500×500. Left one shows first ply stresses, using FSDT, plotted against  $b/t$ . Right one is from ANSYS.

Still, the results differ from each other. From Matlab, the first ply failure occurs at 90 degrees' ply every time, while ANSYS at ply 0 degree. Now ANSYS have got stress values much higher than Matlab. Fiture (6.47) shows us Tsai-wu stress distribution for case B5. Maximum stress occurs at centre.

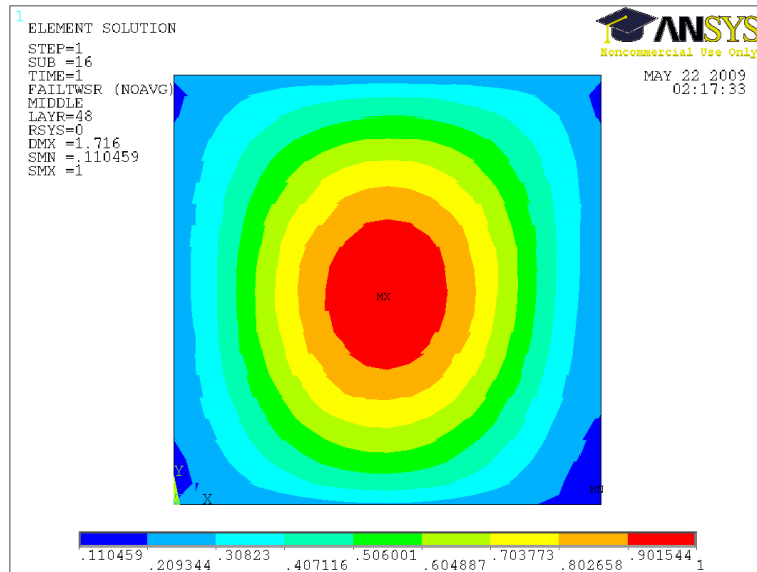


Figure 6.47: Tsai-Wu stress distribution for case B5 with imperfection 2 % of the width. Plotted in ANSYS.

Maybe it is not surprising that failure first occurs at 90 degrees' plies since the tension is perpendicular to the fibres. This may be the explanation for the low stress values. But like case A the main reasons why the results did not match are computation in ANSYS includes nonlinear effects and is based on 3D Tsai-Wu modeling. This means that 90 degrees' plies withstand more and are not subjected to tension in the same way as 90 degrees' plies under 2D Tsai-Wu model.

## 7 Conclusion

### 7.1 Conclusion of Results

This thesis results a method of simplified approaches to the estimation of failure loads for composite plates. The method is based on FSDT, which includes out-of-plane shear deformation.

For uniaxial load in  $x$  direction, both simply supported and clamped edges have been considered. In addition, both thin and thick plate theories (CLPT and FSDT) have been applied. By solving the buckling equation or equation set, we are now able to estimate the critical buckling load for simply supported plates. We see that the results based on FSDT are closer to ANSYS results than the results from CLPT. For a relatively thin plate, the CLPT provides a useful result. But for a thicker plate, including of out-of-plane deformation makes a noticeable difference. But still for cases with large thicknesses, not even FSDT is a good enough method. Here we will have benefit of higher order deformation theories. For clamped edges, Rayleigh-Ritz method has been chosen to determine the critical buckling load. The FSDT gives a better approximation than the CLPT. But the model gives us a higher failure percentage compared with the cases with simply supported edges. It has something to do with the effective length, which is reduced by 1/2 compared to simply supported edges. But this needs more investigation.

Simply supported plates subjected to biaxial load have been investigated using FSDT. By solving the buckling equation set, it is now possible to estimate the critical buckling load using Matlab. Again, the results are acceptable for thin and medium thick plates, while the discrepancy is too large for the thickest plates. It is been computed a several values for  $\zeta$  (relation between  $N_x$  and  $N_y$ ) to construct interaction diagrams. For a  $500 \times 500$  plate from any case, the interaction diagram shows us a straight, sloped line. This indicates that the relationship between  $N_{xcr}/N_{x0cr}$  and  $N_{ycr}/N_{y0cr}$  is inversely proportional. For  $2000 \times 500$  plates, it is more complicated. Here the interaction diagrams are more parable shaped.

For simply supported plates subjected to in-plane shear, the solution is based on FSDT. Rayleigh-Ritz method has been chosen to estimate the buckling load. The accuracy of the results depends on the number of  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  terms. We see that for square plates, it is enough to consider  $3MN = 3 \cdot 3 \cdot 3 = 27$  terms, but preferably  $M = N = 5$ , while it is required minimum  $3MN = 3 \cdot 5 \cdot 5 = 75$  terms for long plates.  $M = N = 7$  or even higher number of  $M$  and  $N$  will no doubt give us more accurate answer. But calculating the determinant of a over  $100 \times 100$  matrix in Matlab takes eternity (the matrixes are programmed

in "forloops", which makes heavy demand on Matlab). Results and time taken into account,  $M = N = 5$  give us a acceptable answer. So for cases A, B and C,  $M = 5$  and  $N = 5$  have been chosen to be the standard values. We have got really good approximations compared with ANSYS. But for case B the model developed in section 4.4 is not good enough, not even including a higher number of the  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  terms. Primarily, this is caused by neglecting of  $D_{16}$  and  $D_{26}$ . A thin plate with many plies has been tested to confirm this suspicion.

For combined loads, the model has been developed in the same way as for pure in-plane shear loading. The investigation has been confined to shear combined with uniaxial compressive loading and to the case of simply supported plates. Again, Rayleigh-Ritz method has been used to solve the buckling problem. The accuracy of the results depends on the number of  $x_{mn}$ ,  $y_{mn}$  and  $w_{mn}$  terms.  $M = N = 5$  give us a acceptable answer. So for cases A-C,  $M = 5$  and  $N = 5$  have been chosen to be the standard values. We see that we have got really good approximations compared with ANSYS except for case B. The inaccuracy partially caused by neglecting of  $D_{16}$  and  $D_{26}$ .

For plates with an initial geometric imperfection, it has been confined to small-deflection (linearised) buckling theory. The load-displacement curves from Matlab will never exceed the current critical buckling loads. The analysis in ANSYS includes nonlinear effects. Thus, it's load-displacement curves show the post-buckling behavior. The curve will continue growing although the current critical buckling load is reached. We also notice that load-displacement curve becomes more straight for larger value of imperfection. Sometimes it also shows us the change of the buckling modes.

Further, the model developed for simply supported plates with an initial geometric imperfection has been applied to establish the onset of first ply failure. Only case A and B have been investigated. The method is unsuccessful. There are large discrepancies between ANSYS results and Matlab results. The deviation can be explained that computation in ANSYS is based on large-deflection theory and 3D Tsai-Wu modeling, while Matlab is based on small-deflection theory and 2D Tsai-Wu modeling.

Finally, the methods based on FSDT are better than CLPT. It is best suited for thin and moderately thick plates. Higher order deformation theories should be considered for really thick plates. For case A and C, the results are good and acceptable. For case B, the assumption of specially orthotropic laminates will affect the results in a bad way. The method is also limited to linear cases.



## 7.2 Suggestions for Futher Work

In general, continue the investigation of simplified approaches to the estimation of failure loads for composite plates under in-plane loading. Extend the analysis method based on FSDT to cover cases with clamped edges. It is also interesting to investigate higher order deformation theories. For thick plates, it will cerntainly give a better estimation of critical buckling load. For plates with an initial geometric imperfection, extend the approach to include large-deflection (nonlinear) buckling theory. It is also necessary to take a closer check on the first ply failure part.

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# Appendix

## A Parameter Definitions [9]

### A.1 CASE A

**Triaxial Lay-up:**  $[-45/ + 45/0/ + 45/ - 45/0]_{X,S}$

- Plate aspect ratio:  $a \times b$ 
  1. AR1:  $500 \times 500mm$
  2. AR2:  $2000 \times 500mm$   
(Sides  $b$  are the loaded edges)
- Plate thicknesses:
  1.  $X = 1 \Rightarrow t = 8.0mm \Rightarrow b/t = 62.5$
  2.  $X = 2 \Rightarrow t = 16.0mm \Rightarrow b/t = 31.25$
  3.  $X = 3 \Rightarrow t = 24.0mm \Rightarrow b/t = 20.83$
  4.  $X = 4 \Rightarrow t = 32.0mm \Rightarrow b/t = 15.63$
  5.  $X = 6 \Rightarrow t = 48.0mm \Rightarrow b/t = 10.42$
- Ply thicknesses:
  1.  $t_{45} = t_{-45} = 0.143mm$
  2.  $t_0 = 1.714mm$
- Imperfection magnitude at center: (half sin-wave shaped)
  1. 0.1% of  $b = 0.5mm$
  2. 1% of  $b = 5mm$
  3. 2% of  $b = 10mm$
  4. 3% of  $b = 15mm$
- Imperfection amplitude equal to full plate width and length.
- Material definitions:[ $MPa$ ]

$$\begin{aligned} E_1 &= 49627 \quad , \quad E_2 = 15430 \quad , \quad E_3 = 15430 \\ \nu_{12} &= 0.272 \quad , \quad \nu_{13} = 0.272 \quad , \quad \nu_{23} = 0.607 \\ G_{12} &= 4800 \quad , \quad G_{13} = 4800 \quad , \quad G_{23} = 4800 \\ X_t &= 968 \quad , \quad X_c = 915 \quad , \quad Y_t = 24 \quad , \quad Y_c = 118 \quad , \quad S_{12} = 65 \quad , \quad S_{23} = 65 \end{aligned}$$

## A.2 CASE B

**Quasi-isotropic:**  $[0/ + 45/90/ - 45]_{X,S}$

- Plate aspect ratio:  $a \times b$ 
  1. AR1:  $500 \times 500mm$
  2. AR2:  $2000 \times 500mm$   
(Sides  $b$  are the loaded edges)
- Plate thicknesses:
  1.  $X = 1 \Rightarrow t = 8.0mm \Rightarrow b/t = 62.5$
  2.  $X = 2 \Rightarrow t = 16.0mm \Rightarrow b/t = 31.25$
  3.  $X = 3 \Rightarrow t = 24.0mm \Rightarrow b/t = 20.83$
  4.  $X = 4 \Rightarrow t = 32.0mm \Rightarrow b/t = 15.63$
  5.  $X = 6 \Rightarrow t = 48.0mm \Rightarrow b/t = 10.42$
- Ply thicknesses: ( $t$  is the same for all plies)
  1.  $t = 1mm$
- Imperfection magnitude at center: (half sin-wave shaped)
  1. 0.1% of  $b = 0.5mm$
  2. 1% of  $b = 5mm$
  3. 2% of  $b = 10mm$
  4. 3% of  $b = 15mm$
- Imperfection amplitude equal to full plate width and length.
- Material definitions:[ $MPa$ ] (same as in case A)

$$\begin{aligned} E_1 &= 49627 \quad , \quad E_2 = 15430 \quad , \quad E_3 = 15430 \\ \nu_{12} &= 0.272 \quad , \quad \nu_{13} = 0.272 \quad , \quad \nu_{23} = 0.607 \\ G_{12} &= 4800 \quad , \quad G_{13} = 4800 \quad , \quad G_{23} = 4800 \\ X_t &= 968 \quad , \quad X_c = 915 \quad , \quad Y_t = 24 \quad , \quad Y_c = 118 \quad , \quad S_{12} = 65 \quad , \quad S_{23} = 65 \end{aligned}$$

### A.3 CASE C

**Woven Roving:**  $[0]_X$

- Plate aspect ratio:  $a \times b$ 
  1. AR1:  $500 \times 500mm$
  2. AR2:  $2000 \times 500mm$   
(Sides  $b$  are the loaded edges)
- Plate thicknesses:
  1.  $X = 51 \Rightarrow t = 49.98mm \Rightarrow b/t = 10.00$
  2.  $X = 43 \Rightarrow t = 42.14mm \Rightarrow b/t = 11.87$
  3.  $X = 29 \Rightarrow t = 28.42mm \Rightarrow b/t = 17.59$
  4.  $X = 22 \Rightarrow t = 21.56mm \Rightarrow b/t = 23.19$
  5.  $X = 17 \Rightarrow t = 16.66mm \Rightarrow b/t = 30.01$
  6.  $X = 10 \Rightarrow t = 9.80mm \Rightarrow b/t = 51.02$
- Ply thicknesses: ( $t$  is the same for all plies)
  1.  $t = 0.98mm$   
(tip: Ply thickness can be equal to plate thickness using only one ply and increasing the number of integration points)
- Imperfection magnitude at center: (half sin-wave shaped)
  1. 0.1% of  $b = 0.5mm$
  2. 1% of  $b = 5mm$
  3. 2% of  $b = 10mm$
  4. 3% of  $b = 15mm$
- Imperfection amplitude equal to full plate width and length.
- Material definitions:[ $MPa$ ]

$$\begin{aligned} E_1 &= 17180 \quad , \quad E_2 = 17180 \quad , \quad E_3 = 10800 \\ \nu_{12} &= 0.170 \quad , \quad \nu_{13} = 0.270 \quad , \quad \nu_{23} = 0.270 \\ G_{12} &= 3520 \quad , \quad G_{13} = 5150 \quad , \quad G_{23} = 5150 \\ X_t &= 238.6 \quad , \quad X_c = 324.5 \quad , \quad Y_t = 238.6 \quad , \quad Y_c = 324.5 \\ S_{12} &= 80.9 \quad , \quad S_{23} = 60.7 \end{aligned}$$

## B Definitions and Equations

### B.1 Matrixes

Compliance matrix [2]:

$$[S] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_L} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix},$$

Transformation matrix [2]:

$$[T] = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix},$$

where  $\theta$  is the orientation angle.

Stiffness matrix [2]:

$$[Q] = [S]^{-1}$$
$$[\bar{Q}] = [T]^{-1}[Q][T]$$

Extensional stiffness matrix, A [2]:

$$A_{ij} = \sum_{k=1}^n (\bar{Q}_{ij})_k (h_k - h_{k-1})$$

Coupling stiffness matrix, B [2]:

$$B_{ij} = \frac{1}{2} \sum_{k=1}^n (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2)$$

Bending stiffness matrix, D [2]:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^n (\bar{Q}_{ij})_k (h_k^3 - h_{k-1}^3),$$

For FSDT we have to include out of plane shear. In composite laminated plates the transverse shear stress varies almost quadratically through the thickness.

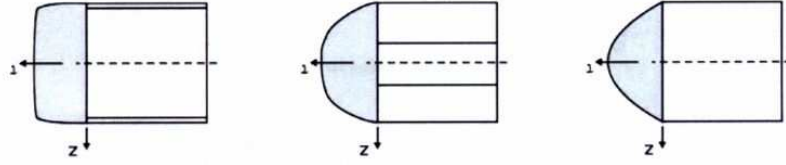


Figure B.1: Example on shear stress variations for different materials [10].

Materialtype	Homogen	Komposit	Sandwich
Tykkplate	5/6	5/6	$\sim 1$

Table B.1: Typical shear correction coefficient [10].

Stiffness matrix for shear [3]:

$$[Q]_{shr} = \begin{bmatrix} G_{23} & 0 \\ 0 & G_{13} \end{bmatrix},$$

where  $Q_{44} = G_{23}$ ,  $Q_{55} = G_{13}$  and  $Q_{45} = Q_{54} = 0$ .

Transformation matrix for shear [3]:

$$[T]_{shr} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Extensional stiffness matrix for shear [3]:

$$A_{ij} = k \sum_{k=1}^n (\bar{Q}_{ij})_k (h_k - h_{k-1}),$$

for  $i = 4, 5$  and  $j = 4, 5$ .  $[\bar{Q}]_{shr} = [T]_{shr}^{-1} [Q]_{shr} [T]_{shr}$ .

## B.2 Governing Equations for Plates

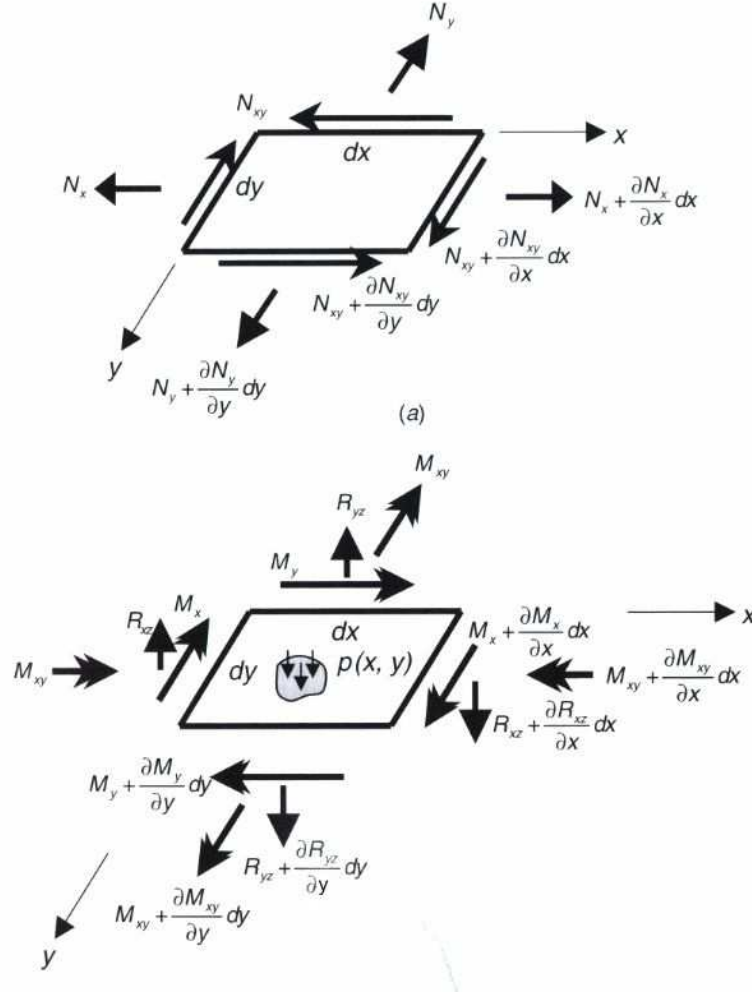


Figure B.2: Above - A differential element with in-plane force resultants. Under - A differential element with moment resultants, shear force resultants and applied transverse forces [2].

**Equilibrium of forces in  $x$  direction (Figure B.2):**

$$\begin{aligned}
 -N_x dy + (N_x + \frac{\partial N_x}{\partial x} dx) dy - N_{xy} dx + (N_{xy} + \frac{\partial N_{xy}}{\partial y} dy) dx &= 0 \\
 \Rightarrow \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0
 \end{aligned} \tag{B.1}$$

**Equilibrium of forces in  $y$  direction (Figure B.2):**



$$\begin{aligned}
-N_y dx + (N_y + \frac{\partial N_y}{\partial y} dx) dx - N_{xy} dy + (N_{xy} + \frac{\partial N_{xy}}{\partial x} dx) dy &= 0 \\
\Rightarrow \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= 0
\end{aligned} \tag{B.2}$$

**Equilibrium of forces in  $z$  direction:**

1) Without force projections (Figure B.2):

$$\begin{aligned}
-R_{xz} dy + (R_{xz} + \frac{\partial R_{xz}}{\partial x} dx) dy - R_{yz} dx + (R_{yz} + \frac{\partial R_{yz}}{\partial y} dy) dx + p dx dy &= 0 \\
\Rightarrow \frac{\partial R_{xz}}{\partial x} + \frac{\partial R_{yz}}{\partial y} + p &= 0
\end{aligned} \tag{B.3}$$

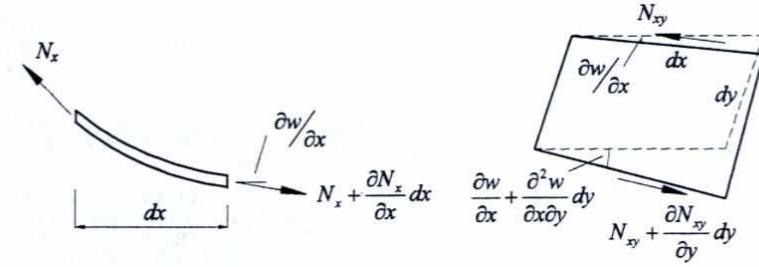


Figure B.3: Force projections [11].

2) Including force projections (Figure B.3):

$$\begin{aligned}
-N_x \frac{\partial w}{\partial x} dy + (N_x + \frac{\partial N_x}{\partial x} dx) dy (\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx) &= N_x \frac{\partial^2 w}{\partial x^2} dx dy + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} dx dy \\
-N_{yx} \frac{\partial w}{\partial x} dx + (N_{yx} + \frac{\partial N_{yx}}{\partial y} dy) dx (\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x \partial y} dy) &= N_{yx} \frac{\partial^2 w}{\partial x \partial y} dx dy + \frac{\partial N_{yx}}{\partial y} \frac{\partial w}{\partial x} dx dy \\
-N_y \frac{\partial w}{\partial y} dx + (N_y + \frac{\partial N_y}{\partial y} dy) dx (\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2} dy) &= N_y \frac{\partial^2 w}{\partial y^2} dx dy + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} dx dy
\end{aligned}$$

$$-N_{xy} \frac{\partial w}{\partial y} dy + (N_{xy} + \frac{\partial N_{xy}}{\partial x} dx) dy (\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} dx) = N_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} dx dy$$

Summing all contributions for the in plane forces and we arrive at:

$$\begin{aligned} & N_x \frac{\partial^2 w}{\partial x^2} dx dy + N_{yx} \frac{\partial^2 w}{\partial x \partial y} dx dy + N_y \frac{\partial^2 w}{\partial y^2} dx dy + N_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy \\ & + \frac{\partial w}{\partial x} dx dy (\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y}) + \frac{\partial w}{\partial y} dx dy (\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}) \\ & = N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \end{aligned}$$

We have used equations (B.1) and (B.2) to get rid of parentheses.

Now with equation (B.3), the final expression for the force equilibrium in  $z$  direction is then:

$$\frac{\partial R_{xz}}{\partial x} + \frac{\partial R_{yz}}{\partial y} + p + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_{yx} \frac{\partial^2 w}{\partial x \partial y} - \rho^* \frac{\partial^2 w}{\partial t^2} = 0 \quad (\text{B.4})$$

where  $\rho^*$  is the surface weight or mass of the plate.

**By summing moments about x axis (Figure B.2):**

$$\begin{aligned} & M_y dx - (M_y + \frac{\partial M_y}{\partial y} dy) dx + M_{xy} dy - (M_{xy} + \frac{\partial M_{xy}}{\partial x} dx) dy + (R_{yz} + \frac{\partial R_{yz}}{\partial y} dy) dx dy \\ & + (R_{xz} + \frac{\partial R_{xz}}{\partial x} dx) dy \frac{dy}{2} - R_{xz} dy \frac{dy}{2} + p dx dy \frac{dy}{2} = 0 \end{aligned}$$

When higher-order terms are neglected, this equation simplifies to:

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - R_{yz} = 0 \quad (\text{B.5})$$

**By summing moments about y axis (Figure B.2):**

$$M_x dy - (M_x + \frac{\partial M_x}{\partial x} dx) dy + M_{xy} dx - (M_{xy} + \frac{\partial M_{xy}}{\partial y} dy) dx + (R_{xz} + \frac{\partial R_{xz}}{\partial x} dx) dx dy$$

$$+(R_{yz} + \frac{\partial R_{yz}}{\partial y} dy) dx \frac{dx}{2} - R_{yz} dx \frac{dx}{2} + p dx dy \frac{dx}{2} = 0$$

Similarly, we neglect the higher-order terms:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - R_{xz} = 0 \quad (\text{B.6})$$

From equations (B.5) and (B.6), we get the expressions for shear:

$$R_{xz} = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$R_{yz} = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}$$

Now, substitution of equations above in equation (B.4) gives:

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2}$$

$$+ p + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} - \rho^* \frac{\partial^2 w}{\partial t^2} = 0 \quad (\text{B.7})$$

## C Program Codes

### C.1 Matlab

#### Simply supported CLPT (uniaxial), Case A

```
% Frittøplagt kompositt tynnplate Masteroppgaven
% Case A
clear all;

%Materialdata
EL=49627;
ET=15430;
vLT=0.272;
GLT=4800;
a=500;
b=500;
n=1;
m=1;

% Orientering av lag i radianer og tykkelse av lagene
%R=[-45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
```

```

%h=[-4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4];

R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45];
h=[-8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857
 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45
 0 45 -45 0 -45 45 0 45 -45];
%h=[-12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857
-3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8
9.714 9.857 10 11.714 11.857 12];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45
0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45];
%h=[-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
-7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
13.857 14 15.714 15.857 16];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0
45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45
0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45];
%h=[-24 -23.857 -23.714 -22 -21.857 -21.714 -20 -19.857 -19.714 -18 -17.857 -17.714
-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
-7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
13.857 14 15.714 15.857 16 17.714 17.857 18 19.714 19.857 20 21.714 21.857 22
23.714 23.857 24];

%Initielverdi
A(1:3,1:3)=0;
B(1:3,1:3)=0;
D(1:3,1:3)=0;
Q(1:3,1:3)=0;
S(1:3,1:3)=0;

%Compliance (S-matirx)
S(1,1)=1/EL;
S(2,2)=1/ET;
S(1,2)=-vLT/EL;
S(3,3)=1/GLT;
S(2,1)=S(1,2);

%Stiffness (Q-matrix)
Q=inv(S);

Q(3,3)=2*Q(3,3); %for tensorielle tøyninger ved rotasjon

for i=1:length(R) %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

%Transformation matrix (T-matrix)
T(1,1)=cos(R(i))^2;
T(2,2)=cos(R(i))^2;
T(1,2)=sin(R(i))^2;
T(2,1)=sin(R(i))^2;
T(3,1)=-sin(R(i))*cos(R(i));
T(3,2)=sin(R(i))*cos(R(i));
T(1,3)=2*sin(R(i))*cos(R(i));
T(2,3)=-2*sin(R(i))*cos(R(i));
T(3,3)=cos(R(i))^2-sin(R(i))^2;

Qk(:, :, i)=inv(T)*Q*T; %Stivhetsmatrise for tensorielle tøyninger
Qk(:, 3, i)=Qk(:, 3, i)/2; %Går tilbake til ingeniørtøyninger etter rotasjon
A=A+Qk(:, :, i)*(h(i+1)-h(i)); %Beregner A-matrisen
B=B+.5*Qk(:, :, i)*(h(i+1)^2-h(i)^2); %Beregner B-matrisen

```

```

D=D+1/3*Qk(:, :, i)*(h(i+1)^3-h(i)^3); %Beregner D-matrisen
end

A
B
D

P=pi^2*(D(1,1)*(m/a)^2+(2*D(1,2)+4*D(3,3))*(n/b)^2+D(2,2)*(a/m)^2*(n/b)^4)%Kritisk verdi for P

% Beregning av knekningskoeffisient

knekkoeff=P*b^2/(pi^2*(D(1,1)*D(2,2))^(1/2)) % Knekningskoeffisient

DD=(D(1,2)+2*D(3,3))/((D(1,1)*D(2,2))^(1/2))

data=knekkoeff-2*DD

```

## Clamped CLPT (uniaxial), Case A

```

% Fastinnspent kompositt tynnplate Masteroppgaven
% Case A
clear all;

%Materialdata
EL=49627;
ET=15430;
vLT=0.272;
GLT=4800;
a=2000;
b=500;
n=3;
m=5;

% Orientering av lag i radianer og tykkelse av lagene
%R=[-45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
%h=[-4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714 13.857 14 15.714 15.857 16];

R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
h=[-24 -23.857 -23.714 -22 -21.857 -21.714 -20 -19.857 -19.714 -18 -17.857 -17.714 -16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714 13.857 14 15.714 15.857 16 17.714 17.857 18 19.714 19.857 20 21.714 21.857 22 23.714 23.857 24];

```

```

%Initielverdi
A(1:3,1:3)=0;
B(1:3,1:3)=0;
D(1:3,1:3)=0;
Q(1:3,1:3)=0;
S(1:3,1:3)=0;

%Compliance (S-matrix)
S(1,1)=1/EL;
S(2,2)=1/ET;
S(1,2)=-vLT/EL;
S(3,3)=1/GLT;
S(2,1)=S(1,2);

%Stiffness (Q-matrix)
Q=inv(S);

Q(3,3)=2*Q(3,3); %for tensorielle tøyninger ved rotasjon

for i=1:length(R) %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

%Transformation matrix (T-matrix)
T(1,1)=cos(R(i))^2;
T(2,2)=cos(R(i))^2;
T(1,2)=sin(R(i))^2;
T(2,1)=sin(R(i))^2;
T(3,1)=-sin(R(i))*cos(R(i));
T(3,2)=sin(R(i))*cos(R(i));
T(1,3)=2*sin(R(i))*cos(R(i));
T(2,3)=-2*sin(R(i))*cos(R(i));
T(3,3)=cos(R(i))^2-sin(R(i))^2;

Qk(:, :, i)=inv(T)*Q*T; %Stivhetsmatrise for tensorielle tøyninger
Qk(:, 3, i)=Qk(:, 3, i)/2; %Går tilbake til ingeniørtøyninger etter rotasjon
A=A+Qk(:, :, i)*(h(i+1)-h(i)); %Beregner A-matrisen
B=B+.5*Qk(:, :, i)*(h(i+1)^2-h(i)^2); %Beregner B-matrisen
D=D+1/3*Qk(:, :, i)*(h(i+1)^3-h(i)^3); %Beregner D-matrisen
end

A
B
D

if m==n
P=(4*pi^2*(3*D(1,1)*m^4*b^4+3*D(2,2)*a^4+4*m^2*D(3,3)*a^2*b^2+2*m^2*D(1,2)*a^2*b^2))/(3*b^4*m^2*a^2)
else
P=(pi^2/a^2*(n^4+6*m^2*n^2+m^4)*D(1,1)+16/3*a^2/b^4*pi^2*D(2,2)+16/3*pi^2/b^2*(n^2+m^2)*(1/2*D(1,2)
+D(3,3)))/(n^2+m^2)
end

L1=pi^4/(4*a^3*b^3)*(4*D(3,3)*a^2*b^2+2*D(1,2)*a^2*b^2+3*D(2,2)*a^4+3*D(1,1)*b^4);
L2=3*pi^2*b/(16*a);
L3=pi^4*a^4*D(2,2)/(2*a^3*b^3);
L4=pi^4/(4*a^3*b^3)*(8*D(1,2)*a^2*b^2+3*a^4*D(2,2)+16*D(3,3)*a^2*b^2+48*D(1,1)*b^4);
L5=3*pi^2*b/(4*a);

P2=((L1*L5+L2*L4)-sqrt((L1*L5+L2*L4)^2-4*L2*L5*(L1*L4-L3*L3)))/(2*L2*L5)

```

## Simply supported FSDT (uniaxial+biaxial+imperfection), Case A

```

% Frittopplagt kompositt tykkplate Masteroppgaven
% Case A

```

```

clear all;

%Materialdata
EL=49627;
ET=15430;
vLT=0.272;
GLT=4800;
G23=4800;
G13=4800;
a=2000;
b=500;
k=5/6;
n=1;
m=3;
ni=1;
mi=3;
alpha=m*pi/a;
beta=n*pi/b;
alpha_i=mi*pi/a;
gamma=0.9;
x=325;
y=b/2;
Wi=5;
N1=0:1:31300;

% Orientering av lag i radianer og tykkelse av lagene
%R=[-45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
%h=[-4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4];
%t=8;

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857
  2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8];
%t=16;

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45
  0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857
  -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8
  9.714 9.857 10 11.714 11.857 12];
%t=24;

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45
  0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
  -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
  3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
  13.857 14 15.714 15.857 16];
%t=32;

R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45
  -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45
  45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
h=[-24 -23.857 -23.714 -22 -21.857 -21.714 -20 -19.857 -19.714 -18 -17.857 -17.714 -16
  -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857
  -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857
  4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714 13.857 14 15.714
  15.857 16 17.714 17.857 18 19.714 19.857 20 21.714 21.857 22 23.714 23.857 24];
t=48;

%Initielverdi
A(1:3,1:3)=0;
B(1:3,1:3)=0;
D(1:3,1:3)=0;
Q(1:3,1:3)=0;

```

```

S(1:3,1:3)=0;
A1(1:2,1:2)=0;   %A-matrisen med skjær
Q1(1:2,1:2)=0;   %Q-matrisen med skjær

%Compliance (S-matrix)
S(1,1)=1/EL;
S(2,2)=1/ET;
S(1,2)=-vLT/EL;
S(3,3)=1/GLT;
S(2,1)=S(1,2);

%Stiffness (Q-matrix)
Q=inv(S);

Q1(1,1)=G23;
Q1(2,2)=G13;
Q1(1,2)=0;
Q1(2,1)=Q1(1,2);

Q(3,3)=2*Q(3,3);      %for tensorielle tøyninger ved rotasjon

for i=1:length(R)      %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

    %Transformation matrix (T-matrix)
    T(1,1)=cos(R(i))^2;
    T(2,2)=cos(R(i))^2;
    T(1,2)=sin(R(i))^2;
    T(2,1)=sin(R(i))^2;
    T(3,1)=-sin(R(i))*cos(R(i));
    T(3,2)=sin(R(i))*cos(R(i));
    T(1,3)=2*sin(R(i))*cos(R(i));
    T(2,3)=-2*sin(R(i))*cos(R(i));
    T(3,3)=cos(R(i))^2-sin(R(i))^2;
    %Transformasjon matrise for skjær
    T1(1,1)=cos(R(i));
    T1(1,2)=-sin(R(i));
    T1(2,1)=sin(R(i));
    T1(2,2)=cos(R(i));

    Qk(:,:,i)=inv(T)*Q*T;          %Stivhetsmatrise for tensorielle tøyninger
    Qk(:,3,i)=Qk(:,3,i)/2;        %Går tilbake til ingeniørtøyninger etter rotasjon
    Qk1(:,:,i)=inv(T1)*Q1*T1;     %Skjær

    A=A+Qk(:,:,i)*(h(i+1)-h(i));   %Beregner A-matrisen
    B=B+.5*Qk(:,:,i)*(h(i+1)^2-h(i)^2); %Beregner B-matrisen
    D=D+1/3*Qk(:,:,i)*(h(i+1)^3-h(i)^3); %Beregner D-matrisen
    A1=A1+Qk1(:,:,i)*(h(i+1)-h(i)); %Beregner A1-matrisen
end

A
B
D
A1

C1=-D(1,1)*alpha^2-D(3,3)*beta^2-k*A1(2,2);
C2=-D(1,2)*alpha*beta-D(3,3)*alpha*beta;
C3=-k*A1(2,2)*alpha;
C4=-D(2,2)*beta^2-D(3,3)*alpha^2-k*A1(1,1);
C5=-k*A1(1,1)*beta;

P=(C1*(C5)^2+alpha*C3*(C2)^2+beta*(C2)^2*C5+(C3)^2*C4-alpha*C1*C3*C4
-beta*C1*C4*C5-2*C2*C3*C5)/(alpha^2*C1*C4-alpha^2*(C2)^2) %Knekningslast i x-retning

P2=(C1*(C5)^2+alpha*C3*(C2)^2+beta*(C2)^2*C5+(C3)^2*C4-alpha*C1*C3*C4
-beta*C1*C4*C5-2*C2*C3*C5)/(beta^2*C1*C4-beta^2*(C2)^2) %Knekningslast i y-retning

```



```

N=(C1*C5*C5+alpha*C2*C2*C3+beta*C2*C2*C5+C3*C3*C4-alpha*C1*C3*C4-beta*C1*C4*C5-2*C2*C3*C5)
/(alpha*alpha*C1*C4+gamma*beta*beta*C1*C4-alpha*alpha*C2*C2-gamma*beta*beta*C2*C2) %Biaksial

% Imperfeksjon

W=-N1.*alpha.^2.*Wi./((C5-C2.*C3./C1).*(C2.*C3./C1-C5)./(C4-C2.*C2./C1)-C3.*C3./C1+alpha.*C3+beta.*C5+N1.*alpha.^2);

def=W.*sin(m.*pi.*x./a).*sin(n.*pi.*y./b)+Wi.*sin(mi.*pi.*x./a).*sin(ni.*pi.*y./b);
afaktor=(Wi.*sin(mi.*pi.*x./a).*sin(ni.*pi.*y./b))./(1-(N1./P));

plot(def,N1,'r',afaktor,N1,'--')
xlabel('Displacement, mm')
ylabel('Applied load, N/mm')
legend('w_{tot} from sec. 4.6.1', 'w_{tot} with magn. factor', 0)

% Beregning av knekningskoeffisient

Pansys=30494; % Knekningsverdi fra ANSYS (input)

knekkoeff=P*b^2/(pi^2*(D(1,1)*D(2,2))^(1/2)) % Knekningskoeffisient
knekkoeff2=Pansys*b^2/(pi^2*(D(1,1)*D(2,2))^(1/2)) % Knekningskoeffisient for ANSYS

DD=(D(1,2)+2*D(3,3))/(D(1,1)*D(2,2))^(1/2);

data=knekkoeff-2*DD % den modifiserte knek.koeff
data2=knekkoeff2-2*DD % den modifiserte knek.koeff for ANSYS

```

## Clamped edges FSDT (uniaxial), Case A

```

% Fastinnspent kompositt tykkplate Masteroppgaven
% Case A
clear all;

%Materialdata
EL=49627;
ET=15430;
vLT=0.272;
GLT=4800;
G23=4800;
G13=4800;
a=2000;
b=500;
k=5/6;
n=5;
m=5;
alpha=m*pi/a;
beta=n*pi/b;

% Orientering av lag i radianer og tykkelse av lagene
%R=[-45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
%h=[-4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
%h=[-8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857
2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 0 -45 45 0 45 -45 0 -45 45
0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857
-3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8
9.714 9.857 10 11.714 11.857 12];

```

```

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45
    0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
    -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
    3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
    13.857 14 15.714 15.857 16];

R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45
    -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45
    45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
h=[-24 -23.857 -23.714 -22 -21.857 -21.714 -20 -19.857 -19.714 -18 -17.857 -17.714 -16
    -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857
    -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857
    4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714 13.857 14 15.714
    15.857 16 17.714 17.857 18 19.714 19.857 20 21.714 21.857 22 23.714 23.857 24];

%Initielverdi
A(1:3,1:3)=0;
B(1:3,1:3)=0;
D(1:3,1:3)=0;
Q(1:3,1:3)=0;
S(1:3,1:3)=0;
A1(1:2,1:2)=0;    %A-matrisen med skjær
Q1(1:2,1:2)=0;    %Q-matrisen med skjær

%Compliance (S-matirx)
S(1,1)=1/EL;
S(2,2)=1/ET;
S(1,2)=-vLT/EL;
S(3,3)=1/GLT;
S(2,1)=S(1,2);

%Stiffness (Q-matrix)
Q=inv(S);

Q1(1,1)=G23;
Q1(2,2)=G13;
Q1(1,2)=0;
Q1(2,1)=Q1(1,2);

Q(3,3)=2*Q(3,3);    %for tensorielle tøyninger ved rotasjon

for i=1:length(R)    %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

    %Transformation matrix (T-matrix)
    T(1,1)=cos(R(i))^2;
    T(2,2)=cos(R(i))^2;
    T(1,2)=sin(R(i))^2;
    T(2,1)=sin(R(i))^2;
    T(3,1)=-sin(R(i))*cos(R(i));
    T(3,2)=sin(R(i))*cos(R(i));
    T(1,3)=2*sin(R(i))*cos(R(i));
    T(2,3)=-2*sin(R(i))*cos(R(i));
    T(3,3)=cos(R(i))^2-sin(R(i))^2;
    %Transformasjon matrise for skjær
    T1(1,1)=cos(R(i));
    T1(1,2)=-sin(R(i));
    T1(2,1)=sin(R(i));
    T1(2,2)=cos(R(i));

    Qk(:, :, i)=inv(T)*Q*T;    %Stivhetsmatrise for tensorielle tøyninger
    Qk(:, 3, i)=Qk(:, 3, i)/2;    %Går tilbake til ingeniørtøyninger etter rotasjon
    Qk1(:, :, i)=inv(T1)*Q1*T1;    %Skjær

```

```

A=A+Qk(:, :, i)*(h(i+1)-h(i));           %Beregner A-matrisen
B=B+.5*Qk(:, :, i)*(h(i+1)^2-h(i)^2);    %Beregner B-matrisen
D=D+1/3*Qk(:, :, i)*(h(i+1)^3-h(i)^3);   %Beregner D-matrisen
A1=A1+Qk1(:, :, i)*(h(i+1)-h(i));        %Beregner A1-matrisen
end

A
B
D
A1

%L1=pi^4/(4*a^3*b^3)*(3*D(1,1)*b^4+4*D(3,3)*a^2*b^2+2*D(1,2)*a^2*b^2+3*D(2,2)*a^4);
%L2=k*pi^2/(4*a*b)*(A1(1,1)*a^2+A1(2,2)*b^2);

%P2=(18*(36*L1*pi^2+27*pi^2*L2-sqrt(1296*L1*L1*pi^4-1944*L1*pi^4*L2+729*pi^4*L2*L2+262144*L1*L2))*a)
/(b*(243*pi^4-16384))

C1=pi^4/(8*a^3*b^3)*(6*a^4*D(2,2)+4*D(1,2)*a^2*b^2+8*D(3,3)*a^2*b^2+6*D(1,1)*b^4);
C2=270*pi^2*b/(1440*a);
C3=pi^4/(8*a^3*b^3)*4*a^4*D(2,2);
C4=-2560*b/(1440*a);
C5=pi^4/(8*a^3*b^3)*(6*a^4*D(2,2)+32*D(3,3)*a^2*b^2+96*D(1,1)*b^4+16*D(1,2)*a^2*b^2);
C6=1080*pi^2*b/(1440*a);
C7=-2048*b/(1440*a);
C8=k*pi^2/(4*a*b)*(A1(1,1)*a^2+A1(2,2)*b^2);
C9=360*pi^2*b/(1440*a);

U1=C4^2*C6+C7^2*C2-C2*C6*C9
U2=C2*C6*C8+C1*C6*C9+C2*C5*C9+2*C3*C4*C7-C1*C7^2-C4^2*C5
U3=C3^2*C9-C1*C6*C8-C2*C5*C8-C1*C5*C9
U4=C1*C5*C8-C3*C3*C8

F1=pi^4/(32*a^3*b^3)*(18*m^2*D(1,1)*b^4+8*D(1,2)*a^2*b^2+16*D(3,3)*b^2*a^2+16*D(3,3)*b^2*m^2*a^2
+8*m^2*D(1,2)*a^2*b^2+3*D(1,1)*b^4+3*m^4*D(1,1)*b^4+16*a^4*D(2,2));
F2=(-b/192*(18*pi^2*m^6+18*pi^2+18*pi^2*n^4-18*pi^2*m^4-18*pi^2*m^2+18*pi^2*m^2*n^4-72*pi^2*m^2*n^2
-36*pi^2*n^2-36*pi^2*m^4*n^2))/(a*(1-2*n^2+m^4-2*m^2*n^2-2*m^2+n^4));
F3=(-b/192*(-512*n^3*m+512*(-1)^(m+n+1)*m*n^3))/(a*(1-2*n^2+m^4-2*m^2*n^2-2*m^2+n^4));
F4=k*pi^2/(4*a*b)*(A1(1,1)*a^2+A1(2,2)*n^2*b^2);
F5=(-b/192*(48*pi^2*n^2+48*pi^2*n^6-96*pi^2*m^2*n^4+48*pi^2*m^4*n^2-96*pi^2*n^4-96*pi^2*m^2*n^2)
/(a*(1-2*n^2+m^4-2*m^2*n^2-2*m^2+n^4)));

G1=pi^4/(4*a^3*b^3)*(3*D(1,1)*b^4+2*D(1,2)*a^2*b^2+3*D(2,2)*a^4+4*D(3,3)*a^2*b^2);
G2=- (pi^2*b*(3*n^4-12*n^2))/(16*a*(n^4-4*n^2));
G3=- (8*b*(-n^3+(-1)^(n+2)*n^3))/(3*a*(n^4-4*n^2));
G4=k*pi^2/(4*a*b)*(A1(1,1)*a^2+A1(2,2)*n^2*b^2);
G5=- (pi^2*b*(n^6-4*n^4))/(4*a*(n^4-4*n^2));

if m==1
P=(-(G1*G5+G2*G4)-sqrt((G1*G5+G2*G4)^2-4*(G2*G5-G3*G3)*G1*G4))/(2*(G2*G5-G3*G3))
else
P=(-(F1*F5+F2*F4)-sqrt((F1*F5+F2*F4)^2-4*(F2*F5-F3*F3)*F1*F4))/(2*(F2*F5-F3*F3))
end

```

## In-plane shear FSDT, Case A

```

% Frittoplagt kompositt tykkplate Masteroppgaven
% Case A
clear all;

%Materialdata
EL=49627;
ET=15430;
vLT=0.272;

```

```

GLT=4800;
G23=4800;
G13=4800;
a=2000;
b=500;
k=5/6;
M=5;
N=5;

% Orientering av lag i radianer og tykkelse av lagene
%R=[-45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
%h=[-4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857
 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8];

R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45
 0 45 -45 0 -45 45 0 45 -45]*pi/180;
h=[-12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857
-3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8
9.714 9.857 10 11.714 11.857 12];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 0 -45 45
 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
-7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
13.857 14 15.714 15.857 16];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0
 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45
 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-24 -23.857 -23.714 -22 -21.857 -21.714 -20 -19.857 -19.714 -18 -17.857 -17.714
-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
-7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
13.857 14 15.714 15.857 16 17.714 17.857 18 19.714 19.857 20 21.714 21.857 22
23.714 23.857 24];

%Initielverdi
A(1:3,1:3)=0;
B(1:3,1:3)=0;
D(1:3,1:3)=0;
Q(1:3,1:3)=0;
S(1:3,1:3)=0;
A1(1:2,1:2)=0; %A-matrisen med skjær
Q1(1:2,1:2)=0; %Q-matrisen med skjær

%Compliance (S-matirx)1/4*k*A1(1,1)*pi^2*n^2*a/b+1/4*k*A1(2,2)*m^2*pi^2*b/a
S(1,1)=1/EL;
S(2,2)=1/ET;
S(1,2)=-vLT/EL;
S(3,3)=1/GLT;
S(2,1)=S(1,2);

%Stiffness (Q-matrix)
Q=inv(S);

Q1(1,1)=G23;
Q1(2,2)=G13;
Q1(1,2)=0;
Q1(2,1)=Q1(1,2);

Q(3,3)=2*Q(3,3); %for tensorielle tøyninger ved rotasjon

```

```

for i=1:length(R)          %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

    %Transformation matrix (T-matrix)
    T(1,1)=cos(R(i))^2;
    T(2,2)=cos(R(i))^2;
    T(1,2)=sin(R(i))^2;
    T(2,1)=sin(R(i))^2;
    T(3,1)=-sin(R(i))*cos(R(i));
    T(3,2)=sin(R(i))*cos(R(i));
    T(1,3)=2*sin(R(i))*cos(R(i));
    T(2,3)=-2*sin(R(i))*cos(R(i));
    T(3,3)=cos(R(i))^2-sin(R(i))^2;
    %Transformasjon matrise for skjær
    T1(1,1)=cos(R(i));
    T1(1,2)=-sin(R(i));
    T1(2,1)=sin(R(i));
    T1(2,2)=cos(R(i));

    Qk(:, :, i)=inv(T)*Q*T;          %Stivhetsmatrise for tensorielle tøyninger
    Qk(:, 3, i)=Qk(:, 3, i)/2;      %Går tilbake til ingeniørtøyninger etter rotasjon
    Qk1(:, :, i)=inv(T1)*Q1*T1;     %Skjær

    A=A+Qk(:, :, i)*(h(i+1)-h(i));   %Beregner A-matrisen
    B=B+.5*Qk(:, :, i)*(h(i+1)^2-h(i)^2); %Beregner B-matrisen
    D=D+1/3*Qk(:, :, i)*(h(i+1)^3-h(i)^3); %Beregner D-matrisen
    A1=A1+Qk1(:, :, i)*(h(i+1)-h(i)); %Beregner A1-matrisen
end

A;
B;
D;
A1;

matrise1=zeros(M*N,M*N);
matrise2=zeros(M*N,M*N);
matrise3=zeros(M*N,M*N);
matrise4=zeros(M*N,M*N);
matrise5=zeros(M*N,M*N);
matrise6=zeros(M*N,M*N);
matrise7=zeros(M*N,M*N);
matrise8=zeros(M*N,M*N);
matrise9=zeros(M*N,M*N);
matrise10=sym(zeros(M*N,M*N));
Nxy=sym('Nxy');

% dpi/dx med x-ledd
for m=1:M
    for n=1:N
        matrise1(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*m^2*b/a*D(1,1)
            +1/4*pi^2*n^2*a/b*D(3,3)+1/4*k*a*b*A1(2,2);
    end
end

matrise1;

% dpi/dx med y-ledd
for m=1:M
    for n=1:N
        matrise2(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*m*n*D(1,2)
            +1/4*pi^2*m*n*D(3,3);
    end
end

matrise2;

```

```

% dpi/dx med w-ledd
for m=1:M
    for n=1:N
        matrise3(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(2,2)*b*m*pi;
    end
end
matrise3;

% dpi/dy med x-ledd
for m=1:M
    for n=1:N
        matrise4(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*m*n*D(1,2)+1/4*pi^2*m*n*D(3,3);
    end
end
matrise4;

% dpi/dy med y-ledd
for m=1:M
    for n=1:N
        matrise5(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*n^2*a/b*D(2,2)
        +1/4*pi^2*m^2*b/a*D(3,3)+1/4*k*A1(1,1)*a*b;
    end
end
matrise5;

% dpi/dy med w-ledd
for m=1:M
    for n=1:N
        matrise6(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(1,1)*n*pi*a;
    end
end
matrise6;

% dpi/dw med x-ledd
for m=1:M
    for n=1:N
        matrise7(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(2,2)*m*pi*b;
    end
end
matrise7;

% dpi/dw med y-ledd
for m=1:M
    for n=1:N
        matrise8(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(1,1)*n*pi*a;
    end
end
matrise8;

% dpi/dw med w-ledd
for m=1:M
    for n=1:N
        matrise9(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(1,1)*pi^2*n^2*a/b
        +1/4*k*A1(2,2)*m^2*pi^2*b/a;
    end
end
matrise9;

```

```

% dpi/dw med w-leddpq
for m=1:M
    for n=1:N
        for p=1:M
            for q=1:N
                if m==p
                elseif n==q
                elseif rem(m-p,2)==0
                elseif rem(n-q,2)==0
                elseif rem(m+p,2)==0
                elseif rem(n+q,2)==0
                else
                    i=(m-1)*N+n;
                    j=(p-1)*N+q;
                    matrise10(i,j)=-8*Nxy*m*n*p*q/((m^2-p^2)*(n^2-q^2));
                end
            end
        end
    end
end
end

matrise10;

matrisetot=[matrise1 matrise2 matrise3; matrise4 matrise5
            matrise6; matrise7 matrise8 matrise9+matrise10];

Nxy=solve(det(matrisetot));

double(Nxy)    % velger ut den minste verdien, fortegn spiller ingen rolle

%Beregning av knekningskoeff.

Pansys=35106; % Knekningsverdi fra ANSYS (input)
P=36140;     % Knekningsverdi fra Matlab (input)

knekkoeff=P*b^2/(pi^2*(D(1,1)*D(2,2))^(1/2)) % Knekningskoeffisient
knekkoeff2=Pansys*b^2/(pi^2*(D(1,1)*D(2,2))^(1/2)) % Knekningskoeffisient for ANSYS

DD=(D(1,2)+2*D(3,3))/((D(1,1)*D(2,2))^(1/2))

data=knekkoeff-2*DD          % den modifiserte knek.koeff
data2=knekkoeff2-2*DD       % den modifiserte knek.koeff for ANSYS

```

## Combined load FSDT, Case A

```

% Fritttopplagt kompositt tykkplate Masteroppgaven
% Case A
clear all;

%Materialdata
EL=49627;
ET=15430;
vLT=0.272;
GLT=4800;
G23=4800;
G13=4800;
a=500;
b=500;
k=5/6;
mu=1;

```

```

M=3;
N=3;

% Orientering av lag i radianer og tykkelse av lagene
R=[-45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
h=[-4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857
  2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45
  0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857
  -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8
  9.714 9.857 10 11.714 11.857 12];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 0 -45 45
  0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
  -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
  3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
  13.857 14 15.714 15.857 16];

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0
  45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45
  0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-24 -23.857 -23.714 -22 -21.857 -21.714 -20 -19.857 -19.714 -18 -17.857 -17.714
  -16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
  -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
  3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
  13.857 14 15.714 15.857 16 17.714 17.857 18 19.714 19.857 20 21.714 21.857 22
  23.714 23.857 24];

%Initielverdi
A(1:3,1:3)=0;
B(1:3,1:3)=0;
D(1:3,1:3)=0;
Q(1:3,1:3)=0;
S(1:3,1:3)=0;
A1(1:2,1:2)=0; %A-matrisen med skjær
Q1(1:2,1:2)=0; %Q-matrisen med skjær

%Compliance (S-matrix)1/4*k*A1(1,1)*pi^2*n^2*a/b+1/4*k*A1(2,2)*m^2*pi^2*b/a
S(1,1)=1/EL;
S(2,2)=1/ET;
S(1,2)=-vLT/EL;
S(3,3)=1/GLT;
S(2,1)=S(1,2);

%Stiffness (Q-matrix)
Q=inv(S);

Q1(1,1)=G23;
Q1(2,2)=G13;
Q1(1,2)=0;
Q1(2,1)=Q1(1,2);

Q(3,3)=2*Q(3,3); %for tensorielle tøyninger ved rotasjon

for i=1:length(R) %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

%Transformation matrix (T-matrix)
T(1,1)=cos(R(i))^2;
T(2,2)=cos(R(i))^2;

```



```

T(1,2)=sin(R(i))^2;
T(2,1)=sin(R(i))^2;
T(3,1)=-sin(R(i))*cos(R(i));
T(3,2)=sin(R(i))*cos(R(i));
T(1,3)=2*sin(R(i))*cos(R(i));
T(2,3)=-2*sin(R(i))*cos(R(i));
T(3,3)=cos(R(i))^2-sin(R(i))^2;
%Transformasjon matrise for skjær
T1(1,1)=cos(R(i));
T1(1,2)=-sin(R(i));
T1(2,1)=sin(R(i));
T1(2,2)=cos(R(i));

Qk(:, :, i)=inv(T)*Q*T;           %Stivhetsmatrise for tensorielle tøyninger
Qk(:, 3, i)=Qk(:, 3, i)/2;       %Går tilbake til ingeniørtøyninger etter rotasjon
Qk1(:, :, i)=inv(T1)*Q1*T1;      %Skjær

A=A+Qk(:, :, i)*(h(i+1)-h(i));    %Beregner A-matrisen
B=B+.5*Qk(:, :, i)*(h(i+1)^2-h(i)^2); %Beregner B-matrisen
D=D+1/3*Qk(:, :, i)*(h(i+1)^3-h(i)^3); %Beregner D-matrisen
A1=A1+Qk1(:, :, i)*(h(i+1)-h(i)); %Beregner A1-matrisen
end

A;
B;
D;
A1;

matrise1=zeros(M*N,M*N);
matrise2=zeros(M*N,M*N);
matrise3=zeros(M*N,M*N);
matrise4=zeros(M*N,M*N);
matrise5=zeros(M*N,M*N);
matrise6=zeros(M*N,M*N);
matrise7=zeros(M*N,M*N);
matrise8=zeros(M*N,M*N);
matrise9=sym(zeros(M*N,M*N));
matrise10=sym(zeros(M*N,M*N));
Nk=sym('Nk');

% dpi/dx med x-ledd
for m=1:M
    for n=1:N
        matrise1(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*m^2*b/a*D(1,1)
            +1/4*pi^2*n^2*a/b*D(3,3)+1/4*k*a*b*A1(2,2);
    end
end

matrise1;

% dpi/dx med y-ledd
for m=1:M
    for n=1:N
        matrise2(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*m*n*D(1,2)
            +1/4*pi^2*m*n*D(3,3);
    end
end

matrise2;

% dpi/dx med w-ledd
for m=1:M
    for n=1:N
        matrise3(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(2,2)*b*m*pi;
    end
end

```

```

end

matrise3;

% dpi/dy med x-ledd
for m=1:M
    for n=1:N
        matrise4(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*m*n*D(1,2)+1/4*pi^2*m*n*D(3,3);
    end
end

matrise4;

% dpi/dy med y-ledd
for m=1:M
    for n=1:N
        matrise5(N*(m-1)+n,N*(m-1)+n)=1/4*pi^2*n^2*a/b*D(2,2)
        +1/4*pi^2*m^2*b/a*D(3,3)+1/4*k*A1(1,1)*a*b;
    end
end

matrise5;

% dpi/dy med w-ledd
for m=1:M
    for n=1:N
        matrise6(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(1,1)*n*pi*a;
    end
end

matrise6;

% dpi/dw med x-ledd
for m=1:M
    for n=1:N
        matrise7(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(2,2)*m*pi*b;
    end
end

matrise7;

% dpi/dw med y-ledd
for m=1:M
    for n=1:N
        matrise8(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(1,1)*n*pi*a;
    end
end

matrise8;

% dpi/dw med w-ledd
for m=1:M
    for n=1:N
        matrise9(N*(m-1)+n,N*(m-1)+n)=1/4*k*A1(1,1)*pi^2*n^2*a/b
        +1/4*k*A1(2,2)*m^2*pi^2*b/a-1/4*pi^2*m^2*Nk*b/a;
    end
end

matrise9;

% dpi/dw med w-leddpq
for m=1:M
    for n=1:N
        for p=1:M
            for q=1:N

```

```

        if m==p
        elseif n==q
        elseif rem(m-p,2)==0
        elseif rem(n-q,2)==0
        elseif rem(m+p,2)==0
        elseif rem(n+q,2)==0
        else
            i=(m-1)*N+n;
            j=(p-1)*N+q;
            matrise10(i,j)=-8*mu*Nk*m*n*p*q/((m^2-p^2)*(n^2-q^2));
        end
    end
end
end
end
end

matrise10;

matrisetot=[matrise1 matrise2 matrise3; matrise4 matrise5
            matrise6; matrise7 matrise8 matrise9+matrise10];

Nk=solve(det(matrisetot));

double(Nk)

```

## First ply failure FSDT, Case A

```

% Frittopplagt kompositt tykkplate Masteroppgaven
% Case A
clear all;

%Materialdata
EL=49627;
ET=15430;
vLT=0.272;
GLT=4800;
G23=4800;
G13=4800;
a=500;
b=500;
k=5/6;
n=1;
m=1;
ni=1;           % initiell n
mi=1;           % initiell m
alpha=m*pi/a;
beta=n*pi/b;
alphai=mi*pi/a; % initiell
x=a/2;          % x-koordinat
y=b/2;          % y-koordinat
Wi=0.5;         % initielle geometrisk feil
% Feilkriterier
Xt=968;
Xc=915;
Yt=24;
Yc=118;
S12=65;

% Orientering av lag i radianer og tykkelse av lagene
%R=[-45 45 0 45 -45 0 0 -45 45 0 45 -45]*pi/180;
%h=[-4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4];
%t=8;

```

```

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857
 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8];
%t=16;

R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45
 0 45 -45 0 -45 45 0 45 -45]*pi/180;
h=[-12 -11.857 -11.714 -10 -9.857 -9.714 -8 -7.857 -7.714 -6 -5.857 -5.714 -4 -3.857
-3.714 -2 -1.857 -1.714 0 1.714 1.857 2 3.714 3.857 4 5.714 5.857 6 7.714 7.857 8
9.714 9.857 10 11.714 11.857 12];
t=24;

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 0 -45 45
 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
-7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
13.857 14 15.714 15.857 16];
%t=32;

%R=[-45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0
 45 -45 0 -45 45 0 45 -45 0 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45
 0 -45 45 0 45 -45 0 -45 45 0 45 -45 0 -45 45 0 45 -45]*pi/180;
%h=[-24 -23.857 -23.714 -22 -21.857 -21.714 -20 -19.857 -19.714 -18 -17.857 -17.714
-16 -15.857 -15.714 -14 -13.857 -13.714 -12 -11.857 -11.714 -10 -9.857 -9.714 -8
-7.857 -7.714 -6 -5.857 -5.714 -4 -3.857 -3.714 -2 -1.857 -1.714 0 1.714 1.857 2
3.714 3.857 4 5.714 5.857 6 7.714 7.857 8 9.714 9.857 10 11.714 11.857 12 13.714
13.857 14 15.714 15.857 16 17.714 17.857 18 19.714 19.857 20 21.714 21.857 22
23.714 23.857 24];
%t=48;

%Initielverdi
A(1:3,1:3)=0;
B(1:3,1:3)=0;
D(1:3,1:3)=0;
Q(1:3,1:3)=0;
S(1:3,1:3)=0;
A1(1:2,1:2)=0; %A-matrisen med skjær
Q1(1:2,1:2)=0; %Q-matrisen med skjær

%for First Ply Failure
sigma(1:3,1)=0; % Spenning
Nk(1:3,1)=0; % Påførte laster
toyning(1:3,1)=0; % Tøyning
krumning(1:3,1)=0; % Krumning

%Compliance (S-matirx)
S(1,1)=1/EL;
S(2,2)=1/ET;
S(1,2)=-vLT/EL;
S(3,3)=1/GLT;
S(2,1)=S(1,2);

%Stiffness (Q-matrix)
Q=inv(S);

Q1(1,1)=G23;
Q1(2,2)=G13;
Q1(1,2)=0;
Q1(2,1)=Q1(1,2);

Q(3,3)=2*Q(3,3); %for tensorielle tøyninger ved rotasjon

for i=1:length(R) %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

```

```

%Transformation matrix (T-matrix)
T(1,1)=cos(R(i))^2;
T(2,2)=cos(R(i))^2;
T(1,2)=sin(R(i))^2;
T(2,1)=sin(R(i))^2;
T(3,1)=-sin(R(i))*cos(R(i));
T(3,2)=sin(R(i))*cos(R(i));
T(1,3)=2*sin(R(i))*cos(R(i));
T(2,3)=-2*sin(R(i))*cos(R(i));
T(3,3)=cos(R(i))^2-sin(R(i))^2;
%Transformasjon matrise for skjær
T1(1,1)=cos(R(i));
T1(1,2)=-sin(R(i));
T1(2,1)=sin(R(i));
T1(2,2)=cos(R(i));

Qk(:, :, i)=inv(T)*Q*T; %Stivhetsmatrise for tensorielle tøyninger
Qk(:, 3, i)=Qk(:, 3, i)/2; %Går tilbake til ingeniørtøyninger etter rotasjon
Qk1(:, :, i)=inv(T1)*Q1*T1; %Skjær

A=A+Qk(:, :, i)*(h(i+1)-h(i)); %Beregner A-matrisen
B=B+.5*Qk(:, :, i)*(h(i+1)^2-h(i)^2); %Beregner B-matrisen
D=D+1/3*Qk(:, :, i)*(h(i+1)^3-h(i)^3); %Beregner D-matrisen
A1=A1+Qk1(:, :, i)*(h(i+1)-h(i)); %Beregner A1-matrisen
end

A;
B;
D;
A1;

C1=-D(1,1)*alpha^2-D(3,3)*beta^2-k*A1(2,2); % Variablene i Matrisen
C2=-D(1,2)*alpha*beta-D(3,3)*alpha*beta;
C3=-k*A1(2,2)*alpha;
C4=-D(2,2)*beta^2-D(3,3)*alpha^2-k*A1(1,1);
C5=-k*A1(1,1)*beta;

Nk(1)=3672; % Påførte lasten i x-retning

W=-Nk(1)*alpha^2*Wi/((C5-C2*C3/C1)*(C2*C3/C1-C5)
/(C4-C2*C2/C1)-C3*C3/C1+alpha*C3+beta*C5+Nk(1)*alpha^2);
% Beregne konstanten W i w(x,y). har allerede tatt hensyn til minus-tegnet i Nk(1).

Y=(C2*C3/C1-C5)/(C4-C2*C2/C1)*W; % Beregne konstanten i phi_y(x,y)

X=(-C2*Y-C3*W)/C1; % Beregne konstanten i phi_x(x,y)

krumning(1)=-X*alpha*sin(m*pi*x/a)*sin(n*pi*y/b);
krumning(2)=-Y*beta*sin(m*pi*x/a)*sin(n*pi*y/b);
krumning(3)=(X*beta+Y*alpha)*cos(m*pi*x/a)*cos(n*pi*y/b);

toeyning=inv(A)*(-1*Nk) % tøyning, ikke tatt hensyn til minus-tegnet
% i Nk(1), dermed må multiplisere med -1

% initielle verdi for de nye S- og Q-matrisen, ikke platen, men LAG nr i
QQ(1:3,1:3)=0;
SS(1:3,1:3)=0;

%Compliance (S-matirx)
SS(1,1)=1/EL;
SS(2,2)=1/ET;
SS(1,2)=-vLT/EL;
SS(3,3)=1/GLT;
SS(2,1)=SS(1,2);

```

```

%Stiffness (Q-matrix)
QQ=inv(S);

QQ(3,3)=2*QQ(3,3);          %for tensorielle tøyninger ved rotasjon

for i=37                    %Beregner T og Q for hvert lag 'k'. Summerer A, B og D

    %Transformation matrix (T-matrix)
    TT(1,1)=cos(R(i))^2;
    TT(2,2)=cos(R(i))^2;
    TT(1,2)=sin(R(i))^2;
    TT(2,1)=sin(R(i))^2;
    TT(3,1)=-sin(R(i))*cos(R(i));
    TT(3,2)=sin(R(i))*cos(R(i));
    TT(1,3)=2*sin(R(i))*cos(R(i));
    TT(2,3)=-2*sin(R(i))*cos(R(i));
    TT(3,3)=cos(R(i))^2-sin(R(i))^2;

    QQk=inv(TT)*QQ*TT;      %Stivhetsmatrise for tensorielle tøyninger
    QQk(:,3)=QQk(:,3)/2;   %Går tilbake til ingeniørtøyninger etter rotasjon

    z=(h(i+1)-h(i))/2+h(i); %høyden fra nullpunkt til midten av laget

end

QQk;
z;

sigma=QQk*toyning+z*QQk*krumning

% Feilkriterier
F11=1/(Xt*Xc);
F1=1/Xt-1/Xc;
F22=1/(Yt*Yc);
F2=1/Yt-1/Yc;
F66=1/(S12*S12);
F12star=-0.5;
F12=F12star*sqrt(F11*F22);

% Beregner Tsai-wu
faktor1=F11*(sigma(1))^2+F22*(sigma(2))^2+F66*(sigma(3))^2+2*F12*sigma(1)*sigma(2);
faktor2=F1*sigma(1)+F2*sigma(2);

tsaiwu=1/((-faktor2+sqrt((faktor2)^2+4*faktor1))/(2*faktor1))

```

## C.2 ANSYS

In ANSYS, it is been used script and ANSYS main menu.

**Simply supported with uniaxial**

```

/prep7

!Geometri
k,1,0,0          !koordinat til knutepunkt1
k,2,500,0
k,3,500,500

```

```

k,4,0,500
a,1,2,3,4      !areal laget av knutepunktene 1-2-3-4

!mesh
type,1        !shell91 eller shell181
real,1        !referer til tykkelsen
aesize,all,25 !deklarerer elementstørrelse
amesh,all     !mesh areal

!boundary conditions
nset,s,loc,y,0
d,all,uz,0
d,all,uy,0

nset,s,loc,x,500
d,all,uz,0

nset,s,loc,y,500
d,all,uz,0

nset,s,loc,x,0
d,all,uz,0
d,all,ux,0

!load
nset,s,loc,x,500
cp,1,ux,all
nset,r,loc,y,500
f,all,fx,-1

finish

/solu
antype,static
allsel
pstres,on
solve
finish

/solu
antype,buckle
bucopt,subsp,4
solve
finish

/post1

set,list

finish

```

## Clamped with uniaxial

```

/prep7

!Geometri
k,1,0,0      !koordinat til knutepunkt1
k,2,500,0
k,3,500,500
k,4,0,500
a,1,2,3,4    !areal laget av knutepunktene 1-2-3-4

!mesh

```

```
type,1          !shell91 eller ahell181
real,1          !referer til tykkelsen
aesize,all,25   !deklarerer elementstørrelse
amesh,all       !mesh areal
```

```
!boundary conditions
```

```
nselect,s,loc,y,0
d,all,uz,0
d,all,uy,0
d,all,rotx,0
d,all,roty,0
d,all,rotz,0
```

```
nselect,s,loc,x,500
d,all,uz,0
d,all,rotx,0
d,all,roty,0
d,all,rotz,0
```

```
nselect,s,loc,y,500
d,all,uz,0
d,all,rotx,0
d,all,roty,0
d,all,rotz,0
cp,2,uy,all
```

```
nselect,s,loc,x,0
d,all,ux,0
d,all,uz,0
d,all,rotx,0
d,all,roty,0
d,all,rotz,0
```

```
!load
nselect,s,loc,x,500
cp,1,ux,all
nselect,r,loc,y,500
f,all,fx,-1
```

```
finish
```

```
/solu
antype,static
allsel
pstres,on
solve
finish
```

```
/solu
antype,buckle
bucopt,subsp,4
solve
finish
```

```
/post1
```

```
set,list
```

```
finish
```

**Simply supported with biaxial**

```
/prep7
```



```

!Geometri
k,1,0,0      !koordinat til knutepunkt1
k,2,500,0
k,3,500,500
k,4,0,500
a,1,2,3,4    !areal laget av knutepunktene 1-2-3-4

!mesh
type,1       !shell91
real,1       !referer til tykkelsen
aesize,all,25 !deklarerer elementstørrelse
amesh,all    !mesh areal

!boundary conditions
nset,s,loc,y,0
d,all,uz,0
d,all,uy,0

nset,s,loc,x,500
d,all,uz,0

nset,s,loc,y,500
d,all,uz,0
cp,2,uy,all

nset,s,loc,x,0
d,all,uz,0
d,all,ux,0

!load
nset,s,loc,x,500
cp,1,ux,all
nset,r,loc,y,500
f,all,fx,-1
f,all,fy,-2    !gamma=1 => -1, gamma=0.5 => -0.5, gamma=2 => -2

finish

/solu
antype,static
allsel
pstres,on
solve
finish

/solu
antype,buckle
bucopt,subsp,4
solve
finish

/post1

set,list

finish

```

**Simply supported with in-plane shear**

```
/prep7
```

```

!Geometri
k,1,0,0      !koordinat til knutepunkt1
k,2,500,0
k,3,500,500
k,4,0,500
a,1,2,3,4    !areal laget av knutepunktene 1-2-3-4

!mesh
type,1       !shell191 eller shell281
real,1       !referer til tykkelsen
aesize,all,25 !deklarerer elementstørrelse
amesh,all    !mesh areal

!boundary conditions
nset,s,loc,x,0
d,all,uz,0

nset,s,loc,y,0
d,all,uz,0

nset,s,loc,x,500
d,all,uz,0

nset,s,loc,y,500
d,all,uz,0

nset,s,loc,x,0
nset,r,loc,y,0
d,all,ux,0
d,all,uy,0

nset,s,loc,x,0
nset,r,loc,y,500
d,all,ux,0

!load
nset,s,loc,x,0
nset,r,loc,y,0
f,all,fy,-1/60
f,all,fx,-1/60

nset,s,loc,x,0
nset,r,loc,y,500
f,all,fx,1/60
f,all,fy,-1/60

nset,s,loc,x,500
nset,r,loc,y,500
f,all,fy,1/60
f,all,fx,1/60

nset,s,loc,x,500
nset,r,loc,y,0
f,all,fx,-1/60
f,all,fy,1/60

nset,s,node,,3,41,2      !x=12.5 til x=487.5 , y=0
f,all,fx,-1/60

nset,s,node,,4,40,2
f,all,fx,-2/60

nset,s,node,,43,81,2    !x=500 , y=12.5 til y=487.5
f,all,fy,1/60

```

```

nset,s,node,,44,80,2
f,all,fy,2/60

nset,s,node,,83,121,2      !x=487.5 til x=12.5 , y=500
f,all,fx,1/60

nset,s,node,,84,120,2
f,all,fx,2/60

nset,s,node,,122,160,2    !x=0 , y=487.5 til y=12.5
f,all,fy,-1/60

nset,s,node,,123,159,2
f,all,fy,-2/60

finish

/solu
antype,static
allsel
pstres,on
solve
finish

/solu
antype,buckle
bucopt,subsp,4
solve
finish

/post1

set,list

finish

```

## Simply supported with combined

```

/prep7

!Geometri
k,1,0,0      !koordinat til knutepunkt1
k,2,500,0
k,3,500,500
k,4,0,500
a,1,2,3,4    !areal laget av knutepunktene 1-2-3-4

!mesh
type,1      !shell91 eller shell281
real,1      !referer til tykkelsen
aesize,all,25 !deklarerer elementstørrelse
amesh,all   !mesh areal

!boundary conditions
nset,s,loc,x,0
d,all,uz,0
d,all,ux,0

nset,s,loc,y,0
d,all,uz,0

```

```

nset,s,loc,x,500
d,all,uz,0

nset,s,loc,y,500
d,all,uz,0

nset,s,loc,x,0
nset,r,loc,y,0
d,all,uy,0

!shear load

nset,s,loc,x,0           !punktlast i de fire hjørnene
nset,r,loc,y,0
f,all,fy,-1/60
f,all,fx,-1/60

nset,s,loc,x,0
nset,r,loc,y,500
f,all,fx,1/60
f,all,fy,-1/60

nset,s,loc,x,500
nset,r,loc,y,500
f,all,fy,1/60
f,all,fx,1/60

nset,s,loc,x,500
nset,r,loc,y,0
f,all,fx,-1/60
f,all,fy,1/60

nset,s,node,,3,41,2     !x=12.5 til x=487.5 , y=0
f,all,fx,-1/60

nset,s,node,,4,40,2
f,all,fx,-2/60

nset,s,node,,43,81,2   !x=500 , y=12.5 til y=487.5
f,all,fy,1/60

nset,s,node,,44,80,2
f,all,fy,2/60

nset,s,node,,83,121,2  !x=487.5 til x=12.5 , y=500
f,all,fx,1/60

nset,s,node,,84,120,2
f,all,fx,2/60

nset,s,node,,122,160,2 !x=0 , y=487.5 til y=12.5
f,all,fy,-1/60

nset,s,node,,123,159,2
f,all,fy,-2/60

!uniax load

nset,s,loc,x,500
cp,1,ux,all
nset,r,loc,y,250
f,all,fx,-1

finish

```

```

/solu
antype,static
allsel
pstres,on
solve
finish

/solu
antype,buckle
bucopt,subsp,4
solve
finish

/post1

set,list

finish

```

## Simply supported with imperfection

```

/prep7

!Geometri
k,1,0,0      !koordinat til knutepunkt1
k,2,500,0
k,3,500,500
k,4,0,500
a,1,2,3,4    !areal laget av knutepunktene 1-2-3-4

!mesh
type,1      !shell191 eller shell1281
real,1      !referer til tykkelsen
aesize,all,25 !deklarerer elementstørrelse
amesh,all   !mesh areal

!nummrg,all

!boundary conditions
nset,s,loc,y,0
d,all,uz,0
d,all,uy,0

nset,s,loc,x,500
d,all,uz,0

nset,s,loc,y,500
d,all,uz,0

nset,s,loc,x,0
d,all,uz,0
d,all,ux,0

!load
nset,s,loc,x,500
cp,1,ux,all
nset,r,loc,y,500
f,all,fx,-1

finish

/solu

```

```

antype,static
allsel
pstres,on
solve
finish

/solu
antype,buckle
bucopt,subsp,4
mxpand,4
solve
finish

/post1

set,1,1
pldisp,1
upcoord,5,on

finish

/solu

fdele,all,all      !slette den gamle lasten
nsel,s,loc,x,500
!cp,1,ux,all
nsel,r,loc,y,500
f,all,fx,-825000  !setter på en ny last
allsel

antype,static      !statisk analyse
ngeom,on          !ikke-lineær analyse, inkluderer store deformasjonseffekter
nsubst,100        !lastinkrement(antall steg)
neqit,10          !begrensning på antall iterasjon
arclen,on         !buelengde-alternativ, setter på last gradvis
outres,all,all    !resultater for alle lastinkremerter
allsel
solve

finish

/post26

nsol,2,721,u,z,uzmaks
xvar,2            !bytter om x- og y-aksen
/grid,1
/axlabel,x,Displacement
/axlabel,y,Applied load
/title,nodenr.721(x=250,y=250)
/plopts,info,off
plvar,1,         !viser last-forskyvningskurve

finish

```

## First ply failure

```

/prep7

!Geometri
k,1,0,0          !koordinat til knutepunkt1
k,2,500,0
k,3,500,500
k,4,0,500

```

```

a,1,2,3,4      !areal laget av knutepunktene 1-2-3-4

!mesh
type,1         !shell191 eller shell181
real,1         !referer til tykkelsen
aesize,all,25  !deklarerer elementstørrelse
amesh,all      !mesh areal

!nummrg,all

!boundary conditions
nset,s,loc,y,0
d,all,uz,0
d,all,uy,0

nset,s,loc,x,500
d,all,uz,0

nset,s,loc,y,500
d,all,uz,0

nset,s,loc,x,0
d,all,uz,0
d,all,ux,0

!load
nset,s,loc,x,500
cp,1,ux,all
nset,r,loc,y,500
f,all,fx,-1

finish

/solu
antype,static
allsel
pstres,on
solve
finish

/solu
antype,buckle
bucopt,subsp,4
mxpand,4
solve
finish

/post1

set,1,1
pldisp,1
upcoord,5,on

finish

/solu

fdele,all,all      !slette den gamle lasten
nset,s,loc,x,500
!cp,1,ux,all
nset,r,loc,y,500
f,all,fx,-757000  !setter på en ny last
allsel

antype,static      !statisk analyse

```

```
nlgeom,on
nsubst,100
neqit,25
arclen,on
outres,all,all
allsel
solve

finish
```