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Higher Order Polars and Dual Forms

With Applications to Power Sum Decompositions

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Abstract

We review the classical theory of apolarity and investigate its applications in relation to power sum decompositions. Higher order polars admits, in a natural way, a duality between graded symmetric algebras. This duality can be expressed via a matrix called the catalecticant and we present its close relation to the Waring rank. Finite, zero-dimensional schemes corresponding to Artinian Gorenstein rings are studied, and techniques for finding so-called apolar schemes are presented. For any homogeneous form of even degree one can construct a dual form via apolarity. We investigate how such forms behave in relation to their dual forms. We look at apolar schemes and present precise criteria for determining when the catalecticant and cactus rank for a ternary homogeneous form differ. Lastly, we develop a method for computing explicit power sum decompositions of ternary homogeneous forms of even degree.

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Chapter 1

Introduction

The main goal of this thesis is to review the classical theory of apolarity and show how it relates to decomposing homogeneous polynomials into sums of powers of linear forms. Edward Waring stated in 1770 that every integer is a sum of at most 9 positive cubes. Later, Jacobi and others addressed the problem of determining in how many unique ways an integer can be written in this way. Since then, many problems related to additive decomposition are named after Waring [MM13]. In the 19th century, Sylvester was interested in finding a canonical form for homogeneous polynomials via additive decompositions. A well known classical result is that a homogeneous polynomial $F \in S_d = K[x_0, \ldots, x_n]_d$ can be written on the form

$$F = l_1^d + l_2^d + \ldots + l_s^d,$$

where $l_i \in S_1 = K[x_0, \ldots, x_n]_1$. Today, this is called a *power sum decomposition*, or just a *decomposition* for short, and finding such decompositions for homogeneous forms has become known as the Waring problem for polynomials.

Let $R = K[y_0, \ldots, y_n]$ and $S = K[x_0, \ldots, x_n]$ be two polynomial rings acting upon each other via derivation. This action is explained in detail in Section 2.2. Macaulay showed in 1916 that apolarity gives a bijection between homogeneous polynomials $F \in S_d$ up to scaling, and graded Artinian Gorenstein rings with socle in degree d (see Lemma 2.4.2). The annihilator $F^\perp$ of F under apolarity is named the *apolar ideal* and is defined as the set of forms $G \in R$ such that the derivative of F with respect to G is zero. Via apolarity theory, a decomposition of F into a sum of powers of linear forms corresponds to so-called apolar schemes $X = \{[l_1], \ldots, [l_s]\} \subset \mathbb{P}(S_1)$. These are schemes whose defining ideals are contained in the apolar ideal $F^\perp$. In other words, the coordinates of the points of X correspond to the coefficients of the linear forms of a decomposition. Hence, given a specific F, one may concretely find equations defining apolar schemes corresponding to decompositions $F = l_1^d + \ldots + l_s^d$. This leads us to the first research question of this thesis:

Question 1. *Can apolarity be used to find explicit decompositions?*

With the above question there is no requirement on the number of linear forms s to be minimal. Determining the minimal number of linear forms required, usually called the *rank* of F, was only proven as recently as 1995 for general forms by Alexander and Hirschowitz (see Theorem 2.4.6). Due to the Macaulay correspondence one may equivalently define rank in terms of apolar schemes:

Definition 1.0.1. The *rank* of F is defined as

$$\mathrm{r}(F) = \min\{\text{length of a scheme } X \mid X \subset \mathbb{P}(S_1) \text{ smooth}, \dim X = 0, I_X \subset F^\perp\}.$$

This rather naturally leads to another type of rank called the *cactus* rank, initially studied under the name *scheme length* by Iarrabino and Kanev in 1999 [IK99], but later renamed by Buczynska and Buczsynski in their study of secant and cactus varieties [BB11]. Here the requirement on the schemes to be be smooth is dropped.

Definition 1.0.2. The cactus rank is defined as

$$\mathrm{cr}(F) = \min\{\text{length of a scheme } X \mid X \subset \mathbb{P}(S_1), \dim X = 0, I_X \subset F^{\perp}\}.$$

By definition, the cactus rank is bounded above by the rank. For a homogenous form $F \in S_d$ apolarity yields nontrivial maps

$$\begin{aligned} \mathrm{ap}_F^k : R_k &\to S_{d-k} \\ G &\mapsto G(F), \end{aligned} \tag{1.1}$$

where $G(F)$ denotes the derivative of F with respect to G, for each k between 1 and d. The matrix representation of this operation is called the catalecticant $\mathrm{Cat}_k(F)$ of F, and the maximum of the ranks of these matrices yields a lower bound on the rank of F (see Lemma 2.3.2). For forms of relatively low degree and few variables the cactus and catalecticant rank coincide, but they divert as the degree and number of variables increase. This leads us to the second research question:

Question 2. *Can one find an explicit ternary form $F \in S_{2k}$ such that $\mathrm{rank}\ \mathrm{Cat}_k(F) < \mathrm{cr}\, F$?*

A quadratic form Q on a vector space V can be viewed as a linear map from V to its dual space $V^\vee$ via the map given by first order partial derivatives. When Q is non-singular, this induces an inverse map defined by a quadratic form $Q^\vee$ from $V^\vee$ to V, also defined by first order partial derivatives. This notion of duality might be extended to forms of any even degree. Apolarity induces a duality between R and S in a natural way: For an even homogeneous form $F \in S_{2k}$ the apolarity map ap_F^k defines a linear map $R_k \to S_k$ and a bilinear map

$$\begin{aligned} \Omega_F : R_k \times R_k &\to K \\ (G, H) &\mapsto H(G(F)). \end{aligned} \tag{1.2}$$

To the inverse map $\Omega_F^\vee : S_k \to R_k$ one can associate a polar dual form $F^\vee \in R_{2k}$. In general, $\Omega_F^\vee$ is not an apolarity map with respect to some $F^\vee$, i.e., it is not defined via differentiation. This leads us to the last, rather open ended, research question:

Question 3. *How does a polar dual form $F^\vee$ behave in relation to F?*

1.1 Contribution and results

Before we start with concrete results, we would like to bring attention to the overarching contribution of this thesis. Namely, that this text in its entirety is a modern review of apolarity. As such it is an amalgamation of several books and papers. We have distilled the most relevant parts of apolarity from various sources relating to the Waring problem as described above. We have put great effort into making the entire text as easily readable and understandable as possible. Especially, the translation of the ideas of Dolgachev into a simpler, more easily digestible format, is considered a significant contribution. A consequence of this is that there are not a great number of deep and novel results in this thesis.

We start our treatment of the material with answering the third research question. This is done in Chapter 3. The discussion and results regarding higher order polars and dual forms will fuel much of the theory used to answer the two other research questions. In this chapter, we precisely define the notion of a polar dual form and discuss some of its properties. We present what we call self-polarity (see Definition 3.1.6) and show some results of when binary forms are self-polar. For instance, we show the following proposition:

Proposition 1.1.1. *Let $k \leq 5$ and $F = \sum_{i=0}^{2k} a_i x_0^{2k-i} x_1^i$ be a binary form in S_{2k}. If there exists a factorization*

$$F = (x_0^{k+1} + \lambda x_1^{k+1})(a_0 x_0^{k-1} + \ldots + a_{k-1} x_1^{k-1}) + a_k x_0^k x_1^k,$$

for some scalar λ, then F is self-polar.

The answer to the second question is affirmative. In Chapter 4 we develop an approach for analysing apolar schemes with the use of techniques such as dehomogenization, Hilbert-Burch and Buchsbaum-Eisenbud matrices. These concepts are general and well known, but the way in which they are utilized in this thesis is novel. The approach can in theory generate examples where catalecticant rank and cactus rank differ, but due to computational complexities an explicit example was not found. We define the *isotropy* ideal I of F, which is an ideal determined by the coefficients of F (see Definition 4.2.10). We concretely prove our approach with respect to ternary forms of degree 10, yielding a method for finding forms with catalecticant rank 21 and cactus rank 22.

Theorem 1.1.2. *Let F be a ternary form of degree* 10 *such that* $\operatorname{rank} \operatorname{Cat}_5(F) = 21$*. Then* $\operatorname{cr} F = 22$ *if and only if $V(I) = \emptyset$, where I is the isotropy ideal of F.*

Lastly, in Chapter 5 the first research question is considered and answered affirmatively. Here we define pole schemes, prove that they are apolar schemes (Lemma 5.1.4) and that their defining ideals follow a specific pattern (Proposition 5.1.3). We develop an approach which can be used to find explicit, relatively small, but not minimal, decompositions for ternary forms via the following theorem:

Theorem 1.1.3. *Let $F \in S_{2k}$ be a ternary homogenous form of degree $2k$. Then there is a constructable 3-dimensional family of pole schemes $X = \{[l_1], [l_2], \ldots, [l_{k^2+2}]\} \subset \mathbb{P}(R_1)$ corresponding to (not necessarily minimal) decompositions of F.*

By *constructable* we mean that the defining ideals can explicitly be written down in terms of polynomial equations. For a precise definition of pole schemes see Definition 5.1.1. Furthermore, we investigate if one can expect to find minimal decompositions among pole scheme decompositions with respect to ternary sextics.

1.2 Motivation and impact

The work presented in this project is chiefly concerned with symmetric algebras and zero-dimensional finite schemes. The analysis of these objects are naturally motivated by and of themselves. However, the application of apolarity to the Waring problem is especially rewarding. Furthermore, it can be viewed as an application to tensors which are ubiquitous in electrical engineering, computer science, statistics, quantum physics etc. A regular problem is that of decomposing a tensor into simpler constituents. For example, this task frequently surfaces within Antenna Array Processing, Telecommunications and Statistics, to name a few [Bra+09].

In a space of tensors $V_1 \otimes \ldots \otimes V_d$ of vector spaces V_i over the same ground field, a tensor T on the form $T = v_1 \otimes \ldots \otimes v_d$, where $v_i \in V_i$, is said to be a *rank one* tensor. Given a tensor T it is a frequent occurrence to wish to find a decomposition of T into a sum of rank one tensors. Furthermore, one often wishes for the decomposition to be *minimal*. The minimal length of such a decomposition is called the *rank* of T. This is a generalization of the notion of the rank of a matrix. An important family of tensors are the symmetric tensors. These are the elements which are invariant under the action of the permutation group $\mathfrak{S}_d$ on the tensor space $V^{\otimes^d}$ by permuting the factors. Symmetric tensors can naturally be identified with homogeneous polynomials of degree d in $n+1$ variables. Additive decomposition of a symmetric tensor into sums of rank one symmetric tensors is also naturally identifiable with decomposing homogeneous forms into sums of powers of linear forms [MO20].

Hence, our treatment here of homogeneous forms has direct applications to several applied, scientific fields.

1.3 Assumptions and notation

Unless otherwise specified, the following always applies. We let K denote an algebraically closed field of characteristic zero. More often than not, we will use $\mathbb{C}$ for simplicity, but all our results work over any algebraically closed field of characteristic zero. In a similar vein, one can think of the symbols R and S as symmetric graded algebras. However for convenience, we will frequently refer to R and S as polynomial rings in order to express our results in coordinates. In other words, we think of R as $K[y_0, \ldots, y_n]$ and S as $K[x_0, \ldots, x_n]$. Furthermore, whenever R and S admit bases, they are always assumed to be monomial and lexicographically ordered. The word *form* is used to mean a homogeneous polynomial of positive degree. Quadric, cubic, quartic et cetera, pertains to forms of degree 2, 3 and 4 respectively. Binary and ternary forms are forms of 2 and 3 variables respectively. We use capital letters like F and G for homogeneous forms, and lowercase f and g for inhomogeneous forms, or when homogeneity does not matter. Lastly, whenever we say a polynomial is unique we implicitly mean up to scalar.

1.4 Thesis outline

The structure of the thesis is such that general concepts are introduced first, interspersed with some educational examples, and then proper examples and applications follow afterwards. In Chapter 2 we present the necessary background knowledge needed to read this thesis, provided an already rudimentary understanding of classical algebraic geometry. We present an exposition on apolarity, Artinian Gorenstein algebras, Buchsbaum-Eisenbud and Hilbert-Burch matrices. Chapters 2, 3 and 5 form the main body of this thesis and can be read in any order, given an already detailed knowledge of the subject matter. In Chapter 3 we review higher order polars and generalized dual forms with respect to apolarity and their applications to decompositions. Chapter 4 is about the cactus and catalecticant rank for even homogeneous forms. A novel approach for finding forms where these two notions of rank do not coincide is presented. In Chapter 5 we present a new technique using polarity to find explicit decompositions for homogeneous forms of even degree. In the final chapter we conclude our efforts and discuss further relevant research.

Chapter 2

Apolar rings, catalecticants and power sum decompositions

In this chapter we introduce notational conventions and the basic objects which will be studied in the subsequent chapters. Apolarity is the foundation upon which nearly all of the following material relies. Hence, we treat it very thoroughly. We discuss Artinian Gorenstein rings and their correspondence to apolar schemes and we show how this relates to power sum decompositions. Lastly, we present some techniques for finding apolar schemes via Hilbert-Burch and Buchsbaum-Eisenbud matrices and dehomogenization.

2.1 Classical pole and polar

The concept of pole and polar was known about in classical Euclidean geometry around year 300 BC. It got a renaissance with the rise of projective geometry in France in the 17th century. In the 19th century by the works of Plücker among others, pole and polar got an analytic foundation and were generalized to higher dimensions.

In the plane, pole and polar denotes a correspondence between points and lines with respect to a conic. Consider at first the affine plane with a conic. For any point outside the conic, one can uniquely draw two tangents from the conic intersecting in the selected point. The intersections of the tangents with the conic defines two points, which yields a secant to the conic. The original point is what is referred to as a pole, while the secant is the corresponding polar.

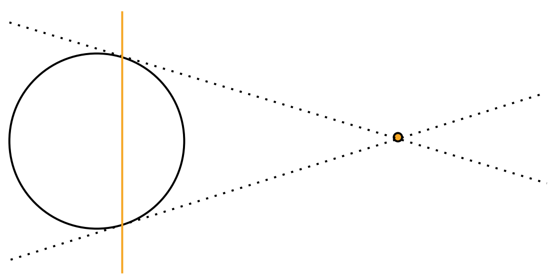


Figure 2.1: Pole and polar with respect to a conic. The orange point to the right is the pole and the orange, vertical secant is the polar.

Any conic in the plane can be expressed via a quadratic form

$$q = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0. \tag{2.1}$$

In affine space there are lines that do not intersect, i.e., they are parallel. This implies that there are polars in affine space that do not correspond to poles. To remedy this problem, one introduces a third coordinate z, yielding a homogeneous quadratic form

$$Q = ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = 0. \tag{2.2}$$

This can be rewritten into a more compact form

$$Q = x^{\mathrm{T}}Cx, \tag{2.3}$$

where C is the symmetric matrix

$$C = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}. \tag{2.4}$$

For a point $p \in \mathbb{P}^2$ one can compute the corresponding polar via the quadratic relation

$$x^{\mathrm{T}}Cp = 0. \tag{2.5}$$

Generalizing, one can take a homogeneous form of any even degree and form a matrix as above. Hence, a homogeneous form of even degree $2k$ on a vector space V naturally defines a quadratic form on the space of forms of degree k on the dual space $V^\vee$. We will study this generalization in much greater detail in Chapter 3. Before that is possible some formal language must be introduced.

2.2 Apolarity

Let R and S be the graded polynomial rings $R = K[y_0, \ldots, y_n]$ and $S = K[x_0, \ldots, x_n]$ over an algebraically closed field K of characteristic 0. We let R act on S by means of differentiation

$$y^\beta(x^\alpha) = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!}x^{\alpha-\beta} & \text{if } \alpha - \beta \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Here, α and β are multi-indices, i.e., $\alpha = \{(a_0, \ldots, a_n) \mid a_i \in \mathbb{N}_0\}$ and we use the following vector notation

$$\alpha! = a_0! \ldots a_n!, \qquad \binom{k}{\alpha} = \frac{k!}{\alpha!}, \qquad |\alpha| = a_0 + \ldots + a_n.$$

A key fact to observe is the correspondence between evaluation and derivation. For two linear forms $\psi = b_0y_0 + \ldots + b_ny_n \in R_1$ and $l = a_0x_0 + \ldots a_nx_n \in S_1$, derivation corresponds to evaluation,

$$\psi(l) = \sum_{i=0}^{n} a_ib_i = \psi(a).$$

Whenever ambiguity might appear, we use subscripts for linear forms $l_a \in S_1$, to mean that the coefficients of l_a is expressed in terms of a_is. Hence, $\psi(l_a)$ denotes the derivative of l_a with respect to ψ, while $\psi(a)$ denotes the evaluation of ψ in the coefficients of l_a. We identify $\mathbb{P}(S_1) = \mathbb{P}^n$ by letting a projective coordinate $(a_0 : \ldots : a_n)$ be mapped to a linear form $l_a = a_0y_0 + \ldots + a_ny_n \in S_1$. We have a few basic results of derivation whenever powers of linear forms and involved.

Lemma 2.2.1. *Let ψ and l be linear forms in R_1 and S_1 respectively. For $k \leq d \in \mathbb{N}$ we have that*

$$\psi^k(l^d) = d(d-1)\cdots(d-k+1)l^{d-k}(\psi(a))^k. \tag{2.7}$$

Proof. For a simple linear form, derivation yields

$$\psi(l^d) = dl^{d-1}\psi(l) = dl^{d-1}\psi(a).$$

Furthermore, we have that

$$\psi^k(l^d) = \psi^{k-1}(\psi(l^d)),$$

which implies the result. □

The order of operations does not matter for linear forms of the same degree.

Lemma 2.2.2. *Let ψ_b and l_a be linear forms in R_1 and S_1 respectively. Then*

$$l_a^d(\psi_b^d) = \psi_b^d(l_a^d).$$

Proof. By Lemma 2.2.1 and the fact that $\psi_b(a) = l_a(b)$ the result follows immediately. □

The correspondence between differentiation and evaluation can be extended further:

Lemma 2.2.3. *Let $g \in R_k$ and $l_a \in S_1$. Then for all $m \geq k$ we have that*

$$g(l_a^m) = 0 \iff g(a) = 0. \tag{2.8}$$

Combining the previous lemma with Lemma 2.2.2, we have the following corollary.

Corollary 2.2.4. *If $g \in R_k$ and $l \in S_1$, then*

$$g(l^k) = 0 \iff g(a) = 0 \iff l^k(g) = 0. \tag{2.9}$$

Proof. Any $g \in R_k$ can be written as a sum of s powers of linear forms for some $s \in \mathbb{N}$. Hence,

$$g(l^k) = (\psi_1^k + \ldots + \psi_s^k)(l^k) = \psi_1^k(l^k) + \ldots + \psi_s^k(l^k).$$

Applying Lemma 2.2.2 we can switch the order of operations, yielding the desired result

$$g(l^k) = l^k(\psi_1^k) + \ldots + l^k(\psi_s^k) = l^k(g).$$

□

Moving forward, we use the following definition for the map given by Equation (2.6), following the notation of Dolgachev [Dol12].

Definition 2.2.5. Let $F \in S_d$ be a homogeneous form. Let ap_F^k denote the map

$$\begin{aligned} \mathrm{ap}_F^k : R_k &\to S_{d-k} \\ G &\mapsto G(F), \end{aligned} \tag{2.10}$$

where $G(F)$ denotes the derivative of F with respect to G. We call ap_F^k the *apolarity map* of F.

Consider the following example, showing how ap_F^k maps elements in R_k to elements in S_{d-k}, as well as the correspondence between evaluation and derivation.

Example 2.2.6. Let $F = x_0^4 + x_0^2x_1^2 \in S_4$ and $G = y_0^2 \in R_2$. Then

$$\mathrm{ap}_F^2(G) = G(F) = y_0^2(x_0^4+x_0^2x_1^2) = y_0^2(x_0^4)+y_0^2(x_0^2x_1^2) = \frac{\partial}{\partial x_0^2}(x_0^4)+\frac{\partial}{\partial x_0^2}(x_0^2x_1^2) = 12x_0^2+2x_1^2.$$

The correspondence between evaluation and derivation tells us that since $F(0,b) = 0$, for all $b \in K$, then any linear form $H = (0y_0 + by_1)^4$ yields zero when taking the derivative of F with respect to it. This is easily checked

$$H(F) = b^4\frac{\partial}{\partial x_1^4}(x_0^4 + x_0^2x_1^2) = 0.$$

♣

Apolarity is by definition closely related to the concept of pole and polar. Formally, we define a polar with respect to a point (a pole) as:

Definition 2.2.7. Let $X = V(f)$ be a hypersurface of degree d in $\mathbb{P}^n$ and $p = [l]$ be a point in $\mathbb{P}^n$. The hypersurface

$$P_{a^k}(X) := V(l^k(f))$$

of degree $d - k$ is called the k-th polar hypersurface of the point p with respect to the hypersurface $V(f)$ (or of the hypersurface with respect to the point).

Perhaps the most important consequence of pole and polar is the reciprocity theorem: Given a pole l_a and its corresponding polar P_a, any pole l_b lying on the the polar P_b admits another polar P_b which contains the original pole l_a.

Theorem 2.2.8 (Polar reciprocity). *Let F be a homogeneous polynomial in S_d in $n+1$ variables. Let a and b be two points in $\mathbb{P}^n$. Then*

$$b \in P_{a^k}(X) \iff a \in P_{b^{d-k}}(X) \tag{2.11}$$

Proof. As points in $\mathbb{P}^n$, a and b correspond to linear forms l and l' in R_1 respectively. We have that $a \in P_{b^{d-k}}(X)$ means that $l^k(l'^{d-k}(F)) = 0$. By Corollary 2.2.4, we get that $l'^{d-k}(l^d(F)) = 0$ which means that $b \in P_{a^k}(X)$ and we are done. □

This theorem was classically known, but first stated in the generality and form presented here by Dolgachev [Dol12].

2.3 The catalecticant

Let $\mathrm{Cat}_k(F)$ denote the matrix of ap_F^k with respect to monomial lexicographic bases of R and S. This matrix is called the kth *catalecticant*[1] of F. It was first described by Sylvester in 1852 [Syl52]. The entries of $\mathrm{Cat}_k(F)$ are linear forms in the coefficients of F and the size of the matrix is $\binom{k+n}{k} \times \binom{n+d-k}{d-k}$. The entries $c_{\mathbf{uv}}$ are parameterized by pairs $(\mathbf{u},\mathbf{v}) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$ with $|\mathbf{u}| = d - k$ and $|\mathbf{v}| = k$. If one writes

$$F = \sum_{|\mathbf{i}|=d} \binom{d}{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}},$$

[1] Originally the term catalecticant was termed by Sylvester and was used to denote the determinant of the catalecticant matrix [Syl04]. In other words the catalecticant meant a polynomial in the coefficients of F. However, we here simply use the word catalecticant to refer to the catalecticant matrix.

then

$$c_{\mathbf{uv}} = a_{\mathbf{u}+\mathbf{v}}.$$

Furthermore, the kernel of the catalecticant is the space $\mathrm{AP}_k(F)$ of forms of degree k which are apolar to F. For catalecticants of degree k, where $d = 2k$, the size of the matrix coincides with the dimension of the space of hypersurfaces of degree k in $\mathbb{P}^n$.

Example 2.3.1. If F is a binary polynomial of the following form

$$F = \sum_{i=0}^{d} \binom{d}{i} a_i x_0^{d-i} x_1^i,$$

then the catalecticant is given by

$$\mathrm{Cat}_k(F) = \begin{pmatrix} a_0 & a_1 & \dots & a_k \\ a_1 & a_2 & \dots & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-k} & a_{d-k+1} & \dots & a_d \end{pmatrix}. \tag{2.12}$$

♣

A fundamental aspect of the catalecticant is its relation to the Waring rank. The following lemma shows that the catalecticant rank is bounded above by the Waring rank.

Lemma 2.3.2. *If $F = l_1^d + \ldots + l_s^d$ where $l_i \in S_1$ then* $\operatorname{rank} \mathrm{Cat}_k(F) \leq \operatorname{rank} F = s$.

Proof. For any $\psi \in R_k$ we have that

$$D_\psi(F) = D_\psi\left(\sum_{i=1}^{s} l_i^d\right) = \sum_{i=1}^{s} \psi(l_i) l_i^{[d-k]}.$$

This shows that $\mathrm{ap}_F^k(R_k) \subset \left\langle l_1^{d-k}, \ldots, l_s^{d-k} \right\rangle$ and hence the desired result. □

In the special case that F is a power of a single linear form, then we have that the catalecticant and Waring rank coincide exactly.

Lemma 2.3.3. *A homogeneous polynomial $F \in R_{2k}$ admits a k-th catalecticant of rank 1 if and only if $F = l^{2k}$, where $l \in R_1$.*

Proof. For any $G \in R_k$ we have that

$$G(F) = G(l^{2k}) = l^{[2k-k]} G(l).$$

Hence, l^k forms a basis for the image of the catalecticant and thus the rank is 1. □

The catalecticant is a symmetric matrix, but in addition it has some extra symmetry along the diagonals going from bottom left to top right.

Example 2.3.4. If F is a ternary quartic of the following form

$$F = \sum_{|\mathbf{i}|=4} \binom{4}{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}}$$

then the catalecticant is

$$\mathrm{Cat}_2(F) = \begin{pmatrix} a_{400} & a_{310} & a_{301} & a_{220} & a_{211} & a_{202} \\ a_{310} & a_{220} & a_{211} & a_{130} & a_{121} & a_{112} \\ a_{301} & a_{211} & a_{202} & a_{031} & a_{112} & a_{013} \\ a_{220} & a_{130} & a_{031} & a_{040} & a_{031} & a_{022} \\ a_{211} & a_{121} & a_{112} & a_{031} & a_{022} & a_{013} \\ a_{202} & a_{112} & a_{013} & a_{022} & a_{013} & a_{004} \end{pmatrix}. \tag{2.13}$$

♣

2.4 Artinian Gorenstein rings, apolar schemes and power sum decompositions

Definition 2.4.1. For a homogeneous polynomial $F \in S_d$ the *apolar ideal* $F^\perp$ is the ideal of forms annihilating F,

$$F^\perp = \{G \in R \mid G(F) = 0\}. \tag{2.14}$$

The ideal $F^\perp$ is a homogeneous ideal. We call the quotient $A_F = S/F^\perp$ the *apolar ring*. The apolar ring is Artinian and Gorenstein. There are many equivalent definitions of Gorenstein. For example Eisenbud defines a zero-dimensional local ring A to be Gorenstein if $A \cong \omega_A$, where ω_A is the canonical module [Eis95]. Our interest in Gorenstein rings however does not stem from such technical properties, but rather from the following correspondence by Macaulay:

Lemma 2.4.2 (Macaulay 1916 [Mac16]). *The map $F \mapsto A_F$ is a bijection between homogeneous forms $F \in R_d$ and graded Artinian Gorenstein quotient rings $A_F = S/I$ of S with socle degree d.*

Hence, any ring S/I where I is an apolar ideal to some homogeneous form F is Gorenstein. We will now see that Artinian Gorenstein rings correspond to a certain type of schemes called *apolar* schemes.

Definition 2.4.3. A subscheme $X \subset \mathbb{P}^n = \mathbb{P}(R_1)$ is said to be *apolar* to F if the homogeneous ideal $I_X \subset F^\perp \subset S$.

Furthermore, we can interpret apolarity via the Veronese map

$$\begin{aligned} v_d : \mathbb{P}(R_1) &\to \mathbb{P}(R_d) \\ [l] &\mapsto [l^d]. \end{aligned} \tag{2.15}$$

If $X \subset \mathbb{P}^n$ one can observe that

$$\langle v_d(X) \rangle = (I_X)_d^\perp \subset \mathbb{P}(R_d). \tag{2.16}$$

In other words the linear span of $v_d(X)$ is determined by the apolar ideal $(I_X)_d^\perp$. Hence, we can correspond an apolar scheme X to its d-th graded apolar ideal. This leads to a very fundamental result, coined the apolarity lemma, connecting apolar schemes and power sum decompositions.

Lemma 2.4.4 (Apolarity). *$F = l_1^d + \ldots + l_s^d$ where $X = \{[l_1], \ldots, [l_s]\} \subset \mathbb{P}(R_1)$ if and only if X is apolar to F.*

Proof. Let $F = l_1^d + \ldots + l_s^d \in S_d$ be a sum of powers of s linearly independent forms $l_i \in S_1$. For any $G \in R_d$ we have that

$$\begin{aligned} G(F) &= G\left(\sum_{i=1}^{s} l_i^d\right) \\ &= \sum_{i=1}^{s} G(\mathbf{a_i}). \end{aligned}$$

By definition the apolar ideal of F is

$$F^\perp = \{H \in R \mid H(F) = 0\}.$$

It is clear from the above expression that all G such that each $G(\mathbf{a_i}) = 0$ is a subset of $F^\perp$. This can be written as the following ideal

$$(I_X)_d = \Big\{G \in R_d \mid G(l_i^d) = 0, i = 1, \ldots, s\Big\} \subset F^\perp. \tag{2.17}$$

Let X be the closed reduced subscheme of points $\{[l_1], \ldots, [l_s]\} \subset \mathbb{P}(R_1)$. Its defining ideal is precisely the one above. Hence, we see that $I_X \subset F^\perp$. □

Alternative proofs can be found in [Ber+17, §3] and [Tei14, §4]. Let us quickly look at a concrete example of how power sum decompositions act in relation with apolar schemes.

Example 2.4.5. Let $F = x_0^3 + x_1^3 \in R_3$. This polynomial consists of two linear forms, hence the corresponding apolar scheme contains two elements. Explicitly, we have

$$X = \{[l_1], [l_2]\} = \{(1:0), (0:1)\} \subset \mathbb{P}^1. \tag{2.18}$$

There exists many ideals that correspond to this set of points, strictly considered as a set. However, since we want a scheme in $\mathbb{P}^1$ the defining ideal must be homogeneous and we have that

$$I_X = (y_0 y_1). \tag{2.19}$$

The apolar ideal $F^\perp$ can be computed directly

$$F^\perp = \{G \in S \mid G(x^3 + y^3) = 0\} = \left(y_0 y_1, y_0^4, y_1^4, y_0^3 - y_1^3\right). \tag{2.20}$$

Clearly, $I_X \subset F^\perp$. ♣

There are two specific questions regarding decompositions that have been extensively studied classically, and still to this day drive further research:

1. Determine the rank of a homogeneous form F.
2. Given the rank s, of a homogeneous form, determine the size of the family of decompositions of length s.

Finding the minimal s was solved for *general* homogeneous polynomials by Alexander and Hirschowitz in 1995 [AH95].

Theorem 2.4.6 (Alexander-Hirschowitz). *A general form F of degree d in $n+1$ variables is a sum of $s = \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil$ powers of linear forms, unless*

$$\begin{aligned} &d = 2, \ \textit{where } s = n+1 \textit{ instead of } \lceil \frac{n+2}{2} \rceil \\ &d = 4 \textit{ and } n = 2,3,4, \ \textit{where } s = 6, 10, 15 \textit{ instead of } 5, 9, 14, \ \textit{respectively} \\ &d = 3 \textit{ and } n = 4, \ \textit{where } s = 8 \textit{ instead of } 7. \end{aligned}$$

The exceptions were classically known, but Alexander and Hirschowitz were the first to rigorously prove that these are indeed all possible exceptions [Kle95]. Naturally, when F is not general, the number of linear forms required can be both larger or smaller than that indicated by the theorem.

The second question is about how many different ways it is possible to write an F as a sum of powers of s linear forms. A compactification of this is usually called the Variety of Sums of Powers (VSP). There is a vast amount of research done by authors like Sylvester, Massarenti and Mella, Mukai, Dolgachev, Ranestad and Schreier, to name a few [MM09]. Much is known about the VSP for general ternary forms of relatively low degree, but for forms in four variables and more, and for forms of higher degree, much is still uncharted. For instance, for a general ternary form of degree 10 it has only *numerically* been shown that such a form admits 320 minimal decompositions [Hau+16].

2.5 Apolar rings and catalecticants

Since an apolar scheme X is such that its defining ideal I is a subideal of the apolar ideal $F^\perp$, then $F^\perp$ and hence A_F carry a lot of important information. In this section we will see that the Hilbert function of A_F and the rank of the catalecticants $\mathrm{Cat}_k(F)$ coincide.

Definition 2.5.1. We define $\mathrm{Diff}\, F$ to be the space of partial derivatives of $F \in S_d$

$$\mathrm{Diff}\, F = \{\psi(F) \mid \psi \in R\}.$$

The space of partial derivatives $\mathrm{Diff}\, F$ is naturally isomorphic to the apolar ring A_F. Additionally we let $\mathrm{ldiff}\, F$ denote the maximum dimension of the space of kth order partial derivatives as k runs from 0 to $\deg F$.

Whenever we mention the Hilbert function H_{A_F} we mean this to be the Hilbert function of the apolar ring A_F. We let H_F denote the sequence whose entries are the rank of the catalecticant for each degree between 0 and d. In other words,

$$H_F = (\mathrm{rank}\, \mathrm{Cat}_0(F), \dots, \mathrm{rank}\, \mathrm{Cat}_d(F)).$$

Proposition 2.5.2. *The dimension of the apolar ring $(A_F)_k$ corresponds to the rank of the catalecticant $\mathrm{Cat}_k(F)$,*

$$\dim(A_F)_k = \mathrm{rank}\, \mathrm{Cat}_k(F).$$

Proof. Recall that $F_k^\perp$ is precisely the kernel of the apolarity map $\mathrm{ap}_F^k : R_k \to S_{d-k}$ which defines the matrix $\mathrm{Cat}_k(F)$. An element $v \in R_k$ is in $\ker \mathrm{Cat}_k(F)$ if and only if the derivative of F with respect to v is zero. By definition we have that $v \in \ker \mathrm{Cat}_k(F)$ is equivalent with $v \in F_k^\perp$. Thus, $F_k^\perp = \ker \mathrm{Cat}_k(F)$. The elements not in the kernel of $\mathrm{Cat}_k(F)$ are precisely the elements in $(A_F)_k$. Since the rank of the catalecticant is nothing but the dimension of its image, the result is clear. □

This gives us an easy way to find examples of homogeneous forms with any catalecticant rank.

Example 2.5.3. Let $F \in S_4$ be a binary polynomial. We have that either

$$H_{A_F} = (1,1,1,1,1), H_{A_F} = (1,2,2,2,1) \text{ or } H_{A_F} = (1,2,3,2,1).$$

The sizes of the first and third catalecticant are 2×4 and 4×2 respectively. The size of the second catalecticant is 3×3. As stated earlier, we say that F is general when the square catalecticant has maximal rank. As such, only $H_{A_F} = (1, 2, 3, 2, 1)$ corresponds to a general polynomial. In this case we have that A_F contains two elements of degree 1 and 3, and three elements of degree 2. ♣

The previous example alludes to a more general fact; the Hilbert polynomial of an Artinian Gorenstein ring is rather simplistic looking.

Corollary 2.5.4. *The Hilbert polynomial $H_F(t)$ is a reciprocal monic polynomial.*

Proof. Since the rank of the catalecticant and its transpose are the same, then $H_F(t)$ is a reciprocal monic polynomial. □

This provides a very useful tool in studying the different possible Hilbert functions for homogeneous forms. Additionally, the previous lemma gives us that

$$H_F(t) = \sum_{k=0}^{d} \operatorname{rank} \operatorname{Cat}_k(F) t^k.$$

This means that the coefficient at t^k in $H_F(t)$ is equal to the rank of $\operatorname{Cat}_k(f)$. Frequently it will be convenient to write the Hilbert function as a sequence which is finite due to A_F being Artinian, and in our notation we will frequently omit the trailing zeroes.

Definition 2.5.5. We call the sequence H_F the *Hilbert sequence.*

2.6 Subideals of the apolar ideal

As we have seen, there is a correspondence between Artinian Gorenstein rings and zero-dimensional finite schemes, given via apolarity. Hence, in our study to come, a frequent problem will be that of finding subideals of the apolar ideal $F^\perp$. Here we present some techniques for finding such subideals. First we need some commutative algebra and we will use some definitions following Eisenbud [Eis04; Eis95].

Definition 2.6.1. A ring such that $\operatorname{depth} I = \operatorname{codim} I$ for every maximal prime ideal I of R is called *Cohen-Macaulay.*

Definition 2.6.2. A projective variety (scheme) $X \subset \mathbb{P}^n$ is called *arithmetically Cohen-Macaulay* if the homogeneous coordinate ring $S_X = \frac{K[x_0,\ldots,x_n]}{I(X)}$ is Cohen-Macaulay.

Localization preserves the Cohen-Macaulay property. The Auslander-Buchsbaum formula [Eis04] states that given a local ring R and a finitely generated R-module I with finite projective dimension, that

$$\operatorname{depth} I = \operatorname{depth} R - \operatorname{pd} I,$$

where pd denotes the projective dimension (which is the minimal length of a projective resolution). If R is a Noetherian ring, I an ideal in R, and M a finitely generated R-module then we say that the depth of I on M, written $\operatorname{depth}(I, M)$, is the supremum of the lengths of all M-regular sequences of elements of I. Any scheme corresponding to a finite set of points in $\mathbb{P}^2$ is Cohen-Macaulay. Such schemes also have a free resolution of length 1. We give a reformulation of the proof from [Eis04][Proposition 3.1].

Proposition 2.6.3. *If $I \subset S$ is the homogeneous ideal of a finite set of points in $\mathbb{P}^2$, then I has a free resolution of length* 1.

Proof. The Auslander-Buchsbaum formula tells us that

$$\operatorname{pd} S/I = \operatorname{depth} S - \operatorname{depth} S/I.$$

The depth of S is 3 since the coordinates form a maximal homogeneous regular sequence. We have that $\operatorname{depth} S/I \leq \dim S/I = 1$. However, $\operatorname{depth} S/I$ can not be 0, since the maximal homogeneous ideal m of S is not associated to I. Hence, $\operatorname{pd} S/I = 3 - 1 = 2$. Because I is the first module of syzygies in the free resolution of S/I, then we must have that $\operatorname{pd} I = 1$ completing the proof. □

2.6.1 Hilbert-Burch and Buchsbaum-Eisenbud matrices

The preceding discussion leads us to two important practical tools for finding subideals of the apolar ideal $F^{\perp}$. We largely follow the notation and ideas presented in [Bru23]. The first tool stems from the following theorem called the Hilbert-Burch theorem.

Theorem 2.6.4 ([Eis95], Theorem 20.15). *Let $X \subset \mathbb{P}^2$ be a finite scheme. Then I_X is generated by a $(\beta - 1) \times \beta$-matrix A and the resolution of I_X is*

$$0 \longrightarrow S^{\beta-1} \xrightarrow{A} S^{\beta} \longrightarrow I_X \longrightarrow 0.$$

Conversely, if A is a $(\beta - 1) \times \beta$-matrix where the $(\beta - 1)$-minors have no common factor, then the minors generate the ideal of a finite scheme.

We call the matrices A appearing in such a resolution for *Hilbert-Burch* matrices. This result gives a correspondence between matrices which are easy to work with and ideals I contained in the apolar ideal $F^{\perp}$.

Let M be an R-module. We say that $f : M^{\vee} \to M$ is an alternating map if there exists a basis such that f is presentable as a skew symmetric matrix. The pfaffian $\mathrm{Pf}(A)$ of a matrix A is the square root of the determinant. The $(n-1)$th order pfaffians Pf_{n-1} are the square roots of the determinants of a matrix having removed one row and its corresponding column. We let $\mathrm{Pf}_{n-1}(A)$ denote the ideal generated by the $(n-1)$th order pfaffians of A. By the works of Buchsbaum and Eisenbud, we have the Buchsbaum-Eisenbud theorem:

Theorem 2.6.5 ([BE77], Theorem 2.1). *Let R be a Noetherian local ring with maximal ideal J.*

1. *Let $n \geq 3$ be an odd integer and let M be a free R-module of rank n. Let $f : M^{\vee} \to M$ be an alternating map whose image is contained in JM. Suppose $Pf_{n-1}(f)$ has codimension 3. Then $Pf_{n-1}(f)$ is a Gorenstein ideal, minimally generated by n elements.*
2. *Every Gorenstein ideal of codimension 3 arises as above.*

We are chiefly interested in graded polynomial rings and homogeneous ideals, and the Buchsbaum-Eisenbud theorem holds in this case. Hence, the following corollary is more easily applicable in our setting:

Corollary 2.6.6 ([Bru23], Corollary 2.3.4). *Let $n \geq 3$ be an odd integer and let R be a graded polynomial rings in three variables.*

1. *Let $B = (b_{ij})$, where $b_{ij} \in R$ is homogeneous and $b_{ij} \notin \mathbb{C}^{\star}$, be a skew symmetric matrix of dimension n. Assume $Pf_{n-1}(B)$ has codimension 3. Then $Pf_{n-1}(B)$ is the apolar ideal of a homogeneous $F \in R$ minimally generated by n elements.*

2. *Let $I \subset S$ be a Gorenstein ideal of codimension 3. Then I is minimally generated by $Pf_{n-1}(B)$, where B is a skew symmetric matrix whose columns form a minimal basis for the syzygies of I.*

We call the matrices B appearing above Buchsbaum-Eisenbud matrices. A useful fact is that Hilbert-Burch matrices appear as sub-matrices of Buchsbaum-Eisenbud matrices.

Lemma 2.6.7 ([Bru23], Lemma 5.2.1). *Let $F^{\perp} \subset R$ be an apolar ideal and X a finite scheme. Let A denote the Hilbert-Burch matrix of the ideal I_X corresponding to X and let B the Buchsbaum-Eisenbud matrix of $F^{\perp}$. If the generators of I_X are linear combinations of the generators in $F^{\perp}$, then A is a submatrix of B.*

Proof. Theorem 2.6.5 and Theorem 2.6.4 yield two exact sequences connected by inclusions:

$$\begin{tikzcd} & 0 \arrow[r] & R^{\beta-1} \arrow[r, "A"] \arrow[d, hook, "\psi"] & R^{\beta} \arrow[r] \arrow[d, hook, "\phi"] & I_X \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & R \arrow[r] & R^n \arrow[r, "B"] & R^n \arrow[r] & F^{\perp} \arrow[r] & 0. \end{tikzcd}$$

Clearly, the diagram commutes and thus A is a submatrix of B. □

2.6.2 Dehomogenization

Another method for finding subideals of the apolar ideal was discovered by Bernardi and Ranestad in 2013 and involves investigating dehomogenizations of homogeneous forms with respect to linear forms [BR13]. Furthermore, they showed that this technique can be used to give bounds on the cactus rank, which we present and utilize in Chapter 4. Here we present the dehomogenization procedure.

Let l be a linear form in S and let $F \in S_d$. Naturally, l can be included in a basis for S_1. We denote such a basis of S_1 by $\{l, l_1, \ldots, l_n\}$. Dually, R_1 has basis $\{l', l'_1, \ldots, l'_n\}$. The locus $V(F)$ is a hypersurface in $\mathbb{P}(R_1)$. The linear form l can naturally be viewed as a point $[l] \in \mathbb{P}(S_1)$. Let $I \subset R$ be the homogenous ideal corresponding to $[l]$, i.e., the collection of all hypersurfaces passing through this point. Equivalently, I is generated by all hyperplanes intersecting in $[l]$. Each $[l'_i]$ is a point in $\mathbb{P}(R_1)$ and hence defines a hyperplane in $\mathbb{P}(R_1)^{\vee}$. In other words I is generated by the elements $\{l'_1, \ldots, l'_n\}$.

Let ϕ be the identity map

$$\begin{aligned} \phi : \mathbb{P}(R_1) &\to \mathbb{P}(S_1) \\ (y_0 : \ldots : y_n) &\mapsto (x_0 : \ldots : x_n). \end{aligned} \tag{2.21}$$

In a symmetrical way, we have that $\phi([l]) \in \mathbb{P}(S_1)$ corresponds to an ideal $J \subset S$ which is generated by $\{l_1, \ldots, l_n\}$. Note that if $[l] \in V(F)$ then $F \in J$.

Now, since $F \in S_d$ defines a hypersurface $V(F) \subset \mathbb{P}(R_1)$ and $\phi([l]) \in \mathbb{P}(R_1)$ we may take the Taylor expansion of F with respect to $\phi([l])$. There exists $a_0, \ldots, a_n \in \mathbb{C}$ such that

$$F = a_0 l^d + a_1 l^{d-1} f_1(l_1, \ldots, l_n) + \ldots + a_d f_d(l_1, \ldots, l_n).$$

We denote the dehomogenization of F with respect to $l \in S_1$ by F_l

$$F_l = a_0 + a_1 f_1(l_1, \ldots, l_n) + \ldots + a_d f_d(l_1, \ldots, l_n).$$

There are two distinct types of subscript present here: F_l denotes the dehomogenization with respect to l while f_i is a polynomial of degree i. The symbol R_l will mean the subring of R generated by $\{l'_1, \ldots, l'_n\}$. It is the natural coordinate ring of the affine subspace $\{l' \neq 0\} \subset \mathbb{P}(S_1)$.

The most important property of a dehomogenized polynomial F_l is that it is apolar to $F^\perp$.

Lemma 2.6.8 ([BR13], Lemma 2). *The Artininan Gorenstein scheme $X(F_l)$ defined by $F_l^\perp \subset S_{l'}$ is apolar to F, i.e., the homogenization*

$$(F_l^\perp)^h \subset F^\perp \subset R.$$

We give a simple example showing how the above lemma works in practise.

Example 2.6.9. Let $F = x_0^2 x_1^2 + x_0^2 x_2^2 \in S_4$. Then $F^\perp = \langle y_1 y_2, y_1^2 - y_2^2, y_0^3 \rangle \subset R$. We let $\{x_0, x_1, x_2\}$ and $\{y_0, y_1, y_2\}$ be bases for S_1 and R_1 respectively. One may look at dehomogenization with respect to any linear form. Consider for example dehomogenizing with respect to x_0 yielding

$$F_{x_0} = x_1^2 + x_2^2.$$

We get the following apolar ideal

$$F_{y_0}^\perp = (F_{y_0}^\perp)^h = \left\langle y_1 y_2, y_1^2 - y_2^2 \right\rangle \subset R_{y_0},$$

which clearly is a subideal of $F^\perp$. ♣

Chapter 3

Higher order polars and dual forms

In this chapter we extend the theory of apolarity presented in the preliminaries. This is an exposition on, and extension of, the classical theory of apolarity which is nowadays an almost forgotten chapter within multilinear algebra [Dol04]. We primarily follow the methodology of Dolgachev and his textbook *Classical Algebraic Geometry* [Dol12]. We present dual forms with respect to apolarity, which is a generalization of the polarity presented in the preliminaries, extending it to higher orders. We show that higher order polars have direct applications to power sum decompositions. Furthermore, we discuss a concept which we call self-polarity: when an even homogeneous form corresponding to an apolarity map admits a polar dual form which also corresponds to an apolarity map.

3.1 Dual homogeneous forms

As already shown, a homogeneous polynomial F in S_d defines a pairing between R_k and S_{d-k}, which is coined the apolarity pairing. We will now discuss how this pairing naturally induces a dual $F^\vee \in R_d$ to $F \in S_d$. Recall that a closed subvariety X admits a dual variety $\check{X}$ which is defined to be the closure in the dual space of the locus of hyperplanes which are tangent to X at some nonsingular point of X. For a hypersurface $X = V(F)$ the dual $\check{X}$ is the image of X under the rational map given by the first polars. In the same way that a variety is defined by a homogeneous polynomial $X = V(F)$, the dual $\check{X}$ is defined by a dual form $\check{F}$. Furthermore, this dual satisfies reflexivity, i.e., that $\check{\check{X}} = X$.

In this section we show how the theory of polarity can be used to analogously express the dual form with respect to polarity for any homogeneous form of even degree, i.e., $F \in S_{2k}$.

3.1.1 Quadratic forms

To motivate, consider the case of dual quadrics.

Example 3.1.1. Let $X = V(F)$ be a nonsingular quadric in $\mathbb{P}^n$ and $A = (a_{ij})$ be the symmetric matrix defining F. Then,

$$F = xAx^{\mathrm{T}} = \sum_{j=0}^{n} a_{0j}x_0x_j + \ldots + \sum_{j=0}^{n} a_{nj}x_nx_j.$$

The tangent hyperplane, or rather the first polar, at a point $[p] = (b_0 : \ldots : b_n) \in \mathbb{P}^n$ is given by

$$b_0 \sum_{j=0}^{n} a_{0j} x_j + \ldots + b_n \sum_{j=0}^{n} a_{nj} x_j = bAx^{\mathrm{T}} = 0.$$

In other words, a tangent hyperplane is given by $V(Ax^{\mathrm{T}})$. Let $y = (y_0, \ldots, y_n)$ be the vector of coordinates of such a hyperplane. Then, since A is invertible, $x^{\mathrm{T}} = A^{-1}y^{\mathrm{T}}$. We get that

$$0 = xAx^{\mathrm{T}} = yA^{-1}y^{\mathrm{T}} = 0.$$

Finally, recalling that $A^{-1} = \det(A)^{-1} \operatorname{adj}(A)$, we have that the dual variety of X is also a quadric and is given by the adjugate matrix of A. ♣

To extend the nice behaviour of quadrics with respect to dual forms to higher orders, we now express the apolarity map in a more general setting. Consider the pairing

$$\begin{aligned} \Omega_F : R_k \times R_k &\to \mathbb{C} \\ (G_1, G_2) &\mapsto G_2(G_1(F)), \end{aligned} \tag{3.1}$$

where we identify R_k and $(S_k)^{\vee}$. The pairing can be considered as a bilinear form and its matrix with respect to monomial bases is the catalecticant $\operatorname{Cat}_k(F)$. Furthermore, one can identify Ω_F with a quadratic form on S_k yielding a map

$$\begin{aligned} \Omega : S_{2k} &\to (S_k)_2 \\ F &\mapsto \Omega_F. \end{aligned} \tag{3.2}$$

We say that Ω_F is the *polar quadratic form* of F. Its dual, $\Omega_F^{\vee}$, is the *dual polar quadratic form* of F. We will often refer to Ω_F as simply a quadratic form, omitting the polar specifier. Viewing Ω_F as a quadratic form in S_k, determined by the catalecticant matrix $\operatorname{Cat}_k(F)$, the dual quadratic form $\Omega_F^{\vee}$ is defined to be the adjugate of $\operatorname{Cat}_k(F)$.

Definition 3.1.2. For a homogeneous polynomial of degree $2k$ we define the *dual polar quadratic form* $\Omega_F^{\vee}$ of F to be

$$\Omega_F^{\vee} = \operatorname{adj} \operatorname{Cat}_k(F). \tag{3.3}$$

The dual $\Omega_F^{\vee}$ is a quadric in R_k defining a bilinear map

$$\Omega_F^{\vee} : S_k \times S_k \to \mathbb{C}. \tag{3.4}$$

The apolarity map with respect to an even homogenous form F can be considered in two equivalent ways: First as a map $\operatorname{ap}_F : R_k \to S_k$ and second as a bilinear form $\Omega_F : R_k \times R_k \to \mathbb{C}$. Similarly the dual $\Omega_F^{\vee}$ can be viewed as a map $\operatorname{ap}_F^{\vee} : S_k \to R_k$ and as a bilinear map $\Omega_F^{\vee} : S_k \times S_k \to \mathbb{C}$. Describing the dual map $\Omega_F^{\vee}$ is more delicate in the sense that the dual of ap_F is not necessarily an apolarity map, i.e., it does not correspond to derivation. However, since Ω_F is a linear map it naturally has an inverse when the determinant of the catalecticant is nonzero. We are chiefly interested in the case when Ω_F is invertible. We call forms that admit invertible catalecticants nondegenerate.

Definition 3.1.3. A quadratic homogeneous form Ω_F is called *nondegenerate* if its determinant is nonzero.

Definition 3.1.4. A homogeneous form $F \in S_{2k}$ is called *nondegenerate* if Ω_F is a nondegenerate quadratic form in S_k.

The result of multiplying an invertible matrix with its adjugate produces the identity matrix multiplied by the determinant,

$$\Omega_F \circ \Omega_F^\vee = (\det \mathrm{Cat}_k(F)) \cdot \mathbf{1}.$$

Since we consider forms equivalent up to scalar we will often refer to $\Omega_F \circ \Omega_F^\vee$ simply as the identity, ignoring the scaling.

In the following example we illustrate concretely a case where the dual quadratic form $\Omega_F^\vee$ does not correspond to an apolarity map.

Example 3.1.5. Let $F = \sum_{i=0}^4 \binom{d}{i} a_i x_0^{d-i} x_1^i$. This corresponds to the following catalecticant

$$\mathrm{Cat}_2(F) = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}. \tag{3.5}$$

The dual quadric $\Omega_F^\vee : S_k \times S_k \to \mathbb{C}$ is given by

$$\mathrm{adj}(C) = \begin{pmatrix} a_2a_4 - a_3^2 & a_1a_4 - a_2a_3 & a_1a_3 - a_2^2 \\ a_1a_4 - a_2a_3 & a_0a_4 - a_2^2 & a_0a_3 - a_1a_2 \\ a_1a_3 - a_2^2 & a_0a_3 - a_1a_2 & a_0a_2 - a_1^2 \end{pmatrix}. \tag{3.6}$$

It is clear that the anti-diagonal, the diagonal running from the bottom left to the top right, contains different elements, i.e., the middle element is not equal to the elements in the bottom left and top right corners of the matrix. Hence, by Example 2.3.1 this matrix can not be a catalecticant. The image of the basis monomials (x_0^2, x_1^2) and (x_0x_1, x_0x_1) are

$$\Omega_F^\vee(x_0^2, x_1^2) = a_1a_3 - a_2^2$$

and

$$\Omega_F^\vee(x_0x_1, x_0x_1) = a_0a_4 - a_2^2.$$

Clearly, there can exist no homogenous polynomial in R_k which yield different results when being differentiated with respect to $x_0^2x_1^2$ and $(x_0x_1)^2$. ♣

Definition 3.1.6. If a polynomial $F \in S_{2k}$ is such that $\Omega_F^\vee = \Omega_G$ for some $G \in R_{2k}$ then we call F *self-polar*.

We leave self-polarity for now and return to it later in the chapter.

3.1.2 The polar dual

Recall from the preliminaries that derivation corresponds to evaluation for linear forms with respect to a homogeneous form of even degree. Points in the zero locus of F correspond to linear forms l^d apolar to F. That is,

$$V(F) = \{p \in \mathbb{P}^n \mid F(p) = 0\} \cong \{\psi \in R_1 \mid F(\psi^d) = 0\}. \tag{3.7}$$

Dually, one can use this to define the *polar dual* $F^\vee$ corresponding to the dual quadratic form $\Omega_F^\vee$.

Definition 3.1.7. Let $F^\vee \in R_{2k}$ be such that

$$V(F^\vee) = \{p \in \mathbb{P}^{n\vee} \mid F^\vee(p) = 0\} \cong \{l \in S_1 \mid \Omega_F^\vee(l^k, l^k) = 0\}.$$

We call $F^\vee$ the *polar dual*.

Instead of defining the zero set one could also define the polar dual directly. If F is a quadratic form in S_{2k} it can be written as xCx^{T}, where C is the catalecticant matrix $\mathrm{Cat}_k(F)$ and x is the coordinate vector of S_k. This gives rise to a polynomial in R_{2k} in the following way:

Definition 3.1.8. Given an $F = xCx^{\mathrm{T}}$, the *polar dual* $F^\vee$ of F is the polynomial

$$F^\vee = y\,\mathrm{adj}(C)y^{\mathrm{T}},$$

where y is the coordinate vector in R_k.

These two definitions are equivalent. To see this, fix an $F \in S_{2k}$ and let $l \in S_1$ be a linear form such that $\Omega_F^\vee(l^k, l^k) = 0$. By the definition of the dual quadratic form this is equivalent to $x_l\,\mathrm{adj}(C)x_l^{\mathrm{T}} = 0$, where x_l is the vector for l^k with respect to a basis of S_k. Since S_k and R_k are isomorphic as vector spaces, x_l corresponds to an element y_l in R_k. Hence, we see that the zero set of $F^\vee$ is identical to the set of linear forms in S_1 such that $\Omega_F^\vee(l^k, l^k) = 0$.

The following basic result from linear algebra is helpful to keep in mind when thinking about the rank of adjugate matrices.

Lemma 3.1.9. *Let $A \in M_m(\mathbb{C})$ and let $B = \mathrm{adj}\,A$. Then the following holds*

- *If A is invertible so is B.*
- *If A has rank $m-1$ then B has rank 1.*
- *If A has at most rank $m-2$ then $B = 0$.*

A form F and its polar dual $F^\vee$ are by definition quite similar. The following example displays an F and its corresponding $F^\vee$.

Example 3.1.10. Let F be a binary quartic on the form

$$F = a_0x_0^4 + 4a_1x_0^3x_1 + 6a_2x_0^2x_1^2 + 4a_3x_0x_1^3 + a_4x_1^4.$$

The dual quadratic $\Omega_F^\vee$ is then defined by the matrix

$$\mathrm{adj}(\mathrm{Cat}_2(F)) = \begin{pmatrix} b_0 & b_1 & b_2 \\ b_1 & b_5 & b_3 \\ b_2 & b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_2a_4 - a_3^2 & a_1a_4 - a_2a_3 & a_1a_3 - a_2^2 \\ a_1a_4 - a_2a_3 & a_0a_4 - a_2^2 & a_0a_3 - a_1a_2 \\ a_1a_3 - a_2^2 & a_0a_3 - a_1a_2 & a_0a_2 - a_1^2 \end{pmatrix}. \tag{3.8}$$

Hence, the dual quadric $F^\vee$ is

$$\begin{aligned} F^\vee &= y\,\mathrm{adj}(\mathrm{Cat}_2(F))y^{\mathrm{T}} \\ &= b_0y_0^4 + 4b_1y_0^3y_1 + 6\left(\frac{1}{3}b_2 + \frac{2}{3}b_5\right)y_0^2y_1^2 + 4b_3y_0y_1^3 + b_4y_1^4. \end{aligned} \tag{3.9}$$

Note for instance that if each $a_i = 1$ then $F = (x_0 + x_1)^4$. The catalecticant $\mathrm{Cat}_2(F)$ is then of rank 1 and the adjugate $\mathrm{adj}\,\mathrm{Cat}_2(F)$ has rank 0. In this case the dual form $F^\vee$ is identically zero. Lemma 3.1.9 clearly motivates the fact that it is only when F is nondegenerate that $F^\vee$ is interesting. ♣

The polynomial $F^\vee$ in the previous example is an even homogenous polynomial in R_{2k}. This means that it again gives rise to a catalecticant matrix which does correspond to an apolarity map. We denote this quadratic form by $\Omega_{F^\vee}$. In the next example we will see an example of $\Omega_{F^\vee}$ being not equal to $\Omega_F^\vee$.

Example 3.1.11. Continuing with the previous example we let

$$F^\vee = b_0y_0^4 + 4b_1y_0^3y_1 + 6\left(\frac{1}{3}b_2 + \frac{2}{3}b_5\right)y_0^2y_1^2 + 4b_3y_0y_1^3 + b_4y_1^4.$$

The catalecticant is

$$\mathrm{Cat}_2(F^\vee) = \begin{pmatrix} b_0 & b_1 & \frac{1}{3}b_2 + \frac{2}{3}b_5 \\ b_1 & \frac{1}{3}b_2 + \frac{2}{3}b_5 & b_3 \\ \frac{1}{3}b_2 + \frac{2}{3}b_5 & b_3 & b_4 \end{pmatrix}. \tag{3.10}$$

Clearly, $\mathrm{Cat}_2(F^\vee)$ is not equal to $\mathrm{adj}(\mathrm{Cat}_2(F))$. In other words $\Omega_{F^\vee}$ does not correspond to the same map as $\Omega_F^\vee$. ♣

Furthermore, we have that the polar dual is not in general reflexive, i.e., taking the dual of the dual does not yield the original object. This is illustrated nicely by the following example.

Example 3.1.12. For notational convenience let $2b_2 + 4b_5$ be denoted by b_6. The adjugate of $\mathrm{Cat}_2(F^\vee)$ is given by

$$\mathrm{adj}(\mathrm{Cat}_2(F^\vee)) = \begin{pmatrix} b_6b_4 - b_3^2 & b_1b_4 - b_3b_6 & b_1b_3 - b_6^2 \\ b_1b_4 - b_3b_6 & b_0b_4 - b_6^2 & b_0b_3 - b_1b_6 \\ b_1b_3 - b_6^2 & b_0b_3 - b_1b_6 & b_0b_6 - b_1^2 \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 \\ c_1 & c_5 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix}$$

which again is not a catalecticant. However it does correspond to a polynomial $(F^\vee)^\vee \in S_{2k}$. This polynomial can be seen to be

$$F^{\vee\vee} = c_0x_0^4 + 4c_1x_0^3x_1 + (2c_2 + 4c_5)x_0^2x_1^2 + 4c_3x_0x_1^3 + c_4x_1^4.$$

One can write out each c_i in terms of the a_is it depends on and directly verify that $(F^\vee)^\vee$ is not equal to F. ♣

Combined, the previous few examples shows the following result.

Lemma 3.1.13. *Forms F and $(F^\vee)^\vee$ are in general not equivalent.*

A natural question to ask is whether continuing to take duals of duals ad infinitum ever terminates.

Definition 3.1.14. Let $F[0] = F$, $F[1] = F^{\vee\vee}$ and so on, for all $n \in \mathbb{N}$.

We do not look further into this here, but based on computed examples we give the following conjecture:

Conjecture 3.1.15. *Given a general $F \in S_{2k}$ there exist no $n \in \mathbb{N}$ such that $F = F[n]$.*

3.1.3 Properties of the polar dual

By Lemma 2.2.3, a point $p \in \mathbb{P}^n$ in the zero locus $V(F)$ corresponds to a linear form $\psi \in R_1$ such that $F(\psi^{2k}) = 0$. This can be written

$$F(\psi^k\psi^k) = \psi^k\psi^k(F) = 0,$$

which by definition is the same as $\Omega_F(\psi^k, \psi^k) = 0$. Hence, $\Omega_F(G, G) = 0$ if and only if $G = \psi^k$ for some ψ such that $F(\psi^{2k}) = 0$. The form $\Omega_F^\vee(l^k)$ is classically known as the *anti-polar* of l^k. We will sometimes use inner product notation to make arguments easier to follow. It is given via differentiation, i.e., $\langle G, F\rangle = G(F)$. Immediately from definitions, we have the following corollary:

Corollary 3.1.16. *Let $F^\vee \in R_{2k}$ and let G and H be two homogeneous forms in S_k. Then*

$$\Omega_{F^\vee}(G,H) = \langle G, \mathrm{ap}_{F^\vee}(H)\rangle .$$

Proof. By definition, we have that

$$\langle G, \mathrm{ap}_{F^\vee}(H)\rangle = G(\mathrm{ap}_{F^\vee}(H)) = G(H(F^\vee)) = GH(F^\vee) = \Omega_{F^\vee}(G,H), \tag{3.11}$$

and we are done. □

Turning our attention to Waring decompositions, we see that the dual homogeneous form $\Omega_F^\vee$ can be used to confirm whenever F admits linear forms.

Definition 3.1.17. We say that two linear forms $l_1, l_2 \in S_1$ are *conjugate* with respect to a nondegenerate $F \in S_{2k}$ if

$$\Omega_F^\vee(l_1^k, l_2^k) = 0.$$

Proposition 3.1.18. *Let $F = l_1^{2k} + \ldots + l_s^{2k}$ and let the l_i^ks be linearly independent in S_k. Then any pair (l_i^k, l_j^k) are conjugate with respect to F.*

Proof. Note that since the l_i^ks are linearly independent in S_k they are also linearly independent in S_{2k}. Since the l_i^ks are independent in S_k we can include them in a basis for S_k. Similarly, one can use the induced basis on R_k such that the catalecticant becomes

$$\mathrm{Cat}_k(F) = \begin{pmatrix} \mathbf{1_{s\times s}} & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \tag{3.12}$$

where B is some matrix of size $\left(\binom{n+k}{n} - s\right) \times \left(\binom{n+k}{n} - s\right)$. The adjugate matrix $\mathrm{adj}(\mathrm{Cat}_k(F)$ will necessarily have a similar form. Hence $\Omega_F^\vee$ is the identity matrix in the upper left $(s \times s)$ submatrix. Finally, it is clear that

$$\Omega_F^\vee(l_i^k, l_j^k) = 0,$$

whenever $i \neq j$. □

3.2 Self-polarity

Theorem 2.3 in [Dol04] states, for a nondegenerate $F \in S_{2k}$, that there exists a unique homogeneous form $F^\vee \in R_{2k}$ such that $\Omega_{F^\vee} = \Omega_F^\vee$. In the proof, the authors had assumed that the adjugate of a catalecticant matrix is itself a catalecticant which is a mistake first pointed out by Bart van den Dries [Dol12]. Here, we would like to present a version of the same theorem circumventing this issue. The proof is a rewording of the one given in [Dol04] except for the additional assumption of self-polarity.

Theorem 3.2.1. *Assume that $F \in S_{2k}$ is nondegenerate and self-polar. Then there exists a unique homogeneous form $F^\vee \in R_{2k}$ such that*

$$\Omega_{F^\vee} = \Omega_F^\vee. \tag{3.13}$$

Proof. The dual quadric $\Omega_F^\vee$ is defined by the adjugate matrix $\mathrm{adj}\,\mathrm{Cat}_k(F)$. Since F is self-polar then $\mathrm{adj}\,\mathrm{Cat}_k(F) = (b_{\mathbf{uv}})$ is a catalecticant. We have that

$$\Omega_F^\vee = y\,\mathrm{adj}(\mathrm{Cat}_k(F))y^{\mathrm{T}},$$

where y is the coordinate vector in R_k. We change notation to

$$\Omega_F^\vee = \sum_{|u|=k,|v|=k} b_{\mathbf{uv}} y^{\mathbf{u}+\mathbf{v}},$$

where $(\mathbf{u},\mathbf{v}) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$. Since $\operatorname{adj}\operatorname{Cat}_k(F)$ is a catalecticant, it uniquely induces a polynomial $F^\vee \in R_{2k}$. Let

$$F^\vee = \sum_{|\mathbf{u}+\mathbf{v}|=2k} \frac{2k!}{(\mathbf{u}+\mathbf{v})!} b_{\mathbf{uv}} y^{\mathbf{u}+\mathbf{v}}.$$

Now we need to check that the map defined by differentiation of $F^\vee$ with respect to forms in S_k is the same as the map defined by the catalecticant $\Omega_F^\vee$. For any monomial $x^{\mathbf{i}} \in S_k$, we have

$$D_{x^{\mathbf{i}}}(F^\vee) = \sum_{\mathbf{u}+\mathbf{v}\geq \mathbf{i}} \frac{2k!}{(\mathbf{u}+\mathbf{v})!} b_{\mathbf{uv}} \frac{(\mathbf{u}+\mathbf{v})!}{(\mathbf{u}+\mathbf{v}-\mathbf{i})!} y^{\mathbf{u}+\mathbf{v}-\mathbf{i}} = \sum_{\mathbf{u}+\mathbf{v}\geq \mathbf{i}} \frac{2k!}{(\mathbf{u}+\mathbf{v}-\mathbf{i})!} b_{\mathbf{uv}} y^{\mathbf{u}+\mathbf{v}-\mathbf{i}},$$

where we use $D_{x^{\mathbf{i}}}(F^\vee)$ to mean the derivative of $F^\vee$ with respect to $x^{\mathbf{i}}$. Changing indices one gets that

$$D_{x^{\mathbf{i}}}(F^\vee) = \sum_{|j|=k} \frac{2k!}{\mathbf{j}!} b_{\mathbf{ij}} y^{\mathbf{j}}.$$

Furthermore, $D_{x^{\mathbf{i}}}(F^\vee)$ is an element in R_k. The basis for R_k with respect to the catalecticant above is given by the elements $\frac{2k!}{\mathbf{l}!} y^{\mathbf{l}}$ for all possible $|\mathbf{l}| = k$. Since this holds for every basis element we have that the matrix of the linear map $S_k \to R_k$ defined by $\Omega_{F^\vee}$ is equal to the matrix $\operatorname{adj}\operatorname{Cat}_k(F)$, and we are done. □

For self-polar forms we have the following equality:

Lemma 3.2.2. *Let $F \in S_{2k}$ and $G \in R_k$ be two homogeneous forms where F is nondegenerate and self-polar. Then*

$$\Omega_{F^\vee}(\operatorname{ap}_F(G), \operatorname{ap}_F(G)) = \Omega_F(G,G). \tag{3.14}$$

Proof. Since F is nondegenerate and self-polar we have that $\operatorname{ap}_{F^\vee}(\operatorname{ap}_F(G)) = \lambda G$, where λ is some scalar. Hence,

$$\begin{aligned}\Omega_{F^\vee}(\operatorname{ap}_F(G), \operatorname{ap}_F(G)) &= \langle \operatorname{ap}_F(G), \operatorname{ap}_F^\vee(\operatorname{ap}_F(G))\rangle \\ &= \langle \operatorname{ap}_F(G), G\rangle \\ &= \langle G, \operatorname{ap}_F(G)\rangle \\ &= \Omega_F(G,G),\end{aligned} \tag{3.15}$$

showing the desired result. □

Let us investigate when one can expect the adjugate of the catalecticant to be a catalecticant. For quadrics the case is trivial, as seen in Example 3.1.1. It is also straight forward to explicitly verify that the adjugate catalecticant is a catalecticant:

Example 3.2.3. Let $F \in S_2$ be of the form $F = a_0x_0^2 + a_1x_0x_1 + a_2x_1^2$. A computation yields the inverse form $F^\vee = a_2y_0^2 - a_1y_0y_1 + a_0y_1^2$. The catalecticants of these two polynomials are

$$\operatorname{Cat}(F) = \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \text{ and } \begin{pmatrix} a_2 & -a_1 \\ -a_1 & a_0 \end{pmatrix} = \operatorname{Cat}(F^\vee). \tag{3.16}$$

Clearly, these are inverses of one another. ♣

In fact, this implies that all general binary quadrics are self-polar.

Lemma 3.2.4. *Let X be the collection of binary quartics on the form $F = \sum_{i=0}^{4} \binom{4}{i} a_i x_0^{4-i} x_1^i$. Then each quartic in the subvariety $V(a_1a_3 - a_0a_4)$ is self-polar.*

Proof. Any F on the form above has the following catalecticant

$$\mathrm{Cat}_2(F) = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}. \tag{3.17}$$

The dual quadratic form $\Omega_F^\vee$ is given by the adjugate of $\mathrm{Cat}_2(F)$

$$\mathrm{adj}(\mathrm{Cat}_2(F)) = \begin{pmatrix} a_2a_4 - a_3^2 & a_1a_4 - a_2a_3 & a_1a_3 - a_2^2 \\ a_1a_4 - a_2a_3 & a_0a_4 - a_2^2 & a_0a_3 - a_1a_2 \\ a_1a_3 - a_2^2 & a_0a_3 - a_1a_2 & a_0a_2 - a_1^2 \end{pmatrix}. \tag{3.18}$$

The inverse of the catalecticant is a symmetric matrix, but not in general a catalecticant, as easily seen from Example 2.3.1. In order for $\Omega_F^\vee$ to be an apolarity map, the anti-diagonal must be constant. In other words we require that $a_0a_4 - a_2^2 = a_1a_3 - a_2^2$. Hence, for any F with $a_0a_4 = a_1a_3$ we have that $\Omega_F^\vee = \Omega_{F^\vee}$. $\square$

The space of binary quartics can be thought of as $\mathbb{P}^4$. Since the space of binary self-polar quartics is in codimension 1, there is a $\mathbb{P}^3$ of binary self-polar quartics.

Proposition 3.2.5. *Let A be the collection of binary forms on the form $F = \sum_{i=0}^{d} \binom{d}{i} a_i x_0^{d-i} x_1^i$. Then any self-polar form lies in a subvariety of at most codimension $\binom{k}{2}$.*

Proof. In general in $\mathbb{P}^1$ we have that the catalecticant is of the form

$$\mathrm{Cat}_k(F) = \begin{pmatrix} a_0 & a_1 & \dots & a_k \\ a_1 & a_2 & \dots & a_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_{k+1} & \dots & a_{2k} \end{pmatrix}. \tag{3.19}$$

In order for the adjugate matrix to be a catalecticant each anti-diagonal must be constant. The catalecticant has size $(k+1) \times (k+1)$. Let the adjugate matrix $\mathrm{adj}\,\mathrm{Cat}_k(F) = (c_{ij})$ be indexed such that c_{ij} denotes the determinant of $\mathrm{Cat}_k(F)$ having removed row i and column j. For example

$$c_{00} = \det \begin{pmatrix} a_2 & \dots & a_{k+1} \\ \vdots & \ddots & \vdots \\ a_{k+1} & \dots & a_{2k} \end{pmatrix}.$$

Since $\mathrm{Cat}_k(F)$ is a symmetric matrix, so is $\mathrm{adj}\,\mathrm{Cat}_k(F)$. The first two anti-diagonals introduce zero constraints on $\mathrm{Cat}_k(F)$ being a catalecticant, counting from top left, or equivalently, from bottom right. Since the matrix is symmetric, the number of equations is equal to the number of 2×2 minors lying strictly above the diagonal. In total this yields $(k-1) + (k-2) + \ldots + 1 = \binom{k}{2}$ equations. $\square$

A binary sextic $F \in S_6$ lives, as an element, in $\mathbb{P}^6$. For F to be self-polar there are $\binom{3}{2} = 3$ equations that must be satisfied. We denote the variety given by these equations by X. By the previous proposition, X is at least a variety of dimension 3. The catalecticant of a binary sextic has size 4×4. The determinant of the catalecticant is a hypersurface of degree 4 in $\mathbb{P}^6$ which we denote D_F.

Computing the primary decomposition of I_X one sees that X has two components $X = U \cup V$. The first component U has dimension 4 and degree 3 and is defined by the ideal

$$I_U = (a_2a_5 - a_1a_6, a_2a_4 - a_0a_6, a_1a_4 - a_0a_5),$$

whose generators are the 2×2 minors of the 2×3 matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 \\ a_4 & a_5 & a_6 \end{pmatrix}.$$

The second component V is of dimension 3 and degree 10. Its defining ideal can be seen to be generated by 10 minors of size 3×3 of $\mathrm{Cat}_k(F)$. Since $\mathrm{Cat}_k(F)$ is symmetric, these are precisely all the unique 3×3 minors. For a form to be self-polar, the rank of the catalecticant must be maximal, so it is only the intersection $X \cap D_F^c$ which corresponds to self-polar binary forms. If all the 3×3 minors in the second component is zero then the determinant of the catalecticant is zero. This implies that $V \subset D_F$. Hence, the dimension of self-polar forms are solely determined by the first component U.

A similar pattern emerges when looking at binary forms of degree 8 and 10. In both cases X admits two components. If F is of degree 8 then one component has dimension 5 and degree 4 and the other is of dimension 5 and degree 20. If F is of degree 10, then one component has dimension 6 and degree 5 and the other is of dimension 7 and degree 35. The second component is always contained in the hypersurface defined by the determinant of the catalecticant. This is because the second component is defined by the $k \times k$ minors of the catalecticant matrix. Naturally, when every $k \times k$ minor is zero, then so is the determinant of the catalecticant.

In all cases, we have that the first component is defined by the 2×2 minors of a matrix

$$\begin{pmatrix} a_0 & a_1 & \dots & a_{k-1} \\ a_{k+1} & a_{k+2} & \dots & a_{2k} \end{pmatrix}.$$

Since this matrix must have rank 1, the rows must be scalar multiples of each other. Let F be written on the following form:

$$F = a_0x_0^{2k} + \ldots + a_{k-1}x_0^{k+1}x_1^{k-1} + a_kx_0^kx_1^k + a_{k+1}x_0^{k-1}x_1^{k+1} + \ldots + a_{2k}x_1^{2k}.$$

We can factorize F using the fact $\lambda(a_0, \ldots, a_{k-1}) = (a_{k+1}, \ldots, a_{2k})$. We get that

$$F = a_0x_0^{2k} + \ldots + a_{k-1}x_0^{k+1}x_1^{k-1} + a_kx_0^kx_1^k + \lambda a_0x_0^{k-1}x_1^{k+1} + \ldots + \lambda a_{k-1}x_1^{2k}.$$

Pulling out a common factor from the first $k-1$ terms and last $k-1$ yields

$$F = x_0^{k+1}(a_0x_0^{k-1} + \ldots + a_{k-1}x_1^{k-1}) + a_kx_0^kx_1^k + \lambda x_1^{k+1}(a_0x_0^{k-1} + \ldots + a_{k-1}x_1^{k-1}).$$

Grouping terms, F can be factorized into the form

$$F = (x_0^{k+1} + \lambda x_1^{k+1})(a_0x_0^{k-1} + \ldots + a_{k-1}x_1^{k-1}) + a_kx_0^kx_1^k.$$

We summarize the preceding discussion in the following proposition:

Proposition 3.2.6. *Let $k \leq 5$ and $F = \sum_{i=0}^{2k} a_i x_0^{2k-i} x_1^i$ be a binary form in S_{2k}. If there exists a factorization*

$$F = (x_0^{k+1} + \lambda x_1^{k+1})(a_0 x_0^{k-1} + \ldots + a_{k-1} x_1^{k-1}) + a_k x_0^k x_1^k,$$

for some scalar λ, then F is self-polar.

Example 3.2.7. Let F be a binary quartic on the form

$$F = (x_0^3 + \lambda x_1^3)(a_0 x_0 + a_1 x_1) + a_2 x_0^2 x_1^2 = a_0 x_0^4 + a_1 x_0^3 x_1 + a_2 x_0^2 x_2^2 + \lambda a_0 x_0^1 x_1^3 + \lambda a_1 x_1^4$$

The catalecticant of F is then of the form

$$\mathrm{Cat}_2(F) = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_0 \\ a_2 & a_0 & a_1 \end{pmatrix}, \tag{3.20}$$

yielding an adjugate matrix with constant anti diagonals:

$$\mathrm{adj}(\mathrm{Cat}_2(F)) = \begin{pmatrix} a_1 a_2 - a_0^2 & a_1^2 - a_2 a_0 & a_0 a_1 - a_2^2 \\ a_1^2 - a_2 a_0 & a_0 a_1 - a_2^2 & a_0^2 - a_1 a_2 \\ a_0 a_1 - a_2^2 & a_0^2 - a_1 a_2 & a_0 a_2 - a_1^2 \end{pmatrix}. \tag{3.21}$$

Since the anti diagonals are constant, the adjugate matrix corresponds to the apolarity map defined by an $F^\vee \in R_4$. This is in accordance with the definition of self-polar forms. ♣

This discussion warrants more research. We conjecture that Proposition 3.2.6 holds for all k, but this is not immediately evident. We have not in any way proven that the variety X always has a decomposition into two components, nor that the components are always on the form observed here. One potential issue is that the number of components might increase as the degree increases. Another issue is that the first component U might be contained in the hypersurface defined by the determinant of the catalecticant.

Chapter 4

Cactus rank vs catalecticant rank

In Chapter 2 we introduced the catalecticant and proved that it provides a lower bound for the Waring rank. Several other closely related notions of rank appear when studying Waring decompositions. A lot of work has been done in the last century to determine good bounds for the different notions of rank [BR13]. The cactus rank is a fairly recent object of study and coincides with the scheme length introduced by Iarrabino and Kanev in 1999 [IK99]. The name *cactus* rank was first introduced by Buczynska and Buczynski in 2010 in their study of secant and cactus varieties [BB11].

In this chapter we take a closer look at the relationship between the cactus and catalecticant rank. We shall shortly see that there are distinct cases where these two notions of rank coincide and divert. The main motivating question for this chapter is:

Question. *Can we develop a procedure for finding forms whose cactus rank is strictly larger than their catalecticant rank?*

There most definitely exist such examples, but finding them explicitly is a rather challenging task.

4.1 Cactus rank

As mentioned in the introduction, the rank of a homogeneous form F is the minimum length of smooth schemes that are apolar with respect to F. The cactus rank is defined identically, except for the fact that the schemes are not required to be smooth.

Definition 4.1.1. The cactus rank is defined as

$$\operatorname{cr}(F) = \min\{\text{length of a scheme } X \mid X \subset \mathbb{P}(S_1), \dim X = 0, I_X \subset F^{\perp}\}.$$

The cactus rank fits in between the catalecticant and waring rank.

$$\operatorname{rank} \operatorname{Cat} F \leq \operatorname{cr} F \leq \operatorname{rk} F.$$

That the cactus rank is bounded above by the Waring rank follows immediately from their definitions. The other inequality, that the cactus rank is bigger than the catalecticant rank, is something that will become clear in the next few sections. For now, note that the three notions of rank coincide for polynomials of relatively low degree and number of variables. However, as the degree and number of indeterminates increase, this is no longer the case.

Definition 4.1.2. For a polynomial $F \in S_d$ we say that the *natural rank* of F is the minimal length of a scheme $X(F_l)$ for some $l \in S_1$.

Let

$$\mathcal{N}_d = \begin{cases} 2\binom{n+k}{n}, & \text{when } d = 2k+1 \\ \binom{n+k-1}{n} + \binom{n+k}{n}, & \text{otherwise.} \end{cases}$$

The following theorem by Bernardi and Ranestad can be used to find an upper bound on the cactus rank.

Theorem 4.1.3 ([Ber+17], Theorem 3). *Let $F \in S$ be a homogeneous form of degree d, and let $l \in S_1$ be any linear form. Let F_l be a dehomogenization of F with respect to l. Then*

$$\operatorname{cr} F \leq \dim_k \operatorname{Diff}(F_l).$$

In particular

$$\operatorname{cr}(F) \leq \mathcal{N}_d.$$

Example 4.1.4. Let $F = x_0^5 x_1^5 + x_0^5 x_2^5 + x_1^5 x_2^5 + x_0^4 x_1^3 x_2^3$. This gives the Hilbert sequence

$$H_F = (1, 3, 6, 10, 15, 21, 15, 10, 6, 3, 1).$$

The middle entry is 21 meaning that the catalecticant $\operatorname{Cat}_6(F)$ has rank 21. Dehomogenizing at x_0 yields $F_{x_0} = x_1^5 + x_2^5 + x_1^5 x_2^5 + x_1^3 x_2^3$ which has the Hilbert sequence

$$H_{F_{x_0}} = (1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1).$$

The natural rank of F is thus less than or equal to the sum of the sequence $H_{F_{x_0}}$. We have that $\mathcal{N}_{10} \leq 36$. We can compute the upper bound of the rank directly:

$$\mathcal{N}_{10} = \binom{2+5-1}{2} + \binom{2+5}{2} = 15 + 21 = 36.$$

Hence, dehomogenizing with respect to any linear form will never yield a sum greater than 36 for H_{F_l}[1]. However, finding the dehomogenization corresponding to the natural rank is a difficult task, which we will not investigate further here. From the previous lemma we have that $21 \leq \operatorname{cr} F \leq 36$. Simultaneously, we know that F is a general polynomial so it has a Waring rank of 22. Thus, F is a polynomial such that

$$\operatorname{rank} \operatorname{Cat} F = 21 \leq \operatorname{cr} F \leq \operatorname{rank} F = 22.$$

In the next section we will investigate a criterion to check whether the cactus rank is actually 21 or 22. ♣

4.2 An approach to finding examples with smaller catalecticant rank than cactus rank

In this section we attempt to find a ternary form whose cactus rank is strictly larger than its catalecticant rank. According to Lemma 1.17 in [IK99], the rank of the catalecticant is equal to the Waring rank precisely when $s \leq \min\left(\binom{k+n}{k} \times \binom{n+d-k}{d-k}\right)$. It is interesting to note what happens with the catalecticant and Waring rank as the degree increases. Let again $d = 2k$ and consider the square kth catalecticant $\operatorname{Cat}_k(F)$ for a homogeneous polynomial $F \in S_d$ in $\mathbb{P}^2$. In this case, the catalecticant has maximal rank $\binom{k+2}{2}$, while the Waring rank is given by $\lceil \frac{1}{3}\binom{d+2}{2} \rceil$. For even degrees we get the following table:

[1] As a fun side note, we see that this is, in fact, an example of the maximal possible cactus rank, as shown by Theorem 1.2 in [BBG19].

d	rank Cat F	rank F	cr F
$\vdots$	$\vdots$	$\vdots$	$\vdots$
6	10	10	10
8	15	15	15
10	21	22	21-22
12	28	31	28-31
14	36	40	36-40
16	45	51	45-51
$\vdots$	$\vdots$	$\vdots$	

Table 4.1: Comparison of the catalecticant and Waring rank for even homogeneous forms.

For instance we see that the maximal catalecticant rank for a 10th degree polynomial is 21 while the Waring rank is 22. The cactus rank must either be 21 or 22. This suggests that it might be possible to find an example of a 10th degree polynomial with cactus rank strictly larger than the catalecticant rank.

4.2.1 Minimal apolar scheme of a ternary decic

In this section, let F denote a ternary homogeneous form of degree 10. Two important tools when investigating cactus rank are Hilbert-Burch and Buchsbaum-Eisenbud matrices. We here give a criterion, on the Buchsbaum-Eisenbud matrix of an apolar ideal $F^\perp$, guaranteeing that the smallest possible apolar scheme has length 22. In other words, that the cactus rank of F is 22.

In the preliminaries it was shown that any Hilbert-Burch matrix corresponding to an apolar subideal $I \subset F^\perp$ appears as a submatrix of a Buchsbaum-Eisenbud matrix of the apolar ideal $F^\perp$.

Lemma 4.2.1. *Let $F \in S_{2k}$ be a ternary homogeneous form such that $\mathrm{Cat}_k(F)$ has full rank. Then the apolar ideal $F^\perp$ is generated in at least degree $k+1$.*

Proof. The apolar map is $\mathrm{ap}_F^i : R_i \to S_{d-i}$. We have that

$$\begin{aligned} \dim R_i &= \dim \ker \mathrm{ap}_F^i + \dim \mathrm{im}\, \mathrm{ap}_F^i \\ &= \dim F_i^\perp + \dim \mathrm{im}\, \mathrm{ap}_F^i \,. \end{aligned} \tag{4.1}$$

In the plane, the catalecticant has size $\binom{2+k}{2} \times \binom{2+d-k}{2}$. Thus, the kth catalecticant $\mathrm{Cat}_k(F)$ is square of size $\binom{2+k}{2}$. By definition, the image of ap_F^k is the rank of the kth catalecticant. Hence, we get that

$$\dim F_k^\perp = \binom{2+k}{2} - \binom{2+k}{2} = 0. \tag{4.2}$$

Naturally, there can not be any elements in $F^\perp$ of degree less than k. This proves our claim. □

Furthermore, we have that the Buchsbaum-Eisenbud matrices have an easily predictable size for ternary forms.

Lemma 4.2.2. *The Buchsbaum-Eisenbud matrix for a ternary $F \in S_{2k}$ has size $n \times n$, where $n = 2k+3$.*

Proof. The number of minimal generators of $F^\perp$ is equal to the size of the corresponding Buchsbaum-Eisenbud matrix. By Lemma 4.2.1, $F^\perp$ contains no elements of degree $\leq k$. Hence, the generators are all of degree $\geq k+1$. In fact,

$$\begin{aligned} \dim F^\perp_{k+1} &= \binom{3+k}{2} - \binom{1+d-k}{2} \\ &= 2k+3. \end{aligned} \tag{4.3}$$

Hence there are at least $2k+3$ generators in $F^\perp$. As a Buchsbaum-Eisenbud matrix consist of linear entries there cannot be more. □

In specific scenarios we have that the Buchsbaum-Eisenbud matrix corresponding to a form F is on a special form, readily yielding a submatrix which is a Hilbert-Burch matrix and hence corresponding to a subideal $I \subset F^\perp$.

Lemma 4.2.3. *Let $F^\perp \subset R$ be minimally generated by $n = 2k+3$ elements. Let B be the corresponding $n \times n$ Buchsbaum-Eisenbud matrix. If there exists a basis such that B is representable as*

$$B = \begin{pmatrix} B_0 & -{B_1}^T \\ B_1 & B_2 \end{pmatrix},$$

where B_2 is a $(k+1) \times (k+1)$ zero block, then the maximal minors of B_1 is the Hilbert-Burch matrix of a subideal of $F^\perp$.

Proof. The ideal $F^\perp$ is generated by the principal minors of B. These are computed via removing the ith row and column, taking the determinant of the resulting submatrix, and then doing this for all i. We want to show that the principal minors of B_1 are the same as the maximal minors of B. Let B_i be B with the ith row and column removed. Since $F^\perp$ contains an odd number of generators, B is a square matrix of odd size. Hence B_i has even size. Dividing B_i into even square pieces

$$B_i = \begin{pmatrix} B_{0i} & -{B_{1i}}^T \\ B_{1i} & B_{2i} \end{pmatrix},$$

we get

$$\begin{aligned} \det B_i &= \det(B_{0i})\det(B_{2i}) - \det(B_{1i})\det\left(-{B_{1i}}^T\right) \\ &= \det(B_1)^2 \end{aligned} \tag{4.4}$$

for all i. This completes the proof. □

Lemma 4.2.2 yields that a form $F \in S_{10}$ has a Buchsbaum-Eisenbud matrix of size 13×13.

Lemma 4.2.4. *Schemes of length 21, where not all points lie on a quintic, correspond to Hilbert-Burch matrices of dimension 6×7 with linear entries.*

Proof. If M is a Hilbert-Burch matrix of dimension 6×7 with linear entries, then by the Hilbert-Burch theorem the 6×6-minors generate the ideal I_X of a finite scheme X. We have the following exact sequence

$$0 \longrightarrow R^6(-7) \xrightarrow{A} R^7(-6) \longrightarrow I_X \longrightarrow 0.$$

We compute the Hilbert function

$$\begin{aligned} H_{I_X} &= H_{R^7(-6)} - H_{R^6(-7)} \\ &= 7\binom{t-6+2}{2} - 6\binom{t-7+2}{2} \\ &= 7\binom{t-4}{2} - 6\binom{t-5}{2}. \end{aligned} \tag{4.5}$$

Thus, I_X contains 0 quintics, 7 sextics, 15 septics and so on. The standard sequence

$$0 \longrightarrow I_X \xrightarrow{A} R \longrightarrow R/I_X \longrightarrow 0$$

gives the following Hilbert function

$$\begin{aligned} H_{R/I_X} &= H_R - H_{I_X} \\ &= \binom{t+2}{2} - 7\binom{t-4}{2} + 6\binom{t-5}{2}. \end{aligned} \tag{4.6}$$

Hence, the Hilbert polynomial is $HP_{R/I_X} = 21$ proving that X is a scheme of length 21, i.e., $\dim_K H^0(X, \mathcal{O}_X) = 21$.

Conversely, let $X \subset \mathbb{P}^2$ be a scheme of length 21 with defining ideal I_X, where not all points lie on a quintic. Recall that there is a 1-1 correspondence between projective closed subschemes and saturated homogeneous ideals. The homogeneous ideal consisting of all curves passing through 21 such points is necessarily saturated and hence corresponds to a closed subscheme. Since not all points lie on a quintic, I_X must be generated in degree 6 or higher. The 21 points impose at most 21 linear conditions, so I_X contains at least 7 sextics, 15 septics and so on. Let J_6 be the vector space of sextics containing X. As just stated,

$$\dim J_6 \geq 28 - 21 = 7.$$

Any minimal system of generators of I_X must contain a basis for J. The Hilbert-Burch theorem provides a resolution

$$0 \longrightarrow R^{\beta-1} \xrightarrow{A} R^{\beta} \longrightarrow I_X \longrightarrow 0,$$

where the $(\beta-1)\times(\beta-1)$-minors of A generate I_X. The number of rows of A corresponds to the number of syzygies, while the number of columns corresponds to the number of generators of I_X. The entries of A lie in the maximal ideal (x_0, x_1, x_2). It follows that if an ideal is minimally generated by β generators, then the generators must each be of degree $\geq \beta - 1$. Since I_X has at least one generator of degree 6 and no generators of lower degree, it must have ≤ 7 generators, which implies that $\dim J_6 = 7$. Now, any basis for J_6 is a set of generators for I_X. One such basis is given by the maximal minors of a 6×7 matrix with all linear entries, and due to the Hilbert-Burch correspondence this is in fact the only one, and we are done.

□

Lemma 4.2.5. *Let B be a Buchsbaum-Eisenbud matrix corresponding to a ternary form $F \in S_{10}$ such that $\mathrm{Cat}_5(F)$ has full rank. Then there is a correspondence between Buchsbaum-Eisenbud matrices containing a 6×6 zero block and schemes of length 21.*

Proof. We prove the two directions independently. First assume that B has a zero block. By Lemma 4.2.3, we have that the 6×7 neighbouring submatrix of the 6×6 zero block is a Hilbert-Burch matrix of a subideal $I \subset F^{\perp}$. Since the Buchsbaum-Eisenbud matrix consists of linear entries, so does this Hilbert-Burch submatrix. Hence, by Lemma 4.2.4, I corresponds to an apolar scheme of length 21. The maximal catalecticant rank of F is 21. Finally, since the catalecticant rank is a lower bound for the cactus rank there cannot be any apolar subschemes of length less than 21.

Conversely, given that $\operatorname{cr} F = 21$ we get a scheme X of length 21 which corresponds to a Hilbert-Burch matrix of size 6×7 with linear entries. The generators of ideal I_X are naturally given by linear combinations of the generators of the apolar ideal $F^{\perp}$. The rows of the Hilbert-Burch matrix are linear syzygies on I_X and hence linear syzygies on $F^{\perp}$ as well. As the Buchsbaum-Eisenbud matrix is the matrix of all such syzygies we have that the Hilbert-Burch matrix must necessarily be a submatrix of the Buchsbaum-Eisenbud matrix. Thus, there exists a choice of bases arranging the Buchsbaum-Eisenbud matrix with a 6×6 zero block. □

Finally, the result which we are the most interested in follows immediately as a corollary.

Corollary 4.2.6. *Let B be a Buchsbaum-Eisenbud matrix corresponding to a ternary form $F \in S_{10}$ such that $\operatorname{Cat}_5(F)$ has full rank. If B does not contain a 6×6 zero block for any choices of bases, then*

$$\operatorname{cr} F = 22.$$

4.2.2 Zero block in the Buchsbaum-Eisenbud matrix

We now answer exactly when a Buchsbaum-Eisenbud matrix admits a zero block. By Corollary 2.6.6, the Buchsbaum-Eisenbud matrix always has a skew symmetric presentation. In practical applications it may be difficult to find this skew symmetric presentation, but as for the moment we assume that it is given. In the following section let m be an odd integer.

Definition 4.2.7. Let B be an $m \times m$ matrix with entries in $\mathbb{C}[x_0, \ldots, x_n]_1$. We say that the *variable decomposition* is the decomposition

$$B = B_0 x_0 + \ldots + B_n x_n,$$

where the entries in all B_is are scalars.

Definition 4.2.8. Let B be an $m \times m$ skew symmetric matrix. An *isotropic* subspace to B is a subspace $U \subset \mathbb{C}^m$, such that for every $u, v \in U$ we have $uBv^T = 0$.

Recall from linear algebra that a matrix is skew symmetric if

$$xBx^{\mathrm{T}} = 0 \quad \text{for all } x \in \mathbb{C}^m.$$

Let A be a $k \times m$ matrix with entries in $\mathbb{C}$. When we say that the rowspace of A is isotropic to a matrix B, we mean that the rows of A form a basis for a subspace $U \in \mathbb{C}^m$ which is isotropic to B.

Lemma 4.2.9. *Let B_0 be a $m \times m$ skew symmetric matrix with entries in $\mathbb{C}$ and let $k = \lfloor \frac{m}{2} \rfloor$. If there exists a nonzero matrix A of size $k \times m$ and rank k whose rowspace is isotropic to B_0 then B_0 has a zero block of size $k \times k$ under some coordinate change.*

Proof. To see that B_0 has a zero block, consider the following. Let A be a $k \times m$ matrix with scalar entries consisting of all zeroes on the first $k+1$ columns and arbitrary entries elsewhere. Furthermore, let A be isotropic to B_0. Take $\hat{A}$ to be the matrix

$$\hat{A} = \begin{pmatrix} C \\ A, \end{pmatrix}$$

where C is a matrix of size $(k+1) \times m$ such that $\hat{A}$ has size $m \times m$ with entries in $\mathbb{C}$. A basis change of B_0 relative to $\hat{A}$ yields

$$\hat{B} = \hat{A} B_0 \hat{A}^{\mathrm{T}},$$

which is skew symmetric because B_0 is. Also, since A is isotropic to B_0 we have that the entire bottom right $k \times k$ submatrix of $\hat{B}$ is identically zero. □

This lemma provides a concrete solution to the question of when a Buchsbaum-Eisenbud matrix has a zero block. However, finding such an isotropic space equates to solving a large number of quadratic equations in several indeterminates. Let $B = B_0 x_0 + \ldots + B_n x_n$ be as above. Then Lemma 4.2.9 applies for each B_i in the variable decomposition of B. Let $A = (a_{ij})$ be a $k \times m$ matrix with entries in $\mathbb{C}$ and denote its rows by u_i. For every possible pair, consider the equations of the form

$$f_l = u_i^T B_l u_j \text{ where } i \neq j.$$

Denote the ideal formed by all such equations

$$I = \langle f_0, \ldots, f_n \rangle .$$

Definition 4.2.10. We call the ideal I above the *isotropy ideal* of F.

The zero locus $V(I)$ represents $n \cdot \binom{k}{2}$ quadratic equations in km indeterminates. For a polynomial $F \in S_{10}$ this amounts to solving 45 quadratic equations in 78 unknowns. We can reduce the number of unknowns by using the fact that two different isotropic subspaces are the same if they admit the same Plücker coordinates up to scalars.

Lemma 4.2.11. *Let B be an $m \times m$ skew symmetric matrix and let U be a k-dimensional isotropic subspace of B. Let A be a $k \times m$ matrix representing a basis of U and and let $\mathcal{A}$ be the set of all such matrices A. The Plücker coordinates of A are equivalent up to scalars for all A.*

Hence, without loss of generality, we only consider the isotropic subspaces given by letting the first Plücker coordinate be equal to 1 and 0. We can thus give a version of Lemma 4.2.9 which is easily computable.

Lemma 4.2.12. *Let B be an $m \times m$ skew symmetric matrix with entries in $k[x_0, \ldots, x_n]_1$ and consider its variable decomposition $B = B_0 x_0 + \ldots + B_n x_n$. Let $k = \lfloor \frac{m}{2} \rfloor$. Then B has a zero block of size $k \times k$ if and only if there exists a nonzero matrix A representing a basis for a isotropic subspace of B which is either of the form*

$$A = \begin{pmatrix} \mathbf{1}_{k \times k} & \hat{A}_{k \times k+1} \end{pmatrix},$$

or

$$A = \begin{pmatrix} \mathbf{0}_{k \times k} & \hat{A}_{k \times k+1} \end{pmatrix},$$

where $\hat{A} = (a_{ij})$ is a matrix of coefficients in $\mathbb{C}$.

Example 4.2.13. When $F \in S_{10}$ then the related Buchsbaum-Eisenbud matrix B has size 13×13. The corresponding A is then of size 6×7

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & 1 & 0 & 0 & 0 & 0 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} \\ 0 & 0 & 1 & 0 & 0 & 0 & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ 0 & 0 & 0 & 1 & 0 & 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & 0 & 0 & 1 & 0 & a_{28} & a_{29} & a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} & a_{40} & a_{41} \end{pmatrix}.$$

♣

This does however come with the drawback that I is no longer homogeneous. We can remedy this fact, and also include the case that the first Plücker coordinate is zero, by introducing variables along the first diagonal.

Example 4.2.14. When $F \in S_{10}$ then the related Buchsbaum-Eisenbud matrix B has size 13×13. The corresponding A is then of size 6×7

$$A = \begin{pmatrix} c_0 & 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & c_1 & 0 & 0 & 0 & 0 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} \\ 0 & 0 & c_2 & 0 & 0 & 0 & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ 0 & 0 & 0 & c_3 & 0 & 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ 0 & 0 & 0 & 0 & c_4 & 0 & a_{28} & a_{29} & a_{30} & a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 & 0 & c_5 & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} & a_{40} & a_{41} \end{pmatrix}.$$

♣

We summarize the preceding discussion in the following theorem:

Theorem 4.2.15. *Let F be a ternary form of degree 10 such that $\operatorname{rank} \operatorname{Cat}_5(F) = 21$. Then $\operatorname{cr} F = 22$ if and only if $V(I) = \emptyset$, where I is the isotropy ideal of F.*

4.3 Computations and examples

In practical computational terms, the most challenging part of the preceding sections is that of actually proving that a Buchsbaum-Eisenbud matrix has no zero block of appropriate size. This boils down to solving a large amount of equations in many indeterminates. A lot of work was put into finding examples of polynomials $F \in S_{10}$ where there exists no isotropic subspace. In other words that the zero set of I is empty. This equates to solving 45 quadratic equations in 42 variables which we were not able to achieve with Macaulay2 on a normal desktop computer. We give here a general outline of our approach and the relevant code is given in Appendix A.2.

Due to Theorem 4.1.3 the cactus rank is bounded above by the natural rank. Hence, for a form $F \in S_d$ to be able to admit cactus rank 22 then every dehomogenization F_l of F must be such that the Hilbert sequence of A_{F_l} sums to at least 22. Thus, one may start with binary forms $f \in S$ of degree less than or equal to 10 whose natural rank is at least 22 and then homogenize (we denote the homogenization of f by F here). From this collection of forms one can extract all forms where the rank of $\operatorname{Cat}_5(F)$ is 21. At this point, one can compute the Buchsbaum-Eisenbud matrix for each $F^{\perp}$, and set up the isotropy ideal I of F. By Theorem 4.2.15, we then have that the cactus rank of F is 22 if and only if there are no solutions to the equations defined by the isotropy ideal. Summarized we have:

1. Consider all binary forms $f \in S$ of degree less than 10.
2. Homogenize all forms with respect to a third variable if their natural rank is 22.
3. Extract all F with catalecticant rank 21.
4. Compute the Buchsbaum-Eisenbud matrix for each such $F^\perp$ and find their skew-symmetric representation.
5. Solve the isotropy ideal.

It is virtually impossible to represent such a large set of forms on a computer. Hence, we started the search by looking at increasingly larger and larger sets of forms. For instance, we computed via brute force code in Appendix A the following three results.

Lemma 4.3.1. *There are no monomials of the form $F = x_0^{\alpha_0} x_1^{\alpha_1}$, where $\alpha_i \leq 10$, with cactus rank 22.*

Lemma 4.3.2. *Consider all polynomials of the form $F_{x_2} = x_0^a + x_1^b + x_0^c x_1^d$, where $a, b, c, d \leq 10$. There are no homogenized polynomials F with catalecticant rank 21.*

Lemma 4.3.3. *Consider all polynomials of the form $F_{x_2} = x_0^a + x_1^b + x_0^c x_1^d + x_0^3 x_1^4$, where $a, b, c, d \leq 10$. There are some homogenized polynomials F with catalecticant rank 21.*

Some examples of the polynomials found satisfying Lemma 4.3.3 are

$$\begin{aligned}
&x_0^8 x_1 x_2 + x_0^3 x_1^4 x_2^3 + x_0^6 x_2^4 + x_1^5 x_2^5 + x_2^{10}, \\
&x_1^{10} + x_0^3 x_1^4 x_2^3 + x_0^6 x_2^4 + x_0^4 x_1 x_2^5 + x_2^{10}, \\
&x_1^9 x_2 + x_0^7 x_2^3 + x_0^3 x_1^4 x_2^3 + x_0^5 x_1 x_2^4 + x_2^{10}, \\
&x_0^8 x_2^2 + x_1^8 x_2^2 + x_0^3 x_1^4 x_2^3 + x_0^4 x_1 x_2^5 + x_2^{10}, \\
&x_0^9 x_2 + x_0^3 x_1^4 x_2^3 + x_1^6 x_2^4 + x_0^4 x_1 x_2^5 + x_2^{10}, \\
&x_0^9 x_2 + x_0^3 x_1^4 x_2^3 + x_0^5 x_1 x_2^4 + x_1^6 x_2^4 + x_2^{10}, \\
&x_0^9 x_2 + x_0^3 x_1^4 x_2^3 + x_1^7 x_2^3 + x_0^4 x_1^2 x_2^4 + x_2^{10}.
\end{aligned} \tag{4.7}$$

This yields a handful of interesting examples. The following polynomials have a maximal catalecticant rank of 21 and exactly 22 partial derivatives when dehomogenizing with respect to x_2.

$$\begin{aligned}
&x_0^8 x_1 x_2 + x_0^3 x_1^4 x_2^3 + x_0^6 x_2^4 + x_1^5 x_2^5 \\
&x_1^{10} + x_0^3 x_1^4 x_2^3 + x_0^6 x_2^4 + x_0^4 x_1 x_2^5 \\
&x_0^8 x_2^2 + x_1^8 x_2^2 + x_0^3 x_1^4 x_2^3 + x_0^4 x_1 x_2^5 \\
&x_0^9 x_2 + x_0^3 x_1^4 x_2^3 + x_1^6 x_2^4 + x_0^4 x_1 x_2^5.
\end{aligned} \tag{4.8}$$

We do not know exactly what the natural rank of the polynomials above are. Dehomogenizing with respect to x_2 is just an arbitrary linear form. It is not conceivable to compute dehomogenizations with respect to all linear forms.

Example 4.3.4. Consider the homogeneous form

$$F = x_1^{10} + x_0^3 x_1^4 x_2^3 + x_0^6 x_2^4 + x_0^4 x_1 x_2^5.$$

Localizing on x_2 gives a Hilbert sum of 22. However, localizing on x_0 yields a Hilbert sum of 21. So, this polynomial has catalecticant and cactus rank 21. ♣

In the case that there are solution to the isotropy ideal, we have that there is a Hilbert-Burch submatrix of size 6×7 of the Buchsbaum-Eisenbud matrix. When F has catalecticant rank 21 this Hilbert-Burch matrix corresponds to an apolar scheme which has cactus rank 21. This is verified in the following example.

Example 4.3.5. The following homogeneous form has cactus rank 21:

$$F = x_1^{10} + x_0^3 x_1^4 x_2^3 + x_0^6 x_2^4 + x_0^4 x_1 x_2^5.$$

We can compute the resolution. The Buchsbaum-Eisenbud matrix is of size 13×13 and we can find a skew symmetric representation with a 6×6 zero block. The minors of the block vertically adjacent to the zero block yields the ideal

$$\begin{aligned} I = (&12855x_1^3 - 14872x_0^2 x_2 - 6347x_1 x_2^2, 12855x_1^2 x_2 - 13124x_2^3, \\ &- 14872x_0 x_2^2, 14872x_1 x_2^2, -14872x_2^3). \end{aligned}$$

Consider the variety $X = V(I)$. Computing the points of X over $\mathbb{C}$ is unstable in Macaulay2. Solving it over $\mathbb{Q}$ yields that $\dim X = 0$ and $\deg X = 21$. ♣

Chapter 5

Poles and polars: explicit decompositions

The Waring problem for general forms was proven by Alexander and Hirschowitz at the end of the 20th century [AH95]. However, knowing the rank of a homogeneous form is not sufficient in order to find an explicit decomposition, a challenge that is prevalent in several applied fields. In this chapter, we want to showcase how the theory of apolarity can be used to compute decompositions restricted to the plane. The method does not aim at producing a minimal decomposition, but rather a sufficiently small one. Furthermore, we devote a section to doing some dimensional analysis of when one can expect to find a minimal decomposition via polarity.

Before we begin in earnest, let us demonstrate the symmetry of poles and polars induced by the duality of apolarity. Let $F \in S_{2k}$ be an even homogeneous form and consider the apolarity map

$$\begin{aligned} \mathrm{ap}_F^k : R_k &\to S_k \\ G &\mapsto G(F). \end{aligned} \tag{5.1}$$

In general, the apolarity map has an inverse and throughout this chapter we will always assume F to be general, in the sense that the square catalecticant has maximal rank. This is vital in order to make sense of the inverse apolarity map. If $a \in R_k$ is a pole then $\mathrm{ap}_F^k(a)$ is naturally a polar in S_k. Dually, if $b \in S_k$, then $\mathrm{ap}_F^k(b)$ is classically referred to as an *anti polar* residing in R_k.

Example 5.0.1. Consider, with respect to $F \in S_4$, the apolarity map $\mathrm{ap}_F^2 : R_2 \to S_2$. Let $L \subset S_2$ be the space of simple powers of linear forms $L = \{l^2 \mid l \in S_1\}$. An element $l^2 \in L$ can be considered as a point in the projective space $\mathbb{P}(S_1) \cong \mathbb{P}^2$. The preimage ${\mathrm{ap}_F^k}^{-1}(l^2)$ is a quadric in R_2. We denote the locus of the preimage $V({\mathrm{ap}_F^k}^{-1}(l^2))$ by Q. We recognize Q as the anti polar of l^2 with respect to $X = V(F)$. ♣

Due to the symmetry of polars and anti polars with respect to the pole and polarity correspondence, we will usually omit specifically referring to an anti polar as an anti polar, and rather just call it a polar. This is illustrated by the following lemma:

Lemma 5.0.2. *Let $F \in S_{2k}$ and let a and b be two points in $\mathbb{P}^2$. The anti polars of linear forms $l^k \in S_k$ satisfy the pole and polar correspondence of Theorem 2.2.8.*

Proof. Let Q_a and Q_b be the inverse polars of a and b respectively. We need to show that $Q_b(a) = 0 \iff Q_a(b) = 0$. We have that $a^k = Q_a(F)$ and $b^k = Q_b(F)$. Using the fact that a and b correspond to linear forms we can use Corollary 2.2.4 to interchange the order of operations, giving the desired result

$$Q_b(a) = Q_b(Q_a(F)) = Q_a(Q_b(F)) = Q_a(b).$$

□

5.1 Pole schemes

The concept of poles and polars induces a family of apolar schemes relative to an even form $F \in S_{2k}$ in the plane in the following way. Pick a point $l_1 \in S_1$ and denote its corresponding polar Q_1. Picking a second point l_2 on Q_1 again admits a polar which we denote Q_2. Under the apolarity map $\mathrm{ap}_k(F)$, the polars Q_1 and Q_2 are both of degree k. The second polar Q_2 passes, by Lemma 5.0.2, through l_1, and by Bezout's theorem the two polars Q_1 and Q_2 intersect k^2 times. Denote the intersection points $l_{p_1}, \dots, l_{p_{k^2}}$.

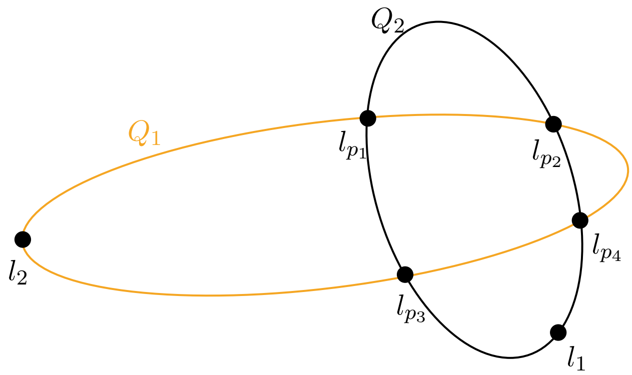


Figure 5.1: Poles and polars for a quartic in the plane

Definition 5.1.1. A scheme X corresponding to the collection of points $\{l_1, l_2, l_{p_1}, \dots, l_{p_{k^2}}\}$ is called a *pole scheme* with respect to F. We write

$$X = \{[l_1], [l_2], [l_{p_1}], \dots, [l_{p_{k^2}}]\}.$$

Whenever we need to be explicit, we will denote a pole scheme X_F with F as a subscript to show that it is related to F. By definition every pole scheme is zero dimensional.

Corollary 5.1.2. *A ternary even homogenous form F corresponds to a 3 dimensional family of pole schemes.*

Proof. This follows immediately from the construction: First, one chooses a linear form l_1 in $\mathbb{P}^2$ and then a second form l_2 lying on the anti polar of l_1. These two choices yield a 3 dimensional space of pole schemes. □

From the construction it immediately follows that there exists a generator set for the defining ideal of a pole scheme consisting of four forms.

Proposition 5.1.3. *Let X be a pole scheme for a ternary even form $F \in S_{2k}$ and let I be an ideal on the form*

$$I = (C_1L_1, C_1L_1', C_2L_2, C_2L_2'), \tag{5.2}$$

where each C_i is a polar of degree k of l_i^k and each L_i is a line such that L_i and L_i' intersect in l_i^k. Then I generates the ideal of X.

Proof. The polars C_1, C_2 are curves of degree k intersecting k^2 times. We show that I corresponds to a scheme of length k^2+2.

By construction there exists a curve of degree $k+1$ passing through all points of P. Without loss of generality, we assume that there are no curves of degree less than $k+1$ passing through P. If there were, this would simply imply that we could find a decomposition of F into fewer than k^2+2 forms. The Hilbert-Burch matrix of I is

$$M = \begin{pmatrix} L_1' & L_1 & 0 & 0 \\ 0 & 0 & L_2' & L_2 \\ D_1 & D_2 & D_3 & D_4 \end{pmatrix},$$

where D_i is a curve of degree $k-1$. Via the Hilbert-Burch theorem, we get the following graded resolution

$$0 \longrightarrow S(-k-2)^2 \bigoplus S(-2k) \xrightarrow{M} S(-k-1)^4 \longrightarrow I \longrightarrow 0.$$

Combining this with the exact sequence

$$0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

yields the Hilbert polynomial

$$HP_{S/I} = \binom{t+2}{2} - 4\binom{t-k+1}{2} + \binom{t-2k+1}{2} + 2\binom{t-k}{2} = k^2+2. \tag{5.3}$$

Thus, I corresponds to a zero dimensional scheme of length k^2+2. □

Lemma 5.1.4. *For a ternary even form F, the corresponding pole scheme X is apolar to F, i.e., $I \subset F^{\perp}$.*

Proof. The pole and polar correspondence yields

$$C_i(F) = l_i^k, \quad \text{for } i = 0, 1,$$

and

$$L_i(l_i^k) = 0 \text{ and } L_i'(l_i^k) = 0 \quad \text{for } i = 0, 1.$$

This directly implies the desired result

$$C_i L_i(F) = L_i(C_i(F)) = L_i(l_i^k) = 0.$$

Since every generator is apolar to F we have that $I \subset F^{\perp}$. □

Since pole schemes X_F are apolar to F they correspond to decompositions of degree $2k$ forms into k^2+2 linear forms. In the following example we demonstrate the previous two results in a concrete setting:

Example 5.1.5. Let F be a ternary quartic $F \in S_4$. The corresponding pole scheme ideal can be written in the form

$$I = (Q_1 L_1, Q_1 L_1', Q_2 L_2, Q_2 L_2'), \tag{5.4}$$

where each Q_i is of degree 2 and each L_i is a line such that L_i and L_i' meet transversely. Let us illustrate this in more detail.

Let J be the ideal consisting of all possible curves passing through 6 general points $P = \{l_1, l_2, l_{p_1}, \ldots, l_{p_4}\}$, where $l_{p_1}, \ldots, l_{p_4}$ lie in a complete intersection. We have that the dimension of the graded pieces of J as a vector space over K is given by

$$\dim_K J_d = \binom{2+d}{2} - 6.$$

Clearly, J contains no lines or quadrics. We will show that the ideal

$$I = \langle Q_1 L_1, Q_1 L_1', Q_2 L_2, Q_2 L_2' \rangle \tag{5.5}$$

defines a scheme of length 6 such that $I = J$. Consider the quadric Q_1 which passes through $P \setminus l_1$. There is a 2-dimensional space of lines passing through l_1 in $\mathbb{P}^2$. Denote two linearly independent lines through l_1 by L_1 and L_1'. Mutatis mutandis for Q_2. Thus, the ideal spans a 4-dimensional space of cubics. The space of all cubics in $\mathbb{P}^2$ is 10-dimensional. Since 6 points introduces 6 linear conditions, the space of cubics through 6 general points is 4-dimensional. This means that the above collection are all possible cubics passing through P. The space of quartics passing through 6 points is 9-dimensional. Since we are working in $\mathbb{P}^2$ there are potentially $4(n+1) = 12$ quartics in I. At most 9 of these can be linearly independent, but there might be fewer. If there were to be exactly 3 syzygies among the generators of I, this would imply that every possible quartic is in I.

For any ideal corresponding to a finite set of points in $\mathbb{P}^2$ the ring R/I is Cohen-Macaulay of codimension 2. The Hilbert-Burch theorem yields the following finite free resolution of length 1

$$0 \longrightarrow S^3 \xrightarrow{M} S^4 \longrightarrow I \longrightarrow 0.$$

Furthermore, M is a 3×4-matrix where the four 3×3-minors are the generators of I up to scalar. The rows of M are precisely the syzygies of I. Since there are 3 rows in M there are exactly 3 syzygies, and consequently the quartics of I are contained in J. The resolution is of length 1 so there are no second syzygies and thus all quintics, sextics, etc. of I must also lie in J. We have shown that $I = J$. Hence, we can use the generators of I as a generator set of J.

Furthermore, we can easily verify that X_F indeed is an apolar scheme. We do this by showing that every generator of I is apolar to F. For $Q_1 L_1$ to be apolar to F we must have that $Q_1 L_1(F) = 0$. Using the key fact that Q_1 corresponds to the inverse image of a simple linear form and that this linear form lies on L_1 we get

$$Q_1 L_1(F) = L_1(Q_1(F)) = L_1(l_1^2) = 0.$$

Similar computations for the other generators shows that each generator is apolar to F and thereby $I_X \subset F^{\perp}$. We have found a homogeneous subideal of $F^{\perp}$ corresponding to 6 points. By Lemma 2.4.4, we have that F can be written as a sum of powers of 6 linear forms.

♣

We summarise the preceding discussion in the following theorem.

Theorem 5.1.6. *Let $F \in S_{2k}$ be a ternary homogenous form of degree $2k$. Then there is a constructable 3-dimensional family of pole schemes $X = \{[l_1], [l_2], \ldots, [l_{k^2+2}]\} \subset \mathbb{P}(R_1)$ corresponding to (not necessarily minimal) decompositions of F.*

By *constructable* we mean that the defining ideals can explicitly be written down in terms of polynomial equations.

5.2 Explicit decompositions

The problem of finding a minimal decomposition of a homogeneous form is in general very challenging. However, if one permits oneself to drop the requirement of the decomposition being minimal, the problem becomes more feasible. The pole and polar construction displayed in the previous section can be used to find such (usually not minimal) decompositions. The general procedure can be outlined as follows:

1. Choose an arbitrary linear form $l_1^k \in S_k$.
2. Compute the polar $Q_1 = \mathrm{ap}_F^{-1}(l_1^k)$.
3. Choose a linear form $l_2^k \in V(Q_1)$.
4. Compute the polar $Q_2 = \mathrm{ap}_F^{-1}(l_2^k)$.
5. Compute the intersections, P, of Q_1 and Q_2.
6. The set $\{\ l_1, l_2, P\}$ corresponds to a decomposition.

The following example illustrates the procedure.

Example 5.2.1. Consider, for example, the Fermat quartic

$$F = x_0^4 + x_1^4 + x_2^4 \in S_4.$$

It is evident that this polynomial is the sum of powers of three linear forms, but assume for now this fact to be unknown. We want to find two linear forms in S_1 and their corresponding polars such that each pole lie on the other pole's polar. In this example, F is *special* in the sense that the apolarity map is not an isomorphism and caution must be exercised when choosing a linear form in S. For example, for the form

$$l^2 = (x_0 + x_1 + x_2)^2,$$

there does not exist an inverse under the apolarity map. There is no quadric $Q \in R_2$ such that $\mathrm{ap}_F^2(Q) = l^2$. Choose instead the first polar to be

$$l_1^2 = x_0^2.$$

There exist an infinite number of polars to this pole. Let, for example, the polar of l_1 with respect to F be

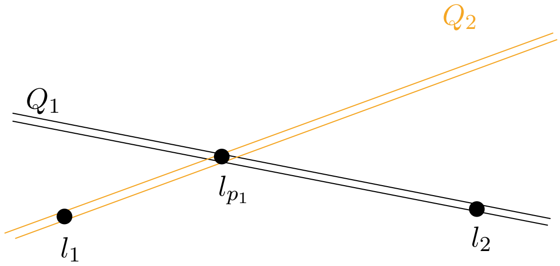
$$Q_1 = y_0^2.$$

One solution to $V(Q_1)$ is $p = (0 : 0 : 1)$. Hence, let

$$l_2^2 = x_2^2.$$

The second polar is then

$$Q_2 = y_2^2.$$



These two quadrics are degenerate and correspond to double lines. Hence Q_1 and Q_2 intersect in a single point l_{p_1} with multiplicity 4. Together with l_1 and l_2, these 3 points correspond to a decomposition of F into a sum of powers of 3 linear forms. This follows immediately from the fact that the ideal

$$I = (Q_1 L_1, Q_1 L_1', Q_2 L_2, Q_2 L_2')$$

is apolar to F, i.e., $I \subset F^{\perp}$, as was shown in the previous lemma. In other words we have the three points

$$\{(1:0:0), (0:1:0), (0:0:1)\} = V(I),$$

and we have have that F can be written

$$F = x_0^4 + x_1^4 + x_2^4.$$

Of course, the choice of degree 4 in this example does not matter whatsoever. The procedure would indeed be completely identical for any Fermat polynomial of arbitrary degree. ♣

5.3 Dimension analysis

In this section we analyse if it is reasonable to believe that there might exist a minimal Waring decomposition among the decompositions given by pole schemes. Let F be a planar sextic. A corresponding pole scheme X_F has length $3^2 + 2 = 11$ and from its construction, 9 among these 11 points lie on a complete intersection. Furthermore, an F has a 3 dimensional family of pole schemes. The Alexander-Hirschowitz theorem tells us that a general $F \in S_6$ has a minimal decomposition into 10 linear forms. There are however many such minimal decompositions. In fact, a result by Mukai [Muk92], yields that the $\mathrm{VSP}(F, 10)$ is a K3 surface of genus 20. Hence, there is a 2 dimensional family of minimal decompositions for any given form of degree 6.

We wish to check if it is reasonable to believe that there exists a choice of 10 points among the 11 points in a pole scheme which corresponds to a minimal decomposition. Let the schemes Y_{10} and X_{11} represent 10 arbitrary points and 11 points wherein 9 lie in a complete intersection respectively. We have the following maps

$$\begin{tikzcd}
 & \{(Y_{10}, X_{11}) \mid Y_{10} \subset X_{11}\} \arrow[dl, "\binom{10}{2}"'] \arrow[dr, "\binom{9}{1}"] \arrow[rr] & & \mathrm{Hilb}_{10} \times \mathrm{Hilb}_{11}^{CI} \\
\mathrm{Hilb}_{10} & & \mathrm{Hilb}_{11}^{CI} &
\end{tikzcd}$$

where Hilb_{10} is the Hilbert scheme of 10 points in general linear position and Hilb_{11} is the Hilbert scheme of 11 points wherein 9 lie in complete intersection. Note that both $\dim \mathrm{Hilb}_{10} = \dim \mathrm{Hilb}_{11}^{CI} = 20$. The leftmost projection to Hilb_{10} is $\binom{10}{2}$ because of the following: If one starts with 10 general points one may remove 2 points and observe that there are 2 cubics through 8 points. These cubics intersect in 9 points. Adding back the 2 removed points we have 11 points wherein 9 lie in a complete intersection.

Since F is a binary sextic it has 28 coordinates. Hence, it can be viewed as an element in $\mathbb{P}^{27}$. We have the following maps

$$\begin{tikzcd} & \{(F, X_{11}) \mid F \in \langle X_{11} \rangle\} \arrow[r, hook] \arrow[dl, "3 \text{ dim}"] \arrow[dr, "\mathbb{P}^{10}"'] & \mathbb{P}^{27} \times \mathrm{Hilb}_{11}^{CI} \\ \mathbb{P}^{27} & & \mathrm{Hilb}_{11}^{CI} \end{tikzcd}$$

The left projection is 3 dimensional because for every F there is a 3 dimensional choice of corresponding pole schemes. The right projection is 11 dimensional, or rather a $\mathbb{P}^{10}$, since F lies in the span of X_{11}. From this, one sees that the space of forms of rank 11 arising in the pole and polar construction is 30 dimensional. Simultaneously, we have that

$$\begin{tikzcd} & \{(F, Y_{10}) \mid F \in \langle Y_{10} \rangle\}. \arrow[r, hook] \arrow[dl, "2 \text{ dim}"] \arrow[dr, "\mathbb{P}^{9}"'] & \mathbb{P}^{27} \times \mathrm{Hilb}_{10} \\ \mathbb{P}^{27} & & \mathrm{Hilb}_{10} \end{tikzcd}$$

The leftmost projection is 2 dimensional, since $\mathrm{VSP}(F, 10)$ is a K3 surface. The space of forms of rank 10 is 27 dimensional and so the incidence $\{(F, Y_{10}) \mid F \in \langle Y_{10} \rangle\}$ is 29 dimensional. Lastly, we have the following diagram:

$$\begin{tikzcd} \{(F, Y_{10}, X_{11}) \mid F \in \langle Y_{10} \rangle \text{ and } Y_{10} \subset X_{11}\} \arrow[r, hook] \arrow[d, hook] \arrow[dr, "\binom{10}{2}"] & \mathbb{P}^{27} \times \mathrm{Hilb}_{10} \times \mathrm{Hilb}_{11}^{CI} \\ \{(F, X_{11}) \mid F \in \langle X_{11} \rangle\} & \{(F, Y_{10}) \mid F \in \langle Y_{10} \rangle\} \end{tikzcd}$$

Thus, given a ternary sextic $F \in S_6$, there is a 3 dimensional family of pole schemes among which there is a 2 dimensional family of minimal schemes. These minimal schemes correspond to 10 points whose span contains F.

Chapter 6

Summary and Conclusion

We have, in this thesis, presented an exposition on the theory of apolarity, and put it into a contemporary setting, tallying it up against modern problems. It is an approach that has somewhat fallen out of modern textbooks, but which still merits research as it has far reaching consequences, especially within the topics of Artinian Gorenstein rings and zero-dimensional finite schemes. We hope to have motivated the reader in this direction throughout this thesis.

Our efforts have largely been that of understanding the theory of apolarity and applying it to concrete problems. As such, this thesis has not chiefly been concerned with developing new results within the field. Attempts in this direction were made, but to little avail. As a forefront in the contemporary study of apolarity, the VSP has been central. It especially gained traction in and after the late 90s due to novel results by Mukai [Muk92]. Another slew of popular questions stem from the study of the Waring rank. Determining the rank of general forms was solved by Alexander and Hirschowitz in 1995 [AH95]. However, for special forms, not much is known. For instance, the lowest upper bound on the Waring rank is not known when considering all possible forms. Several researchers utilize apolarity to find explicit decompositions, but, as far as we are aware, there are no contemporary studies using what we in this thesis have called pole schemes. The same is true for what we call self-polarity. Furthermore, there are several contemporaries studying secant and cactus varieties (which were introduced as recently as 2010). It is motivated by this that we looked for examples of forms with differing cactus and catalecticant rank. Hence, it would have been beneficial to present some concrete examples where this was true, but the development of the procedure took precedence. The body of this thesis is hence, in addition to being an exposition on apolarity, a contribution to several distinct facets regarding apolarity; not tackling the most popular questions within the field, but rather a supplementary selection.

6.1 Limitations, weaknesses and future work

A large but necessary limitation of this thesis was working in the projective plane, i.e., working with forms in 3 variables. It is natural to wonder how the techniques we have utilized generalize to $\mathbb{P}^n$. In order to achieve this, one would have to deal with schemes in $\mathbb{P}^n$ as opposed to schemes in $\mathbb{P}^2$. This introduces several challenges. For example, one does no longer have that these schemes would necessarily be Cohen-Macaulay. Techniques relying on Hilbert-Burch and Buchsbaum-Eisenbud matrices, used to find apolar subideals, would have had to be generalized. Despite this our intuition tells us that this should be possible. In particular, it would be rewarding to generalize Theorem 5.1.6

to forms of arbitrary number of variables. In $\mathbb{P}^2$ we take two poles and two polars from which we construct a pole scheme. In $\mathbb{P}^n$ it might be possible to take n poles and polars and make a similar construction. This would be interesting to look further into.

In Chapter 3 we discussed the dual forms which arise via higher order polars. There, we briefly touched upon the matter of reflexivity of polar duals. In general, we do not have reflexivity, but it would be interesting to look further into the behaviour of these dual forms. For instance, if one were to keep on taking duals of duals, we wonder what would happen. Does there exists a limit under certain circumstances and if not, can one categorize the divergent behaviour?

The procedure developed in Chapter 4 can be used to find ternary forms F in S_{2k} such that $\operatorname{rank} \operatorname{Cat}_k(F) < \operatorname{cr} F$. However, the method involves solving a degree k polynomial in 3 variables ($n+1$ in general), something which is notoriously difficult. We made no great strides in developing efficient algorithms in our work, but we believe that doing so could be a valuable future contribution. Especially, as this has consequences to applied fields relying on tensor decompositions.

Some effort was lost in trying to determine when one can expect to find a minimal decomposition among the decompositions produced by the pole schemes. At the end of Chapter 5 we saw that for ternary sextics that one expects there to be a 2 dimensional family corresponding to minimal decompositions among the 3 dimensional family of pole schemes. However, it would be advantageous to be more rigorous in this study. For instance:

- Does one expect similar behaviour for ternary forms of any even degree?
- Can one explicitly find algebraic or geometric criteria for determining which pole schemes admit such minimal decompositions?

Lastly, we would like to note that algebraic geometry is a vast field, and most literature therein requires significant preliminary knowledge. In this regard the author had to put significant effort into reviewing material in order to understand the relevant concept. The primary source for the material in this thesis is the work done by Dolgachev, which demands a rather sophisticated knowledge of algebraic geometry. A more thorough understanding of the preliminary material from the outset would have been an advantage. Had the author had to write this entire body of text again more focus would have been put on the fundamentals. More rigour would have been exerted early on and more "basic" questions would have been asked.

Appendix A

Macaulay2 code

In this appendix we present some of the Macaulay2 code used in this thesis. The full overview of our code can be found in our github repository "hersta/master_uio" here. We do not claim that this code is any way optimized or designed for readability, and can certainly be improved upon. However, should someone want to try to extend the analysis done in this thesis, then the following code is a good place to start.

A.1 Ternary sextic

The following is an example of how to apply the theory of zero blocks in Buchsbaum-Eisenbud matrices in the case where F is a ternary sextic.

```
-- load the package used to find a skew symmetrix
-- representation of the Buschbaum-Eisenbud matrix
loadPackage "ResLengthThree"
-- define our system and compute the resolution of the apolar ideal
kk=QQ[x,y,z]
F = x^6 + y^6 + z^6
Fperp = inverseSystem(F)
betti res Fperp
J = res Fperp
B = J.dd_2
-- Find a skew symmetrix representation
A = resLengthThreeAlg J
netList multTableOneTwo A
H = sub(((matrix((multTableOneTwo(A))_{1..5}))_{1..5}), g_1=>1)
-- X is the skew-symmetric matrix corresponding to B
X = transpose(H)*B
-- Extract the ideal to the side of the zero block
-- corresponding to a subideal of the apolar ideal
subM = X^{0,1}_{2,3,4}
myIdeal = minors(2,subM)
-- make a basis change back to the original ring kk
use kk
myIdeal = substitute(myIdeal, kk)
-- compute the dimension and degree
v = variety myIdeal
dim v
degree v
```

A.2 Apolar schemes with catalecticant and cactus rank of 21

The following algorithm tests if a ternary polynomial $F \in S_{10}$ has a skew symmetric Buchsbaum-Eisenbud matrix with a 6×6 zero block.

```
polyHasDim22 = (F) -> (
    Fperp := inverseSystem F;
    M := res Fperp;
    B := M.dd_2;
    A := resLengthThreeAlg M;
    netList multTableOneTwo A;
    some := ((matrix((multTableOneTwo(A))_{1..13}))_{1..13});
    H:=sub(some, g_1=>1);
    X:=transpose(H)*B;
    potZero := X^{7..12}_{7..12};
    if (potZero ==  0) then
    (
        subM := X^{7..12}_{0..6};
        print "Zero block found!";
        print F;
        myIdeal := minors(6,subM);
        use S;
        myIdeal = substitute(myIdeal, S);
        v := variety myIdeal;
        if (degree v == 21) then return true;
    )
    else false;
    false);
```

The following code finds forms F in three variables and of degree 10 with maximal catalecticant rank equal to 21 and cactus rank 21 over $\mathbb{Q}$. It checks a rather small subset of all polynomials, but it can easily be extended.

```
d := 10;
S = QQ[x0,x1];
for j from 2 to (d-1) list
(
    for k from 2 to (d-1) list
    (
        spice := {x0^(j)*x1^(k) ,x0^3*x1^4};
        -- pols is the collection of polynomials to analyze
        -- all of which has natural rank 21
        pols := findPolys(d, spice);
        S = QQ[x0,x1,x2];
        g := {x0^(j)*x1^(k) , x0^3*x1^4};
        -- change basis of pols to a more standard form
        special := apply(pols, p -> specialize p);
        -- homogenize
        homs := apply(special, s -> getHomPoly(s, g));
        num := length pols - 1;
        for i from 0 to num list (
            -- check if a polynomial has maximal catalecticant rank
            if (maxHilbert(homs_i) == 21) then
            (
                print toString(homs_i);
                -- Any polynomial in here has
                -- natural rank 22 and catalecticant rank 21
                if (polyHasDim22 homs_i) then
                (
                    -- This polynomial has cactus rank 21,
                    -- natural rank at most 22,
```

```
                    -- and catalecticant rank 21
                    print homs_i;
                    );
                );
            );
        );
    );
```

A.2.1 Helper methods

```
-- Produces a collection of polynomials of degree 10
-- Takes a degree and a seed polynomial as input
-- The seed polynomial is used to manually extend the polynomial list
-- with specific terms
-- Only returns polynomials with natural rank equal to 22
findPolys = (d, g) -> (
    var := ();
    numPartDiff := ceiling (binomial(2+d, 2)/3);
    for i from 0 to d list -- 0 to 10
    (for j from 0 to d list -- 0 to 10
    (for k from 0 to d list -- 1 to 9
    (for l from 0 to (d-k) list -- 1 to 9
        if (hilbertSum(x0^i + x1^j + x0^k * x1^l + g_0 + g_1) ==  numPartDiff)
        then var = append(var, (i,j,k,l))
        else ""))));
    var);
```

```
-- Technical method to write a form in a standard form
-- Expects orgininal basis in order x0, x1, x0*x1
-- Returns basis in order x0, x1, x0x1, x0x2, x1x2, x0x1x2
specialize = A -> (
        a:= A_0;
        b:= A_1;
        c:= A_2;
        d:= A_3;
        l0:=0;
        l1:=0;
        l2:=0;
        l3:=0;
        l4:=0;
        l5:=0;
        l6:=0;
        l7:=0;
        l8:=0;
        l9:=0;
        l10:=0;
        z1 := 10 - a;
        z2 := 10 - b;
        z3 := 10 - c - d;
        if a == 10 then l0=10;
        if b == 10 then l1=10;
        if c+d == 10 then (l2=c; l3=d);
        if a < 10 then (l4=a; l5=z1);
        if b < 10 then (l6=b; l7=z2);
        if c+d < 10 then(
            l8=c;
            l9=d;
            l10=z3;);
        (l0, l1, (l2, l3), (l4, l5), (l6, l7), (l8, l9, l10))
    )
```

```
getHomPoly = (a, g) -> (
    x0^(a_0) +
    x1^(a_1) +
    x0^((a_2)_0) * x1^((a_2)_1) +
    x0^((a_3)_0) * x2^((a_3)_1) +
    x1^((a_4)_0) * x2^((a_4)_1) +
    x0^((a_5)_0) * x1^((a_5)_1) * x2^((a_5)_2) +
    g_0 * x2^(10 - (degree g_0)_0) +
    g_1 * x2^(10 - (degree g_1)_0)
    );
```

A.3 Points in the isotropy ideal

In order for a ternary form of degree 10 to not have a zero block the isotropy ideal must have no solutions. In fact, this equation set consists of 45 quadratic equations in 42 unknowns. In this case, the Buchsbaum-Eisenbud matrix is of size 13×13. The following example is of a 9×9 matrix, and already here a normal desktop computer is not powerful enough to find a solution. Matrices of size 7×7 were the biggest we managed to compute on a normal desktop computer without timing out. The bottleneck of the proceeding code is line 37; computing the dimension of the isotropy ideal.

```
1 S = QQ[x0,x1,x2]
2 -- An example polynomial
3 F = x1^(10)+x0^3 * x1^4 * x2^3+x0^6 *x2^4+x0^4 *x1* x2^5
4 Fperp = inverseSystem F
5 betti res Fperp
6 M = res Fperp
7 M.dd
8 B = M.dd_2
9 S = QQ[x0,x1,x2]
10 m = 9; -- number of rows
11 n = 9; -- number of columns
12 T=random(S^m, S^{n:-1}) -- {a:-b} means a cols, degree b
13 B=T-transpose T
14 numVars = 4*5
15 R = QQ[c,a_0..a_numVars, MonomialOrder=>Lex]
16 eqM = matrix{{c,0,0,0,a_0,  a_1,  a_2,  a_3, a_4},
17             {0,c,0,0,a_5,  a_6,  a_7,  a_8, a_9},
18             {0,0,c,0,a_10, a_11, a_12, a_13, a_14},
19             {0,0,0,c,a_15, a_16, a_17, a_18, a_19}}
20 B0 = sub(sub(sub(sub(B, x1=>0), x2=>0), x0=>1), QQ)
21 B1 = sub(sub(sub(sub(B, x0=>0), x2=>0), x1=>1), QQ)
22 B2 = sub(sub(sub(sub(B, x1=>0), x0=>0), x2=>1), QQ)
23  -- verifying that the decompositions looks as expected
24 B0 + B1 + B2
25 eqs = {}
26 for i from 0 to 3 list
27 (
28     for j from (i+1) to 3 list
29     (
30         eqs = append(eqs,eqM^{i} * B0 * transpose(eqM^{j}));
31         eqs = append(eqs,eqM^{i} * B1 * transpose(eqM^{j}));
32         eqs = append(eqs,eqM^{i} * B2 * transpose(eqM^{j}));
33     )
34 )
35 eqs
36 I = ideal(eqs)
37 dim I
```

```
v = variety(I)
degree v
dim v
```

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