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Cones of divisors on Calabi-Yau varieties

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Chapter 1

Introduction

Given a projective variety X, we wish to classify all morphisms from X to other projective varieties. This is equivalent to classifying all morphisms to projective spaces. Given such a morphism $\phi\colon X \to \mathbb{P}^N$, we can produce a line bundle $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^N}(1)$ and a collection of global sections $\phi^*x_0, \ldots, \phi^*x_N \in \Gamma(X, \mathcal{L})$. Conversely we can attempt to create morphisms from X to projective spaces from line bundles by choosing some finite set of global sections and 'inserting colons between them' — we succeed if the sections do not all vanish at a common point. This sets up an imperfect correspondence between the maps from X to projective spaces and the line bundles on X, and thus the classification of maps from X to projective spaces is closely tied to the classification of line bundles on X.

The line bundles on a projective variety that arise as pullbacks of $\mathcal{O}(1)$ through such morphisms are precisely the globally generated ones. An interesting feature of globally generated line bundles is that they are *nef* — they have nonnegative intersection products with every curve in the variety. Not all nef line bundles are globally generated, but a consequence of the Abundance conjecture [KM98, Conjecture 3.12] is that all nef line bundles on Calabi-Yau varieties have some globally generated tensor power.

Unlike global generation, nefness is preserved by tensor products and a tensor power of a nef line bundle is nef if and only if the line bundle itself is nef. Thus the collection of nef line bundles, viewed as a subset of the Picard group, satisfies conditions analogous to those of a convex cone in a rational vector space. This analogy can be made literal by formally extending the intersection product from $\operatorname{Pic} X$ to the vector space $\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q}$ and defining a class in $\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Q}$ to be nef if it intersects all curves in X nonnegatively. The nef classes now form a convex cone in this possibly infinite-dimensional vector space. In fact we are better off if we modify this construction a little by quotienting out by the subgroup of those line bundles that have zero intersection product with all curves, called *numerically trivial* line bundles, and tensor with $\mathbb{R}$ in place of $\mathbb{Q}$.

So far, we have simplified the problem of classifying morphisms from X into projective spaces to the problem of computing a cone in a finite-dimensional real vector space, admittedly losing some information along the way. This cone is in general hard to compute since it is a priori described by infinitely many inequalities, one for each irreducible curve in X. Many of these inequalities are redundant and can be discarded using convex geometry as follows. We first define the *Néron-Severi space of curves* $N_1(X)_{\mathbb{R}}$ in a similar manner to the Néron-Severi space of line bundles described above and notice that the intersection pairing descends to a perfect pairing $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$. Next we identify those classes in this space that correspond to actual curves. An element of $N^1(X)_{\mathbb{R}}$ will be nonnegative on these classes if and only if it is nonnegative on the closure of the convex cone they generate, called the *closed cone of effective curves* in X. Being nonnegative on this cone is in turn equivalent to being nonnegative on any set of generators for it.

This naturally leads us to look for a small set of generators for the closed cone of effective curves. For general varieties no such set can be found — the closed cone of effective curves may for example be a circular cone, in which case every ray in the boundary is required to generate it. There are two cases in which we have a good chance at computing the cone of effective curves: Fano varieties and Calabi-Yau varieties.

The three main classes of varieties of interest in birational geometry are Fano, Calabi-Yau and canonically polarised varieties. These are roughly speaking varieties whose canonical divisors are respectively 'negative', 'zero' and 'positive'. Fano varieties are fairly well understood in low dimension, while very little is understood about canonically polarised varieties. Since Calabi-Yau varieties sit 'in between' these two classes, they tend to exhibit very interesting behaviour.

For Fano varieties, Mori's *Cone theorem* (Theorem 2.13) tells us that the closed cone of effective curves is rational polyhedral; that is, it is generated by finitely many classes of positive rational (equivalently, integral) linear combinations of curves. Since the nef cone and the closed cone of effective curves are dual under the intersection pairing, the nef cone is rational polyhedral too. Hence we can compute both the closed cone of effective curves and the nef cone simply by finding their generators.

The Cone theorem gives no useful information for Calabi-Yau varieties, but in its place we have the *Morrison-Kawamata cone conjecture* (Conjecture 2.22(i)), which loosely speaking says that the only way the nef cone of a Calabi-Yau variety can fail to be rational polyhedral is if the variety has a large automorphism group.

Another divisor cone attached to a projective variety is the *movable cone*. This too is known to be rational polyhedral for Fano varieties (combine [Bir+10, Corollary 1.3.2]) with the description of Mori dream spaces in [HK00, Definition 1.10 and Proposition 2.9]). For Calabi-Yau varieties there is another Morrison-Kawamata cone conjecture (Conjecture 2.22(ii)) which roughly predicts that the only way the movable cone of a Calabi-Yau variety can fail to be rational polyhedral is if its birational automorphism group is large.

1.1 Outline

This thesis consists of three further chapters. Chapter 2 contains mostly statements of definitions and results that will be used in later computations. We also show an example computation of the nef, movable and effective divisor cones for the blowup of the projective plane in one and two points. The other two chapters are devoted to explicit computations of divisor cones on families of Calabi-Yau varieties.

Chapter 3 begins by studying the complete intersection of five general bilinear forms on $\mathbb{P}^4 \times \mathbb{P}^4$. We show that such varieties are Calabi-Yau threefolds and compute their cones of nef, movable and effective divisors. The effective and movable cones coincide and are irrational. Using ideas from linear algebra, we construct a countably infinite number of small $\mathbb{Q}$-factorial modifications and show that the pullbacks of nef cones through these modifications cover the movable cone. We infer that our list of small $\mathbb{Q}$-factorial modifications is exhaustive and prove that the automorphism group is finite. These results generalise to higher dimensions. We then study the degenerate case of where the bilinear forms are general amongst symmetric bilinear forms. In this case we only get three small $\mathbb{Q}$-factorial modifications and rational polyhedral divisor cones. We verify that the Morrison-Kawamata cone conjectures hold in all cases.

Chapter 4 is devoted to general sections of the anticanonical divisor in the blowup of $\mathbb{P}^4$ in two lines. Here we find that the nef, effective and movable divisor cones are all rational polyhedral. The nef and effective divisor cones are similar to those of the blowup of the projective plane in two points, but an unexpected divisor shows up in the movable cone. We suggest a possible construction of a small $\mathbb{Q}$-factorial modification explaining this divisor.

1.2 Acknowledgements

I want to take this opportunity to thank my supervisor, prof. John Christian Ottem, for an amazing introduction to the field of birational geometry. I would not have been able to produce this work without his careful guidance and stimulating supervisions. I also want to extend my gratitude to all my fellow students in the algebra and geometry section at the University of Oslo for a wonderful social and academic environment.

Chapter 2

Background

All schemes considered in this thesis are defined over the field complex numbers.

2.1 Minimal models

The minimal model program very roughly aims to deconstruct 'sufficiently non-singular' projective varieties into three different types: Fano, Calabi-Yau and canonically polarised. We record some basic facts from minimal model theory here.

Definition 2.1. Let X be a nonsingular projective variety.

- X is *Fano* if the canonical sheaf is anti-ample.
- X is *Calabi-Yau* if the canonical sheaf is trivial and $H^1(X, \mathcal{O}_X) = \cdots = H^{n-1}(X, \mathcal{O}_X) = 0$ where $n = \dim X$.
- X is *canonically polarised* if the canonical sheaf is ample.

Definition 2.2. A nonsingular projective variety is *minimal* if its canonical divisor is nef.

In particular, Calabi-Yau and Fano varieties are minimal.

Proposition 2.3 ([Mat02, Proposition 12-1-2])**.** *A birational map $\phi \colon X \dashrightarrow Y$ between two minimal varieties is an isomorphism in codimension one.*

2.2 Néron-Severi spaces

We are interested in classifying line bundles on projective varieties according to their 'positivity properties'. This term refers to a loosely defined class of properties that seek to characterise the different ways a line bundle may admit many global sections, generally inspired by the observation that the line bundles $\mathcal{O}_{\mathbb{P}^n}(m)$ behave very differently for m positive and m negative.

We postpone the definitions of the precise properties of interest to us to the next chapter. They will mostly be stable under tensor products, and a line bundle $\mathcal{L}$ should enjoy the property if and only if any positive tensor power $\mathcal{L}^{\otimes m}$ does, if and only if all positive tensor powers do. Thus the line bundles satisfying a given positivity property should essentially form a convex cone. In order to make this precise, we will map the Picard group into a finite-dimensional vector space called the *Néron-Severi space.*

Let X be a projective variety. Taking the degree of restrictions of line bundle to curves in X induces an intersection form

$$\operatorname{Pic} X \times Z_1(X) \to \mathbb{Z}$$

between line bundles and one-cycles (formal sums of integral curves) in X.

Definition 2.4.

- A line bundle on X is *numerically trivial* if its intersection product with every integral curve is zero. A curve is *numerically trivial* if its intersection product with every line bundle is zero.
- The *Néron-Severi group of divisors* on X is the quotient $N^1(X)$ of $\operatorname{Pic} X$ by the numerically trivial line bundles. The *Néron-Severi group of curves* in X is the quotient $N_1(X)$ of $Z_1(X)$ by numerically trivial curves.
- The *real Néron-Severi spaces* of respectively divisors and curves are the spaces $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$ obtained from $N^1(X)$ and $N_1(X)$ by tensoring with $\mathbb{R}$. *Rational Néron-Severi spaces* are defined similarly.

The first step in the formation of the Néron-Severi space of divisors, quotienting out by the numerically trivial divisors, can be very drastic in certain cases. However in practice there are many cases where the Néron-Severi space is equal to the Picard group modulo torsion; this will be the case for all varieties considered in this thesis. Since this first step already kills all the torsion elements, the second step, tensoring with $\mathbb{R}$ or $\mathbb{Q}$, gives an injective map. Thus for the purposes of this thesis we may think of the Néron-Severi space of divisors as essentially the space of isomorphism classes of line bundles with coefficients in the appropriate field.

Remark 2.5. If $\mathcal{L}_1, \ldots, \mathcal{L}_k$ are line bundles on a variety and V is a k-dimensional subvariety then the intersection number $\mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_k \cdot V$ is completely determined by the numerical equivalence classes of the $\mathcal{L}_i$ on X. Hence we can make sense of intersection numbers of Néron-Severi classes with arbitrary subvarieties.

Theorem 2.6 (Theorem of the base [Laz04, Theorem 1.1.16])**.** *The Néron-Severi group of a projective variety is a finitely generated abelian group. Equivalently, the rational and real Néron-Severi spaces are finite-dimensional vector spaces.*

2.3 Positivity properties of line bundles

In this section we introduce the properties of line bundles that will occupy us for the rest of this thesis, starting with those directly related to morphisms to projective space, *amplitude* and *global generation*.

Definition 2.7. Let $\mathcal{L}$ be a line bundle on a projective variety.

- $\mathcal{L}$ is *very ample* if its global sections determine a closed embedding into some projective space; equivalently, if there exists a closed embedding $\phi\colon X \hookrightarrow \mathbb{P}^n$ such that $\mathcal{L}$ is isomorphic to $\phi^*\mathcal{O}_{\mathbb{P}^n}(1)$.
- $\mathcal{L}$ is *globally generated* if its global sections determine a map to projective space; equivalently if for every point in X we can find a global section of $\mathcal{L}$ not vanishing there.

Neither of these are 'stable under division' — there are line bundles that are neither very ample nor globally generated but still admit a positive tensor power which is very ample and hence globally generated. Stability under division will be forced upon us when we pass to cones in Néron-Severi space, so we introduce the stable properties already at the level of line bundles.

Definition 2.8. Let X be a projective variety.

- A line bundle on X is *ample* if some positive tensor power of it is ample and *semiample* if some positive tensor power of it is globally generated.
- A class in the Néron-Severi space of divisors on a projective variety is respective *ample* or *semiample* if it can be written as a linear combination of classes of ample or semiample line bundles with positive coefficients.

This definition demonstrates the most common way to transport properties from the Picard group to Néron-Severi spaces. Amplitude of line bundles is characterised numerically by the *Nakai-Moishezon criterion* [Laz04, Theorem 1.2.23] which states that a line bundle $\mathcal{L}$ is ample if and only if it has positive self-intersection numbers $\mathcal{L}^{\dim V} \cdot V$ on every subvariety V of X. This inspires an obvious alternative definition of amplitude of Néron-Severi classes which is easily checked to be equivalent to the given one for rational coefficients and surprisingly hard to check [Laz04, Theorem 2.3.18] is equivalent for real coefficients (real classes in Néron-Severi space have the disadvantage that we are not guaranteed that some multiple can be represented as the class of a line bundle).

Weakening this numerical condition for amplitude produces the crucial notion of *nef* line bundles and Néron-Severi classes:

Definition 2.9. A line bundle or Néron-Severi class on a projective variety is *nef* if its intersection product with every curve is nonnegative.

Notice that our definition allows for a Néron-Severi class to be nef even if it cannot be written as a convex combination of classes of nef line bundles. The property of being nef is much more flexible than amplitude; for example, the pullback of a nef class along a morphism is nef. The projection formula from intersection theory shows that pseudoample line bundles are nef. Conversely the *Abundance conjecture* predicts that every nef divisor on a Calabi-Yau is semiample. Thus computing the nef cones on a Calabi-Yau threefold will take us very close to classifying its morphisms to projective space.

Definition 2.10. The *ample cone* of a projective variety X is the subset $\operatorname{Amp}^1 X$ of $N^1(X)_{\mathbb{R}}$ consisting of ample classes. Similarly the *cone of nef divisors* is the subset $\operatorname{Nef}^1 X$ of nef classes.

We will introduce many more divisor cones shortly. The benefit of working with real coefficients is that convex cones may have irrational extremal rays, even if generated by (infinitely many) rational rays.

The nef cone is closed and contains the ample cone by the above. In fact their relationship is much stronger than this:

Theorem 2.11 (Kleiman's theorem [Laz04, Theorem 1.4.23])**.** *Let X be a projective variety. The cone of nef divisors in $N^1(X)_{\mathbb{R}}$ is the closure of the cone of ample divisors and the cone of ample divisors is the interior of the cone of nef divisors.*

We next recall the definition of the closed cone of effective curves from the introduction:

Definition 2.12.

- The *cone of effective curves* is the cone $\operatorname{Eff}_1 X \subset N_1(X)_{\mathbb{R}}$ generated by the classes of effective curves. Its closure $\overline{\operatorname{Eff}}_1 X$ is the *closed cone of effective curves.*
- The *cone of nef curves* is the cone $\operatorname{Nef}_1 X \subset N_1(X)_{\mathbb{R}}$ consisting of classes that have nonnegative intersection with every effective line bundle on X.

As explained in the introduction, the cone of nef divisors is dual to the closed cone of effective curves by definition, and so determining one is equivalent to determining the other. Both of these cones can be complicated in general, but Mori's Cone theorem tells us the part of $\overline{\operatorname{Eff}}_1 X$ consisting on curves which are negative on the canonical is easy:

Theorem 2.13 (Cone theorem [Laz04, Theorem 1.5.33])**.** *Let X be a smooth projective variety whose canonical divisor K_X is not nef.*

(i) *The closed cone $\overline{\operatorname{Eff}}_1 X$ of effective curves in X is generated by all curves that intersect K_X nonnegatively together with a countable family of rational curves C satisfying $0 \leq -K_X \cdot C \leq \dim X + 1$.*

(ii) *For a fixed $\epsilon > 0$ and ample divisor H on X, only finitely many of the rational curves in the previous part will satisfy $-K_X \cdot C \leq \epsilon H \cdot C$.*

Definition 2.14.

- A line bundle $\mathcal{L}$ is *movable* if its base locus is of codimension at least two. The *cone of movable divisors* is convex cone $\operatorname{Mov}^1 X \subset N^1(X)_{\mathbb{R}}$ spanned by classes of movable divisors. The *closed cone of movable divisors* is its closure $\overline{\operatorname{Mov}}^1 X$.
- A line bundle is *effective* if it admits global sections. The *cone of effective divisors* is the convex cone $\operatorname{Eff}^1 X \subset N^1(X)_{\mathbb{R}}$ generated by classes of effective line bundles. The *closed cone of effective divisors* is its closure $\overline{\operatorname{Eff}}^1 X$.
- A line bundle is *big* if the global sections of some positive tensor power defines a rational map to projective space whose image is of the same dimension as that of X. The *big cone* is the convex cone $\operatorname{Big}^1 X \subset N^1(X)$ spanned by classes of big line bundles.

A divisor can be shown to be big if and only if the global sections of some tensor power defines a birational map to its image in projective space [Laz04, Corollary 2.2.7]. The big cone is the interior of the effective cone and the effective cone is the closure of the big cone [Laz04, Theorem 2.2.26].

Lemma 2.15. *For any nonsingular projective variety X we have*

$$\operatorname{Nef}^1 X \subset \overline{\operatorname{Mov}}^1 X \subset \overline{\operatorname{Eff}}^1 X.$$

Proof. The nef cone is the closure of the ample cone, which is spanned by classes of very ample line bundles. The first containment thus follows from that ample line bundles are movable. The second containment follows from that movable line bundles are effective. □

Lemma 2.16. *Let X be an n-dimensional projective variety. The intersection of $n-1$ nef divisor classes is a nef curve class.*

Proof. We first show the result for ample divisor classes, so let $A_1, \dots, A_{n-1}$ be ample divisors and let D be an effective divisor on X. The claim is that the intersection number $A_1 \cdots A_{n-1} \cdot D$ is positive. After scaling we may replace the A_i with very ample divisors corresponding to closed embeddings $\phi_1, \dots, \phi_{n-1}$ into projective spaces. Let $\phi\colon X \hookrightarrow \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_{n-1}}$ be the product of these maps. Then A_i is the pullback along ϕ of $\mathcal{O}(0, \dots, 0, 1, 0, \dots, 0)$, with the one in the ith position. Now the push-pull formula for ϕ together with the description of the Chow ring of products of projective spaces give that $A_1 \cdots A_{n-1} \cdot E$ is nonnegative. The intersection product of n divisors is a multilinear map on finite-dimensional Néron-Severi spaces and in particular are continuous. Hence we are done since the nef cone is the closure of the ample cone. □

Lemma 2.17. *The closed effective cone of a nonsingular projective variety does not contain any lines.*

Proof. [Cas+14, Lemma 2.3] shows that every nonzero divisor in the *closed* effective cone $\overline{\mathrm{Eff}}^1 X$ is strictly positive on the product of $\dim X - 1$ ample classes. Hence the only way both D and $-D$ may lie in this cone is if $D = 0$ in $N^1(X)_{\mathbb{R}}$. □

2.4 Examples

In order to get a feel for the divisor cones introduced above, we compute them for the blowup of the projective plane in one and two points.

2.4.1 blowup of one point in the plane

Let X be the blowup of one point p in $\mathbb{P}^2$. Since this is a nonsingular variety, we may identify the Picard group with the Weil class group. Since it is also a surface, we may further identify these with the group of one-cycles up to rational equivalence. It is well known that $\mathrm{Cl}\, X = \mathbb{Z}\{H, E\}$ where H is the class of a pullback of a general hyperplane section and E is the exceptional divisor. We have the following intersection numbers [Har77, Proposition V.3.2]:

$$H^2 = 1, \qquad H \cdot E = 0, \qquad E^2 = -1.$$

Hence we get identifications $\mathrm{Pic}\, X = \mathrm{Cl}\, X = N^1(X) = N_1(X)$.

Clearly E is an effective divisor, as is the strict transform of a line through the point in the centre of the blowup. This latter can be computed to be $H - E$ [Har77, Proposition V.3.6]. Thus we have that

$$\mathrm{Eff}_1 X = \mathrm{Eff}^1 X \supset \mathbb{R}_+\{H - E, E\}.$$

The global sections of the line bundle associated to H determines the blowup morphism $\mathrm{Bl}_1 \mathbb{P}^2 \to \mathbb{P}^2$ and the global sections of the line bundle associated to $H - E$ gives the morphism to $\mathbb{P}^1$ that 'maps a point to the slope of the line through it and p'. Thus these are nef as divisors, hence also as curves:

$$\mathrm{Nef}_1 X = \mathrm{Nef}^1 X \supset \mathbb{R}_+\{H, H - E\}.$$

We claim that we have found all nef divisors. Indeed let $aH + bE$ be a nef divisor. Then it must have positive intersection with all effective curves. Its intersection products with $H - E$ and E are respectively $a + b$ and $-b$, and the inequalities $a + b \geq 0$ and $-b \geq 0$ cut out precisely the cone $\mathbb{R}_+\{H, H - E\}$. Since the cone of nef divisors both contain and is contained in this cone, it must be equal to it. Exactly the same argument shows that we have found all effective divisors.

Lemma 2.18. *The closed movable cone and the cone of nef divisors coincide on surfaces.*

Proof. The nef cone is always contained in the closed movable cone. Conversely if a movable divisor has negative intersection with some effective curve, all linearly equivalent divisors must contain this curve. Hence that curve is a divisor in the base locus, contradicting movability. Hence movable divisors are nef. □

In our case we see that the divisors H and $H - E$ on the nef boundary correspond to globally generated sheaves and hence are movable. Thus we have

$$\operatorname{Nef}^1 X = \operatorname{Mov}^1 X = \mathbb{R}_+\{H, H - E\}$$
$$\operatorname{Eff}^1 X = \mathbb{R}_+\{E, H - E\}.$$

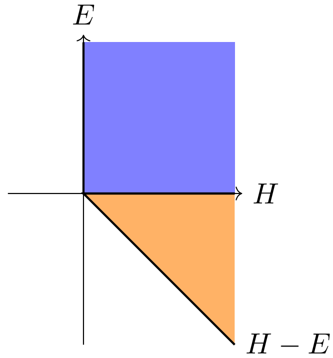


Figure 2.1: Divisor cones for the blowup of $\mathbb{P}^2$ in one point

The nef cone is the orange cone and the effective cone is the union of the orange and the blue cones.

2.4.2 blowup of two points in the plane

Let Y be the blowup of two points p_1 and p_2 in $\mathbb{P}^2$. This time we have

$$\operatorname{Pic} Y = \operatorname{Cl} Y = N^1(Y) = N_1(Y) = \{\mathbb{Z}, E_1, E_2\}$$

where H is still the strict transform of a hyperplane and E_1 and E_2 are the two exceptional divisors, with intersection numbers

$$H^2 = 1, \qquad H \cdot E_1 = H \cdot E_2 = E_1 \cdot E_2 = 0, \qquad E_1^2 = E_2^2 = -1.$$

Similar to the above case, we spot some effective cycles

$$\operatorname{Eff}^1 X = \operatorname{Eff}_1 X \supset \mathbb{R}_+\{E_1, E_2, H - E_1 - E_2\},$$

where $H - E_1 - E_2$ corresponds to the strict transform of the line through both points. We also spot some morphisms to projective space and see that

$$\operatorname{Nef}^1 X = \operatorname{Nef}_1 X \supset \mathbb{R}_+\{H, H - E_1, H - E_2\}$$

where H corresponds to the blowup morphism and $H - E_1$ and $H - E_2$ correspond to the two maps to $\mathbb{P}^1$ that 'map points to the slopes of the lines through it and respectively p_1 and p_2'. By a similar argument as above we can use intersection numbers to conclude that we have found the entire nef and effective cones. By Lemma 2.18 we have also found the closed movable cone, and this is equal to the movable cone itself since the extremal rays correspond to globally generated line bundles.

$$\begin{aligned} \operatorname{Nef}^1 X = \operatorname{Mov}^1 X &= \mathbb{R}_+\{H, H - E_1, H - E_2\} \\ \operatorname{Eff}^1 X &= \mathbb{R}_+\{E_1, E_2, H - E_1 - E_2\}. \end{aligned}$$

Since these cones are three-dimensional, it is easier to draw their intersection with an appropriate plane than to draw the entire cones. In our case we draw the intersection with the plane of divisors $aH + bE_1 + cE_2$ for which $3a + b + c = 1$.

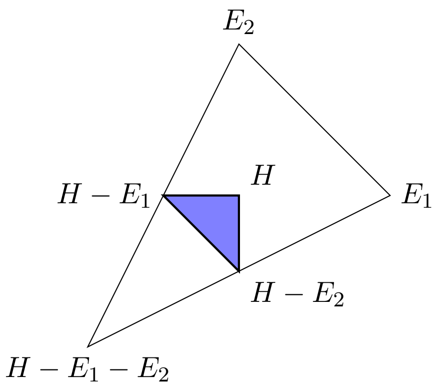


Figure 2.2: Divisor cones for the blowup of two points in the plane

2.5 Pullbacks of line bundles through isomorphisms in codimension one

Lemma 2.19. *Let $U \hookrightarrow X$ be an open subscheme of a nonsingular projective variety whose complement is of codimension at least two.*

(i) *Every line bundle on U extends uniquely to a line bundle on X.*

(ii) *If $\mathcal{L}$ is a line bundle on X then every section in $\mathcal{L}(U)$ extends uniquely to a global section of $\mathcal{L}$.*

Proof.

(i) Since U and X schemes are nonsingular, separated and noetherian, there are canonical isomorphisms $\operatorname{Pic} X = \operatorname{Cl} X$ and $\operatorname{Pic} U = \operatorname{Cl} U$. Combining this with the class group sequence gives the following diagram

$$\begin{tikzcd} 0 = \operatorname{Cl}(X \setminus U) \arrow[r] & \operatorname{Cl} X \arrow[r] & \operatorname{Cl} U \arrow[r] & 0 \\ & \operatorname{Pic} X \arrow[r] \arrow[u, dash, "\sim"] & \operatorname{Pic} U \arrow[u, dash, "\sim"] & \end{tikzcd}$$

where the top row is exact and the bottom row is the restriction map. Hence restriction is an isomorphism on Picard groups, so every line bundle on U extends uniquely to a line bundle on X.

(ii) Given a section $s \in \mathcal{L}(U)$ and a point $x \in X \setminus U$, let $V = \operatorname{Spec} A$ be an affine open subset around x trivialising $\mathcal{L}$. Then $U \cap V$ contains all height one primes of $\operatorname{Spec} A$ so by Hartogs' theorem [Har77, Proposition II.6.3A] the restriction map $\Gamma(V, \mathcal{L}) \to \Gamma(U \cap V, \mathcal{L})$ is an isomorphism. Thus we may extend s to a section of $U \cup V$. Repeatedly extending like this must eventually terminate since the underlying topological space of our scheme is noetherian. Thus we eventually produce the desired global section. $\square$

Definition 2.20. Let $\phi\colon X \dashrightarrow Y$ be an isomorphism in codimension one between nonsingular projective varieties. Then we may find inclusions $i\colon U \hookrightarrow X$ and $j\colon U \hookrightarrow Y$ of the same scheme U as an open subscheme in both X and Y whose complement is of codimension at least two such that $\phi = j \circ i^{-1}$. Using the above lemma, define the induced *pullback map* of line bundles to be the isomorphism $(i^*)^{-1} \circ j^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$. We define the *pullback* of a global section of a line bundle through ϕ by restricting it to U and extending it by the uniqueness statement above.

One checks that the above gives us well-defined notions of pullbacks that interact nicely with intersection products.

Corollary 2.21. *Let $\phi\colon X \dashrightarrow Y$ be an isomorphism in codimension one between nonsingular projective varieties.*

(i) *The pullback through ϕ of an effective line bundle is effective.*

(ii) *The pullback through ϕ of a movable line bundle is movable.*

2.6 The Morrison-Kawamata cone conjectures

The cone conjectures were introduced by Morrison and Kawamata in [Col93; Mor94; Kaw97]. The versions we cite are simplified from the statement appearing in [Tot10, Conjecture 2.1].

Conjecture 2.22 (Morrison-Kawamata cone conjectures)**.** *Let X be smooth Calabi-Yau.*

(i) *There exists a rational polyhedral cone Π which is a fundamental domain for the action of $\operatorname{Aut} X$ on the nef effective cone $\operatorname{Nef}^1 X \cap \operatorname{Eff}^1 X$.*

(ii) *There exists a rational polyhedral cone Π' which is a fundamental domain for the action of $\operatorname{BirAut} X$ on the movable effective cone $\overline{\operatorname{Mov}}^1 X \cap \operatorname{Eff}^1 X$.*

2.7 Further background results

Lemma 2.23. *Let X be a projective variety and let $\{\phi_\alpha\colon X \dashrightarrow X_\alpha\}$ be a family of isomorphisms in codimension one such that the movable cone of X is covered by the pullbacks of nef divisor cones through these maps. Then every isomorphism in codimension one $\phi\colon X \dashrightarrow Y$ is isomorphic to one of the isomorphisms in the given family.*

Proof. The pullback of the ample cone on Y to X is an open cone contained in the movable cone. Hence it intersects the pullback of the ample cone of some X_α in an open subset. Replacing X with X_α and ϕ with the composition $\phi \circ \phi_\alpha^{-1}$, we now have an isomorphism in codimension one $\phi\colon X \dashrightarrow Y$ where the pullback of the ample cone of Y meets the ample cone of X. Thus we can choose a line bundle $\mathcal{L}$ on Y so that both it and its pullback $\phi^*\mathcal{L}$ to X are very ample. We thus see that X and Y embed as the same closed subvariety of projective space through the map determined by this line bundle. $\square$

2.7.1 Bertini, Lefschetz and Kawamata-Viehweg

The main examples studied in this thesis are defined as general elements of a suitable basepoint free linear system on a nonsingular projective variety. Bertini's theorem shows that these are themselves nonsingular.

Theorem 2.24 (Bertini's theorem [Har77, Corollary III.10.9]). *The general element of a linear system on a nonsingular projective variety over an algebraically closed field of characteristic zero is nonsingular away from the base points of the system.*

Before we try to compute divisor cones on such varieties we must understand the ambient Néron-Severi space. In the cases considered in this thesis all the hard work is done for us by the Lefschetz theorem.

Theorem 2.25 (Lefschetz theorem for Picard groups [Laz04, Example 3.1.25]). *Let X be a smooth projective complex variety of dimension at least four and let $D \subset X$ be a reduced effective ample divisor. Then the restriction map $\operatorname{Pic} X \to \operatorname{Pic} D$ is an isomorphism.*

We will need the following vanishing theorem as a technical ingredient in some cohomology calculations later on.

Theorem 2.26 (Kawamata-Viehweg vanishing [Laz04, Theorem 4.3.1]). *Let D be a nef and big divisor on a smooth complex projective variety. Then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.*

Chapter 3

Determinantal quintic threefolds

In this chapter we study the divisor cones for the complete intersection X of five general sections of $\mathcal{O}(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^4$. Such varieties are studied from the point of view of mirror symmetry in [HT14]. We can think of such a variety as being given by a general five-dimensional subspace of the vector space $\Gamma\mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^4}(1,1)$ or, choosing a basis for this subspace, by five bihomogeneous polynomials in $\mathbb{C}[x_0,\ldots,x_4][y_0,\ldots,y_4]$ of bidegree $(1,1)$. We may write these equations as

$$\sum_{i,j} A_{ijk}x_iy_j = 0$$

where A_{ijk} are coefficients in $\mathbb{C}$ and k ranges from zero to four. Thus if we think of A_{ijk} as a $5\times5\times5$-tensor then the five-dimensional space of global sections of $\mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^4}(1,1)$ cutting out X can be represented by contracting A against the vectors $(x_0,\ldots,x_4)$, $(y_0,\ldots,y_4)$ and $(z_0,\ldots,z_4)$ in the indices i,j and k respectively, where $(z_0,\ldots,z_4)$ ranges over vectors in k^5. At this point we have already involved three distinct copies of $\mathbb{P}^4$, coordinatised by the x_i, the y_j and the z_k respectively. For the sake of clarity we will write $\mathbb{P}^4_x$, $\mathbb{P}^4_y$ and $\mathbb{P}^4_z$ to distinguish these.

Changing the roles of the indices, a general $5\times5\times5$-tensor A_{ijk} gives rise to three varieties in this way:

$$\begin{aligned}
X_1 &= \{(y_0:\cdots:y_4)\times(z_0:\cdots:z_4) : \sum_{j,k} A_{ijk}y_jz_k = 0 \text{ for all } i\} \subset \mathbb{P}^4_y\times\mathbb{P}^4_z\\
X_2 &= \{(x_0:\cdots:x_4)\times(z_0:\cdots:z_4) : \sum_{i,k} A_{ijk}x_iz_k = 0 \text{ for all } j\} \subset \mathbb{P}^4_x\times\mathbb{P}^4_z\\
X = X_3 &= \{(x_0:\cdots:x_4)\times(y_0:\cdots:y_4) : \sum_{i,j} A_{ijk}x_iy_j = 0 \text{ for all } k\} \subset \mathbb{P}^4_x\times\mathbb{P}^4_y.
\end{aligned}$$

The image of X under the first projection to $\mathbb{P}^4_x$ consists of those points $(x_0:\cdots:x_4)$ such that the 5×5-tensor $\sum_i A_{ijk}x_i$ contracts along the j-index against some $(y_0:\cdots:y_4)$ to the zero vector. In other words, the image is the vanishing locus of the determinant of $\sum_i A_{ijk}x_i$ viewed as a matrix indexed by j and k. Notice that by symmetry this is also the image of X_2 under its projection to $\mathbb{P}^4_x$. So in this way we obtain three more varieties

$$\begin{aligned}
Y_1 &= \{(x_0:\cdots:x_4) : \det(\sum_i A_{ijk}x_i)_{jk} = 0\} \subset \mathbb{P}^4_x\\
Y_2 &= \{(y_0:\cdots:y_4) : \det(\sum_j A_{ijk}y_j)_{ik} = 0\} \subset \mathbb{P}^4_y\\
Y_3 &= \{(z_0:\cdots:z_4) : \det(\sum_k A_{ijk}z_k)_{ij} = 0\} \subset \mathbb{P}^4_z.
\end{aligned}$$

These six varieties together with their projection morphisms fit into an infinite commutative diagram:

$$
\begin{tikzcd}
& X_1 \arrow[rr, dashed, bend left, "\psi"] \arrow[dl, "p"'] \arrow[dr, "q"] & & X_2 \arrow[ll, dashed, bend left, "\phi"] \arrow[rr, dashed, bend left, "\psi"] \arrow[dl, "p"'] \arrow[dr, "q"] & & X_3 \arrow[ll, dashed, bend left, "\phi"] \arrow[rr, dashed, bend left, "\psi"] \arrow[dl, "p"'] \arrow[dr, "q"] & & X_1 \arrow[ll, dashed, bend left, "\phi"] \arrow[rr, dashed, bend left, "\psi"] \arrow[dl, "p"'] \arrow[dr, "q"] & & X_2 \arrow[ll, dashed, bend left, "\phi"] \arrow[dl, "p"'] \arrow[dr, "q"] & \\
Y_2 & & Y_3 & & Y_1 & & Y_2 & & Y_3 & & Y_1
\end{tikzcd}
$$

Figure 3.1: Birational models of X

By abuse of notation we denote all projection maps to the left by p and all projection maps to the right by q. In Lemma 3.3 below we show that p and q are birational. Thus we get birational maps $\phi = q^{-1} \circ p$ and $\psi = p^{-1} \circ q$, again abusing notation. Composing three ϕ's or three ψ's give birational automorphisms of X which will turn out to have an interesting action on the Néron-Severi space of X; see Lemma 3.8 below. In particular, these are not the identity automorphism of X_3.

Lemma 3.1. *Let M be a singular $n \times n$ matrix. Then the columns of* $\operatorname{adj} M$ *all lie in the kernel of M and the jth column is nonzero if and only if all rows of M except the jth form a linearly independent set. In particular* $\operatorname{adj} M$ *has a nonzero column if and only if M is of corank one, and in this case we moreover have that the columns of* $\operatorname{adj} M$ *are all proportional.*

Proof. It is a well-known linear algebra fact that $M \operatorname{adj} M = I \det M$ for any matrix, and the RHS is zero since M is singular. This shows that the columns of $\operatorname{adj} M$ all lie in the kernel of M. The condition for the jth column to be nonzero follows immediately from the definition of adjugate matrices. The 'moreover' part holds because the identity $(\operatorname{adj} M)M = 0$ imposes $n-1$ linearly independent conditions on the columns of $\operatorname{adj} M$. □

Lemma 3.2. *Let A_{ijk} be a general $5 \times 5 \times 5$-tensor. The loci in respectively Y_1, Y_2 and Y_3 where the matrices $(\sum_i A_{ijk} x_i)_{jk}$, $(\sum_j A_{ijk} y_j)_{ik}$ and $(\sum_k A_{ijk} z_k)_{ij}$ are of rank three are nonempty and zero-dimensional. The loci where the matrices are rank at most two are empty.*

Proof. By symmetry, it suffices to check the rank three locus of $(\sum_i A_{ijk} x_i)_{jk}$ in Y_1. We have a morphism $\mathbb{A}^5_x \to \mathrm{Mat}_{5\times 5} \cong \mathbb{A}^{25}$ given by $(x_0, \dots, x_4) \mapsto (\sum_i A_{ijk} x_i)_{jk}$. Observe that this is an embedding of vector spaces, hence induces a closed embedding $\mathbb{P}^4_x \hookrightarrow \mathbb{P}(\mathrm{Mat}_{5\times 5}) \cong \mathbb{P}^{24}$. We want to compute the dimension of the preimage of the locus $Z \subset \mathbb{P}(\mathrm{Mat}_{5\times 5})$ consisting of rank three matrices. Since any linear map $\mathbb{A}^5 \to \mathrm{Mat}_{5\times 5}$ can be obtained by choosing A and we only care about the general A, it suffices to prove that Z is a closed subset of $\mathbb{P}(\mathrm{Mat}_{5\times 5})$ of codimension four.

The locus $Z \subset \mathbb{P}(\mathrm{Mat}_{5\times 5})$ consisting of matrices of rank three is described as the vanishing locus of the four by four minors, hence is a closed subset. For every choice of three out of the five columns, we can ask for the subset of Z consisting of matrices where these three chosen columns are linearly independent. This gives an open cover of Z consisting of $\binom{5}{3} = 10$ open sets, and it suffices to compute the dimension of each of these. A choice of a rank three 5×5-matrix whose first three columns are linearly independent is the same as first choosing the three linearly independent columns (a Zariski open subset of a 15-dimensional affine space) and then choosing the final two columns as linear combinations of the first three columns (a 6-dimensional affine space).

Hence after quotienting by scaling we get that the open sets covering Z are of dimension $15 + 6 - 1 = 20$. Hence Z is of codimension four in $\mathbb{P}^{24}$.

Finally the loci where the matrices are rank two are empty by the same argument since the corresponding Z is codimension nine. □

Lemma 3.3. *The projection morphisms p and q in Figure 3.1 are small contractions; that is, they are birational morphisms and isomorphisms in codimension one.*

Proof. For each $j = 0, \dots, 4$ let U_j be the open subscheme of Y_1 consisting of those $(x_0 : \dots : x_4)$ such that all but the jth row of the matrix $(\sum_i A_{ijk}x_i)_{jk}$ forms a linearly independent subset. Then the union of $U_0, \dots, U_4$ consists of all the points $(x_0 : \dots : x_4)$ for which $(\sum_i A_{ijk}x_i)_{jk}$ is of rank four. By Lemma 3.2 the complement of $U_0 \cup \dots \cup U_4$ is a zero-dimensional closed subscheme of Y_1 and its preimage is a union of $\mathbb{P}^1$'s, one for each point in the complement of $U_0 \cup \dots \cup U_4$. Thus it suffices to construct an inverse over $U_0 \cup \dots \cup U_4$. By Lemma 3.1 the jth column of the adjugate matrix of $(\sum_i A_{ijk}x_i)_{jk}$ serves as an inverse over U_j, and since the columns of the adjugate are proportional they agree on overlaps. We thus get the desired inverse over $U_0 \cup \dots \cup U_4$. □

Remark 3.4. It follows from this proof that the singular loci of Y_1, Y_2 and Y_3 are contained in the rank three loci of the matrices in Lemma 3.2. In fact one can show that the singular loci are equal to the respective rank three loci. We compute the degrees of these determinantal rank loci using the formula by Harris and Tu [HT84, Proposition 12(a)] and find that the singular loci of Y_1, Y_2 and Y_3 are zero-dimensional schemes of degree 50.

Lemma 3.5. *The varieties X_1, X_2, X_3 are smooth Calabi-Yau threefolds.*

Proof. That X is smooth follows from Bertini's theorem 2.24 applied five times. The canonical divisor of X is trivial by five applications of the adjunction formula. It remains to check that $H^1(X, \mathcal{O}_X)$ vanishes. The Koszul complex for X is exact since X is locally Cohen-Macaulay [Laz04, Appendix B.2]:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^4}(-5,-5)^{\oplus\binom{5}{5}} \xrightarrow{d_5} \cdots \xrightarrow{d_2} \mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^4}(-1,-1)^{\oplus\binom{5}{1}} \xrightarrow{d_1} \mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^4} \xrightarrow{d_0} \mathcal{O}_X \longrightarrow 0.$$

Breaking this up into short exact sequences and using the Künneth theorem to compute cohomology of sheaves on $\mathbb{P}^4 \times \mathbb{P}^4$, we get that the dimensions of the cohomology groups of $\mathcal{O}_X$ are $(h^0, \dots, h^3) = (1, 0, 0, 1)$. In particular $H^1(X, \mathcal{O}_X) = 0$. By symmetry the same applies to X_1 and X_2. □

Lemma 3.6. *The Picard group of X is $\operatorname{Pic} X = \mathbb{Z}\{p^*H, q^*H\}$, generated by the pullbacks of hyperplane sections through the projections $p\colon X \to Y_1$ and $q\colon X \to Y_2$. The intersection numbers of divisors on X are*

$$(p^*H)^3 = 5, \qquad (p^*H)^2 \cdot q^*H = 10, \qquad p^*H \cdot (q^*H)^2 = 10, \qquad (q^*H)^3 = 5.$$

By symmetry the same is true for X_1 and X_2.

Proof. The statement about the Picard group holds by five applications of the Lefschetz theorem for Picard groups (Theorem 2.25) and the fact that $\operatorname{Pic}(\mathbb{P}^4_x \times \mathbb{P}^4_y)$ is the free abelian group on the corresponding two hyperplane sections, which we also denote by p^*H and q^*H by slight abuse of notation. The push-pull formula for the inclusion map $X \hookrightarrow \mathbb{P}^4_x \times \mathbb{P}^4_y$ gives that

$$\int_X (p^*H)^3 = \int_{\mathbb{P}^4\times\mathbb{P}^4} (p^*H)^3 \cdot [X] = \int_{\mathbb{P}^4\times\mathbb{P}^4} (p^*H)^3 \cdot (p^*H + q^*H)^5 = 5.$$

The other intersection products are similar. □

3.1 Nef cone

Proposition 3.7. *The cone of nef divisors on X is* $\operatorname{Nef}^1 X = \mathbb{R}_+\{p^*H, q^*H\}$.

Proof. Since the projection map p contracts any curve C in the preimage of a point in the singular locus of Y_1 we have that $p^*H \cdot C = 0$. On the other hand, C projects to a curve in Y_2 so $q^*H \cdot C$ is positive. Hence any nef divisor $aH_1 + bH_2$ must satisfy $a \geq 0$, and by symmetry also $b \geq 0$. So the nef cone is contained in the one specified in the statement. Conversely p^*H and q^*H are semiample and in particular nef. □

3.2 Pullbacks of divisor classes through birational maps

Lemma 3.8. *The maps $\phi\colon X_l \dashrightarrow X_{l-1}$ and $\psi\colon X_l \dashrightarrow X_{l+1}$ in Figure 3.1 induce the following pullback maps on divisor class groups*

$$\begin{aligned} \phi^*\colon \operatorname{Cl} X_{l-1} &\to \operatorname{Cl} X_l \\ p^*H &\mapsto 4p^*H - q^*H \\ q^*H &\mapsto p^*H \end{aligned} \qquad\qquad \begin{aligned} \psi^*\colon \operatorname{Cl} X_{l+1} &\to \operatorname{Cl} X_l \\ p^*H &\mapsto q^*H \\ q^*H &\mapsto -p^*H + 4q^*H \end{aligned}$$

Here H denotes the class of a hyperplane section in the appropriate copy of $\mathbb{P}^4$.

Proof. By Lemma 3.3 these birational maps are isomorphisms in codimension one, so they do indeed induce pullbacks on class groups. By Lemma 3.6 the expressions given for ϕ^* and ψ^* completely determine them.

That ϕ^* sends $q^*H \mapsto p^*H$ follows from the fact that $q \circ \phi = p$. To check that ϕ^* also sends $p^*H \mapsto 4p^*H - q^*H$ we must compute the pullback of a hyperplane section through the map $p \circ \phi\colon X_l \dashrightarrow Y_l$. By symmetry it suffices to do this for $l = 3$. Hence we are pulling back a hyperplane section through the map $X_3 \dashrightarrow Y_3$ obtained by composing maps in the following commutative diagram:

$$\begin{tikzcd} & X_2 \arrow[dl, "p"'] \arrow[dr, "q"] & & X_3 \arrow[ll, dashed, "\phi"'] \arrow[dl, "p"] \\ Y_3 & & Y_1 & \end{tikzcd}$$

We construct the map $X_3 \dashrightarrow Y_3$ explicitly using similar ideas as in the proof of Lemma 3.3. Given a point $(x_0 : \cdots : x_4) \times (y_0 : \cdots : y_4) \in X_3$, we first project it down to $(x_0 : \cdots : x_4)$, then take the generally unique lift $(x_0 : \cdots : x_4) \times (z_0 : \cdots : z_4) \in X_2$ and project it down to $(z_0 : \cdots : z_4) \in Y_3$. That is, the desired map takes a general point $(x_0 : \cdots : x_4) \times (y_0 : \cdots : y_4) \in X$ to the one-dimensional kernel of the matrix $M = (\sum_i A_{ijk}x_i)_{jk}$. Lemma 3.1 tells us that this can be described as the one-dimensional column space of $\operatorname{adj} M$. Hence taking z to be the jth column c_j of $\operatorname{adj} M$ would describe the map over the locus in X where this jth column is nonzero.

Unfortunately the above description only gives the desired rational map over the complement of the divisor $\{y_j = 0\} \subset X$ and hence fails to be an isomorphism in codimension one. Luckily we can glue these five rational maps together as follows. Recall that $(\operatorname{adj} M)M = \det M = 0$ when $(x_0 : \cdots : x_4) \times (y_0 : \cdots : y_4) \in X_3$ and observe that $(y_0, \ldots, y_4)M = 0$. Since $\operatorname{adj} M$ is rank one and M is rank four, it follows that $\frac{c_0}{y_0}, \ldots, \frac{c_4}{y_4}$ are defined on the opens $\{y_0 \neq 0\}, \ldots, \{y_4 \neq 0\} \subset X$ and agree on overlaps. Hence they glue to a vector of five global sections of $\mathcal{O}_{X_3}(4p^*H - q^*H)$ which is nonzero over the complement of the singular locus of Y_1. These five sections in turn give the desired rational map $X_3 \dashrightarrow Y_3$. Hence the class of hyperplane sections of Y_3 pulls back to the divisor class $4p^*H - q^*H$ on X_3, completing our computation of ϕ^*. The expression for ψ^* follows either by symmetry or by inverting ϕ^*. $\square$

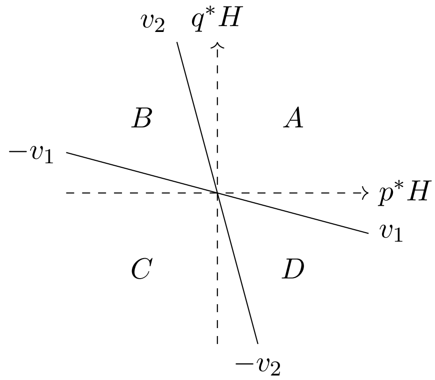
We can restate the above lemma as saying that the pullback maps through ϕ and ψ are represented by the matrices

$$[\phi^*] = \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}, \qquad [\psi^*] = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}$$

with respect to the ordered bases p^*H, q^*H for each $\operatorname{Cl} X_i$. We diagonalise these matrices in order to understand the asymptotic behaviour of pulling back divisors through repeated compositions. Since ϕ^* and ψ^* are inverse, they share the same eigenvectors with eigenvalues given in the following table:

eigenvector	eigenvalue for ϕ^*	eigenvalue for ψ^*
$v_1 = (2+\sqrt{3})\, p^*H - q^*H$	$2+\sqrt{3}$	$2-\sqrt{3}$
$v_2 = (2+\sqrt{3})\, q^*H - p^*H$	$2-\sqrt{3}$	$2+\sqrt{3}$

Beware that we are here implicitly using that ϕ and ψ are isomorphisms in codimension one and hence induce isomorphism on Néron-Severi spaces to identify the Néron-Severi spaces of X_1, X_2 and X_3. We now subdivide these Néron-Severi spaces according to the eigenspaces as shown in the following picture:



Pulling back a divisor in one of the four regions A, B, C or D through ϕ or ψ gives another divisor in the same region since the eigenvalues of ϕ^* and ψ^* are all positive. Repeatedly pulling back a ray in A or D through ϕ will produce a sequence of rays converging to the ray spanned by v_1 and repeatedly pulling back a ray in B or C will produce a sequence converging to the ray spanned by $-v_1$. Similarly repeatedly pulling back through ψ will produce a sequence of rays converging to $\pm v_2$ depending on whether the original ray lies in $A \cup B$ or $C \cup D$. Using this description of repeated pullbacks, we can now easily show the following:

Proposition 3.9. *The closed cones of effective and movable divisors on X are*

$$\overline{\operatorname{Mov}}^1 X = \overline{\operatorname{Eff}}^1 X = \mathbb{R}_+\{(2+\sqrt{3})\, p^*H - q^*H, (2+\sqrt{3})\, q^*H - p^*H\}.$$

This is the closure of the region labelled A in the above diagram.

Proof. By Lemma 2.15 it suffices to show that the divisor classes in A are movable and that those outside $\bar{A}$ fail to be effective. Nef divisors are movable and pullbacks of a movable divisors through isomorphisms in codimension on are movable by Corollary 2.21. Since every divisor class in A is obtained as the pullback of a nef divisor class through an appropriate number of ϕ's or ψ's, it follows that divisor classes in A are movable.

It remains to check that divisors outside $\bar{A}$ fail to be effective. Since effective classes are preserved under pullbacks through isomorphisms in codimension one by Corollary 2.21, we see that if any divisor in one of the regions B, C or D is effective then so is every divisor in this region. The closed cone of effective divisors cannot contain a line by Lemma 2.17, so no divisor class outside of $\bar{A}$ can be effective. □

Remark 3.10. The effective cone was computed by [Fry01, page 4] by similar methods.

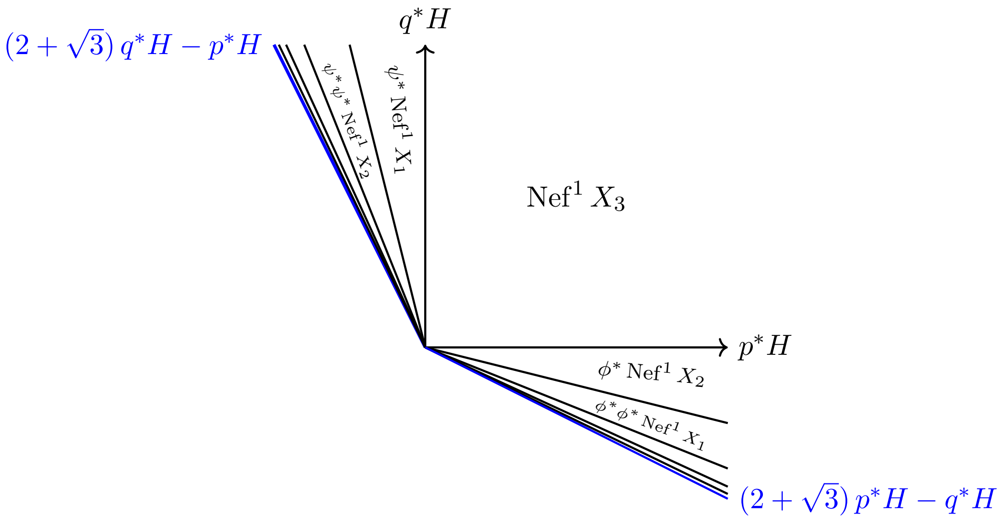


Figure 3.2: Divisor cones of a general determinantal quintic threefold

Corollary 3.11. *X satisfies the Morrison-Kawamata cone conjectures (Conjecture 2.22).*

Proof. The nef effective cone of X is just the nef cone. Since pullbacks of nef divisors through automorphisms are nef, the automorphism group of X must act on $N^1(X)_{\mathbb{R}}$ in a way that preserves the nef cone. Moreover the action must send ray generators to ray generators, hence either fixes or swaps p^*H and q^*H. If every automorphism fixes the ray generators, $\mathrm{Nef}^1 X$ is itself a rational polyhedral fundamental domain for the action. If there exists some automorphism which swaps the two generators, we may divide this cone in half along the diagonal ray generated by $p^*H + q^*H$ and choose one of the two halves as our rational polyhedral fundamental domain. In either case the conjecture holds for the nef cone.

Since the pullbacks of nef cones cover the movable cone, we know from Lemma 2.23 that we have all isomorphisms in codimension one from X to other varieties up to isomorphism. Since X is a Calabi-Yau variety and thus a minimal model, every birational automorphism of X is an isomorphism in codimension one (Proposition 2.3), hence is among the maps already discovered. Thus the birational automorphism group acts on the rays of the nef cones in the above figure and preserves adjacency of two rays; in other words, we have a group homomorphism

$$\mathrm{BirAut}\, X \to D_\infty = \langle r, s : s^2 = 1, srs = r^{-1} \rangle$$

from the birational automorphism group to the infinite dihedral group such that the action of X is taken to the obvious action of D_∞ on the set of rays in the above figure. The rational polyhedral fundamental domain we choose depends only on the image of this group homomorphism. Notice that ϕ^3 and ψ^3 are birational automorphisms of X, hence the image of this group homomorphism contains the subgroup generated by r^3. This is a normal subgroup of index six whose quotient is the ordinary dihedral group $D_6 \cong S_3$. Thus we have narrowed the possible images of the above group homomorphism down to the six different subgroups of D_6. We split into cases depending on which of the six subgroups appear.

- The trivial subgroup of D_6 corresponds to $\langle r^3 \rangle \subset D_\infty$, which acts by 'shifting the nef cones multiples of three steps along' in the above diagram. Thus we can take the union $\psi^* \operatorname{Nef}^1 X_1 \cup \operatorname{Nef}^1 X_3 \cup \phi^* \operatorname{Nef}^1 X_2$ as a fundamental domain for the action.

- The unique subgroup of D_6 of order three corresponds to $\langle r \rangle \subset D_\infty$, which acts by 'shifting nef cones an integral number of steps along' in the above diagram. Thus we can take $\operatorname{Nef}^1 X_3$ to be fundamental domain in this case.

- The group D_6 itself corresponds to the action of all of D_∞, thus we can take for example the half of $\operatorname{Nef}^1 X_3$ below the diagonal ray generated by $p^*H + q^*H$ as our rational polyhedral fundamental domain.

- The order two subgroup $\langle s \rangle \subset D_6$ corresponds to the subgroup $\langle r^3, s \rangle \subset D_6$, thus we can take for example the union of $\phi^* \operatorname{Nef}^1 X_2$ and the lower half of $\operatorname{Nef}^1 X_3$ as our rational polyhedral fundamental domain. The other two order two subgroups are similar.

In any case we find a rational polyhedral fundamental domain for the action of the birational automorphism group of X on its movable effective cone. □

3.3 Automorphism group

As an application of our knowledge of the divisor cones of X, we give a partial computation of its automorphism group. The key observation is that an automorphism of X must induce an automorphism of Néron-Severi spaces preserving the nef cones and the lattice structure. Hence the pullback on Picard groups either fixes the isomorphism classes of $\mathcal{O}_X(1,0)$ and $\mathcal{O}_X(0,1)$ or swaps them.

Lemma 3.12. *The natural maps $\Gamma\mathcal{O}_{\mathbb{P}^4_x}(1) \to \Gamma\mathcal{O}_X(1,0)$ and $\Gamma\mathcal{O}_{\mathbb{P}^4_y}(1) \to \Gamma\mathcal{O}_X(0,1)$ are isomorphisms.*

Proof. For the first isomorphism, notice that we can factor it as $\Gamma\mathcal{O}_{\mathbb{P}^4_x}(1) \to \Gamma\mathcal{O}_{\mathbb{P}^4_x \times \mathbb{P}^4_y}(1,0) \to \Gamma\mathcal{O}_X(1,0)$ and that the first of these maps is an isomorphism. Hence it suffices to show that the second arrow is also an isomorphism. The Koszul complex is exact. Splitting it up into short exact sequences we see that the map $\mathcal{O}_{\mathbb{P}^4 \times \mathbb{P}^4}(1,0) \to \mathcal{O}_X(1,0)$ induces isomorphisms in all cohomology groups and in particular induces isomorphisms on global sections. The second isomorphism follows by symmetry. □

Lemma 3.13. *Every automorphism of X extends to an automorphism of $\mathbb{P}^4_x \times \mathbb{P}^4_y$.*

Proof. Let α be an automorphism of X. As remarked above, pullback through α must either fix or swap the isomorphism classes of $\mathcal{O}_X(1,0)$ and $\mathcal{O}_X(0,1)$. Suppose α fixes these; the other case is similar. Then $\alpha^*\mathcal{O}_X(1,0)$ admits an isomorphism to $\mathcal{O}_X(1,0)$. Moreover an automorphism of a line bundle corresponds to an element of $(\Gamma\mathcal{O}_X)^* = k^*$, so this isomorphism ϕ is unique up to scaling by an element of k^*. Composing the natural map $\mathcal{O}_X(1,0) \to \alpha_*\alpha^*\mathcal{O}_X(1,0)$ with $\alpha^*\phi$ gives a map $\mathcal{O}_X(1,0) \to \alpha_*\mathcal{O}_X(1,0)$. Now applying the global sections functor gives a map $\Gamma\mathcal{O}_X(1,0) \to \Gamma\mathcal{O}_X(1,0)$, again defined up to scaling. One can check that this association is contravariant functorial. Therefore since α is an automorphism, so are the maps $\Gamma\mathcal{O}_X(1,0) \to \Gamma\mathcal{O}_X(1,0)$. By the previous lemma, we can interpret this as an automorphism of $\Gamma\mathcal{O}_{\mathbb{P}^4_x}(1)$ which in turn induces an automorphism of $\mathbb{P}^4_x$. Similarly by considering the pullback of $\mathcal{O}_X(0,1)$ we get an automorphism of $\mathbb{P}^4_y$, and together these determine an automorphism of $\mathbb{P}^4_x \times \mathbb{P}^4_y$ (automorphisms which swap $\mathcal{O}_X(1,0)$ and $\mathcal{O}_X(0,1)$ induce automorphisms of $\mathbb{P}^4_x \times \mathbb{P}^4_y$ that swap the two factors). One then checks that this restricts to the original automorphism of X. □

Proposition 3.14. *The automorphism group of X is finite.*

Proof. The subgroup consisting of automorphisms which fix $\mathcal{O}_X(1,0)$ and $\mathcal{O}_X(0,1)$ is of index at most two, so it suffices to show that it is finite. This subgroup is in turn a closed subscheme of the compact algebraic group $\mathrm{PGL}_5 \times \mathrm{PGL}_5$, so it suffices to show that it is discrete or equivalently that the component containing the identity consists only of the identity. If this was not the case, there the automorphism group would contain a curve containing the identity. Small translations along this curve produces a nonzero global section of the tangent bundle of X. We show that $H^0(X, \mathcal{T}_X) = 0$ to reach a contradiction.

We have a perfect pairing $\Omega_X \times \Omega^2_X \to \Omega^3_X = \omega_X \cong \mathcal{O}_X$ given by the wedge product. This gives an isomorphism $\Omega^2_X \cong \mathcal{T}_X$, so

$$H^0(X, \mathcal{T}_X) \cong H^0(X, \Omega^2) \cong H^2(X, \mathcal{O}_X) = 0$$

where the second isomorphism is by Hodge symmetry and the final equality is because X is Calabi-Yau. □

We expect that the automorphism group is trivial, although it is easy to find $5 \times 5 \times 5$-tensors A_{ijk} for which the automorphism group becomes nontrivial.

3.4 Generalising to higher dimension

We observe that the calculations we have done so far work the same in higher dimensions.

Proposition 3.15. *Let $n \geq 4$. The complete intersection X of $n+1$ general $\mathcal{O}(1,1)$-divisors in $\mathbb{P}^n \times \mathbb{P}^n$ is a smooth Calabi-Yau $(n-1)$-fold. The Néron-Severi space of X is $N^1(X)_{\mathbb{R}} = \mathbb{R}\{H_1, H_2\}$ where H_1 and H_2 are the pullbacks of the hyperplane sections of the two copies of $\mathbb{P}^4$. The closed cones of nef, movable and effective divisors on X are*

$$\mathrm{Nef}^1 X = \mathbb{R}_+\{H_1, H_2\}$$
$$\overline{\mathrm{Mov}}^1 X = \overline{\mathrm{Eff}}^1 X = \mathbb{R}_+\{(\tfrac{n}{2} + \tfrac{1}{2}\sqrt{n^2-4})H_1 - H_2, (\tfrac{n}{2} + \tfrac{1}{2}\sqrt{n^2-4})H_2 - H_1\}.$$

Proof. This calculation is essentially the same as the case $n = 4$ done above, so we only highlight the differences. Notice that we need $n \geq 4$ in order for the Lefschetz theorem on Picard groups to apply. X still fits into a diagram such as Figure 3.1. The projection maps p and q contract n-cycles (in fact copies of $\mathbb{P}^n$) to points over loci in the Y_i of codimension $n^2 + 2n$. In particular there are curves that contract to points (again using that $n \geq 4$) and no divisors are contracted. The curves that contract to points impose conditions on nef divisor classes, proving that $\operatorname{Nef}^1 X$ is as stated. That no divisors are contracted shows that ϕ and ψ are still isomorphisms in codimension one, hence pullbacks under these preserve nef and movable divisor classes. The pullback maps induced by ϕ and ψ this time are represented by the matrices

$$[\phi^*] = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}, \qquad\qquad [\psi^*] = \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix}.$$

As before we use the symmetry of the problem to identify the source and target spaces of these pullbacks and diagonalise with respect to $p^*H = H_1$ and $q^*H = H_2$.

eigenvector	eigenvalue for ϕ^*	eigenvalue for ψ^*
$v_1 = (\frac{n}{2} + \frac{1}{2}\sqrt{n^2-4})H_1 - H_2$	$\frac{n}{2} + \frac{1}{2}\sqrt{n^2-4}$	$\frac{n}{2} - \frac{1}{2}\sqrt{n^2-4}$
$v_2 = (\frac{n}{2} + \frac{1}{2}\sqrt{n^2-4})H_2 - H_1$	$\frac{n}{2} - \frac{1}{2}\sqrt{n^2-4}$	$\frac{n}{2} + \frac{1}{2}\sqrt{n^2-4}.$

Hence by the same argument as before we get the specified movable and closed effective cones. $\square$

Notice that the extremal rays in the movable and closed effective cones are always irrational. By exactly the same argument as before, we have that:

Corollary 3.16. *X in the above proposition satisfies the Morrison-Kawamata cone conjecture.*

3.5 Symmetric matrices

Next we study the subvariety of $\mathbb{P}^4 \times \mathbb{P}^4$ cut out by five general *symmetric* bilinear forms. That is, we take A to be a $5\times5\times5$-tensor satisfying the condition $A_{ijk} = A_{jik}$ for all i, j, k and study the corresponding variety X. Again we form the diagram in Figure 3.1. The crucial difference is that the closed subset of Y_3 consisting of those points $(z_0 : \cdots : z_4)$ such that the matrix $(\sum_k A_{ijk}z_k)_{ij}$ is of rank at most three becomes a one-dimensional scheme.

Lemma 3.17. *The closed subset of Y_3 consisting of points $(z_0 : \cdots : z_4)$ whose associated matrix $(\sum_k A_{ijk})_{ij}$ is of rank at most three is one-dimensional and of degree 20. The corresponding subsets of Y_1 and Y_2 are still zero-dimensional and the rank two loci here remain empty.*

Proof. The argument is similar to the proof of Lemma 3.2. We have a map $\mathbb{A}^5_z \to \operatorname{SymMat}_{5\times5} \cong \mathbb{A}^{15}$ given by mapping $(z_0, \ldots, z_4) \mapsto (\sum_k A_{ijk}z_k)_{ij}$. This is a general embedding of vector spaces (since A is general amongst tensors symmetric in the first two indices) so induces a closed embedding of projective spaces. We want to compute the dimension and degree of the intersection with the closed subset Z of symmetric matrices of rank at most three.

The dimension of Z can be computed on the open subset of symmetric matrices of rank at most three whose first three columns are linearly independent. We have a twelve-dimensional family of choices in the first three columns (fifteen entries with three symmetry relations). Now symmetry and the requirement that the last two columns be linear combinations of the first three specify the rest of the matrix. Projectivising we get that Z is an eleven-dimensional closed subset of $\mathbb{P}(\mathrm{SymMat}_{5\times 5})$. Hence it intersects the copy of $\mathbb{P}^4_z$ in a codimension three, equivalently one-dimensional closed subset. The degree of Z is easily computed to be 20 by the formula in [HT84, Proposition 12(b)].

The statements for Y_1 and Y_2 are essentially proved in Lemma 3.2. $\square$

Lemma 3.18. *Let A be general among $5 \times 5 \times 5$-tensors symmetric in swapping the two first indices. Then the projections in Figure 3.1 down to Y_1 and Y_2 are isomorphisms in codimension one which contract a finite number of curves each. The projections down to Y_3 contract a divisor, namely the preimages of the rank three locus in Y_3. Moreover the birational equivalence of X_1 and X_2 extends to an isomorphisms as in the following diagram:*

$$\begin{tikzcd}
X_3 \arrow[rr, dashed, bend left, "\psi"] \arrow[rrrrrr, bend left=40, "\mathrm{swap}"] \arrow[dr, "q"'] & & X_1 \arrow[ll, dashed, bend left, "\phi"] \arrow[rr, leftrightarrow, "\sim"] \arrow[dl, "p"] \arrow[dr, "q"'] & & X_2 \arrow[rr, dashed, bend left, "\psi"] \arrow[dl, "p"] \arrow[dr, "q"'] & & X_3 \arrow[ll, dashed, bend left, "\phi"] \arrow[rr, dashed, bend left, "\psi"] \arrow[dl, "p"] \arrow[dr, "q"'] & & X_1 \arrow[ll, dashed, bend left, "\phi"] \arrow[rr, leftrightarrow, "\sim"] \arrow[dl, "p"] \arrow[dr, "q"'] & & X_2 \arrow[dl, "p"] \arrow[dr, "q"'] & \\
& Y_2 & & Y_3 & & Y_1 & & Y_2 & & Y_3 & & Y_1
\end{tikzcd}$$

The birational automorphism of X_3 obtained by composing from the X_3 on the left to the X_3 in the middle is the automorphism given by $(x_0 : \cdots : x_4) \times (y_0 : \cdots : y_4) \mapsto (y_0 : \cdots : y_4) \times (x_0 : \cdots : x_4)$ which swaps the coordinates. That tracing through from the left to the middle produces the birational automorphism equal to the automorphism which swaps coordinates is an easy exercise in composing maps.

Proof. The first two statements follow directly from the previous lemma. To see that the birational equivalence extends to an isomorphism, just notice that $p^{-1} \circ q \colon X_1 \dashrightarrow X_2$ is the map which sends $(y_0 : \cdots : y_4) \times (z_0 : \cdots : z_0)$ to itself over the complement of the rank three locus in Y_3. $\square$

Lemma 3.19. *The preimages $D_1 \subset X_1$ and $D_2 \subset X_2$ of the rank three locus in Y_3 are divisors of class*

$$D_1 = 4q^*H - 2p^*H \quad (\text{in } X_1), \qquad\qquad D_2 = 4p^*H - 2q^*H \quad (\text{in } X_2).$$

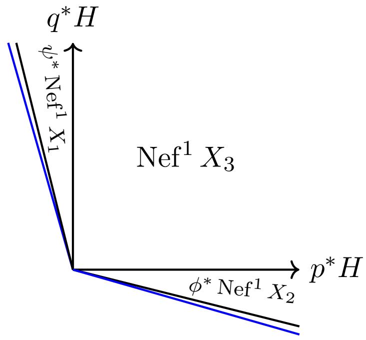
Proof. Write C for the rank three locus in Y_3. We compute D_2 using the method of undetermined coefficients. Write $D_2 = ap^*H + bq^*H$. Now intersecting both sides with p^*H^2 and $p^*H \cdot q^*H$ and applying Lemma 3.6 gives

$$\begin{aligned} 5a + 10b &= D_2 \cdot p^*H^2 = 0 \\ 10a + 10b &= D_2 \cdot p^*H \cdot q^*H = 20. \end{aligned}$$

In the first equation we have used that $D_2 \cdot p^*H^2 = p^*(C \cdot H^2)$ and that planes in $\mathbb{P}^4$ can be chosen to miss the curve C. In the second equation we used that degrees can be computed on pushforwards by q and the projection formula

$$q_*(D_2 \cdot p^*H \cdot q^*H) = q_*p^*(C \cdot H) \cdot H = 20.$$

This last equality follows from that C is a curve of degree 20, so that $q_*p^*(C \cdot H)$ is the class of the lines in $Y_1 \subset \mathbb{P}^4_x$ that contract to the 20 points on $C \cdot H$. Solving the above equations give $a = 4$ and $b = -2$ as desired. $\square$



Proposition 3.20. *The nef, effective and movable cones of X are*

$$\begin{aligned}\mathrm{Nef}^1 X &= \mathbb{R}_+\{p^*H, q^*H\}\\ \overline{\mathrm{Mov}}^1 X &= \mathbb{R}_+\{4p^*H - q^*H, 4q^*H - p^*H\}\\ \overline{\mathrm{Eff}}^1 X &= \mathbb{R}_+\{7p^*H - 2q^*H, 7q^*H - 2p^*H\}.\end{aligned}$$

Proof. Since the projections p and q are morphisms to projective space which contract curves, p^*H and q^*H are on the boundary of the nef cone. The same argument holds for the birational models X_1 and X_2, and since the maps ϕ and ψ are isomorphisms in codimension one, the pullbacks of the nef cones of X_1 and X_2 remain movable divisors on X. Hence the movable cone contains $\mathbb{R}_+\{4p^*H - q^*H, 4q^*H - p^*H\}$. To see that $4p^*H - q^*H$ is extremal in the movable cone, note that it is the pullback of p^*H from X_2. Since the fibre f of the map $p\colon X_2 \to Y_3$ over a general point in the rank three locus has intersection numbers $p^*H \cdot f = 0$ and $q^*H \cdot f = 1$ on X_2, any divisor $ap^*H + bq^*H$ on X_2 with $b < 0$ must contain this fibre and hence must contain the entire divisor D_2 from Lemma 3.19. In particular such divisors are not movable, so p^*H is extremal in the movable cone on X_2 and hence its pullback $4p^*H - q^*H$ is extremal in the movable cone on X. By symmetry, so is $4q^*H - p^*H$, hence we have now computed the movable cone on X.

The ray $7p^*H - 2q^*H$ is the pullback of (half of) the divisor $D_2 = 4p^*H - 2^*H$ on X_2. Since this is the preimage under p of the rank three locus in Y_3, it is an effective divisor, and since it vanishes on the nonzero nef curve p^*H^2 on X_2 it is extremal in the effective cone. By symmetry this computes the effective cone of X. $\square$

Chapter 4

Quintic hypersurface in the blowup of $\mathbb{P}^4$ in two lines

Let l_1 and l_2 be two disjoint lines in $\mathbb{P}^4$ and consider the blowup $Y = \mathrm{Bl}_{l_1,l_2}\mathbb{P}^4$ together with its blowup morphism $\pi\colon Y \to \mathbb{P}^4$. We define H to be the divisor class of the pullback $\pi^*\mathcal{O}_{\mathbb{P}^4}(1)$ and E_1 and E_2 to be the exceptional divisors over the two lines. In this chapter we show that the line bundle associated to $5H - 2E_1 - 2E_2$ admits global sections and that its general section is a Calabi-Yau threefold. We then compute the cones of nef and effective divisors for this threefold.

4.1 Preliminaries on the blowup of two lines

For the rest of this chapter, we assume that the coordinates on $\mathbb{P}^4$ are chosen so that $l_1 = \mathbb{V}(x_0, x_1, x_2)$ and $l_2 = \mathbb{V}(x_2, x_3, x_4)$.

Lemma 4.1.

(i) *Y is isomorphic to the closed subscheme of $\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^2$ cut out by the maximal minors of the two matrices*

$$\begin{pmatrix} u_0 & u_1 & u_2 \\ x_0 & x_1 & x_2 \end{pmatrix}, \qquad \begin{pmatrix} v_2 & v_3 & v_4 \\ x_2 & x_3 & x_4 \end{pmatrix}.$$

Under this isomorphism, the blowup morphism π is identified with the projection onto the $\mathbb{P}^4$.

(ii) *The exceptional divisors E_1 and E_2 are both isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$. The morphisms $E_i \to l_i$ are the projections onto the $\mathbb{P}^1$ factor.*

(iii) *We have the following relation between (the sheaves associated to) the divisor classes H, E_1, E_2 and the pullbacks of $\mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^2\times\mathbb{P}^2}(a,b,c)$:*

$$\begin{aligned} H &= \mathcal{O}_Y(1,0,0) & \mathcal{O}_Y(1,0,0) &= H \\ E_1 &= \mathcal{O}_Y(1,-1,0) & \mathcal{O}_Y(0,1,0) &= H - E_1 \\ E_2 &= \mathcal{O}_Y(1,0,-1) & \mathcal{O}_Y(0,0,1) &= H - E_2. \end{aligned}$$

In particular the maps to $\mathbb{P}^4_x, \mathbb{P}^2_u$ and $\mathbb{P}^2_v$ correspond to respectively $H, H - E_1$ and $H - E_2$.

(iv) *The maps induced on Picard groups by the inclusion maps $E_i \hookrightarrow Y$ satisfy*

$$\begin{aligned} \operatorname{Pic} Y &\to \operatorname{Pic} E_1 & \qquad \operatorname{Pic} Y &\to \operatorname{Pic} E_2 \\ \mathcal{O}_Y(a,b,c) &\mapsto \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(a+c,b) & \qquad \mathcal{O}_Y(a,b,c) &\mapsto \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(a+b,c) \end{aligned}$$

under the identifications of the exceptional divisors with $\mathbb{P}^1 \times \mathbb{P}^2$. (Below we show that these are all elements of $\operatorname{Pic} Y$*, so we have in fact determined the map on Picard groups.) In particular the normal sheaves $\mathcal{O}_{E_1}(E_1)$ and $\mathcal{O}_{E_2}(E_2)$ are both identified with $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,-1)$.*

Proof.

(i) In this proof, a matrix in a denominator should be interpreted as the ideal generated by the maximal minors of that matrix. The statement is equivalent to that the morphism

$$\pi\colon \operatorname{Proj} \frac{\mathbb{C}[x_0,\dots,x_4][u_0,u_1,u_2][v_2,v_3,v_4]}{\left(\begin{smallmatrix} u_0 & u_1 & u_2 \\ x_0 & x_1 & x_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} v_2 & v_3 & v_4 \\ x_2 & x_3 & x_4 \end{smallmatrix}\right).} \to \operatorname{Proj} \mathbb{C}[x_0,\dots,x_4] = \mathbb{P}^4$$

induced by the inclusion of rings satisfies the universal property of a blowup. Here the ring on the left is a trigraded polynomial ring in the obvious way. This universal property can be checked locally on the target. The above morphism is an isomorphism over the complement of the two lines, so we only need to check that it is the blowup morphism over opens covering the two lines. Notice that l_1 is covered by D_+x_3 and D_+x_4 and that both of these are disjoint from l_2. Over D_+x_3 we can identify π with the diagonal map in the following diagram, which we recognise as the blowup of $D_+x_3 \cong \mathbb{A}^4$ in the line $l_1 \cap D_+x_3 = \mathbb{V}(x_{0/3}, x_{1/3}, x_{2/3})$.

$$\begin{tikzcd} \operatorname{Proj} \frac{\mathbb{C}[x_{0/3},\dots,x_{4/3}][u_0,u_1,u_2][v_2,v_3,v_4]}{\left(\begin{smallmatrix} u_0 & u_1 & u_2 \\ x_{0/3} & x_{1/3} & x_{2/3} \end{smallmatrix}\right), \left(\begin{smallmatrix} v_2 & v_3 & v_4 \\ x_{2/3} & 1 & x_{4/3} \end{smallmatrix}\right)} \arrow[r, dash, "\sim"] \arrow[d, "\pi|_{D_+x_3}"] & \operatorname{Proj} \frac{\mathbb{C}[x_{0/3},\dots,x_{4/3}][u_0,u_1,u_2]}{\left(\begin{smallmatrix} u_0 & u_1 & u_2 \\ x_{0/3} & x_{1/3} & x_{2/3} \end{smallmatrix}\right)} \arrow[dl] \\ D_+x_3 & \end{tikzcd}$$

By symmetry we also have that π is the desired blowup morphism over D_+x_0, D_+x_1 and D_+x_4. Hence π is indeed to blowup of $\mathbb{P}^4$ in $l_1 \cup l_2$.

(ii) The exceptional divisor is the preimage of the two lines. The preimage of l_1 is the closed subscheme cut out by x_0, x_1, x_2, hence is naturally identified with the trigraded Proj of the ring

$$\frac{\mathbb{C}[x_3,x_4][u_0,u_1,u_2][v_2,v_3,v_4]}{(v_2x_3, v_2x_4, v_3x_4 - v_4x_3)}.$$

This ring receives an inclusion map from the ring $\mathbb{C}[x_3,x_4][u_0,u_1,u_2]$ which is easily seen to induce an isomorphism on projective schemes, giving the desired isomorphism $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^2$. The projection down to l_1 corresponds to the inclusion of the subring $\mathbb{C}[x_3,x_4]$ which in turn corresponds to the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$. By symmetry the same holds for E_2 and l_2.

(iii) That H corresponds to the sheaf $\mathcal{O}_Y(1,0,0)$ is essentially the definition of H. We compute the ideal sheaf $\mathcal{I}_{E_1}$ of the exceptional divisor E_1 in Y. This is the image of the natural map $\pi^*\mathcal{I}_{l_1} \to \pi^*\mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_Y$. Precomposing with the pullback of the surjection $(x_0, x_1, x_2)\colon \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 3} \to \mathcal{I}_{l_1}$, it suffices to describe the image of $\pi^*\mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 3} \to \mathcal{O}_Y$. This is just the submodule generated by rational sections of the form $\frac{x_i}{x_j}$ where $i \in \{0,1,2\}$ and $j \in \{0,\dots,4\}$ over appropriate opens. But this in turn can be described as the image of the injective map $\frac{x_0}{u_0} = \frac{x_1}{u_1} = \frac{x_2}{u_2}\colon \mathcal{O}_Y(-1,1,0) \to \mathcal{O}_Y$. Hence the ideal sheaf of E_1 is $\mathcal{I}_{E_1} \cong \mathcal{O}_Y(-1,1,0)$ and consequently the sheaf associated to E_1 is $\mathcal{O}_Y(1,-1,0)$. The computation of the sheaf associated to E_2 is similar.

(iv) To compute the map $\operatorname{Pic}(\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^2) \to \operatorname{Pic} E_1$, observe that the inclusion $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^4_x \times \mathbb{P}^2_u \times \mathbb{P}^2_v$ is the map

$$\begin{gathered}\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^2 \\ (x_3 : x_4) \times (u_0 : u_1 : u_2) \mapsto (0:0:0:x_3:x_4) \times (u_0:u_1:u_2) : (0:x_3:x_4).\end{gathered}$$

The description of the restriction map $\operatorname{Pic} Y \to \operatorname{Pic} E_1$ now follows, and we get the restriction map to E_2 by symmetry. □

Lemma 4.2. $\operatorname{Pic} Y = \mathbb{Z}\{H, E_1, E_2\}$. *In particular, the pullback map* $\operatorname{Pic}(\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^2) \to \operatorname{Pic} Y$ *is an isomorphism.*

Proof. Since Y is nonsingular, the Picard group coincides with the Weil class group. We have the exact sequences

$$\begin{tikzcd} \mathbb{Z}\{l_1, l_2\} \arrow[r, "0"] & \operatorname{Cl}\mathbb{P}^4 \arrow[r, "\sim"] & \operatorname{Cl}(\mathbb{P}^4 \setminus (l_1 \cup l_2)) \arrow[r] \arrow[d, dash, "\sim"'] & 0 \\ \mathbb{Z}\{E_1, E_2\} \arrow[r] & \operatorname{Cl} Y \arrow[r] & \operatorname{Cl}(Y \setminus (E_1 \cup E_2)) \arrow[r] & 0. \end{tikzcd}$$

$\operatorname{Cl}\mathbb{P}^4$ is the free abelian group on a hyperplane section. We claim that the map $\mathbb{Z}\{E_1, E_2\} \to \operatorname{Cl} Y$ is injective. It will then follow that the bottom row is a split short exact sequence, giving a canonical isomorphism $\operatorname{Pic} Y = \operatorname{Cl} Y = \mathbb{Z}\{H, E_1, E_2\}$ as desired.

To prove the claim, suppose $aE_1 + bE_2$ is trivial in $\operatorname{Cl} Y$. Then the associated sheaf $\mathcal{O}_Y(a+b, -a, -b)$ is trivial in the Picard group. Pulling this back further to $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^2$ gives that the sheaf $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(a, -a)$ is trivial in $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^2)$, from which we can conclude that $a = 0$. By symmetry we also have $b = 0$, hence the map is injective. □

Lemma 4.3. *The canonical divisor on the blowup of* $\mathbb{P}^4$ *in two lines is* $K_Y = -5H + 2E_1 + 2E_2 = \mathcal{O}_Y(1, 2, 2)$.

Proof. Since the blowup morphism π restricts to an isomorphism over the complement of the two exceptional divisors and Y is nonsingular, the canonical sheaf must be of the form $\omega_Y \cong \pi^*\omega_{\mathbb{P}^4} \otimes \mathcal{O}_Y(aE_1) \otimes \mathcal{O}_Y(bE_2)$ for integers a, b to be determined. Using Lemma 4.1 (iii) we can rewrite this as $\omega_Y \cong \mathcal{O}_Y(-5 + a + b, -a, -b)$. Restricting both sides to E_1 using Lemma 4.1 (iv) now gives

$$\omega_Y|_{E_1} \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-5 + a, -a)$$

under the identification $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^2$. On the other hand, the adjunction formula for E_1 as a divisor in Y says $\omega_{E_1} \cong \omega_Y|_{E_1} \otimes \mathcal{O}_{E_1}(E_1)$, which rearranges to

$$\omega_Y|_{E_1} \cong \omega_{\mathbb{P}^1\times\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-1, 1) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-3, -2)$$

again using Lemma 4.1 to transport line bundles over the identification $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^2$. Combining the two displayed equations above now lets us solve for a, giving $a = 2$. By symmetry we also have that $b = 2$. □

Let X be a general section of the sheaf $\mathcal{O}_Y(-K_Y) = \mathcal{O}_Y(5H - 2E_1 - 2E_2) = \mathcal{O}_Y(1,2,2)$.

Lemma 4.4. *X is a Calabi-Yau threefold whose Picard group is canonically isomorphic to that of Y via the pullback map.*

Proof. X is smooth by Bertini's theorem 2.24 and the pullback map on Picard groups is an isomorphism by the Lefschetz theorem 2.25. The adjunction formula tells us that the canonical divisor is $\omega_X \cong \omega_Y|_X \otimes \mathcal{O}_X(X) \cong \mathcal{O}_X$.

To see that the middle cohomology groups of the structure sheaf vanish we use the ideal sheaf sequence for X as a divisor in Y,

$$0 \longrightarrow \mathcal{O}_Y(-5H + 2E_1 + 2E_2) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Since $5H - 2E_1 - 2E_2$ is an ample divisor (it is the pullback of $\mathcal{O}(1,2,2)$ through the embedding into $\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^2$), the Kawamata-Viehweg vanishing theorem (Theorem 2.26) tells us that the sheaf on the left has no cohomology except possibly in degree four. In particular the map $\mathcal{O}_Y \to \mathcal{O}_X$ induces an isomorphisms on cohomology in degree one and two. Hence it suffices to show that $H^1(Y, \mathcal{O}_Y)$ and $H^2(Y, \mathcal{O}_Y)$ both vanish. This follows from a special case of the Leray spectral sequence [Har77, Exercise III.8.1] for the blowup morphism $\pi\colon Y \to \mathbb{P}^4$ using that $\pi_*\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^4}$ and that the remaining right derived pushforwards $R^i\pi_*\mathcal{O}_Y$ vanish for $i > 0$ (these last two are proved using the Theorem on formal functions similarly to the proof of [Har77, Proposition V.3.4]). Hence X is a Calabi-Yau threefold. $\square$

4.2 Intersection products

Lemma 4.5. *We have the following intersection products on Y*

$$H^4 = 1, \qquad H \cdot E_1^3 = H \cdot E_2^3 = 1, \qquad E_1^4 = E_2^4 = 3.$$

and all other degree four monomials in H, E_1, E_2 vanish.

Proof. This follows from the following calculations:

- $E_1 \cdot E_2 = 0$

  This holds simply because these two divisors are disjoint.

- $H^2 \cdot E_1 = H^2 \cdot E_2 = 0$

  These follow from the push-pull formula for the map $\pi\colon Y \to \mathbb{P}^4$ since $\pi_* E_i = 0$.

- $H^4 = 1$

  Since pullback through maps of smooth quasi-projective varieties is a ring homomorphism, it suffices to compute the pullback of $\mathcal{O}_{\mathbb{P}^4}(1)^4$, which is clearly the class of a reduced point.

- $H \cdot E_1^3 = H \cdot E_2^3 = 1$

  The push-pull formula tells us that the first of these can be computed as the intersection product $\mathcal{O}_{E_1}(H) \cdot \mathcal{O}_{E_1}(E_1)^2$ on E_1. Identifying E_1 with $\mathbb{P}^1 \times \mathbb{P}^2$, this intersection product becomes $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,0) \cdot \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,-1)^2 = 1$. By symmetry we get the same value for $H \cdot E_2^3$.

- $E_1^4 = E_2^4 = 3$

  The push-pull formula gives $E_1^4 = \mathcal{O}_{E_1}(E_1)^3$. This in turn is equal to $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,-1)^3 = 3$. By symmetry we get the same value for E_2^4. □

Lemma 4.6. *The intersection numbers of divisors on X are*

$$H^3 = 5, \qquad H\cdot E_1^2 = H\cdot E_2^2 = -2, \qquad E_1^3 = E_2^3 = -1$$

and all other degree three monomials in H, E_1, E_2 vanish.

Proof. These follow immediately from the intersection numbers on Y and the push-pull formula for the inclusion of X. For example we compute H^3 on X as $H^3\cdot(5H-2E_1-2E_2)$ on Y. Expanding brackets and using the known intersection theory on Y gives $H^3 = 5$. The others are similar. □

4.3 Finding special curves

Lemma 4.7. *The fibres C_1 and C_2 of the restriction of the blowup morphism $\pi|_X : X \to \mathbb{P}^4$ over a general point in respectively l_1 and l_2 are curves with the following intersection numbers with divisors on X:*

$$\begin{aligned} &H\cdot C_1 = 0, && E_1\cdot C_1 = -2, && E_2\cdot C_1 = 0,\\ &H\cdot C_2 = 0, && E_1\cdot C_2 = 0, && E_2\cdot C_2 = -2. \end{aligned}$$

Proof. The intersection of X with E_1 can be obtained as the zero locus of the restriction of the section determining X. This is a section of $\mathcal{O}_Y(1,2,2)$ which restricts to a section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(3,2)$ on $E_1 \cong \mathbb{P}^1\times\mathbb{P}^2$. Let C_1 be the fibre of this over a general point in $\mathbb{P}^1 \cong l_1$. This is clearly an effective curve in X.

By the push-pull formula the intersection numbers of this curve with divisors in X is the same as the intersection numbers with the corresponding divisors on Y. By definition C_1 is a curve in E_1 of type $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(3,2)\cdot\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,0)$. Denote by i the inclusion $E_1 \hookrightarrow Y$. Then we may write the class of C_1 as $i^*\mathcal{O}_Y(1,2,2)\cdot i^*\mathcal{O}_Y(1,0,0)$. Using the push-pull formula for i, the intersection numbers of C_1 with the divisors H, E_1, E_2 on Y equal the intersection numbers of $\mathcal{O}_Y(1,2,2)\cdot\mathcal{O}_Y(1,0,0)\cdot\mathcal{O}_Y(1,-1,0) = (5H-2E_1-2E_2)\cdot H\cdot E_1 = -2H\cdot E_1^2$ with the same divisors:

$$\begin{aligned} H\cdot C_1 &= H\cdot(-2H\cdot E_1^2) = 0\\ E_1\cdot C_1 &= E_1\cdot(-2H\cdot E_1^2) = -2\\ E_2\cdot C_1 &= E_2\cdot(-2H\cdot E_1^2) = 0. \end{aligned}$$

The intersection numbers for C_2 follow by symmetry. □

Lemma 4.8. *X contains an effective curve C such that*

$$H\cdot C = 1, \qquad E_1\cdot C = 1, \qquad E_2\cdot C = 1.$$

Geometrically, C is the strict transform of a line meeting both l_1 and l_2.

Proof. We show that the image of X under the blowup map to $\mathbb{P}^4$ contains a line intersecting both of the given lines. This is a statement about the intersection of X with the strict transform S of the hyperplane in $\mathbb{P}^4$ containing the two lines. Since this strict transform is represented by the divisor $H - E_1 - E_2$ we get the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}_Y(4H - E_1 - E_2) \longrightarrow \mathcal{O}_Y(5H - 2E_1 - 2E_2) \longrightarrow \mathcal{O}_S(5H - 2E_1 - 2E_2) \longrightarrow 0.$$

Writing $4H - E_1 - E_2 = 2H + (H - E_1) + (H - E_2)$, this divisor is clearly nef and big, hence the leftmost sheaf has no higher cohomology by Kawamata-Viehweg vanishing (Theorem 2.26). Hence we see that all global sections of the line bundle associated to $5H - 2E_1 - 2E_2$ restricted to S arise as restrictions of sections of the original line bundle. By functoriality of blowups we can realise S as the blowup of $\mathbb{P}^3$ in two lines, and the new line bundles H, E_1, E_2 will coincide with the restrictions of the old ones. In particular the intersection of X with S is a general section of $\mathcal{O}_{\mathrm{Bl}_{l_1\cup l_2}\mathbb{P}^3}(5H - 2E_1 - 2E_2)$. Since $\pi_*\mathcal{O}_{\mathrm{Bl}_{l_1\cup l_2}\mathbb{P}^3}(5H - 2E_1 - 2E_2) = \mathcal{I}^2_{l_1\cup l_2}(5)$, global sections correspond to quintic polynomials in the ideal $(x_0, x_1)^2(x_2, x_3)^2$. In particular, the image of X intersects the hyperplane through the two lines in the vanishing locus of a general quintic polynomial in this ideal.

For fixed points $(a : b : 0 : 0)$ and $(0 : 0 : c : d)$ on the two lines, the condition that the line joining them is contained in the image of X is equivalent to that the quintic equation in the above ideal vanishes at $(\lambda a : \lambda b : \mu c : \mu d)$ for all λ and μ. We decompose f as a sum of two polynomials in $\mathbb{C}[x_0, x_1][x_2, x_3]$, one of bidegree $(3, 2)$ and one of bidegree $(2, 3)$, and find that the coefficients of $\lambda^3\mu^2$ and $\lambda^2\mu^3$ cut out loci in $\mathbb{P}^1_{a,b} \times \mathbb{P}^1_{c,d}$ of bidegree $(3, 2)$ and $(2, 3)$ respectively. These generally intersect in twelve points. This gives twelve lines in the image of X meeting both of the original lines. The strict transform of any of these serves as the desired C. $\square$

4.4 Computing divisor cones

Theorem 4.9. *The nef, closed movable and closed effective cones of X are*

$$\begin{aligned}
\mathrm{Nef}^1 X &= \mathbb{R}_+\{H, H - E_1, H - E_2\}\\
\overline{\mathrm{Mov}}^1 X &= \mathbb{R}_+\{H, H - E_1, H - E_2, 5H - 3E_1 - 3E_2\}\\
\overline{\mathrm{Eff}}^1 X &= \mathbb{R}_+\{H - E_1 - E_2, E_1, E_2\}.
\end{aligned}$$

Before proving this, we draw the cones by drawing their intersections with a plane in the Néron-Severi space. In our case we choose the hyperplane consisting of divisors $aH + bE_1 + cE_2$ for which $3a + b + c = 0$.

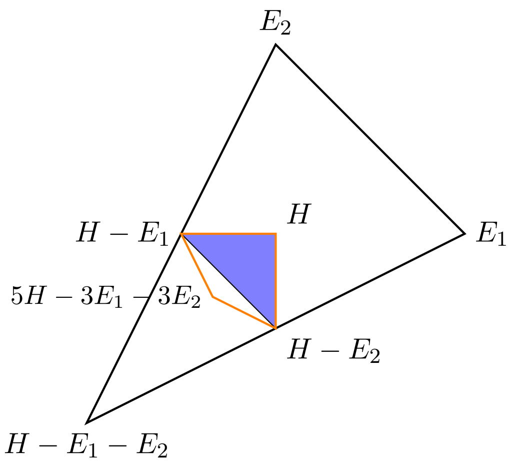


Figure 4.1: Intersection of divisor cones with hyperplane

The big triangle represents the effective cone, the orange quadrilateral bounds the movable cone and the shaded region is the nef cone.

We split the proof into three lemmas.

Lemma 4.10. *The cone of nef divisors on X is* $\operatorname{Nef}^1 X = \mathbb{R}_+\{H, H - E_1, H - E_2\}$.

Proof. The three generators for the specified cone are nef since their associated line bundles determine the projection maps to the three factors of $\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^2$. Conversely suppose $aH + bE_1 + cE_2$ is a nef divisor on X; then it must have nonnegative intersection with all effective curves. In particular it must have nonnegative intersections with the curves C_1, C_2 and C from the previous two lemmas. This translates to the inequalities

$$\begin{aligned} -2b &\geq 0 \\ -2c &\geq 0 \\ a + b + c &\geq 0 \end{aligned}$$

which cut out the specified cone. ☐

Lemma 4.11. *The closed cone of effective divisors on X is*

$$\overline{\operatorname{Eff}}^1 X = \mathbb{R}_+\{H - E_1 - E_2, E_1, E_2\}.$$

Proof. First note that the two exceptional divisors are effective by definition and that $H - E_1 - E_2$ is linearly equivalent to the intersection of the strict transform of the unique hyperplane containing the two lines in $\mathbb{P}^4$ with X. This shows that the cone in the statement is contained in the effective cone. Conversely suppose $aH + bE_1 + cE_2$ lies in the closed cone of effective divisors; then it must have nonnegative intersection with all nef curve classes in X. By Lemma 2.16 the intersection product of two nef divisor classes is a nef curve class, so in particular H^2, $(H - E_1)^2$ and $(H - E_2)^2$ are classes of nef curves. Intersecting these with our divisor gives the three inequalities

$$\begin{aligned} 5a &\geq 0 \\ 3a + 3b &\geq 0 \\ 3a + 3c &\geq 0 \end{aligned}$$

which cut out the specified cone. ☐

Lemma 4.12. *The closed cone of movable divisors on X is*

$$\overline{\operatorname{Mov}}^1 X = \mathbb{R}_+\{H, H - E_1, H - E_2, 5H - 3E_1 - 3E_2\}.$$

Proof. We first show that the closed movable cone is contained in the specified cone, so let $aH + bE_1 + cE_2$ be a movable divisor. If this divisor is negative on the class C_1 of the fibre in X over a general point in l_1, then any linearly equivalent divisor must contain every curve arising as such a fibre. These curves swipe out a divisor, contradicting that our divisor is movable. Hence $aH + bE_1 + cE_2$ must be nonnegative on C_1, and the same is true for the fibre C_2 over a general point in l_2. Hence by the intersection numbers in Lemma 4.7 we get the two equalities

$$\begin{aligned} b &\geq 0 \\ c &\geq 0. \end{aligned}$$

The strict transform S of the hyperplane containing the two lines to the blowup is a divisor of class $H - E_1 - E_2$. By functoriality of blowups we may identify S with the blowup of this hyperplane in the two lines. Now restricting the divisor $H - E_1$ to S produces the class of all strict transforms of planes through l_1 to S. In particular, the divisor $H - E_1$ traces out all of S. Restricting the above to X, we see that the curve class $(H-E_1-E_2)(H-E_1)$ on X swipes out the entire divisor $S \cap X$. Thus if $aH+bE_1+cE_2$ is negative on the curve class $(H-E_1-E_2)(H-E_1)$ then every divisor linearly equivalent to it must contain the entire divisor S, again contradicting movability. The same is true if we swap the roles of E_1 and E_2, so we get two more conditions on our movable divisor:

$$\begin{aligned} 3a + 3b + 2c &\geq 0 \\ 3a + 2b + 3c &\geq 0. \end{aligned}$$

The four displayed inequalities so far cut out the specified movable cone.

It remains to check that the four generators for the specified cone indeed belong to the closed movable cone. H, $H - E_1$ and $H - E_2$ are movable since they are globally generated. To see that $D = 5H - 3E_1 - 3E_2$ is movable requires a bit more work. Rewriting it as $2(H - E_1) + 2(H - E_2) + (H - E_1 - E_2)$ we see that its base locus is contained in the intersection of S with X, so it suffices to find some section which does does not vanish identically on this intersection; equivalently to show that the left map in the following short exact sequence is not surjective on global sections.

$$0 \longrightarrow \mathcal{O}_X(D-S) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_{X\cap S}(D) \longrightarrow 0$$

The sheaf on the left is the line bundle associated to the divisor $4H - 2E_1 - 2E_2 = 2(H - E_1) + 2(H - E_2)$. This line bundle is nef and big since it corresponds to the map $X \to \mathbb{P}^2_u \times \mathbb{P}^2_v$ postcomposed with the Segre embedding followed by the second Veronese embedding. Hence by the Kawamata-Viehweg vanishing theorem 2.26 and that the canonical divisor on X is trivial, the sheaf on the left has no cohomology in nonzero degrees. Thus it suffices to check that the sheaf on the right admits a global section. We show this by considering the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}_S(D-X) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_{X\cap S}(D) \longrightarrow 0$$

It suffices to show that H^1 of the sheaf on the left is one-dimensional and that both H^0 and H^1 of the middle sheaf vanish. These sheaves can be identified with the line bundles $\mathcal{O}_{\mathrm{Bl}_{l_1\cup l_2}\mathbb{P}^3}(-E_1-E_2)$ and $\mathcal{O}_{\mathrm{Bl}_{l_1\cup l_2}\mathbb{P}^3}(5H-3E_1-3E_2)$ on the blowup of $\mathbb{P}^3$ in two disjoint lines (notice that we are now reusing E_1 and E_2 as notation for the exceptional divisors in the blowup of $\mathbb{P}^3$). To compute cohomology of the sheaf on the left, use the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{Bl}_{l_1\cup l_2}\mathbb{P}^3}(-E_1-E_2) \longrightarrow \mathcal{O}_{\mathrm{Bl}_{l_1\cup l_2}\mathbb{P}^3} \longrightarrow \mathcal{O}_{E_1\cup E_2} \longrightarrow 0.$$

The cohomology of the sheaf in the middle is isomorphic to the cohomology of $\mathbb{P}^3$ by derived pushforwards and the Leray spectral sequence as in the proof of Lemma 4.4. The sheaf on the left is the structure sheaf on two disjoint copies of $\mathbb{P}^1 \times \mathbb{P}^1$. The map on the right in the sequence is clearly injective on global sections, so we get in particular that $H^1(S, \mathcal{O}_S(D-X)) = 1$. Finally we compute the cohomology of $\mathcal{O}_S(D)$ by derived pushforwards to $\mathbb{P}^3$, again using that the higher right derived pushforwards vanish (this holds by the Theorem on formal functions essentially because the line bundles on the fibres are $\mathcal{O}_{\mathbb{P}^1}(3)$'s and the degenerate case of Leray spectral sequence. Thus the cohomology of $\mathcal{O}_S(D)$ is isomorphic to that of $\mathcal{I}^3_{l_1\cup l_2}(5)$ on $\mathbb{P}^3$. This sheaf has zero cohomology in all degrees, as can be seen from the ideal sheaf sequence for the subscheme of $\mathbb{P}^3$ cut out by $(x_0, x_1)^3(x_3, x_4)^3$, thus completing the proof. $\square$

Corollary 4.13. *X satisfies the Morrison-Kawamata cone conjecture (Conjecture 2.22).*

Proof. This is a simpler version of the proof of Corollary 3.11. The automorphism group must act on the minimal integral ray generators for the nef cone, and this action determines the action on $N^1(X)_{\mathbb{R}}$. We quotient out by the kernel of the action and separately study the six possible subgroups of $\mathrm{Sym}\{H, H - E_1, H - E_2\} \cong S_3$ that may arise. It is clear that all of these admit rational polyhedral fundamental domains, thus the Morrison-Kawamata cone conjecture for the nef cone holds. Similarly the birational automorphism group acts on the minimal integral ray generators of the movable cone, hence maps into $\mathrm{Sym}\{H, H - E_1, 5H - 3E_1 - 3E_2, H - E_2\} \cong S_4$. Moreover the action must send adjacent generators to adjacent generators, hence the image is contained in $D_8 \subset S_4$. Again considering all ten possible subgroups individually, we easily spot the rational polyhedral fundamental domains. $\square$

Remark 4.14. The set-theoretic difference between the movable cone and nef cone should correspond to a small $\mathbb{Q}$-factorial modification of X. We give an indication of how this can be found. X is cut out from $\mathbb{P}^4 \times \mathbb{P}^2 \times \mathbb{P}^2$ by an equation of the form

$$x_0 f_0 + \cdots + x_4 f_4 = 0$$

where the f_i are bihomogeneous of bidegree $(2, 2)$ in coordinates (u, v) together with the maximal minors of the matrices

$$\begin{pmatrix} u_0 & u_1 & u_2 \\ x_0 & x_1 & x_2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} v_2 & v_3 & v_4 \\ x_2 & x_3 & x_4 \end{pmatrix}.$$

Multiply the first equation by $u_2 v_2$, rewrite using minors and divide by x_2^2:

$$\frac{u_0 v_2 f_0}{x_2} + \frac{u_1 v_2 f_1}{x_2} + \frac{u_2 v_2 f_2}{x_2} + \frac{u_2 v_3 f_3}{x_2} + \frac{u_2 v_4 f_4}{x_2} = 0.$$

Here we notice that the first three terms are all regular as long as one of x_2, x_3, x_4 are nonzero (that is, away from E_2) since we can rewrite $\frac{v_2}{x_2} = \frac{v_3}{x_3} = \frac{v_4}{x_4}$. Similarly the last three terms are all regular as long as one of x_0, x_1, x_2 are nonzero (that is, away from E_1). Notice especially that the middle term is regular over all of X. We therefore have two distinct ways of rearranging this equation

$$\begin{aligned} s &:= \frac{u_0 v_2 f_0}{x_2} + \frac{u_1 v_2 f_1}{x_2} + \frac{u_2 v_2 f_2}{x_2} = -\Big(\frac{u_2 v_3 f_3}{x_2} + \frac{u_2 v_4 f_4}{x_2}\Big) \\ t &:= \frac{u_0 v_2 f_0}{x_2} + \frac{u_1 v_2 f_1}{x_2} = -\Big(\frac{u_2 v_2 f_2}{x_2} + \frac{u_2 v_3 f_3}{x_2} + \frac{u_2 v_4 f_4}{x_2}\Big) \end{aligned}$$

such that at every point, one of the two sides is well-defined. Thus gluing these gives us two distinct sections s, t of the sheaf $\mathcal{O}_X(1, 2, 2) = \mathcal{O}_X(5H - 3E_1 - 3E_2)$ which together determine a rational map to $\mathbb{P}^1$. We predict that the map $X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$ given by

$$(x_0 : \cdots : x_4) \times (u_0 : u_1 : u_2) \times (v_2 : v_3 : v_4) \mapsto (u_0 : u_1 : u_2) \times (v_2 : v_3 : v_4) \times (s : t)$$

is the desired modification.

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