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# Injectivity of Mean Value Mappings Between Convex Polygons

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# Injectivity of Mean Value Mappings Between Convex Polygons

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# Abstract

Mean value mappings is a type of generalized barycentric mappings that are used in for instance computer graphics and geometric modelling. In this thesis we investigate when mean value mappings between convex polygons in the plane are injective. We derive a new proof for why all barycentric mappings, and thereby mean value mappings, are injective between triangles. In addition, we present a new example of why mean value mappings between convex pentagons are not necessarily injective. Our main result is that we make some progress towards a possible proof that shows that mean value mappings between convex quadrilaterals are injective. This proof relies on some assumptions that will need to be proven before this proof is completely analytical. We test the statement numerically, and all tests indicate that mean value mappings are injective between convex quadrilaterals. We are, however, able to prove analytically that mean value mappings between convex quadrilaterals are injective in the intersection between the diagonals in a quadrilateral.

# Acknowledgements

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## Chapter 1

# Introduction

Barycentric coordinates have for a long time been used to represent points inside a triangle (see [1], [2]). However, in recent years there has been growing interest in the construction and applications of various generalizations of barycentric coordinates, for instance how to use generalized barycentric coordinates to deform shapes when modelling and processing geometry. Most of these methods focus on two kinds of generalized barycentric coordinates; Wachspress coordinates and mean value coordinates. Wachspress coordinates are relatively simple functions, in the sense that the coordinates only consist of rational functions. These coordinates are, however, limited to convex polygons. That is, for nonconvex polygons will the denominator in the rational expressions of the Wachspress coordinates become zero at certain points in the polygon. Mean value coordinates, on the other hand, are possible to generalize to nonconvex polygons, and in later years there have been discovered a lot of interesting properties of mean value coordinates that can be used in for instance computer graphics and geometric modelling.

In [3], Warren suggested that barycentric coordinates could be used to deform curves. The coordinates can be used to define a barycentric mapping from one convex polygon to another, and such a mapping will then map, or deform, a curve embedded in the first polygon into a new one, with the vertices of the polygon acting as control points, with an effect similar to those of Bézier and spline curves and surfaces [4]. It was then later shown in [5] by Hormann and Floater, that the curve deformation method could be extended to arbitrary polygons for a generalization of mean value coordinates. In addition, they showed that these coordinates could be applied to for instance image warping, to improve the idea of Phong shading and transfinite interpolation.

In curve deformation we want to guarantee that when we deform a curve we do not introduce any new self-intersections. In this case, this is equivalent to show that the barycentric mappings we use are injective. In 2008, Floater and Kosinka proved the injectivity of Wachspress mappings between convex polygons in the plane [6]. They also proved that mean value mappings between convex polygons with five or more vertices are not necessarily injective in the plane, but it still remains to find out if mean value mappings between convex quadrilaterals are injective.

The goal in this thesis is to show precisely that mean value mappings between convex quadrilaterals are injective. Floater and Kosinka tested this statement numerically in [6], and in this article they conjectured that mean value mapping between convex quadrilaterals are injective, but it still remains to find a proof of this. During the writing of this thesis there has been an attempt to prove that mean value mappings between convex quadrilaterals are injective (see [7]), but the argument does not appear to be complete.

# Outline of Thesis

**Chapter 2** gives an introduction to mean value coordinates and mean value mappings. In addition, we will derive necessary and sufficient conditions for injectivity between barycentric mappings.

**Chapter 3** presents a proof of injectivity of barycentric mappings between triangles, presents a combination of numerical and analytical proof of the injectivity of mean value mappings between convex quadrilaterals and a counterexample for mean value mappings between convex pentagons.

**Chapter 4** gives a discussion and presents potential improvements of the results presented in chapter 3. In this chapter we will also highlight further work.

**Appendix A** presents the code and test runs we use in the analysis of the injectivity of mean value mappings between convex quadrilaterals.

**Appendix B** presents the analysis and test runs of the cases we left out in chapter 3.

Chapter 2

# Mathematical background

This chapter is structured as follows: Section 2.1 gives a brief introduction to barycentric and generalized barycentric coordinates. Here we will give a presentation of mean value coordinates, and also present some useful properties that we will need later. In section 2.2 we define what a barycentric mapping is, and derive necessary and sufficient conditions for a barycentric mapping to be injective.

In this thesis we will use bold letters for real vectors and coordinates $\mathbf{x} \in \mathbb{R}^2$, and italic letters for scalar values $a \in \mathbb{R}$. In addition, we will let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^2$.

## 2.1 Generalized barycentric coordinates

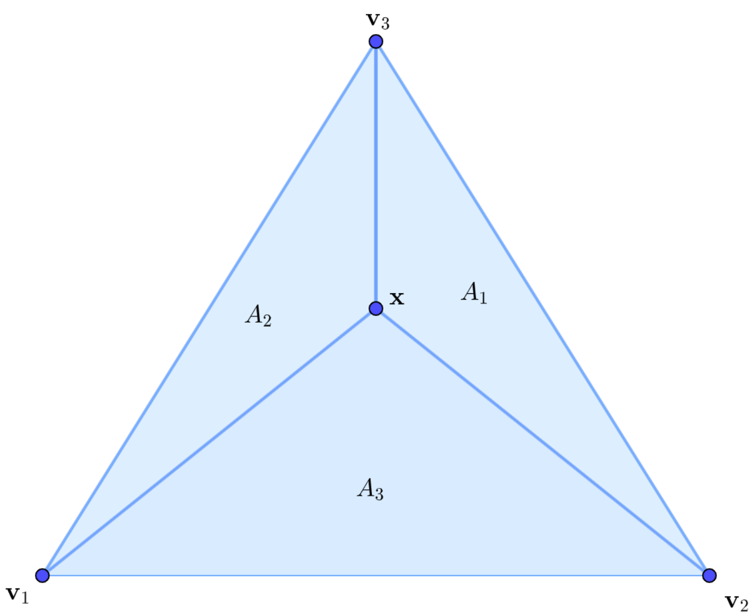


Figure 2.1: Barycentric coordinates in a triangle

Barycentric coordinates are based on how to represent any point $\mathbf{x}$ in a triangle $T \subset \mathbb{R}^2$ with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Any $\mathbf{x} \in T$ partitions $T$ into three sub-triangles $A_1$, $A_2$

and $A_3$ (see figure 2.1). The three barycentric coordinates of $\mathbf{x}$ in $T$ are the ratios of the areas of the sub-triangles $A_i$ and the area $A$ of $T$. So,

$$\mathbf{x} = \phi_1 \mathbf{v}_1 + \phi_2 \mathbf{v}_2 + \phi_3 \mathbf{v}_3,$$

where

$$\phi_i = \frac{A_i}{A}, \text{ for } i = 1, 2, 3.$$

If we want to generalize this representation so we can represent any point $\mathbf{x}$ in any polygon $P$ with more than three vertices, in a similar way, we need to use generalized barycentric coordinates. Let $P \subset \mathbb{R}^2$ be a polygon with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$, where $n \geq 3$, ordered anticlockwise. We view $P$ as an open set of $\mathbb{R}^2$ and denote the boundary by $\partial P$ and the closure by $\bar{P}$. Then the set of functions $\phi_i \colon P \to \mathbb{R}$, $i = 1, \ldots, n$ will be called a set of generalized barycentric coordinates if, for all $\mathbf{x} \in \bar{P}$ and $i = 1, \ldots, n$,

$$\phi_i(\mathbf{x}) \geq 0 \tag{2.1}$$

$$\sum_{i=1}^{n} \phi_i(\mathbf{x}) = 1 \tag{2.2}$$

$$\sum_{i=1}^{n} \phi_i(\mathbf{x}) \mathbf{v}_i = \mathbf{x}. \tag{2.3}$$

In this thesis we will investigate mean value coordinates, so we will now give a presentation of these coordinates.

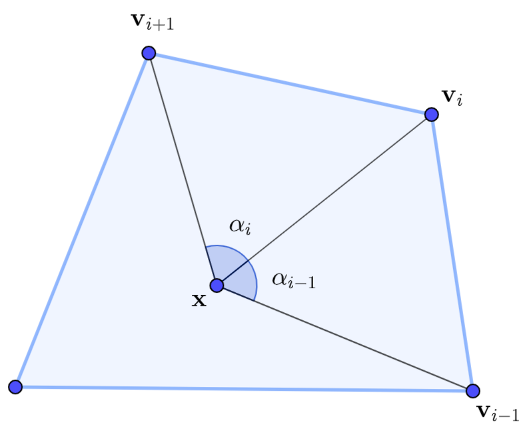


Figure 2.2: Illustration of mean value coordinates

### 2.1.1 Mean value coordinates

In [8] it was derived a set of generalized barycentric coordinates from an application of the mean value theorem for harmonic functions, namely the mean value coordinates. The original motivation for these coordinates was for parameterizing triangular meshes ([9], [10], [11]). It was later observed in [5] that these coordinates also were well-defined, though not necessarily positive, for arbitrary polygons, as opposed to for instance Wachspress coordinates. We will now define a formula for the mean value coordinates (see e.g. [12]). Let $P$ be a strictly convex polygon with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$, ordered anticlockwise, where $n \geq 3$. By strictly, we mean that no three vertices are collinear. Then the mean value coordinates for $\mathbf{x} \in P$ are defined by

$$\phi_i = \phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^n w_j(\mathbf{x})}, \tag{2.4}$$

where

$$w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{\|\mathbf{v}_i - \mathbf{x}\|}. \tag{2.5}$$

Here, $\alpha_i = \alpha_i(\mathbf{x})$ is the angle at $\mathbf{x}$ in the triangle with vertices $\mathbf{x}, \mathbf{v}_i$ and $\mathbf{v}_{i+1}$, as shown in figure 2.2. We note that $0 < \alpha_i < \pi$ for $\mathbf{x} \in P$, since $P$ is convex.

The mean value coordinates $\phi_i$ extend continuously to the boundary $\partial P$ (see e.g. [6]). So, if $\mathbf{x} \in \partial P$ we replace equation (2.5) with

$$w_i(\mathbf{x}) = (r_{i-1}r_{i+1} - \mathbf{d}_{i-1} \cdot \mathbf{d}_{i+1})^{1/2} \prod_{j \neq i-1,i} (r_j r_{j+1} + \mathbf{d}_j \cdot \mathbf{d}_{j+1})^{1/2}, \tag{2.6}$$

where $\mathbf{d}_i = \mathbf{v}_i - \mathbf{x}$ and $r_i = \|\mathbf{v}_i - \mathbf{x}\|$.

Later on, we will see that we need an expression for the gradient $\nabla w_i$ of the weights of the mean value coordinates $\phi_i$ to check if mean value mappings between convex quadrilaterals are injective. We will therefore state an expression for $\nabla w_i$, but before doing this we will need to define some notations. Let $\mathbf{e}_i = \frac{\mathbf{v}_i - \mathbf{x}}{\|\mathbf{v}_i - \mathbf{x}\|}$, $r_i = \|\mathbf{v}_i - \mathbf{x}\|$ and $t_i = \tan(\alpha_i/2)$. Further, define

$$\mathbf{c}_i = \frac{\mathbf{e}_i}{r_i} - \frac{\mathbf{e}_{i+1}}{r_{i+1}},$$

and for a vector $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$, let $\mathbf{a}^\perp := (-a_2, a_1)$. We now let $\mathbf{R}_i := \frac{\nabla w_i}{w_i}$. Then it was shown by Floater [4] that

$$\mathbf{R}_i = \left(\frac{t_{i-1}}{t_{i-1} + t_i}\right) \frac{\mathbf{c}_{i-1}^\perp}{\sin \alpha_{i-1}} + \left(\frac{t_i}{t_{i-1} + t_i}\right) \frac{\mathbf{c}_i^\perp}{\sin \alpha_i} + \frac{\mathbf{e}_i}{r_i}, \tag{2.7}$$

so

$$\nabla w_i = \mathbf{R}_i w_i.$$

For proof see [4].

### 2.1.2 Useful property of generalized barycentric coordinates

Let $P \subset \mathbb{R}^2$ be a strictly convex polygon with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$, where $n \geq 3$. Furthermore, let $\phi_1, \ldots, \phi_n$ be a set of generalized barycentric coordinates and let $\mathbf{x}_{bound}$ be a point on the boundary of $P$, $\partial P$. Then there exists an $l \in \{1, 2, \ldots, n\}$ such that

$$\mathbf{x}_{bound} = (1 - \mu)\mathbf{v}_l + \mu \mathbf{v}_{l+1}, \tag{2.8}$$

for some $\mu \in [0, 1]$, with indexes treated cyclically [6]. Since $\phi_1, \ldots, \phi_n$ are barycentric coordinates they have to satisfy (2.1) - (2.3) which implies that

$$\phi_l(\mathbf{x}_{bound}) = 1 - \mu, \ \phi_{l+1}(\mathbf{x}_{bound}) = \mu, \text{ and } \phi_i(\mathbf{x}_{bound}) = 0 \text{ for } i \neq l, l+1. \tag{2.9}$$

We will need this property later in section 3.2 when we are checking if mean value mappings between convex quadrilaterals are injective.

## 2.2 Barycentric mappings between convex polygons

Let $P, Q \subset \mathbb{R}^2$ be strictly convex polygons with vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ and $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$, respectively, ordered anticlockwise, with $n \geq 3$. We view both polygons as open sets of $\mathbb{R}^2$ and denote their boundaries by $\partial P$ and $\partial Q$ and their closures by $\bar{P}$ and $\bar{Q}$. We now define a barycentric mapping as a smooth mapping $\mathbf{f} \colon \bar{P} \to \bar{Q}$ by

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x}) \mathbf{q}_i, \tag{2.10}$$

where $\phi_1, \ldots, \phi_n$ are a set of barycentric coordinates satisfying (2.1)-(2.3).

We observe that since $\phi_1, \ldots, \phi_n$ satisfy (2.1)-(2.3), $\mathbf{f}(\mathbf{x})$ is a convex combination of the points $\mathbf{q}_i$, so $\mathbf{f}(\bar{P}) \subset \bar{Q}$. Furthermore, by (2.9), if $\mathbf{x}$ is the boundary point (2.8) then

$$\mathbf{f}(\mathbf{x}) = (1 - \mu)\mathbf{q}_l + \mu \mathbf{q}_{l+1}, \tag{2.11}$$

for some $\mu \in [0, 1]$.

Thus, $\mathbf{f}$ maps $\partial P$ to $\partial Q$ in a piecewise linear fashion, mapping vertices and edges of $\partial P$ to corresponding vertices and edges of $\partial Q$.

### 2.2.1 Conditions for injectivty

We will now derive conditions for injectivty of generalized barycentric mappings; both sufficient conditions and necessary conditions. Using basic results of real analysis, see for instance [13], it can be shown that a sufficient condition for the injectivity of $\mathbf{f}$ in (2.10) is that its Jacobian $J(\mathbf{f})$ is strictly positive in $\bar{P}$. Therefore, we will now try to find an expression for $J(\mathbf{f})$ in terms of signed areas of triangles formed by vertices of $Q$.

Lemma 2.2.1 (rendered from [6]). *Let* $\mathbf{f}(\mathbf{x}) = (g(\mathbf{x}), h(\mathbf{x}))$*, and* $\partial_r f(\mathbf{x}) := \frac{\partial f}{\partial x^r}$*,* $r = 1, 2$*. Then, for any set of differentiable functions* $\phi_1, \dots, \phi_n$ *satisfying (2.2)*

$$J(\mathbf{f})(\mathbf{x}) = 2 \sum_{1 \le i < j < k \le n} \mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k), \tag{2.12}$$

*where*

$$J(\mathbf{f}) = \begin{vmatrix} \partial_1 g & \partial_1 h \\ \partial_2 g & \partial_2 h \end{vmatrix},$$

*and*

$$\mathcal{D}(a, b, c) = \begin{vmatrix} a & b & c \\ \partial_1 a & \partial_1 b & \partial_1 c \\ \partial_2 a & \partial_2 b & \partial_2 c \end{vmatrix}. \tag{2.13}$$

*Proof.* Differentiating (2.2) and (2.10) gives the matrix identity

$$\begin{pmatrix} 1 & g & h \\ 0 & \partial_1 g & \partial_1 h \\ 0 & \partial_2 g & \partial_2 h \end{pmatrix} = \begin{pmatrix} \phi_1 & \cdots & \phi_n \\ \partial_1 \phi_1 & \cdots & \partial_1 \phi_n \\ \partial_2 \phi_1 & \cdots & \partial_2 \phi_n \end{pmatrix} \begin{pmatrix} 1 & q_1^1 & q_1^2 \\ \vdots & \vdots & \vdots \\ 1 & q_n^1 & q_n^2 \end{pmatrix}.$$

Applying the Cauchy-Binet theorem (see [14], formula (1.23)) to this equation and using the fact that the determinant of the matrix on the left equals $J(\mathbf{f})$, the result follows. ■

Since $Q$ is strictly convex we have that $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) > 0$, whenever $1 \le i < j < k \le n$, since the triangle with vertices $\mathbf{q}_i, \mathbf{q}_j$ and $\mathbf{q}_k$ is positively oriented. Thus, Lemma 2.2.1 leads to a sufficient condition for injectivity, which we will now state in the theorem below.

Theorem 2.2.2 (rendered from [6]). *If* $\phi_1, \dots, \phi_n$ *are differentiable barycentric coordinates such that for all* $\mathbf{x} \in \bar{P}$*,* $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) \ge 0$ *for all* $i, j, k$ *satisfying* $1 \le i < j < k \le n$ *and* $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) > 0$ *for some* $i, j, k$ *satisfying* $1 \le i < j < k \le n$*, then* $\mathbf{f}$ *is injective.*

In [6] they also derived a set of necessary conditions for injectivity, so we also state this result in a theorem below.

Theorem 2.2.3 (rendered from [6]). *If* $\phi_1, \dots, \phi_n$ *are differentiable barycentric coordinates and* $\mathbf{f}$ *is injective, then for* $r, s, t$ *satisfying* $1 \le r < s < t \le n$*,*

$$\sum_{r \le i < s \le j < t \le k < n+r} \mathcal{D}(\phi_i, \phi_j, \phi_k) \ge 0 \quad \text{in } P. \tag{2.14}$$

*Proof.* (rendering from [6]) If $\mathbf{f}$ is injective then $J(\mathbf{f}) \ge 0$ in $P$. Let $Q$ be the polygon with vertices

$$\mathbf{q}_1 = \cdots = \mathbf{q}_{r-1} = (0,0), \qquad \mathbf{q}_r = \cdots = \mathbf{q}_{s-1} = (1,0),$$
$$\mathbf{q}_s = \cdots = \mathbf{q}_{t-1} = (0,1), \qquad \mathbf{q}_t = \cdots = \mathbf{q}_n = (0,0).$$

Then $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) = \frac{1}{2}$ if $1 \le i < r \le j < s \le k < t$ or $r \le i < s \le j < t \le k \le n$ and zero otherwise. Substituting these values into (2.12) gives

$$\begin{aligned} J(\mathbf{f})(\mathbf{x}) = 2 &\sum_{1 \le i < j < k \le n} \mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) \\ = 2( &\sum_{1 \le i < r \le j < s \le k < t} \mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) + \\ &\sum_{r \le i < s \le j < t \le k \le n} \mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k)). \end{aligned}$$

Since $A(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) = \frac{1}{2}$ when $1 \leq i < r \leq j < s \leq k < t$ or $r \leq i < s \leq j < t \leq k \leq n$ we have that

$$J(\mathbf{f})(\mathbf{x}) = \sum_{1 \leq i < r \leq j < s \leq k < t} \mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) + \sum_{r \leq i < s \leq j < t \leq k \leq n} \mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) \geq 0. \quad (2.15)$$

If we now replace $i, j, k$ by $k, i, j$ in the first sum of (2.15) and use the fact that

$$\mathcal{D}(\phi_k, \phi_i, \phi_j) = \mathcal{D}(\phi_i, \phi_j, \phi_k),$$

the first sum can be written as

$$\sum_{1 \leq i < r \leq j < s \leq k < t} \mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) = \sum_{\substack{r \leq i < s \leq j < t \\ n+1 \leq k < n+r}} \mathcal{D}(\phi_i, \phi_j, \phi_{k-n})(\mathbf{x}).$$

Replacing $\phi_{k-n}$ by $\phi_k$, and combining this second expression with the second sum in (2.15) gives (2.14). ■

We have now derived sets of sufficient conditions and necessary conditions for injectivity of barycentric mappings, based on the signs of the determinants $\mathcal{D}(\phi_i, \phi_j, \phi_k)$. It is worth to notice that we have not found a set of conditions that are both sufficient and necessary for some general $n$. However, Floater and Kosinka showed in [6] that the sufficient conditions of Theorem 2.2.2 were good enough to show that Wachspress mappings are injective, while the necessary conditions of Theorem 2.2.3 were sufficient to find a counterexample for mean value mappings with $n \geq 5$. We will use the sufficient conditions of Theorem 2.2.2 when we investigate if mean value mappings are injective between convex quadrilaterals in section 3.2.

# Chapter 3

# Injectivity of mean value mappings

We recall that our goal in this thesis was to investigate when mean value mappings are injective, and we will present the results that we found in this chapter. This chapter is structured as follows: Section 3.1 proves injectivity of all barycentric mappings between triangles and thereby also mean value mappings between triangles. This proof has already been done in [6], but we will present an alternative proof in this section. In section 3.2 we will investigate the injectivity of mean value mappings between convex quadrilaterals. We are able to prove analytically that mean value mappings between convex quadrilaterals are injective in the intersection between the diagonals, but for the rest of the quadrilateral we will rely on some numerical analysis. This is the main result in this thesis, and this analysis will support the numerical result found by Floater and Kosinka in [6]. Last, in section 3.3 we will present an example that proves that mean value mappings between strictly convex pentagons are not necessarily injective.

## 3.1 Injectivity of barycentric mappings between triangles

Let $T$ be a triangle with vertices $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, ordered anticlockwise, and assume that $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are non-collinear. We know from chapter 2.1 that barycentric coordinates in $\mathbb{R}^2$ have to satisfy equation (2.2) and (2.3) (note: in $\mathbb{R}^2$ (2.3) is two equations). Then, by Cramer's rule [15], we have that the barycentric coordinates $\phi_1, \phi_2$ and $\phi_3$ for $T$ are uniquely determined by

$$\phi_1(\mathbf{x}) = \frac{A(\mathbf{x}, \mathbf{v}_2, \mathbf{v}_3)}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \qquad \phi_2(\mathbf{x}) = \frac{A(\mathbf{v}_1, \mathbf{x}, \mathbf{v}_3)}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \qquad \phi_3(\mathbf{x}) = \frac{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{x})}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \tag{3.1}$$

where $A(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ denotes the signed area of the triangle with vertices $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$. That is,

$$A(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ t_1^1 & t_2^1 & t_3^1 \\ t_1^2 & t_2^2 & t_3^2 \end{vmatrix}, \tag{3.2}$$

where $\mathbf{t}_i = (t_i^1, t_i^2)$, for $i = 1, 2, 3$.

We will now show that a barycentric mapping between triangles is injective.

Theorem 3.1.1. *If $\phi_1, \phi_2, \phi_3$ are the barycentric coordinates (3.1) then the barycentric mapping* $\mathbf{f}$ *between two strictly convex triangles are injective.*

*Proof.* In chapter 2.2.1 we derived Theorem 2.2.2, which tells us that it is sufficient to show that

$$\mathcal{D}(\phi_1, \phi_2, \phi_3)(\mathbf{x}) > 0,$$

for all $\mathbf{x} \in \bar{T}$. Since $\frac{1}{A(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} > 0$ is a common factor in $\phi_1, \phi_2$ and $\phi_3$, it is sufficient to show that

$$\mathcal{D}(A_1, A_2, A_3)(\mathbf{x}) > 0,$$

where $A_1 = A(\mathbf{x}, \mathbf{v}_2, \mathbf{v}_3)$, $A_2 = A(\mathbf{v}_1, \mathbf{x}, \mathbf{v}_3)$ and $A_3 = A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{x})$. To this end, we observe that

$$\mathcal{D}(A_1, A_2, A_3)(\mathbf{x}) = A_1 \boldsymbol{\nabla} A_2 \times \boldsymbol{\nabla} A_3 + A_2 \boldsymbol{\nabla} A_3 \times \boldsymbol{\nabla} A_1 + A_3 \boldsymbol{\nabla} A_1 \times \boldsymbol{\nabla} A_2, \tag{3.3}$$

where $\boldsymbol{\nabla} w := (\partial_1 w, \partial_2 w)$ and $\mathbf{u} \times \mathbf{v} := u^1 v^2 - u^2 v^1$. By differentiating (3.2) with respect to $x^1$ and $x^2$ we get that

$$\boldsymbol{\nabla} A_1 = \frac{1}{2}\begin{pmatrix} \mathbf{v}_2^2 - \mathbf{v}_3^2 \\ \mathbf{v}_3^1 - \mathbf{v}_2^1 \end{pmatrix} = \frac{1}{2}\mathrm{rot}(\mathbf{v}_3 - \mathbf{v}_2), \tag{3.4}$$

$$\boldsymbol{\nabla} A_2 = \frac{1}{2}\begin{pmatrix} \mathbf{v}_3^2 - \mathbf{v}_1^2 \\ \mathbf{v}_1^1 - \mathbf{v}_3^1 \end{pmatrix} = \frac{1}{2}\mathrm{rot}(\mathbf{v}_1 - \mathbf{v}_3), \tag{3.5}$$

$$\boldsymbol{\nabla} A_3 = \frac{1}{2}\begin{pmatrix} \mathbf{v}_1^2 - \mathbf{v}_2^2 \\ \mathbf{v}_2^1 - \mathbf{v}_1^1 \end{pmatrix} = \frac{1}{2}\mathrm{rot}(\mathbf{v}_2 - \mathbf{v}_1), \tag{3.6}$$

where $\mathrm{rot}(v^1, v^2) := (-v^2, v^1)$. Substituting (3.4)-(3.6) into (3.3) gives

$$\begin{aligned}\mathcal{D}(A_1, A_2, A_3)(\mathbf{x}) = \frac{1}{2}\Big(& A_1 \frac{1}{2}\mathrm{rot}(\mathbf{v}_1 - \mathbf{v}_3) \times \mathrm{rot}(\mathbf{v}_2 - \mathbf{v}_1) \\ &+ A_2 \frac{1}{2}\mathrm{rot}(\mathbf{v}_2 - \mathbf{v}_1) \times \mathrm{rot}(\mathbf{v}_3 - \mathbf{v}_2) \\ &+ A_3 \frac{1}{2}\mathrm{rot}(\mathbf{v}_3 - \mathbf{v}_2) \times \mathrm{rot}(\mathbf{v}_1 - \mathbf{v}_3)\Big).\end{aligned}$$

By using the fact that $\mathrm{rot}(\mathbf{a}) \times \mathrm{rot}(\mathbf{b}) = \mathbf{a} \times \mathbf{b}$, and $(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{d}) = (\mathbf{d} - \mathbf{c}) \times (\mathbf{a} - \mathbf{b})$, we can rewrite the expression above as

$$\begin{aligned}\mathcal{D}(A_1, A_2, A_3)(\mathbf{x}) = \frac{1}{2}\Big(& A_1 \frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2) \times (\mathbf{v}_1 - \mathbf{v}_3) + A_2 \frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_3) \times (\mathbf{v}_2 - \mathbf{v}_1) \\ &+ A_3 \frac{1}{2}(\mathbf{v}_3 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_2)\Big).\end{aligned} \tag{3.7}$$

We note that

$$(\mathbf{v}_1 - \mathbf{v}_2) \times (\mathbf{v}_1 - \mathbf{v}_3) = (\mathbf{v}_2 - \mathbf{v}_3) \times (\mathbf{v}_2 - \mathbf{v}_1) = (\mathbf{v}_3 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_2) = 2A, \tag{3.8}$$

where $A$ is the signed area of the triangle spanned by the vertices $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. Since $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are non-collinear and ordered anticlockwise, we observe that $A > 0$.

Substituting (3.8) into (3.7) gives

$$\mathcal{D}(A_1, A_2, A_3)(\mathbf{x}) = \frac{1}{2}(A_1 A + A_2 A + A_3 A) \tag{3.9}$$

$$= \frac{1}{2}A(A_1 + A_2 + A_3) \tag{3.10}$$

$$= \frac{1}{2}A^2 > 0. \tag{3.11}$$

■

Since the barycentric coordinates $\phi_1, \phi_2$ and $\phi_3$ are uniquely determined by (3.1), Theorem 3.1.1 implies that mean value mappings between triangles are also injective.

## 3.2 Injectivity of mean value mappings between convex quadrilaterals

We will now try to prove that mean value mappings between convex quadrilaterals in the plane are injective. This proof is inspired by the proof of injectivity of Wachspress mappings between convex polygons in [6], in the way that we will use the same theorem as Floater and Kosinka used in their article, when we try to prove our statement.

Let $Q \subset \mathbb{R}^2$ be a strictly convex quadrilateral with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$, ordered anticlockwise. In chapter 2.2.1 we derived Theorem 2.2.2, which tells us that it is sufficient to show that for all $\mathbf{x} \in \bar{Q}$

$$\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) \geq 0, \text{ for all } 1 \leq i < j < k \leq 4,$$

and

$$\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) > 0, \text{ for some } 1 \leq i < j < k \leq 4.$$

We will do this proof in two steps; first we will prove for $\mathbf{x} \in Q$, and then we will prove for $\mathbf{x} \in \partial Q$.

### 3.2.1 Step 1

We recall from chapter 2.1.1 that $\phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^4 w_j(\mathbf{x})}$, where $w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}/2)+\tan(\alpha_i/2)}{\|\mathbf{v}_i - \mathbf{x}\|}$. Since $Q$ is a convex quadrilateral, we now that $\alpha_i \in (0, \pi)$, for $i = 1, \ldots, 4$ and $\mathbf{x} \in Q$. This implies that $\tan(\alpha_i/2) > 0$ for all $\mathbf{x} \in Q$ and $i = 1, \ldots, 4$, which again implies that $\frac{1}{\sum_{j=1}^4 w_j(\mathbf{x})} > 0$. Since $\frac{1}{\sum_{j=1}^4 w_j(\mathbf{x})}$ is a common factor in $\phi_1, \phi_2, \phi_3$ and $\phi_4$, it is enough to prove that for all $\mathbf{x} \in Q$

$$\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) \geq 0, \text{ for all } 1 \leq i < j < k \leq 4,$$

and

$$\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) > 0, \text{ for some } 1 \leq i < j < k \leq 4.$$

We observe that we can write $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x})$ as

$$\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) = w_i \boldsymbol{\nabla} w_j \times \boldsymbol{\nabla} w_k + w_j \boldsymbol{\nabla} w_k \times \boldsymbol{\nabla} w_i + w_k \boldsymbol{\nabla} w_i \times \boldsymbol{\nabla} w_j. \tag{3.12}$$

In chapter 2.1.1 we showed that we could write $\boldsymbol{\nabla} w_i = w_i \mathbf{R}_i$ (see (2.7)). If we substitute this into (3.12) we get that

$$\begin{aligned}\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) &= w_i(w_j\mathbf{R}_j) \times (w_k\mathbf{R}_k) + w_j(w_k\mathbf{R}_k) \times (w_i\mathbf{R}_i) + w_k(w_i\mathbf{R}_i) \times (w_j\mathbf{R}_j) \\ &= w_i w_j w_k(\mathbf{R}_j \times \mathbf{R}_k + \mathbf{R}_k \times \mathbf{R}_i + \mathbf{R}_i \times \mathbf{R}_j). \end{aligned} \tag{3.13}$$

So, we will need to prove that $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) \geq 0$ for all combinations of $i$, $j$, $k$ that satisfies $1 \leq i < j < k \leq 4$, that is, $(i,j,k) = (1,2,3), (1,2,4), (1,3,4)$ and $(2,3,4)$. Note that for $\mathbf{x} \in Q$, we have that $w_1, w_2, w_3, w_4 > 0$, so for all $\mathbf{x} \in Q$ and all combinations of $i,j,k$ that satisfy $1 \leq i < j < k \leq 4$, it is always enough to prove that $\mathbf{R}_j \times \mathbf{R}_k + \mathbf{R}_k \times \mathbf{R}_i + \mathbf{R}_i \times \mathbf{R}_j \geq 0$, and strictly greater than zero for some combination of $i,j,k$. Let us start by proving that $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) \geq 0$ for $i = 1$, $j = 2$ and $k = 3$: We now want to check if

$$\mathbf{R}_2 \times \mathbf{R}_3 + \mathbf{R}_3 \times \mathbf{R}_1 + \mathbf{R}_1 \times \mathbf{R}_2 \geq 0,$$

for all $\mathbf{x} \in Q$. Before we continue, we will introduce some notation:

$$A_{n,m} = \frac{(\mathbf{v}_n - \mathbf{x})}{\|(\mathbf{v}_n - \mathbf{x})\|^2} \times \frac{(\mathbf{v}_m - \mathbf{x})}{\|(\mathbf{v}_m - \mathbf{x})\|^2}, \text{ for } n, m \in \{1,2,3,4\}$$

$$P_{n,m} = \frac{(\mathbf{v}_n - \mathbf{x})}{\|(\mathbf{v}_n - \mathbf{x})\|^2} \cdot \frac{(\mathbf{v}_m - \mathbf{x})}{\|(\mathbf{v}_m - \mathbf{x})\|^2}, \text{ for } n, m \in \{1,2,3,4\}$$

$$t_n = \tan(\alpha_n/2), \text{ for } n = 1,2,3,4$$

$$\mathbf{R}_{i,j,k}(\mathbf{x}) = \mathbf{R}_i \times \mathbf{R}_j + \mathbf{R}_k \times \mathbf{R}_i + \mathbf{R}_j \times \mathbf{R}_k, \text{ for } 1 \leq i < j < k \leq 4.$$

We will start by writing down an expression for the sum $\mathbf{R}_{1,2,3}(\mathbf{x})$:

$$\begin{aligned}\mathbf{R}_{1,2,3}(\mathbf{x}) = &\frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_4}\Big(A_{2,4} + A_{1,2} + A_{4,1}\Big) \\ &+ \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{2,1} + A_{1,3} + A_{4,2} + A_{3,4}\Big) \\ &+ \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{1,2} - P_{2,4}\Big) + \frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_2}\Big(A_{1,2} + A_{3,1} + A_{2,3}\Big) \\ &+ \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1}\Big(P_{2,2} - P_{1,2}\Big) + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin\alpha_1}\Big(P_{1,1} - P_{1,2}\Big) \\ &+ \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin\alpha_2}\Big(P_{1,2} - P_{1,3}\Big) + A_{1,2} \\ &+ \frac{t_2 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{4,3} + A_{3,1} + A_{2,4} + A_{1,2}\Big) \\ &+ \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_2}\Big(A_{1,3} + A_{3,2} + A_{2,1}\Big) + \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_2}\Big(P_{1,3} - P_{1,2}\Big) \\ &+ \frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_3 \sin\alpha_4}\Big(A_{4,1} + A_{3,4} + A_{1,3}\Big) \\ &+ \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_3}\Big(A_{1,4} + A_{4,2} + A_{3,1} + A_{2,3}\Big) \\ &+ \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_3}\Big(P_{4,1} - P_{1,3}\Big) + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{3,4} - P_{1,3}\Big)\end{aligned}$$

$$+\frac{t_1}{t_1+t_4}\cdot\frac{1}{\sin\alpha_1}\Big(P_{1,3}-P_{2,3}\Big)+A_{3,1}+\frac{t_1t_2}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin\alpha_1\sin\alpha_2}\Big(A_{2,3}$$
$$+A_{1,2}+A_{3,1}\Big)+\frac{t_1t_3}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin\alpha_1\sin\alpha_3}\Big(A_{3,2}+A_{2,4}+A_{1,3}+A_{4,1}\Big)$$
$$+\frac{t_1}{t_1+t_2}\cdot\frac{1}{\sin\alpha_1}\Big(P_{2,3}-P_{1,3}\Big)+\frac{t_2t_3}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin\alpha_2\sin\alpha_3}\Big(A_{3,4}+A_{2,3}+A_{4,2}\Big)$$
$$+\frac{t_2}{t_1+t_2}\cdot\frac{1}{\sin\alpha_2}\Big(P_{3,3}-P_{2,3}\Big)+\frac{t_2}{t_2+t_3}\cdot\frac{1}{\sin\alpha_2}\Big(P_{2,2}-P_{2,3}\Big)$$
$$+\frac{t_3}{t_2+t_3}\cdot\frac{1}{\sin\alpha_3}\Big(P_{2,3}-P_{2,4}\Big)+A_{2,3}.$$

We observe that several of the terms in $\mathbf{R}_{1,2,3}$ are ambiguous, i.e. that the terms can be both positive and negative depending on $\mathbf{x}$. To illustrate this, lets look an example where this is the case.

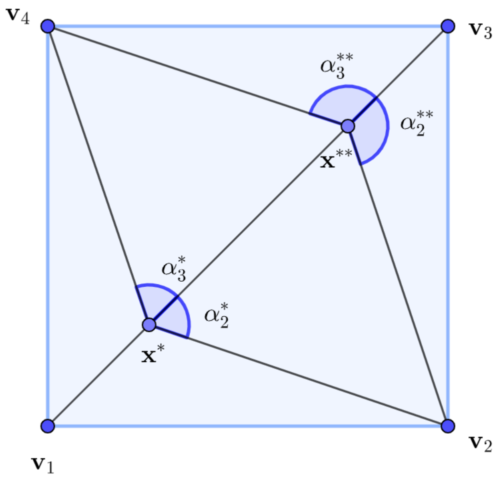


Figure 3.1: Example illustrating that the sign of $A_{2,4}$ depends on the location of x

Example 3.2.1. Let $Q \subset \mathbb{R}^2$ be the unit square, i.e. $\mathbf{v}_1 = (0,0)$, $\mathbf{v}_2 = (1,0)$, $\mathbf{v}_3 = (1,1)$ and $\mathbf{v}_4 = (0,1)$. Furthermore, let $\mathbf{x}^* = (0.25, 0.25)$ and $\mathbf{x}^{**} = (0.75, 0.75)$, and let $\alpha_2^*, \alpha_3^*$ and $\alpha_2^{**}, \alpha_3^{**}$ be the angles that belongs to $\mathbf{x}^*$ and $\mathbf{x}^{**}$ respectively (see figure 3.1). Then we notice that both $\mathbf{x}^*$ and $\mathbf{x}^{**}$ are located at the diagonal between $\mathbf{v}_1$ and $\mathbf{v}_3$. We then observe that

$$A_{2,4}(\mathbf{x}^*) = (\mathbf{v}_2 - \mathbf{x}^*) \times (\mathbf{v}_4 - \mathbf{x}^*) = 0.5,$$

and

$$A_{2,4}(\mathbf{x}^{**}) = (\mathbf{v}_2 - \mathbf{x}^{**}) \times (\mathbf{v}_4 - \mathbf{x}^{**}) = -0.5.$$

So, this is an example where the sign of $A_{2,4}$ depends on the location of x.

We want to avoid the problem illustrated in example 3.1, i.e. we want to make the analysis of $\mathbf{R}_{1,2,3}$ unambiguous. To do this, we will choose to split the quadrilateral $Q$ into four quadrants $Q_1$, $Q_2$, $Q_3$, $Q_4$, and then analyse each quadrant separately. The quadrants will be decided by the diagonals in $Q$. We will split the diagonals into four lines; $d_1$, $d_2$, $d_3$, $d_4$, where $d_i$ is the line between $\mathbf{v}_i$ and the intersection between the diagonals in $Q$ for $i = 1, 2, 3, 4$ (see figure 3.2). We will then begin the proof by showing that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$, for $\mathbf{x} \in d_1 \cup d_2 \cup d_3 \cup d_4$. After doing this, we will analyse each quadrant separately. Since we by this time have checked the diagonal lines $d_1$, $d_2$, $d_3$ and $d_4$, we will view the quadrants as open sets. We will later see that when we view the quadrants as open sets we get some special properties that will help us to prove that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$.

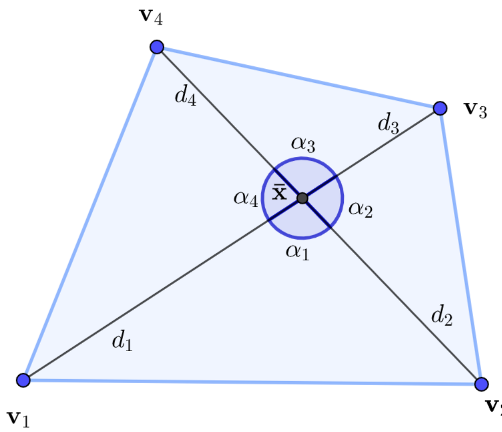


Figure 3.2: Example of $d_1$, $d_2$, $d_3$ and $d_4$ in quadrilateral. Here $\bar{\mathbf{x}}$ is the intersection between the diagonals.

When we are checking if $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$, we will in most cases preform a numerical analysis to check this statement. The code we use in the numerical analysis is written in Appendix A, section A.1. In this section we will also discuss the accuracy of the numerical test we are using. Note that the numerical analysis we are doing in this thesis is different from the one that was preformed in [6]. The reason why we need some numerical analysis in this thesis is that the analytical analysis is quite extensive. This being said, with the right tools, it might be possible to use the outline of this proof and the expressions we have found to make an analytical proof. We will discuss this further in chapter 4.

To prove that $\mathbf{R}_{1,2,3} \geq 0$ we will need a lemma:

**Lemma 3.2.2.** *If* $P_{n,m} = \frac{(\mathbf{v}_n - \mathbf{x})}{\|(\mathbf{v}_n - \mathbf{x})\|^2} \cdot \frac{(\mathbf{v}_m - \mathbf{x})}{\|(\mathbf{v}_m - \mathbf{x})\|^2}$ *and* $A_{n,m} = \frac{(\mathbf{v}_n - \mathbf{x})}{\|(\mathbf{v}_n - \mathbf{x})\|^2} \times \frac{(\mathbf{v}_m - \mathbf{x})}{\|(\mathbf{v}_m - \mathbf{x})\|^2}$, *then* $P_{i,i+1} = \frac{\cos(\alpha_i)}{\sin(\alpha_i)} A_{i,i+1}$, *for* $i = 1, 2, 3, 4$, $\alpha_i \in (0, \pi)$, *and* $P_{i,i+2} = \frac{\cos(\alpha_i + \alpha_{i+1})}{|\sin(\alpha_i + \alpha_{i+1})|} |A_{i,i+2}|$, *for* $(\alpha_i + \alpha_{i+1}) \in (0, \pi) \vee (\pi, 2\pi)$. *Note that when* $i = 4$, *then* $i + 1 = 1$.

*Proof.* We will prove this by using the definition of dot products and cross products (see e.g. [16]).

$$\begin{aligned}
P_{i,i+1} &= \frac{\mathbf{v}_i - \mathbf{x}}{\|\mathbf{v}_i - \mathbf{x}\|^2} \cdot \frac{\mathbf{v}_{i+1} - \mathbf{x}}{\|(\mathbf{v}_{i+1} - \mathbf{x})\|^2} \\
&= \frac{\|\mathbf{v}_i - \mathbf{x}\| \|\mathbf{v}_{i+1} - \mathbf{x}\| \cdot \cos(\alpha_i)}{\|\mathbf{v}_i - \mathbf{x}\|^2 \|\mathbf{v}_{i+1} - \mathbf{x}\|^2} \\
&= \frac{\cos(\alpha_i)}{\sin(\alpha_i)} \frac{\|\mathbf{v}_i - \mathbf{x}\| \|\mathbf{v}_{i+1} - \mathbf{x}\| \cdot \sin(\alpha_i)}{\|\mathbf{v}_i - \mathbf{x}\|^2 \|\mathbf{v}_{i+1} - \mathbf{x}\|^2}
\end{aligned}$$

$$= \frac{\cos(\alpha_i)}{\sin(\alpha_i)} A_{i,i+1}$$

The proof for $P_{i,i+2}$ is similar. ■

We now have everything we need to prove that $\mathbf{R}_{1,2,3}$ is non-negative. We will split the analysis in step 1 into nine cases;

Case 1: $\mathbf{x}$ is the intersection between the diagonals,
Case 2: $\mathbf{x} \in d_1$,
Case 3: $\mathbf{x} \in d_3$,
Case 4: $\mathbf{x} \in d_2$,
Case 5: $\mathbf{x} \in d_4$,
Case 6: $\mathbf{x} \in Q_1$,
Case 7: $\mathbf{x} \in Q_2$,
Case 8: $\mathbf{x} \in Q_3$,
Case 9: $\mathbf{x} \in Q_4$.

Note that a lot of these cases have many similarities, so the analysis in the different cases will be very similar to each other. Due to this similarity, some of the text we write will be quite similar from case to case.

**Case 1:** $\mathbf{x}$ is the intersection between the diagonals

When $\mathbf{x}$ is the intersection between the diagonals, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ in this case. These are

- $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$. This follows from the fact that $\alpha_1$ and $\alpha_3$, and $\alpha_2$ and $\alpha_4$ are vertical angles (see e.g. [17]).
- $\alpha_1 = \pi - \alpha_2$ and $\alpha_3 = \pi - \alpha_4$. This follows from the fact that $\alpha_1$ and $\alpha_2$, and $\alpha_3$ and $\alpha_4$ are supplementary angles (see e.g. [17]). This means that $\sin(\alpha_1) = \sin(\alpha_2) = \sin(\alpha_3) = \sin(\alpha_4)$. Since all the sine values are equal we will denote $\sin(\alpha_i)$ as $s$, for $i = 1, 2, 3, 4$, in this case.
- $A_{2,4}$, $A_{4,2}$, $A_{1,3}$, $A_{3,1} = 0$, since $\sin(\alpha_1 + \alpha_2) = \sin(\pi) = 0$ and $\sin(\alpha_3 + \alpha_4) = \sin(\pi) = 0$.

We observe that we have four expressions in $\mathbf{R}_{1,2,3}$ of the form $A_{n,m}$ that are negative; $A_{1,4}$, $A_{2,1}$, $A_{3,2}$ and $A_{4,3}$. All four of these expressions have positive counterparts, namely $A_{4,1}$, $A_{1,2}$, $A_{2,3}$ and $A_{3,4}$, but since all of these expressions are multiplied with different factors in $\mathbf{R}_{1,2,3}$ we will need to show that the sum of the factors in front of $A_{4,1}$, $A_{1,2}$, $A_{2,3}$ and $A_{3,4}$ are greater than or equal to the sum of the factors in front of $A_{1,4}$, $A_{2,1}$, $A_{3,2}$ and $A_{4,3}$ respectively. If this is the case, we have proven that the sum of all terms containing $A_{n,m}$, for $n, m \in \{1, 2, 3, 4\}$, is greater than or equal to zero, and we are then one step closer to show that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$.

Before we begin the analysis we will define two functions that will be used frequently through the rest of this thesis when we preform an analysis.

Definition 3.2.1. *Let $x_1, x_2, \ldots, x_n$, for $n \geq 1$, be such that $x_i$, for $1 \leq i \leq n$, is a factor. Then, let $f^+$ be a function such that $f^+(x_1, x_2, \ldots, x_n)$ returns the sum of all terms containing the factors $x_1, x_2, \ldots, x_n$, for $n \geq 1$.*

Definition 3.2.2. *Let $x$, $y$ be two factors. Then, let $f^-$ be a function such that $f^-(x,y)$ returns the sum of the factors multiplied with $x$ minus the sum of the factors multiplied with $y$.*

We will now check if $f^-(A_{n,m}, A_{m,n}) \geq 0$, for all $n, m$ that satisfy $A_{n,m} > 0$.

$\underline{f^-(A_{4,1}, A_{1,4}):}$

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

$\underline{f^-(A_{1,2}, A_{2,1}):}$

$$\begin{aligned} f^-(A_{1,2}, A_{2,1}) &= 3 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_2^2}{(t_1+t_2)^2} \frac{1}{s^2} + 1 \\ &- \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

$\underline{f^-(A_{2,3}, A_{3,2}):}$

$$\begin{aligned} f^-(A_{2,3}, A_{3,2}) &= 3 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 - \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

$\underline{f^-(A_{3,4}, A_{4,3}):}$

$$\begin{aligned} f^-(A_{3,4}, A_{4,3}) &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

We have now proven that the sums of all terms containing $A_{n,m}$, for $n, m \in \{1,2,3,4\}$, are greater than or equal to zero. Note that $f^-(A_{2,4}, A_{4,2})$ and $f^-(A_{1,3}, A_{3,1})$ are the only two expressions that are equal to zero. This follows from the fact that $A_{2,4}$, $A_{4,2}$, $A_{1,3}$, $A_{3,1} = 0$. It then remains to prove that the sum of all terms containing $P_{n,m}$ in $\mathbf{R}_{1,2,3}$, for $n, m \in \{1,2,3,4\}$, is greater than or equal to zero.

First, we observe that

$$\begin{aligned}\frac{t_1}{t_1+t_2}\cdot\frac{1}{s}\Big(P_{1,1}-P_{1,2}-P_{1,2}+P_{2,2}\Big) &= \frac{t_1}{t_1+t_2}\cdot\frac{1}{s}\Big(P_{1,1}-2\cdot P_{1,2}+P_{2,2}\Big)\\ &= \frac{t_1}{t_1+t_2}\cdot\frac{1}{s}\Bigg(\left(\frac{\|\mathbf{v}_1-\mathbf{x}\|}{\|\mathbf{v}_1-\mathbf{x}\|^2}\right)^2\\ &-2\cdot\left(\frac{\|\mathbf{v}_1-\mathbf{x}\|\|\mathbf{v}_2-\mathbf{x}\|}{(\|\mathbf{v}_1-\mathbf{x}\|\|\mathbf{v}_2-\mathbf{x}\|)^2}\right)\cdot\cos(\alpha_1)+\left(\frac{\|\mathbf{v}_2-\mathbf{x}\|}{\|\mathbf{v}_2-\mathbf{x}\|^2}\right)^2\Bigg)\\ &\geq \frac{t_1}{t_1+t_2}\cdot\frac{1}{s}\left(\frac{\|\mathbf{v}_1-\mathbf{x}\|}{\|\mathbf{v}_1-\mathbf{x}\|^2}-\frac{\|\mathbf{v}_2-\mathbf{x}\|}{\|\mathbf{v}_2-\mathbf{x}\|^2}\right)^2\\ &> 0.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\frac{t_2}{t_1+t_2}\cdot\frac{1}{s}\Big(P_{3,3}-P_{2,3}-P_{2,3}+P_{2,2}\Big) &= \frac{t_2}{t_1+t_2}\cdot\frac{1}{s}\Big(P_{3,3}-2\cdot P_{2,3}+P_{2,2}\Big)\\ &= \frac{t_2}{t_1+t_2}\cdot\frac{1}{s}\Bigg(\left(\frac{\|\mathbf{v}_3-\mathbf{x}\|}{\|\mathbf{v}_3-\mathbf{x}\|^2}\right)^2\\ &-2\cdot\left(\frac{\|\mathbf{v}_2-\mathbf{x}\|\|\mathbf{v}_3-\mathbf{x}\|}{(\|\mathbf{v}_2-\mathbf{x}\|\|\mathbf{v}_3-\mathbf{x}\|)^2}\right)\cdot\cos(\alpha_2)+\left(\frac{\|\mathbf{v}_2-\mathbf{x}\|}{\|\mathbf{v}_2-\mathbf{x}\|^2}\right)^2\Bigg)\\ &\geq \frac{t_2}{t_1+t_2}\cdot\frac{1}{s}\left(\frac{\|\mathbf{v}_3-\mathbf{x}\|}{\|\mathbf{v}_3-\mathbf{x}\|^2}-\frac{\|\mathbf{v}_2-\mathbf{x}\|}{\|\mathbf{v}_2-\mathbf{x}\|^2}\right)^2\\ &> 0.\end{aligned}$$

If we now collect the rest of the terms containing $P_{n,m}$, for $n, m = 1, 2, 3, 4$, we get the two sums

$$\frac{t_2}{t_1+t_2}\cdot\frac{1}{s}\Big(P_{1,2}-P_{2,4}+P_{3,4}-P_{1,3}\Big),$$

and

$$\frac{t_1}{t_1+t_2}\cdot\frac{1}{s}\Big(P_{4,1}-P_{1,3}+P_{2,3}-P_{2,4}\Big).$$

First, we observe that $-P_{2,4}, -P_{1,3} > 0$. This follows from the fact that $\cos(\alpha_2+\alpha_3) = \cos(\pi) = -1$ and $\cos(\alpha_1+\alpha_2) = \cos(\pi) = -1$. Second, we observe that if $\alpha_1 = \alpha_3 \leq \frac{\pi}{2}$, then $P_{1,2}, P_{3,4} \geq 0$, which means that the first sum would be greater than zero. Similarly, if $\alpha_2 = \alpha_4 \leq \frac{\pi}{2}$, then $P_{4,1}, P_{2,3} \geq 0$, which means that the second sum would be greater than zero. If, on the other hand, $\alpha_1 = \alpha_3 > \frac{\pi}{2}$, we can not guarantee that the first sum is greater than or equal to zero. To prove that $P_{1,2}, P_{3,4}$ in this case do lead to $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$, we will use Lemma 3.2.2. From this lemma we see that we can write $P_{1,2}, P_{3,4}$ as $\frac{\cos(\alpha_1)}{\sin(\alpha_1)}A_{1,2}$, $\frac{\cos(\alpha_3)}{\sin(\alpha_3)}A_{3,4}$, respectively. We will now preform a similar analysis as we did earlier, when we checked if $f^-(A_{n,m}, A_{m,n}) \geq 0$ for $n, m$ satisfying $A_{n,m} > 0$. Since we already made an analysis for both $A_{1,2}$ and $A_{3,4}$ we will use the results from these.

$f^-(A_{1,2}, A_{2,1})$ (new expression marked in blue):

$$\begin{aligned}f^-(A_{1,2}, A_{2,1}) &= 2\cdot\frac{t_1t_2}{(t_1+t_2)^2}\cdot\frac{1}{s^2}+1+\frac{t_2}{t_1+t_2}\cdot\frac{\cos(\alpha_1)}{s^2}\\ &> \frac{-t_2^2+t_1t_2}{(t_1+t_2)^2}\cdot\frac{1}{s^2}+1\\ &> 0.\end{aligned}$$

Since we are looking at the case where $\alpha_1 > \frac{\pi}{2}$, we know that $\alpha_2 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_2 < t_1$, which again implies that $t_2^2 < t_1 t_2$.

$f^-(A_{3,4}, A_{4,3})$ :

$$\begin{aligned} f^-(A_{3,4}, A_{4,3}) &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_2}{t_1+t_2} \cdot \frac{\cos(\alpha_3)}{s^2} \\ &> \frac{-t_2^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

Similarly, if now $\alpha_2 = \alpha_4 > \frac{\pi}{2}$, we can not guarantee that the second sum is greater than or equal to zero.

We will now need to make a similar argument as we did for $P_{1,2}$ and $P_{3,4}$. We know from Lemma 3.2.2 that we can write $P_{4,1}, P_{2,3}$ as $\frac{\cos(\alpha_4)}{\sin(\alpha_4)} A_{4,1}$, $\frac{\cos(\alpha_2)}{\sin(\alpha_2)} A_{2,3}$. We will now use this in the analysis for both $A_{4,1}$ and $A_{2,3}$.

$f^-(A_{4,1}, A_{1,4})$:

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_1}{t_1+t_2} \cdot \frac{\cos(\alpha_4)}{s^2} \\ &> \frac{-t_1^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_2 > \frac{\pi}{2}$, we know that $\alpha_1 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_1 < t_2$, which again implies that $t_1^2 < t_1 t_2$. This in turn implies that $\frac{-t_1^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} > 0$.

We will now make a similar argument for $A_{2,3}$.

$f^-(A_{2,3}, A_{3,2})$ :

$$\begin{aligned} f^-(A_{2,3}, A_{3,2}) &= 2 \cdot \frac{t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 + \frac{t_1}{t_1+t_2} \cdot \frac{\cos(\alpha_2)}{s^2} \\ &> \frac{-t_1^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

We have now proven analytically that all terms in $\mathbf{R}_{1,2,3}(\mathbf{x})$ are strictly greater than zero, when $\mathbf{x}$ is the intersection between the diagonals.

**Case 2: $\mathbf{x} \in d_1$**

When $\mathbf{x} \in d_1$, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ in this case. These are

- $\alpha_2 = \pi - \alpha_1$ and $\alpha_3 = \pi - \alpha_4$. This follows from the fact that $\alpha_1$ and $\alpha_2$, and $\alpha_3$ and $\alpha_4$ are supplementary angles. This means that $\sin(\alpha_1) = \sin(\alpha_2)$ and $\sin(\alpha_3) = \sin(\alpha_4)$.
- $A_{1,3} = A_{3,1} = 0$, since $\sin(\alpha_1 + \alpha_2) = \sin(\pi) = 0$.
- $\alpha_1 \geq \alpha_3$ and $\alpha_4 \geq \alpha_2$. This follows from the fact that $\alpha_1 = \alpha_3$ and $\alpha_4 = \alpha_2$ in the intersection between the diagonals. In addition, both $\alpha_1$ and $\alpha_4$ will increase when $\mathbf{x}$ moves closer to $\mathbf{v}_1$ along $d_1$.

We now observe that we have five expressions in $\mathbf{R}_{1,2,3}$ on the form $A_{n,m}$ that are negative, namely $A_{1,4}$, $A_{1,2}$, $A_{2,3}$, $A_{4,2}$ and $A_{4,3}$. All five of these expressions have positive counterparts; $A_{4,1}$, $A_{2,1}$, $A_{3,2}$, $A_{2,4}$ and $A_{3,4}$, respectively. In addition to these we have some expressions on the form $P_{n,m}$. To make it easier to compare we will convert most of these to be on the form $A_{n,m}$ by using Lemma 3.2.2. We observe that $P_{n,m}$ will vary between being positive and negative, depending on the value of $\alpha_{n,m} \coloneqq \alpha_n + \cdots + \alpha_{m-1}$. Since all the expressions are multiplied with different factors in $\mathbf{R}_{1,2,3}$, we will need to show that the sum of the factors in front of the positive parts are greater than or equal to the sum of the factors in front of the corresponding negative parts. Note that when we convert $P_{n,m}$ into $A_{n,m}$, we get a cosine factor that makes sure that we will get the correct sign in front of the term. In this case we will preform a numerical analysis on each expression to check if the expression is greater than or equal to zero. You can find a description of the numerical method in section A.1, and then the actual test results will be presented in section A.2. For simplicity, we will derive the different expressions below, and then state the result that we find in A.2.1.

$f^-(A_{4,1}, A_{1,4})$ :

First, we convert all $P_{4,1}$ into $A_{1,4}$, by using Lemma 3.2.2. We then get that

$$P_{4,1} = \frac{\cos(\alpha_4)}{\sin(\alpha_4)} A_{4,1}.$$

Using this we get that

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) = & \frac{t_1 t_4}{(t_1 + t_2)(t_1 + t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\ & + \frac{t_3 t_4}{(t_2 + t_3)(t_1 + t_4)} \cdot \frac{1}{\sin(\alpha_3)\sin(\alpha_4)} \\ & + \frac{t_1 t_3}{(t_1 + t_2)(t_2 + t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\ & - \frac{t_1 t_3}{(t_2 + t_3)(t_1 + t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\ & + \frac{t_3}{t_2 + t_3} \cdot \frac{1}{\sin(\alpha_3)} \cdot \frac{\cos(\alpha_4)}{\sin(\alpha_4)}. \end{aligned}$$

By using the properties we derived in the beginning of case 2, we can simplify the expression above as follows

$$\begin{aligned}
f^-(A_{4,1}, A_{1,4}) = & \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& + \frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin^2(\alpha_4)} \\
& + \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& - \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& + \frac{t_3}{t_2+t_3} \cdot \frac{\cos(\alpha_4)}{\sin(\alpha_4)^2} > 0.
\end{aligned}$$

$\underline{f^-(A_{1,2}, A_{2,1}):}$

First, we convert all of the $P_{1,2}$ terms into $A_{1,2}$, by using Lemma 3.2.2. We then get that

$$P_{1,2} = \frac{\cos(\alpha_1)}{\sin(\alpha_1)} A_{1,2}.$$

Using this we get that

$$\begin{aligned}
f^-(A_{1,2}, A_{2,1}) = & \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& + \frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
& + 1 + \frac{t_2 t_4}{(t_1+t_4)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
& + \frac{t_1 t_2}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
& - \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
& - \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
& + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
& + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
& - \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
& - \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
& - \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)}.
\end{aligned}$$

By using the properties we derived in the beginning of case 2, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{1,2}, A_{2,1}) = &\frac{t_4(t_1 - t_2)}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&+ \frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin^2(\alpha_1)} \\
&+ 1 + \frac{t_2 t_4}{(t_1+t_4)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&+ \frac{t_1 t_2}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin^2(\alpha_1)} \\
&- \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin^2(\alpha_1)} \\
&+ \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \left( \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \right) \\
&- \frac{\cos(\alpha_1)}{\sin^2(\alpha_1)} \left( \frac{t_2}{t_2+t_3} + \frac{t_1}{t_1+t_4} + \frac{t_1}{t_1+t_2} \right) > 0.
\end{aligned}
$$

<u>$f^-(A_{2,3}, A_{3,2})$ :</u>
First, we convert all $P_{2,3}$ into $A_{2,3}$, by using Lemma 3.2.2. We then get that

$$
P_{2,3} = \frac{\cos(\alpha_2)}{\sin(\alpha_2)} A_{2,3} = -\frac{\cos(\alpha_1)}{\sin(\alpha_1)} A_{2,3}.
$$

Using this we get that

$$
\begin{aligned}
f^-(A_{2,3}, A_{3,2}) = &\frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_1 t_2}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_2 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_3)} \\
&+ 1 + \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&- \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&+ \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_3)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)}.
\end{aligned}
$$

By using the properties we derived in the beginning of case 2, we can simplify the expression above as follows

$$\begin{aligned}
f^-(A_{2,3}, A_{3,2}) = & \frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin^2(\alpha_1)} \\
& + \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& + \frac{t_1 t_2}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin^2(\alpha_1)} \\
& + \frac{t_2 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} + 1 \\
& - \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin^2(\alpha_1)} \\
& - \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& + \frac{\cos(\alpha_1)}{\sin^2(\alpha_1)} \left( \frac{t_1}{t_1+t_4} + \frac{t_2}{t_1+t_2} + \frac{t_2}{t_2+t_3} \right) \\
& - \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \left( \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_4)} + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \right) > 0.
\end{aligned}$$

$\underline{f^-(A_{2,4}, A_{4,2}):}$
First, we convert all $P_{2,4}$ into $A_{2,4}$, by using Lemma 3.2.2. We then get that

$$P_{2,4} = \frac{\cos(\alpha_2+\alpha_3)}{\sin(\alpha_2+\alpha_3)} A_{2,4} = \frac{\cos(2\pi-\alpha_1-\alpha_4)}{\sin(2\pi-\alpha_1-\alpha_4)} A_{2,4}.$$

Using this we get that

$$\begin{aligned}
f^-(A_{2,4}, A_{4,2}) = & \frac{t_2 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
& + \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
& + \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& - \frac{t_2 t_4}{(t_1+t_4)(t_1+t_2)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
& - \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
& - \frac{t_2 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_3)} \\
& - \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(2\pi-\alpha_1-\alpha_4)}{\sin(2\pi-\alpha_1-\alpha_4)} \\
& - \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_3)} \cdot \frac{\cos(2\pi-\alpha_1-\alpha_4)}{\sin(2\pi-\alpha_1-\alpha_4)}.
\end{aligned}$$

By using the properties we derived in the beginning of case 2, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{2,4}, A_{4,2}) = &\frac{t_2t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&+ \frac{t_1t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&+ \frac{t_1t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&- \frac{t_2t_4}{(t_1+t_4)(t_1+t_2)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&- \frac{t_1t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&- \frac{t_2t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&- \frac{\cos(2\pi-\alpha_1-\alpha_4)}{\sin(2\pi-\alpha_1-\alpha_4)\sin(\alpha_4)}\left(\frac{t_4}{t_1+t_4} + \frac{t_3}{t_2+t_3}\right) > 0.
\end{aligned}
$$

$\underline{f^-(A_{3,4}, A_{4,3}):}$
First, we convert all $P_{3,4}$ into $A_{3,4}$, by using Lemma 3.2.2. We then get that

$$
P_{3,4} = \frac{\cos(\alpha_3)}{\sin(\alpha_3)}A_{3,4} = -\frac{\cos(\alpha_4)}{\sin(\alpha_4)}A_{3,4}.
$$

Using this we get that

$$
\begin{aligned}
f^-(A_{3,4}, A_{4,3}) = &\frac{t_2t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
&+ \frac{t_3t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_3)\sin(\alpha_4)} \\
&+ \frac{t_2t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_3)} \\
&- \frac{t_2t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
&- \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(\alpha_4)}{\sin(\alpha_4)}.
\end{aligned}
$$

By using the properties we derived in the beginning of case 2, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{3,4}, A_{4,3}) = &\frac{t_2t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&+ \frac{t_3t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin^2(\alpha_4)} \\
&+ \frac{t_2t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&- \frac{t_2t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&- \frac{t_4}{t_1+t_4} \cdot \frac{\cos(\alpha_4)}{\sin^2(\alpha_4)} > 0.
\end{aligned}
$$

We have now looked at all terms containing $A_{n,m}$ and most of the terms containing $P_{n,m}$ in $\mathbf{R}_{1,2,3}$, but some terms containing $P_{n,m}$ still remain. We will now show that the sum of the remaining terms in $\mathbf{R}_{1,2,3}$ are positive. We will start by looking at the terms containing $P_{1,3}$. First, we notice that $P_{1,3} < 0$ since $\cos(\alpha_1 + \alpha_2) = \cos(\pi) = -1$. If we now gather all terms containing $P_{1,3}$ we get that

$$\begin{aligned} f^+(P_{1,3}) =& \Big( \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_2} + \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1} - \Big( \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin\alpha_2} \\ &+ \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_3} + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4} + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin\alpha_1} \Big) \Big) P_{1,3}. \end{aligned}$$

We can simplify this to

$$\begin{aligned} f^+(P_{1,3}) =& \Big( \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_1} + \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1} - \Big( \frac{1}{\sin\alpha_1} + \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_4} \\ &+ \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4} \Big) \Big) P_{1,3}. \end{aligned}$$

Second, we observe that we can use the terms containing $P_{1,1}$ and $P_{3,3}$ in the analysis of $P_{1,3}$, since both of these are greater than zero, and

$$P_{1,1} + P_{3,3} \geq -P_{1,3}.$$

The inequality above holds since

$$\begin{aligned} P_{1,1} + P_{1,3} + P_{3,3} &\geq P_{1,1} + 2P_{1,3} + P_{3,3} \\ &= \left( \frac{\|\mathbf{v}_1 - \mathbf{x}\|}{\|(\mathbf{v}_1 - \mathbf{x})\|^2} - \frac{\|\mathbf{v}_3 - \mathbf{x}\|}{\|(\mathbf{v}_3 - \mathbf{x})\|^2} \right)^2 \\ &\geq 0. \end{aligned}$$

From $\mathbf{R}_{1,2,3}$ we see that $P_{1,1}$ is multiplied with $\frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)}$, and $P_{3,3}$ is multiplied with $\frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} = \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)}$. We then observe that

$$\frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{1,1} + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{3,3} \geq \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \Big( P_{1,1} + P_{3,3} \Big),$$

when $\alpha_1 \in (0, \frac{\pi}{2})$. Similarly,

$$\frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{1,1} + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{3,3} \geq \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \Big( P_{1,1} + P_{3,3} \Big),$$

when $\alpha_1 \in (\frac{\pi}{2}, \pi)$. This means that when $\alpha_1 \in (0, \frac{\pi}{2})$,

$$\begin{aligned} & f^+(P_{1,3}) + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{1,1} + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{3,3} \\ &\geq f^+(P_{1,3}) + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \Big( P_{1,1} + P_{3,3} \Big) \\ &\geq \Big( \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_1} + \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1} - \Big( \frac{1}{\sin\alpha_1} + \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_4} \\ &\quad + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4} \Big) - \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \Big) P_{1,3}. \end{aligned} \tag{3.14}$$

Similarly, when $\alpha_1 \in (\frac{\pi}{2}, \pi)$,

$$
\begin{aligned}
&f^+(P_{1,3}) + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{1,1} + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{3,3} \\
&\geq f^+(P_{1,3}) + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \Big( P_{1,1} + P_{3,3} \Big) \\
&\geq \Bigg( \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_1} + \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1} - \Bigg( \frac{1}{\sin\alpha_1} + \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_4} \\
&\quad + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4} \Bigg) - \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \Bigg) P_{1,3}. \qquad (3.15)
\end{aligned}
$$

We need to show that both of the expressions (3.14) and (3.15), are greater than or equal to zero. Since $P_{1,3} < 0$ we will need to show that the expressions in front of $P_{1,3}$ in (3.14) and (3.15) are less than or equal to zero. If this is the case, we know that the two expressions multiplied with $P_{1,3}$ is greater than or equal to zero. When we preform the numerical test, we choose to change the signs of the two expressions in front of $P_{1,3}$, that is, we check if the expression in front of $-P_{1,3}$ in (3.14) and (3.15) is greater than zero. We then observe by numerical testing that both of these expressions are strictly greater than zero. You can find the tests in Appendix A, subsection A.2.1.

We are then left with two terms, namely $\frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} P_{2,2}$ and $\frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} P_{2,2}$. These are both positive since

$$P_{2,2} = \frac{(\mathbf{v}_2 - \mathbf{x})^2}{\|(\mathbf{v}_2 - \mathbf{x})\|^4} \cdot \cos(0) > 0,$$

and the factors in front of both expressions are also always greater than zero.

We have now, by numerical testing, showed that all the negative terms in $\mathbf{R}_{1,2,3}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in d_1$. This means that $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ when $\mathbf{x} \in d_1$.

**Case 3: $\mathbf{x} \in d_3$**

When $\mathbf{x} \in d_3$, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ in this case. These are

- $\alpha_2 = \pi - \alpha_1$ and $\alpha_3 = \pi - \alpha_4$. This follows from the fact that $\alpha_1$ and $\alpha_2$, and $\alpha_3$ and $\alpha_4$ are supplementary angles. This means that $\sin(\alpha_1) = \sin(\alpha_2)$ and $\sin(\alpha_3) = \sin(\alpha_4)$.
- $A_{1,3} = A_{3,1} = 0$, since $\sin(\alpha_1 + \alpha_2) = \sin(\pi) = 0$.
- $\alpha_3 \geq \alpha_1$ and $\alpha_2 \geq \alpha_4$. This follows from the fact that $\alpha_3 = \alpha_1$ and $\alpha_2 = \alpha_4$ in the intersection between the diagonals. In addition, both $\alpha_2$ and $\alpha_3$ will increase when $\mathbf{x}$ moves closer to $\mathbf{v}_3$ along $d_3$.

We observe that case 3 is quite similar to case 2. The only differences is bullet point 3, and the fact that $A_{2,4}$ is negative and $A_{4,2}$ is positive. This means that we can modify the analysis from case 2, to get an analysis for case 3. The only difference in the numerical analysis is that we will only test for values of $\alpha_1$ and $\alpha_4$ that satisfy $0 < \alpha_1 + \alpha_4 < \pi$. For the expression containing $A_{2,4}$ and $A_{4,2}$, we will also change the sign of each term in the expression. When we make these changes and analyse these numerically (see Appendix

A, subsection A.2.1), we observe that all expressions are strictly positive. This means that $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ when $\mathbf{x} \in d_3$.

Note, since we have only made some minor changes, we chose to not write out the analysis in detail as we did for $\mathbf{x} \in d_1$. Since we have explained the differences between these two cases above, we leave it up to the reader to do the detailed analysis in this case.

**Case 4: $\mathbf{x} \in d_2$**

When $\mathbf{x} \in d_2$, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3} \geq 0$ in this case. These are

- $\alpha_3 = \pi - \alpha_2$ and $\alpha_4 = \pi - \alpha_1$. This follows from the fact that $\alpha_1$ and $\alpha_4$, and $\alpha_2$ and $\alpha_3$ are supplementary angles. This means that $\sin(\alpha_1) = \sin(\alpha_4)$ and $\sin(\alpha_2) = \sin(\alpha_3)$.
- $A_{2,4} = A_{4,2} = 0$, since $\sin(\alpha_2 + \alpha_3) = \sin(\pi) = 0$.
- $\alpha_1 \geq \alpha_3$ and $\alpha_2 \geq \alpha_4$. This follows from the fact that $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$ in the intersection between the diagonals. In addition, both $\alpha_1$ and $\alpha_2$ will increase when $\mathbf{x}$ moves closer to $\mathbf{v}_2$ along $d_2$.

We observe that we have five expressions in $\mathbf{R}_{1,2,3}$ on the form $A_{n,m}$ that are negative, namely $A_{1,3}$, $A_{1,4}$, $A_{2,1}$, $A_{3,2}$ and $A_{4,3}$. All five of these expressions have positive counterparts; $A_{3,1}$, $A_{4,1}$, $A_{1,2}$, $A_{3,2}$ and $A_{3,4}$. In addition to these we have some expressions on the form $P_{n,m}$. To make it easier to compare we will convert most of these to be on the form $A_{n,m}$ by using Lemma 3.2.2. We recall from the previous cases that $P_{n,m}$ will vary between being positive and negative, depending on the value of $\alpha_{n,m}$. Since all the expressions are multiplied with different factors, we will need to show that the factors in front of the positive parts are greater than or equal to the factors in front of the corresponding negative parts. In this case we will preform a numerical analysis on each expression to check if the expression is greater than or equal to zero. You can find the tests and test results in section A.2.2. For simplicity, we will derive the different expressions below, and then directly state the result that we find in A.2.2.

$f^-(A_{3,1}, A_{1,3})$ :

First, we convert all $P_{1,3}$ into $A_{3,1}$. We then get that

$$P_{1,3} = -\frac{\cos(\alpha_1 + \alpha_2)}{\sin(\alpha_1 + \alpha_2)} A_{3,1}.$$

The reason why we multiply the expression above with $-1$, is that $\alpha_1 + \alpha_2 \in (\pi, 2\pi)$ when $\mathbf{x} \in d_2$, which implies that $\sin(\alpha_1 + \alpha_2) < 0$. Since we want to convert $P_{1,3}$ into $A_{3,1}$, we will need to counteract the negative sine value by multiplying with $-1$.
Using this we get that

$$\begin{aligned} f^-(A_{3,1}, A_{1,3}) = & \frac{t_1 t_2}{(t_1 + t_4)(t_1 + t_2)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\ & + \frac{t_2 t_4}{(t_2 + t_3)(t_1 + t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\ & + \frac{t_1 t_3}{(t_2 + t_3)(t_1 + t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\ & + \frac{t_1 t_2}{(t_1 + t_2)(t_2 + t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} + 1 \end{aligned}$$

$$
\begin{aligned}
&- \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
&- \frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_3)\sin(\alpha_4)} \\
&- \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} \\
&+ \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_3)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} \\
&+ \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} \\
&+ \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} \\
&- \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} \\
&- \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)}.
\end{aligned}
$$

By using the properties we derived in the beginning of case 4, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{3,1}, A_{1,3}) = &\frac{t_2(t_1-t_4)}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_2(t_4-t_1)}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_3(t_1-t_4)}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_1(t_2-t_3)}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} + 1 \\
&+ \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} \left( \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \right) \\
&- \frac{\cos(\alpha_1+\alpha_2)}{\sin(\alpha_1+\alpha_2)} \left( \frac{t_2-t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} + \frac{t_1-t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \right) > 0.
\end{aligned}
$$

$f^-(A_{4,1}, A_{1,4})$ :

First, we convert all $P_{4,1}$ into $A_{4,1}$, by using Lemma 3.2.2. We then get that

$$
P_{4,1} = \frac{\cos(\alpha_4)}{\sin(\alpha_4)} A_{4,1} = -\frac{\cos(\alpha_1)}{\sin(\alpha_1)} A_{4,1}.
$$

Using this we get that

$$
\begin{aligned}
f^-(A_{4,1}, A_{1,4}) = &\frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_3)\sin(\alpha_4)} \\
&+ \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&- \frac{t_3}{t_2+t_3} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)\sin(\alpha_3)}.
\end{aligned}
$$

By using the properties we derived in the beginning of case 4, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{4,1}, A_{1,4}) = &\frac{t_3(t_4-t_1)}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin^2(\alpha_1)} \\
&- \frac{t_3}{t_2+t_3} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)\sin(\alpha_2)} > 0.
\end{aligned}
$$

$f^-(A_{1,2}, A_{2,1})$ :
First, we convert all $P_{1,2}$ into $A_{1,2}$, by using Lemma 3.2.2. We then get that

$$
P_{1,2} = \frac{\cos(\alpha_1)}{\sin(\alpha_1)} A_{1,2}.
$$

Using this we get that

$$
\begin{aligned}
f^-(A_{1,2}, A_{2,1}) = &\frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&+ \frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_2 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} + 1 \\
&- \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&- \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
&+ \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&+ \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)}.
\end{aligned}
$$

By using the properties we derived in the beginning of case 4, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{1,2},A_{2,1}) = &\frac{t_1t_4}{(t_1+t_2)(t_1+t_4)}\cdot\frac{1}{\sin^2(\alpha_1)}\\
&+\frac{t_2(t_1-t_4)}{(t_1+t_4)(t_1+t_2)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}\\
&+\frac{t_2(t_4-t_1)}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}+1\\
&+\frac{\cos(\alpha_1)}{\sin(\alpha_1)}\left(\frac{t_4-t_1}{t_1+t_4}\cdot\frac{1}{\sin(\alpha_1)}+\frac{t_2}{t_1+t_2}\cdot\frac{1}{\sin(\alpha_2)}\right)\\
&-\frac{\cos(\alpha_1)}{\sin(\alpha_1)}\left(\frac{t_1}{t_1+t_2}\cdot\frac{1}{\sin(\alpha_1)}+\frac{t_2}{t_2+t_3}\cdot\frac{1}{\sin(\alpha_2)}\right)>0.
\end{aligned}
$$

<u>$f^-(A_{2,3},A_{3,2})$ :</u>
First, we convert all $P_{2,3}$ into $A_{2,3}$, by using Lemma 3.2.2. We then get that

$$
P_{2,3}=\frac{\cos(\alpha_2)}{\sin(\alpha_2)}A_{2,3}.
$$

Using this we get that

$$
\begin{aligned}
f^-(A_{2,3},A_{3,2}) = &\frac{t_1t_2}{(t_1+t_2)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}\\
&+\frac{t_1t_3}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_3)}\\
&+\frac{t_1t_2}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}+1\\
&-\frac{t_1t_2}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}\\
&-\frac{t_1t_3}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_3)}\\
&+\frac{t_1}{t_1+t_2}\cdot\frac{1}{\sin(\alpha_1)}\cdot\frac{\cos(\alpha_2)}{\sin(\alpha_2)}\\
&+\frac{t_3}{t_2+t_3}\cdot\frac{1}{\sin(\alpha_3)}\cdot\frac{\cos(\alpha_2)}{\sin(\alpha_2)}\\
&-\frac{t_1}{t_1+t_4}\cdot\frac{1}{\sin(\alpha_1)}\cdot\frac{\cos(\alpha_2)}{\sin(\alpha_2)}\\
&-\frac{t_2}{t_1+t_2}\cdot\frac{1}{\sin(\alpha_2)}\cdot\frac{\cos(\alpha_2)}{\sin(\alpha_2)}\\
&-\frac{t_2}{t_2+t_3}\cdot\frac{1}{\sin(\alpha_2)}\cdot\frac{\cos(\alpha_2)}{\sin(\alpha_2)}.
\end{aligned}
$$

By using the properties we derived in the beginning of case 4, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{2,3},A_{3,2}) = &\frac{t_1t_2}{(t_1+t_2)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}\\
&+\frac{t_1(t_3-t_2)}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t_1(t_2-t_3)}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}+1\\
&-\frac{t_1t_2}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}\\
&+\frac{\cos(\alpha_2)}{\sin(\alpha_2)}\Big(\frac{t_3-t_2}{t_2+t_3}\cdot\frac{1}{\sin(\alpha_2)}+\frac{t_1}{t_1+t_2}\cdot\frac{1}{\sin(\alpha_1)}\Big)\\
&-\frac{\cos(\alpha_2)}{\sin(\alpha_2)}\Big(\frac{t_1}{t_1+t_4}\cdot\frac{1}{\sin(\alpha_1)}+\frac{t_2}{t_1+t_2}\cdot\frac{1}{\sin(\alpha_2)}\Big)>0.
\end{aligned}
$$

$f^-(A_{3,4},A_{4,3}):$

First, we convert all $P_{3,4}$ into $A_{4,3}$, by using Lemma 3.2.2. We then get that

$$
P_{3,4}=\frac{\cos(\alpha_3)}{\sin(\alpha_3)}A_{3,4}=-\frac{\cos(\alpha_2)}{\sin(\alpha_2)}A_{3,4}.
$$

Using this we get that

$$
\begin{aligned}
f^-(A_{3,4},A_{4,3})=&\frac{t_2t_4}{(t_1+t_2)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_2)\sin(\alpha_4)}\\
&+\frac{t_3t_4}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_3)\sin(\alpha_4)}\\
&+\frac{t_2t_3}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin(\alpha_2)\sin(\alpha_3)}\\
&-\frac{t_2t_4}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_2)\sin(\alpha_4)}\\
&-\frac{t_4}{t_1+t_4}\cdot\frac{1}{\sin(\alpha_1)}\cdot\frac{\cos(\alpha_2)}{\sin(\alpha_2)}.
\end{aligned}
$$

By using the properties we derived in the beginning of case 4, we can simplify the expression above as follows

$$
\begin{aligned}
f^-(A_{3,4},A_{4,3})=&\frac{t_2t_4}{(t_1+t_2)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}\\
&+\frac{t_4(t_3-t_2)}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_1)\sin(\alpha_2)}\\
&+\frac{t_2t_3}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin^2(\alpha_2)}\\
&-\frac{\cos(\alpha_2)}{\sin(\alpha_1)\sin(\alpha_2)}\cdot\frac{t_4}{t_1+t_4}>0.
\end{aligned}
$$

We have now looked at all terms containing $A_{n,m}$ and most of the terms containing $P_{n,m}$ in $\mathbf{R}_{1,2,3}$, but there still remain some terms containing $P_{n,m}$. We will now show that the sum of the remaining terms in $\mathbf{R}_{1,2,3}$ is positive. We will start by looking at the terms containing $P_{2,4}$. First, we notice that $P_{2,4}<0$ since $\alpha_2+\alpha_3=\pi$ and $\cos(\pi)=-1$. If we now gather all terms containing $P_{2,4}$ we get that

$$
f^+(P_{2,4})=\Big(-\frac{t_4}{t_1+t_4}\cdot\frac{1}{\sin\alpha_4}-\frac{t_3}{t_2+t_3}\cdot\frac{1}{\sin\alpha_3}\Big)P_{2,4}.
$$

This expression has to be greater than zero, since $-\frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4} - \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_3} < 0$ and $P_{2,4} < 0$. So, we have now proven that the sum of all terms containing $P_{2,4}$ is strictly greater than 0.

We are then left with the terms containing $P_{1,1}, P_{2,2}$ and $P_{3,3}$. First, we observe that all of these expressions are greater than zero, since $\cos(\alpha_{n,n}) = \cos(0) = 1$. Second, we know that all these expressions are multiplied with factors that are greater than zero in $d_2$. Since both the factors and $P_{1,1}, P_{2,2}$ and $P_{3,3}$ are strictly greater than zero, we know that all the terms containing $P_{1,1}, P_{2,2}$ and $P_{3,3}$ are greater than zero.

By numerical testing, we have showed that all the negative terms in $\mathbf{R}_{1,2,3}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in d_2$. This means that $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ when $\mathbf{x} \in d_2$.

**Case 5: $\mathbf{x} \in d_4$**

When $\mathbf{x} \in d_4$, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ in this case. These are

- $\alpha_3 = \pi - \alpha_2$ and $\alpha_4 = \pi - \alpha_1$. This follows from the fact that $\alpha_1$ and $\alpha_4$, and $\alpha_2$ and $\alpha_3$ are supplementary angles. This means that $\sin(\alpha_1) = \sin(\alpha_4)$ and $\sin(\alpha_2) = \sin(\alpha_3)$.
- $A_{2,4} = A_{4,2} = 0$, since $\sin(\alpha_2 + \alpha_3) = \sin(\pi) = 0$.
- $\alpha_3 \geq \alpha_1$ and $\alpha_4 \geq \alpha_2$. This follows from the fact that $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$ in the intersection between the diagonals. In addition, both $\alpha_3$ and $\alpha_4$ will increase when $\mathbf{x}$ moves closer to $\mathbf{v}_4$ along $d_4$.

We observe that case 5 is quite similar to case 4. The only differences are bullet point 3, and the fact that $A_{3,1}$ is negative and $A_{1,3}$ is positive. This means that we can just modify the analysis from case 4. The only difference in the numerical analysis is that we will only test for values of $\alpha_1$ and $\alpha_2$ that satisfy $0 < \alpha_1 + \alpha_2 < \pi$. For the analysis containing $A_{1,3}$ and $A_{3,1}$ we will also change the sign of each term in the expression. When we make these changes and analyse these numerically (see Appendix A, subsection A.2.2), we observe that all expressions are strictly positive. This means that $\mathbf{R}_{1,2,3} > 0$ when $\mathbf{x} \in d_4$.

Note, since we have only made some minor changes, we chose not to write out the analysis in detail as we did for $\mathbf{x} \in d_2$. Since we have explained the differences between these two cases above, we leave it up to the reader to do the detailed analysis in this case.

**Case 6: $\mathbf{x} \in Q_1$**

When $\mathbf{x} \in Q_1$, we have some special properties that we can use to check if $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ in this case. Instead of listing them as bullet points, as was done in the previous cases, we will list them as properties and write out the relevant proofs.

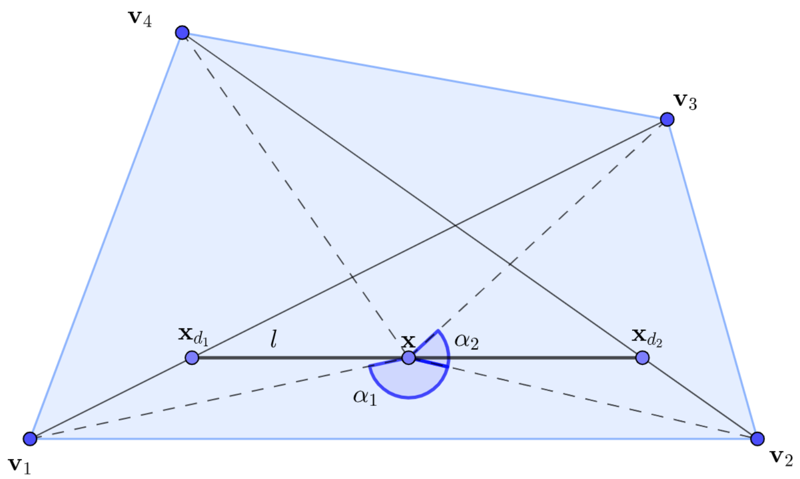


Figure 3.3: Example of how we can prove that $\alpha_1 + \alpha_2 > \pi$ when $\mathbf{x} \in Q_1$

**Property 1:** $\alpha_1 + \alpha_2 > \pi$, when $\mathbf{x} \in Q_1$.

*Proof.* Assume $\mathbf{x} \in Q_1$. Then, let $l$ be the line that is parallel with the line that connects $\mathbf{v}_1$ and $\mathbf{v}_2$, and that goes through $\mathbf{x}$. Furthermore, let $\mathbf{x}_{d_1}$ be the point where $l$ intersects $d_1$, and $\mathbf{x}_{d_2}$ be the point where $l$ intersects $d_2$ (see figure 3.3). First, we observe that along $l$, $\alpha_2$ has its minimum in $\mathbf{x}_{d_1}$. This follows from the fact that $\mathbf{x}_{d_1}$ is the point on $l$ that is furthest away from the line between $\mathbf{v}_2$ and $\mathbf{v}_3$, which implies that this is the point on $l$ that gives the most acute $\alpha_2$. By the same explanation, we note that $\alpha_2$ has its maximum in $\mathbf{x}_{d_2}$. Second, we observe that $\alpha_1$ has a local minimum in both $\mathbf{x}_{d_1}$ and $\mathbf{x}_{d_2}$. This follows from the fact that the distance between $l$ and the line that connects $\mathbf{v}_1$ and $\mathbf{v}_2$ is the same for every point on $l$. Therefore, $\alpha_1$ has its maximum when the length of line between $\mathbf{x}$ and $\mathbf{v}_1$ is equal to the length of the line between $\mathbf{x}$ and $\mathbf{v}_2$. Since the biggest difference in length of the lines between $\mathbf{x}$ and $\mathbf{v}_1$ and $\mathbf{x}$ and $\mathbf{v}_2$ is in $\mathbf{x}_{d_1}$ and $\mathbf{x}_{d_2}$, $\mathbf{x}_{d_1}$ and $\mathbf{x}_{d_2}$ are the local minimums of $\alpha_1$ along $l$.

Since $\alpha_2$ has its minimum in $\mathbf{x}_{d_1}$, and $\alpha_1$ has a local minimum in both $\mathbf{x}_{d_1}$ and $\mathbf{x}_{d_2}$, we know that $\alpha_1 + \alpha_2$ has a minimum in $\mathbf{x}_{d_1}$. We remember from case 2 that when $\mathbf{x}_{d_1} \in d_1$, then $\alpha_1 + \alpha_2 = \pi$. That means that the minimum of $\alpha_1 + \alpha_2$ is $\pi$. Since $d_1$ is on the boundary of $Q_1$, and we are just interested in the inner points, this implies that $\alpha_1 + \alpha_2 > \pi$, when $\mathbf{x} \in Q_1$. ■

**Property 2:** $\alpha_1 > \alpha_3$, when $\mathbf{x} \in Q_1$.

*Proof.* We know from case 1, that if $\bar{\mathbf{x}}$ is the intersection between the diagonals, then $\alpha_1 = \alpha_3$. If we now assume that $\mathbf{x} \in Q_1$, we know that $\mathbf{x}$ will be closer to the line that connects $\mathbf{v}_1$ and $\mathbf{v}_2$, and further from the line that connects $\mathbf{v}_3$ and $\mathbf{v}_4$, than $\bar{\mathbf{x}}$. This means that $\alpha_1$ will be more obtuse for $\mathbf{x}$, than for $\bar{\mathbf{x}}$, and $\alpha_3$ will be more acute for $\mathbf{x}$, than for $\bar{\mathbf{x}}$. Therefore, $\alpha_1 > \alpha_3$, when $\mathbf{x} \in Q_1$. ∎

**Property 3:** $\alpha_2 + \alpha_3 < \pi$, when $\mathbf{x} \in Q_1$.

*Proof.* In the proof for property 1 and property 2, we saw that when $\mathbf{x} \in \partial Q_1$, then $\alpha_2$ has it maximum on $d_2$, and $\alpha_3$ has its maximum in the intersection between the diagonals. This means that the maximum of $\alpha_2 + \alpha_3$ has to be on $d_2$. We know from case 4, that when $\mathbf{x} \in d_2$, then $\alpha_2 + \alpha_3 = \pi$. Therefore, $\alpha_2 + \alpha_3 < \pi$, when $\mathbf{x} \in Q_1$. ∎

**Property 4:** $\alpha_3 + \alpha_4 < \pi$, when $\mathbf{x} \in Q_1$.

*Proof.* This follows directly from property 1 and the fact that the sum of all the angles is equal to $2\pi$. ∎

**Property 5:** $\alpha_4 + \alpha_1 > \pi$, when $\mathbf{x} \in Q_1$.

*Proof.* This follows directly from property 3 and the fact that the sum of all the angles is equal to $2\pi$. ∎

**Property 6:** $\alpha_4 = 2\pi - \alpha_1 - \alpha_2 - \alpha_3$.

*Proof.* This follows from the fact that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$. ∎

We have now proven that all the properties above are valid, so we can now proceed to show that when $\mathbf{x} \in Q_1$, then $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$. We will do this by first collecting all terms in $\mathbf{R}_{1,2,3}$ that contain $A_{n,m}$, $A_{m,n}$ and $P_{n,m}$, for a given $n, m = 1, 2, 3, 4$, where $A_{n,m} > 0$. After doing this, we will convert all $P_{n,m}$ into $A_{n,m}$, and then analyse if $f^-(A_{n,m}, A_{m,n}) \geq 0$. We will do this analysis by writing a code that tests for a reasonable large amount of $\alpha_i$'s, that satisfy the properties above. You can find the code in Appendix A, section A.1.2. If all of these tests passes, we will check if the remaining terms of $\mathbf{R}_{1,2,3}$, the terms containing $P_{n,n}$, are greater than or equal to zero. If all this holds, we have showed that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ when $\mathbf{x} \in Q_1$. For simplicity, we will derive the different expressions below, and then directly state the results that we find in A.2.3.

$\underline{f^-(A_{1,2}, A_{2,1}):}$
First, we convert all $P_{1,2}$ into $A_{1,2}$, by using Lemma 3.2.2. We then get that

$$P_{1,2} = \frac{\cos(\alpha_1)}{\sin(\alpha_1)} A_{1,2}.$$

Using this we get that

$$\begin{aligned}
f^-(A_{1,2}, A_{2,1}) = &\frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
&+ \frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_2 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} + 1 \\
&- \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&- \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
&+ \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&+ \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} \\
&- \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)} > 0.
\end{aligned}$$

<u>$f^-(A_{3,1}, A_{1,3})$ :</u>
First, we convert all $P_{1,3}$ into $A_{3,1}$. We then get that

$$P_{1,3} = \frac{\cos(\alpha_1+\alpha_2)}{|\sin(\alpha_1+\alpha_2)|} |A_{3,1}|.$$

Using this we get that

$$\begin{aligned}
f^-(A_{3,1}, A_{1,3}) = &\frac{t_1 t_2}{(t_1+t_4)(t_1+t_2)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_2 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
&+ \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_1 t_2}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} + 1 \\
&- \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
&- \frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_3)\sin(\alpha_4)} \\
&- \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{|\sin(\alpha_1+\alpha_2)|} \\
&+ \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{|\sin(\alpha_1+\alpha_2)|}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{|\sin(\alpha_1+\alpha_2)|} \\
&- \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_3)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{|\sin(\alpha_1+\alpha_2)|} \\
&- \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{|\sin(\alpha_1+\alpha_2)|} \\
&- \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_1+\alpha_2)}{|\sin(\alpha_1+\alpha_2)|} > 0.
\end{aligned}
$$

$f^{-}(A_{2,3}, A_{3,2}):$

First, we convert all $P_{2,3}$ into $A_{2,3}$, by using Lemma 3.2.2. We then get that

$$
P_{2,3} = \frac{\cos(\alpha_2)}{\sin(\alpha_2)} A_{2,3}.
$$

Using this we get that

$$
\begin{aligned}
f^{-}(A_{2,3}, A_{3,2}) = &\frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&+ \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_1 t_2}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} + 1 \\
&- \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_2)} \\
&- \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
&+ \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_2)}{\sin(\alpha_2)} \\
&+ \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_3)} \cdot \frac{\cos(\alpha_2)}{\sin(\alpha_2)} \\
&- \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} \cdot \frac{\cos(\alpha_2)}{\sin(\alpha_2)} \\
&- \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_2)}{\sin(\alpha_2)} \\
&- \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} \cdot \frac{\cos(\alpha_2)}{\sin(\alpha_2)} > 0.
\end{aligned}
$$

$f^{-}(A_{2,4}, A_{4,2}):$

First, we convert all $P_{2,4}$ into $A_{2,4}$, by using Lemma 3.2.2. We then get that

$$
P_{2,4} = \frac{\cos(\alpha_2+\alpha_3)}{|\sin(\alpha_2+\alpha_3)|} |A_{2,4}|.
$$

Using this we get that

$$\begin{aligned}
f^-(A_{2,4}, A_{4,2}) = & \frac{t_2 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
& + \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
& + \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& - \frac{t_2 t_4}{(t_1+t_4)(t_1+t_2)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
& - \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
& - \frac{t_2 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_3)} \\
& - \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_4)} \cdot \frac{\cos(\alpha_2+\alpha_3)}{|\sin(\alpha_2+\alpha_3)|} \\
& - \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_3)} \cdot \frac{\cos(\alpha_2+\alpha_3)}{|\sin(\alpha_2+\alpha_3)|} > 0.
\end{aligned}$$

<u>$f^-(A_{4,1}, A_{1,4})$ :</u>

First, we convert all $P_{4,1}$ into $A_{4,1}$, by using Lemma 3.2.2. We then get that

$$P_{4,1} = \frac{\cos(\alpha_4)}{\sin(\alpha_4)} A_{4,1}.$$

Using this we get that

$$\begin{aligned}
f^-(A_{4,1}, A_{1,4}) = & \frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_3)\sin(\alpha_4)} \\
& + \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
& + \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)} \\
& - \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_3)} \\
& - \frac{t_3}{t_2+t_3} \cdot \frac{\cos(\alpha_1)}{\sin(\alpha_1)\sin(\alpha_3)} > 0.
\end{aligned}$$

<u>$f^-(A_{3,4}, A_{4,3})$ :</u>

First, we convert all $P_{3,4}$ into $A_{4,3}$, by using Lemma 3.2.2. We then get that

$$P_{3,4} = \frac{\cos(\alpha_3)}{\sin(\alpha_3)} A_{3,4}.$$

Using this we get that

$$\begin{aligned}
f^-(A_{3,4}, A_{4,3}) = & \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_2)\sin(\alpha_4)} \\
& + \frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_3)\sin(\alpha_4)}
\end{aligned}$$

$$+\frac{t_2t_3}{(t_1+t_2)(t_2+t_3)}\cdot\frac{1}{\sin(\alpha_2)\sin(\alpha_3)}$$
$$-\frac{t_2t_4}{(t_2+t_3)(t_1+t_4)}\cdot\frac{1}{\sin(\alpha_2)\sin(\alpha_4)}$$
$$-\frac{t_4}{t_1+t_4}\cdot\frac{1}{\sin(\alpha_1)}\cdot\frac{\cos(\alpha_2)}{\sin(\alpha_2)}.$$

We have now looked at all terms containing $A_{n,m}$ and all of the terms containing $P_{n,m}$ in $\mathbf{R}_{1,2,3}$, where $n \neq m$. We are then left with the terms containing $P_{1,1}, P_{2,2}$ and $P_{3,3}$. First, we observe that all of these expressions are greater than zero, since $\cos(\alpha_{n,n}) = \cos(0) = 1$. Second, we know that all these are multiplied with factors that are greater than zero in $Q_1$. Since both the factors and $P_{1,1}, P_{2,2}$ and $P_{3,3}$ are strictly greater than zero, we know that all the terms containing $P_{1,1}, P_{2,2}$ and $P_{3,3}$ are strictly greater than zero.

By numerical analysis, we have now showed that all the negative terms in $\mathbf{R}_{1,2,3}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in Q_1$. This means that $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ when $\mathbf{x} \in Q_1$.

**Case 7: $\mathbf{x} \in Q_2$**

When $\mathbf{x} \in Q_2$, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ in this case.

- $\alpha_1 + \alpha_2 > \pi$.
- $\alpha_2 + \alpha_3 > \pi$.
- $\alpha_3 + \alpha_4 < \pi$.
- $\alpha_4 + \alpha_1 < \pi$.
- $\alpha_2 > \alpha_4$.
- $\alpha_1 = 2\pi - \alpha_2 - \alpha_3 - \alpha_4$.

The proofs of these bullet points are similar to the proofs of the properties in case 6.

The analysis of the terms in $\mathbf{R}_{1,2,3}$ will be quite similar as in case 6. The main difference is that we have to change the values of all $\alpha_i$, for $i = 1, \ldots, 4$, that we are testing for. We also observe that bullet point 2 causes $A_{1,3}$ and $A_{3,1}$ to change signs from case 6. We are handling this in the code in section A.2.3. Since the analysis is so similar to case 6, we will refer the reader to the tests in A.2.3. We will then see, by numerical testing, that the sum of all terms in $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ when $\mathbf{x} \in Q_2$.

**Case 8: $\mathbf{x} \in Q_3$**

When $\mathbf{x} \in Q_3$, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3}(\mathbf{x}) \geq 0$ in this case.

- $\alpha_1 + \alpha_2 < \pi$.
- $\alpha_2 + \alpha_3 > \pi$.
- $\alpha_3 + \alpha_4 > \pi$.
- $\alpha_4 + \alpha_1 < \pi$.
- $\alpha_3 > \alpha_1$.
- $\alpha_4 = 2\pi - \alpha_1 - \alpha_2 - \alpha_3$.

The proofs of these bullet points are similar to the proofs of the properties in case 6.

The analysis of the terms in $\mathbf{R}_{1,2,3}$ will be quite similar as in case 6. The main difference is that we have to change the values of all $\alpha_i$, for $i = 1, \ldots, 4$, that we are testing for. We also observe that bullet point 1 and 2 causes $A_{1,3}$, $A_{3,1}$, $A_{2,4}$ and $A_{4,2}$ to change signs from case 6. We are handling this in the code in section A.2.3. Since the analysis is so similar to case 6, we will refer the reader to the tests in A.2.3. We will then see, by numerical testing, that the sum of all terms in $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ when $\mathbf{x} \in Q_3$.

**Case 9: $\mathbf{x} \in Q_4$**

When $\mathbf{x} \in Q_4$, we have some special properties that we can use to prove that $\mathbf{R}_{1,2,3} \geq 0$ in this case.

- $\alpha_1 + \alpha_2 < \pi$.
- $\alpha_2 + \alpha_3 < \pi$.
- $\alpha_3 + \alpha_4 > \pi$.
- $\alpha_4 + \alpha_1 > \pi$.
- $\alpha_4 > \alpha_2$.
- $\alpha_3 = 2\pi - \alpha_1 - \alpha_2 - \alpha_4$.

The proofs of these bullet points are similar to the proofs of the properties in case 6.

The analysis of the terms in $\mathbf{R}_{1,2,3}$ will be quite similar as in case 6. The main difference is that we have to change the values of all $\alpha_i$, for $i = 1, \ldots, 4$, that we are testing for. We also observe that bullet point 1 causes $A_{2,4}$ and $A_{4,2}$ to change signs from case 6. We are handling this in the code in section A.2.3. Since the analysis is so similar to case 6, we will refer the reader to the tests in A.2.3. We will then see, by numerical testing, that the sum of all terms in $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ when $\mathbf{x} \in Q_4$.

We have now gone through all nine cases, and in each case we have proven, with a combination of analytical and numerical analysis, that $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$. This means that $\mathbf{R}_{1,2,3}(\mathbf{x}) > 0$ for all $\mathbf{x} \in Q$, which again implies that

$$\mathcal{D}(w_1, w_2, w_3)(\mathbf{x}) > 0$$

for all $\mathbf{x} \in Q$.

We recall that our goal was to show that $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) \geq 0$, for $1 \leq i < j < k \leq 4$, and $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) > 0$ for one combination of $i, j, k$. So, it then remains to show that

$$\mathcal{D}(w_1, w_2, w_4)(\mathbf{x}), \mathcal{D}(w_1, w_3, w_4)(\mathbf{x}), \mathcal{D}(w_2, w_3, w_4)(\mathbf{x}) \geq 0,$$

for all $\mathbf{x} \in Q$. In Appendix B we make an analysis of the three remaining expressions $\mathbf{R}_{1,2,4}$, $\mathbf{R}_{1,3,4}$ and $\mathbf{R}_{2,3,4}$, and we then discover that all these expressions are strictly greater than zero. This means that $\mathcal{D}(w_1, w_2, w_4)(\mathbf{x})$, $\mathcal{D}(w_1, w_3, w_4)(\mathbf{x})$, $\mathcal{D}(w_2, w_3, w_4)(\mathbf{x}) > 0$ for all $\mathbf{x} \in Q$. We note that in all the cases when we are checking the intersection between the diagonals, we are able to prove analytically that $\mathcal{D}(w_i, w_j, w_k)(\mathbf{x}) > 0$ for $1 \leq i < j < k \leq 4$. This result will therefore become a theorem.

Theorem 3.2.3. *Let $Q \subset \mathbb{R}^2$ be a convex quadrilateral with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$, ordered anticlockwise, and assume that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are non-collinear. Furthermore, let $\mathbf{f}$ be a mean value mapping, and let $\bar{\mathbf{x}} \in Q$ be the intersection between the diagonals in $Q$. Then $\mathbf{f}$ is injective in $\bar{\mathbf{x}}$.*

*Proof.* See case 1 in section 3.2, appendix B.1, B.2 and B.3. ■

Note, the reason why we are doing the analysis of $\mathbf{R}_{1,2,4}$, $\mathbf{R}_{1,3,4}$ and $\mathbf{R}_{2,3,4}$ in Appendix B is that the analysis of these three expressions will be quite similar to the analysis of $\mathbf{R}_{1,2,3}$. We therefore think that it would be redundant to have a detailed analysis of these expressions in this thesis, so we will leave the detailed analysis in these cases to the reader to avoid too much repetition.

### 3.2.2 Step 2

It remains to show that $\mathcal{D}(\phi_i, \phi_j, \phi_k) \geq 0$ for all $1 \leq i < j < k \leq 4$ and $\mathcal{D}(\phi_i, \phi_j, \phi_k) > 0$ for some $1 \leq i < j < k \leq 4$, when $\mathbf{x} \in \partial Q$. If we assume that the numerical analysis in step 1 is correct, that is that $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) > 0$ for all $\mathbf{x} \in Q$ and for all $1 \leq i < j < k \leq 4$, then we know that $\mathcal{D}(\phi_i, \phi_j, \phi_k) \geq 0$, when $\mathbf{x} \in \partial Q$, for all $1 \leq i < j < k \leq 4$. This follows from the fact that $\mathcal{D}$ and $\phi_n$ are continuous on the whole of $\bar{Q}$. Since we already know that $\mathcal{D}(\phi_i, \phi_j, \phi_k) \geq 0$ for all $1 \leq i < j < k \leq 4$, we know from Theorem 2.2.2 that it is sufficient to find one combination of $i, j, k$ such that $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) > 0$ for all $\mathbf{x} \in \partial Q$.

First, we recall from section 2.1.2 that every $\mathbf{x} \in \partial Q$ can be written as

$$\mathbf{x} = \mu \mathbf{v}_l + (1 - \mu)\mathbf{v}_{l+1}, \tag{3.16}$$

for some $\mu \in [0, 1]$ and $l \in \{1, 2, 3, 4\}$. We now want to prove that when $\mathbf{x}$ is as in (3.16) for a given $l$, then $\mathcal{D}(\phi_{l-1}, \phi_l, \phi_{l+1}) > 0$. To simplify this, we will again look at

$\mathcal{D}(w_{l-1}, w_l, w_{l+1})$, but this time we need to use the form of $w_i$ that holds for all of $\bar{Q}$ (see equation (2.6)). We observe that

$$\mathcal{D}(w_{l-1}, w_l, w_{l+1}) = w_{l-1}\boldsymbol{\nabla} w_l \times \boldsymbol{\nabla} w_{l+1} + w_l \boldsymbol{\nabla} w_{l+1} \times \boldsymbol{\nabla} w_{l-1} + w_{l+1}\boldsymbol{\nabla} w_{l-1} \times \boldsymbol{\nabla} w_l. \tag{3.17}$$

We recall from section 2.1.2 that when $\mathbf{x} = \mu\mathbf{v}_l + (1-\mu)\mathbf{v}_{l+1}$, then $w_{l-1} = 0$. This means that we can rewrite (3.17) as

$$\begin{aligned}\mathcal{D}(w_{l-1}, w_l, w_{l+1}) &= w_l \boldsymbol{\nabla} w_{l+1} \times \boldsymbol{\nabla} w_{l-1} + w_{l+1}\boldsymbol{\nabla} w_{l-1} \times \boldsymbol{\nabla} w_l \\ &= w_l w_{l+1}\left(\frac{\boldsymbol{\nabla} w_{l+1}}{w_{l+1}} - \frac{\boldsymbol{\nabla} w_l}{w_l}\right) \times \boldsymbol{\nabla} w_{l-1}. \end{aligned} \tag{3.18}$$

We observe that (3.18) holds when $w_l, w_{l+1} \neq 0$. Note, that in this case we only need to show that $\mathcal{D}(w_{l-1}, w_l, w_{l+1}) \neq 0$, since this implies that $\mathcal{D}(w_{l-1}, w_l, w_{l+1}) > 0$. This follows from the fact that $\mathcal{D}$ and $w_i$ are continuous and $\mathcal{D}(w_{l-1}, w_l, w_{l+1}) > 0$ when $\mathbf{x} \in Q$. We will now find expressions for $\boldsymbol{\nabla} w_{l-1}$, $\boldsymbol{\nabla} w_l$ and $\boldsymbol{\nabla} w_{l+1}$, but before doing this we will introduce some notation:

- $\mathbf{d}_i = \mathbf{v}_i - \mathbf{x}$
- $r_i = \|\mathbf{v}_i - \mathbf{x}\|$
- $\mathbf{e}_i = \frac{\mathbf{d}_i}{r_i}$
- $\mathbf{c}_{i,j} = r_i \mathbf{e}_j$
- $s_i = (r_{i+1}r_{i-1} - \mathbf{d}_{i+1} \cdot \mathbf{d}_{i-1})^{1/2}$
- $k_i = (r_i r_{i+1} + \mathbf{d}_i \cdot \mathbf{d}_{i+1})^{1/2}$
- $\mathbf{g}_{i,j} = \begin{pmatrix} x(\mathbf{d}_i^x + \mathbf{d}_j^x) \\ y(\mathbf{d}_i^y + \mathbf{d}_j^y) \end{pmatrix}$, where $(x, y) = \mathbf{x}$ and $(\mathbf{d}_i^x, \mathbf{d}_i^y) = \mathbf{d}_i$

Then we have that

$$\boldsymbol{\nabla} w_i = \boldsymbol{\nabla} s_i k_{i+1} k_{i+2} + s_i(\boldsymbol{\nabla} k_{i+1} k_{i+2} + k_{i+1}\boldsymbol{\nabla} k_{i+2}),$$

where

$$\boldsymbol{\nabla} s_i = \frac{1}{2s_i}\Big(\mathbf{g}_{i-1,i+1} - \mathbf{c}_{i+1,i-1} - \mathbf{c}_{i-1,i+1}\Big),$$

and

$$\boldsymbol{\nabla} k_i = \frac{1}{2k_i}\Big(-\mathbf{g}_{i,i+1} - \mathbf{c}_{i,i+1} - \mathbf{c}_{i+1,i}\Big),$$

for $i = l-1, l, l+1$.

We know that $\mathcal{D}(w_{l-1}, w_l, w_{l+1}) \neq 0$ if $\left(\frac{\boldsymbol{\nabla} w_{l+1}}{w_{l+1}} - \frac{\boldsymbol{\nabla} w_l}{w_l}\right) \times \boldsymbol{\nabla} w_{l-1} \neq 0$. This is the case if there does not exist an $a \in \mathbb{R}\backslash\{0\}$ such that

$$\frac{\boldsymbol{\nabla} w_{l+1}}{w_{l+1}} - \frac{\boldsymbol{\nabla} w_l}{w_l} = a\boldsymbol{\nabla} w_{l-1},$$

for all $\mu \in (0,1)$. We will now check if this is the case.

$$\begin{aligned}\frac{\nabla w_{l+1}}{w_{l+1}} - \frac{\nabla w_l}{w_l} - a\nabla w_{l-1} &= \frac{\nabla s_{l+1}}{s_{l+1}} + \frac{\nabla k_{l-1}}{k_{l-1}} - \frac{\nabla s_l}{s_l} - \frac{\nabla k_{l+1}}{k_{l+1}} \\ &\quad - a \cdot (k_l k_{l+1} \nabla s_{l-1} + s_{l-1}(k_{l+1}\nabla k_l + k_l \nabla k_{l+1})) \\ &= -\frac{\nabla s_l}{s_l} + \frac{\nabla k_{l-1}}{k_{l-1}} - a'\nabla s_{l-1} \\ &\quad - s_{l-1}((k_{l+1} a \nabla k_l + a'' \nabla k_{l+1})), \end{aligned} \tag{3.19}$$

where $a' = k_l k_{l+1} a - \frac{s_{l-1}}{s_{l+1}^2}$ and $a'' = k_l a + (\frac{1}{s_{l-1}k_{l+1}})$. It turns out that (3.19) is different from zero for all $a \in \mathbb{R}\backslash\{0\}$ and $\mu \in (0,1)$. You can see this if you write out (3.19) and use the fact that $\mathbf{d}_i \neq b \cdot \mathbf{d}_j$, for all $b \in \mathbb{R}$ and $\mu \in (0,1)$, when $(i,j) \neq (l, l+1)$. Note, that this follows from the fact that we have assumed that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are non-collinear.

### 3.2.3 Conclusion

We have in two steps checked if $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) \geq 0$ for all $1 \leq i < j < k \leq 4$ and $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) > 0$ for some $1 \leq i < j < k \leq 4$, for all $\mathbf{x} \in \bar{Q}$. In step 1 we checked for all $\mathbf{x} \in Q$, and we were able to prove analytically that $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\bar{\mathbf{x}}) > 0$ for all $1 \leq i < j < k \leq 4$ when $\bar{\mathbf{x}}$ is the intersection between the diagonals in $Q$. In all the other cases in step 1 we relied on some numerical analyses, but these analyses indicated that $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x}) > 0$, for all $1 \leq i < j < k \leq 4$, when $\mathbf{x} \in Q$. In step 2 we proved that $\mathcal{D}(\phi_{l-1}, \phi_l, \phi_{l+1})(\mathbf{x}) > 0$, for some $l \in \{1,2,3,4\}$, when $\mathbf{x} \in \partial Q$, given that the numerical analysis in step 1 was correct. With these assumptions we can therefore conclude that mean value mappings between convex quadrilaterals are injective.

## 3.3 Injectivity of mean value mappings between convex pentagons

In [6], Floater and Kosinka showed by example that mean value mappings between convex pentagons were not injective. We will now present an example of our own that shows that mean value mappings between convex pentagons are not injective. So, let $P$ be a strictly convex polygon with five vertices, $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_5$, ordered anticlockwise. Furthermore, let $\phi_1$, $\phi_2$, ..., $\phi_5$ be the mean value coordinates as described in (2.4). We recall from section 2.2.1, Theorem 2.2.3 that a necessary condition for injectivity was that

$$\sum_{r \leq i < s \leq j < t \leq k < n+r} \mathcal{D}(\phi_i, \phi_j, \phi_k) \geq 0 \quad \text{in } P,$$

for $r, s, t$ satisfying $1 \leq r < s < t \leq n$. Note that in this case, $n = 5$. If we can find an example where this condition is not fulfilled, then we have proven that mean value mappings between convex pentagons are not necessarily injective. To find one example of this we wrote a code that first generated a random, convex pentagon. Then we generated some points $\mathbf{x}$ in that given pentagon, and then we checked if $\sum_{r \leq i < s \leq j < t \leq k < 5+r} \mathcal{D}(\phi_i, \phi_j, \phi_k) \geq 0$, for $r, s, t$ satisfying $1 \leq r < s < t \leq 5$. The reader may find the code that we used to find a counterexample in this git repository[1]. By using this code, we were able to find an example where the condition in Theorem 2.2.3 was not fulfilled. We will now present this example.

[1] https://github.uio.no/agnethhk/Tests-mean-value-mappings.git

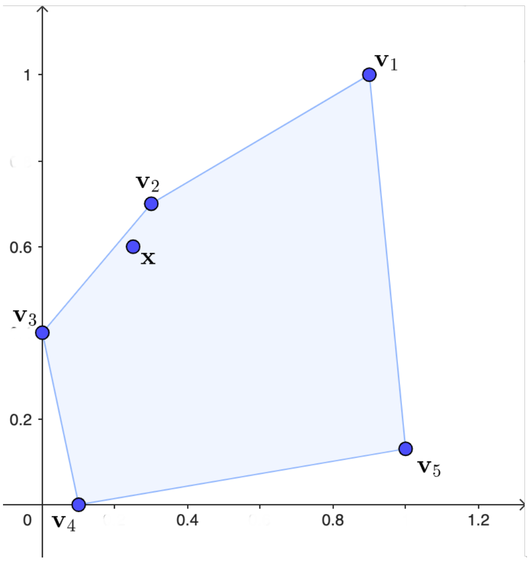


Figure 3.4: Pentagon with vertices $\mathbf{v}_1 = (0.9, 1)$, $\mathbf{v}_2 = (0.3, 0.7)$, $\mathbf{v}_3 = (0, 0.4)$, $\mathbf{v}_4 = (0.1, 0)$ and $\mathbf{v}_5 = (1, 0.13)$, where $\mathbf{x} = (0.25, 0.6)$

Example 3.3.1. Let $P \subset \mathbb{R}^2$ be a pentagon with vertices $\mathbf{v}_1 = (0.9, 1)$, $\mathbf{v}_2 = (0.3, 0.7)$, $\mathbf{v}_3 = (0, 0.4)$, $\mathbf{v}_4 = (0.1, 0)$ and $\mathbf{v}_5 = (1, 0.13)$ (see figure 3.4). Moreover, let $\mathbf{x} = (0.25, 0.6) \in P$. By evaluating $\mathcal{D}(\phi_i, \phi_j, \phi_k)(\mathbf{x})$ for the mean value coordinates of $P$ we obtain that

$$\mathcal{D}(\phi_1, \phi_2, \phi_5)(\mathbf{x}) = -19.61590501,$$
$$\mathcal{D}(\phi_1, \phi_3, \phi_5)(\mathbf{x}) = 7.01491292,$$
$$\mathcal{D}(\phi_1, \phi_4, \phi_5)(\mathbf{x}) = 0.68785952.$$

We then observe that

$$\mathcal{D}(\phi_1, \phi_2, \phi_5)(\mathbf{x}) + \mathcal{D}(\phi_1, \phi_3, \phi_5)(\mathbf{x}) + \mathcal{D}(\phi_1, \phi_4, \phi_5)(\mathbf{x}) = -11.913132561694452.$$

As this value is negative, it violates the necessary condition of Theorem 2.2.3 for $n = 5$, $r = 1$, $s = 2$ and $t = 5$. This again implies that mean value mappings between strictly convex pentagons are not necessarily injective.

# Chapter 4

# Concluding remarks

In this thesis we have investigated when mean value mappings between convex polygons are injective. We recall that it was already shown by Floater and Kosinka [6] that all barycentric mappings between triangles are injective, and they also found examples where mean value mappings between convex pentagons were not injective, but it still remained to find proof of whether mean value mappings between convex quadrilaterals are injective. In this thesis we have been able to present an alternative proof of why barycentric mappings between triangles in the plane are injective, and we have also found a new example for why mean value mappings between convex pentagons in the plane are not necessarily injective.

However, we recall that our main goal was to show that mean value mappings between convex quadrilaterals in the plane are injective. In the case where $\mathbf{x}$ is the intersection between the diagonals, we have been able to prove injectivity analytically, but in all other cases we have relied on some numerical analysis. This analysis is still a step forward from what was done in [6], and as we discuss in A.1.1, the numerical analysis is likely good enough to conclude that mean value mappings between convex quadrilaterals in the plane are injective. It is also worth noticing that we have been able to make an outline of a proof that only depends on the angles $\alpha_i$ and not the sizes of $A_{n,m}$ and $P_{n,m}$. Therefore, it might be possible to use this outline to make an analytical proof later on. During this thesis we discovered some properties and we got some ideas on how it might be possible to make an analytical proof. We will state these potential improvements and ideas:

1. Check if the analysis becomes more obvious if we rewrite some of the expressions. In this thesis we have been operating with $\alpha_i$ and $\frac{\alpha_i}{2}$, for $i = 1, 2, 3, 4$, but it might be easier to analyse if we use the formulas for double angles on $\sin(\alpha_i)$ and $\cos(\alpha_i)$, so we only need to operate with $\frac{\alpha_i}{2}$. In addition, it would be interesting to see if we could find some more trigonometric properties that we could use in the analysis. For instance, in this analysis, we could have used the fact that $t_1 t_2 = \frac{\sin(\alpha_1/2)}{\cos(\alpha_1/2)} \cdot \frac{\cos(\alpha_1/2)}{\sin(\alpha_1/2)} = 1$, when $\mathbf{x} \in d_1$. The analysis might become easier if we implement these changes. In addition, there might be possible to find even more properties that could be used to simplify the analysis, so it would be interesting to investigate this further.

2. Check if the positive terms in $f^-$ dominates the negative terms by using the properties we derived in case 2-9 in step 1. Especially for the diagonals, it might be possible to show analytically that all the positive terms in $f^-$ dominates all the negative terms. If we for instance take $f^-(A_{1,2}, A_{2,1})$ in $\mathbf{R}_{1,2,3}$ for $\mathbf{x} \in d_1$, we

can observe that $\frac{t_2 t_4}{(t_1+t_4)(t_2+t_3)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)}$ dominates $-\frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin(\alpha_1)\sin(\alpha_4)}$, since $\alpha_1 > \alpha_3$ when $\mathbf{x} \in d_1$. Since we are trying to analyse a combination of dot products and cross products, some analyses become a bit tricky, but by using some trigonometric properties, as for instance formulas for double angles, it might be possible to get an analytical result with this method.

3. Check if it is possible to find out where $f^-$ has its minimum for different $A_{n,m}$, where $n, m \in \{1, 2, 3, 4\}$. If we can find out where $f^-(A_{n,m}, A_{m,n})$ has its minimum, when $A_{n,m} > 0$, we can check if $f^-$ is positive in this case, and we would then be able to prove that mean value mappings are injective between convex quadrilaterals. It might be a bit tricky to find the minimum of $f^-$ analytically, especially for $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$, but it might be easier to find the minimum of $f^-$ on the diagonals, since $f^-$ only depends on two variables in this case, so this could be a good starting point.

It is also of great interest to analyse closer how $f^-$ behaves. By doing this it might be possible to prove that the numerical method used in this thesis is good enough to conclude that mean value mappings between convex quadrilaterals in the plane are injective.

## Future research

In the future it might be of interest to check if it is possible to make a completely analytical proof that mean value mappings between convex quadrilaterals are injective. In addition, it would be interesting to investigate if mean value mappings between non-convex quadrilaterals are injective. The main advantage with mean value coordinates versus for instance Wachspress coordinates, is that mean value coordinates are also well-defined for star-shaped polygons, so it could therefore be of interest to do some analysis to check if mean value mappings between star-shaped quadrilaterals could be injective.

Appendix A

# Presentation of numerical method and test results for $\mathbf{R}_{1,2,3}$

In this appendix we will present the code and the tests from the numerical analysis we presented in section 3.2. This appendix is structured as follows: Section A.1 presents the numerical method we have used in our tests and section A.2 presents the results of the numerical test runs.

## A.1 Numerical test method

Before we present the numerical method that we have used to analyse the different expressions in section 3.2, we will recall some notation from that section:

- $\mathbf{R}_{1,2,3} = \mathbf{R}_2 \times \mathbf{R}_3 + \mathbf{R}_3 \times \mathbf{R}_1 + \mathbf{R}_1 \times \mathbf{R}_2$, where $\mathbf{R}_i$ is as in (2.7).
- $d_i$ is the line from $\mathbf{v}_i$ to the intersection between the diagonals, for $i = 1, 2, 3, 4$.
- $A_{n,m} = \frac{(\mathbf{v}_n - \mathbf{x})}{\|(\mathbf{v}_n - \mathbf{x})\|^2} \times \frac{(\mathbf{v}_m - \mathbf{x})}{\|(\mathbf{v}_m - \mathbf{x})\|^2}$, for $n, m \in \{1, 2, 3, 4\}$.
- $P_{n,m} = \frac{(\mathbf{v}_n - \mathbf{x})}{\|(\mathbf{v}_n - \mathbf{x})\|^2} \cdot \frac{(\mathbf{v}_m - \mathbf{x})}{\|(\mathbf{v}_m - \mathbf{x})\|^2}$, for $n, m \in \{1, 2, 3, 4\}$.
- $f^-(x, y)$ is a function that return the factors in front of expression x minus the factors in front of expression y.

### A.1.1 Idea behind the numerical method

Our goal is to make a numerical method that can check if $f^-$ is positive. Before we describe our method, we observe that $f^-(A_{n,m}, A_{m,n})$, for $n, m \in \{1, 2, 3, 4\}$, is a continuous function. This follows from the fact that $f^-(A_{n,m}, A_{m,n})$ is a sum of terms containing a combination of trigonometric functions. Since it is a well known fact that trigonometric functions are continuous, and that the product and quotient of continuous functions are continuous, it follows that $f^-(A_{n,m}, A_{m,n})$ is continuous for all $n, m \in \{1, 2, 3, 4\}$.

We will begin our numerical analysis by writing a function that finds the numerical derivative to $f^-$. We will then use this function to calculate the numerical derivative to $f^-$ for given values of $\alpha_i$, for $i = 1, \ldots, 4$. We will then for a given $i$ change $\alpha_i$ with some small interval, and then save the derivatives in a list. When we have calculated three consecutive, numerical derivatives $g_1$, $g_2$ and $g_3$ in the points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, we will check if $g_1$, $g_2 \leq 0$ and $g_3 \geq 0$. If this is the case, we know that in the interval $[\mathbf{p}_2, \mathbf{p}_3]$,

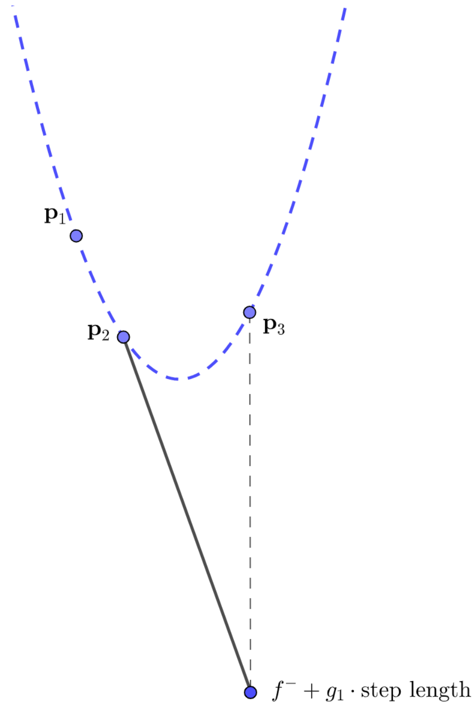


Figure A.1: Visualization of how we use the numerical derivative to check if $f^-$ is positive

$f^-$ will have at least one minima, since we know that $f^-$ is a continuous function. If we make an assumption that $f^-$ has exactly one minima in the interval $[\mathbf{p}_2, \mathbf{p}_3]$, and assume that the derivative of $f^-$ is increasing in the interval $[\mathbf{p}_1, \text{local minima}]$, then we can make a numerical test that guarantees that $f^-$ is greater than or equal to zero. We note that if this assumption holds, then $g_1 \leq g_2$ when $g_1$, $g_2 \leq 0$ and $g_3 \geq 0$. Then we know that the minima have to be in the interval $[\mathbf{p}_2, \mathbf{p}_3]$. In addition, we know from assumption that the derivative in every point in this interval must be greater than $g_1$. Therefore, if $f^-(A_{n,m}, A_{m,n})(\mathbf{p}_2) \geq g_1 \cdot |\mathbf{p}_3 - \mathbf{p}_2|$, for some $n, m \in \{1, 2, 3, 4\}$ that satisfy $A_{n,m} > 0$, this implies that the minima of $f^-$ in the interval $[\mathbf{p}_2, \mathbf{p}_3]$ is greater than or equal to zero. You can see an illustartion of this in figure A.1. In the code we will actually test if $f^-$ is strictly greater than zero, since we need this to be fulfilled for at least one combination of $i, j, k$, where $1 \leq i < j < k \leq 4$. We will also make some modifications to this idea when we implement the code, in case our assumption does not hold. We will explain these modifications in A.1.2.

The question now is if we can guarantee that all intervals on the form $[\mathbf{p}_i, \mathbf{p}_{i+2}]$, where $g_i, g_{i+1} \leq 0$ and $g_{i+2} \geq 0$, have exactly one minima, and that the derivative in the interval $[\mathbf{p}_i,$ local minima$]$ is increasing. We know that all the expressions that we are going to analyse are a combination of trigonometric functions, so intuitively it seems reasonable to assume that all intervals on the form $[\mathbf{p}_i, \mathbf{p}_{i+2}]$, where $g_i, g_{i+1} \leq 0$ and $g_{i+2} \geq 0$, have exactly one minima, and that the derivative in the interval $[\mathbf{p}_i,$ local minima$]$ is increasing, if we choose small enough intervals. Nevertheless, it is easier said than done to prove this analytically. We will therefore in this thesis assume that this claim holds. In figure A.2 you can see a plot of $f^-(A_{1,2}, A_{2,1})$ in $\mathbf{R}_{1,2,3}$, and we observe that this plot seems to support our claim.

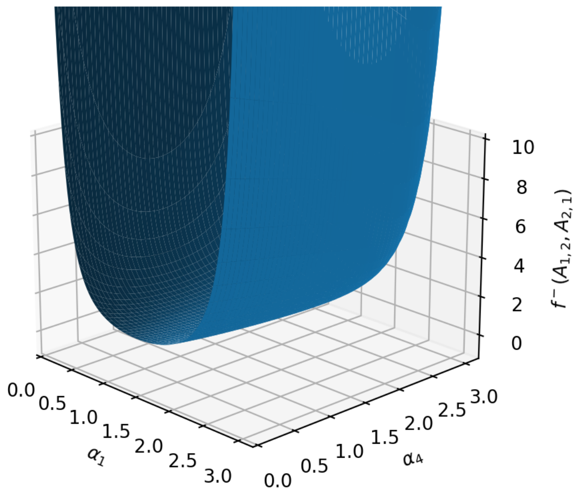


Figure A.2: Plot of $f^-(A_{1,2}, A_{2,1})$ in $\mathbf{R}_{1,2,3}$

### A.1.2 Code

All code is written in Python version 3.8.5. You can find the code below, and the test runs will be documented in the section A.2.

For all code we begin with importing some functionality from NumPy, version 1.23.3:

```
import numpy as np
from numpy import cos, tan, sin, pi
```

Code for $\mathbf{x} \in \{d_1 \cup d_3\}$:
We start by defining two functions that calculate the numerical derivative of $f^-$ with respect to $\alpha_1$ and $\alpha_4$ respectively.

```
def derivative_a1(f, a4, a1_list):
    return (f(a1_list[1], a4)-f(a1_list[0], a4))/(a1_list[1]-a1_list[0])

def derivative_a4(f, a1, a4_list):
    return (f(a1, a4_list[1])-f(a1, a4_list[0]))/(a4_list[1]-a4_list[0])
```

Both functions use the first-order backward difference approximation (see e.g. [18]) to approximate the derivative of $f^-$ with respect to $\alpha_1$ and $\alpha_4$.

Second, we define a function that will append the the derivative we calculate to a list, while ensuring that the list only contains three consecutive derivatives.

```
def append_derivates(derivates, g):
    if len(derivates) < 3:
        derivates.append(g)
        return derivates
    else:
        del derivates[0]
        derivates.append(g)
        return derivates
```

Next, we will need a function that checks if the derivatives in the list satisfy the claim from A.1.1, and then returns the derivative we are going use. Note that when $g_2 \leq 0$ and

$g_3 \geq 0$, we will return $\min(g_1, g_2)$. According to the idea we explained earlier we should return $g_1$, but if we have not chosen a small enough interval, we might get some cases where $g_1, g_3 > 0$ and $g_2 < 0$. If this is the case, we still know that there exists at least one minima between the points with derivative $g_2$ and $g_3$, so in this case we would like our function to return $g_2$. In addition, we would like our function to return $g_2$ if it turns out that the derivative is not increasing in the interval $[\mathbf{p}_1$, local minima$]$, as we have assumed, so to make our code a bit more robust, we will check if $g_2 < g_1$.

If all the derivatives in the list are negative, we assume that $f^-$ does not have a minima in this interval, and we will then return $g_3$. If both $g_2, g_3 \geq 0$ we have that $f^-$ is increasing, so in this case we are only interested in checking the function value of $f^-$, so our function will in this case return zero.

```
def find_derivative(derivates):
    if len(derivates) == 1:
        g1 = derivates[0]; g2 = 0; g3 = 0;
    elif len(derivates) == 2:
        g1 = derivates[0]; g2 = derivates[1]; g3 = 0;
        if g1 <=0 and g2 >= 0:
            return g1
        elif g2<=0:
            return g2
        else:
            return 0
    else:
        g1 = derivates[0]; g2 = derivates[1]; g3 = derivates[2];

    if g2<=0 and g3>=0:
        if g1 < g2:
            return g1

        else:
            return g2

    elif g1<0 and g2<0 and g3<0:
        return g3

    else:
        return 0
```

To save computation time, we will write two functions that partition problematic intervals into smaller intervals. We will call these functions `warninga1` and `warninga4`. These functions will be called if $f^- + \min(0, g_1, g_2, g_3) \cdot \texttt{step size} \leq 0$. We will then partition the given interval into smaller intervals, and check if the claim is still invalidated.

```
def warninga1(f, a1_start, a1_end, a4, step = 100):
    a1_list = np.linspace(a1_start, a1_end, step)
    stepsize = (a1_end-a1_start)/(step-1)
    derivatives = []

    foundException = False
    errorMessage = []
    for i in range(1, len(a1_list)-1):
        derivate = derivative_a1(f, a4, [a1_list[i-1], a1_list[i+1]])
        derivatives = append_derivates(derivatives, derivate)
        minDerivative = find_derivative(derivatives)
        if f(a1_list[i-1], a4)+minDerivative*stepsize <= 0:
            if step <= 800:
```

```
                foundException, errorMessage = warninga1(f, a1_start, a1_end, a4, 2*step)
                return foundException, errorMessage
            else:
                errorMessage.append(f'{minDerivative*stepsize} {f(a1_list[i], a4)}, {a1_list[i]}, {a4}')
                foundException = True
    return foundException, errorMessage

def warninga4(f, a4_start, a4_end, a1, step = 100):
    a4_list = np.linspace(a4_start, a4_end, step)
    stepsize = (a4_end-a4_start)/(step-1)
    derivatives = []

    foundException = False
    errorMessage = []
    for i in range(1, len(a4_list)-1):
        derivate = derivative_a4(f, a1, [a4_list[i-1], a4_list[i+1]])
        derivatives = append_derivates(derivatives, derivate)
        minDerivative = find_derivative(derivatives)
        if f(a1, a4_list[i-1])+minDerivative*stepsize <= 0:
            if step <= 800:
                foundException, errorMessage = warninga4(f, a4_start, a4_end, a1, 2*step)
                return foundException, errorMessage
            else:
                errorMessage.append(f'{minDerivative*stepsize} {f(a1, a4_list[i])}, {a1}, {a4_list[i]}')
                foundException = True
    return foundException, errorMessage
```

Last, we need two functions that check if $f^- \geq$ `find_derivative`$\cdot$`step size`, where `step size` is the length between each $\alpha_i$, for $i = 1, 4$ respectively. This function will raise a warning if we might have some problems when we approach the limits of $\alpha_1$ or $\alpha_4$. If these warnings are printed in the terminal we will check the limits analytically in the test section.

```
def a1_check(f, step, a4_list, diagonal):
    for a4 in a4_list[1:-1]:
        if diagonal == 'd1':
            a1_list = np.linspace(pi-a4, pi, step)
        elif diagonal == 'd3':
            a1_list = np.linspace(0, pi-a4, step)

        first_i = True
        last_i = True

        derivatives = []
        stepsize = (a1_list[-1]-a1_list[0])/(step-1)
        for i in range(2, len(a1_list[2:-1])):
            if a1_list[i] != 0 and a1_list[i] != pi:
                derivate = derivative_a1(f, a4, [a1_list[i-1], a1_list[i+1]])
                derivatives = append_derivates(derivatives, derivate)
                minDerivative = find_derivative(derivatives)
                if f(a1_list[i-1], a4)+minDerivative*stepsize <= 0:
                    foundException, errorMessage = warninga4(f, a1_list[i-1], a1_list[i+1], a4)
                    if foundException:
                        if i == 2:
                            if first_i:
```

```
                                print("For x in", diagonal, ": Check limits when a1 -> 0")
                                first_i = False
                        elif i == len(a1_list[2:])-1:
                            if last_i:
                                print("For x in", diagonal, ": Check limits when a1 -> pi")
                                last_i = False
                        elif a4 == a4_list[1]:
                             print("For x in", diagonal, ": Check limits when a4 -> 0")
                             break
                        elif a4 == a4_list[-2]:
                             print("For x in", diagonal, ": Check limits when a4 -> pi")
                             break
                        else:
                            print(errorMessage)
```

```
def a4_check(f, step, a1_list, diagonal):
    for a1 in a1_list[1:-1]:
        if diagonal == 'd1':
            a4_list = np.linspace(pi - a1, pi, step)
        elif diagonal == 'd3':
            a4_list = np.linspace(0, pi-a1, step)

        first_i = True
        last_i = True

        derivatives = []
        stepsize = (a4_list[-1]-a4_list[0])/(step-1)
        for i in range(2, len(a4_list[2:-1])):
            if a4_list[i] != 0 and a4_list[i] != pi:
                derivate = derivative_a4(f, a1, [a4_list[i-1], a4_list[i+1]])
                derivatives = append_derivates(derivatives, derivate)
                minDerivative = find_derivative(derivatives)
                if f(a1, a4_list[i-1])+minDerivative*stepsize <= 0:
                    foundException, errorMessage = warninga1(f, a4_list[i-1], a4_list[i+1], a1)
                    if foundException:
                         if i == 2:
                            if first_i:
                                print("For x in", diagonal, ": Check limits when a1 -> 0")
                                first_i = False
                         elif i == len(a1_list[2:])-1:
                             if last_i:
                                print("For x in", diagonal, ": Check limits when a1 -> pi")
                                last_i = False
                         elif a4 == a4_list[1]:
                              print("For x in", diagonal, ": Check limits when a4 -> 0")
                              break
                         elif a4 == a4_list[-2]:
                              print("For x in", diagonal, ": Check limits when a4 -> pi")
                              break
                         else:
                             print(errorMessage)
```

Now it remains to make a function that tests for different values of $\alpha_1$ and $\alpha_4$.

```
def testd1d3(f, twoAngles = False):
    step = 1000
    alpha1_list = np.linspace(0, pi, step)

    a4_check(f, step, alpha1_list, 'd1')

    alpha4_list = np.linspace(0, pi, step)
    a1_check(f, step, alpha4_list, 'd1')

    alpha1_list = np.linspace(0, pi, step)
    if twoAngles:
        a4_check(lambda a1, a4: -f(a1, a4), step, alpha1_list, 'd3')
    else:
        a4_check(f, step, alpha1_list, 'd3')

    alpha4_list = np.linspace(0, pi, step)
    if twoAngles:
        a1_check(lambda a1, a4: -f(a1, a4), step, alpha4_list, 'd3')
    else:
        a1_check(f, step, alpha4_list, 'd3')

    return 'Done'
```

Code for $\mathbf{x} \in \{d_2 \cup d_4\}$:
We will reuse some of the code from when $\mathbf{x} \in \{d_1 \cup d_3\}$, namely the functions `derivative_a1`, `append_derivative` and `find_derivative`. So, the first code we need for $\mathbf{x} \in \{d_2 \cup d_4\}$ is a way of finding the numerical derivative with respect to $\alpha_2$.

```
def derivative_a2(f, a1, a2_list):
    return (f(a2_list[1], a1)-f(a2_list[0], a1))/(a2_list[1]-a2_list[0])
```

As for the case where $\mathbf{x} \in \{d_1 \cup d_3\}$, we need two warning functions.

```
def warninga1(f, a1_start, a1_end, a2, step = 100):
    a1_list = np.linspace(a1_start, a1_end, step)
    stepsize = (a1_end-a1_start)/(step-1)
    derivatives = []

    foundException = False
    errorMessage = []
    for i in range(1, len(a1_list)-1):
        derivate = derivative_a1(f, a2, [a1_list[i-1], a1_list[i+1]])
        derivatives = append_derivates(derivatives, derivate)
        minDerivative = find_derivative(derivatives)
        if f(a1_list[i-1], a2)+minDerivative*stepsize <= 0:
            if step <= 800:
                foundException, errorMessage = warninga1(f, a1_start,
    a1_end, a2, 2*step)
                return foundException, errorMessage
            else:
                errorMessage.append(f'{minDerivative*stepsize} {f(a1_list
    [i], a2)}, {a1_list[i]}, {a2}')
                foundException = True
    return foundException, errorMessage

def warninga2(f, a2_start, a2_end, a1, step=100):
    a2_list = np.linspace(a2_start, a2_end, step)
    stepsize = (a2_end-a2_start)/(step-1)
    derivatives = []

```

```
    foundException = False
    errorMessage = []
    for i in range(1, len(a2_list)-1):
        derivate = derivative_a2(f, a1, [a2_list[i-1], a2_list[i+1]])
        derivatives = append_derivates(derivatives, derivate)
        minDerivative = find_derivative(derivatives)
        if f(a1, a2_list[i-1])+minDerivative*stepsize <= 0:
            if step <= 800:
                foundException, errorMessage = warninga2(f, a2_start, a2_end, a1, 2*step)
                return foundException, errorMessage
            else:
                errorMessage.append(f'{minDerivative*stepsize} {f(a1, a2_list[i])}, {a1}, {a2_list[i]}')
                foundException = True
    return foundException, errorMessage
```

Next, we will need a way of checking if $f^{-} \geq$ `find_derivative` $\cdot$ `step size`.

```
def a1_check(f, step, a2_list, diagonal):
    first_i = True
    last_i = True

    for a2 in a2_list[1:-1]:
        if diagonal == 'd2':
            a1_list = np.linspace(pi-a2, pi, step)
        elif diagonal == 'd4':
            a1_list = np.linspace(0, pi-a2, step)

        derivatives = []
        stepsize = (a1_list[-1]-a1_list[0])/(step-1)
        for i in range(2, len(a1_list[2:-1])):
            if a1_list[i] != 0 and a1_list[i] != pi:
                derivate = derivative_a1(f, a2, [a1_list[i-1], a1_list[i+1]])
                derivatives = append_derivates(derivatives, derivate)
                minDerivative = find_derivative(derivatives)
                if f(a1_list[i-1], a2)+minDerivative*stepsize <= 0:
                    foundException, errorMessage = warninga1(f, a1_list[i-1], a1_list[i+1], a2)
                    if foundException:
                        if i == 2:
                            if first_i:
                                print("For x in", diagonal, ": Check limits when a1 -> 0")
                                first_i = False
                        elif i == len(a1_list[2:])-1:
                            if last_i:
                                print("For x in", diagonal, ": Check limits when a1 -> pi")
                                last_i = False
                        elif a2 == a2_list[1]:
                            print("For x in", diagonal, ": Check limits when a2 -> 0")
                            break
                        elif a2 == a2_list[-2]:
                            print("For x in", diagonal, ": Check limits when a2 -> pi")
                            break
                        else:
                            print(errorMessage)
```

```python
def a2_check(f, step, a1_list, diagonal):
    first_i = True
    last_i = True
    for a1 in a1_list[1:-1]:
        if diagonal == 'd2':
            a2_list = np.linspace(pi - a1, pi, step)
        elif diagonal == 'd4':
            a2_list = np.linspace(0, pi-a1, step)

        derivatives = []
        stepsize = (a2_list[-1]-a2_list[0])/(step-1)
        for i in range(2, len(a2_list[2:-1])):
            if a2_list[i-1] != 0 and a2_list[i] != pi:
                derivate = derivative_a2(f, a1, [a2_list[i-1], a2_list[i+1]])
                derivatives = append_derivates(derivatives, derivate)
                minDerivative = find_derivative(derivatives)
                if f(a1, a2_list[i-1])+minDerivative*stepsize <= 0:
                    foundException, errorMessage = warninga2(f, a2_list[i-1], a2_list[i+1], a1)
                    if foundException:
                        if i == 2:
                            if first_i:
                                print("For x in", diagonal, ": Check limits when a2 -> 0")
                                first_i = False
                        elif i == len(a2_list[2:])-1:
                            if last_i:
                                print("For x in", diagonal, ": Check limits when a2 -> pi")
                                last_i = False
                        elif a1 == a1_list[1]:
                            print("For x in", diagonal, ": Check limits when a1 -> 0")
                            break
                        elif a1 == a1_list[-2]:
                            print("For x in", diagonal, ": Check limits when a1 -> pi")
                            break
                        else:
                            print(errorMessage)
```

Last, we will make a function that calls the functions `a1_check` and `a2_check` for different values of $\alpha_2$ and $\alpha_1$, respectively.

```python
def testd2d4(f, twoAngles = False):
    step = 1000
    alpha1_list = np.linspace(0, pi, step)

    a2_check(f, step, alpha1_list, 'd2')

    alpha2_list = np.linspace(0, pi, step)
    a1_check(f, step, alpha2_list, 'd2')

    alpha1_list = np.linspace(0, pi, step)
    if twoAngles:
        a2_check(lambda a1, a2: -f(a1, a2), step, alpha1_list, 'd4')
    else:
        a2_check(f, step, alpha1_list, 'd4')

```

```
    alpha2_list = np.linspace(0, pi, step)
    if twoAngles:
        a1_check(lambda a1, a2: -f(a1, a2), step, alpha2_list, 'd4')
    else:
        a1_check(f, step, alpha2_list, 'd4')

    return 'Done'
```

Code for $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$:
The code for $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$ will build on the same principles as the code for $\mathbf{x} \in \{d_1 \cup d_3\}$ and $\mathbf{x} \in \{d_2 \cup d_4\}$, but the code will be quite a lot longer than for the previous cases. Since these principles already have been described in detail for $\mathbf{x} \in \{d_1 \cup d_3\}$ and $\mathbf{x} \in \{d_2 \cup d_4\}$, we will skip commenting this case further. The reader may find the code that is used in this case in this git repository[1].

## A.2 Testing

All tests were done on an Intel Core i5 with 1.1 GHz and 4 Cores on a macOS Server with 8GB RAM.

### A.2.1 Runs for $\mathbf{x} \in d_1$ and $\mathbf{x} \in d_3$

Numerical analysis of $f^-(A_{4,1}, A_{1,4})$:
Code:

```
def testA41(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term2 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a4)**2)
    term3 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term4 = t3/(t2+t3)*cos(a4)/(sin(a4)**2)
    term5 = (t1*t3)/((t1+t4)*(t2+t3))*1/(sin(a1)*sin(a4))

    return term1 + term2+ term3 + term4 - term5

print(testd1d3(testA41))
```

Output: Done

Numerical analysis of $f^-(A_{1,2}, A_{2,1})$:
Code:

```
def testA12(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = t4*(t1-t2)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a4))
    term2 = (t1*t2)/((t1+t2)*(t1+t4)) *1/(sin(a1)**2)
    term3 = 1
    term4 = (t2*t4)/((t1+t4)*(t2+t3)) *1/(sin(a1)*sin(a4))
    term5 = (t1*t2)/((t1+t2)*(t2+t3)) *1/(sin(a1)**2)
    term6 = t4/(t1+t4)*cos(a1)/(sin(a1)*sin(a4))
    term7 = t2/(t1+t2)*cos(a1)/(sin(a1)**2)
```

[1] https://github.uio.no/agnethhk/Tests-mean-value-mappings.git

```
    term8 = (t1*t2)/((t2+t3)*(t1+t4)) *1/(sin(a1)**2)
    term9 = t2/(t2+t3)*cos(a1)/(sin(a1)**2)
    term10 = t1/(t1+t4)*cos(a1)/(sin(a1)**2)
    term11 = t1/(t1+t2)*cos(a1)/(sin(a1)**2)
    return term1 + term2 + term3 + term4 - term5 + term6 + term7 - term8
    - term9 - term10 - term11

print(testd1d3(testA12))
```

Output: Done

Numerical analysis of $f^{-}(A_{2,3}, A_{3,2})$:
Code:

```
def testA23(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)**2)
    term2 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)**2)
    term4 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term5 = 1
    term6 = cos(a1)/(sin(a1)**2)*(t1/(t1+t4)+t2/(t1+t2)+t2/(t2+t3))
    term7 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)**2)
    term8 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term9 = cos(a1)/sin(a1)*(t3/(t2+t3)*1/sin(a4)+t1/(t1+t2)*1/sin(a1))

    return term1 + term2 + term3 + term4 + term5 + term6 - term7 -term8 -
     term9

print(testd1d3(testA23))
```

Output: Done

Numerical analysis of $f^{-}(A_{2,4}, A_{4,2})$:
Code:

```
def testA24(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))
    term2 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term3 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term4 = (t2*t4)/((t1+t4)*(t1+t2))*1/(sin(a1)*sin(a4))
    term5 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))
    term6 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term7 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))*t4/(t1+t4)
    term8 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))*t3/(t2+t3)

    return term1 + term2 + term3 - term4 - term5 - term6 - term7 - term8

print(testd1d3(testA24, True))
```

Output: Done

Numerical analysis of $f^-(A_{3,4}, A_{4,3})$:
Code:

```
def testA34(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term2 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a4)**2)
    term3 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term4 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))
    term5 = cos(a4)/(sin(a4)**2)*t4/(t1+t4)

    return term1 + term2 + term3 - term4 - term5

print(testd1d3(testA34))
```

Output: Done

Numerical analysis of $f^+(P_{1,3})$:
Code:

```
def testP13(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = t2/(t2+t3)*1/sin(a1)
    term2 = t1/(t1+t4)*1/sin(a1)
    term3 = 1/sin(a1)
    term4 = t3/(t2+t3)*1/sin(a4)
    term5 = t4/(t1+t4)*1/sin(a4)

    if a1 < pi/2:
        term6 = 1/sin(a1)*t1/(t1+t2)
    else:
        term6 = 1/sin(a1)*t2/(t1+t2)

    return -(term1 + term2 - term3 - term4 - term5) + term6

print(testd1d3(testP13))
```

Output: Done

### A.2.2 Runs for $\mathbf{x} \in d_2$ and $\mathbf{x} \in d_4$

Numerical analysis of $f^-(A_{3,1}, A_{1,3})$ and $f^-(A_{1,3}, A_{3,1})$:
Code:

```
def testA31(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = t2*(t1-t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term2 = t2*(t4-t1)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term3 = t3*(t1-t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = t1*(t2-t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term5 = 1
    term6 = cos(a1+a2)/sin(a1+a2)*t2/(t1+t2)*1/sin(a2)
    term7 = cos(a1+a2)/sin(a1+a2)*t1/(t1+t2)*1/sin(a1)
    term8 = cos(a1+a2)/sin(a1+a2)*(t2-t3)/(t2+t3)*1/sin(a2)
    term9 = cos(a1+a2)/sin(a1+a2)*(t1-t4)/(t1+t4)*1/sin(a1)
```

```
    return term1 + term2 + term3 + term4 + term5 + term6 + term7 - term8
    - term9

print(testd2d4(testA31, True))
```

Output: Done

Numerical analysis of $f^-(A_{4,1}, A_{1,4})$:
Code:

```
def testA41(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = t3*(t4-t1)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term2 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term3 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)**2)
    term4 = cos(a1)/(sin(a1)*sin(a2))*t3/(t2+t3)

    return term1 + term2 + term3 - term4

print(testd2d4(testA41))
```

Output: Done

Numerical analysis of terms containing $A_{1,2}$ and $A_{2,1}$:
Code:

```
def testA12(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t1*t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)**2)
    term2 = t2*(t1-t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term3 = 1
    term4 = t2*(t4-t1)/((t2+t3)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term5 = cos(a1)/(sin(a1)*sin(a2))*(t4-t1)/(t1+t4)
    term6 = cos(a1)/(sin(a1)*sin(a2))*t2/(t1+t2)
    term7 = cos(a1)/(sin(a1)**2)*t1/(t1+t2)
    term8 = cos(a1)/(sin(a1)*sin(a2))*t2/(t2+t3)

    return term1 + term2 + term3 + term4 + term5 + term6 - term7 - term8

print(testd2d4(testA12))
```

Output: Done

Numerical analysis of terms containing $A_{2,3}$ and $A_{3,2}$:
Code:

```
def testA23(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t1*t2)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term2 = t1*(t3-t2)/((t2+t3)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term3 = t1*(t2-t3)/((t1+t2)*(t2+t3)) *1/(sin(a1)*sin(a2))
    term4 = (t2*t3)/((t1+t2)*(t2+t3)) *1/(sin(a2)**2)
    term5 = 1
    term6 = cos(a2)/(sin(a2)*sin(a1))*t1/(t1+t2)
```

```
    term7 = cos(a2)/(sin(a2)**2)*(t3-t2)/(t2+t3)
    term8 = cos(a2)/(sin(a2)*sin(a1))*t1/(t1+t4)
    term9 = cos(a2)/(sin(a2)**2)*t2/(t1+t2)

    return term1 + term2 + term3 + term4 + term5 + term6 + term7 - term8
    - term9

print(testd2d4(testA23))
```

Output: Done

Numerical analysis of $f^{-}(A_{3,4}, A_{4,3})$:
Code:

```
def testA34(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term2 = t4*(t3-t2)/((t2+t3)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term3 = (t2*t3)/((t1+t2)*(t2+t3)) *1/(sin(a2)**2)
    term4 = cos(a2)/(sin(a1)*sin(a2))*t4/(t1+t4)

    return term1 + term2 + term3 - term4

print(testd2d4(testA34))
```

Output: Done

### A.2.3 Runs for $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$

Numerical analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term2 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term3 = 1
    term4 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a2)*sin(a4))
    term5 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term6 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a2)*sin(a4))
    term7 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term8 = cos(a1)/(sin(a1)*sin(a4))*t4/(t1+t4)
    term9 = cos(a1)/(sin(a1)*sin(a2))*t2/(t1+t2)
    term10 = cos(a1)/(sin(a1)**2)*t1/(t1+t4)
    term11 = cos(a1)/(sin(a1)**2)*t1/(t1+t2)
    term12 = cos(a1)/(sin(a1)*sin(a2))*t2/(t2+t3)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7 \
    +term8 + term9 - term10 - term11 - term12

print(testQ1(testA12))
print(testQ2(testA12))
print(testQ3(testA12))
print(testQ4(testA12))
```

Output:
Done
Done

Done
Done

Numerical analysis of terms containing $A_{1,3}$, $A_{3,1}$ and $P_{1,3}$:

```
def testA13(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    #A_(3,1) - terms
    term1 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term2 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a2)*sin(a4))
    term3 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))
    term4 = 1
    term5 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    sumA31 = term1 + term2 + term3 + term4 + term5

    #A_(1,3) - terms
    term6 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a2)*sin(a4))
    term7 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term8 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a3)*sin(a4))
    term9 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))
    sumA13 = term6 + term7 + term8 + term9

    #Converted P_(1,3)-terms
    term9 = cos(a1+a2)/(sin(a1+a2)*sin(a2))*t2/(t2+t3)
    term10 = cos(a1+a2)/(sin(a1+a2)*sin(a1))*t1/(t1+t4)
    term11 = -cos(a1+a2)/(sin(a1+a2)*sin(a2))*t2/(t1+t2)
    term12 = -cos(a1+a2)/(sin(a1+a2)*sin(a3))*t3/(t2+t3)
    term13 = -cos(a1+a2)/(sin(a1+a2)*sin(a4))*t4/(t1+t4)
    term14 = -cos(a1+a2)/(sin(a1+a2)*sin(a1))*t1/(t1+t2)
    sumP13 = term9 + term10 + term11 + term12 + term13 + term14

    if a1+a2 < pi:
        return sumA13 - sumA31 + sumP13

    elif a1+a2 > pi:
        return sumA31 - sumA13 - sumP13

print(testQ1(testA13))
print(testQ2(testA13))
print(testQ3(testA13))
print(testQ4(testA13))
```

Output:
Done
Done
Done
Done

Numerical analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term2 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))
    term3 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term4 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)*sin(a3))
```

```
    term5 = 1
    term6 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term7 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))
    term8 = cos(a2)/(sin(a2)*sin(a1))*t1/(t1+t2)
    term9 = cos(a2)/(sin(a2)*sin(a3))*t3/(t2+t3)
    term10 = cos(a2)/(sin(a2)*sin(a1))*t1/(t1+t4)
    term11 = cos(a2)/(sin(a2)*sin(a2))*t2/(t1+t2)
    term12 = cos(a2)/(sin(a2)*sin(a2))*t2/(t2+t3)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7 \
    +term8 + term9 - term10 - term11 - term12

print(testQ1(testA23))
print(testQ2(testA23))
print(testQ3(testA23))
print(testQ4(testA23))
```

Output:
Done
Done
Done
Done

Numerical analysis of terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$:

```
def testA24(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    #A_(2,4)-terms:
    term1 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term2 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a2)*sin(a4))
    term3 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))

    #A_(4,2)-terms:
    term4 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a2)*sin(a4))
    term5 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))
    term6 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)*sin(a3))

    #Converted P_(2,4)-terms
    term7 = -cos(a2+a3)/(sin(a2+a3)*sin(a4))*t4/(t1+t4)
    term8 = -cos(a2+a3)/(sin(a2+a3)*sin(a3))*t3/(t2+t3)

    if a2+a3 < pi:
        return term1 + term2 + term3 - term4 - term5 - term6 + term7 +
    term8
    elif a2+a3 > pi:
        return -(term1 + term2 + term3 - term4 - term5 - term6 + term7 +
    term8)

print(testQ1(testA24))
print(testQ2(testA24))
print(testQ3(testA24))
print(testQ4(testA24))
```

Output:
Done
Done
Done

Done

Numerical analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t1+t4)*(t1+t2))*1/(sin(a2)*sin(a4))
    term2 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a3)*sin(a4))
    term3 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)*sin(a3))
    term4 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a2)*sin(a4))
    term5 = cos(a3)/(sin(a3)*sin(a4))*t4/(t1+t4)

    return term1 + term2 + term3 - term4 + term5

print(testQ1(testA34))
print(testQ2(testA34))
print(testQ3(testA34))
print(testQ4(testA34))
```

Output:
Done
Done
Done
Done

Numerical analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term2 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a3)*sin(a4))
    term3 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))
    term4 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))
    term5 = cos(a4)/(sin(a4)*sin(a3))*t3/(t2+t3)
    return term1 + term2 + term3 - term4 + term5

print(testQ1(testA41))
print(testQ2(testA41))
print(testQ3(testA41))
print(testQ4(testA41))
```

Output:
Done
Done
Done
Done

Appendix B

# Test results for $\mathbf{R}_{1,2,4}$, $\mathbf{R}_{1,3,4}$ and $\mathbf{R}_{2,3,4}$

In this appendix we will preform an analysis on the remaining expressions from step 1 in section 3.2 that we will need to complete the proof of injectivity of mean value mappings between convex quadrilaterals in the plane. We will use some notation in this appendix that we introduced in section 3.2. This appendix is structured as follows: In section B.1 we will check if $\mathbf{R}_{1,2,4} > 0$ for $\mathbf{x} \in Q$, in section B.2 we will check if $\mathbf{R}_{1,3,4} > 0$ for $\mathbf{x} \in Q$ and in section B.3 we will check if $\mathbf{R}_{2,3,4} > 0$ for $\mathbf{x} \in Q$.

## B.1 Analysis of $\mathbf{R}_{1,2,4}$

We will start by writing down the expression for $\mathbf{R}_{1,2,4}(\mathbf{x})$:

$$
\begin{aligned}
\mathbf{R}_{1,2,4}(\mathbf{x}) = & \frac{t_1 t_3}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_3}\Big(A_{3,2}+A_{2,4}+A_{1,3}+A_{4,1}\Big) \\
&+ \frac{t_1 t_4}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_4}\Big(A_{4,2}+A_{2,1}+A_{1,4}\Big) \\
&+ \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin\alpha_1}\Big(P_{2,4}-P_{1,4}\Big) + \frac{t_2 t_3}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_3}\Big(A_{3,4}+A_{2,3}+A_{4,2}\Big) \\
&+ \frac{t_2 t_4}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{4,3}+A_{3,1}+A_{2,4}+A_{1,2}\Big) \\
&+ \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin\alpha_2}\Big(P_{3,4}-P_{2,4}\Big) + \frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin\alpha_3}\Big(P_{2,3}-P_{2,4}\Big) + A_{2,4} \\
&+ \frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{2,4}-P_{1,2}\Big) + \frac{t_3 t_4}{(t_1+t_4)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_3 \sin\alpha_4}\Big(A_{4,1}+A_{3,4}+A_{1,3}\Big) \\
&+ \frac{t_1 t_3}{(t_1+t_4)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_3}\Big(A_{1,4}+A_{4,2}+A_{3,1}+A_{2,3}\Big) \\
&+ \frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin\alpha_3}\Big(P_{1,4}-P_{1,3}\Big) + \frac{t_1 t_4}{(t_1+t_4)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_4}\Big(A_{1,2}+A_{4,1}+A_{2,4}\Big) \\
&+ \frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{1,1}-P_{1,4}\Big) + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{4,4}-P_{1,4}\Big) + A_{4,1} \\
&+ \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1}\Big(P_{1,4}-P_{2,4}\Big) + \frac{t_1 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_4}\Big(A_{2,4}+A_{1,2}+A_{4,1}\Big) \\
&+ \frac{t_2 t_4}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{2,1}+A_{1,3}+A_{4,2}+A_{3,4}\Big) \\
&+ \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{1,2}-P_{2,4}\Big) + \frac{t_1 t_2}{(t_1+t_2)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_2}\Big(A_{1,2}+A_{3,1}+A_{2,3}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1}\Big(P_{2,2}-P_{1,2}\Big) + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin\alpha_2}\Big(P_{1,1}-P_{1,2}\Big) \\
&+ \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin\alpha_2}\Big(P_{1,2}-P_{1,3}\Big) + A_{1,2}.
\end{aligned}
$$

Since the analysis of $\mathbf{R}_{1,2,4}(\mathbf{x})$ is so similar to the analysis we did for $\mathbf{R}_{1,2,3}(\mathbf{x})$ in section 3.2, we will only present the results in this chapter. In the cases where we need to make an numerical analysis we will use the same code as we presented in section A.1.2. Note that we will use the properties we derived in section 3.2 when we preform the different analyses here. We will now present the results from the different cases that were listed in step 1, section 3.2.

### B.1.1 Analysis for when $\mathbf{x}$ is the intersection between the diagonals

$f^-(A_{1,2}, A_{2,1})$:

$$
\begin{aligned}
f^-(A_{1,2}, A_{2,1}) &= \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{3t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\
&- \frac{t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\
&> 0.
\end{aligned}
$$

$f^-(A_{2,3}, A_{3,2})$:

$$
\begin{aligned}
f^-(A_{2,3}, A_{3,2}) &= \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&- \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&> 0.
\end{aligned}
$$

$f^-(A_{3,4}, A_{4,3})$:

$$
\begin{aligned}
f^-(A_{3,4}, A_{4,3}) &= \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&- \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&> 0.
\end{aligned}
$$

<u>$f^-(A_{4,1}, A_{1,4})$:</u>

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{3t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &- \frac{t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

<u>Analysis of terms containing $P_{1,1}$, $P_{1,4}$ and $P_{4,4}$:</u>
We will now check if $f^+(P_{1,1}, P_{1,4}, P_{4,4}) \geq 0$, but we will leave out two terms, namely $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,4}$ and $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,1}$. As we will see later, these terms will be used in other analyses, so we will leave them out here so we do not use them twice. It is also worth noticing that we have two terms containing $P_{1,4}$ that are cancelling each other out, namely $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,4}$ and $-(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,4}$.

$$\begin{aligned} f^+(P_{1,1}, P_{1,4}, P_{4,4}) - \frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,1} + P_{1,4}\Big) &= \frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,1} - 2P_{1,4} + P_{4,4}\Big) \\ &= \frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,1} - 2P_{1,4} + P_{4,4}\Big) \\ &> 0. \end{aligned}$$

This follows from the definition of $P_{n,m}$, and the fact that $\cos(\alpha_4) \leq 1$.

<u>Analysis of terms containing $P_{1,1}$, $P_{1,2}$ and $P_{2,2}$:</u>
We will now check if $f^+(P_{1,1}, P_{1,2}, P_{2,2}) \geq 0$, but we will leave out two term, namely $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{1,2}$ and $(\frac{t_2}{t_1+t_2})P_{1,1}$. As we will see, we leave out $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{1,2}$ so we can use it in an analysis later on, while we leave out $(\frac{t_2}{t_1+t_2})P_{1,1}$ because we have used this term in an earlier analysis. It is also worth noticing that we have two terms containing $P_{1,2}$ that are cancelling each other out, namely $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{1,2}$ and $-(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{1,2}$.

$$\begin{aligned} f^+(P_{1,1}, P_{1,2}, P_{2,2}) - \frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,1} + P_{1,2}\Big) &= \frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,1} - 2P_{1,2} + P_{2,2}\Big) \\ &> 0. \end{aligned}$$

This follows from the definition of $P_{n,m}$, and the fact that $\cos(\alpha_1) \leq 1$.

<u>Analysis of the remaining terms containing $P_{n,m}$:</u>
We observe that the remaining terms containing $P_{n,m}$ gives us the two sums

$$\frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{2,3} + P_{1,4} - P_{2,4} - P_{1,3}\Big), \tag{B.1}$$

and

$$\frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{3,4} + P_{1,2} - P_{2,4} - P_{1,3}\Big). \tag{B.2}$$

First, we observe that $-P_{1,3}, -P_{2,4} > 0$, since $\cos(\pi) = -1$. Second, we observe that (B.1) is greater than zero when $\alpha_2 = \alpha_4 \leq \frac{\pi}{2}$, while (B.2) is greater than zero when

$\alpha_1 = \alpha_3 \leq \frac{\pi}{2}$. Let us now look at the case where $\alpha_2 = \alpha_4 > \frac{\pi}{2}$. In this case, we can not guarantee that (B.1) is greater than or equal to zero. We will therefore use Lemma 3.2.2 to convert $P_{2,3}, P_{1,4}$ into $A_{2,3}, A_{4,1}$, and then we will put these expressions into the analyses of the terms containing $A_{2,3}$ and $A_{1,4}$ respectively.

$f^-(A_{2,3}, A_{3,2})$ (new expression marked in blue):

$$\begin{aligned} f^-(A_{2,3}, A_{3,2}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_1}{t_1+t_2} \cdot \frac{\cos(\alpha_2)}{s^2} \\ &> \frac{-t_1^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_2 > \frac{\pi}{2}$, we know that $\alpha_1 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_1 < t_2$, which again implies that $t_1^2 < t_1t_2$, which then implies that $\frac{-t_1^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} > 0$.

We will now make a similar argument for $f^-(A_{4,1}, A_{1,4})$.

$f^-(A_{4,1}, A_{1,4})$:

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 + \frac{t_1}{t_1+t_2} \cdot \frac{\cos(\alpha_4)}{s^2} \\ &> \frac{-t_1^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_2 > \frac{\pi}{2}$, we know that $\alpha_1 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_1 < t_2$, which again implies that $t_1^2 < t_1t_2$, which then implies that $\frac{-t_1^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 > 0$.

Similarly, when $\alpha_1 = \alpha_3 > \frac{\pi}{2}$, we get the following analysis:

$f^-(A_{1,2}, A_{2,1})$:

$$\begin{aligned} f^-(A_{1,2}, A_{2,1}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 + \frac{t_2}{t_1+t_2} \cdot \frac{\cos(\alpha_1)}{s^2} \\ &> \frac{-t_2^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_1 > \frac{\pi}{2}$, we know that $\alpha_2 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_2 < t_1$, which again implies that $t_2^2 < t_1t_2$, which then implies that $\frac{-t_2^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 > 0$.

$f^-(A_{3,4}, A_{4,3})$:

$$
\begin{aligned}
f^-(A_{3,4}, A_{4,3}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_2}{t_1+t_2} \cdot \frac{\cos(\alpha_3)}{s^2} \\
&> \frac{-t_2^2 + t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&> 0.
\end{aligned}
$$

We have now proven analytically that all negative terms in $\mathbf{R}_{1,2,4}(\mathbf{x})$ are strictly dominated by the positive terms in $\mathbf{R}_{1,2,4}(\mathbf{x})$, when $\mathbf{x}$ is the intersection between the diagonals.

### B.1.2 Analysis for $\mathbf{x} \in \{d_1 \cup d_3\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term2 = (t1*t4)/((t1+t4)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term3 = (t1*t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a4))
    term4 = (t1*t2)/((t1+t2)*(t1+t4)) *1/(sin(a1)**2)
    term5 = 1
    term6 = (t1*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term7 = (t2*t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a4))

    term8 = t4/(t1+t4)*cos(a1)/(sin(a1)*sin(a4))
    term9 = t2/(t1+t2)*cos(a1)/(sin(a1)**2)
    term10 = t4/(t3+t4)*cos(a1)/(sin(a1)*sin(a4))
    term11 = t1/(t1+t4)*cos(a1)/(sin(a1)**2)
    term12 = t1/(t1+t2)*cos(a1)/(sin(a1)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 - term10 - term11 - term12

print(testd1d3(testA12))
```

Output: Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)**2)
    term4 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))

    term5 = t3/(t3+t4)*cos(a1)/(sin(a1)*sin(a4))

    return term1 + term2 + term3 - term4 - term5

print(testd1d3(testA23))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$:

```
def testA24(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = 1
    term3 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term4 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term5 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term6 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term8 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term9 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))

    term10 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))*t1/(t1+t2)
    term11 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))*t4/(t3+t4)
    term12 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))*t2/(t1+t2)
    term13 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))*t3/(t3+t4)
    term14 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))*t1/(t1+t4)
    term15 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))*t4/(t1+t4)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7 - term8
   - term9\
    + term10 + term11 - term12 - term13 - term14 - term15

print(testd1d3(testA24, True))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term4 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))

    term5 = t2/(t1+t2)*cos(a4)/(sin(a4)*sin(a1))

    return term1 + term2 + term3 - term4 - term5

print(testd1d3(testA34))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a4)**2)
    term3 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term4 = 1
    term5 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term6 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))

    term8 = t1/(t1+t4)*cos(a4)/(sin(a4)*sin(a1))
    term9 = t3/(t3+t4)*cos(a4)/(sin(a4)**2)
    term10 = t1/(t1+t2)*cos(a4)/(sin(a4)*sin(a1))
    term11 = t4/(t3+t4)*cos(a4)/(sin(a4)**2)
    term12 = t4/(t1+t4)*cos(a4)/(sin(a4)**2)


    return term1 + term2+ term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 - term10 - term11 - term12

print(testd1d3(testA41))
```

Output: Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $P_{1,3}$:
We observe that we have two terms containing $P_{1,3}$;

$$-\frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin(\alpha_3)} P_{1,3}, \tag{B.3}$$

and

$$-\frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)} P_{1,3}. \tag{B.4}$$

We know that $\cos(\alpha_1 + \alpha_3) = \cos(\pi) = -1$, when $\mathbf{x} \in \{d_1 \cup d_3\}$, which implies that $P_{1,3} < 0$. Since both $\frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin(\alpha_3)}$ and $\frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_2)}$ are positive, this implies that both (B.3) and (B.4) are positive.

Analysis of terms containing $P_{1,1}$, $P_{2,2}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 1, 2, 4$. Since the factors in front of $P_{n,n}$, for $n = 1, 2, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{1,1}$, $P_{2,2}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis, proven that all the negative terms in $\mathbf{R}_{1,2,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{d_1 \cup d_3\}$. This means that $\mathbf{R}_{1,2,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{d_1 \cup d_3\}$.

### B.1.3 Analysis for $\mathbf{x} \in \{d_2 \cup d_4\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a2))
    term2 = (t1*t4)/((t1+t4)*(t3+t4)) *1/(sin(a1)**2)
    term3 = (t1*t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)**2)
    term4 = (t1*t2)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term5 = 1
    term6 = (t1*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)**2)
    term7 = (t2*t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a2))

    term8 = t4/(t1+t4)*cos(a1)/(sin(a1)**2)
    term9 = t2/(t1+t2)*cos(a1)/(sin(a1)*sin(a2))
    term10 = t4/(t3+t4)*cos(a1)/(sin(a1)**2)
    term11 = t1/(t1+t4)*cos(a1)/(sin(a1)**2)
    term12 = t1/(t1+t2)*cos(a1)/(sin(a1)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 - term10 - term11 - term12

print(testd2d4(testA12))
```

Output: Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{3,1}$, $A_{1,3}$ and $P_{1,3}$:

```
def testA31(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))
    term5 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term6 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))

    term7 = t3/(t3+t4)*1/sin(a2)*cos(a1+a2)/sin(a1+a2)
    term8 = t2/(t1+t2)*1/sin(a2)*cos(a1+a2)/sin(a1+a2)

    return term1 + term2 + term3 - term4 - term5 - term6 \
    + term7 + term8

print(testd1d3(testA31, True))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,1}$, $A_{1,3}$ and $P_{1,3}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)**2)
    term4 = 1
    term5 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)**2)
    term6 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)**2)
    term7 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))

    term8 = t1/(t1+t2)*cos(a1)/(sin(a1)*sin(a1))
    term9 = t4/(t3+t4)*cos(a1)/(sin(a1)**2)
    term10 = t4/(t1+t4)*cos(a1)/(sin(a1)**2)
    term11 = t1/(t1+t4)*cos(a1)/(sin(a1)**2)
    term12 = t3/(t3+t4)*cos(a1)/(sin(a1)*sin(a2))
    return term1 + term2 + term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 + term10 - term11 - term12

print(testd2d4(testA41))
```

Output: Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)**2)
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))

    term5 = t3/(t3+t4)*cos(a2)/(sin(a2)**2)

    return term1 + term2 + term3 - term4 + term5

print(testd2d4(testA23))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)**2)
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))

    term5 = t2/(t1+t2)*cos(a2)/(sin(a2)**2)

    return term1 + term2 + term3 - term4 - term5

print(testd2d4(testA34))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $P_{2,4}$:
We observe that we have six terms containing $P_{2,4}$. Since $P_{2,4} < 0$ (follows from the fact that $\cos(\pi) = -1$), we need to check if the sum of the factors in front of $-P_{2,4}$ are greater than the sum of the factors in front of $P_{2,4}$. We will do this analysis in the code below.

```
def testP24(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = t2/(t1+t2)*1/sin(a2)
    term2 = t3/(t3+t4)*1/sin(a2)
    term3 = t1/(t1+t4)*1/sin(a1)
    term4 = t4/(t1+t4)*1/sin(a1)

    term5 = t1/(t1+t2)*1/sin(a1)
    term6 = t4/(t3+t4)*1/sin(a1)

    return term1 + term2 + term3 + term4 - term5 - term6

print(testd2d4(testP24))
```

Output: Done

This test indicates that the sum of all terms containing $P_{2,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $P_{1,1}$, $P_{2,2}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 1, 2, 4$. Since the factors in front of $P_{n,n}$, for $n = 1, 2, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{1,1}$, $P_{2,2}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis, proven that all the negative terms in $\mathbf{R}_{1,2,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{d_2 \cup d_4\}$. This means that $\mathbf{R}_{1,2,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{d_2 \cup d_4\}$.

### B.1.4 Analysis for $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t1+t2)*(t3+t4)) *1/(sin(a2)*sin(a4))
    term2 = (t1*t4)/((t1+t4)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term3 = (t1*t4)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a4))
    term4 = (t1*t2)/((t1+t2)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term5 = 1
    term6 = (t1*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term7 = (t2*t4)/((t1+t2)*(t1+t4)) *1/(sin(a2)*sin(a4))

    term8 = t4/(t1+t4)*cos(a1)/(sin(a1)*sin(a4))
    term9 = t2/(t1+t2)*cos(a1)/(sin(a1)*sin(a2))
    term10 = t4/(t3+t4)*cos(a1)/(sin(a1)*sin(a4))
    term11 = t1/(t1+t4)*cos(a1)/(sin(a1)**2)
    term12 = t1/(t1+t2)*cos(a1)/(sin(a1)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 - term10 - term11 - term12

print(testQ1(testA12))
print(testQ2(testA12))
print(testQ3(testA12))
print(testQ4(testA12))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{1,3}$, $A_{3,1}$ and $P_{1,3}$:

```
def testA13(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    #A_(1,3) - terms
    term1 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a2)*sin(a4))
    sumA13 = term1 + term2 + term3

    #A_(3,1) - terms
    term4 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a4))
    term5 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))
    term6 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    sumA31 = term4 + term5 + term6
```

```

    #Converted P_(1,3)-terms
    term7 = -t3/(t3+t4)*1/sin(a3)*cos(a1+a2)/sin(a1+a2)
    term8 = -t2/(t1+t2)*1/sin(a2)*cos(a1+a2)/sin(a1+a2)
    sumP13 = term7 + term8

    if a1+a2 < pi:
        return sumA13 - sumA31 + sumP13

    elif a1+a2 > pi:
        return sumA31 - sumA13 - sumP13

print(testQ1(testA13))
print(testQ2(testA13))
print(testQ3(testA13))
print(testQ4(testA13))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{1,3}$, $A_{3,1}$ and $P_{1,3}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a3))
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))
    term3 = (t1*t2)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))

    term5 = t3/(t3+t4)*cos(a2)/(sin(a2)*sin(a3))

    return term1 + term2 + term3 - term4 + term5

print(testQ1(testA23))
print(testQ2(testA23))
print(testQ3(testA23))
print(testQ4(testA23))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$:

```
def testA24(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)
```

```

    #A_(2,4)-terms:
    term1 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))
    term2 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a4))
    term3 = 1
    term4 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term5 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    sumA24 = term1 + term2 + term3 + term4 + term5

    #A_(4,2)-terms:
    term6 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a3))
    term8 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))
    term9 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a2)*sin(a4))
    sumA42 = term6 + term7 + term8 + term9

    #Converted P_(2,4)-terms
    term10 = t1/(t1+t2)*cos(a2+a3)/(sin(a2+a3)*sin(a1))
    term11 = t4/(t3+t4)*cos(a2+a3)/(sin(a2+a3)*sin(a4))
    term12 = t2/(t1+t2)*cos(a2+a3)/(sin(a2+a3)*sin(a2))
    term13 = t3/(t3+t4)*cos(a2+a3)/(sin(a2+a3)*sin(a3))
    term14 = t1/(t1+t4)*cos(a2+a3)/(sin(a2+a3)*sin(a1))
    term15 = t4/(t1+t4)*cos(a2+a3)/(sin(a2+a3)*sin(a4))
    sumP24 = term10 + term11 - term12 - term13 - term14 - term15

    if a2+a3 < pi:
        return sumA24 - sumA42 + sumP24
    elif a2+a3 > pi:
        return sumA42 - sumA24 -sumP24

print(testQ1(testA24))
print(testQ2(testA24))
print(testQ3(testA24))
print(testQ4(testA24))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a3))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a3)*sin(a4))
    term3 = (t2*t4)/((t1+t2)*(t1+t4))*1/(sin(a2)*sin(a4))
    term4 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a4))

    term5 = t2/(t1+t2)*cos(a3)/(sin(a3)*sin(a2))

    return term1 + term2 + term3 - term4 + term5

print(testQ1(testA34))
print(testQ2(testA34))
```

```
print(testQ3(testA34))
print(testQ4(testA34))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a3)*sin(a4))
    term3 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term4 = 1
    term5 = (t1*t4)/((t1+t2)*(t1+t4))*1/(sin(a1)*sin(a4))
    term6 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))

    term8 = t3/(t3+t4)*cos(a4)/(sin(a4)*sin(a3))
    term9 = t1/(t1+t4)*cos(a4)/(sin(a4)*sin(a1))
    term10 = t1/(t1+t2)*cos(a4)/(sin(a4)*sin(a1))
    term11 = t4/(t3+t4)*cos(a4)/(sin(a4)**2)
    term12 = t4/(t1+t4)*cos(a4)/(sin(a4)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 - term10 - term11 - term12

print(testQ1(testA41))
print(testQ2(testA41))
print(testQ3(testA41))
print(testQ4(testA41))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{1,2,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $P_{1,1}$, $P_{2,2}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 1, 2, 4$. Since the factors in front of $P_{n,n}$, for $n = 1, 2, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{1,1}$, $P_{2,2}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis, proven that all the negative terms in $\mathbf{R}_{1,2,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$. This means that $\mathbf{R}_{1,2,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

### B.1.5 Conclusion

We have now showed that for every $\mathbf{x} \in Q$,

$$\mathbf{R}_{1,2,4}(\mathbf{x}) > 0.$$

Since $w_1, w_2, w_4 > 0$ for all $\mathbf{x} \in Q$, this implies that

$$\mathcal{D}(w_1, w_2, w_4)(\mathbf{x}) > 0,$$

for all $\mathbf{x} \in Q$.

## B.2 Analysis of $\mathbf{R}_{1,3,4}$

We will start by writing down the expression for $\mathbf{R}_{1,3,4}(\mathbf{x})$:

$$\begin{aligned}
\mathbf{R}_{1,3,4}(\mathbf{x}) = {} & \frac{t_2 t_3}{(t_2+t_3)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_3}\Big(A_{3,4} + A_{2,3} + A_{4,2}\Big) \\
& + \frac{t_2 t_4}{(t_2+t_3)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{4,3} + A_{3,1} + A_{2,4} + A_{1,2}\Big) \\
& + \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_2}\Big(P_{3,4} - P_{2,4}\Big) + \frac{t_3 t_4}{(t_2+t_3)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_3 \sin\alpha_4}\Big(A_{4,1} + A_{3,4} + A_{1,3}\Big) \\
& + \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_3}\Big(P_{4,4} - P_{3,4}\Big) + \frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin\alpha_3}\Big(P_{3,3} - P_{3,4}\Big) + A_{3,4} \\
& + \frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{3,4} - P_{1,3}\Big) + \frac{t_3 t_4}{(t_1+t_4)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_3 \sin\alpha_4}\Big(A_{4,1} + A_{3,4} + A_{1,3}\Big) \\
& + \frac{t_1 t_3}{(t_1+t_4)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_3}\Big(A_{1,4} + A_{4,2} + A_{3,1} + A_{2,3}\Big) \\
& + \frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin\alpha_3}\Big(P_{1,4} - P_{1,3}\Big) + \frac{t_1 t_4}{(t_1+t_4)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_4}\Big(A_{1,2} + A_{4,1} + A_{2,4}\Big) \\
& + \frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{1,1} - P_{1,4}\Big) + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{4,4} - P_{1,4}\Big) + A_{4,1} \\
& + \frac{t_2 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{3,4} + A_{1,3} + A_{4,2} + A_{2,1}\Big) \\
& + \frac{t_1 t_2}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_2}\Big(A_{3,1} + A_{2,3} + A_{1,2}\Big) + \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_2}\Big(P_{1,2} - P_{1,3}\Big) \\
& + \frac{t_3 t_4}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_3 \sin\alpha_4}\Big(A_{1,4} + A_{4,3} + A_{3,1}\Big) + \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1}\Big(P_{1,4} - P_{2,4}\Big) \\
& + \frac{t_1 t_3}{(t_2+t_3)(t_1+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_3}\Big(A_{4,1} + A_{2,4} + A_{1,3} + A_{3,2}\Big) + A_{1,3} \\
& + \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_3}\Big(P_{1,3} - P_{1,4}\Big) + \frac{t_4}{t_1+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{1,3} - P_{3,4}\Big) \\
& + \frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin\alpha_1}\Big(P_{2,3} - P_{1,3}\Big).
\end{aligned}$$

Since the analysis of $\mathbf{R}_{1,3,4}(\mathbf{x})$ is so similar to the analysis we did for $\mathbf{R}_{1,2,3}(\mathbf{x})$ in section 3.2, we will only present the results in this section. When we are preforming a numerical analysis we will use the code we presented in section A.1.2. Note that we will use the properties we derived in section 3.2 when we preform the different analyses here.

### B.2.1 Analysis for when $\mathbf{x}$ is the intersection between the diagonals

$f^-(A_{1,2}, A_{2,1})$:

$$\begin{aligned} f^-(A_{1,2}, A_{2,1}) &= \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &- \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

$f^-(A_{2,3}, A_{3,2})$:

$$\begin{aligned} f^-(A_{2,3}, A_{3,2}) &= \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &- \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

$f^-(A_{3,4}, A_{4,3})$:

$$\begin{aligned} f^-(A_{3,4}, A_{4,3}) &= \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{3t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &- \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

$f^-(A_{4,1}, A_{1,4})$:

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{3t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &- \frac{t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

Analysis of terms containing $P_{3,3}$, $P_{3,4}$ and $P_{4,4}$:

We will now check if $f^+(P_{3,3}, P_{3,4}, P_{4,4}) \geq 0$, but we will leave out two terms, namely $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{3,4}$ and $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{4,4}$. As we will see later, these terms will be used in other analyses, so we will leave them out here so we do not use them twice. It is also worth noticing that we have two terms containing $P_{3,4}$ that are cancelling each other out, namely $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{3,4}$ and $-(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{3,4}$.

$$
\begin{aligned}
f^+(P_{3,3}, P_{3,4}, P_{4,4}) - \frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{3,4} + P_{4,4}\Big) &= \frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{3,3} - 2P_{3,4} + P_{4,4}\Big) \\
&> 0.
\end{aligned}
$$

This follows from the definition of $P_{n,m}$, and the fact that $\cos(\alpha_3) \leq 1$.

Analysis of terms containing $P_{1,1}$, $P_{1,4}$ and $P_{4,4}$:

We will now check if $f^+(P_{1,1}, P_{1,4}, P_{4,4}) \geq 0$, but we will leave out two terms, namely $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,4}$ and $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{4,4}$. As we will see, we leave out $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,4}$ so we can use it in another analysis later on, while we leave out $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{4,4}$ because we have used this term in an earlier analysis. It is also worth noticing that we have two terms containing $P_{1,4}$ that are cancelling each other out, namely $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,4}$ and $-(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{1,4}$.

$$
\begin{aligned}
f^+(P_{1,1}, P_{1,4}, P_{4,4}) - \frac{t_1}{t_1+t_2} \cdot \frac{1}{s}P_{1,4} &= \frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,1} - 2P_{1,4} + P_{4,4}\Big) \\
&> 0.
\end{aligned}
$$

This follows from the definition of $P_{n,m}$, and the fact that $\cos(\alpha_4) \leq 1$.

Analysis of the remaining terms containing $P_{n,m}$:

We observe that the remaining terms containing $P_{n,m}$ gives us the two sums

$$
\frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{2,3} + P_{1,4} - P_{2,4} - P_{1,3}\Big), \tag{B.5}
$$

and

$$
\frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,2} + P_{3,4} - P_{2,4} - P_{1,3}\Big). \tag{B.6}
$$

First, we observe that $-P_{1,3}, -P_{2,4} > 0$, since $\cos(\pi) = -1$. Second, we observe that (B.5) is greater than zero when $\alpha_2 = \alpha_4 \leq \frac{\pi}{2}$, while (B.6) is greater than zero when $\alpha_1 = \alpha_3 \leq \frac{\pi}{2}$. Let us now look at the case where $\alpha_2 = \alpha_4 > \frac{\pi}{2}$. In this case, we can not guarantee that (B.5) is greater than or equal to zero. We will therefore use Lemma 3.2.2 to convert $P_{2,3}, P_{1,4}$ into $A_{2,3}, A_{4,1}$, and then we will put these expressions into the analyses of the terms containing $A_{2,3}$ and $A_{4,1}$ respectively.

$f^-(A_{2,3}, A_{3,2})$ (new expression marked in blue):

$$
\begin{aligned}
f^-(A_{2,3}, A_{3,2}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_1}{t_1+t_2} \cdot \frac{\cos\alpha_2}{s^2} \\
&> \frac{-t_1^2 + t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\
&> 0.
\end{aligned}
$$

Since we are looking at the case where $\alpha_2 > \frac{\pi}{2}$, we know that $\alpha_1 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_1 < t_2$, which again implies that $t_1^2 < t_1 t_2$, which then implies that $\frac{-t_1^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} > 0$.

We will now make a similar argument for $f^-(A_{4,1}, A_{1,4})$.

$f^-(A_{4,1}, A_{1,4})$:

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= \frac{2t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 + \frac{t_1}{t_1+t_2} \cdot \frac{\cos\alpha_4}{s^2} \\ &> \frac{-t_1^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_2 > \frac{\pi}{2}$, we know that $\alpha_1 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_1 < t_2$, which again implies that $t_1^2 < t_1 t_2$, which then implies that $\frac{-t_1^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 > 0$.

Similarly, when $\alpha_1 = \alpha_3 > \frac{\pi}{2}$, we get the following analysis:

$f^-(A_{1,2}, A_{2,1})$:

$$\begin{aligned} f^-(A_{1,2}, A_{2,1}) &= \frac{2t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_2}{t_1+t_2} \cdot \frac{\cos\alpha_1}{s^2} \\ &> \frac{-t_2^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_1 > \frac{\pi}{2}$, we know that $\alpha_2 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_2 < t_1$, which again implies that $t_2^2 < t_1 t_2$, which then implies that $\frac{-t_2^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} > 0$.

$f^-(A_{3,4}, A_{4,3})$:

$$\begin{aligned} f^-(A_{3,4}, A_{4,3}) &= \frac{2t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 + \frac{t_2}{t_1+t_2} \cdot \frac{\cos\alpha_3}{s^2} \\ &> \frac{-t_2^2 + t_1 t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

We have now proven analytically that $\mathbf{R}_{1,3,4}(\mathbf{x}) > 0$, when $\mathbf{x}$ is the intersection between the diagonals in $Q$.

### B.2.2 Analysis for $\mathbf{x} \in \{d_1 \cup d_3\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term2 = (t1*t4)/((t1+t4)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term3 = (t1*t2)/((t2+t3)*(t1+t4)) *1/(sin(a1)**2)
    term4 = (t2*t4)/((t2+t3)*(t1+t4)) *1/(sin(a1)*sin(a4))

    term5 = t2/(t2+t3)*cos(a1)/(sin(a1)**2)

    return term1 + term2 + term3 - term4 + term5

print(testd1d3(testA12))
```

Output: Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)**2)
    term4 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))

    term5 = t1/(t1+t4)*cos(a1)/(sin(a1)**2)

    return term1 + term2 + term3 - term4 - term5

print(testd1d3(testA23))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$:

```
def testA24(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))
    term4 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term5 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term6 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))

    term7 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))*t2/(t2+t3)
```

```python
    term8 = cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))*t1/(t1+t4)

    return term1 + term2 + term3 - term4 - term5 - term6 \
    - term7 - term8

print(testd1d3(testA24, True))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```python
def testA34(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a4)**2)
    term3 = 1
    term4 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a4)**2)
    term5 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))
    term6 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a4)**2)

    term8 = t3/(t2+t3)*cos(a4)/(sin(a4)**2)
    term9 = t3/(t3+t4)*cos(a4)/(sin(a4)**2)
    term10 = t4/(t1+t4)*cos(a4)/(sin(a4)**2)
    term11 = t2/(t2+t3)*cos(a4)/(sin(a4)*sin(a1))
    term12 = t4/(t3+t4)*cos(a4)/(sin(a4)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    + term8 + term9 + term10 - term11 - term12

print(testd1d3(testA34))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```python
def testA41(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a4)**2)
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a4)**2)
    term3 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term4 = 1
    term5 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a4))
    term6 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a4)**2)

    term8 = t3/(t3+t4)*cos(a4)/(sin(a4)**2)
    term9 = t1/(t1+t4)*cos(a4)/(sin(a4)*sin(a1))
    term10 = t4/(t3+t4)*cos(a4)/(sin(a4)**2)
    term11 = t4/(t1+t4)*cos(a4)/(sin(a4)**2)
```

```
    term12 = t3/(t2+t3)*cos(a4)/(sin(a4)**2)


    return term1 + term2+ term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 - term10 - term11 - term12

print(testd1d3(testA41))
```

Output: Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $P_{1,3}$:
We observe that we have six terms containing $P_{1,3}$. Since $P_{1,3} < 0$ (follows from the fact that $\cos(\pi) = -1$), we need to check if the sum of the factors in front of $-P_{1,3}$ are greater than the sum of the factors in front of $P_{1,3}$. We will do this analysis in the code below.

```
def testP13(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = t4/(t3+t4)*1/sin(a4)
    term2 = t3/(t3+t4)*1/sin(a4)
    term3 = t2/(t2+t3)*1/sin(a1)
    term4 = t1/(t1+t4)*1/sin(a1)

    term5 = t3/(t2+t3)*1/sin(a4)
    term6 = t4/(t1+t4)*1/sin(a4)

    return term1 + term2 + term3 + term4 - term5 - term6

print(testd1d3(testP13))
```

Output: Done

This test indicates that the sum of all terms containing $P_{1,3}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $P_{1,1}$, $P_{3,3}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 1, 3, 4$. Since the factors in front of $P_{n,n}$, for $n = 1, 3, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{1,1}$, $P_{3,3}$ and $P_{4,4}$ are greater than zero.

We have now numerically proven that all the negative terms in $\mathbf{R}_{1,3,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{d_1 \cup d_3\}$. This means that $\mathbf{R}_{1,3,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{d_1 \cup d_3\}$.

### B.2.3 Analysis for $\mathbf{x} \in \{d_2 \cup d_4\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4)) *1/(sin(a1)*sin(a2))
    term2 = (t1*t4)/((t1+t4)*(t3+t4)) *1/(sin(a1)**2)
    term3 = (t1*t2)/((t2+t3)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term4 = (t2*t4)/((t2+t3)*(t1+t4)) *1/(sin(a1)*sin(a2))

    term5 = t2/(t2+t3)*cos(a1)/(sin(a1)*sin(a2))

    return term1 + term2 + term3 - term4 + term5

print(testd2d4(testA12))
```

Output: Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{3,1}$, $A_{1,3}$ and $P_{1,3}$:

```
def testA31(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term5 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term6 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term7 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term8 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term9 = 1

    term10 = t4/(t3+t4)*1/sin(a1)*cos(a1+a2)/sin(a1+a2)
    term11 = t3/(t3+t4)*1/sin(a2)*cos(a1+a2)/sin(a1+a2)
    term12 = t2/(t2+t3)*1/sin(a2)*cos(a1+a2)/sin(a1+a2)
    term13 = t1/(t1+t4)*1/sin(a1)*cos(a1+a2)/sin(a1+a2)
    term14 = t3/(t2+t3)*1/sin(a2)*cos(a1+a2)/sin(a1+a2)
    term15 = t4/(t1+t4)*1/sin(a1)*cos(a1+a2)/sin(a1+a2)

    return term1 + term2 + term3 + term4 - term5 - term6 \
    - term7 - term8  - term9 + term10 + term11 + term12\
    +term13 - term14 - term15


print(testd2d4(testA12, True))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,1}$, $A_{1,3}$ and $P_{1,3}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```python
def testA41(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)**2)
    term4 = 1
    term5 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term6 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term7 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))

    term8 = t4/(t3+t4)*cos(a1)/(sin(a1)**2)
    term9 = t4/(t1+t4)*cos(a1)/(sin(a1)**2)
    term10 = t3/(t2+t3)*cos(a1)/(sin(a1)*sin(a2))
    term11 = t3/(t3+t4)*cos(a1)/(sin(a1)*sin(a2))
    term12 = t1/(t1+t4)*cos(a1)/(sin(a1)**2)

    return term1 + term2+ term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 + term10 - term11 - term12

print(testd2d4(testA41))
```

Output: Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```python
def testA23(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)**2)
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))

    term5 = t1/(t1+t4)*cos(a2)/(sin(a1)*sin(a2))

    return term1 + term2 + term3 - term4 + term5

print(testd2d4(testA23))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```python
def testA34(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)**2)
    term2 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = 1
    term4 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a2))
```

```
    term5 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term6 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term7 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))

    term8 = t3/(t2+t3)*cos(a2)/(sin(a2)**2)
    term9 = t3/(t3+t4)*cos(a2)/(sin(a2)**2)
    term10 = t4/(t1+t4)*cos(a2)/(sin(a2)*sin(a1))
    term11 = t2/(t2+t3)*cos(a2)/(sin(a2)**2)
    term12 = t4/(t3+t4)*cos(a2)/(sin(a2)*sin(a1))

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    + term8 + term9 + term10 - term11 - term12

print(testd2d4(testA34))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $P_{2,4}$:
We observe that we have two terms containing $P_{2,4}$;

$$-\frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)} P_{2,4}, \tag{B.7}$$

and

$$-\frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)} P_{2,4}. \tag{B.8}$$

We know that $\cos(\alpha_2 + \alpha_3) = \cos(\pi) = -1$, when $\mathbf{x} \in \{d_2 \cup d_4\}$, which implies that $P_{2,4} < 0$. Since both $\frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin(\alpha_2)}$ and $\frac{t_1}{t_1+t_4} \cdot \frac{1}{\sin(\alpha_1)}$ are positive, this implies that both (B.7) and (B.8) are positive.

Analysis of terms containing $P_{1,1}$, $P_{3,3}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 1, 3, 4$. Since the factors in front of $P_{n,n}$, for $n = 1, 3, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{1,1}$, $P_{3,3}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis, proven that all the negative terms in $\mathbf{R}_{1,3,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{d_2 \cup d_4\}$. This means that $\mathbf{R}_{1,3,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{d_2 \cup d_4\}$.

### B.2.4 Analysis for $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4)) *1/(sin(a2)*sin(a4))
    term2 = (t1*t4)/((t1+t4)*(t3+t4)) *1/(sin(a1)*sin(a4))
```

```
    term3 = (t1*t2)/((t2+t3)*(t1+t4)) *1/(sin(a1)*sin(a2))
    term4 = (t2*t4)/((t2+t3)*(t1+t4)) *1/(sin(a2)*sin(a4))

    term5 = t2/(t2+t3)*cos(a1)/(sin(a1)*sin(a2))

    return term1 + term2 + term3 - term4 + term5

print(testQ1(testA12))
print(testQ2(testA12))
print(testQ3(testA12))
print(testQ4(testA12))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{1,3}$, $A_{3,1}$ and $P_{1,3}$:

```
def testA13(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    #A_(3,1) - terms
    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a4))
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))
    term3 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a3)*sin(a4))
    sumA31 = term1 + term2 + term3 + term4

    #A_(1,3) - terms
    term5 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a3)*sin(a4))
    term6 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a3)*sin(a4))
    term7 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a2)*sin(a4))
    term8 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))
    term9 = 1
    sumA13 = term5 + term6 + term7 + term8 + term9

    #Converted P_(1,3)-terms
    term10 = t3/(t2+t3)*1/sin(a3)*cos(a1+a2)/sin(a1+a2)
    term11 = t4/(t1+t4)*1/sin(a4)*cos(a1+a2)/sin(a1+a2)
    term12 = -t4/(t3+t4)*1/sin(a4)*cos(a1+a2)/sin(a1+a2)
    term13 = -t3/(t3+t4)*1/sin(a3)*cos(a1+a2)/sin(a1+a2)
    term14 = -t2/(t2+t3)*1/sin(a2)*cos(a1+a2)/sin(a1+a2)
    term15 = -t1/(t1+t4)*1/sin(a1)*cos(a1+a2)/sin(a1+a2)
    sumP13 = term10 + term11 +term12 + term13 + term14 + term15

    if a1+a2 < pi:
        return sumA13 - sumA31 + sumP13

    elif a1+a2 > pi:
        return sumA31 - sumA13 - sumP13

print(testQ1(testA13))
print(testQ2(testA13))
print(testQ3(testA13))
print(testQ4(testA13))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{1,3}$, $A_{3,1}$ and $P_{1,3}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a3))
    term2 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))
    term3 = (t1*t2)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a2))
    term4 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))

    term5 = t1/(t1+t4)*cos(a2)/(sin(a2)*sin(a1))

    return term1 + term2 + term3 - term4 - term5

print(testQ1(testA23))
print(testQ2(testA23))
print(testQ3(testA23))
print(testQ4(testA23))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$:

```
def testA24(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    #A_(2,4)-terms:
    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a4))
    term2 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))
    sumA24 = term1 + term2 + term3

    #A_(4,2)-terms:
    term4 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a3))
    term5 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))
    term6 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a2)*sin(a4))
    sumA42 = term4 + term5 + term6

    #Converted P_(2,4)-terms
    term7 = -t2/(t2+t3)*cos(a2+a3)/(sin(a2+a3)*sin(a2))
    term8 = -t1/(t1+t4)*cos(a2+a3)/(sin(a2+a3)*sin(a1))
    sumP24 = term7 + term8
```

```
    if a2+a3 < pi:
        return sumA24 - sumA42 + sumP24

    elif a2+a3 > pi:
        return sumA42 - sumA24 - sumP24

print(testQ1(testA24))
print(testQ2(testA24))
print(testQ3(testA24))
print(testQ4(testA24))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a3))
    term2 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a3)*sin(a4))
    term3 = 1
    term4 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a3)*sin(a4))
    term5 = (t2*t4)/((t2+t3)*(t1+t4))*1/(sin(a2)*sin(a4))
    term6 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a4))
    term7 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a3)*sin(a4))

    term8 = t2/(t2+t3)*cos(a3)/(sin(a3)*sin(a2))
    term9 = t4/(t3+t4)*cos(a3)/(sin(a3)*sin(a4))
    term10 = t3/(t2+t3)*cos(a3)/(sin(a3)**2)
    term11 = t3/(t3+t4)*cos(a3)/(sin(a3)**2)
    term12 = t4/(t1+t4)*cos(a3)/(sin(a3)*sin(a4))

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    + term8 + term9 - term10 - term11 - term12

print(testQ1(testA34))
print(testQ2(testA34))
print(testQ3(testA34))
print(testQ4(testA34))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a3)*sin(a4))
    term2 = (t3*t4)/((t1+t4)*(t3+t4))*1/(sin(a3)*sin(a4))
    term3 = (t1*t4)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a4))
    term4 = 1
    term5 = (t1*t3)/((t2+t3)*(t1+t4))*1/(sin(a1)*sin(a3))
    term6 = (t1*t3)/((t1+t4)*(t3+t4))*1/(sin(a1)*sin(a3))
    term7 = (t3*t4)/((t2+t3)*(t1+t4))*1/(sin(a3)*sin(a4))

    term8 = t3/(t3+t4)*cos(a4)/(sin(a4)*sin(a3))
    term9 = t1/(t1+t4)*cos(a4)/(sin(a4)*sin(a1))
    term10 = t4/(t3+t4)*cos(a4)/(sin(a4)**2)
    term11 = t4/(t1+t4)*cos(a4)/(sin(a4)**2)
    term12 = t3/(t2+t3)*cos(a4)/(sin(a4)*sin(a3))


    return term1 + term2+ term3 + term4 + term5 - term6 - term7 \
    + term8 + term9 - term10 - term11 - term12

print(testQ1(testA41))
print(testQ2(testA41))
print(testQ3(testA41))
print(testQ4(testA41))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{1,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $P_{1,1}$, $P_{3,3}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 1, 3, 4$. Since the factors in front of $P_{n,n}$, for $n = 1, 3, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{1,1}$, $P_{3,3}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis, proven that all the negative terms in $\mathbf{R}_{1,3,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$. This means that $\mathbf{R}_{1,3,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

### B.2.5 Conclusion

We have now showed that for every $\mathbf{x} \in Q$,

$$\mathbf{R}_{1,3,4}(\mathbf{x}) > 0.$$

Since $w_1, w_3, w_4 > 0$ for all $\mathbf{x} \in Q$, this implies that

$$\mathcal{D}(w_1, w_3, w_4)(\mathbf{x}) > 0,$$

for all $\mathbf{x} \in Q$.

## B.3 Analysis of $\mathbf{R}_{2,3,4}$

We will start by writing down the expression for $\mathbf{R}_{2,3,4}(\mathbf{x})$:

$$\begin{aligned}
\mathbf{R}_{2,3,4}(\mathbf{x}) = {} & \frac{t_2 t_3}{(t_2+t_3)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_3}\Big(A_{3,4} + A_{2,3} + A_{4,2}\Big) \\
& + \frac{t_2 t_4}{(t_2+t_3)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{4,3} + A_{3,1} + A_{2,4} + A_{1,2}\Big) \\
& + \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_2}\Big(P_{3,4} - P_{2,4}\Big) + \frac{t_3 t_4}{(t_2+t_3)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_3 \sin\alpha_4}\Big(A_{4,1} + A_{3,4} + A_{1,3}\Big) \\
& + \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_3}\Big(P_{4,4} - P_{3,4}\Big) + \frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin\alpha_3}\Big(P_{3,3} - P_{3,4}\Big) \\
& + A_{3,4} + \frac{t_1 t_3}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_3}\Big(A_{2,3} + A_{4,2} + A_{3,1} + A_{1,4}\Big) \\
& + \frac{t_1 t_4}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_4}\Big(A_{2,4} + A_{1,2} + A_{4,1}\Big) + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin\alpha_1}\Big(P_{1,4} - P_{2,4}\Big) \\
& + \frac{t_2 t_3}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_3}\Big(A_{4,3} + A_{3,2} + A_{2,4}\Big) + \frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{3,4} - P_{1,3}\Big) \\
& + \frac{t_2 t_4}{(t_1+t_2)(t_3+t_4)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_4}\Big(A_{3,4} + A_{1,3} + A_{4,2} + A_{2,1}\Big) \\
& + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin\alpha_2}\Big(P_{2,4} - P_{3,4}\Big) + \frac{t_3}{t_3+t_4} \cdot \frac{1}{\sin\alpha_3}\Big(P_{2,4} - P_{2,3}\Big) + A_{4,2} \\
& + \frac{t_1 t_2}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_2}\Big(A_{2,3} + A_{1,2} + A_{3,1}\Big) + \frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin\alpha_4}\Big(P_{1,2} - P_{2,4}\Big) \\
& + \frac{t_1 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin\alpha_1 \sin\alpha_3}\Big(A_{3,2} + A_{2,4} + A_{1,3} + A_{4,1}\Big) \\
& + \frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin\alpha_1}\Big(P_{2,3} - P_{1,3}\Big) + \frac{t_2 t_3}{(t_1+t_2)(t_2+t_3)} \cdot \frac{1}{\sin\alpha_2 \sin\alpha_3}\Big(A_{3,4} + A_{2,3} + A_{4,2}\Big) \\
& + \frac{t_2}{t_1+t_2} \cdot \frac{1}{\sin\alpha_2}\Big(P_{3,3} - P_{2,3}\Big) + \frac{t_2}{t_2+t_3} \cdot \frac{1}{\sin\alpha_2}\Big(P_{2,2} - P_{2,3}\Big) \\
& + \frac{t_3}{t_2+t_3} \cdot \frac{1}{\sin\alpha_3}\Big(P_{2,3} - P_{2,4}\Big) + A_{2,3}.
\end{aligned}$$

Since the analysis of $\mathbf{R}_{2,3,4}(\mathbf{x})$ is so similar to the analysis we did for $\mathbf{R}_{1,2,3}(\mathbf{x})$ in section 3.2, we will only present the results in this section. When we are preforming a numerical analysis we will use the code we presented in section A.1.2. Note that we will use the properties we derived in section 3.2 when we preform the different analyses here.

### B.3.1 Analysis for when $\mathbf{x}$ is the intersection between the diagonals

$f^-(A_{1,2}, A_{2,1})$:

$$\begin{aligned} f^-(A_{1,2}, A_{2,1}) &= \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &- \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

$f^-(A_{2,3}, A_{3,2})$:

$$\begin{aligned} f^-(A_{2,3}, A_{3,2}) &= \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{3t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &- \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

$f^-(A_{3,4}, A_{4,3})$:

$$\begin{aligned} f^-(A_{3,4}, A_{4,3}) &= \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{3t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &- \frac{t_2^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} - \frac{t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

$f^-(A_{4,1}, A_{1,4})$:

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &- \frac{t_1^2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

Analysis of terms containing $P_{3,3}$, $P_{3,4}$ and $P_{4,4}$:
We will now check if $f^+(P_{3,3}, P_{3,4}, P_{4,4}) \geq 0$, but we will leave out two terms, namely $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{3,3}$ and $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{3,4}$. As we will see later, these terms will be used in other analyses, so we will leave them out here so we do not use them twice. It is also worth noticing that we have two terms containing $P_{3,4}$ that are cancelling each other out, namely $(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{3,4}$ and $-(\frac{t_2}{t_1+t_2} \cdot \frac{1}{s})P_{3,4}$.

$$f^+(P_{3,3}, P_{3,4}, P_{4,4}) - \frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{3,3} + P_{3,4}\Big) = \frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{3,3} - 2P_{3,4} + P_{4,4}\Big)$$
$$> 0.$$

This follows from the definition of $P_{n,m}$, and the fact that $\cos(\alpha_3) \leq 1$.

Analysis of terms containing $P_{2,2}$, $P_{2,3}$ and $P_{3,3}$:
We will now check if $f^+(P_{2,2}, P_{2,3}, P_{3,3}) \geq 0$, but we will leave out two terms, namely $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{2,3}$ and $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{3,3}$. As we will see, we leave out $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{2,3}$ so we can use it in another analysis later on, while we leave out $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{3,3}$ because we have used this term in an earlier analysis. It is also worth noticing that we have two terms containing $P_{2,3}$ that are cancelling each other out, namely $(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{2,3}$ and $-(\frac{t_1}{t_1+t_2} \cdot \frac{1}{s})P_{3,4}$.

$$f^+(P_{2,2}, P_{2,3}, P_{3,3}) - \frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{2,3} + P_{3,3}\Big) = \frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{2,2} - 2P_{2,3} + P_{3,3}\Big)$$
$$> 0.$$

This follows from the definition of $P_{n,m}$, and the fact that $\cos(\alpha_2) \leq 1$.

Analysis of the remaining terms containing $P_{n,m}$:
We observe that the remaining terms containing $P_{n,m}$ gives us the two sums

$$\frac{t_1}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{2,3} + P_{1,4} - P_{2,4} - P_{1,3}\Big), \tag{B.9}$$

and

$$\frac{t_2}{t_1+t_2} \cdot \frac{1}{s}\Big(P_{1,2} + P_{3,4} - P_{2,4} - P_{1,3}\Big). \tag{B.10}$$

First, we observe that $-P_{1,3}, -P_{2,4} < 0$, since $\cos(\pi) = -1$. Second, we observe that (B.9) is greater than zero when $\alpha_2 = \alpha_4 \leq \frac{\pi}{2}$, while (B.10) is greater than zero when $\alpha_1 = \alpha_3 \leq \frac{\pi}{2}$. Let us now look at the case where $\alpha_2 = \alpha_4 > \frac{\pi}{2}$. In this case, we can not guarantee that (B.9) is greater than or equal to zero. We will therefore use Lemma 3.2.2 to convert $P_{2,3}, P_{1,4}$ into $A_{2,3}, A_{4,1}$, and then we will put these expressions into the analyses of the terms containing $A_{2,3}$ and $A_{4,1}$ respectively.

$f^-(A_{2,3}, A_{3,2})$ (new expression marked in blue):

$$\begin{aligned} f^-(A_{2,3}, A_{3,2}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 + \frac{t_1}{t_1+t_2} \cdot \frac{\cos\alpha_2}{s^2} \\ &> \frac{-t_1^2 + t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_2 > \frac{\pi}{2}$, we know that $\alpha_1 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_1 < t_2$, which again implies that $t_1^2 < t_1t_2$, which then implies that $\frac{-t_1^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 > 0$.

We will now make a similar argument for $f^-(A_{4,1}, A_{1,4})$.

$f^-(A_{4,1}, A_{1,4})$:

$$\begin{aligned} f^-(A_{4,1}, A_{1,4}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_1}{t_1+t_2} \cdot \frac{\cos\alpha_4}{s^2} \\ &> \frac{-t_1^2 + t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_2 > \frac{\pi}{2}$, we know that $\alpha_1 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_1 < t_2$, which again implies that $t_1^2 < t_1t_2$, which then implies that $\frac{-t_1^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} > 0$.

Similarly, when $\alpha_1 = \alpha_3 > \frac{\pi}{2}$, we get the following analysis:

$f^-(A_{1,2}, A_{2,1})$ :

$$\begin{aligned} f^-(A_{1,2}, A_{2,1}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + \frac{t_2}{t_1+t_2} \cdot \frac{\cos\alpha_1}{s^2} \\ &> \frac{-t_2^2 + t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} \\ &> 0. \end{aligned}$$

Since we are looking at the case where $\alpha_1 > \frac{\pi}{2}$, we know that $\alpha_2 < \frac{\pi}{2}$ (since $\alpha_1 + \alpha_2 = \pi$). This means that $t_2 < t_1$, which again implies that $t_2^2 < t_1t_2$, which then implies that $\frac{-t_2^2+t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} > 0$.

$f^-(A_{3,4}, A_{4,3})$:

$$\begin{aligned} f^-(A_{3,4}, A_{4,3}) &= \frac{2t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 + \frac{t_2}{t_1+t_2} \cdot \frac{\cos\alpha_3}{s^2} \\ &> \frac{-t_2^2 + t_1t_2}{(t_1+t_2)^2} \cdot \frac{1}{s^2} + 1 \\ &> 0. \end{aligned}$$

We have now proven analytically that that $\mathbf{R}_{2,3,4}(\mathbf{x}) > 0$, when $\mathbf{x}$ is the intersection between the diagonals.

### B.3.2 Analysis for $\mathbf{x} \in \{d_1 \cup d_3\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term2 = (t1*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term3 = (t1*t2)/((t1+t2)*(t2+t3)) *1/(sin(a1)**2)
    term4 = (t2*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a4))
```

```python

    term5 = t4/(t3+t4)*cos(a1)/(sin(a1)*sin(a4))

    return term1 + term2 + term3 - term4 + term5

print(testd1d3(testA12))
```

Output: Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```python
def testA23(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)**2)
    term4 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term5 = 1
    term6 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))

    term8 = t2/(t1+t2)*cos(a1)/(sin(a1)**2)
    term9 = t2/(t2+t3)*cos(a1)/(sin(a1)**2)
    term10 = t3/(t3+t4)*cos(a1)/(sin(a1)*sin(a4))
    term11 = t1/(t1+t2)*cos(a1)/(sin(a1)**2)
    term12 = t3/(t2+t3)*cos(a1)/(sin(a1)*sin(a4))

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    +term8 + term9 + term10 - term11 - term12

print(testd1d3(testA23))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$:

```python
def testA24(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term4 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term5 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term6 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term8 = 1
    term9 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))

    term10 = t2/(t1+t2)*cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))
    term11 = t3/(t3+t4)*cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))
```

```
    term12 = t2/(t2+t3)*cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))
    term13 = t1/(t1+t2)*cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a1))
    term14 = t4/(t3+t4)*cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))
    term15 = t3/(t2+t3)*cos(2*pi-a1-a4)/(sin(2*pi-a1-a4)*sin(a4))

    return term1 + term2 + term3 + term4 - term5 - term6 \
    - term7 - term8 - term9 + term10 + term11 - term12 \
    - term13 - term14 - term15

print(testd1d3(testA24, True))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term2 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a4)**2)
    term3 = 1
    term4 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term5 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term6 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a4))
    term7 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))

    term8 = t3/(t2+t3)*cos(a4)/(sin(a4)**2)
    term9 = t3/(t3+t4)*cos(a4)/(sin(a4)**2)
    term10 = t2/(t1+t2)*cos(a4)/(sin(a4)*sin(a1))
    term11 = t2/(t2+t3)*cos(a4)/(sin(a4)*sin(a1))
    term12 = t4/(t3+t4)*cos(a4)/(sin(a4)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    + term8 + term9 + term10 - term11 - term12

print(testd1d3(testA34))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a4):
    t1 = tan(a1/2); t2 = tan(pi/2-a1/2)
    t3 = tan(pi/2-a4/2); t4 = tan(a4/2)

    term1 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a4)**2)
    term2 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a4))
    term4 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))

    term5 = t1/(t1+t2)*cos(a4)/(sin(a4)*sin(a1))

    return term1 + term2+ term3 - term4 + term5
```

```
print(testd1d3(testA41))
```

Output: Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_1 \cup d_3\}$.

Analysis of terms containing $P_{1,3}$:
We observe that we have two terms containing $P_{1,3}$;

$$-\frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin(\alpha_4)} P_{1,3}, \tag{B.11}$$

and

$$-\frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)} P_{1,3}. \tag{B.12}$$

We know that $\cos(\alpha_1 + \cos(\alpha_3)) = \cos(\pi) = -1$, when $\mathbf{x} \in \{d_1 \cup d_3\}$, which implies that $P_{1,3} < 0$. Since both $\frac{t_4}{t_3+t_4} \cdot \frac{1}{\sin(\alpha_4)}$ and $\frac{t_1}{t_1+t_2} \cdot \frac{1}{\sin(\alpha_1)}$ are positive, this implies that both (B.11) and (B.12) are positive.

Analysis of terms containing $P_{2,2}$, $P_{3,3}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 2, 3, 4$. Since the factors in front of $P_{n,n}$, for $n = 2, 3, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{1,1}$, $P_{3,3}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis, proven that all the negative terms in $\mathbf{R}_{2,3,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{d_1 \cup d_3\}$. This means that $\mathbf{R}_{2,3,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{d_1 \cup d_3\}$.

### B.3.3 Analysis for $\mathbf{x} \in \{d_2 \cup d_4\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4)) *1/(sin(a1)*sin(a2))
    term2 = (t1*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)**2)
    term3 = (t1*t2)/((t1+t2)*(t2+t3)) *1/(sin(a1)*sin(a2))
    term4 = (t2*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a2))

    term5 = t4/(t3+t4)*cos(a1)/(sin(a1)**2)

    return term1 + term2 + term3 - term4 + term5

print(testd2d4(testA12))
```

Output: Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$

is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{3,1}$, $A_{1,3}$ and $P_{1,3}$:

```
def testA31(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term2 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term4 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term5 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))
    term6 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))

    term7 = t4/(t3+t4)*cos(a1+a2)/(sin(a1+a2)*sin(a1))
    term8 = t1/(t1+t2)*cos(a1+a2)/(sin(a1+a2)*sin(a1))

    return term1 + term2 + term3 - term4 - term5 - term6 \
    + term7 + term8

print(testd2d4(testA13), True)
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,1}$, $A_{1,3}$ and $P_{1,3}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)**2)
    term2 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term4 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)**2)
    term5 = 1
    term6 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)**2)
    term7 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))

    term8 = t1/(t1+t2)*cos(a2)/(sin(a2)*sin(a1))
    term9 = t3/(t2+t3)*cos(a2)/(sin(a2)**2)
    term10 = t3/(t3+t4)*cos(a2)/(sin(a2)**2)
    term11 = t2/(t1+t2)*cos(a2)/(sin(a2)**2)
    term12 = t2/(t2+t3)*cos(a2)/(sin(a2)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    +term8 + term9 - term10 - term11 - term12

print(testd2d4(testA23))
```

Output: Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)**2)
    term2 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term3 = 1
    term4 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))
    term5 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)**2)
    term6 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term7 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)**2)

    term8 = t3/(t2+t3)*cos(a2)/(sin(a2)**2)
    term9 = t3/(t3+t4)*cos(a2)/(sin(a2)**2)
    term10 = t2/(t1+t2)*cos(a2)/(sin(a2)**2)
    term11 = t2/(t2+t3)*cos(a2)/(sin(a2)**2)
    term12 = t4/(t3+t4)*cos(a2)/(sin(a2)*sin(a1))

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    + term8 + term9 + term10 - term11 - term12

print(testd2d4(testA34))
```

Output: Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a1)*sin(a2))
    term2 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)**2)
    term3 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term4 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a2))

    term5 = t1/(t1+t2)*cos(a1)/(sin(a1)**2)

    return term1 + term2+ term3 - term4 - term5

print(testd2d4(testA41))
```

Output: Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $P_{2,4}$:
We observe that we have six terms containing $P_{2,4}$. Since $P_{2,4} < 0$ (follows from the fact that $\cos(\pi) = -1$), we need to check if the sum of the factors in front of $-P_{2,4}$ are greater than the sum of the factors in front of $P_{2,4}$. We will do this analysis in the code below.

```
def testP24(a1, a2):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(pi/2-a2/2); t4 = tan(pi/2-a1/2)

    term1 = t2/(t2+t3)*1/sin(a2)
    term2 = t1/(t1+t2)*1/sin(a1)
    term3 = t4/(t3+t4)*1/sin(a1)
    term4 = t3/(t2+t3)*1/sin(a2)

    term5 = t2/(t1+t2)*1/sin(a2)
    term6 = t3/(t3+t4)*1/sin(a2)

    return term1 + term2 + term3 + term4 - term5 - term6

print(testd2d4(testP24))
```

Output: Done

This test indicates that the sum of all terms containing $P_{2,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{d_2 \cup d_4\}$.

Analysis of terms containing $P_{2,2}$, $P_{3,3}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 2, 3, 4$. Since the factors in front of $P_{n,n}$, for $n = 2, 3, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{2,2}$, $P_{3,3}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis proven that all the negative terms in $\mathbf{R}_{2,3,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{d_2 \cup d_4\}$. This means that $\mathbf{R}_{2,3,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{d_2 \cup d_4\}$.

### B.3.4 Analysis for $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$

Analysis of terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$:

```
def testA12(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t4)/((t2+t3)*(t3+t4)) *1/(sin(a2)*sin(a4))
    term2 = (t1*t4)/((t1+t2)*(t3+t4)) *1/(sin(a1)*sin(a4))
    term3 = (t1*t2)/((t1+t2)*(t2+t3)) *1/(sin(a1)*sin(a2))
    term4 = (t2*t4)/((t1+t2)*(t3+t4)) *1/(sin(a2)*sin(a4))

    term5 = t4/(t3+t4)*cos(a1)/(sin(a1)*sin(a4))

    return term1 + term2 + term3 - term4 + term5

print(testQ1(testA12))
print(testQ2(testA12))
print(testQ3(testA12))
print(testQ4(testA12))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{1,2}$, $A_{2,1}$ and $P_{1,2}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{1,3}$, $A_{3,1}$ and $P_{1,3}$:

```
def testA13(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    #A_(3,1) - terms
    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a4))
    term2 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))
    term3 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    sumA31 = term1 + term2 + term3

    #A_(1,3) - terms
    term4 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a3)*sin(a4))
    term5 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a4))
    term6 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))
    sumA13 = term4 + term5 + term6

    #Converted P_(1,3)-terms
    term7 = -t4/(t3+t4)*cos(a1+a2)/(sin(a1+a2)*sin(a4))
    term8 = -t1/(t1+t2)*cos(a1+a2)/(sin(a1+a2)*sin(a1))
    sumP13 = term7 + term8

    if a1+a2 < pi:
        return sumA13 - sumA31 + sumP13

    elif a1+a2 > pi:
        return sumA31 - sumA13 - sumP13

print(testQ1(testA13))
print(testQ2(testA13))
print(testQ3(testA13))
print(testQ4(testA13))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{1,3}$, $A_{3,1}$ and $P_{1,3}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$:

```
def testA23(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a3))
    term2 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))
    term3 = (t1*t2)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a2))
    term4 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)*sin(a3))
    term5 = 1
    term6 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a3))
    term7 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))

    term8 = t1/(t1+t2)*cos(a2)/(sin(a2)*sin(a1))
    term9 = t3/(t2+t3)*cos(a2)/(sin(a2)*sin(a3))
    term10 = t3/(t3+t4)*cos(a2)/(sin(a2)*sin(a3))
    term11 = t2/(t1+t2)*cos(a2)/(sin(a2)**2)
    term12 = t2/(t2+t3)*cos(a2)/(sin(a2)**2)

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    +term8 + term9 - term10 - term11 - term12

print(testQ1(testA23))
print(testQ2(testA23))
print(testQ3(testA23))
print(testQ4(testA23))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{2,3}$, $A_{3,2}$ and $P_{2,3}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$:

```
def testA24(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    #A_(2,4)-terms:
    term1 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a4))
    term2 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a3))
    term4 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))
    sumA24 = term1 + term2 + term3 + term4

    #A_(4,2)-terms:
    term5 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a3))
    term6 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))
    term7 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a4))
    term8 = 1
    term9 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)*sin(a3))
    sumA42 = term5 + term6 + term7 + term8 + term9

    #Converted P_(2,4)-terms
    term10 = t2/(t1+t2)*cos(a2+a3)/(sin(a2+a3)*sin(a2))
    term11 = t3/(t3+t4)*cos(a2+a3)/(sin(a2+a3)*sin(a3))
```

```
    term12 = -t2/(t2+t3)*cos(a2+a3)/(sin(a2+a3)*sin(a2))
    term13 = -t1/(t1+t2)*cos(a2+a3)/(sin(a2+a3)*sin(a1))
    term14 = -t4/(t3+t4)*cos(a2+a3)/(sin(a2+a3)*sin(a4))
    term15 = -t3/(t2+t3)*cos(a2+a3)/(sin(a2+a3)*sin(a3))
    sumP24 = term10 + term11 + term12 + term13 + term14 + term15

    if a2+a3 < pi:
        return sumA24 - sumA42 + sumP24
    elif a2+a3 > pi:
        return sumA42 - sumA24 - sumP24

print(testQ1(testA24))
print(testQ2(testA24))
print(testQ3(testA24))
print(testQ4(testA24))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{2,4}$, $A_{4,2}$ and $P_{2,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$:

```
def testA34(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t2*t3)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a3))
    term2 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a3)*sin(a4))
    term3 = 1
    term4 = (t2*t4)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a4))
    term5 = (t2*t3)/((t1+t2)*(t2+t3))*1/(sin(a2)*sin(a3))
    term6 = (t2*t4)/((t2+t3)*(t3+t4))*1/(sin(a2)*sin(a4))
    term7 = (t2*t3)/((t1+t2)*(t3+t4))*1/(sin(a2)*sin(a3))

    term8 = t2/(t2+t3)*cos(a3)/(sin(a3)*sin(a2))
    term9 = t4/(t3+t4)*cos(a3)/(sin(a3)*sin(a4))
    term10 = t3/(t2+t3)*cos(a3)/(sin(a3)**2)
    term11 = t3/(t3+t4)*cos(a3)/(sin(a3)**2)
    term12 = t2/(t1+t2)*cos(a3)/(sin(a3)*sin(a2))

    return term1 + term2 + term3 + term4 + term5 - term6 - term7\
    + term8 + term9 - term10 - term11 - term12

print(testQ1(testA34))
print(testQ2(testA34))
print(testQ3(testA34))
print(testQ4(testA34))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{3,4}$, $A_{4,3}$ and $P_{3,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$:

```
def testA41(a1, a2, a3, a4):
    t1 = tan(a1/2); t2 = tan(a2/2)
    t3 = tan(a3/2); t4 = tan(a4/2)

    term1 = (t3*t4)/((t2+t3)*(t3+t4))*1/(sin(a3)*sin(a4))
    term2 = (t1*t4)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a4))
    term3 = (t1*t3)/((t1+t2)*(t2+t3))*1/(sin(a1)*sin(a3))
    term4 = (t1*t3)/((t1+t2)*(t3+t4))*1/(sin(a1)*sin(a3))

    term5 = t1/(t1+t2)*cos(a4)/(sin(a4)*sin(a1))

    return term1 + term2 + term3 - term4 + term5

print(testQ1(testA41))
print(testQ2(testA41))
print(testQ3(testA41))
print(testQ4(testA41))
```

Output: Done
Done
Done
Done

This test indicates that the sum of all terms containing $A_{4,1}$, $A_{1,4}$ and $P_{1,4}$ in $\mathbf{R}_{2,3,4}(\mathbf{x})$ is strictly greater than zero when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

Analysis of terms containing $P_{2,2}$, $P_{3,3}$ and $P_{4,4}$:
We know that $\cos(\alpha_{n,n}) = \cos(0) = 1$, which implies that $P_{n,n} > 0$, for $n = 2, 3, 4$. Since the factors in front of $P_{n,n}$, for $n = 2, 3, 4$, are greater than zero (this follows from the definition of mean value coordinates for convex polygons), we know that all the terms containing $P_{2,2}$, $P_{3,3}$ and $P_{4,4}$ are greater than zero.

We have now, with the help of some numerical analysis, proven that all the negative terms in $\mathbf{R}_{2,3,4}(\mathbf{x})$ are strictly dominated by their positive counterparts when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$. This means that $\mathbf{R}_{2,3,4}(\mathbf{x}) > 0$ when $\mathbf{x} \in \{Q_1 \cup Q_2 \cup Q_3 \cup Q_4\}$.

### B.3.5 Conclusion

We have now showed that for every $\mathbf{x} \in Q$,

$$\mathbf{R}_{2,3,4}(\mathbf{x}) > 0.$$

Since $w_2, w_3, w_4 > 0$ for all $\mathbf{x} \in Q$, this implies that

$$\mathcal{D}(w_2, w_3, w_4)(\mathbf{x}) > 0,$$

for all $\mathbf{x} \in Q$.

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