Risk Measures and Differential Games

by

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Abstract

We study risk measures in relation to stochastic differential games in a Lévy -market. We minimize a risk measure to get a min-max problem. The problem is to find an optimal solution for a convex risk measure in zero-sum games with a 3-dimensional controller. To verify a solution we develop a Hamilton-Jacobi-Bellman-Isacs (HJBI) equation and prove it. Moreover we provide a Nash-equilibrium game that includes scenario optimization. These results are illustrated by entropic risk measure and more general cases. Further, a HJBI equation for dynamic risk measures are shown and proven. We extend our convex risk measure model to include stopping control. Last, a theorem for viscosity solutions are shown and proven.

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CONTENTS

GLOSSARY

$:=, \equiv$	Defined as	10
$L^p(\Omega, F, P)$	The space of all p-power P-integrable $B \times F$-measurable functions	10
$N(A)$	The Jump measure	13
$P \ll Q$	The measure P is absolute continuous w.r.t the measure Q	17
$\mathcal{B}$	The Borel σ-algebra	12
$\mathcal{M}$	Sett of all probability measures, see definition 2.6.1	17
$\mathcal{M}_a$	Set of all probability measures of Girsanov transformations (2.6.2)	18
$\nu(A)$	The intensity measure for a jump measure	12
$\omega \in \Omega$	Scenario of randomness	10
$\mathbb{R}^n$	The n-dimensional Euclidean space	20
$\sigma(A)$	The smallest σ-algebra so that A is measurable	14
τ_A	The first exit time from the set A of X_t: $\tau_A = \inf\{t > 0 : X_t \notin A\}$	36
$\tilde{N}(A)$	Compensated Poisson random measure; $\tilde{N}(A) = N(A) - \nu(A)$	13
$\varphi_1, \varphi_{11}, \varphi_2, \varphi_{22}, \varphi_{12}$	$\frac{\partial\varphi}{\partial y_1}, \frac{\partial^2\varphi}{\partial^2 y_1}, \frac{\partial\varphi}{\partial y_2}, \frac{\partial^2\varphi}{\partial^2 y_2}$ and $\frac{\partial^2\varphi}{\partial y_1 \partial y_2}$ respectively	48
$ess \inf f$	$\sup\{K \in \mathbb{R} : f \geq K a.s.\}$	24
càdlàg	Right-continuous with left limits	13, 18
càglàd	Left-continuous with right limits	18
a.a.	Almost always	13
a.s.	Almost surely	18
BSDE	Backward stochastic differential equation	14, 15
ELMM	Equivalent local martingale measure	18
EMM	Equivalent martingale measure	18
FBSDE	Forward-Backward stochastic differential equation	16
P&L	Profit and loss statement	10
r.v.	Random variable	63
VaR	Value at Risk, a non-coherent risk measure	19

CHAPTER 1

INTRODUCTION

> "Uncertainty and mystery are energies of life. Don't let them scare you unduly, for they keep boredom at bay and spark creativity" - R. I. Fitzhenry

Life is unpredictable and we forced to accept the fact that the future is uncertain. The ability to recognize, quantify and calculate risk has proven vital for the evolvement and survival of mankind. We are daily dependent on our ability to calculate risks. You may for example decide that the risk of running a red light is acceptable if you are in a hurry. As well as being able to change our lives in matter of second's, uncertainty are one of the elements that provide meaning to our lives. Uncertainty fascinates and games of chance have existed almost as long as human civilization. Since uncertainty is such a vital part of our lives we need to be able to understand, represent and quantify it.

Measuring and managing risks is one of the key disciplines in the financial world. The ability to analyze and measure a positions exposure to risk provides not only managers, but also regulators, with powerful information and insight. Risk management provides methods to determine how to best handle different risk exposures and identify acceptable positions.

The last year or so has shaken the very foundation of modern economics. Keynesian economics has convinced many right-winged, no-market intervention fundamentalist. Henry Paulson, who was a firm believer in non-market intervention, ended up as the treasury secretary that has performed the greatest market interventions in the history of the US. The bank run that led to the fall of Northern Rock, the acquiring of Bear Stearns by JP Morgan Chase, the overtaking of Fannie Mae and Freddie Mac by the U.S. government, the fall of Lehman Brothers, the sell-off of Merrill Lynch, the collapse of Iceland and the saving of A.I.G. started a chain of events that would affect every aspect of the economy. It was a liquidity crisis that required an injection of vast amount of capital into the financial market by institutions like the United States Federal Reserve, Bank of England and the European Central Bank. A $700 billion bailout bill was rushed into law by the United States government, and billions more pumped into struggling companies. Several banks have, in effect, been nationalized (Northern Rock was nationalized by an embarrassed British government) real estate prices are tumbling and it is harder to secure a home loan. This is the reality and everyone is asking, or should be asking; what went wrong?

The constant search for higher yields has lead to a high demand for exotic instruments. This has resulted in a rapid development of complex and often poorly tested structured products. Using these new mathematical models rare events could be seen as "Black Swans". Financial risks were "normalized" and suddenly rare events were non-existing. The industry wide embracement of David X. Li's Gaussian copula model function, that assumed that the price of Credit Default Swaps was correlated with mortgage backed securities, strengthened this camouflage. All of this allowed US banks to lower their requirements for sub-prime loans. "Ninjas", people with "no income, no job or assets", were generously given loans even though they had no hope of repaying them. These loans were then packaged into collateralized debt obligations (CDO's) and sold off. This took the loans off the bank's balance sheets and the banks were able to lend out even more money. As it turns out this complex financial instruments constructed by the large investment banks and other financial institutions were economic bombs waiting to go off.

In an extensive article in the New York Times, January 4. 2009, Joe Nocera [2009] discussed the role risk measures, especially value at risk (VaR), played in the financial crisis. As he states: "the fact that risk measures, such as VaR, do not measure the possibility of an extreme event was a blessing to the executives. It made the black swans all the easier to ignore." Everyone slept easy as long as the VaR value was acceptable. Some people, like Taleb [2007], tried to point out our blindness with respect to randomness. But few stopped to listen.

Regulators such as the U.S. Securities and Exchange Commission and the Financial Services Authority are supposed to put a restrain on greedy executives, investors, analyst and other financial players. They seemed to trust that the banks and investment firms were run by people that understand and adhered to the financial risks. With all the mathematical formulas, complex instruments and leveraged deals they trusted them to have control. But it may seems like many of these institutions were driven by the search for bonuses and higher returns and that the concept of social responsibility and risk management were neglected or had a low priority. It is now painfully clear that the risks in the largely unregulated collateralized debt obligation and credit default swap markets was catastrophically underestimated. As the financial system is based on credit creation this was a high stake game. But it was in everyone's interest to pretend the boom could go on forever, and that securitization had taken the risk out of lending money. As the former Citigroup chief executive Charles Prince said, "As long as the music is playing, you've got to get up and dance."

Finally, regulators were left in no doubt of the perils hiding in the financial system. It became unavoidable obvious that the risks taken by these banks and investment firms were excessive and non-neglectable. After the collapse of Lehman Brothers institutions all over the world realized the scale of the threat. They promptly initiated stimulations to the economy that today seems to have had a great effect.

Recent events are blamed on an extremely indebted US economy. It seems that, even with precautions like the Glass-Steagall Act, history keeps repeating itself and debt deflations are an unavoidable flaw within the financial system.

In light of the recent event, risk management has proven more valuable than ever before. It has shown that it is of great importance for banks and investment firms to review their risk management procedures and controls. The new market conditions and FSA expectations require that the risk frameworks must be reconsidered and adapted to reflect the new economic reality. Risk measures, such as VaR the most commonly used in the industry, is not structured in an axiomatic theory and do not adhere to a mathematic approach. A well-defined mathematically theory for risk measures that adheres to financial reality is vital. This axiomatic way of defining a risk measure is provided in the papers by Artzner et al. [1997], Artzner et al. [1999] and Delbaen [2000]. This represented a breakthrough in financial mathematics as well as risk management. It was the first attempt at a definition of a quantitative theory. To establish this mathematically sound approach to risk measures Artzner et al. [1999] list four axioms that are inspired by a supervisors point of view and based on financial theory;

(I) Translation invariance. For all $x \in X$ and all real numbers $\alpha, \rho(x + \alpha r) + \rho(x) - \alpha$,
(II) Sub additivity. For all x_1 and $x_2 \in X$, $\rho(x_1 + x_2) \leq \rho(x_1) + \rho(x_2)$,
(III) Positive homogeneity. For all $\lambda \geq 0$ and all $x \in X$, $\rho(\lambda x) = \lambda \rho(x)$,
(IV) Monotonicity. For all x and $y \in X'$ with $x \leq y, \rho(y) \leq \rho(x)$.

While these are reasonable assumptions, it can be argued that a position that is large relative to the market could be less liquid, and therefore more risky, than that it of smaller positions. Föllmer and Leukert [1999] took this into account and constructed the convex risk measure. The idea was thoroughly defined by Föllmer and Penner [2006], Föllmer and Schied [2002], Föllmer and Schied [2002] and Frittelli and Gianin [2002]. Convex risk measures were defined by replacing the requirement of sub additivity in coherent risk measures with the requirement;

(II') Convexity. $\rho(\lambda x + (1 - \lambda) y) \leq \lambda \rho(x) + (1 - \lambda) \rho(y)$ for any $\lambda \in [0, 1]$.

Risk managers work in a time-changing world, thus a static measure seems restrictive. Föllmer and Leukert [1999] gives a construction for a dynamic risk measure that takes into account the flexibility and dynamics of time. Before we give a thorough review of the existing work on the subject of risk measures in chapter 3, we will in chapter 2 go through the necessary notations and mathematical background.

Finding an optimal investment strategy for an investor with a given utility function and a fixed initial endowment is s well studied and frequent problem in financial mathematics. To solve optimization problems one could try find a set of necessary conditions that an optimal solution must satisfy, but this is often very complex and difficult to solve. An important result given by Pontryagin is the maximum principle. This principle states that any optimal control must solve the Hamiltonian system (a forward-backward differential equation) and a maximum condition for the function called the Hamiltonian. Another powerful approach for optimal control problems is the method of dynamic programming. This approach was pioneered by R. Bellman in the 1950's. He considered a family of optimal controls with varying initial times and states to give a relationship among them known as the Hamilton-Jacobi-Bellman equation (HJB-equation). The HJB-equation is a nonlinear second-order partial differential equation (in the stochastic case). This is a verification technique which provides a solution to the whole family of problems. Bellman [1957][p.83] describes the principle of optimality as,

> "An optimal policy has the property that whatever the initial stat and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state from the first decision."

Optimizing a portfolio is one of the classical problems in finance. Merton applied classical stochastic control methods to reduce the portfolio problem to a matter of solving a HJB-equation using dynamic programming. This has later been extended to more realistic models like the jump-diffusion model. Inspired by the game theoretic approach in Mataramvura and Øksendal [2008] we will investigate the problem of optimizing a performance functional which extends on risk measures. This leads to a min-max problem of stochastic optimization. Then our problem will be to find an optimal portfolio process while the market controls the scenarios and a control process. This results in an asymmetric game with a performance functional that extends the case of convex risk measures, and where the solution is characterized by a HJBI equation. The relation from risk measures to game theory has been investigated by Delbaen [2002]. In chapter 4 we show a zero-sum stochastic differential game between an agent and a market, this extends Mataramvura and Øksendal [2008] to a 3-dimensional case. We establish a connection between the two dimensional and the three dimensional problem. Moving on from zero-sum games to Nash-equilibrium we will, in chapter 5, construct and prove a HJBI equation for a Nash-equilibrium. This is an extension from the case in Mataramvura and Øksendal [2008] to a setting for two players where the market plays a role through scenarios, a scenario driven Nash-equilibrium HJBI. Next, in chapter 6, we will step out of the static setting and show and prove a HJBI equation for dynamic risk measures. Our problem will be to optimize a dynamic risk measures constructed from a g-expectation.

Optimal stopping problems are a class of mathematical problems in which a player may stop a randomly moving process, such as a Levy process, in order to claim a prize equal in value to some predefined function of the random process at the time of stopping. A fundamental problem is to establish an optimal stopping strategy according to some optimization criteria. To allow our model to extend to optimal stopping problems, we will in chapter 7 include stopping control. This is an useful extension of our model and has many real-life applications.

In the classical dynamic programming we require that the HJB equation admits a classical solution, i.e. a smooth solution. As this is not always the case Crandall and Lions introduced the viscosity solution in the 1980's. The requirement of a smooth solution is replaced by a super/sub differential. Under some mild conditions the uniqueness of the solution is guaranteed. In chapter 8 we show and prove that the value function Φ under some conditions is a viscosity solution to our HJBI equation.

In the last chapter we will review our findings and discuss the vital parts of our paper. Finally we look at some possibilities for further research.

Part 1

THEORETICAL BACKGROUND AND NOTATION

CHAPTER 2

NOTATION AND MATHEMATICAL BACKGROUND

In this chapter we review notations and technical language used throughout this paper. We will construct our market model and look at the basics of a Lévy market. Further, we will discuss the theory behind backward differential equations which will prove useful for us in chapter 6.

2.1. Defintions

First, let us define a topological vector space as in Pedersen [1995], definition 2.4.1;

Definition 2.1.1. A *topological vector space is a vector space X equipped with a Hausdorff topology such that the vector operations are continuous, i.e.*

$$(x, y) \to x + y,$$
$$(\alpha, x) \to \alpha x,$$

are continuous with respect to the product topology.

In this paper we let X be a normed topological vector space, understanding that X represents the space of financial positions who's risk we need to measure. We let X' be the dual space (which is a Banach space, $B(X, \mathbb{R})$) consisting of real functionals on X with the weak *-topology (2.4.2 in Pedersen [1995]). Thus $x \in X$ represent the portfolios profit and loss statements (P&L). However, with some abuse of terminology x could represent the portfolios themselves. We let X'

be the space of our risk measures. Our state space will be the d-dimensional Euclidean space equipped with the σ-field of Borel sets. We will assume that

$$X = L^p(\Omega, \mathcal{F}, \mathcal{P}); 1 \leq p \leq +\infty$$
$$X' = L^q(\Omega, \mathcal{F}, \mathcal{P}); 1 \leq q \leq +\infty.$$

One example would be to let p and q be conjugate ($\frac{1}{p} + \frac{1}{q} = 1$) and $\tau = \sigma(L^p, L^q)$. In this paper we will consider $p = 2$ and define a scalar product between two elements in X as $(x, y)_{L_2} := E[xy]$ so that this becomes a Hilbert space. We say that two variables x, y in X is orthogonal if $(x, y)_{L_2} = 0$. Let X'_+ be the set of all positive bounded linear functionals on X,

$$X'_+ \equiv \{x' \in X' | x'(x) \leq 0 \forall x \in X : x \leq 0\}$$

For a fixed point $\omega \in \Omega$, $t \to x_t(\omega)$ represents a sample path associated with ω. We will often use the notion of *càdlàg* and *càglàd* processes. For two elements in X, let

$$f(t-) = \lim_{s \to t, s<t} f(s),$$

and

$$f(t+) = \lim_{s \to t, s>t} f(s).$$

If a function $f : [0, T] \to \mathbb{R}^d$ is right-continuous with left limits, e.g for each $t \in [0, T]$ $f(t-)$,$f(t+)$ exists and $f(t) = f(t-)$, we denote

$$\Delta f(t) = f(t) - f(t-)$$

as the jump of f at t.

Definition 2.1.2. We say a process X is *càdlàg* if it is right-continuous with left limits, and that a process X is *càglàd* if it is left-continuous with right limits.

Definition 2.1.3. We denote the space $D(E, M)$ of all *càdlàg* process from E to M the Skorokhod space.

For any $F \subseteq E$, we let

$$w_f(F) := \sup_{s,t \in F} |f(s) - f(t)|$$

and, for $\delta > 0$, define the *càdlàg* modulus as

$$\varpi'_f(\delta) := \inf_{\Pi} \max_{1 \leq i \leq k} w_f([t_{i-1}, t_i)),$$

with $max_i(t_i - t_{i-1}) < \delta$. It can be shown that f is *càdlàg* if and only if $\varpi'_f(\delta) \to 0$ as $\delta \to 0$. Now let Λ denote the set of all strictly increasing, continuous bijections from E to itself and let

$$\|f\| := \sup_{t \in E} |f(t)|$$

denote the uniform norm on functions on E. We then define the Skorokhod metric σ on D by

$$\sigma(f, g) := \inf_{\lambda \in \Lambda} \max\{\|\lambda - I\|, \|f - g \circ \lambda\|\},$$

where $I : E \to E$ is the identity function. The topology Σ generated by σ is called the Skorokhod topology on D.

The space C of continuous functions on E is a subspace of D. Although D is not a complete space with respect to the Skorokhod metric σ, there is a topologically equivalent metric σ_0 to which D is complete. With respect to either σ or σ_0, D is a separable space. Thus, Skorokhod space is a Polish space, see Billingsley [1995] and Billingsley [1999].

The choice between *càdlàg* and *càglàd* is based on the jump time. Since we for a *càdlàg* process define $f(t)$ as the value after the jump it is unpredictable. On the other hand, the jump of a *càglàd* process is foreseeable and can be predicted by following the path of f. We will in this paper encounter predictable processes. While adapted processes are a function of time for fixed ω we will consider both time and randomness by looking at X as a function on $[0, T] \times \Omega$. A natural σ-algebra would be the algebra generated by the section $A \times B \in [0, T] \times \Omega$. In this approach we may end up with the previously defined non-anticipating *càdlàg* process as non-measurable. To solve this we take the σ-algebra generated by the non-anticipating *càdlàg* processes.

Definition 2.1.4 (Optional processes). The σ-algebra $\mathcal{A}$ generated on $[0, T] \times \Omega$ by all non anticipating *càdlàg* processes is called the optional σ-algebra. A process $x : [0, T] \to \mathbb{R}^d$ which is measurable with respect to $\mathcal{A}$ is called an optional process.

By definition, any non anticipating *càdlàg* process is optional.

Definition 2.1.5 (Predictable processes). The σ-algebra $\mathcal{P}$ generated on $[0, T] \times \Omega$ by all non anticipating left continuous processes is called the optional σ-algebra. A process $x : [0, T] \to \mathbb{R}^d$ which is measurable with respect to $\mathcal{P}$ is called an predictable process.

We often need to compare to processes to see if they are the same. For two stochastic processes we say they are the same if $X_t(\omega) = Y_t(\omega)$ for all $t \in T$. This is a very strong condition so we have some weaker concepts:

Definition 2.1.6. We say Y is a modification of X if, for every $t \in T$ we have that

$$P[X_t = Y_t] = 1.$$

Definition 2.1.7. X and Y have the same finite-dimensional distribution if for any integer $n \geq 1$, $0 \leq t_1 < \cdots < t_n < \infty$, and $A \in B(\mathbb{R}^{nd})$, we have

$$P[(X_{t1}, \ldots, X_{tn}) \in A] = P[(Y_{t1}, \ldots, Y_{tn}) \in A].$$

Definition 2.1.8. X and Y are indistinguishable if almost all their sample paths agree, i.e.

$$P[X_t = Y_t; \forall 0 \leq t < \infty] = 1.$$

Where the third definition is the strongest one and imply the first one, which again implies the second, see Karatzas and Shreve [2000].

DEFINITION 2.1.9. We say that a stochastic process X is measurable if, for every $A \in B(R^d)$ the set $\{(t,\omega); X_t(\omega) \in A\}$ belongs to the product σ-field $B(T) \times \mathcal{F}$.

An important consequence of Fubini's theorem, pointed out in Karatzas and Shreve [2000], is that the trajectories of a measurable stochastic function are Borel-measurable. If the components of X have defined expectation then the same is true for the functional EX_t. We will often use that if $\int_I E|X_t|dt < \infty$ where I is a subinterval of T then

$$\int_I |X_t|dt < \infty \text{ a.s. and } \int_I EX_t dt = E\int_I X_t dt.$$

In this paper our model will allow discontinuities in the trajectories so we need to review the theory behind Lévy processes.

2.2. The Lévy Model

Levy processes can be seen as a family of models that describe the path of a randomly moving particle. These particles may diffuse or undergo independent random jumps whose order of magnitude is arbitrarily.

DEFINITION 2.2.1 (Lévy process, Cont and Tankov [2004] Definition 3.1). A *càdlàg* stochastic process $(X_t)_{t\in T}$ on $(\Omega,\mathcal{F},P)$ with values in the state space such that $X_0 = 0$ is called a Lévy process if it has the following properties;

1. Independent increments: for every increasing sequence of times $t_0,\dots,t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_n-1}$are independent.
2. Stationary increments: the law of $X_{t_j} - X_{t_j-1}$ does not depend on t.
3. Stochastic continuity: $\forall\epsilon > 0$, $\lim_{\mathcal{H}\to 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$.

REMARK 2.2.1. Item 3 does not mean that we cannot have jumps, or the sample paths are continuous, it ensures that we cannot predict when the jumps or discontinuities occur.

We will work with a Lévy marked where $\eta(t) = \eta(t,\omega)$ is a Lévy process on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathcal{P})$. Let $N(dt,dz)$ be the jump measure of $\eta(\cdot)$ and

$$\nu(V) = E\left[N\left((0,1],V\right)\right]; V \subset \mathbb{R}\backslash\{0\} \text{ Borel set},$$

be the Lévy measure of $\eta(\cdot)$. We know by the Lévy - Itô decomposition that

$$\eta(t) = a(t)t + b(t)B(t) + \int_{|z|<R} \gamma(t,z)\tilde{N}(t,dz) + \int_{|z|\geq R} zN(t,dz) \tag{2.2.1}$$

for some constant R where

$$\tilde{N}(dr,dz) = N(dt,dz) - \nu(dz)dt,$$

is the compensated Poisson process of $\eta(\cdot)$. Lets assume that $E[|\eta_t|] < \infty$ for all $t \geq 0$, then

THEOREM 2.2.1. *If $E[|\eta_t|] < \infty$ for all $t \geq 0$ we have that*

$$\int_{|z|\geq 1} |z|\nu(dz) < \infty,$$

and we can choose $R = \infty$ and write

$$\eta(t) = a(t)t + b(t)B(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(t,dz)$$

(See e.g Øksendal and Sulem [2007] Theorem 1.8). We can then define

$$\int_0^t H(s)d\eta_s$$

for an adapted càglàd processes, $H(\cdot)$, in the space equipped with the topology of uniform convergence on compacts in probability. From Itô -Lévy decomposition we can consider the general stochastic integral on the form

$$X(t) = X(0) + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(t,dz)$$

where α, β and γ are $\mathcal{F}_t$-predictable processes such that

$$\int_0^T \left[|\alpha(t)| + \beta^2(t) + \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)\right] dt < \infty \text{ for all } T < \infty.$$

2.3. The Market Model

Fix $T > 0$ and let α,β and γ be as above. Let $r(t) = r(t,\omega)$ be adapted such that $\int_0^T |r(t)|dt < \infty$ a.s. We let the marked be described by;

(i) a risk free asset

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt \\ S_0(0) = 1, \end{cases} \tag{2.3.2}$$

(ii) and a risky asset

$$\begin{cases} dS_1(t) = S_1(t^-)\left[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right] \\ S_1(0) > 0. \end{cases} \tag{2.3.3}$$

where $\gamma(t,z) > -1$ for a.a. t, z and where we also require that

$$E\Big[\int_0^T \{|r(s)| + \int_{\mathbb{R}} |\log(1+\gamma(s,z)) - \gamma(s,z)|\nu(dz)\}ds\Big] < \infty.$$

Then from Øksendal and Sulem [2007] we get that

$$S_1(t) = S_1(0)\exp\Big[\int_0^t \{\alpha(s) - \frac{1}{2}\beta^2(s) + \int_{\mathbb{R}} \log(1+\gamma(s,z)) - \gamma(s,z)\nu(dz)\}ds$$
$$+ \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz)\Big].$$

We let $\pi(t)$ be a portfolio and $V(t) = V^{\pi}(t)$ be the wealth processes given by π with dynamics

$$\begin{cases} dV(t) = V(t^-)\left[\{(1-\pi(t))r(t) + \pi(t)\alpha(t)\}dt + \pi(t)\beta(t)dB(t) + \pi(t^-)\int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right] \\ V(0) > 0. \end{cases}$$

where π is a *càglàd* predictable process, $\pi(t^-)\gamma(t,z) > -1$ for a.a. t and for z a.s. and

$$\int_0^T \left[|(1-\pi(t))r(t)| + |\pi(t)\alpha(t)| + \pi(t)^2\beta^2(t) + \pi(t^-)^2\int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)\right] dt < \infty$$

for all $T < \infty$.

2.3.1. Geometric Lévy Processes. Assume we have a probability space $(\Omega; F; P)$ and a filtration $\{\mathcal{F}_t;\ 0 \leq t \leq T\}$. We let the price process $S_t = S_0 e^{Z_t}$ be defined on this probability space, Z_t is a Lévy process. We call such a process St the geometric Lévy process (GLP). Throughout this paper we assume that $\mathcal{F}_t = \sigma(S_s; 0 \leq s \leq t) = \sigma(Z_s; 0 \leq s \leq t)$ and $\mathcal{F} = \mathcal{F}_T$.

2.4. Backward Stochastic Differential Equations

In chapter 6 we will use g-expectations defined by the solution to a Backward Stochastic Differential equation (BSDE) to construct a dynamic risk measure optimization model. Therefore we will review some essential theory about BSDE.

First some notation used

- $L_T^{2,d}(\mathbb{R}^d)$: is the space of all $\mathcal{F}_t$-measurable r.v. $X : \Sigma \to \mathbb{R}^d$ s.t. $E[|X|^2] < \infty$.
- $H_T^2(L_2)$: space of all predictable processes $\phi : \Sigma \times [0,T] \to \mathbb{R}^d$ s.t

$$||\phi||^2 = E[\int_0^T |\phi_s|^2 ds] < \infty.$$

Next, for $\beta > 0$ we define $||\phi||_\beta = E[\int_0^T e^{\beta t}|\phi_s|^2 ds]$ so that;

- $H_{T,\beta}^2$: is the space H_T^2 endowed with the nor $||\cdot||_\beta$. (Its easily seen that $||\cdot||_\beta$ and $||\cdot||$ are equivalent.)
- $L^2\mathcal{F}_T(\Omega, W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m))$ is the set of all functions $f : [0,T] \times M \times \Omega \to N$, such that for any fixed $\theta \in M$, $(t,\omega) \to f(t,\omega)$ is $\{\mathcal{F}_t\}_{t\geq 0^-}$ progressively measurable with $f(t,0,\omega) \in L^2_{\mathcal{F}}([0,T];N)$, and there exists a constant $L >$, such that

$$|f(t,\theta,\omega) - f(t,\bar{\theta},\omega)| \leq L|\theta - \bar{\theta}|, \forall \theta, \bar{\theta} \in M, \text{ a.e. } t \in [0,T], \text{ a.s.};$$

Definition 2.4.1. A backward stochastic differential equations (BSDE) is equation on the form

$$\begin{cases} -dY(t) = g(t, Y(t), Z(t))dt - Z(t)dB(t). \\ Y(T) = \xi, \end{cases} \tag{2.4.4}$$

and where (Y, Z) is a solution to the BSDE such that $Y(t), t \in [0, T]$ is a continuous adapted process and $Z(t), t \in [0, T]$ is a predictable process satisfying $\int_0^T |Z(t)|^2 ds < \infty$ P-a.s. and

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dB(s),\ P\text{-a.s.} \tag{2.4.5}$$

DEFINITION 2.4.2. We say that (f, ξ) is a pair of standard parameters for the BSDE if they satisfy

- $\xi \in L_T^2$.
- $f(\cdot, 0, 0) \in H_T^2$.
- f is uniformly Lipschitz.

THEOREM 2.4.1 (Existence and uniqueness of solution, Zhang [2009] and also in Ma and Yong [2007] section 4). *Let (f, ξ) be a pair of standard parameters for the BSDE* (2.4.5)*, then there exists an unique pair $(Y, Z) \in H_T^2 \times H_T^2$ which solves the BSDE* (2.4.5).

THEOREM 2.4.2 (Theorem 3.1 in Øksendal and Zhang [2001]). *Assume $E[|\phi|_H^2] < \infty$. Then there exists an unique $H \times L_2(K, H)$-valued progressively measurable process (Y_t, Z_t) such that*

(i) $E[\int_0^T |Y_t|_H^2] < \infty$ *and* $E[\int_0^T |Z_t|_H^2] < \infty$
(ii) $\phi = Y_t + \int_t^T AY_s ds + \int_t^T b(s, Y_S, Z_S)ds + \int_t^T Z_s dB_s;\ 0 \le t \le T.$

2.4.1. BSDE with Concave Generator. Assume the generator $f(t, y, z)$ is concave w.r.t. (y, z). Define

$$F(t, \beta, \gamma) = \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^n} [f(t, y, z) - \beta y - \gamma z].$$

Since f is concave and continuous, we have from concave analysis that

$$f(t, y, z) = \sup_{(\beta,\gamma)\in D_F} \{F(t, \beta, \gamma) + \beta y + \gamma z\},$$

where we let, for a pair of predictable processes

$$f^{\beta,\gamma}(t, y, z) := F(t, \beta, \gamma) + \beta y + \gamma z \text{ which is linear in } (y, z),$$

and

$$D_F = \{(\beta, \gamma) : F(t, \beta, \gamma) < \infty\} \subset [-C, C]^{n+1}.$$

Let

$$A = \{(\beta_t, \gamma_t) : E[\int_0^T F(t, \beta_t, \gamma_t)^2 dt] < \infty\},$$

be the set of admissible controls. Then we have

THEOREM 2.4.3. *There exists an optimal control $(\bar\beta, \bar\gamma) \in A$ s.t.*

$$f(t, Y_t, Z_t) = f^{\bar\beta,\bar\gamma}(t, Y_t, Z_t).$$

PROPOSITION 2.4.4. *Let f be a concave standard parameter and define*

$$f^{\beta,\gamma}(t, y, z) = F(t, \beta_t, \gamma_t) + \beta_t y + \gamma_t z.$$

Then for any $t \geq 0$,

$$Y_t = \underset{(\beta,\gamma)\in A}{ess\inf}\{Y^{\beta,\gamma}(t)\},$$

where $Y^{\beta,\gamma}(t)$ is the solution of the BSDE associated with the linear generator $f^{\beta,\gamma}$

2.4.2. Forward-Backward Stochastic Differential Equations. We define a Forward-Backward stochastic differential equation (FBSDE) as follows:

$$(2.4.6)\qquad \begin{cases} dX(t) & = b(X(t), u_0(t), u_1(t))dt + \sigma(X(t), u_0(t), u_1(t))dB(t) \\ & \quad + \int_{\mathbb{R}^k} \gamma(X(t^-), u_2(t,z), z)\tilde{N}(dt, dz). \\ dY(t) & = -g(t, Y(t), Z(t), K(t))dt + Z(t)dB(t) + \int_{\mathbb{R}^k} K(t,z)\tilde{N}(dt, dz). \\ X(0) & = x_0, Y(\tau_S) = -X(\tau_S). \end{cases}$$

THEOREM 2.4.5 (Ma and Yong [2007] Theorem 5.1). *Let $M = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$, and b, σ, h and g satisfy*

$$(2.4.7)\qquad \begin{cases} b \in L^2_{\mathcal{F}}([0,T], W^{1,\infty}(M; \mathbb{R}^n)) \\ g \in L^2\mathcal{F}([0,T], W^{1,\infty}(M; \mathbb{R}^m)) \\ h \in L^2\mathcal{F}_T(\Omega, W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)) \\ \sigma \in L^2\mathcal{F}([0,T], W^{1,\infty}(M; \mathbb{R}^{n\times d})) \end{cases}$$

Moreover, we assume that

$$(2.4.8)\qquad \begin{cases} |\sigma(t;x;y;z;\omega) - \sigma(t;x;y;z;\omega)| \leq L_0|z - \bar{z}|; \\ \forall(x,y) \in \mathbb{R}^n \times \mathbb{R}^n; z; \bar{z} \in \mathbb{R}^{m\times d};\ a.e.\ t > 0;\ a.s. \\ |g(x;\omega) - gx;\omega)| \leq L_1|x - \bar{x}|; \forall x; \bar{x} \in \mathbb{R}^n a.s. \end{cases}$$

with

$$L_0 L_1 < 1.$$

Then there exists a $T_0 > 0$, such that for any $T \in (0, T_0]$ and any $x \in \mathbb{R}^d$ (2.4.6) admits an unique adapted solution $(X;Y;Z) \in M[0,T]$.

2.4.3. Backward Stochastic Differential Equations and g-Expectations. From the above Peng [1997] and Gianin [2002a] defined the g-expectation and the conditional g-expectation as follows:

DEFINITION 2.4.3. For every $x \in L^2_T(\mathcal{F}_t)$ and for every $t \in [0,T]$ the conditional g-expectation of x under $\mathcal{F}_t$ is defined by

$$(2.4.9)\qquad \varepsilon_g[x|\mathcal{F}_t] := Y(t),$$

where $Y(t)$ is the first component of the solution to the FBSDE (2.4.5) with terminal condition $\xi = x$. For $t = 0$ we have

$$(2.4.10)\qquad \varepsilon_g[x] := Y(0),$$

which is called the g-expectation.

2.5. Stopping Times

A concept vital to this paper is stopping times. If we want to look at a particular time, $\tau \in [0,T]$, when an event occurs we would need the information concerning the event, $\{\omega : \tau(\omega) \leq t\}$, to be included into our filtration.

Definition 2.5.1 (Karatzas and Shreve [2000] definition 2.1). Let $(\Omega, \mathcal{F}, P)$be given, we call random time τ a stopping time of the filtration if the event $\{\tau \leq t\}$ belongs to the σ-algebra $\mathcal{F}_t$ for all t. A random time is an optional time of the filtration if $\{\tau < t\} \in \mathcal{F}_t$, for every t.

We define the first exit time as

$$\tau_S = \inf\{t > 0 : X_t \notin S\},$$

which is a stopping time w.r.t. $\mathcal{F}_t$ since

$$\{\omega; \tau_S \leq t\} = \bigcap_m \bigcup_{r \in Q, r<t} \{\omega; X_r \notin K_m\} \in \mathcal{F}_t,$$

where $\{K_m\}$ is an increasing sequence of closed sets such that $U = \bigcup_m K_m$.

2.6. Probability Measures

Definition 2.6.1. We denote the set of all probability measures that are absolutely continuous w.r.t P ($Q \ll P$) by $\mathcal{M}$.

Let $\theta_0(t) = \theta_0(t,\omega) \in \mathbb{R}^m$ and $\theta_1(t,z) = \theta_1(t,z,\omega) \in \mathbb{R}^l$ be predictable processes, then let $\mathcal{Q}_\theta$ be on the form

$$dQ_\theta = \exp\Big(-\int_0^T \theta_0(s)dB(s) - \frac{1}{2}\int_0^T \theta_0^2(s)ds + \sum_{j=1}^{l}\int_0^t \int_{\mathbb{R}} \log[1-\theta_{1j}(s,z)] + \sum_{j=1}^{l}\int_0^t \int_{\mathbb{R}} \big(\log[1-\theta_{1j}(s,z)] + \theta_{1j}(s,z)\big)\nu(dz)ds\Big)dP(\omega) = Z_\theta(T)dP. \tag{2.6.11}$$

such that it is well defined and $\theta_1(t,z) < 1$ for a.a t,z, $Z_\theta(0) = 1$ and

$$\int_0^T \left[\theta_0^2(t) + \int_{real} \theta_1^2(t,z)\nu(dz)\right] < \infty \text{ a.s.}$$

Using Ito's formula for Lévy processes, see Øksendal and Sulem [2007] Theorem 1.14, we get

$$dZ_\theta(t) = -Z(t)\theta_0(t)dB(t) - Z(t)\int_{\mathbb{R}} \theta_1(s,z)\tilde{N}(ds,dz)$$

REMARK 2.6.1. Note that from Øksendal and Sulem [2007] if we let $\theta_0(t)$ and $\theta_1(t,z)$ be such that

$$E[Z(T)] = 1,$$

then $Q_\theta(\Omega) = 1$, i.e. Q_θ is a probability measure. If θ_0 and θ_1 is such that

$$\sigma(t)\theta_0(t) + \int_{\mathbb{R}^l} \gamma(t,z)\theta_1(t,z)\nu(dz) = \alpha(t) - r(t) \text{ for a.a.}(t,\omega) \in [0,T] \times \Omega.$$

then the probability measure Q on $(\Omega; F)$ is called an equivalent local martingale measure (ELMM). If $X(t)$ is a martingale w.r.t. Q then Q is called an equivalent martingale measure (EMM).

DEFINITION 2.6.2. In this paper we will let $\mathcal{M}_a$ consist of all measures $Q = Q_\theta$ of Girsanov transformations given above.

We let $E_{\mathcal{Q}}[x]$ denote the integral of x with respect to $\mathcal{Q} \in \mathcal{M}_a$ and $E_{\mathcal{Q}}[x] = E[\frac{d\mathcal{Q}}{dP}x]$ where $\frac{d\mathcal{Q}}{dP}$ is the Radon-Nikodym derivative. ,

We think of a risk measure as a functional in the space X' which takes values in the space X. The next chapter will show that a functional should satisfy certain conditions of consistency.

CHAPTER 3

A INTRODUCTION TO RISK MEASURES

We will now give a thorough axiomatic approach to risk measures and review some central theorems regarding coherent risk measures. We will look at an extension to coherent risk measures, the convex risk measure. At the end of the chapter we turn our attention to the dynamic setting, which give rise to dynamic risk measures.

3.1. The Evolution of Risk Measures

Risk management is a key concept in modern finance. It is a discipline where the aim is to analyze, identify, control and minimize unacceptable risks. According to McNeil et al. [2005] financial institutions "manage risk by repacking them and transferring them to markets in customized ways". In order to manage risk we need to be able to measure risk. A probabilistic measure would use the distribution of the position to measure the risk by moments or quantiles. Moment-measures such as variance, which was first proposed by Markowitz, does not take into account the asymmetric interpretation of the risk of a portfolio, the downside. Another traditional risk measure, that takes this asymmetry into account, is the Value at Risk (VaR) introduced in 1994 by the leading investment bank JP Morgans. VaR captures asymmetry by measuring the quantiles for the lower tail. According to Morgan Guarantee Trust Company [2005] VaR is widely accepted in the financial industry and endorsed by regulatory agencies. VaR is defined as follows:

$$\mathrm{VaR}^{\alpha}(L) = -\inf\{x \in \mathbb{R} : P[L \leq x] > \alpha\}$$

Despite its easy computation and interpretation there has been raised several questions to the use of VaR to quantify risk. In the Jorion-Taleb debate, Jorion and Taleb [1997] argued about VaR. Taleb claimed that VaR:

1. Is charlatanism because it claimed to estimate the risks of rare events, which is impossible.
2. Gives false confidence.
3. Could be exploited by traders.

David Einhorn goes as far as saying "VaR is like an airbag that works all the time except when you have a car accident." In Brown and Einhorn [2008], David Einhorn also claims that VaR

1. Is potentially catastrophic, as it can create a false sense of security among executives and regulators.
2. Leeds to excessive risk-taking and use of leverage.
3. Created an incentive to take remote but excessive risk.
4. Focuses on the manageable risks near the center of the distribution and ignored the tails.

In December 2006 Goldman's various models, including VaR, gave a indication that something was wrong. Goldman decided to get closer to home, meaning reining in the risk. The risk was hedged and in the summer of 2007, Goldman Sachs avoided the faith that had fallen so brutally upon giants such as Merrill Lynch and Lehman Brothers. In this example VaR proved to be of value. However one cannot avoid the fact that there are some critical issues with VaR (see Artzner et al. [1997]). First, VaR is completely ignorant of the seriousness of the worst cases, which could create a false sense of security. By investing in asymmetric positions that generate small gains and very rarely have losses, VaR could be constructed to underestimate the risk. An example is the credit-default swap that generates steady income. If the probability of default is less than 1%, 99% Var is 0, but on the offset of a default a substantial loss could be incurred. Second, a very critical flaw is that VaR can generate scenarios where risk is decreased under decentralization. To overcome these shortcomings, an axiomatic approach to risk measure has been a key point for development in risk management and mathematical finance in the recent years. The concept of coherent measures and more generally convex measures follows a axiomatic approach and are now well developed. Initial research into constructing a solid groundwork for risk measures was initiated by Artzner, Delbaen, Eber, and Heath. Artzner and Delbaen provided the path-breaking axiomatic definition of coherent risk measures in the papers Artzner et al. [1997] and Artzner et al. [1999]. They provided four axioms for a coherent risk measure,

(I) Translation invariance. For all $x \in X$ and all real numbers $\alpha, \rho(x + \alpha r) + \rho(x) - \alpha$,
(II) Sub additivity. For all x_1 and $x_2 \in X$, $\rho(x_1 + x_2) \leq \rho(x_1) + \rho(x_2)$,
(III) Positive homogeneity. For all $\lambda \geq 0$ and all $x \in X$, $\rho(\lambda x) = \lambda \rho(x)$,
(IV) Monotonicity. For all x and $y \in X'$ with $x \leq y, \rho(y) \leq \rho(x)$.

In financial theory hedging and pricing of claims has been given much attention. In a complete market, if the payoff of a claim can be constructed from the payoff of basis assets, this value is uniquely determined and other prices would lead to an arbitrage opportunity. Whenever this perfect replication is unattainable, due to market frictions, transaction costs and sources of unhedgeable risk, we get an incomplete market. In an incomplete market, the standard Black-Scholes pricing methodology fails because the price of the focus asset is no longer unique. This can be solved by the representative agent equilibrium, where the pricing functional is obtained from the marginal utility of the optimized representative agent's consumption. Another way to solve this problem is to construct good deal bounds so that the price of a non-redundant contingent claim is not unique. As Černý and Hodges [2000] points out, no arbitrage is a rather weak requirement as can be seen in their example of a claim with zero price that either earns 1000 or loses 1 with equal probability. Good deals are defined as an opportunity to buy a desirable claim at no cost. To construct a good deal bound given a risk measure, one hedges the given claim with a portfolio of self-financed, basic assets so to maximize w.r.t. the given risk measure. Then one remove prices that give an undesirable good hedging strategy. This provides a price interval for every contingent claim called generalized arbitrage bounds or good deal bounds by Černý and Hodges [2000]. The connection to coherent risk measures were made by Jaschke and Küchler [2001], who proved that coherent risk measures are essentially equivalent to generalized arbitrage bounds.

A valuable property for risk measure is that they have a close relation to utility function and asset pricing. If we consider the classical (μ, σ) portfolio optimization theory of Markowitz we can use (μ, ρ), where ρ is coherent risk measure. This enables us to consider the (μ, ρ) problem, which can be viewed as the problem of maximizing $U = \mu - \lambda\rho$. When the preference function $\Phi(X) = E[U(X)]$, where U denotes the utility function specific to each decision maker, do not separate risk or value, Jia and Dyer [1996] proved that it is possible to derive an explicit risk measures: $\rho(X) = -E[U(X - E(X))]$.

So from the connection to utility functions and good deal bounds, we can conclude that coherent risk measures are consistent with economic theories of arbitrage as well as utility maximization.

The sub linearity axiom of coherent risk measures gives us that $\rho(\lambda X) \leq \lambda\rho(X)$ but we may want to model cases where a single position (λX) could be less liquid, and therefore more risky, than that of λ smaller positions, so convex risk measure was later extended to convex risk measures by Föllmer and Schied [2002] and Frittelli and Gianin [2002]. Although these new risk measures are very useful in risk management, they are static. Static risk measures only quantify risk at a single instance in the future and was by Föllmer and Leukert [1999] generalized to a dynamic setting. To construct a dynamic risk measure Riedel [2003] considered the changes of a position and availability of new information with time. As additional information about the position may be released with time and changes may occur in the position or there may be an intermediate cash flow, there may be a need for a reassessment of the position under this new information. Changes in the position are to be taken into account by recalculating the (stochastic) present value of future positions. Information is processed in a Bayesian way on a set of generalize scenarios. Dynamic risk measures allow a manager the flexibility of adjusting a position. They take a random cash flow and return a random process as a function of time.

We will now review the axioms for coherent risk measures in depth.

3.2. An Axiomatic Approach

Capturing reality necessitates a minimum set of requirements or proposition we consider self-evident. This has been the starting point for every mathematical approach to model reality since Euclid. For financial risk, capturing these propositions has been an area of neglect. Concepts have been intuitively developed as they seem to give a logical measure of risk but no formal requirement has been given. These intuitions need to be made concrete and unambiguous. To answer this challenge Artzner et al. [1997] proposed a set of axioms that a risk measure needs to fulfill;

Axiom 3.2.1 (Translation invariance). []

For all $x \in X$ and all real numbers α,

$$\rho(x + \alpha r) + \rho(x) - \alpha.$$

This axiom, translation invariance, ensures that risk measures are given in the same units as the final net worth. We see that by adding a sure return m to a position X the risk $\rho(X)$ decrease by m. The next axiom is probably the one that seems most intuitive

Axiom 3.2.2 (Sub additivity). For all x_1 and $x_2 \in X$,

$$\rho(x_1 + x_2) \leq \rho(x_1) + \rho(x_2).$$

One would naturally think that if two independent financial institutions that separately have adequate reserves to cover extreme scenarios would be in no greater risk after a merger. Therefore the risk of a portfolio should be no more than the sum of its components. This is where VaR often fails. It is coherent in the case of unimodal distributions like the normal- and t-distribution. A perfectly diversified portfolio, due to the Central Limit theorem, is normal distributed. Most portfolios are not perfectly diversified and have significant deviation from the normal distribution. Diversification risk should be monitored as they can lead to inadequate capital reserves and unexpected losses. Market risk factors such as equity indices, foreign exchange rates, commodity prices and interest rates are continuously distributed. They exhibit properties of skewness and excess kurtosis so they are not normally distributed. Many market risk return distributions are not normally distributed but they are very often unimodal so that VaR is coherent. Credit rating migrations and insurance risk are typical cases where the return distributions are not unimodal and VaR is not coherent. In this case VaR can be constructed to artificially lower risk. This is mainly due to the large loses of low-probability events.

EXAMPLE 3.2.1. A zero-coupon bond pays 100$ in 2 year if the issuer does not default. Let the probability of default be 0.11%. Then VaR with 99% confidence is zero. On the other hand, if we construct a portfolio with 10 similar bonds which pays 10$ on the coupon date, issued by independent counter parties with the same credit rating, VaR with 99% confidence is $10. This is due to the fact that the probability of one party defaulting is larger than 1%. Both position has the same expected payoff and are as financial portfolios identical (liquidy issues aside), but they poses different risk when measured using VaR.

The above axiom gives us that $\rho(\lambda x) \leq \lambda\rho(x)$, but we should not be able to lower the risk of multiple identical portfolios by merging them.

AXIOM 3.2.3 (Positive homogeneity). For all $\lambda \geq 0$ and all $x \in X$,

$$\rho(\lambda x) = \lambda\rho(x).$$

This axiom ensures us that multiplying a position multiply the risk by the same amount. Another intuition we have about risk measures is that it seems clear that when a position is surely larger than another position, the risk of the first should be lower than the risk of the last.

AXIOM 3.2.4 (Monotonicity). For all x and $y \in X'$ with $x \leq y$,

$$\rho(y) \leq \rho(x).$$

REMARK 3.2.1. The axioms are motivated from a supervising agency's point of view. From this perspective a risk measure is viewed as capital requirement, the amount needed to make the position acceptable.

We can now define a coherent risk measure.

DEFINITION 3.2.1. A coherent risk measure is a functional $\rho : X \longmapsto \Re$, that satisfies axioms 1.1.1 - 1.1.4.

DEFINITION 3.2.2. We say a risk measure is continuous from below (resp. above) if for any increasing (resp. decreasing) sequence X_n of elements of $L^\infty(\Omega, \mathcal{F}, P)$such that $X = \lim X_n$ a.s., the sequence $\rho(X_n)$ has the limit $\rho(X)$ a.s.

Let us now look at some examples of a coherent risk measures.

EXAMPLE 3.2.2 (Spectral risk measure). The measure $M_\phi : \mathbb{R}^S \to \mathbb{R}$ defined by $M_\phi(X) = -\delta \sum_{s=1}^{S} \phi_s X_{s:S}$ is a spectral measure of risk if $\phi \in \mathbb{R}^S$ satisfies the conditions

1. Non-negativity: $\phi_s \geq 0$ for all $s = 1, \ldots, S$,
2. Normalization: $\sum_{s=1}^{S} \phi_s = 1$,
3. Monotonicity : ϕ_s is non-increasing, that is $\phi_{s_1} \geq \phi_{s_2}$ if $s_1 < s_2$ and $s_1, s_2 \in \{1, \ldots, S\}$.

EXAMPLE 3.2.3 (Expected Tail Loss (ETL)). Expected Tail Loss is more sensitive to the shape of the loss distribution in the tail of the distribution than VaR, and defined as;

$$ESq = E(x|x < \mu),$$

where μ is determined by $Pr(x < \mu) = q$. Expected Tail Loss is a spectral risk measure.

EXAMPLE 3.2.4 (EVaR).

$$EVaR^\alpha(X) = -rE[X|X \leq -VaR^\alpha(X)], \tag{3.2.1}$$

where r is a normalization constant.

See Lüthi and Doege [2005] for more examples.

In Pedersen [1995] (2.3.1) we find the following definition;

DEFINITION 3.2.3. Let X be a vector space. A Minkowski functional on X is a function $\rho : X \to \mathbb{R}$ such that

(a) $\rho(x + y) \leq \rho(x) + \rho(y)$, $x, y \in X$
(b) $\rho(\alpha x) = \alpha\rho(x)$, $x \in X$, $\alpha \geq 0$,

We can see that a coherent risk measure is a Minkowski functional (a semi-norm if $F = \mathbb{R}$).

REQUIREMENT 3.2.1. In most cases there will be no loss of generality by assuming ρ is normalized in the sense that

$$\rho(0) = 0.$$

If $\rho(x)$ is ≤ 0 we say the position is acceptable, if not $\rho(x)$ is the amount that must be added to the position to make it acceptable.

REMARK 3.2.2. It can be, by definition of axioms, argued that the stated axioms are not correct ones, but they are nevertheless required as basis for a thorough mathematical approach.

It can be shown that every coherent risk measure can be represented in a specific form.

THEOREM 3.2.1. *[Representation theorem Föllmer and Penner [2006]] A functional $\rho : X' \to \mathbb{R}$ is a coherent risk measure iff there exists a closed convex set $\mathcal{Q}$ of P-absolutely continuous probability measures on a set of states of nature, such that*

$$\rho(x) = \sup_{\mathcal{Q}} E_{\mathcal{Q}}[-x], \forall x \in L^{\infty}$$

This tells us that any coherent risk measure can be represented as the supremum of the expected loss over a set of scenarios.

3.2.1. Axioms on Acceptance Sets.

We have discussed the axioms that a risk measure should arguably satisfy. We will now review the related concept of defining an acceptable position through acceptance sets. As we will see later there is a strong correlation between these two approaches.

DEFINITION 3.2.4. We let $\mathcal{A}_{i,j}, j \in J_i$ be a set of final net worth. in currency i, which is accepted by regulator j.

We now state some axioms on acceptance sets;

AXIOM 3.2.5. The Acceptance set $\mathcal{A}$ contains X'_+.

AXIOM 3.2.6. The Acceptance set $\mathcal{A}$ do not intersect the set X'_{--} where

$$X'_{--} = \{X| \text{ for each } x \in X, X(x) < 0\} \tag{3.2.2}$$

The meaning of these two axioms is that a non-negative net worth do not require additional capital to be acceptable. On the other hand a strictly negative one needs additional security.

AXIOM 3.2.7. The acceptance set $\mathcal{A}$ is convex.

This captures the risk aversion of the regulator, exchange director or trading room supervisor. Artzner,Eber and Heath also suggest the less natural requirement of

AXIOM 3.2.8. The acceptance set $\mathcal{A}$ is a positively homogeneous cone.

The correlation between risk measures and acceptance sets will be made clear by using the two definitions given below.

Definition 3.2.5 (Risk measure associated with an acceptance set). The risk measure associated with the acceptance set $\mathcal{A}$ is the mapping from X into $\mathbb{R}$ defined by

$$\rho_{\mathcal{A}}(x) = \inf\{m|m + x \in \mathcal{A}\}. \tag{3.2.3}$$

Definition 3.2.6 (Acceptance set associated with a risk measure). The acceptance set associated with a risk measure ρ is the set

$$\mathcal{A}_{\rho} = \{x \in X|\rho(x) \leq 0\} \tag{3.2.4}$$

The next theorem given by Artzner et al. [1997], links acceptance sets to risk measures.

Theorem 3.2.2. *If the set $\mathcal{A}$ satisfy the four axioms 3.2.5-3.2.8, the risk measure $\rho_{\mathcal{A}}$ is coherent. Moreover $\mathcal{A}_{\rho_{\mathcal{A}}} = \bar{\mathcal{A}}$.*

Earlier we gave a representation theorem for coherent risk measures. This representation has a natural connection to acceptance sets, as shown by Delbaen [2002].

Theorem 3.2.3. *Let $\rho : L^{\infty}(\Sigma, \mathcal{F}, P) \to \mathbb{R}$ be a coherent risk measure. Then there exist a closed convex set $\mathcal{P}$ of P-absolutely continuous probability measures such that*

$$\rho(x) = \sup_{\mathcal{Q}\in\mathcal{P}} E_{\mathcal{Q}}[-x], \forall x \in L^{\infty}$$

$$\Leftrightarrow$$

$$\textit{the acceptance set } \{x \in L^{\infty} : \rho(x) \leq 0\} \textit{is a } \sigma(L^{\infty}, L^1) - \textit{closed convex cone}$$

This thorough approach is a breakthrough to risk management and provides a mathematical understanding of risk. When coherent risk measure was first introduced in 1997 it created a heated debate since VaR, which had a strong position amongst regulators and practioners, was seen to not be coherent and not even convex (see below).

3.3. A Generalization to Convex Risk Measures

By looking at liquidity issues for large portfolios or bonds issued by firms as opposed to treasury bonds or bills, Frittelli and Gianin [2002] argues against the requirement for sub-additivity and positive homogeneity. There could be liquidity risks if an investor holds a portfolio that is large relative to the marked depth. If the market is not able to absorb a sudden sell-off of a large position, doubling the investment in the position increases the risk by more than the double. To account for this the concept of convex risk measures was proposed by Föllmer and Penner [2006]. Heath considered risk measures in finite settings which was later extended to the infinite case by Föllmer and Schied [2002] and Frittelli and Gianin [2002]. They propose to relax the conditions of positive homogeneity and of sub additivity. They required instead;

Axiom 3.3.1. Convexity. $\rho(\lambda x + (1-\lambda)y) \leq \lambda\rho(x) + (1-\lambda)\rho(y)$ for any $\lambda \in [0,1]$.

This axiom ensures that the risk is not increased by diversification of the portfolio.

Definition 3.3.1. A map $\pi : X' \longrightarrow \mathbb{R}$ will be called a convex measure of risk if it satisfies the condition of: translation invariance (axiom 3.2.1), monotonicity (axiom 3.2.4), and convexity (axiom 3.3.1).

Let the risk measure, π, be normalized, meaning $\pi(0) = 0$. Then we can se that the axiom of convexity gives

$$\pi(\delta x) \leq \delta\pi(x), \forall\delta \in [0,1], \forall x \in X$$
$$\pi(\delta x) \geq \delta\pi(x), \forall\delta \geq 1, \forall x \in X$$

In light of the liquidity discussion these two axioms seem reasonable since large portfolios will suffer from liquidity issues. The convexity requirement encourages diversification of risk, which is in coherence with our understanding of portfolio management.

As we showed, coherent risk measures have a representation form as the supremum of the expected loss. It turns out that this representation depends only on the sub-linearity axiom. Frittelli and Gianin [2002] gives a representation for the larger class of convex risk measures.

Definition 3.3.2. Let $\pi : X \mapsto \mathbb{R}$. If it exists a convex functional $F : X' \mapsto \mathbb{R} \cup +\infty$ satisfying $\inf_{x' \in X'} F(x') = 0$ such that

$$\pi(x) = \sup_{x' \in \mathcal{P}} \{x'(x) - F(x')\} < +\infty, \text{ for all } x \in X,$$

where $\mathcal{P} = \{x' \in X' : F(x') < \infty\}$ is the effective domain of F. Then we say that π is representable or that $\rho(x) = \pi(-x)$ is a convex risk measure.

THEOREM 3.3.1 (Frittelli and Gianin [2002]). (1) *A functional* $\pi : X \mapsto \mathbb{R}$ *is representable if and only if it is convex and lower semi-continuous.*

(2) *A functional* $\pi : X \mapsto \mathbb{R}$ *is representable with* $F = 0$ *on* $\mathcal{P}$ *if and only if it is sublinear and lower semi-continuous.*

The financial interpretation is that F represents a correction term and $x'(x)$ the expected loss, ρ is the supremum over a set of generalized scenarios where the correction term is dependent on the scenario.

One type of penalty function $F(\mathcal{Q})$ is the relative entropy of $\mathcal{Q}$ with respect to $\mathcal{P}$, defined as

$$F(\mathcal{Q}) \equiv I(\mathcal{Q};\mathcal{P}) \equiv E_{\mathcal{Q}}\left[\log\frac{d\mathcal{Q}}{d\mathcal{P}}\right] = E\left[\frac{d\mathcal{Q}}{d\mathcal{P}}\log\frac{d\mathcal{Q}}{d\mathcal{P}}\right].$$

See Grandits and Rheinlander [2002]. Entropy is not a metric because $d(p,q) \neq d(q,p)$ (but $d(p,q) \geq 0$,$d(p,q) = 0$ iff $p = q$.). Relative entropy, also called the Kullback-Leibler, gives the proximity of two measures. In Cont and Tankov [2004] we see that if a measure are generated by an exponential Lévy model, the relative entropy can be expressed in terms of Lévy measures:

PROPOSITION 3.3.2. *Let* P *and* Q *be two equivalent measures on* (Ω, F) *generated by an exponential Lévy model with Lévy triplet* $(\sigma^2, \nu^P, \gamma^P)$ *and* $(\sigma^2, \nu^Q, \gamma^Q)$. *Assume* $\sigma > 0$. *The relative entropy* $\xi(Q,P)$ *is given by*

$$\begin{aligned}\xi(Q,P) =& \frac{T}{2\sigma^2}\Big\{\gamma^Q - \gamma^P - \int_{-1}^{1} x(\nu^Q - \nu^P)(dx)\Big\}^2 \\ &+ T\int_{-\infty}^{\infty}(\frac{d\nu^Q}{d\nu^P}\log\frac{d\nu^Q}{d\nu^P} + 1 - \frac{d\nu^Q}{d\nu^P})\nu^P(dx).\end{aligned}$$

We can then see that the first term penalizes the difference in drifts while the second one penalizes the difference in Lévy measures.

Another example is the quadratic distance:

$$E\left[\left(\frac{dQ}{dP}\right)^2\right].$$

In Föllmer and Schied [2002] theorem 4.12 we have that the penalty function have a given representation.

THEOREM 3.3.3. *Any convex risk measure on X is on the form*

$$\rho(x) = \sup_{Q \in M_a} E_Q[-x] - F_{\min}(Q),$$

where M_a as above and the penalty function $F_{\min}$ is given by

$$F_{\min} := \sup_{x \in \mathcal{A}_\rho} E_Q[-x].$$

Moreover, $F_{\min}(Q)$ is the minimal penalty function for ρ, i.e. for any penalty function F, $F(Q) \geq F_{\min}(Q)$ for all $Q \in M_a$.

This functional representation has a connection to pricing functionals in incomplete markets. For each non-attainable claim $x \in X$ there is an interval of prices that gives absence of arbitrage. The maximum price $\hat{x}$ in this interval is given by

$$\hat{x} = \sup_{\mathcal{P}'} E_{\mathcal{P}'}[x],$$

see Lüthi and Doege [2005].

REMARK 3.3.1. As pointed out by Acerbi [2004], the issue of designing a convex measure that allow for sub additive violations solely due to liquidity is very difficult, and if the measure is allowed to break the sub additivity in general cases, not just to model liquidity, there is a possibility for loss effect.

3.4. An Extension to Dynamic Risk Measures

The risk measures discussed all quantify risk at a single point in the future. They are static risk measures. Most investors are making portfolio decisions dynamically and usually at discrete times. As a consequence Föllmer and Leukert [1999] came up with the concept of dynamic risk measures. Important research and development of dynamic risk measure include; Peng [1997], Peng [2003], Frittelli and Gianin [2004] and Föllmer and Penner [2006],.

We now define a dynamic (convex or not) risk measure, $(\rho_t)_{t \in A}$, where A is not necessarily countable, and the set of ρ_t is a net. At an instance $t \in A$, ρ_t represent the riskiness of our position at time t. We also need the boundary requirement that ρ_0 is a static risk measure.

Definition 3.4.1 (Gianin [2002a] and Gianin [2002b]). A dynamic risk measure is a net $(\rho_t)_{t\in A}$ such that

- $\rho_t : L^p(\mathcal{F}_t) \to L^0(\Sigma, \mathcal{F}_t, P)$, for all t.
- ρ_0 is a static risk measure.
- $\rho_T(x) = -x$ P-a.s. for all $x \in X'$

As before we will continue the axiomatic approach by listing some desirable properties for $(\rho_t)_{t\in T}$.

Axiom 3.4.1. Convexity: ρ_t is convex for all $t \in [0, T]$ P-a.s.

Axiom 3.4.2. Positivity: $x \geq 0 \Rightarrow \forall t \in [0, T], \rho_t(x) \leq \rho_t(0)$ P-a.s.

Axiom 3.4.3. monotonicity: $x \geq y \Rightarrow \forall t \in [0, T], \rho_t(x) \leq \rho_t(y)$ P-a.s.

Axiom 3.4.4. Sub-additivity: $\forall x, y \in L^p(\mathcal{F}_t), \forall t \in [0, T]\rho_t(x + y) \leq \rho_t(x) + \rho_t(y)$ P-a.s.

Axiom 3.4.5. Positive homogeneity: $\forall \alpha \geq 0, \forall x \in L^p(\mathcal{F}_t), \forall t \in [0, T]\rho_t(\alpha x) = \alpha\rho_t(x)$ P-a.s.

Axiom 3.4.6. Translation-invariance: $\forall t \in [0, T], \forall$ $\mathcal{F}_t$-measurable $a \in L^p(\mathcal{F}_t), \forall$ $x \in L^p(\mathcal{F}_t)$ $\rho_t(x + a) \leq \rho_t(x) - a$ P-a.s.

Axiom 3.4.7. Constancy: $\forall c \in \mathbb{R}, \forall t \in [0, T]\rho_t(c) = -c$ P-a.s.

Definition 3.4.2 (Frittelli and Gianin [2002]). Let us now define convex and coherent risk measures respectively as;

1. A dynamic risk measure,$(\rho_t)_{t\in T}$, is called coherent if it satisfy the axiom of positivity, sub-additivity, positive homogeneity and translation invariance.
2. A dynamic risk measure,$(\rho_t)_{t\in T}$, is called convex if it satisfy the axiom of convexity and $\rho_t(0) = 0$.

Definition 3.4.3. Two important properties of dynamic risk measures are;

1. $(\rho_t)_{t\in T}$ is said to be time consistent if
$$\rho_0[x\mathbf{1}_A] = \rho_0[-\rho_t(x)\mathbf{1}_A], \forall t \in [0, T], \forall x \in L^p(\mathcal{F}_t), \forall A \in \mathcal{F}_t.$$
2. A dynamic risk measure is continuous from below (resp. above) if each ρ_t is continuous from below (resp. above).

3.4.1. Dynamic Risk Measures from g-Expectations. We now look at the connection between g-expectations, introduced by Peng [1997] as a nonlinear expectations, and risk measures. Let the BSDE be given as;

$$\begin{cases} dY(t) &= g(t, Y(t), Z(t))dt - Z(t)dB(t). \\ Y(T) &= x, \end{cases} \tag{3.4.5}$$

then

DEFINITION 3.4.4 (Peng [1997] definition 36.1-36.5). The conditional g-expectation of x under $\mathcal{F}_t$ for every $x \in L^2(\mathcal{F}_T)$ is defined as

$$\varepsilon_g[-x|\mathcal{F}_t] := Y_t,$$

where Y_t is (the first component of) the solution to the BSDE (3.4.5) with terminal condition x. In particular, for $t = 0$,

$$\varepsilon_g[-x] := Y_0$$

is called a g-expectation.

DEFINITION 3.4.5. Let $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and let $\rho : L^2(\mathcal{F}_T) \to L^2(\mathbb{R})$ be defined as

$$\rho = (\rho_t)_{t\in[0,T]}, \rho_t(x) := \varepsilon_g[-x|\mathcal{F}_t], \forall x \in L^2(\mathcal{F}_t). \tag{3.4.6}$$

THEOREM 3.4.1. *Using the above definition we get two implications to dynamic risk measures;*

1. *If the functional g is convex in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, then $(\rho_t)_{t\in[0,T]}$ defined as in Definition 3.4.5 is a dynamic convex risk measure. Moreover $(\rho_t)_{t\in[0,T]}$ is time-consistent and satisfies the axioms positivity, translation-invariance and constancy.*
2. *If the functional g is sub-linear in $(y, z) \in \mathbb{R}\times\mathbb{R}^d$, then $(\rho_t)_{t\in[0,T]}$ defined as in Definition 3.4.5 is a dynamic coherent risk measure which is time-consistent.*

Another dynamic risk measure that is time-consistent is the dynamic entropic risk measure with threshold, see Nadal [2008].

With the arise of a solid theory for risk measures, how to allocate risk capital by selecting a proper risk measure has become an important issue for further research.

Part 2

HJBI THEOREMS

CHAPTER 4

WORST CASE SCENARIO VERSION OF THE HJBI EQUATION

Now, we look at a result obtained by Mataramvura and Øksendal [2008], and prove a generalization to the 3 dimensional case. We will define and prove an associated HJBI equation for a zero-sum game. At the end of the chapter we will go through several examples that make use of the theorem that we establish.

4.1. Worst Case Minimizing

We let the state of our financial position, $X^u(t) = X(t) \in \mathbb{R}^k$, be given at time t by

$$\begin{aligned} dX^u(t) &= b(X^u(t), u_0(t), u_1(t))dt + \sigma(X^u(t), u_0(t), u_1(t))dB(t) \\ &\quad + \int_{\mathbb{R}^k} \gamma(X^u(t^-), u_2(t,z), z)\tilde{N}(dt, dz), \\ X^u(0) &= x \in \mathbb{R} \end{aligned} \tag{4.1.1}$$

Where $b : \mathbb{R}^k \times U \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times U \to \mathbb{R}^{k\times k}$ and $\gamma : \mathbb{R}^k \times U \times \mathbb{R}^k \to \mathbb{R}^{k\times m}$. $B(t)$ is a k-dimensional Brownian motion, $\tilde{N}(\cdot,\cdot) = (\tilde{N}_1(\cdot,\cdot), \dots, \tilde{N}_k(\cdot,\cdot))$ are a k-independent compensated Poisson random measure and U a Polish space. The processes $u_0(t) = u_0(t,\omega)$, $u_1(t) = u_1(t,\omega)$ and $u_2(t,z) = u_2(t,z,\omega)$ are the control processes, *càdlàg* and adapted to the filtration $\mathcal{F}_t$ generated by the driving processes $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$, with $u_0(t) \in U$, $u_1(t) \in U$ and $u_2(t,z) \in U$ for a.a. t, a.s. Let $u = (u_0, u_1, u_2)$ and $X^u(t)$ be the controlled jump diffusion.

We then look at the problem of minimizing the risk of the portfolio π associated to the financial position $X^u(t)$;

PROBLEM 4.1.1. Find the portfolio $\pi(t)$ that minimize the worst case risk of the terminal wealth $X^u(T)$.

Now, we let $g : \mathbb{R}^k \to \mathbb{R}$ be a given function called the bequest function. We assume a given family, $\mathcal{A}$, of admissible controls such that 4.1.1 has a unique strong solution and that

$$E^x\left[|g(X(\tau_s))|\right] < \infty, \tag{4.1.2}$$

for all $y \in \mathcal{S}$, where $\mathcal{S} \subset \mathbb{R}^k$ is an open set (called the solvency region) and where

$$\tau_s = \inf\{t > 0; X(t) \notin \mathcal{S}\} \tag{4.1.3}$$

is the bankruptcy time. Let the controls have the form

$$\begin{aligned}
u_0(t) &= \delta(t) \\
u_1(t) &= (\theta_0(t), \pi(t)); \\
u_2(t) &= (\theta_1(t,z), \pi(t,z));
\end{aligned}$$

We will try to minimize the risk by the viewpoint of a regulatory agent. So we use a risk measure, namely the generalization of coherent risk measures, convex risk measure. From the representation theorem for convex risk measure we have the general form

$$\rho(X) = \sup_{\mathcal{Q} \in \mathcal{M}} \{E_{\mathcal{Q}}[-X] - \zeta(\mathcal{Q})\}$$

for some family $\mathcal{M}$ of measures $\mathcal{Q}$ which are absolutely continuous with respect to P and some penalty function $\zeta : \mathcal{M} \to \mathbb{R}$. For ρ a given convex risk measure and $u \in \mathcal{A}$ we get from the representation theorem the following performance functional;

$$J^{\delta,\pi,\theta}(y) = E^y[g_\theta(X^{\delta,\pi,\theta}(\tau_s))].$$

Here, $g_\theta(x) = -x - \xi_0(\theta)$, the bequest function, and ξ_0, the penalty function, is given by the representation theorem. Then, we have that $\sup_\theta J^{\delta,\pi,\theta}(y)$ is a convex risk measure. The problem that we try to solve then gets the form

PROBLEM 4.1.2. Given a convex risk measure, ρ, find the portfolio, π, which minimizes

$$\sup_\delta \left[\inf_\pi \left(\rho(X^{\delta,\pi}(T))\right)\right].$$

From the above we get that this is equal to solving

$$\sup_\delta \left[\inf_\pi \left(\sup_\theta J^{\delta,\pi,\theta}(y)\right)\right].$$

Here

$$J^{\delta,\pi,\theta}(y) = E^y[g_\theta(X^{\delta,\pi,\theta}(\tau_s))],$$

is the corresponding performance functional given by the representation theorem.

To make things a little more intuitive we would use monetary utility functions.

Definition 4.1.1 (Mataramvura and Øksendal [2008] definition 2.3). A monetary utility function is a map $U : \mathbb{F} \to \mathbb{R}$ such that

- Concavity: $U(\lambda X + (1-\lambda)Y) \geq \lambda U(X) + (1-\lambda)U(Y)$, for all $X, Y \in \mathbb{F}$.
- Monotonicity: If $X \leq Y, X, Y \in \mathbb{F}$. then $U(X) \leq U(Y)$.
- Translation invariance: If $X \in \mathbb{F}$ and $m \in \mathbb{R}$ then $U(X+m) = U(X) + m$.

It follows from this that if ρ is a convex risk measure, then $U(X) := -\rho(X)$ is a monetary utility function and conversely. So we have the following version of our problem:

Problem 4.1.3. Find

$$\Phi := \inf_{\delta}\left[\sup_{\pi}\left(U(X^{\delta,\pi,\theta}(T))\right)\right],$$

where

$$U(X^{\delta,\pi,\theta}(T)) = \inf_{Q\in M_a}\{E_Q[X^{\delta,\pi,\theta}(T)] + \zeta(Q)\} = -\rho(X^{\delta,\pi,\theta}(T)),$$

is a monetary utility function as in Definition 4.1.1. Further, find optimal $\hat{\delta}, \hat{\pi}, \hat{\theta}$ such that

$$\Phi := U(X^{\hat{\delta},\hat{\pi},\hat{\theta}}(T)).$$

In the following we will use a generalization that includes convex risk measure by allowing for a function $f : \mathbb{R}^k \times U \to \mathbb{R}$, the profit rate, in the performance functional.

Problem 4.1.4. Find

$$\Phi := \inf_{\delta}\left[\sup_{\pi}\left(\sup_{\pi} J^{\delta,\pi\theta}\right)\right],$$

where

$$J^{\delta,\pi,\theta}(y) = E^y\left[\int_0^{\tau_s} f(X(t), u_0(t)) + g(X(\tau_s))\right].$$

Assume a given family A of admissible controls contained in the set U of controls u such that (4.1.1) has a unique strong solution and

$$E^y\left[\int_0^{\tau_s} |f(X^u(t), u_0(t))|\right] < \infty, \tag{4.1.4}$$

for all $y \in S$. Further, find optimal $\hat{\delta}, \hat{\pi}, \hat{\theta}$ such that

$$\Phi := J^{\hat{\delta},\hat{\pi},\hat{\theta}}(y).$$

This problem clearly includes convex risk measures as a special case. We now go on to formulate the zero-sum game.

4.2. The Zero-Sum Game

We can think of controllers θ, π and δ as the control of players 1,2 and 3 respectively. Let $A = \Delta \times \Pi \times \Theta$ be our familie of admissible controls. We can den formulate the convex zero-sum differential game problem as

PROBLEM 4.2.1. Find $\Phi(y)$ and $(\delta^*, \pi^*, \theta^*) \in \Delta \times \Pi \times \Theta$ such that

$$\Phi(y) = \inf_{\delta} \left[\sup_{\pi} \left(\inf_{\theta} J^{\delta,\pi,\theta}(y) \right) \right]$$

where

$$J^{\delta,\pi,\theta}(y) = E^y \left[\int_0^{\tau_s} f(X(t), u_0(t))dt + g(X(\tau_s)) \right],$$

REMARK 4.2.1. We will show that under some conditions the problem can be seen as

$$\sup_{\pi} \left(\inf_{(\delta,\theta)} J^{\delta,\pi,\theta}(y) \right),$$

which is a two player game with a two dimensional controller.

REMARK 4.2.2. Under some conditions we can look at the problem as a minimax problem where we can apply the minimax theorem from Delbaen [2002].

THEOREM 4.2.1 (Minimax Theorem). *Let K be a compact convex subset of a locally convex space F. Let L be a convex set of an arbitrary vector space E. Suppose that u is bilinear function $u : E \times F \to \mathbb{R}$. For each $l \in L$ we suppose that the partial (linear) function $u(l, \cdot)$ is continuous on F. Then we have that*

$$\inf_{l \in L} \left(\sup_{k \in K} u(l,k) \right) = \sup_{k \in K} \left(\inf_{l \in L} u(l,k) \right)$$

As in Øksendal [2007] we use Markov controls since under mild conditions Markov controls can give just as good performance as more general adapted controls. When we use Markov controls we get that the generator $A^{\delta,\pi,\theta}$ becomes

$$\begin{aligned}
A^{\delta,\pi,\theta}\varphi(y) &= \sum_{i=1}^{k} b_i(y,\theta_0(y),\pi_0(y),\delta(y))\frac{\partial\varphi}{\partial y_i}(y) \\
&+ \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{ij}(y,\theta_0(y),\pi_0(y),\delta(y))\frac{\partial^2\varphi}{\partial y_i\partial y_j}(y) \\
&+ \sum_{j=1}^{k}\int_{\mathbb{R}}\{\varphi(y+\gamma^{(j)}(y,\theta_1(y,z_j),\pi_1(y,z_j),z_j)-\varphi(y) \\
&- \nabla\varphi(y)\gamma^{(j)}(y,\theta_1(y,z_j),\pi_1(y,z_j),z_j)\}v_j(dz_j);
\end{aligned}$$

where $\varphi \in C_0^2(\mathbb{R}^k)$ and $\nabla\varphi$ is the gradient of φ. We let $\mathcal{T}$ be the set of all $\mathcal{F}_t$ -stopping times $\tau \leq \tau_s$.

4.3. A HJBI equation for zero-sum differential games with convex risk measures

We are now ready to state the main theorem.

THEOREM 4.3.1. *Suppose $\varphi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ and a Markov control $(\delta,\pi,\theta) \in \Delta\times\Pi\times\Theta$ such that*

(i) $A^{\delta,\hat{\pi},\theta}\varphi(y) + f(y,\delta,\hat{\pi},\theta) \geq 0$ *for all* $\delta \in K_1$ *and all* $\theta \in K_3$.

(ii) $A^{\hat{\delta},\pi,\hat{\theta}}\varphi(y) + f(y,\hat{\delta},\pi,\hat{\theta}) \leq 0$ *for all* $\pi \in K_2$, *for all* y.

(iii) $A^{\hat{\delta},\hat{\pi},\hat{\theta}}\varphi(y) + f(y,\hat{\delta},\hat{\pi},\hat{\theta}) = 0$ *for all* y

(iv) $X^{\delta,\pi,\theta}(\tau_s) \in \partial\mathcal{S}$ *a.s. on* $\{\tau_s < \infty\}$ *and* $\lim_{t\to\tau_s^-}\varphi(X^{\delta,\pi,\theta}(t)) = g(X^{\delta,\pi,\theta}(\tau_S))\chi_{\{\tau_s<\infty\}}$ *a.s. for all* $(\delta,\pi,\theta) \in \Delta\times\Pi\times\Theta$, $y \in \mathcal{S}$.

(v) *The family* $\{\varphi(X^{\delta,\pi,\theta}(\tau))\}_{\tau\in\mathcal{T}}$ *is uniformly integrable, for all* $(\delta,\pi\,\theta) \in \Delta\times\Pi\times\Theta$, $y \in \mathcal{S}$.

Then

$$
\begin{aligned}
\varphi(y) &= \Phi(y) = J^{\hat{\delta},\hat{\pi},\hat{\theta}}(y) \\
&= \inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right] = \sup_{\pi}\left[\inf_{\delta}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right] \\
&= \inf_{\delta}\left[\inf_{\theta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right] = \inf_{\theta}\left[\inf_{\delta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right] \\
&= \sup_{\pi}\left[\inf_{\theta}\left(\inf_{\delta} J^{\delta,\pi,\theta}(y)\right)\right] = \inf_{\theta}\left[\sup_{\pi}\left(\inf_{\delta} J^{\delta,\pi,\theta}(y)\right)\right] \\
&= \sup_{\pi}\left[\inf_{\theta} J^{\hat{\delta},\pi,\theta}(y)\right] = \sup_{\pi}\left[\inf_{\delta} J^{\delta,\pi,\hat{\theta}}(y)\right] = \inf_{\delta}\left[\inf_{\theta} J^{\delta,\hat{\pi},\theta}(y)\right] \\
&= \inf_{\delta}\left[\sup_{\pi} J^{\delta,\pi,\hat{\theta}}(y)\right] = \inf_{\theta}\left[\sup_{\pi} J^{\hat{\delta},\pi,\theta}(y)\right] = \inf_{\theta}\left[\inf_{\delta} J^{\delta,\hat{\pi},\theta}(y)\right] \\
&= \sup_{\pi} J^{\hat{\delta},\pi,\hat{\theta}}(y) = \inf_{\delta} J^{\delta,\hat{pi},\hat{\theta}}(y) = \inf_{\theta} J^{\hat{\delta},\hat{\pi},\theta}(y)
\end{aligned}
$$

and

$(\hat{\delta},\hat{\pi},\hat{\theta})$ *is an optimal (Markov) control.*

Proof. Step1. First let us prove that

$$\Phi(y) = \varphi(y) = J^{\hat{\delta},\hat{\pi},\hat{\theta}} = \inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right].$$

(a) First, from Dynkin's formula for jump processes (see Øksendal and Sulem [2007] theorem 1.24);

$$E^y[\varphi(Y(\tau_s^N))] = \varphi(y) + E^y[\int_0^{\tau_s^N} A^{\delta,\phi,\theta}\varphi(Y(t))dt], \tag{4.3.5}$$

then, from (i)

$$\varphi(y) \leq E^y[\int_0^{\tau_s^N} f(Y(t),\delta(Y(t)),\pi(Y(t)),\theta(Y(t)))dt + \varphi(Y(\tau_s^N))].$$

Let $N\to\infty$ and (iv) and (v) to obtain

$$\varphi(y) \leq J^{\delta,\hat{\pi},\theta}(y).$$

Since this holds for all δ and all θ we have that

$$\varphi(y) \leq \inf_{\delta}\left(\inf_{\theta} J^{\delta,\hat{\pi},\theta}(y)\right),$$

and on the other hand we see that

$$\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right) \geq \inf_{\theta} J^{\delta,\hat{\pi},\theta}(y),$$

for every δ, so we can take infimum over δ on both sides

$$\inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right] \geq \inf_{\delta}\left(\inf_{\theta} J^{\delta,\hat{\pi},\theta}(y)\right).$$

This leaves us to conclude

$$\varphi(y) \leq \inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right] = \Phi(y). \tag{4.3.6}$$

(b) Again, using 4.3.5 and (ii) we get that

$$\varphi(y) \geq E^{y}[\varphi(X(\tau_s^N))].$$

It then follows that

$$\varphi(y) \geq J^{\hat{\delta},\pi,\hat{\theta}}(y) \geq \inf_{\theta} J^{\hat{\delta},\pi,\theta}.$$

This holds for all π, so

$$\varphi(y) \geq \sup_{\pi} J^{\hat{\delta},\pi,\hat{\theta}}(y) \geq \sup_{\pi}\left(\inf_{\theta} J^{\hat{\delta},\pi,\theta}\right)$$

Taking infimum over δ gives

$$\varphi(y) \geq \inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right]. \tag{4.3.7}$$

Combining (4.3.6) and (4.3.7) we have that

$$\Phi(y) = \inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right] \leq \varphi(y) \leq \inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right] = \Phi(y). \tag{4.3.8}$$

(c) Using 4.3.5 to $\hat{\delta},\hat{\pi},\hat{\theta} \in \Delta,\Pi,\Theta$ and (iii) we get that

$$\varphi(y) = J^{\hat{\delta},\hat{\pi}\hat{\theta}} = \Phi(y). \tag{4.3.9}$$

Combining (4.3.6),(4.3.7) and (4.3.9) we get

$$\Phi(y) = \varphi(y) = J^{\hat{\delta},\hat{\pi},\hat{\theta}} = \inf_{\delta}\left[\sup_{\pi}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right].$$

Step2. Next let us prove that

$$\varphi(y) = \inf_{\theta}\left[\inf_{\delta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right].$$

Using (4.3.5) with $\delta, \hat{\pi}, \theta \in \Delta \times \hat{\pi} \times \Theta$ and (i), we have

$$\varphi(y) \le J^{\delta,\hat{\pi},\theta}(y).$$

Clearly,

$$J^{\delta,\hat{\pi},\theta}(y) \le \sup_{\pi} J^{\delta,\pi,\theta}(y).$$

This holds for all δ, θ so

$$\varphi(y) \le \inf_{\theta}\left[\inf_{\delta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right].$$

Further, we have using (ii)

$$\varphi(y) \ge \sup_{\pi} J^{\hat{\delta},\pi,\hat{\theta}} \ge \inf_{\theta}\left[\inf_{\delta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right]$$

so we conclude that

$$\varphi(y) = \inf_{\theta}\left[\inf_{\delta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right].$$

Step 3. Now let us prove that

$$\varphi(y) = \sup_{\pi}\left[\inf_{\theta}\left(\inf_{\delta} J^{\delta,\pi,\theta}(y)\right)\right].$$

We have from Dynkin with (i)

$$\varphi(y) \le \inf_{\theta}\left[\inf_{\delta} J^{\delta,\hat{\pi},\theta}\right] \le \sup_{\pi}\left[\inf_{\theta}\left(\inf_{\delta} J^{\delta,\pi,\theta}(y)\right)\right]$$

and using (ii) we get that

$$\varphi(y) \ge J^{\delta,\hat{\pi},\theta} \ge \inf_{\theta}\left[\inf_{\delta} J^{\delta,\pi,\theta}(y)\right].$$

Since this holds for all π, we have that

$$\varphi(y) \ge \sup_{\pi}\left[\inf_{\theta}\left(\inf_{\delta} J^{\delta,\pi,\theta}(y)\right)\right].$$

This leaves us to conclude that

$$\varphi(y) = \sup_{\pi}\left[\inf_{\theta}\left(\inf_{\delta} J^{\delta,\pi,\theta}(y)\right)\right].$$

Step 4. Now let us prove that

$$\varphi(y) = \inf_{\delta}\left[\inf_{\theta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right].$$

As before, we have from Dynkin with (i)

$$\varphi(y) \leq J^{\delta,\hat{\pi},\theta}.$$

It follows that

$$\varphi(y) \leq \sup_{\pi} J^{\delta,\pi,\theta}.$$

Since this holds for all π

$$\varphi(y) \leq \inf_{\delta}\left[\inf_{\theta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right].$$

we have that from (ii)

$$\varphi(y) \geq J^{\hat{\delta},\pi,\hat{\theta}}(y).$$

Since it holde fr all π

$$\varphi(y) \geq \inf_{\delta}\left[\inf_{\theta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right].$$

This leaves us to conclude that

$$\varphi(y) = \inf_{\delta}\left[\inf_{\theta}\left(\sup_{\pi} J^{\delta,\pi,\theta}(y)\right)\right].$$

Step5. The same approach gives us

$$\varphi(y) = \sup_{\pi}\left[\inf_{\delta}\left(\inf_{\theta} J^{\delta,\pi,\theta}(y)\right)\right] = \inf_{\theta}\left[\sup_{\pi}\left(\inf_{\delta} J^{\delta,\pi,\theta}(y)\right)\right]$$

Step6. Next thing we prove is that

$$\varphi(y) = \sup_{\pi}\left[\inf_{\theta} J^{\hat{\delta},\pi,\theta}(y)\right].$$

We do as before: Using Dynkin and (i

$$\varphi(y) \geq J^{\hat{\delta},\pi,\hat{\theta}} \geq \inf_{\theta} J^{\hat{\delta},\pi,\theta}.$$

Since it holds for all π, we get

$$\varphi(y) \geq \sup_{\pi}\left[\inf_{\theta} J^{\delta,\pi,\theta}\right].$$

On the other hand usin g Dynkin and (i)

$$\varphi(y) \leq J^{\delta,\hat{\pi},\theta}.$$

Since this holds for all θ

$$\varphi(y) \leq \inf_{\theta} J^{\delta,\hat{\pi},\theta}.$$

So we get

$$\varphi(y) \leq \sup_{\pi} \left[\inf_{\theta} J^{\hat{\delta},\pi,\theta}(y) \right].$$

And so we conclude

$$\varphi(y) = \sup_{\pi} \left[\inf_{\theta} J^{\hat{\delta},\pi,\theta}(y) \right].$$

Step7. The same approach gives us:

$$\begin{aligned}\varphi(y) &= \sup_{\pi} \left[\inf_{\delta} J^{\delta,\pi,\hat{\theta}}(y) \right] = \inf_{\delta} \left[\inf_{\theta} J^{\delta,\hat{\pi},\theta}(y) \right] = \inf_{\delta} \left[\sup_{\pi} J^{\delta,\pi,\hat{\theta}}(y) \right] \\ &= \inf_{\theta} \left[\sup_{\pi} J^{\hat{\delta},\pi,\theta}(y) \right] = \inf_{\theta} \left[\inf_{\delta} J^{\delta,\hat{\pi},\theta}(y) \right]\end{aligned}$$

Step8. Lets prove that

$$\varphi(y) = \sup_{\pi} J^{\hat{\delta},\pi,\hat{\theta}}.$$

Using Dynkin and (ii)

$$\varphi(y) \geq J^{\hat{\delta},\pi,\hat{\theta}}.$$

Since it holds for all π, we get

$$\varphi(y) \geq \sup_{\pi} J^{\hat{\delta},\pi,\hat{\theta}}.$$

On the other hand

$$\varphi(y) = J^{\hat{\delta},\hat{\pi},\hat{\theta}} \leq \sup_{\pi} J^{\hat{\delta},\pi,\hat{\theta}}.$$

And so we conclude

$$\varphi(y) = \sup_{\pi} J^{\hat{\delta},\pi,\hat{\theta}}.$$

Step9. Lets prove that

$$\varphi(y) = \inf_{\delta} J^{\delta,\hat{\pi},\hat{\theta}}.$$

Using Dynkin and (i)

$$\varphi(y) \leq J^{\delta,\hat{\pi},\theta}.$$

Since it holds for all θ, we get

$$\varphi(y) \leq \inf_{\delta} J^{\delta,\hat{\pi},\hat{\theta}}.$$

It also holds for all δ, so we get

$$\varphi(y) \leq J^{\delta,\hat{\pi},\hat{\theta}}.$$

On the other hand

$$\varphi(y) = J^{\hat{\delta},\hat{\pi},\hat{\theta}} \geq \inf_{\delta} J^{\delta,\hat{\pi},\hat{\theta}}.$$

And so we conclude

$$\varphi(y) = \inf_{\delta} J^{\delta,\hat{\pi},\hat{\theta}}.$$

Step10. The same approach gives us:

$$\varphi(y) = \inf_{\theta} J^{\hat{\delta},\hat{\pi},\theta}(y).$$

$\square$

4.4. Examples

We will now apply the theorem to some examples. First, we look at the problem of optimizing when the investor has a consumption function. We then try to maximize the portfolio and the consumption process while the marked minimize through a scenario. Next, we look at a similar example but where we let the investors consumption function be given and we maximize a portfolio and minimize the marked by a scenario and the drift. We then extend this to a Lévy setting. Then we will optimize a portfolio for a utility function where the drift is given a posteriori. We will also give an example of a scenario optimization in a Lévy -market. Further we give two example of optimization using convex risk measure one in a standard Black-Cox market while the other one in a Lévy -market. Finally we look at a mean square hedging problem.

We consider the marked given by (2.3.2) and (2.3.3).

Example 4.4.1 (Consumption). Let us try to solve problem 4.2.1 by using the HJBI equation. We will use an investor with consumption, who is controlling his rate of consumption. The market is minimizing the investor expected return over a set of scenarios while the investor tries to control his consumption and portfolio to maximize his expected return. Let M_a be as before and let $\Gamma(t)$ be a cumulative income process as in Karatzas and Shreve [1998], where $\Gamma(t) = \int_0^t c(u)du$ for a non-negative function $c(\cdot)$ such that $\int_0^T c(u)du < \infty$ a.s. Further, let

$$\begin{cases} dY(t) = (dY_0(t), dY_1(t), dY_2(t)); \\ Y(0) = y = (s, y_1, y_2), \end{cases}$$

where

$$\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}. \\
&dY_1(t) = dV^\pi(t) = -d\Gamma(t) + Y_1(t)[\alpha(t)\pi(t)dt + \beta(t)\pi(t)dB(t)]; && Y_1(0) = y_1 > 0. \\
&dY_2(t) = -\theta(t)Y_2(t)dB(t); && Y_2(0) = y_2 > 0.
\end{aligned}$$

Let

$$J^{c,\pi,\theta}(y) = E^y\left[Y_2(T-s)\int_0^{T-s} U_1(c(t))dt + Y_2(T-s)U_2(Y_1(T-s))\right],$$

where U_1 and U_2 are utility functions be our performance functional. Then

$$\begin{aligned}
J^{c,\pi,\theta}(y) &= E^y\left[Y_2(T-s)\int_0^{T-s} U_1(c(t))dt + Y_2(T-s)U_2(Y_1(T-s))\right] \\
&= E^y\left[\int_0^{T-s} Y_2(t)U_1(c(t))dt + Y_2(T-s)U_2(Y_1(T-s))\right].
\end{aligned}$$

PROOF. We will show this in the general case for a $s \in [0,T]$. By the definition of conditional expectation, see e.g. Øksendal [2007] Appendix B, we have that

$$\begin{aligned}\int_F E[Y(T)\int_s^T u(t)dt|\mathcal{F}_s]dP &= \int_F\int_s^T E[Y(T)u(t)|\mathcal{F}_s]dtdP \\ &= \int_F\int_s^T Y(T)u(t)dtdP \text{ for all } F\in\mathcal{F}_s \text{ for some s } \in [0,T].\end{aligned}$$

Since (Ω, P) and $([0,T],\lambda)$ are σ-finite we can use Fubini and get

$$\int_F\int_s^T E[Y(T)u(t)|\mathcal{F}_s]dtdP = \int_s^T\int_F E[Y(T)u(t)|\mathcal{F}_s]dPdt$$

and Y_t is a martingale so by using the tower property we get

$$\begin{aligned}\int_s^T\int_F E[Y(T)u(t)|\mathcal{F}_s]dPdt &= \int_s^T\int_F E[E[Y(T)u(t)|\mathcal{F}_t|\mathcal{F}_s]dPdt \\ &= \int_s^T\int_F Y(t)E[u(t)|\mathcal{F}_s]dPdt \\ &= \int_s^T\int_F E[Y(t)u(t)|\mathcal{F}_s]dPdt \\ &= \int_s^T\int_F Y(t)u(t)dPdt,\end{aligned}$$

by definition. Again by Fubini

$$\int_s^T\int_F Y(t)u(t)dPdt = \int_F\int_s^T Y(t)u(t)dtdP = \int_F E[\int_s^T Y(t)u(t)dt|\mathcal{F}_s]dP.$$

Since $E[\int_s^T Y(t)u(t)dt|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable we get that

$$E[Y(T)\int_s^T u(t)dt|\mathcal{F}_s] = E[\int_s^T Y(t)u(t)dt|\mathcal{F}_s].$$

Letting $s=0$ we get the result. $\square$

We need a constraint on the admissibility of the pair (c,π) to ensure that $Y_1(t)\geq 0$ so that we don't get negative wealth. We therefor let the consumption process be a relative consumption process, e.g. $c(t)=\lambda(t)Y_1(t)$, where $E_{\mathcal{Q}_\theta}[\int_0^T\lambda(t)dt]<\infty$, $\mathcal{Q}_\theta$-a.s. so that

$$\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s\in\mathbb{R}.\\
&dY_1(t) = dV^\pi(t) = Y_1(t)[(\alpha(t)\pi(t)-\lambda(t))dt+\beta(t)\pi(t)dB(t)]; && Y_1(0)=y_1>0.\\
&dY_2(t) = -\theta(t)Y_2(t)dB(t); && Y_2(0)=y_2>0.
\end{aligned}$$

The problem can then be represented as

$$\Phi(y) = \Phi(s, y_1, y_2) = \sup_c \left[\inf_\theta \left(\sup_\pi J^{c,\pi,\theta}(y) \right) \right] = \inf_\theta \left[\sup_\pi \left(\sup_c J^{c,\pi,\theta}(y) \right) \right].$$

We have that the generator of $Y(\cdot)$ is

$$A^{c,\pi,\theta}\varphi(y) = \frac{\partial \varphi}{\partial s} + y_1(\alpha\pi - \lambda)\frac{\partial \varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1 \partial y_2};$$

The corresponding HJBI equation is

$$(4.4.10) \qquad \begin{cases} \inf_\theta \left[\sup_\pi \left(\sup_c A^{c,\pi,\theta}(y) \right) \right] + U_1(c)y_2 = 0 \\ \varphi(T, y_1, y_2) = U_2(y_1)y_2 \end{cases}$$

Fix π and λ and minimize

$$h(\theta) := -y_1y_2\beta\theta\pi\varphi_{12} + \frac{1}{2}y_2^2\theta^2\varphi_{22}$$

with respect to θ.Minimum is attained at

$$\theta = \hat{\theta}(y) = \frac{y_1\beta\pi\varphi_{12}}{y_2\varphi_{22}}$$

Substitute this and maximize

$$k(\pi) := y_1\alpha\pi\varphi_1 + \frac{1}{2}y_1^2\beta^2\pi^2(\varphi_{11} - \frac{\varphi_{12}^2}{\varphi_{22}})$$

with respect to π. The maximum is attained at

$$\pi = \hat{\pi}(y) = \frac{\alpha\varphi_1\varphi_{22}}{y_1\beta^2(\varphi_{12}^2 - \varphi_{11}\varphi_{22})}.$$

Substituting this, we have

$$\begin{aligned} &\frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} + \frac{\alpha^2\varphi_1^2\varphi_{22}^2\varphi_{11}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2} + \frac{\alpha^2\delta^2\varphi_1^2\varphi_{22}^2\varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2} \\ &- \frac{\alpha^2\varphi_1^2\varphi_{22}^2\varphi_{22}}{\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2} = \frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} + \frac{\alpha^2\varphi_1^2\varphi_{22}(\varphi_{22}\varphi_{11} - \varphi_{12}^2)}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2} \\ &= \frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} \end{aligned}$$

Now, the condition

$$\inf_\theta \left[\sup_\pi \left(\sup_c A^{c,\pi,\theta}(y) \right) \right] + U_1(c)y_2 = 0$$

gives us

$$\frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} + U_1(\lambda y_1)y_2$$

to maximize with respect to λ. When then get

$$U_1'(\lambda y_1) = \frac{\varphi_1}{y_2}$$

Which means that

$$\lambda y_1 = c = I_1(\frac{\varphi_1}{y_2}),$$

where I_1 is the inverse of U_1. We get a partial differential equation for φ

$$\begin{cases} \frac{\partial \varphi}{\partial s} - I_1(\frac{\varphi_1}{y_2})\varphi_1 + \frac{\alpha^2 \varphi_1^2 \varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} = 0 \\ \varphi(T, y_1, y_2) = U_2(y_1)y_2 \end{cases}$$

Let us try some a specific case where $U_1(x) = \log(x)$, $U_2(x) = log(x)$,so that

$$J^{c,\pi,\theta}(y) = E^y\left[\int_0^T \log(c(t))Y_2(t)dt + \log(Y_1(T))Y_2(T)\right].$$

We will then have that the HJBI equation is

$$\begin{cases} \inf_\theta \left[\sup_\pi \left(\sup_c A^{c,\pi,\theta}(y)\right)\right] + \log(c)y_2 = 0, \quad s < T. \\ \varphi(T, y_1, y_2) = \log(y_1)y_2, \end{cases} \tag{4.4.11}$$

and the generator of $Y(\cdot)$ is

$$A^{c,\pi,\theta}\varphi(y) = \frac{\partial \varphi}{\partial s} - \lambda y_1 \frac{\partial \varphi}{\partial y_1} + y_1 \alpha \pi \frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \pi^2 \frac{\partial^2 \varphi}{\partial^2 y_1} + \frac{1}{2} y_2^2 \theta^2 \frac{\partial^2 \varphi}{\partial^2 y_2} - y_1 y_2 \beta \theta \pi \frac{\partial^2 \varphi}{\partial y_1 \partial y_2};$$

Let us try a function on the form

$$\varphi(s, y_1, y_2) = h(s)\log(y_1)y_2.$$

Then the corresponding generator becomes

$$A^{c,\pi,\theta}\varphi(y) = h'(s)\log(y_1)y_2 - \lambda y_2 h(s) + \alpha \pi y_2 h(s) - \frac{1}{2}\beta^2 \pi^2 y_2 h(s) - y_2 \beta \theta \pi h(s);$$

Fix π and θ and maximize

$$h(\lambda) := log(\lambda y_1) - \lambda y_2$$

with respect to λ. This is clearly obtained at $\hat{\lambda} = \frac{1}{y_2}$. So we now maximize the function

$$f(\pi) = \alpha \pi - \frac{1}{2}\beta^2 \pi^2.$$

So $\hat{\pi} = \frac{\alpha}{\beta^2}$.Then finally we minimize

$$-y_1 y_2 \beta\theta\pi.$$

Then we get that $\hat{\theta} = 0$. Then from requirement (iv)

$$A^{\hat{\lambda},\hat{\pi},\hat{\theta}}\varphi(y) = h'(s)\log(y_1)y_2 - 1 + \frac{1}{2}\frac{\alpha^2}{\beta^2}y_2 = 0;$$

So

$$h'(s) = \log^{-1}(y_1)[1 - \frac{1}{2\beta^2}\alpha^2].$$

and we have that

$$\Phi(s, y_1, y_2) = \varphi(s, y_1, y_2) = [1 - \frac{1}{2\beta^2}\alpha^2]sy_2.$$

Example 4.4.2 (Worst Case with Consumption). Let the setting be as above but now assume the investor has a preferred consumption process where we want to maximize the value process by choosing the optimal portfolio where the drift therm is given posteriori. This problem can be represented as

$$\Phi(y) = \Phi(s, y_1, y_2) = \inf_{\alpha}\left[\sup_{\pi}\left(\inf_{\theta} J^{\alpha,\pi,\theta}(y)\right)\right].$$

We have that the generator of $Y(\cdot)$ is

$$A^{c,\pi,\theta}\varphi(y) = \frac{\partial\varphi}{\partial s} + y_1(\alpha\pi - \lambda)\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1 y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2};$$

The corresponding HJBI equation is

$$\begin{cases} \inf_{\alpha}\left[\sup_{\pi}\left(\inf_{\theta} A^{\alpha,\pi,\theta}(y)\right)\right] + U_1(c)y_2 = 0 \\ \varphi(T, y_1, y_2) = U_2(y_1)y_2 \end{cases} \tag{4.4.12}$$

Fix π and α and minimize

$$h(\theta) := -y_1 y_2\beta\theta\pi\varphi_{12} + \frac{1}{2}y_2^2\theta^2\varphi_{22}$$

with respect to θ.Minimum is attained at

$$\theta = \hat{\theta}(y) = \frac{y_1\beta\pi\varphi_{12}}{y_2\varphi_{22}}$$

Substitute this and maximize

$$k(\pi) := y_1\alpha\pi\varphi_1 + \frac{1}{2}y_1^2\beta^2\pi^2(\varphi_{11} - \frac{\varphi_{12}^2}{\varphi_{22}})$$

with respect to π. The maximum is attained at

$$\pi = \hat{\pi}(y) = \frac{\alpha\varphi_1\varphi_{22}}{y_1\beta^2(\varphi_{12}^2 - \varphi_{11}\varphi_{22})}.$$

Substituting this, we have

$$\begin{aligned}&\frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} + \frac{\alpha^2\varphi_1^2\varphi_{22}^2\varphi_{11}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2} + \frac{\alpha^2\delta^2\varphi_1^2\varphi_{22}^2\varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2}\\ &- \frac{\alpha^2\varphi_1^2\varphi_{22}^2\varphi_{22}}{\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2} = \frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} + \frac{\alpha^2\varphi_1^2\varphi_{22}(\varphi_{22}\varphi_{11} - \varphi_{12}^2)}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})^2}\\ &= \frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})}\end{aligned}$$

Now, the condition

$$\inf_\theta \left[\sup_\pi \left(\sup_c A^{c,\pi,\theta}(y)\right)\right] + U_1(c)y_2 = 0$$

gives us

$$\frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 + \frac{\alpha^2\varphi_1^2\varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} + U_1(\lambda y_1)y_2$$

to minimize with respect to α. When then get

$$\alpha = 0.$$

We get a partial differential equation for φ

$$\begin{cases}\frac{\partial\varphi}{\partial s} - \lambda y_1\varphi_1 = 0\\ \varphi(T, y_1, y_2) = U_2(y_1)y_2\end{cases}$$

EXAMPLE 4.4.3 (Consumption in a Lévy Market). We now try to optimize in a setting where the investor has the choice between two investments in a Lévy market. The market are given as (2.3.2) and (2.3.3). Let M_a be as before. We will let the market control (θ_0, θ_1) and the investor will control λ. As before, we let

$$\begin{cases}dY(t) = (dY_0(t), dY_1(t), dY_2(t));\\ Y(0) = y = (s, y_1, y_2),\end{cases}$$

where

$$
\begin{aligned}
dY_0(t) &= dt; && Y_0(0) = s \in \mathbb{R}.\\
dY_1(t) &= dV^{\pi}(t) = Y_1(t)[(\alpha(t)\pi(t) - \lambda(t))dt + \beta(t)\pi(t)dB(t)]\\
&\quad + Y_1(t^-)\pi(t^-)\int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz); && Y_1(0) = y_1 > 0.\\
dY_2(t) &= -Y_2(t)\theta_0(t)dB(t) - Y_2(t)\int_{\mathbb{R}} \theta_1(s,z)\tilde{N}(ds,dz); && Y_2(0) = y_2 > 0.
\end{aligned}
$$

Further, let our performance functional be

$$J^{\lambda,\theta_0,\theta_1}(y) = E^y\left[\int_0^{T-s} Y_2(t)U_1(c(t))dt + Y_2(T-s)U_2(Y_1(T-s))\right].$$

The problem can then be represented as

$$\Phi(y) = \Phi(s,y_1,y_2) = \inf_{\theta_0}\left[\sup_{\lambda}\left(\inf_{\theta_1} J^{\lambda,\theta_0,\theta_1}(y)\right)\right].$$

We will then have that the HJBI equation is

$$
\begin{cases}
\sup_{\lambda}\left[\inf_{\theta_0}\left(\inf_{\theta_1} A^{\lambda,\theta_0,\theta_1}(y)\right)\right] + U_1(\lambda y_1)y_2 = 0,\\
\varphi(T,y_1,y_2) = U_2(y_1)y_2,
\end{cases}
\tag{4.4.13}
$$

and the generator of $Y(\cdot)$ is

(4.4.14)

$$
\begin{aligned}
A^{\delta,\pi,\theta}\varphi(y) &= \frac{\partial\varphi}{\partial s} + y_1(\alpha\pi - \lambda)\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta_0^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta_0\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2}\\
&+ \int_{\mathbb{R}}\left[\varphi(s,y_1 + y_1\pi\gamma(t,z), y_2 - y_2\theta_1(t,z)) - \varphi(s,y_1,y_2) - y_1\pi\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2}\right]v(dz).
\end{aligned}
$$

Let λ and θ_0 be fixed and minimize

$$
\begin{aligned}
f(\theta_1) := \int_{\mathbb{R}}\Big[&\varphi(s,y_1 + y_1\pi z, y_2 - y_2\theta_1(t,z)) - \varphi(s,y_1,y_2)\\
&- y_1\pi\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(t,z)\frac{\partial\varphi}{\partial y_2}\Big]v(dz),
\end{aligned}
$$

for functions $\theta(t,z)$. We minimize pointwise and find minimum

$$\frac{\partial\varphi}{\partial y_2}(s, y_1(1+\pi\gamma(t,z)), y_2(1-\hat{\theta}_1)) = \frac{\partial\varphi}{\partial y_2}(s,y_1,y_2). \tag{4.4.15}$$

We then use

$$g(\lambda) := y_1(\alpha\pi - \lambda)\frac{\partial\varphi}{\partial y_1} + U_1(\lambda y_1)y_2,$$

to maximize over λ to get

$$\hat{\lambda} = \frac{1}{y_1} I_1(\frac{\varphi_1}{y_2}).$$

Further, for θ_0, we let

$$l(\theta_0) = \frac{1}{2} y_2^2 \theta_0^2 \pi^2 \frac{\partial^2 \varphi}{\partial^2 y_2} - y_1 y_2 \beta \theta_0 \pi \frac{\partial^2 \varphi}{\partial y_1 \partial y_2},$$

and find

$$\hat{\theta_0} = \frac{y_1}{y_2} \frac{\beta}{\pi} \frac{\varphi_{12}}{\varphi_{22}}.$$

when $\varphi_{22} \neq 0$. Then we have an optimal trippel $(\hat{\lambda}, \hat{\theta_0}, \hat{\theta_1})$ which is substituted into (4.4.14) to give

$$\begin{aligned}
&\frac{\partial \varphi}{\partial s} + y_1(\alpha\pi - \frac{1}{y_1} I_1(\frac{\varphi_1}{y_2}))\frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \pi^2 \frac{\partial^2 \varphi}{\partial^2 y_1} + \frac{1}{2} y_2^2 \hat{\theta_0}^2 \pi^2 \frac{\partial^2 \varphi}{\partial^2 y_2} - y_1 y_2 \beta \hat{\theta_0} \pi \frac{\partial^2 \varphi}{\partial y_1 \partial y_2} \\
&+ \int_{\mathbb{R}} \Big[\varphi(s, y_1 + y_1 \pi \gamma(t,z), y_2 - y_2 \hat{\theta_1}(t,z)) - \varphi(s, y_1, y_2) \\
&- y_1 \pi \gamma(t,z) \frac{\partial \varphi}{\partial y_1} + y_2 \hat{\theta_1}(z) \frac{\partial \varphi}{\partial y_2} \Big] v(dz) = 0,
\end{aligned}$$

by requirement (iii).

Example 4.4.4 (Portfolio Optimization in Worst Case). We will now try to optimize in a setting where the investor has the choice between two investments in a non-jump market. We assume the drift is given a priori and minimized by the marked. The market are given as (2.3.2) and (2.3.3) with $\gamma(t,z) = 0$. Let M_a be as above with $\theta_1(t,z) = 0$ Now,let

$$\begin{cases} dY(t) = (dY_0(t), dY_1(t), dY_2(t)); \\ Y(0) = y = (s, y_1, y_2), \end{cases}$$

where

$$\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}. \\
&dY_1(t) = dV^{\pi}(t) = Y_1(t)[(r(t) + (\alpha(t) - r(t))\pi(t))dt + \beta\pi(t)dB(t)]; && Y_1(0) = y_1 > 0. \\
&dY_2(t) = -Y_2(t)\theta(t)dB(t); && Y_2(0) = y_2 > 0.
\end{aligned}$$

Let

$$J^{\pi,\alpha,\theta}(y) = E^y \left[U(Y_1(\tau_s)\xi_0(Y_2(\tau_s))) \right].$$

The problem can then be represented as

$$\Phi(y) = \Phi(s, y_1, y_2) = \inf_{\alpha} \left[\sup_{\pi} \left(\inf_{\theta} J^{\alpha,\pi,\theta_1}(y) \right) \right].$$

We will then have that the HJBI equation is

$$\begin{cases} \inf_{\alpha}\left[\sup_{\pi}\left(\sup_{\theta} A^{\alpha,\pi,\theta}(y)\right)\right] = 0, \\ \varphi(T, y_1, y_2) = U(y_1)\xi_0(y_2), \end{cases} \tag{4.4.16}$$

and the generator of $Y(\cdot)$ is

(4.4.17)

$$A^{\delta,\pi,\theta}\varphi(y) = \frac{\partial\varphi}{\partial s} + y_1(r + (\alpha - r)\pi)\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2}.$$

Let π and α be fixed and minimize

$$f(\theta) := \frac{1}{2}y_2^2\theta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2},$$

for functions $\theta(t, z)$. We minimize and find minimum

$$\hat{\theta} = \frac{y_1}{y_2}\frac{\beta}{\pi}\frac{\varphi_{12}}{\varphi_{22}}. \tag{4.4.18}$$

when $\varphi_{22} \neq 0$. We then use

$$g(\pi) := y_1(\alpha - r)\pi\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\hat{\theta}^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\hat{\theta}\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2},$$

to maximize over π we get

$$\hat{\pi} = \frac{r-\alpha}{y_1\beta^2}\frac{\varphi_1}{\varphi_{11}}.$$

Further, for α, we let

$$l(\alpha) = \frac{1}{\beta^2}y_1^2(r-\alpha)^2(\frac{1}{2}y_1^2 - 1)\frac{\varphi_1^2}{\varphi_{11}}$$

and find

$$\hat{\alpha} = r.$$

So

$$\hat{\pi} = 0.$$

Then we have an optimal triple $(\hat{\alpha}, \hat{\pi}, \hat{\theta})$ which is substituted into (4.4.17) to give

$$\frac{\partial\varphi}{\partial s} + y_1(r + (\hat{\alpha} - r)\hat{\pi})\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\hat{\pi}^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\hat{\theta}^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\hat{\theta}\hat{\pi}\frac{\partial^2\varphi}{\partial y_1\partial y_2}$$

With requirement (iii), this gives

$$\frac{\partial \varphi}{\partial s} + y_1 r \frac{\partial \varphi}{\partial y_1} = 0,$$

and we see that

$$\varphi = \frac{-\ln(y_1)}{r} \frac{\partial \varphi}{\partial y_1}.$$

So the investor puts everything into the risk free asset.

EXAMPLE 4.4.5 (Scenario Optimization in a Lévy Marked). We now try to optimize in a setting where the investor has the choice between two investments in a Lévy market. The market are given as (2.3.2) and (2.3.3). Let M_a be as before. We will let the market control (θ_0, θ_1) and the investor will control π. First let

$$\begin{cases} dY(t) = (dY_0(t), dY_1(t), dY_2(t)); \\ Y(0) = y = (s, y_1, y_2), \end{cases}$$

where

$$\begin{aligned} dY_0(t) &= dt; && Y_0(0) = s \in \mathbb{R}. \\ dY_1(t) &= dV^\pi(t) = Y_1(t)[(r(t) + (\alpha(t) - r(t))\pi(t))dt + \beta\pi(t)dB(t)] \\ &\qquad + Y_1(t^-)\pi(t^-)\int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz); && Y_1(0) = y_1 > 0. \\ dY_2(t) &= -Y_2(t)\theta_0(t)dB(t) - Y_2(t)\int_{\mathbb{R}} \theta_1(s,z)\tilde{N}(ds,dz); && Y_2(0) = y_2 > 0. \end{aligned}$$

Let

$$J^{\pi,\theta_0,\theta_1}(y) = E^y\left[U(X(\tau_s)\xi_0(Y_2(\tau_s)))\right].$$

The problem can then be represented as

$$\Phi(y) = \Phi(s, y_1, y_2) = \sup_\pi \left[\inf_{\theta_0}\left(\inf_{\theta_1} J^{\pi,\theta_0,\theta_1}(y)\right)\right].$$

We will then have that the HJBI equation is

$$\begin{cases} \sup_\pi \left[\inf_{\theta_0}\left(\inf_{\theta_1} A^{\pi,\theta_0,\theta_1}(y)\right)\right] = 0, \\ \varphi(T, y_1, y_2) = U(y_1)\xi_0(y_2), \end{cases} \tag{4.4.19}$$

and the generator of $Y(\cdot)$ is

(4.4.20)

$$A^{\delta,\pi,\theta}\varphi(y) = \frac{\partial\varphi}{\partial s} + y_1(r+(\alpha-r)\pi)\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta_0^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta_0\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2}$$
$$+ \int_{\mathbb{R}}\left[\varphi(s, y_1 + y_1\pi\gamma(t,z), y_2 - y_2\theta_1(t,z)) - \varphi(s,y_1,y_2) - y_1\pi\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2}\right]v(dz).$$

Let π and θ_0 be fixed and minimize

$$\begin{aligned} f(\theta_1) := \int_{\mathbb{R}}\Big[&\varphi(s, y_1 + y_1\pi z, y_2 - y_2\theta_1(t,z)) - \varphi(s,y_1,y_2) \\ &- y_1\pi\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(t,z)\frac{\partial\varphi}{\partial y_2}\Big]v(dz), \end{aligned}$$

for functions $\theta(t,z)$. We minimize pointwise and find minimum

$$\frac{\partial\varphi}{\partial y_2}(s, y_1(1+\pi\gamma(t,z)), y_2(1-\hat{\theta_1})) = \frac{\partial\varphi}{\partial y_2}(s,y_1,y_2). \tag{4.4.21}$$

We then use

$$\begin{aligned} g(\pi) := {}& y_1(\alpha-r)\pi\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\hat{\theta_0}^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\hat{\theta_0}\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2} \\ &+ \int_{\mathbb{R}}\left[\varphi(s, y_1 + y_1\pi z, y_2 - y_2\hat{\theta_1}(t,z)) - y_1\pi\gamma(t,z)\frac{\partial\varphi}{\partial y_1}\right]v(dz), \end{aligned}$$

to maximize over π. Further, for θ_0, we let

$$l(\theta_0) = \frac{1}{2}y_2^2\theta_0^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta_0\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2},$$

and find

$$\hat{\theta_0} = \frac{y_1}{y_2}\frac{\beta}{\pi}\frac{\varphi_{12}}{\varphi_{22}}.$$

when $\varphi_{22} \neq 0$. Then we have an optimal trippel $(\hat{\theta_0}, \hat{\pi}, \hat{\theta_1})$ which is substituted into (4.4.20) to give

$$\begin{aligned} &\frac{\partial\varphi}{\partial s} + y_1(r+(\alpha-r)\hat{\pi})\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\hat{\pi}^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\hat{\theta_0}^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\hat{\theta_0}\hat{\pi}\frac{\partial^2\varphi}{\partial y_1\partial y_2} \\ &+ \int_{\mathbb{R}}\Big[\varphi(s, y_1 + y_1\hat{\pi}\gamma(t,z), y_2 - y_2\hat{\theta_1}(t,z)) - \varphi(s,y_1,y_2) \\ &- y_1\hat{\pi}\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\hat{\theta_1}(z)\frac{\partial\varphi}{\partial y_2}\Big]v(dz) = 0, \end{aligned}$$

by requirement (iii). Motivated by requirement (iv) and Øksendal and Sulem [2006] we try a function on the form

$$\varphi(s,y_1,y_2) = y_2 g(f(s)y_1).$$

for some deterministic function f with $f(T)=1$. From (4.4.21) we need that

$$g\left(f(s)y_1(1+\pi\gamma)\right)=\frac{g(f(s)y_1)}{1-\theta_1}.$$

Then, we have that,

$$\begin{aligned}
&A^{\delta,\pi,\theta}\varphi(y)=y_2g'(f(s)y_1)f'(s)y_1+y_1(r+(\alpha-r)\pi)y_2g'(f(s)y_1)f(s)\\
&+\frac{1}{2}y_1^2\beta^2\pi^2y_2g''(f(s)y_1)f^2(s)\\
&-y_1y_2\beta\theta_0\pi g'(f(s)y_1)f(s)\\
&+\int_{\mathbb{R}}\Big[(y_2-y_2\theta_1(z))g(f(s)(y_1+y_1\pi\gamma(z)))-y_2g(f(s)y_1)\\
&-y_1\pi\gamma(t,z)y_2g'(f(s)y_1)f(s)+y_2\theta_1(z)g(f(s)y_1)\Big]v(dz).
\end{aligned}$$

If we minimize $A^{\theta_0,\hat{\pi},\hat{\theta}}$ we get

$$-\hat{\pi}y_1y_2\beta g'(f(s)y_1)f(s)=0$$

So we conclude that

$$\hat{\pi}=0. \tag{4.4.22}$$

We then minimize $A^{\hat{\theta_0},\hat{\pi},\theta}$ we get

$$-\int_{\mathbb{R}}[y_2g(f(s)y_1(1+\hat{\pi}\gamma(y,z)))+y_2g(f(s)y_1)]\nu(dz)=0$$

Next we maximize $A^{\hat{\theta_0},\pi,\hat{\theta_1}}$ over π and get

$$\begin{aligned}
&y_1(\alpha-r(s))y_2g'(f(s)y_1)f(s)+y_1^2\hat{\pi}\beta^2g''(f(s)y_1)f(s)-\hat{\theta_0}\beta g'(f(s)y_1)\\
&+\int_{\mathbb{R}}\{(1-\hat{\theta_1}(z,s))g'(f(s)y_1)1+\hat{\pi}\gamma(y,z)))-g'(f(s)y_1)\}\gamma(y,z)\nu(dz)=0.
\end{aligned}$$

Substituting 4.4.22 into this gives

$$(\alpha-r(s))g'(f(s)y_1)-\hat{\theta_0}\beta g'(f(s)y_1)+\int_{\mathbb{R}}\{(-\hat{\theta_1}(y,z))g'(f(s)y_1)\gamma(y,z)\}\nu(dz)=0.$$

or

$$\hat{\theta_0}(y)\beta(y)+\int_{\mathbb{R}}\{(\hat{\theta_1}(y,z))\gamma(y,z)\}\nu(dz)=\alpha(y)-r(s). \tag{4.4.23}$$

From the HJBI equation we need

$$A^{\hat{\theta_0},\hat{\pi},\hat{\theta_1}}\varphi(y)=0$$

Solving this we get that

$$y_2 g'(f(s)y_1)y_1 f'(s) + y_1 r(s) y_2 g'(f(s)y_1) f(s) +$$
$$\int_{\mathbb{R}} [y_2(1-\hat{\theta}_1(y,z))g(f(s)y_1) - y_2 g(f(s)y_1) + y_2\hat{\theta}_1 g(f(s)y_1)]\nu(dz) = 0.$$

or

$$f'(s) + r(s)f(s) = 0$$

which leads to

$$f(s) = \exp\left(\int_0^{T-s} r(u)du\right). \tag{4.4.24}$$

So the agent is to put everything into the risk free asset i.e.

$$\hat{\pi}(t) = 0,$$

and the market chooses the scenario $\mathcal{Q}_\theta$ where $\hat{\theta} = (\theta_0, \theta_1)$ satisfy

$$\hat{\theta}_0\beta(y) + \int_{\mathbb{R}} \{(\hat{\theta}_1(y,z))\gamma(y,z)\}\nu(dz) = \alpha(y) - r(s). \tag{4.4.25}$$

So the market chooses an equivalent martingale measure.

Example 4.4.6 (Convex Risk Measure in Classic Black-Scholes Marked). Let us try to solve problem 4.1.3 by using the HJBI equation. Let the setting be the Classic Black-Scholes market and assume the rate of return is unknown to the investor and given a posteriori to the portfolio optimization. To keep it simple we assume that in our marked

$$r(t) = 0, \alpha(t) = \alpha(k), \beta(t) = \beta(j), \gamma(t,z) = 0, \text{ where } j,k \in \mathbb{R}. \tag{4.4.26}$$

Let M_a be the set of all probability measures as defined above. Let

$$dY(t) = (dY_0(t), dY_1(t), dY_2(t)); Y(0) = y = (s, y_1, y_2),$$

where

$$\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}.\\
&dY_1(t) = dV^\pi(t) = Y_1(t)[\alpha(t)\delta(t)\pi(t)dt + \beta(t)\pi(t)dB(t)]; && Y_1(0) = y_1 > 0.\\
&dY_2(t) = -\theta(t)Y_2(t)dB(t); && Y_1(0) = y_1 > 0.
\end{aligned}$$

We let the penalty function be on the form

$$\zeta_0(Q_\theta) = E\left[\zeta_0\left(\frac{dQ_\theta}{d\mathcal{P}}\right)\right] = E\left[\zeta_0(Y_2(T))\right] \text{ for a function } \zeta : \mathbb{R} \to \mathbb{R}.$$

Let

$$J^{\delta,\pi,\theta}(y) = E^y\left[g_\theta(Y_1(\tau_s))\right],$$

where $g_\theta(x) = x + \zeta_0(Q_\theta)$. The problem can then be represented as

$$\Phi(y) = \Phi(s, y_1, y_2) = \inf_\delta\left[\sup_\pi\left(\inf_\theta J^{\delta,\pi,\theta}(y)\right)\right] = \inf_\delta\left[\sup_\pi - \rho(y),\right]$$

where $\rho(y) = \inf_\theta(E^y[x] + \zeta_0(\theta))$ is a convex risk measure. Then the generator of $Y(\cdot)$ is

$$A^{\delta,\pi,\theta}\varphi(y) = \frac{\partial\varphi}{\partial s} + y_1\alpha\delta\pi\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2};$$

The corresponding HJBI equation is

$$\begin{cases} \inf_\alpha\left[\sup_\pi\left(\inf_\theta A^{\alpha,\pi,\theta}(y)\right)\right] = 0, \\ \varphi(T, y_1, y_2) = y_1y_2 + \zeta_0(y_2). \end{cases} \tag{4.4.27}$$

Fix π and α and minimize

$$h(\theta) := -y_1y_2\beta\theta\pi\varphi_{12} + \frac{1}{2}y_2^2\theta^2\varphi_{22}$$

with respect to θ.Minimum is attained at

$$\theta = \hat{\theta}(y) = \frac{y_1\beta\pi\varphi_{12}}{y_2\varphi_{22}}$$

Substitute this and maximize

$$k(\pi) := y_1\alpha\pi\varphi_1 + \frac{1}{2}y_1^2\beta^2\pi^2(\varphi_{11} - \frac{\varphi_{12}^2}{\varphi_{22}})$$

with respect to π. The maximum is attained at

$$\pi = \hat{\pi}(y) = \frac{\alpha\varphi_1\varphi_{22}}{y_1\beta^2(\varphi_{12}^2 - \varphi_{11}\varphi_{22})}.$$

Substituting this, we have

$$\begin{aligned} &\frac{\partial\varphi}{\partial s} + \frac{\alpha^2\varphi_1^2\varphi_{22}}{\beta^2 M} + \frac{\alpha^2\varphi_1^2\varphi_{22}^2\varphi_{11}}{2\beta^2M^2} + \frac{\alpha^2\varphi_1^2\varphi_{22}^2\varphi_{22}}{2\beta^2M^2} - \frac{\alpha^2\varphi_1^2\varphi_{22}^2\varphi_{22}}{\beta^2M^2} \\ &= \frac{\partial\varphi}{\partial s} + \frac{\alpha^2\varphi_1^2\varphi_{22}}{\beta^2M} + \frac{\alpha^2\varphi_1^2\varphi_{22}(\varphi_{22}\varphi_{11} - \varphi_{12}^2)}{2\beta^2M^2} \\ &= \frac{\partial\varphi}{\partial s} + \frac{\alpha^2\varphi_1^2\varphi_{22}}{2\beta^2(\varphi_{12}^2 - \varphi_{22}\varphi_{11})} \end{aligned}$$

minimizing with respect to α we see that $\alpha = 0$ gives minimum. Substituting this we get a partial differential equation for φ

$$\begin{cases} \frac{\partial \varphi}{\partial s} = 0 \\ \varphi(T, y_1, y_2) = y_1 y_2 + \zeta_0(y_2) \end{cases}$$

EXAMPLE 4.4.7 (Convex Risk Measure in Itô-Lévy Setting). We will now try to optimize a worst case scenario in a setting where we have a convex risk measure in a Lévy market. The market are given as (2.3.2) and (2.3.3).

Let M_a be First let

$$dY(t) = (dY_0(t), dY_1(t), dY_2(t)); Y(0) = y = (s, y_1, y_2),$$

where

$$\begin{aligned} dY_0(t) &= dt; && Y_0(0) = s \in \mathbb{R}. \\ dY_1(t) &= dV^\pi(t) = Y_1(t)[(r(t) + (\alpha(t) - r(t))\pi(t))dt + \beta\pi(t)dB(t)] \\ &\quad + Y_1(t^-)\pi(t^-)\int_{\mathbb{R}} \gamma(s, z)\tilde{N}(ds, dz); && Y_1(0) = y_1 > 0. \\ dY_2(t) &= -Y_2(t)\theta_0(t)dB(t) - Y_2(t)\int_{\mathbb{R}} \theta_1(s, z)\tilde{N}(ds, dz); && Y_2(0) = y_2 > 0. \end{aligned}$$

We let the penalty function be on the form

$$\zeta_0(Q_\theta) = E\left[\zeta_0\left(\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\right)\right] = E\left[\zeta_0(Y_2(T))\right] \text{ for a function } \zeta : \mathbb{R} \to \mathbb{R}.$$

Let

$$J^{\pi,\theta_0,\theta_1}(y) = E^y\left[g_\theta(Y_1(\tau_s))\right].$$

where $g_\theta(x) = x + \zeta_0(Q_\theta)$. The problem can then be represented as

$$\Phi(y) = \Phi(s, y_1, y_2) = \inf_{\theta_0}\left[\sup_{\pi}\left(\inf_{\theta_1} J^{\theta_0,\pi,\theta_1}(y)\right)\right] = \inf_{\theta_0}\left[\sup_{\pi} - \rho(y)\right]$$

where ρ is a convex risk measure and the generator of $Y(\cdot)$ is

(4.4.28)

$$\begin{aligned} A^{\delta,\pi,\theta}\varphi(y) &= \frac{\partial \varphi}{\partial s} + y_1(r + (\alpha - r)\pi)\frac{\partial \varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1 y_2\beta\theta_0\pi\frac{\partial^2\varphi}{\partial y_1 \partial y_2} \\ &+ \int_{\mathbb{R}}\left[\varphi(s, y_1 + y_1\pi\gamma(t, z), y_2 - y_2\theta_1(t, z)) - \varphi(s, y_1, y_2) - y_1\pi\gamma(t, z)\frac{\partial \varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial \varphi}{\partial y_2}\right] v(dz). \end{aligned}$$

Let π and θ_0 be fixed and minimize

$$f(\theta_1) := \int_{\mathbb{R}} \Big[\varphi(s, y_1 + y_1\pi z, y_2 - y_2\theta_1(t,z)) - \varphi(s, y_1, y_2) \\ - y_1\pi\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(t,z)\frac{\partial\varphi}{\partial y_2}\Big] v(dz),$$

for functions $\theta(t,z)$. We minimize pointwise and find minimum

$$\frac{\partial\varphi}{\partial y_2}(s, y_1(1 + \pi\gamma(t,z)), y_2(1 - \hat{\theta_1})) = \frac{\partial\varphi}{\partial y_2}(s, y_1, y_2).$$

If we substitute into 4.4.28 we get

$$g(\pi) := y_1(\alpha - r)\pi\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} - y_1y_2\beta\hat{\theta_0}\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2} \\ + \int_{\mathbb{R}} \Big[\varphi(s, y_1 + y_1\pi z, y_2 - y_2\hat{\theta_1}(t,z)) - y_1\pi\gamma(t,z)\frac{\partial\varphi}{\partial y_1}\Big] v(dz),$$

to maximize over π. Then we have an optimal trippel $(\hat{\theta_0}, \hat{\pi}, \hat{\theta_1})$ which is substituted into 4.4.20 to give

$$\frac{\partial\varphi}{\partial s} + y_1(r + (\alpha - r)\hat{\pi})\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\hat{\pi}^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\hat{\theta_0}^{\,2}\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\hat{\theta_0}\hat{\pi}\frac{\partial^2\varphi}{\partial y_1\partial y_2} \\ + \int_{\mathbb{R}} \Big[\varphi(s, y_1 + y_1\hat{\pi}\gamma(t,z), y_2 - y_2\hat{\theta_1}(t,z)) - \varphi(s, y_1, y_2) \\ - y_1\hat{\pi}\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\hat{\theta_1}(z)\frac{\partial\varphi}{\partial y_2}\Big] v(dz) = 0.$$

Motivated by requirement (iv) we try a function on the form

$$\varphi(s, y_1, y_2) = f(s)y_1 + \zeta_0(y_2).$$

for some deterministic function f with $f(T) = 1$. Then, we have that,

$$A^{\pi,\theta_0,\theta_1}\varphi(y) = f'(s)y_1 + y_1(r + (\alpha - r)\pi)f(s) \\ + \frac{1}{2}y_2^2\theta_0^2\pi^2\zeta_0''(y_2) \\ + \int_{\mathbb{R}} \Big[f(s)(y_1 + y_1\pi\gamma(z)) + \zeta_0(y_2 - y_2\theta_1) - f(s)y_1 + \zeta_0(y_2) - y_1\pi\gamma(z)f(s) - y_2\theta_1\zeta_0'(y_2)\Big] v(dz).$$

If we minimize $A^{\hat{\pi},\theta_0,\hat{\theta_1}}$ we get

$$y_2\hat{\pi}\xi''(y_2) = 0.$$

So we conclude that

$$\hat{\pi} = 0. \tag{4.4.29}$$

We then minimize $A^{\hat{\pi},\hat{\theta_0},\theta_1}$ we get

$$\int_{\mathbb{R}}[f(s)y_1+\zeta_0(y_2-y_2\theta_1)-f(s)y_1+\zeta_0(y_2)-y_2\theta_1\zeta_0^{'}(y_2)]\nu(dz)=0$$

We then maximize $A^{\pi,\hat{\theta_0},\hat{\theta_1}}$ over π and get

$$\begin{aligned}&y_1(\alpha-r)f(s)+\\&\int_{\mathbb{R}}\Big[f(s)y_1+\zeta_0(y_2-y_2\theta_1)-f(s)y_1+\zeta_0(y_2)-y_2\theta_1\zeta_0^{'}(y_2)\Big]\nu(dz)=0.\end{aligned}$$

or

$$\int_{\mathbb{R}}\Big[f(s)y_1+\zeta_0(y_2-y_2\theta_1)-f(s)y_1+\zeta_0(y_2)-y_2\theta_1\zeta_0^{'}(y_2)\Big]\nu(dz)=y_1(r-\alpha)f(s). \tag{4.4.30}$$

From the HJBI equation we need

$$A^{\hat{\pi},\hat{\theta_0},\hat{\theta_1}}\varphi(y)=0$$

Solving this we get that

$$\begin{aligned}&f'(s)y_1+y_1rf(s)\\&+\int_{\mathbb{R}}\Big[f(s)y_1+\zeta_0(y_2-y_2\theta_1)-f(s)y_1+\zeta_0(y_2)-y_2\theta_1\zeta_0^{'}(y_2)\Big]\nu(dz)=0.\end{aligned}$$

or

$$f'(s)+r(s)f(s)=\frac{1}{y_1}\Big(\zeta_0(y_2-y_2\theta_1)+\zeta_0(y_2)-y_2\theta_1\zeta_0^{'}(y_2)\Big).$$

which leads to

$$f(s)=C\exp\left(-\int_0^s r(u)du\right)+B\int_0^s e^{-\int_u^s r(l)dl}du, \tag{4.4.31}$$

where B $=\frac{1}{y_1}\Big(\zeta_0(y_2-y_2\theta_1)+\zeta_0(y_2)-y_2\theta_1\zeta_0^{'}(y_2)\Big)$. Solving for f(T) =1 gives us

$$f(s)=\int_0^T e^{\int_0^u r(l)dl}du\exp\left(-\int_0^s r(u)du\right)+B\int_0^s e^{-\int_u^s r(l)dl}du.$$

So the agent is to put everything into the risk free asset i.e.

$$\hat{\pi}(t)=0,$$

and the market chooses the scenario $\mathcal{Q}_\theta$ such that

$$\int_{\mathbb{R}}\Big[f(s)y_1+\zeta_0(y_2-y_2\theta_1)-f(s)y_1+\zeta_0(y_2)-y_2\theta_1\zeta_0^{'}(y_2)\Big]\nu(dz)=y_1(r-\alpha)f(s). \tag{4.4.32}$$

EXAMPLE 4.4.8 (Mean-Variance Hedging in a Lévy Market). In this example we would like to find a self-financing hedging strategy that given a claim $x \in X$ we minimize the terminal hedging error by mean-square. Let $x \in X(L^2(\Omega, F, P))$ and

$$\begin{aligned}\rho(x) &= \sup_\theta E[Y_2(T)(V_0 + \int_0^T dY_1(t) - x)^2] \\ &= \sup_\theta E^Q[(V_0 + \int_0^T dY_1(t) - x)^2] \\ &= \sup_\theta E[(V_0\sqrt{Y_2(T)} + \int_0^T \sqrt{Y_2(t)} dY_1(t) - \sqrt{Y_2(T)} x)^2].\end{aligned}$$

and let the problem be: Find $\hat{\pi}$ such that

$$\rho^{\hat{\pi}} \leq \inf_\pi \rho^\pi(x).$$

We can then rewrite this as

$$\inf_\pi \sup_Q E^Q[g(\pi, x)^2] = \inf_{A \in \mathcal{A}} \sup_Q \| x - A \|^2_{L^2(Q)} .$$

Where $\mathcal{A}$ is the set of all attainable payoffs and $g(\pi, x) = V_0 + \int_0^T dY_1^\pi(t) - x$. Then this becomes the problem of finding the orthogonal projections in L^2Q of the payoff x on the set of attainable claims $\mathcal{A}$. Decomposing a r.v. into a stochastic integral and an orthogonal component is known as the Galtchouk-Kunita-Watanabe decomposition, see Cont and Tankov [2004].

CHAPTER 5

HJBI EQUATION FOR NASH-EQUILIBRIA

In this chapter we extend the Nash-equilibria HJBI equation given in Mataramvura and Øksendal [2008] to include the market as a participant who controls a set of scenarios. We will give and prove a HJBI equation for the game.

5.1. Nash Equilibrium

Let the marked be as above and $u = (\pi_1, \pi_2, \theta)$ be admissible controls for player 1,2 and 3. Let there be two performance functionals on the form

$$J_1^u(y) = E^y[\int_0^{\tau_s} f_1(X(t), u(t))dt + g_1(X(\tau_s))], \tag{5.1.1}$$

and

$$J_2^u(y) = E^y[\int_0^{\tau_s} f_2(X(t), u(t))dt + g_2(X(\tau_s))], \tag{5.1.2}$$

Definition 5.1.1. A triple $(\hat{\pi}_1, \hat{\pi}_2, \hat{\theta}) \in \Pi_1 \times \Pi_2 \times \Theta$ is called a Nash equilibrium for the stochastic differential game if the following holds

$$J_1^{\pi_1, \hat{\pi}_2, \hat{\theta}} \leq J_1^{\hat{\pi}_1, \hat{\pi}_2, \hat{\theta}}; \text{ for all } \pi_1 \in \Pi_1$$

$$J_2^{\hat{\pi}_1, \pi_2, \hat{\theta}} \leq J_2^{\hat{\pi}_1, \hat{\pi}_2, \hat{\theta}}; \text{ for all } \pi_2 \in \Pi_2$$

This means that when player 2 uses control π_2^*, it is optimal for player 1 to use π_1^*. So $(\pi_1^*, \pi_2^*, \theta^*)$ is an equilibrium point. As before we use Markov controls. We now have the following problem;

PROBLEM 5.1.1. Find $\Phi(y)_i$, $i = 1, 2$, and $(\pi_1^*, \pi_2^*, \theta^*) \in \ Pi \times \Pi \times \Theta$ such that

$$\Phi(y)_1 = \sup_{\pi_1} \left[\inf_{\theta} J^{\pi_1, \pi_2, \theta}(y) \right]$$

and

$$\Phi(y)_2 = \sup_{\pi_2} \left[\inf_{\theta} J^{\pi_1, \pi_2, \theta}(y) \right]$$

where

$$J_i^{\pi_1, \pi_2, \theta}(y) = E^y \left[\int_0^{\tau_s} f_i(X(t), u(t)) + g_i(X(\tau_s)) \right],$$

5.2. A HJBI for Nash equilibria

To solve the above problem we give a verification theorem for a function, φ, similar to the one given in chapter 4.

THEOREM 5.2.1. *Suppose there exists functions $\varphi_i \in C^2((S)) \cap C(\bar{\mathcal{S}})$; $i = 1, 2$, and a Markov control $(\hat{\pi_1}, \hat{\pi_2}, \hat{\theta}) \in \Pi_1 \times \Pi_2 \times \Theta$ such that*

(i) $A^{\hat{\pi_1}, \hat{\pi_2}, \theta} \varphi(y) + f(y, \hat{\pi_1}, \hat{\pi_2}, \theta) \geq 0$ *for all* $\theta \in K_3$.

(ii) $A^{\pi_1, \hat{\pi_2}, \hat{\theta}} \varphi(y) + f(y, \pi_1, \hat{\pi_2}, \hat{\theta}) \leq 0$ *for all* $\pi_1 \in K_1$.

(iii) $A^{\hat{\pi_1}, \pi_2, \hat{\theta}} \varphi(y) + f(y, \hat{\pi_1}, \pi_2, \hat{\theta}) \leq 0$ *for all* $\pi_2 \in K_2$.

(iv) $A^{\hat{\pi_1}, \hat{\pi_2}, \hat{\theta}} \varphi(y) + f(y, \hat{\pi_1}, \hat{\pi_2}, \hat{\theta}) = 0$.

(v) $X^{\pi_1, \pi_2, \theta}(\tau_s) \in \partial\mathcal{S}$ *a.s. on* $\{\tau_s < \infty\}$ *and* $\lim_{t \to \tau_s^-} \varphi(X^{\pi_1, \pi_2, \theta}(t)) = g(X^{\pi_1, \pi_2, \theta}(\tau_s)) \chi_{\{\tau_s < \infty\}}$ *a.s. for all* $(\pi_1, \pi_2, \theta) \in \Pi_1 \times \Pi_2 \times \Theta$, $y \in \mathcal{S}$.

(vi) *The family* $\{\varphi(X^{\pi_1, \pi_2, \theta}(\tau))\}_{\tau \in \mathcal{T}}$ *is uniformly integrable, for all* $(\pi_1, \pi_2\, \theta) \in \Pi_1 \times \Pi_2 \times \Theta$, $y \in \mathcal{S}$.

Then $(\hat{\pi_1}, \hat{\pi_2}, \hat{\theta})$ is a Nash equilibrium for the game and

$$\varphi_1(y) = \sup_{\pi_1} \left[\inf_{\theta} J_1^{\pi_1, \hat{\pi_2}, \theta}(y) \right] = \inf_{\theta} \left[\sup_{\pi_1} J_1^{\pi_1, \hat{\pi_2}, \theta}(y) \right] = J_1^{\hat{\pi_1}, \hat{\pi_2}, \hat{\theta}}(y), \tag{5.2.3}$$

$$\varphi_2(y) = \sup_{\pi_2} \left[\inf_{\theta} J_2^{\hat{\pi_1}, \pi_2, \theta}(y) \right] = \inf_{\theta} \left[\sup_{\pi_2} J_2^{\hat{\pi_1}, \pi_2, \theta}(y) \right] = J_2^{\hat{\pi_1}, \hat{\pi_2}, \hat{\theta}}(y). \tag{5.2.4}$$

PROOF. To prove 5.2.3 for player 1 we will proceed as in chapter 4.

Step1. Let us first prove that

$$\varphi_1(y) = \sup_{\pi_1}\left[\inf_{\theta} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right] = J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y). \tag{5.2.5}$$

Fist from Dynkin, $Y = Y^{\pi_1,\hat{\pi_2},\hat{\theta}}$ and (ii) to get

$$\begin{aligned} E^y[\varphi_1(Y(\tau_s^N))] &= \varphi_1(y) + E^y[\int_0^{\tau_s^N} A^{\pi_1,\phi,\theta}\varphi_1(Y(t))dt] \\ &\le \varphi_1(y) - E^y[\int_0^{\tau_s^N} f_1(Y(t),u(Y(t)))dt] \end{aligned} \tag{5.2.6}$$

so

$$\varphi_1(y) \ge E^y[\int_0^{\tau_s^N} f_1(Y(t),u(Y(t)))dt + \varphi_1(Y(\tau_s^N))]$$

Letting $N \to \infty$, we get

$$\varphi_1(y) \ge J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y).$$

This holds for every π_1 so

$$\varphi_1(y) \ge \sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y).$$

We also have

$$\inf_{\theta} J_1^{\pi_1,\hat{\pi_2},\theta}(y) \le J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y),$$

for all π_1, so taking supremum on both sides gives us that

$$\sup_{\pi_1}\left[\inf_{\theta} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right] \le \sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y).$$

so

$$\varphi_1(y) \ge \sup_{\pi_1}\left[\inf_{\theta} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right]. \tag{5.2.7}$$

Again using (i), $Y = Y^{\hat{\pi_1},\hat{\pi_2},\theta}$ and Dynkin

$$\varphi_1(y) \le J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y).$$

Since it holds for every θ we have

$$\varphi_1(y) \le \inf_{\theta} J_1^{\hat{\pi_1},\hat{\pi_2},theta}(y).$$

Which implies that

$$\varphi_1(y) \le \sup_{\pi_1}\left[\inf_{\theta} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right]. \tag{5.2.8}$$

If we apply this method to the control $(\hat{\pi_1}, \hat{\pi_2}, \hat{\theta})$ and use (iv) we see that

$$\varphi_1(y) = J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y). \tag{5.2.9}$$

Combining (5.2.7), (5.2.8) and (5.2.9), we get

$$\varphi_1(y) = \sup_{\pi_1}\left[\inf_{\theta} J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y)\right] = J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y). \tag{5.2.10}$$

Step2. Let us now prove that

$$\varphi_1(y) = \inf_{\theta}\left[\sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right] = J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y).$$

Again, from Dynkin and (ii)

$$\varphi_1(y) \geq J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y).$$

Since this holds for all π_1 we get

$$\varphi_1(y) \geq \sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y) \geq \inf_{\theta}\left[\sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right].$$

On the other hand, using Dynkin and (i) we get

$$\varphi_1(y) \leq J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y) \leq \sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\hat{\theta}}(y).$$

Since it holds for every θ we have

$$\varphi_1(y) \leq \inf_{\theta}\left[\sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right].$$

And we conclude that

$$\varphi_1(y) = \inf_{\theta}\left[\sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right]. \tag{5.2.11}$$

Combining (5.2.10) and (5.2.11), we get

$$\varphi_1(y) = \sup_{\pi_1}\left[\inf_{\theta} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right] = J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y) = \inf_{\theta}\left[\sup_{\pi_1} J_1^{\pi_1,\hat{\pi_2},\theta}(y)\right] = J_1^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}(y). \tag{5.2.12}$$

Step3. Using same approach we easily prove statement 5.2.4 for player 2.

□

5.3. Examples

Now we will apply the above result to give some examples. First, we give an example in a Lévy market for two companies with external driven scenarios. Then go on to a non-jump setting. We then look at the same settings except for an internal market factor. Finally, we will give an example of two highly correlated businesses.

We consider the marked given by (2.3.2) and (2.3.3).

EXAMPLE 5.3.1 (Minimizing Worst Case External Factors for two Companies). Suppose we have two companies described by $X(\cdot) = (X_1(\cdot), X_2(\cdot))$ where σ is a 2×2 matrix, $\eta = (\eta_1, \eta_2)$, $\pi = (\pi_1, \pi_2)$ and

$$\begin{cases} dX_1(t) = X_1(t)[\pi_1(t)dt + \mu_1 dB(t)] + \sigma_{11}(t)X_1(t^-)d\eta_1(t) + \sigma_{12}(t)X_1(t^-)d\eta_2(t) \\ X_1(0) = x_1 \in \mathbb{R}. \end{cases}$$

and

$$\begin{cases} dX_2(t) = X_2(t)[\pi_2(t)dt + \mu_2 dB(t)] + \sigma_{21}(t)X_2(t^-)d\eta_1(t) + \sigma_{22}(t)X_2(t^-)d\eta_2(t) \\ X_2(0) = x_2 \in \mathbb{R}. \end{cases}$$

Here $\pi = (\pi_1, \pi_2)$ is the control of company 1 and 2 respectively and $\eta_i(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}_i(ds, dz)$ $i = 1, 2$. Define

$$\begin{aligned} dY_0(t) &= dt; & Y_0(0) &= s \in \mathbb{R}. \\ dY_1(t) &= dX_1(t); & Y_1(0) &= y_1 = x_1. \\ dY_2(t) &= dX_2(t); & Y_2(0) &= y_2 = x_2. \\ dY_3(t) &= -Y_3(t)\theta_0(t)dB(t) - Y_3(t^-)\int_{\mathbb{R}} \theta_1(z)\tilde{N}_3(ds, dz); & Y_3(0) &= y_3 > 0. \end{aligned}$$

Let us try to model some external market factors by $\theta = (\theta_0, \theta_1)$, where we then get the generator of $Y(\cdot)$ as

(5.3.13)

$$
\begin{aligned}
A^{\pi_1,\pi_2,\theta}\varphi(s,y_1,y_2,y_3) &= \frac{\partial\varphi}{\partial s} + x_1\pi_1\frac{\partial\varphi}{\partial x_1} + x_2\pi_2\frac{\partial\varphi}{\partial x_2} + \frac{1}{2}x_1^2\mu_1^2\frac{\partial^2\varphi}{\partial^2 x_1} + \frac{1}{2}x_2^2\mu_2^2\frac{\partial^2\varphi}{\partial^2 x_2} \\
&+ \frac{1}{2}y_3^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_3} - y_1y_2\mu_1\mu_2\frac{\partial^2\varphi}{\partial y_1\partial y_2} - y_1y_3\mu_1\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_3} - y_2y_3\mu_2\theta_0\frac{\partial^2\varphi}{\partial y_2\partial y_3} \\
&+ \int_{\mathbb{R}}\left[\varphi(s,y_1+y_1\sigma_{11}z,y_2+y_2\sigma_{21}z,y_3) - \varphi(s,y_1,y_2,y_3) - y_1\sigma_{11}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{21}z\frac{\partial\varphi}{\partial y_2}\right]v(dz) \\
&+ \int_{\mathbb{R}}\left[\varphi(s,y_1+y_1\sigma_{12}z,y_2+y_2\sigma_{22}z,y_3) - \varphi(s,y_1,y_2,y_3) - y_1\sigma_{12}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{22}z\frac{\partial\varphi}{\partial y_2}\right]v(dz) \\
&+ \int_{\mathbb{R}}\left[\varphi(s,y_1,y_2,y_3+y_3\theta_1(z)) - \varphi(s,y_1,y_2,y_3) - y_3\theta_1 z\frac{\partial\varphi}{\partial y_3}\right]v(dz)
\end{aligned}
$$

The performance functionals to the companies have the form

$$J_1^{\pi,\theta} = \inf_{\theta} E[-\int_0^{T-s}\alpha_1\pi_1^2(t)X_2^2(t)Y_3(t)dt + \gamma_1X_1^2(T)X_2^2(T)Y_3(T)],$$

and

$$J_2^{\pi,\theta} = \inf_{\theta} E[-\int_0^{T-s}\alpha_2\pi_2^2(t)X_1^2(t)Y_3(t)dt + \gamma_2X_1^2(T)X_2^2(T)Y_3(T)].$$

As in Mataramvura and Øksendal [2008] we can interpret π_1 and π_2 as investment rates and the payoff function as describing how the size of each company heats up the economy such that the payoff and energy cost are proportional to the squared size of each other. Here we have

$$\begin{cases} f_1(s,x_1,x_2,y_3,\theta,\pi) = -\alpha_1\pi_1^2x_2^2y_3, \\ g_1(s,x_1,x_2,y_3) = \gamma_1x_1^2x_2^2y_3, \end{cases}$$

and

$$\begin{cases} f_2(s,x_1,x_2,y_3,\theta,\pi) = -\alpha_2\pi_2^2x_1^2y_3, \\ g_2(s,x_1,x_2,y_3) = \gamma_2x_1^2x_2^2y_3. \end{cases}$$

The problem can then, for player 1, be represented as

$$\Phi(y) = \Phi(s,y_1,y_2) = \sup_{\pi_1}\left[\inf_{\theta} J^{\pi_1,\theta}(y)\right].$$

Then from Theorem 5.2.1 and Theorem 4.3.1 we know that

$$\sup_{\pi_1}\left[\inf_{\theta_0}\left(\inf_{\theta_1} J^{\pi_1,\theta_0,\theta_1}(y)\right)\right] = \inf_{\theta_0}\left[\inf_{\theta_1}\left(\sup_{\pi_1} J^{\pi_1,\theta_0,\theta_1}(y)\right)\right].$$

The corresponding HJBI equation is

$$(5.3.14)\qquad \begin{cases} \sup_{\pi_1}\left[\inf_{\theta_0}\left(\inf_{\theta_1} A^{\pi_1,\theta}(y)\right)\right] + f_1 = 0 \\ \varphi(T,y) = g_1(y). \end{cases}$$

To find a Nash equilibrium for we fix $\pi_2 \in \mathbb{R}$ and maximize

$$A^{\delta,\pi_1,\pi_2}\varphi_1(y) + f_1(y)$$

with respect to π_1. So we maximize

$$h_1(\pi_1) := x_1\pi_1\frac{\partial\varphi_1}{\partial x_1} - \alpha_1\pi_1^2x_2^2y_3.$$

This maximum is attained at

$$(5.3.15)\qquad \pi_1 = \hat{\pi_1} = \frac{x_1}{\alpha_1 x_2^2 y_3}\frac{\partial\varphi_1}{\partial x_1}$$

Substituting this into (5.3.13) we get the function

$$g(\theta_1) := \int_{\mathbb{R}}\left[\varphi(s,y_1,y_2,y_3+y_3\theta_1(z)) - y_3\theta_1(z)\frac{\partial\varphi}{\partial y_3}\right]v(dz)$$

We can minimize this point-wise, let

$$\Psi(\theta_1) = \varphi(s,y_1,y_2,y_3+y_3\theta_1(z)) - y_3\theta_1(z)\frac{\partial\varphi}{\partial y_3}.$$

The first order condition for a minimum $\hat{\theta_1}$ of Ψ is

$$\frac{\partial\varphi}{\partial y_3}(s,y_1,y_2,y_3+y_3\hat{\theta}_1(z)) = -2y_3\hat{\theta}_1(z).$$

We then let

$$f(\theta_0) := \frac{1}{2}y_3^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_3} - y_1y_3\mu_1\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_3} - y_2y_3\mu_2\theta_0\frac{\partial^2\varphi}{\partial y_2\partial y_3}.$$

When we maximize f over θ_0 we get

$$\hat{\theta}_0 = \frac{y_1}{y_3}\mu_1\frac{\varphi_{13}}{\varphi_{33}} - \frac{y_2}{y_3}\mu_2\frac{\varphi_{23}}{\varphi_{33}}$$

Then for $(\hat{\pi_1},\hat{\pi_2},\hat{\theta})$ we require that

$$A^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}\varphi(y) + f_1(y) = 0,$$

or

(5.3.16)

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial s}+y_1\frac{1}{2\alpha_1y_2^2}(\frac{\partial\varphi}{\partial y_1})^2+y_2\hat{\pi_2}\frac{\partial\varphi}{\partial y_2}+\frac{1}{2}y_1^2\mu_1^2\frac{\partial^2\varphi}{\partial^2y_1}+\frac{1}{2}y_2^2\mu_2^2\frac{\partial^2\varphi}{\partial^2y_2}\\
&+\frac{1}{2}y_3^2\theta_0^2\frac{\partial^2\varphi}{\partial^2y_3}-y_1y_2\mu_1\mu_2\frac{\partial^2\varphi}{\partial y_1\partial y_2}-y_1y_3\mu_1\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_3}-y_2y_3\mu_2\theta_0\frac{\partial^2\varphi}{\partial y_2\partial y_3}\\
&+\int_{\mathbb{R}}\Big[\varphi(s,y_1+y_1\sigma_{11}z,y_2+y_2\sigma_{21}z,y_3)-\varphi(s,y_1,y_2,y_3)-y_1\sigma_{11}z\frac{\partial\varphi}{\partial y_1}-y_2\sigma_{21}z\frac{\partial\varphi}{\partial y_2}\Big]v(dz)\\
&+\int_{\mathbb{R}}\Big[\varphi(s,y_1+y_1\sigma_{12}z,y_2+y_2\sigma_{22}z,y_3)-\varphi(s,y_1,y_2,y_3)-y_1\sigma_{12}z\frac{\partial\varphi}{\partial y_1}-y_2\sigma_{22}z\frac{\partial\varphi}{\partial y_2}\Big]v(dz)\\
&+\int_{\mathbb{R}}\Big[\varphi(s,y_1,y_2,y_3+y_3\hat{\theta_1}(z))-\varphi(s,y_1,y_2,y_3)-y_3\hat{\theta_1}z\frac{\partial\varphi}{\partial y_3}\Big]v(dz)-\frac{\varphi_1^2}{4\alpha_1x_2^2Y_3^2}=0.
\end{aligned}
$$

We try two functions, φ_i, on the form

$$\varphi_i(s,x_1,x_2)=k_i(s)x_1^2x_2^2y_3;i=1,2.$$

where $k_i(s)$ are some function we need to find. So from (5.3.15) we get that

$$\hat{\pi_1}=\frac{1}{\alpha_1y_3}k_1(s)x_1,$$

and

$$\hat{\theta_1}=-k_1(s)\frac{y_1^2y_2^2}{y_3}$$

and from (5.3.16) we have

(5.3.17)

$$
\begin{aligned}
&k_1'(s)y_1^2y_2^2y_3+\frac{1}{\alpha_1}y_1^2y_2^2y_3^2k_1^2(s)+2y_2^2y_1^2y_3k_1(s)\hat{\pi_2}+y_1^2y_2^2y_3\mu_1^2k_1(s)+y_1^2y_2^2\mu_2^2k_1(s)\\
&-4y_1^2y_2^2y_3\mu_1\mu_2k_1(s)-2y_1^2y_2^2y_3\mu_1\theta_0k_1(s)-2y_1^2y_2^2y_3\mu_2\theta_0k_1(s)\\
&+\int_{\mathbb{R}}\Big[k_1(s)(y_1+y_1\sigma_{11}z)^2(y_2+y_2\sigma_{21}z)^2y_3-k_1(s)y_1^2y_2^2-2y_1^2y_2^2y_3\sigma_{11}zk_1(s)-2y_1^2y_2^2\sigma_{21}zk_1(s)\Big]v(dz)\\
&+\int_{\mathbb{R}}\Big[k_1(s)(y_1+y_1\sigma_{11}z)^2(y_2+y_2\sigma_{21}z)^2y_3-k_1(s)y_1^2y_2^2y_3-2y_1^2y_2^2\sigma_{11}zk_1(s)-2y_1^2y_2^2\sigma_{21}zk_1(s)\Big]v(dz)\\
&+\int_{\mathbb{R}}\Big[k_1(s)(y_1^2y_2^2(y_3+y_3\hat{\theta_1}(z))-k_1(s)y_1^2y_2^2y_3-Y_1^2y_2^2y_3\hat{\theta_1}zk_1(s)\Big]v(dz)=0
\end{aligned}
$$

If we let

$$a = \int_{\mathbb{R}} \Big[(y_1 + y_1\sigma_{11}z)^2(y_2 + y_2\sigma_{21}z)^2 y_3 - y_1^2 y_2^2 - 2y_1^2 y_2^2 y_3 \sigma_{11} z - 2y_1^2 y_2^2 \sigma_{21} z\Big] v(dz)$$
$$+ \int_{\mathbb{R}} \Big[(y_1 + y_1\sigma_{11}z)^2(y_2 + y_2\sigma_{21}z)^2 y_3 - y_1^2 y_2^2 y_3 - 2y_1^2 y_2^2 \sigma_{11} z - 2y_1^2 y_2^2 \sigma_{21} z\Big] v(dz)$$
$$+ \int_{\mathbb{R}} \Big[(y_1^2 y_2^2 (y_3 + y_3\hat{\theta}_1(z)) - y_1^2 y_2^2 y_3 - Y_1^2 y_2^2 y_3 \hat{\theta}_1 z\Big] v(dz)$$

we get that (5.3.17) is

$$k_1'(s) y_1^2 y_2^2 y_3 + \frac{1}{\alpha_1} y_1^2 y_2^2 y_3^2 k_1^2(s) + 2y_2^2 y_1^2 y_3 k_1(s)\hat{\pi_2} + y_1^2 y_2^2 y_3 \mu_1^2 k_1(s) + y_1^2 y_2^2 \mu_2^2 k_1(s)$$
$$- 4y_1^2 y_2^2 y_3 \mu_1 \mu_2 k_1(s) - 2y_1^2 y_2^2 y_3 \mu_1 \hat{\theta}_0 k_1(s) - 2y_1^2 y_2^2 y_3 \mu_2 \hat{\theta}_0 k_1(s) + a k_1(s)$$
$$= k_1'(s) y_1^2 y_2^2 y_3 + k_1^2(s) \frac{1}{\alpha_1} y_1^2 y_2^2 y_3^2 + k_1(s)[2y_2^2 y_1^2 y_3 \hat{\pi_2} + y_1^2 y_2^2 y_3 \mu_1^2 + y_1^2 y_2^2 \mu_2^2$$
$$- 4y_1^2 y_2^2 y_3 \mu_1 \mu_2 - 2y_1^2 y_2^2 y_3 \mu_1 \hat{\theta}_0 - 2y_1^2 y_2^2 y_3 \mu_2 \hat{\theta}_0 + a] = 0$$

Lets do the same for player 2, first we fix $\pi_1 \in \mathbb{R}$ and maximize

$$A^{\delta,\pi,\theta}\varphi_2(y) + f_2(s, x_1, x_2, x_3, \pi, \theta)$$

with respect to π_2 we get that

$$\pi_2 = \hat{\pi_2} = \frac{1}{\alpha_2 y_3} k_2(s) y_2 \tag{5.3.18}$$

Using requirement (v) we have

(5.3.19)

$$k_2'(s) y_1^2 y_2^2 y_3 + \frac{2}{\alpha_1} y_1^2 y_2^2 y_3^2 k_1 k_2(s) + \frac{1}{\alpha_2} y_2 y_1^2 k_2(s) + y_1^2 y_2^2 y_3 \mu_1^2 k_2(s) + y_1^2 y_2^2 \mu_2^2 k_2(s)$$
$$- 4y_1^2 y_2^2 y_3 \mu_1 \mu_2 k_2(s) - 2y_1^2 y_2^2 y_3 \mu_1 \theta_0 k_2(s) - 2y_1^2 y_2^2 y_3 \mu_2 \theta_0 k_2(s)$$
$$+ \int_{\mathbb{R}} \Big[k_1(s)(y_1 + y_1\sigma_{11}z)^2(y_2 + y_2\sigma_{21}z)^2 y_3 - k_1(s) y_1^2 y_2^2 - 2y_1^2 y_2^2 y_3 \sigma_{11} z k_1(s) - 2y_1^2 y_2^2 \sigma_{21} z k_1(s)\Big] v(dz)$$
$$+ \int_{\mathbb{R}} \Big[k_2(s)(y_1 + y_1\sigma_{11}z)^2(y_2 + y_2\sigma_{21}z)^2 y_3 - k_2(s) y_1^2 y_2^2 y_3 - 2y_1^2 y_2^2 \sigma_{11} z k_2(s) - 2y_1^2 y_2^2 \sigma_{21} z k_2(s)\Big] v(dz)$$
$$+ \int_{\mathbb{R}} \Big[k_2(s)(y_1^2 y_2^2 (y_3 + y_3\hat{\theta}_1(z)) - k_2(s) y_1^2 y_2^2 y_3 - Y_1^2 y_2^2 y_3 \hat{\theta}_1 z k_1(s)\Big] v(dz) - \frac{y_1^2 y_2^2 y_3^3 k_2^2}{\alpha_2} = 0$$

Equation 5.3.17 and 5.3.19 leads to the 2-dimensional Riccati equation

Example 5.3.2 (Minimizing Worst Case External Factors for two Companies in a Non-Jump Market). Lets look at the same example as above but in a non-jump setting. $X(\cdot) = (X_1(\cdot), X_2(\cdot))$

where σ is a 2×2 matrix, $\eta = (\eta_1, \eta_2)$, $\pi = (\pi_1, \pi_2)$ and

$$\begin{cases} dX_1(t) = X_1(t)[\pi_1(t)dt + \sigma_{11}(t)dB_1(t) + \sigma_{12}(t)dB_2(t)] \\ X_1(0) = x_1 \in \mathbb{R}. \end{cases}$$

and

$$\begin{cases} dX_2(t) = X_2(t)[\pi_2(t)dt + \sigma_{21}(t)dB_1(t) + \sigma_{22}(t)dB_2(t)] \\ X_2(0) = x_2 \in \mathbb{R}. \end{cases}$$

As before $\pi = (\pi_1, \pi_2)$ is the control of company 1 and 2 respectively. We now define

$$\begin{aligned} dY_0(t) &= dt; & Y_0(0) &= s \in \mathbb{R}. \\ dY_1(t) &= dX_1(t); & Y_1(0) &= y_1 = x_1. \\ dY_2(t) &= dX_2(t); & Y_2(0) &= y_2 = x_2. \\ dY_3(t) &= -Y_3(t)\theta(t)dB(t); & Y_3(0) &= y_3 > 0. \end{aligned}$$

where the generator of $Y(\cdot)$ is

$$\begin{aligned} A^{\pi_1,\pi_2,\theta}\varphi(s, y_1, y_2, y_3) &= \frac{\partial \varphi}{\partial s} + x_1\pi_1\frac{\partial \varphi}{\partial x_1} + x_2\pi_2\frac{\partial \varphi}{\partial x_2} + \frac{1}{2}x_1^2\sigma_1^2\frac{\partial^2 \varphi}{\partial^2 x_1} + \frac{1}{2}x_2^2\sigma_2^2\frac{\partial^2 \varphi}{\partial^2 x_2} \\ &+ \frac{1}{2}y_3^2\theta^2\frac{\partial^2 \varphi}{\partial^2 y_3} + y_1y_2\sigma_1\sigma_2\frac{\partial^2 \varphi}{\partial y_1 \partial y_2} - y_1y_3\sigma_1\theta\frac{\partial^2 \varphi}{\partial y_1 \partial y_3} - y_2y_3\sigma_2\theta\frac{\partial^2 \varphi}{\partial y_2 \partial y_3} \end{aligned} \tag{5.3.20}$$

The performance functionals to the companies have the form

$$J_1^{\pi,\theta} = \inf_\theta E[-\int_0^{T-s} \alpha_1\pi_1^2(t)X_2^2(t)Y_3(t)dt + \gamma_1X_1^2(T)X_2^2(T)Y_3(T)]$$

and

$$J_2^{\pi,\theta} = \inf_\theta E[-\int_0^{T-s} \alpha_2\pi_2^2(t)X_1^2(t)Y_3(t)dt + \gamma_2X_1^2(T)X_2^2(T)Y_3(T)]$$

Here we have

$$\begin{cases} f_1(s, x_1, x_2, y_3, \theta, \pi) = -\alpha_1\pi_1^2x_2^2y_3, \\ g_1(s, x_1, x_2, y_3) = \gamma_1x_1^2x_2^2y_3, \end{cases}$$

and

$$\begin{cases} f_2(s, x_1, x_2, y_3, \theta, \pi) = -\alpha_2\pi_2^2x_1^2y_3, \\ g_2(s, x_1, x_2, y_3) = \gamma_2x_1^2x_2^2y_3. \end{cases}$$

We fix $\pi_2 \in \mathbb{R}$ and maximize

$$A^{\delta,\pi_1,\pi_2}\varphi_1(y) + f_1(y)$$

with respect to π_1. So we maximize

$$h_1(\pi_1) := x_1\pi_1\frac{\partial\varphi_1}{\partial x_1} - \alpha_1\pi_1^2x_2^2.$$

This maximum is attained at

$$\pi_1 = \hat{\pi_1} = \frac{1}{2\alpha_1 x_2^2 y_3}\frac{\partial\varphi_1}{\partial x_1} \tag{5.3.21}$$

Substituting this into (5.3.20) we get the function

$$f(\theta) := \frac{1}{2}y_3^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_3} - y_1y_3\sigma_1\theta\frac{\partial^2\varphi}{\partial y_1\partial y_3} - y_2y_3\sigma_2\theta\frac{\partial^2\varphi}{\partial y_2\partial y_3}.$$

When we maximize f over θ we get

$$\hat{\theta} = \frac{y_1}{y_3}\sigma_1\frac{\varphi_{13}}{\varphi_{33}} - \frac{y_2}{y_3}\sigma_2\frac{\varphi_{23}}{\varphi_{33}} = \frac{1}{y_3\varphi_{33}}(y_1\sigma_1\varphi_{13} - y_2\sigma_2\varphi_{23}).$$

Then for $(\hat{\pi_1}, \hat{\pi_2}, \hat{\theta})$ we require that

$$A^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}\varphi(y) + f_1(s, x_1, x_2, x_3, \pi, \theta) = 0,$$

or

$$\begin{aligned}&\frac{\partial\varphi}{\partial s} + y_1\frac{1}{2\alpha_1 y_2^2}\varphi_1^2 + y_2\hat{\pi_2}\varphi_1 + \frac{1}{2}y_1^2\sigma_1^2\varphi_{11} + \frac{1}{2}y_2^2\sigma_2^2\varphi_{22} + \frac{1}{2}(y_1\sigma_1\varphi_{13} - y_2\sigma_2\varphi_{23})^2\\ &+ y_1y_2\sigma_1\sigma_2\varphi_{12} - y_1\sigma_1\frac{1}{\varphi_{33}}(y_1\sigma_1\varphi_{13} - y_2\sigma_2\varphi_{23})\varphi_{13} - y_2\sigma_2\frac{1}{\varphi_{33}}(y_1\sigma_1\varphi_{13} - y_2\sigma_2\varphi_{23})\varphi_{23} = 0\end{aligned}$$

Inspired by requirement (v) let us try functions, φ_i, on the form

$$\varphi_i(s, x_1, x_2) = k_i(s)x_1^2x_2^2y_3; i = 1, 2.$$

where $k_i(s)$ are some function we need to find. So from (5.3.21) we get that

$$\hat{\pi_1} = \frac{1}{\alpha_1 y_3}k_1(s)x_1$$

$$\begin{aligned}&k_1'(s)y_1^2y_2^2y_3 + \frac{2}{\alpha_1}y_1^2y_2^2y_3^2k_1^2(s) + 2y_2^2y_1^2y_3k_1(s)\hat{\pi_2} + y_1^2y_2^2y_3\sigma_1^2k_1(s) + y_1^2y_2^2\sigma_2^2k_1(s)\\ &+ 4y_1^2y_2^2y_3\sigma_1\sigma_2k_1(s) - 2y_1^2y_2^2y_3\sigma_1\theta_0k_1(s) - 2y_1^2y_2^2y_3\sigma_2\theta_0k_1(s) = 0\end{aligned} \tag{5.3.22}$$

We fix $\pi_1 \in \mathbb{R}$ and maximize

$$A^{\delta,\pi,\theta}\varphi_2(y) + f_2(s, x_1, x_2, x_3, \pi, \theta)$$

with respect to π_2 we get that

$$\pi_2 = \hat{\pi_2} = \frac{1}{\alpha_2 y_3} k_2(s) y_2 \tag{5.3.23}$$

Using requirement (v) we have

$$\begin{aligned} &k_2'(s)y_1^2y_2^2y_3 + \frac{2}{\alpha_1}y_1^2y_2^2y_3^2k_1k_2(s) + \frac{1}{\alpha_2}y_2y_1^2k_2(s) + y_1^2y_2^2y_3\sigma_1^2k_2(s) + y_1^2y_2^2\sigma_2^2k_2(s) \\ &+ 4y_1^2y_2^2y_3\sigma_1\sigma_2k_2(s) - 2y_1^2y_2^2y_3\sigma_1\theta_0k_2(s) - 2y_1^2y_2^2y_3\sigma_2\theta_0k_2(s) = 0 \end{aligned} \tag{5.3.24}$$

Equation 5.3.22 and 5.3.24 leads to the 2-dimensional Riccati equation

Example 5.3.3 (Minimizing Worst Case Internal Market Factors for two Companies). Suppose we have two companies described by $X(\cdot) = (X_1(\cdot), X_2(\cdot))$ where σ is a 2×2 matrix, $\eta = (\eta_1, \eta_2)$, $\pi = (\pi_1, \pi_2)$,

$$\begin{cases} dX_1(t) = X_1(t)[\pi_1(t)dt + \mu_1 dB(t)] + \sigma_{11}(t)X_1(t^-)d\eta_1(t) + \sigma_{12}(t)X_1(t^-)d\eta_2(t) \\ X_1(0) = x_1 \in \mathbb{R}. \end{cases}$$

and

$$\begin{cases} dX_2(t) = X_2(t)[\pi_2(t)dt + \mu_2 dB(t)] + \sigma_{21}(t)X_2(t^-)d\eta_1(t) + \sigma_{22}(t)X_2(t^-)d\eta_2(t) \\ X_2(0) = x_2 \in \mathbb{R}. \end{cases}$$

Here $\pi = (\pi_1, \pi_2)$ is the control of company 1 and 2 respectively and $\eta_i(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}_i(ds, dz)$ $i = 1, 2$. Define

$$\begin{aligned} &dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}. \\ &dY_1(t) = dX_1(t); && Y_1(0) = y_1 = x_1. \\ &dY_2(t) = dX_2(t); && Y_2(0) = y_2 = x_2. \\ &dY_3(t) = -Y_3(t)\theta_0(t)dB(t) - Y_3(t^-)\int_{\mathbb{R}} \theta(z)\tilde{N}(ds, dz); && Y_3(0) = y_3 > 0. \end{aligned}$$

where the generator of $Y(\cdot)$ is

$$
\begin{aligned}
(5.3.25)\qquad A^{\pi_1,\pi_2,\theta}\varphi(s,y_1,y_2,y_3) &= \frac{\partial\varphi}{\partial s} + x_1\pi_1\frac{\partial\varphi}{\partial x_1} + x_2\pi_2\frac{\partial\varphi}{\partial x_2} + \frac{1}{2}x_1^2\mu_1^2\frac{\partial^2\varphi}{\partial^2 x_1} + \frac{1}{2}x_2^2\mu_2^2\frac{\partial^2\varphi}{\partial^2 x_2}\\
&+ \frac{1}{2}y_3^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_3} + y_1y_2\mu_1\mu_2\frac{\partial^2\varphi}{\partial y_1\partial y_2} - y_1y_3\mu_1\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_3} - y_2y_3\mu_2\theta_0\frac{\partial^2\varphi}{\partial y_2\partial y_3}\\
&+ \int_{\mathbb{R}}\Big[\varphi(s,y_1+y_1\sigma_{11}z,y_2+y_2\sigma_{21}z,y_3+y_3\theta_{11}) - \varphi(s,y_1,y_2,y_3)\\
&- y_1\sigma_{11}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{21}z\frac{\partial\varphi}{\partial y_2} - y_3\theta_{11}\frac{\partial\varphi}{\partial y_3}\Big]v(dz)\\
&+ \int_{\mathbb{R}}\Big[\varphi(s,y_1+y_1\sigma_{12}z,y_2+y_2\sigma_{22}z,y_3+y_3\theta_{12}) - \varphi(s,y_1,y_2,y_3)\\
&- y_1\sigma_{12}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{22}z\frac{\partial\varphi}{\partial y_2} - y_3\theta_{12}\frac{\partial\varphi}{\partial y_3}\Big]v(dz)
\end{aligned}
$$

As oppose to the previous case we now see θ as an internal market factor that the market tries to minimize. We continue with the same performance functionals, so we let them be on the form

$$J_1^{\pi,\theta} = \inf_\theta E[-\int_0^{T-s}\alpha_1\pi_1^2(t)X_2^2(t)Y_3(t)dt + \gamma_1X_1^2(T)X_2^2(T)Y_3(T)]$$

and

$$J_2^{\pi,\theta} = \inf_\theta E[-\int_0^{T-s}\alpha_2\pi_2^2(t)X_1^2(t)Y_3(t)dt + \gamma_2X_1^2(T)X_2^2(T)Y_3(T)]$$

Here we have

$$\begin{cases} f_1(s,x_1,x_2,y_3,\theta,\pi) = -\alpha_1\pi_1^2x_2^2y_3,\\ g_1(s,x_1,x_2,y_3) = \gamma_1x_1^2x_2^2y_3,\end{cases}$$

and

$$\begin{cases} f_2(s,x_1,x_2,y_3,\theta,\pi) = -\alpha_2\pi_2^2x_1^2y_3,\\ g_2(s,x_1,x_2,y_3) = \gamma_2x_1^2x_2^2y_3.\end{cases}$$

The problem can then be represented as

$$\Phi(y) = \Phi(s,y_1,y_2) = \sup_\pi\left[\inf_\theta J^{\pi,\theta}(y)\right].$$

The corresponding HJBI equation is

$$(5.3.26)\qquad \begin{cases} \sup_\pi\left[\inf_\theta A^{\pi,\theta}(y)\right] + f_i = 0\\ \varphi(T,y_1,y_2) = g_i.\end{cases}$$

To solve the Nash equilibrium we start by fixing $\pi_2 \in \mathbb{R}$ and maximize

$$A^{\delta,\pi_1,\pi_2}\varphi_1(y) + f_1(s,x_1,x_2,x_3,\pi,\theta)$$

with respect to π_1. So we maximize

$$h_1(\pi_1) := x_1\pi_1\frac{\partial\varphi_1}{\partial x_1} - \alpha_1\pi_1^2 x_2^2 y_3.$$

This maximum is attained at

$$\pi_1 = \hat{\pi}_1 = \frac{x_1}{\alpha_1 x_2^2 y_3}\frac{\partial\varphi_1}{\partial x_1} \tag{5.3.27}$$

Substituting this into (5.3.25) we get the function

$$\begin{aligned}
t(\theta) := \int_{\mathbb{R}} \Big[&\varphi(s, y_1 + y_1\sigma_{11}z, y_2 + y_2\sigma_{21}z, y_3 + y_3\theta_{11}) - \varphi(s, y_1, y_2, y_3) \\
&- y_1\sigma_{11}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{21}z\frac{\partial\varphi}{\partial y_2} - y_3\theta_{11}\frac{\partial\varphi}{\partial y_3}\Big]v(dz) \\
&+ \int_{\mathbb{R}} \Big[\varphi(s, y_1 + y_1\sigma_{12}z, y_2 + y_2\sigma_{22}z, y_3 + y_3\theta_{12}) - \varphi(s, y_1, y_2, y_3) \\
&- y_1\sigma_{12}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{22}z\frac{\partial\varphi}{\partial y_2} - y_3\theta_{12}\frac{\partial\varphi}{\partial y_3}\Big]v(dz).
\end{aligned}$$

We can minimize this point-wise, let

$$\begin{aligned}
\Psi(\theta) = &\varphi(s, y_1 + y_1\sigma_{11}z, y_2 + y_2\sigma_{21}z, y_3 + y_3\theta_{11}) - \varphi(s, y_1, y_2, y_3) \\
&- y_1\sigma_{11}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{21}z\frac{\partial\varphi}{\partial y_2} - y_3\theta_{11}\frac{\partial\varphi}{\partial y_3} \\
&+ \varphi(s, y_1 + y_1\sigma_{12}z, y_2 + y_2\sigma_{22}z, y_3 + y_3\theta_{12}) - \varphi(s, y_1, y_2, y_3) \\
&- y_1\sigma_{12}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{22}z\frac{\partial\varphi}{\partial y_2} - y_3\theta_{12}\frac{\partial\varphi}{\partial y_3}.
\end{aligned}$$

Then for $(\hat{\pi}_1, \hat{\pi}_2)$ we require that

$$A^{\hat{\pi}_1,\hat{\pi}_2,\hat{\theta}}\varphi(y) + f_1(y) = 0,$$

or

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial s} + y_1\frac{1}{2\alpha_1 y_2^2}(\frac{\partial\varphi}{\partial y_1})^2 + y_2\hat{\pi_2}\frac{\partial\varphi}{\partial y_2} + \frac{1}{2}y_1^2\mu_1^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\mu_2^2\frac{\partial^2\varphi}{\partial^2 y_2}\\
&+ \frac{1}{2}y_3^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_3} + y_1y_2\mu_1\mu_2\frac{\partial^2\varphi}{\partial y_1\partial y_2} - y_1y_3\mu_1\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_3} - y_2y_3\mu_2\theta_0\frac{\partial^2\varphi}{\partial y_2\partial y_3}\\
&+ \int_{\mathbb{R}}\Big[\varphi(s, y_1 + y_1\sigma_{11}z, y_2 + y_2\sigma_{21}z, y_3 + y_3\hat{\theta}_{11}) - \varphi(s, y_1, y_2, y_3)\\
&- y_1\sigma_{11}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{21}z\frac{\partial\varphi}{\partial y_2} - y_3\hat{\theta}_{11}\frac{\partial\varphi}{\partial y_3}\Big]v(dz)\\
&+ \int_{\mathbb{R}}\Big[\varphi(s, y_1 + y_1\sigma_{12}z, y_2 + y_2\sigma_{22}z, y_3 + y_3\hat{\theta}_{12}) - \varphi(s, y_1, y_2, y_3)\\
&- y_1\sigma_{12}z\frac{\partial\varphi}{\partial y_1} - y_2\sigma_{22}z\frac{\partial\varphi}{\partial y_2} - y_3\hat{\theta}_{12}\frac{\partial\varphi}{\partial y_3}\Big]v(dz) - \frac{\varphi_1^2}{4\alpha_1 x_2^2 y_2^2} = 0
\end{aligned}
$$

EXAMPLE 5.3.4 (Worst Case Internal Market Factors in a Non-Jump Market). Lets look at the same example as above but in a non-jump setting. $X(\cdot) = (X_1(\cdot), X_2(\cdot))$ where σ is a 2×2 matrix, $\eta = (\eta_1, \eta_2)$, $\pi = (\pi_1, \pi_2)$ and

$$
\begin{cases}
dX_1(t) = X_1(t)[\pi_1(t)dt + \sigma_{11}(t)dB_1(t) + \sigma_{12}(t)dB_2(t)]\\
X_1(0) = x_1 \in \mathbb{R}.
\end{cases}
$$

and

$$
\begin{cases}
dX_2(t) = X_2(t)[\pi_2(t)dt + \sigma_{21}(t)dB_1(t) + \sigma_{22}(t)dB_2(t)]\\
X_2(0) = x_2 \in \mathbb{R}.
\end{cases}
$$

As before $\pi = (\pi_1, \pi_2)$ is the control of company 1 and 2 respectively. We now define

$$
\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}.\\
&dY_1(t) = dX_1(t); && Y_1(0) = y_1 = x_1.\\
&dY_2(t) = dX_2(t); && Y_2(0) = y_2 = x_2.\\
&dY_3(t) = -Y_3(t)\theta_0(t)dB(t); && Y_3(0) = y_3 > 0.
\end{aligned}
$$

where the generator of $Y(\cdot)$ is

$$
\begin{aligned}
A^{\pi_1,\pi_2,\theta}\varphi(s, y_1, y_2, y_3) &= \frac{\partial\varphi}{\partial s} + x_1\pi_1\frac{\partial\varphi}{\partial x_1} + x_2\pi_2\frac{\partial\varphi}{\partial x_2} + \frac{1}{2}y_1^2(\sigma_{11}^2 + \sigma_{12}^2)\frac{\partial^2\varphi}{\partial^2 x_1}\\
&+ \frac{1}{2}y_2^2(\sigma_{21}^2 + \sigma_{22}^2)\frac{\partial^2\varphi}{\partial^2 x_2} + \frac{1}{2}y_3^2(\theta_{11}^2 + \theta_{12}^2)\frac{\partial^2\varphi}{\partial^2 y_3} + y_1y_2(\sigma_{11}\sigma_{21} + \sigma_{11}\sigma_{22})\frac{\partial^2\varphi}{\partial y_1\partial y_2}\\
&- y_1y_3(\sigma_{11}\theta_{11} + \sigma_{12}\theta_{12})\frac{\partial^2\varphi}{\partial y_1\partial y_3} - y_2y_3(\sigma_{11}\theta_{11} + \sigma_{22}\theta_{12})\frac{\partial^2\varphi}{\partial y_2\partial y_3}
\end{aligned} \tag{5.3.28}
$$

The performance functionals to the companies have the form

$$J_1^{\pi,\theta} = \inf_\theta E[-\int_0^{T-s} \alpha_1 \pi_1^2(t) X_2^2(t) Y_3(t) dt + \gamma_1 X_1^2(T) X_2^2(T) Y_3(T)]$$

and

$$J_2^{\pi,\theta} = \inf_\theta E[-\int_0^{T-s} \alpha_2 \pi_2^2(t) X_1^2(t) Y_3(t) dt + \gamma_2 X_1^2(T) X_2^2(T) Y_3(T)]$$

Here we have

$$\begin{cases} f_1(s, x_1, x_2, y_3, \theta, \pi) = -\alpha_1 \pi_1^2 x_2^2 y_3, \\ g_1(s, x_1, x_2, y_3) = \gamma_1 x_1^2 x_2^2 y_3, \end{cases}$$

and

$$\begin{cases} f_2(s, x_1, x_2, y_3, \theta, \pi) = -\alpha_2 \pi_2^2 x_1^2 y_3, \\ g_2(s, x_1, x_2, y_3) = \gamma_2 x_1^2 x_2^2 y_3. \end{cases}$$

We fix $\pi_2 \in \mathbb{R}$ and maximize

$$A^{\delta,\pi_1,\pi_2} \varphi_1(y) + f_1(s, x_1, x_2, x_3, \pi, \theta)$$

with respect to π_1. So we maximize

$$h_1(\pi_1) := x_1 \pi_1 \frac{\partial \varphi_1}{\partial x_1} - \alpha_1 \pi_1^2 x_2^2.$$

This maximum is attained at

$$\pi_1 = \hat{\pi_1} = \frac{1}{2\alpha_1 x_2^2 y_3} \frac{\partial \varphi_1}{\partial x_1} \tag{5.3.29}$$

Substituting this into (5.3.28) we get the function

$$f(\theta) := \frac{1}{2} y_3^2 (\theta_{11}^2 + \theta_{12}^2) \frac{\partial^2 \varphi}{\partial^2 y_3} - y_1 y_3 (\sigma_{11}\theta_{11} + \sigma_{12}\theta_{12}) \frac{\partial^2 \varphi}{\partial y_1 \partial y_3} - y_2 y_3 (\sigma_{11}\theta 11 + \sigma_{22}\theta_{12}) \frac{\partial^2 \varphi}{\partial y_2 \partial y_3}.$$

Then for $(\hat{\pi_1}, \hat{\pi_2}, \hat{\theta})$ we require that

$$A^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}} \varphi(y) + f_1(s, x_1, x_2, x_3, \pi, \theta) = 0,$$

or

$$\frac{\partial \varphi}{\partial s} + y_1 \frac{1}{2\alpha_1 y_2^2} \varphi_1^2 + y_2 \hat{\pi_2} \varphi_1 + \frac{1}{2} y_1^2 \sigma_1^2 \varphi_{11} + \frac{1}{2} y_2^2 \sigma_2^2 \varphi_{22} + \frac{1}{2} (y_1 \sigma_1 \varphi_{13} - y_2 \sigma_2 \varphi_{23})^2$$
$$- y_1 y_2 \sigma_1 \sigma_2 \varphi_{12} - y_1 \sigma_1 \frac{1}{\varphi_{33}} (y_1 \sigma_1 \varphi_{13} - y_2 \sigma_2 \varphi_{23}) \varphi_{13} - y_2 \sigma_2 \frac{1}{\varphi_{33}} (y_1 \sigma_1 \varphi_{13} - y_2 \sigma_2 \varphi_{23}) \varphi_{23} = 0$$

EXAMPLE 5.3.5 (Minimizing Worst Case for two Companies with a Complementary Factor). Lets look at two companies in different industries where a factor is such that if it increases, the profit of company 1 increases while the profit of company 2 decreases. Examples could be an oil producing company and a fertilizer company, as the price of oil increases the firm producing oil increases its revenue while the cost of producing fertilizer, witch is a heavily oil dependent process, increases so the profit of company 2 decreases. The investor has a utility function such that opportunity cost is calculated according to a function U_i. We model the companies by $X(\cdot) = (X_1(\cdot), X_2(\cdot))$ where σ is a 2×2 matrix, $\pi = (\pi_1, \pi_2)$ and

$$\begin{cases} dX_1(t) = X_1(t)[(\pi_1(t) + k_1\lambda(t))dt + \sigma_{11}(t)dB_1(t) + \sigma_{12}(t)dB_2(t)] \\ X_1(0) = x_1 \in \mathbb{R}. \end{cases}$$

and

$$\begin{cases} dX_2(t) = X_2(t)[(\pi_2(t) - k_2\lambda(t))dt + \sigma_{21}(t)dB_1(t) + \sigma_{22}(t)dB_2(t)] \\ X_2(0) = x_2 \in \mathbb{R}. \end{cases}$$

Here $\pi = (\pi_1, \pi_2)$ is the control of company 1 and 2 respectively. Define

$$\begin{aligned} dY_0(t) &= dt; & Y_0(0) &= s \in \mathbb{R}. \\ dY_1(t) &= dX_1(t); & Y_1(0) &= y_1 = x_1. \\ dY_2(t) &= dX_2(t); & Y_2(0) &= y_2 = x_2. \end{aligned}$$

where the generator of $Y(\cdot)$ is

$$\begin{aligned} A^{\pi_1,\pi_2,\theta}\varphi(s, y_1, y_2) &= \frac{\partial\varphi}{\partial s} + x_1(\pi_1 - k_1\lambda)\frac{\partial\varphi}{\partial x_1} + x_2(\pi_2 - k_2\lambda)\frac{\partial\varphi}{\partial x_2} + \frac{1}{2}x_1^2(\sigma_{11}^2 + \sigma_{12}^2)\frac{\partial^2\varphi}{\partial^2 x_1} \\ &+ \frac{1}{2}x_2^2(\sigma_{21}^2 + \sigma_{22}^2)\frac{\partial^2\varphi}{\partial^2 x_2} + y_1 y_2(\sigma_{11}\sigma_{21} + \sigma_{12}^2)\frac{\partial^2\varphi}{\partial y_1 \partial y_2} \end{aligned}$$

(5.3.30)

The performance functionals to the companies have the form

$$J_1^{\pi,\lambda} = \inf_{\lambda} E[\int_0^T U_1(\pi_1(t))dt + X_1^{\lambda}(T)]$$

and

$$J_2^{\pi,\lambda} = \inf_{\lambda} E[\int_0^T U_2(\pi_2(t))dt + X_2^{\lambda}(T)]$$

Here we have

$$\begin{cases} f_1(s, x_1, x_2, \theta, \pi) = U_1(\pi_1), \\ g_1(s, x_1, x_2) = x_1, \end{cases}$$

and

$$\begin{cases} f_2(s, x_1, x_2, \theta, \pi) = U_2(\pi_2), \\ g_2(s, x_1, x_2) = x_2. \end{cases}$$

We fix $\pi_2 \in \mathbb{R}$ and maximize

$$A^{\delta,\pi_1,\pi_2}\varphi_1(y)$$

with respect to π_1. So we maximize

$$h_1(\pi_1) := x_1\pi_1\frac{\partial\varphi_1}{\partial x_1} - U_1(\pi_1).$$

This maximum is attained at

$$\pi_1 = \hat{\pi_1} = I_1(x_1\varphi_1), \tag{5.3.31}$$

where I_1 is the inverse function of U_1 over the strictly increasing domain. Substituting this into (5.3.30) we get the function

$$g_1(\lambda) := y_1k_1\lambda\varphi_1 - y_2k_2\lambda\varphi_2.$$

We get

$$\frac{\varphi_1}{\varphi_2}\frac{k_1}{k_2} = \frac{y_2}{y_1}.$$

Then for $(\hat{\pi_1}, \hat{\pi_2})$ we require that

$$A^{\hat{\pi_1},\hat{\pi_2},\hat{\theta}}\varphi(y) + f_1(s, x_1, x_2, x_3, \pi, \theta) = 0,$$

or

$$\frac{\partial\varphi}{\partial s} + x_1(I_1(y_1\varphi_1) - k_1\lambda)\frac{\partial\varphi}{\partial x_1} + x_2(I_2(y_2\varphi_2) - k_2\lambda)\frac{\partial\varphi}{\partial x_2} + \frac{1}{2}x_1^2(\sigma_{11}^2 + \sigma_{12}^2)\frac{\partial^2\varphi}{\partial^2 x_1}$$
$$+ \frac{1}{2}x_2^2(\sigma_{21}^2 + \sigma_{22}^2)\frac{\partial^2\varphi}{\partial^2 x_2} + y_1y_2(\sigma_{11}\sigma_{21} + \sigma_{12}^2)\frac{\partial^2\varphi}{\partial y_1\partial y_2} = 0$$

CHAPTER 6

DYNAMIC RISK MEASURES AND THE CORRESPONDING HJBI

In this chapter we will extend our static setting to a dynamic one. We will use FBSDE and g-expectation to formulate a dynamic zero-sum game problem. Then we show and prove a HJBI equation for this dynamic zero-sum game.

6.1. The Dynamic Optimization Problem

While we previously have limited ourself to a static setting for our risk measure we will now extend this to a dynamic model. Most investors are not only interested in expressing the riskiness of a future position, but also to continuously monitor and manage the position. To construct such a model, we introduce dynamic risk measure into our equation. Let

$$
\begin{cases}
dX^u(t) &= b(X^u(t), u_0(t), \theta(t))dt + \sigma(X^u(t), u_0(t), \theta(t))dB(t) \\
&\quad + \int_{\mathbb{R}^k} \gamma(X^u(t^-), u_1(t,z), z)\tilde{N}(dt, dz). \\
dY^u(t) &= -g(t, Y^u(t), Z^u(t), K^u(t))dt + Z^u(t)dB(t) + \int_{\mathbb{R}^k} K^u(t,z)\tilde{N}(dt, dz). \\
X(0) &= x_0, Y^u(\tau_S) = -X^u(\tau_S).
\end{cases}
\tag{6.1.1}
$$

be our FBSDE, and assume b, σ and g satisfy (2.4.7). Let S, the solvency region, and τ_S, the bankruptcy time, be as before, then from the above we get that $Y(t) = Y(T) + \int_t^\tau g(s, Y(s), Z(s))ds - \int_t^\tau Z(s)dB(s) = -X(T) + \int_t^\tau g(s, Y(s), Z(s))ds - \int_t^\tau Z(s)dB(s)$. Here, $X(t)$ is our wealth equation and our market is (2.3.2) and (2.3.3). Further let $\rho = (\rho_t)_{t\in[0,T]}$ be a dynamic risk measure where $\rho_t(x) := \varepsilon_g[-x|\mathcal{F}_t] := Y_t, \forall x \in L^2(\mathcal{F}_T)$. Then we get the following optimization problem

PROBLEM 6.1.1. Let our performance functional be

$$J^{\pi,\theta}(x) := E_Q^x\left[\int_0^T \rho_t(X^{\hat{\pi},\hat{\theta}})dt\right] := E_Q^x\left[\int_0^T \epsilon_g(-X^{\hat{\pi},\hat{\theta}}(T)|\mathcal{F}_t]dt\right] := E_Q^x\left[\int_0^T Y_t^{\hat{\pi},\hat{\theta}}dt\right].$$

Find $\Phi(x)$ and $(\pi^*,\theta^*) \in \Pi \times \Theta$ such that

$$\Phi(x) = \sup_{\pi}\left[\inf_{\theta} E_Q\left[\int_0^T \rho_t(X^{\pi,\theta}(T))dt\right]\right] = J^{\pi^*,\theta^*}(x).$$

6.2. A HJBI Equation for a Zero-Sum Game with Dynamic Risk Measures

With our problem as stated above, we give the following theorem.

THEOREM 6.2.1. *Assume that for every $(\pi,\theta) \in \Pi \times \Theta$ there exists a function $v^{\pi,\theta}(t,x)$ Such that*

$$Y^{\pi,\theta}(t) = v^{\pi,\theta}(t, X^{\pi,\theta}(t)).$$

Further, suppose we can find a function $\varphi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ and Markov controls $(\pi,\theta) \in \Pi \times \Theta$ such that

(i) $A^{\pi,\hat{\theta}}\varphi(x) - v^{\pi,\hat{\theta}}(t,x) \geq 0$ *for all t $in[0,T]$, all $x \in S$ and all $\pi \in \Pi$.*

(ii) $A^{\hat{\pi},\theta}\varphi(x) - v^{\hat{\pi},\theta}(t,x) \leq 0$ *for all t $in[0,T]$, $\theta \in \Theta$ and all $x \in S$.*

(iii) $A^{\hat{\pi},\hat{\theta}}\varphi(x) - v^{\hat{\pi},\hat{\theta}}(t,x) = 0$ *for all t $in[0,T]$ and all $x \in S$.*

(iv) $\lim_{t\to T}\varphi(X^{\pi,\theta}(t)) = 0.$

(vii) $Y^{\pi,\theta}(\tau_s) \in \partial\mathcal{S}$ *a.s. on* $\{\tau_s < \infty\}$.

(viii) *The family $\{\varphi(Y^{\pi,\theta}(\tau))\}_{\tau\in\mathcal{T}}$ is uniformly integrable for all $(\pi,\theta) \in \Pi \times \Theta$, $x \in \mathcal{S}$.*

(ix) $X^{\delta,\pi,\theta}(\tau_s) \in \partial\mathcal{S}$ *a.s. on* $\{\tau_s < \infty\}$.

(x) *The family $\{\varphi(X^{\pi,\theta}(\tau))\}_{\tau\in\mathcal{T}}$ is uniformly integrable for all $(\pi,\theta) \in \Pi \times \Theta$, $x \in \mathcal{S}$.*

Then we have;

$$\begin{aligned}\varphi(x) &= \Phi(x) = J^{\hat{\pi},\hat{\theta}}(x)\\ &= \inf_{\theta}\left[\sup_{\pi} J^{\pi,\theta}(x)\right] = \inf_{\theta}\left[\sup_{\pi} J^{\pi,\theta}(x)\right]\\ &= \sup_{\pi} J^{\pi,\hat{\theta}}(x) = \inf_{\theta} J^{,\hat{\pi},\theta}(x),\end{aligned}$$

with

$$(\hat{\pi},\hat{\theta}) \text{ as an optimal (Markov) control.}$$

Further, $v^{\hat{\pi},\hat{\theta}}$ solves the quasi linear PDE:

$$Z^{\hat{\pi},\hat{\theta}} = v_x^{\hat{\pi},\hat{\theta}}(t,x)\sigma(x,\hat{\pi},\hat{\theta}),$$

and

$$\begin{aligned} g^{\hat{\pi},\hat{\theta}} &= -(v_t^{\hat{\pi},\hat{\theta}}(t,x) + \frac{1}{2}v_{xx}^{\hat{\pi},\hat{\theta}}(t,x)\sigma^2(x,\hat{\pi},\hat{\theta}) + v_x^{\hat{\pi},\hat{\theta}}(t,x)b(x,\hat{\pi},\hat{\theta}) \\ &\quad + \int_{\mathbb{R}} \{v^{\hat{\pi},\hat{\theta}}(t,x+\gamma(x,\hat{\pi},\hat{\theta},z)) - v^{\hat{\pi},\hat{\theta}}(t,x) - v_x^{\hat{\pi},\hat{\theta}}(t,x\gamma(x,\hat{\pi},\hat{\theta},z)\}\nu(dz)). \end{aligned}$$

also

$$K^{\hat{\pi},\hat{\theta}} = v^{\hat{\pi},\hat{\theta}}(t,x+\gamma(x,\hat{\pi},\hat{\theta},z)) - v^{\hat{\pi},\hat{\theta}}(t,x).$$

With the boundary value

$$v^{\hat{\pi},\hat{\theta}}(T,x) = x.$$

PROOF. Step1. Let us prove that

$$\varphi(x) = \sup_{\pi\in\Pi}\left(\inf_{\theta\in\Theta} J^{\pi,\theta}(x)\right) = \Phi(x)$$

(a.) Choose $\hat{\theta}$ and let $\pi \in \Pi$. Using Dynkin, (i) and (iv)

$$\begin{aligned} \varphi(x) &= E^x[\int_0^T -A^{\pi,\hat{\theta}}\varphi(X^{\pi,\hat{\theta}})dt] \\ &\geq E^x[\int_0^T v^{\pi,\hat{\theta}}(t,X^{\pi,\hat{\theta}})dt] \\ &= E^x\left[\int_0^T Y^{\pi,\hat{\theta}}(t)dt\right] \\ &= J^{\pi,\hat{\theta}}(x). \end{aligned}$$

So

$$\varphi(x) \geq \inf_{\theta\in\Theta} E^x[\int_0^T Y^{\pi,\theta}(t)dt].$$

Since this holds for all $\pi \in \Pi$ we get

$$\varphi(x) \geq \sup_{\pi\in\Pi}\left[\inf_{\theta\in\Theta} E^x[\int_0^T Y^{\pi,\theta}(t)dt]\right] = \Phi(x).$$

(b.) Using same method and (iii) to $(\hat{\pi},\hat{\theta})$, we get equality so that

$$\varphi(x) = E^x[\int_0^T v^{\hat{\pi},\hat{\theta}}(t,X^{\hat{\pi},\hat{\theta}})dt] = J^{\hat{\pi},\hat{\theta}}(x).$$

(c.) Choose $\hat{\pi}$ and let $\theta \in \Theta$. Using Dynkin, (ii) and (iv)

$$\begin{aligned}
\varphi(x) &= E^x[\int_0^T -A^{\hat{\pi},\theta}\varphi(X^{\hat{\pi},\theta})dt] \\
&\leq E^x[\int_0^T v^{\hat{\pi},\theta}(t, X^{\hat{\pi},\theta})dt] \\
&= E^x\left[\int_0^T Y^{\hat{\pi},\theta}(t)dt\right] \\
&= J^{\hat{\pi},\theta}(x).
\end{aligned}$$

Since this holds for all $\theta \in \Theta$ we get

$$\varphi(x) \leq \inf_{\theta \in \Theta} E^x[\int_0^T Y^{\pi,\theta}(t)dt].$$

So

$$\varphi(x) \leq \sup_{\pi \in \Pi} \left[\inf_{\theta \in \Theta} E^x[\int_0^T Y^{\pi,\theta}(t)dt]\right] = \Phi(x).$$

(d.) Combining a,b,c and d we get that

$$\varphi(x) == J^{\hat{\pi},\hat{\theta}}(x) = \sup_{\pi \in \Pi} \left(\inf_{\theta \in \Theta} J^{\pi,\theta}(x)\right) = \Phi(x)$$

Using $(\hat{\pi}, \hat{\theta})$ and Itô we can find $v^{\hat{\pi},\hat{\theta}}$

$$\begin{aligned}
dY^{\hat{\pi},\hat{\theta}}(t) &= dv^{\hat{\pi},\hat{\theta}}(t, X(t)) = v_t^{\hat{\pi},\hat{\theta}}(t, X(t))dt + v_x^{\hat{\pi},\hat{\theta}}(t, X(t))dX(t) + \frac{1}{2}v_{xx}^{\hat{\pi},\hat{\theta}}(t, X(t))dX^2(t) \\
&+ \int_{\mathbb{R}} \{v^{\hat{\pi},\hat{\theta}}(t, X(t^-) + \gamma(X(t^-), u_1(t,z), z)) - u^{\hat{\pi},\hat{\theta}}(t, X(t^-)) \\
&- v_x^{\hat{\pi},\hat{\theta}}(t, X(t^-)\gamma(X(t^-), u_1(t,z), z)\}\nu(dz)dt \\
&+ \int_{\mathbb{R}} \{v^{\hat{\pi},\hat{\theta}}(t, X(t^-) + \gamma(X(t^-), u_1(t,z), z)) - v^{\hat{\pi},\hat{\theta}}(t, X(t^-))\}\tilde{N}(dt, dz) \\
&= (v_t^{\hat{\pi},\hat{\theta}}(t, X(t)) + \frac{1}{2}v_{xx}^{\hat{\pi},\hat{\theta}}(t, X(t))\sigma^2(X(t), \hat{\pi}, \hat{\theta},) + v_x^{\hat{\pi},\hat{\theta}}(t, X(t))b(X(t), \hat{\pi}, \hat{\theta},) \\
&+ \int_{\mathbb{R}} \{v^{\hat{\pi},\hat{\theta}}(t, X(t^-) + \gamma(X(t^-), \hat{\pi}, \hat{\theta}, z)) - v^{\hat{\pi},\hat{\theta}}(t, X(t^-)) \\
&- v_x^{\hat{\pi},\hat{\theta}}(t, X(t^-)\gamma(X(t^-), \hat{\pi}, \hat{\theta}, z)\}\nu(dz))dt \\
&+ v_x^{\hat{\pi},\hat{\theta}}(t, X(t))\sigma(X(t), u_0(t), \theta(t))dB(t) \\
&+ \int_{\mathbb{R}} \{v^{\hat{\pi},\hat{\theta}}(t, X(t^-) + \gamma(X(t^-), \hat{\pi}, \hat{\theta}, z)) - v^{\hat{\pi},\hat{\theta}}(t, X(t^-))\}\tilde{N}(dt, dz).
\end{aligned}$$

So

$$Z^{\hat{\pi},\hat{\theta}} = v_x^{\hat{\pi},\hat{\theta}}(t, x)\sigma(x, u_0, \theta),$$

and

$$\begin{aligned}g^{\hat{\pi},\hat{\theta}} &= -(v_t^{\hat{\pi},\hat{\theta}}(t,x) + \frac{1}{2}v_{xx}^{\hat{\pi},\hat{\theta}}(t,x)\sigma^2(x),\hat{\pi},\hat{\theta}) + v_x^{\hat{\pi},\hat{\theta}}(t,x)b(x,\hat{\pi},\hat{\theta}) \\ &\quad + \int_{\mathbb{R}}\{v^{\hat{\pi},\hat{\theta}}(t,x+\gamma(x,\hat{\pi},\hat{\theta},z)) - v^{\hat{\pi},\hat{\theta}}(t,x) - v_x^{\hat{\pi},\hat{\theta}}(t,x\gamma(x,\hat{\pi},\hat{\theta},z)\}\nu(dz)).\end{aligned}$$

also

$$K^{\hat{\pi},\hat{\theta}} = v^{\hat{\pi},\hat{\theta}}(t,x+\gamma(x,\hat{\pi},\hat{\theta},z)) - v^{\hat{\pi},\hat{\theta}}(t,x).$$

Further we have that

$$\begin{aligned}v^{\hat{\pi},\hat{\theta}}(T,Y^{\hat{\pi},\hat{\theta}}(T)) &= v^{\hat{\pi},\hat{\theta}}(T,X^{\hat{\pi},\hat{\theta}}(T)) \\ &= Y^{\hat{\pi},\hat{\theta}}(T).\end{aligned}$$

so

$$v^{\hat{\pi},\hat{\theta}}(T,x) = x.$$

Step 3. Using the same approach give us the other equalities.

□

REMARK 6.2.1. This theorem is complicated to apply in practice. The problem of finding v for each controller to find the optimal control would be very complex. This result only shows that there are still work needed on this subject and finding a way to optimize directly on the function v is an area which require further research.

CHAPTER 7

A ZERO-SUM GAME WITH OPTIMAL CONTROL AND STOPPING

Where we in the previous chapters limited ourself to control problems, we will now include stopping into our equations as in chapter 4 in Øksendal and Sulem [2007].

7.1. The Zero-Sum Game

Consider the financial system given in chapter 4;

$$\begin{aligned} dX^u(t) &= b(X^u(t), u_0(t))dt + \sigma(X^u(t), u_0(t))dB(t) \\ &\quad + \int_{\mathbb{R}^k} \gamma(X^u(t^-), u_1(t,z), z)\tilde{N}(dt, dz), \\ X^u(0) &= y \in \mathbb{R} \end{aligned} \tag{7.1.1}$$

Where $b : \mathbb{R}^k \times U \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times U \to \mathbb{R}^{k\times k}$ and $\gamma : \mathbb{R}^k \times U \times \mathbb{R}^k \to \mathbb{R}^{k\times m}$. $B(t)$ is a k-dimensional Brownian motion, $\tilde{N}(\cdot,\cdot) = (\tilde{N}_1(\cdot,\cdot), \ldots, \tilde{N}_k(\cdot,\cdot))$ are a k-independent compensated Poisson random measure and U is a Polish space. The processes $u_0(t) = u_0(t,\omega)$, $u_1(t) = u_1(t,\omega)$ and $u_2(t,z) = u_2(t,z,\omega)$ are the control processes, *càdlàg* and adapted to the filtration $\mathcal{F}_t$ generated by the driving processes $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$, with $u_0(t) \in U$ and $u_1(t) \in U$ for a.a. t, a.s. Let $u = (u_0, u_1)$ and $X^u(t)$ be the controlled jump diffusion and $\tau = \tau(\omega)$, a $\mathcal{F}_t$ -stopping time.

We then look at the problem of minimizing the performance functional of the portfolio π associated to the financial position $X(t)$;

PROBLEM 7.1.1. Find $\Phi(y)$ and $(\tau^*, \pi^*, \theta^*) \in \mathcal{T} \times \Pi \times \Theta$ such that

$$\Phi(y) = \inf_{\theta\in\Theta} \left[\sup_{\tau\in\mathcal{T}} \left(\sup_{\pi\in\Pi} J^{\tau,\pi,\theta}(y) \right) \right],$$

where

$$J^{\tau,\pi,\theta}(y) = E^y \left[\int_0^\tau f(X^{\pi,\theta}(t), u(t))dt + g(X^{\pi,\theta}(\tau))\chi_{\{\tau<\infty\}} \right],$$

where $f : \mathbb{R}^k \times U \to \mathbb{R}$, the profit rate and $g : \mathbb{R}^k \to \mathbb{R}$ the bequest function are given and $u = (\pi, \theta)$.

We assume that the set A of admissible controls are such that (7.1.1) admits a strong solution, and

- $E^y \left[\int_0^\tau |f(X(t), u(t))|dt\right] < \infty$, for all $y \in S$, where $\tau_S = \tau_s(y, u) = \inf\{t > 0; X^u(t) \notin S\}$.
- The family $\{g^-(X^u(\tau)); \tau \in \mathcal{T}\}$ is uniformly P^y-integrable for all $y \in S$, where $g^-(y) = max(0, -g(y))$.

We let $g(X^u(\tau(\omega))) = 0$ for all ω such that $\tau(\omega) = \infty$ and we let $S \in \mathbb{R}^k$ be a fixed Borel set such that

$$S \subset \bar{S^0},$$

(i.e. S has no isolated points). Here, S^0 is the interior of S and $\bar{S}$ is the closure of S.

As in Øksendal [2007] we use Markov controls since under mild conditions Markov controls can give just as good performance as more general adapted controls. When we use Markov controls we get that the generator $A^{\pi,\theta}$ becomes

$$\begin{aligned}
A^{\pi,\theta}\varphi(y) &= \sum_{i=1}^k b_i(y, \theta_0, \pi_0)\frac{\partial\varphi}{\partial y_i}(y) \\
&+ \frac{1}{2}\sum_{i,j=1}^k (\sigma\sigma^T)_{ij}(y, \theta_0, \pi_0)\frac{\partial^2\varphi}{\partial y_i \partial y_j}(y) \\
&+ \sum_{j=1}^k \int_{\mathbb{R}} \{\varphi(y + \gamma^{(j)}(y, \theta_1(y, z_j), \pi_1(y, z_j), z_j) - \varphi(y) \\
&- \nabla\varphi(y)\gamma^{(j)}(y, \theta_1(y, z_j), \pi_1(y, z_j), z_j)\}v_j(dz_j);
\end{aligned}$$

where $\varphi \in C_0^2(\mathbb{R}^k)$ and $\nabla\varphi$ is the gradient of φ. We let $\mathcal{T}$ be the set of all $\mathcal{F}_t$ -stopping times $\tau \leq \tau_s$.

7.2. A HJBI equation with optimal stopping and control

We are now ready to state the our stopping and control theorem.

THEOREM 7.2.1. *Suppose φ is a function $\varphi : \bar{S} \to \mathbb{R}$ and a Markov control $u = (\tau, \pi, \theta) \in \mathcal{T} \times \Pi \times \Theta$ such that*

(i) $\varphi \in C^1(S^0) \cap C(\bar{S})$, *and*

(ii) $\varphi \geq g$ *on* S^0.

Further, define

$$D = \{y \in S; \varphi(y) > g(y)\} \text{ the continuation region.}$$

assume then that,

(iii) $E^y[\int_0^{\tau_s} \chi_{\partial D}(X^u(t))dt] = 0$, *so* $X(t)$ *spends* 0 *time on* ∂D

(iv) ∂D *is a Lipschitz surface*

(v) $\varphi \in C^2(S^0 \backslash \partial D)$ *and the second-order derivatives of* φ *are locally bounded near* ∂D

(vi) $A^{\pi,\hat{\theta}}\varphi(y) + f(y, \pi, \hat{\theta}) \leq 0$ *on* $S^0 \backslash \partial D$ *for all* $\pi \in \Pi$

(vii) $A^{\hat{\pi},\theta}\varphi(y) + f(y, \hat{\pi}, \theta) \geq 0$ *on* $S^0 \backslash D^0$ *for all* $\theta \in \Theta$

(viii) $X^u(\tau_S) \in \partial S$ *a.s. on* $\{\tau_s < \infty\}$ *and*

$$\lim_{t \to \tau_S^-} \varphi(X^u(t)) = g(X^u(t))\chi_{\tau_s < \infty} \text{ a.s.}$$

(ix) $E^y[|\varphi(X^u(\tau))| + \int_0^{\tau_S} |A^u\varphi(X^u(t))|dt] < \infty$ *for all* $u \in U^2$, $\tau \in \mathcal{T}$

(x) $A^{\hat{u}}\varphi(y) + f(y, \hat{u}) = 0$ *for all* $y \in S$

(xi) $\tau_D := \inf\{t > 0; X^{\hat{u}}(t) \notin D\} < \infty$ *for all* $y \in S$

(xii) *the family* $\{\varphi(X^{\hat{u}}(\tau)); \tau \in \mathcal{T}\}$ *is uniformly integrable with respect to* P^y *for all* $y \in D$

Then

$$\begin{aligned}
\varphi(y) &= \Phi(y) = J^{\hat{\tau},\hat{\pi},\hat{\theta}}(y) \\
&= \sup_\tau \left[\sup_\pi \left(\inf_\theta J^{\tau,\pi,\theta}(y)\right)\right] = \sup_\pi \left[\sup_\tau \left(\inf_\theta J^{\tau,\pi,\theta}(y)\right)\right] \\
&= \sup_\tau \left[\inf_\theta \left(\sup_\pi J^{\tau,\pi,\theta}(y)\right)\right] = \inf_\theta \left[\sup_\tau \left(\sup_\pi J^{\tau,\pi,\theta}(y)\right)\right] \\
&= \sup_\pi \left[\inf_\theta \left(\sup_\tau J^{\tau,\pi,\theta}(y)\right)\right] = \inf_\theta \left[\sup_\pi \left(\sup_\tau J^{\tau,\pi,\theta}(y)\right)\right] \\
&= \sup_\pi \left[\inf_\theta J^{\hat{\tau},\pi,\theta}(y)\right] = \sup_\pi \left[\sup_\tau J^{\tau,\pi,\hat{\theta}}(y)\right] = \sup_\tau \left[\inf_\theta J^{\tau,\hat{\pi},\theta}(y)\right] \\
&= \sup_\tau \left[\sup_\pi J^{\tau,\pi,\hat{\theta}}(y)\right] = \inf_\theta \left[\sup_\pi J^{\hat{\tau},\pi,\theta}(y)\right] = \inf_\theta \left[\inf_{\sup} J^{\tau,\hat{\pi},\theta}(y)\right] \\
&= \sup_\pi J^{\hat{\tau},\pi,\hat{\theta}}(y) = \sup_\tau J^{\tau,\hat{pi},\hat{\theta}}(y) = \inf_\theta J^{\hat{\tau},\hat{\pi},\theta}(y)
\end{aligned}$$

and

$$(\hat{\tau}, \hat{\pi}, \hat{\theta}) \text{ is an optimal (Markov) control.}$$

Proof. We will proceed as in the proof for Theorem 4.2 in Øksendal and Sulem [2007], but first we need a supporting theorem:

Theorem 7.2.2 (Approximation theorem 2.1 in Øksendal and Sulem [2007]). *Let D be an open set, $D \subset S$. Assume that $X(\tau_S) \in \partial S$ a.s. on $\{\tau_S < \infty\}$ and ∂D is a Lipschitz surface (i.e. ∂D is locally the graph of a Lipschitz continuous function. and let $\varphi : \bar{S} \to \mathbb{R}$ be a function with the following properties*

$$\varphi \in C^1(S) \cap C(\bar{S}),$$

and

$$\varphi \in C^2(S \backslash \partial D)$$

and that the second order derivatives of φ are locally bounded near ∂D. Then there exits a sequence $\{\varphi_m\}_{m=1}^{\infty} \subset C^2(S) \cap C(\bar{S})$ such that, with the generator A of X_t

1. *$\varphi_m \to \varphi$ pointwise dominatatedly in $\bar{S}$ as $m \to \infty$,*
2. *$\frac{\partial \varphi_m}{\partial x_i} \to \frac{\partial \varphi}{\partial x_i}$ pointwise dominatatedly in $\bar{S}$ as $m \to \infty$,*
3. *$\frac{\partial^2 \varphi_m}{\partial x_i \partial x_j} \to \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ and $A\varphi_m \to A\varphi$ pointwise dominatatedly in $S \backslash \partial D$ as $m \to \infty$.*

Step1. First let us prove that

$$\begin{aligned}
\Phi(y) = \varphi(y) &= E^y[\int_0^{\hat{\tau}} f(Y\hat{(}t))dt + g(Y(\hat{\tau})\chi_{\tau<\infty}] \\
&= J^{\hat{\tau},\hat{\pi},\hat{\theta}} = \inf_{\theta\in\Theta}\left[\sup_{\tau\in\mathcal{T}}\left(\sup_{\pi\in\Pi} J^{\tau,\pi,\theta}(y)\right)\right].
\end{aligned}$$

(a) First, from theorem 7.2.2 above, we can assume that $\varphi \in C^2(S^0) \cap C(\bar{S})$. Then by Dynkin's formula for jump processes applied to $\tau_m := \min(\tau, m), m = 1, 2, \ldots$, by (viii) and (ix) we have

$$E^y[\varphi(Y(\tau \wedge m))] = \varphi(y) + E^y[\int_0^{\tau\wedge m} A^{\pi,\theta}\varphi(Y(t))dt], \tag{7.2.2}$$

Hence by (ii),and Fatous lemma

$$\begin{aligned}
\varphi(y) &= \lim_{m\to\infty} E^y[\int_0^{\tau_s^N} -A^{\pi,\theta}\varphi((Y^{\pi,\theta}(t), \pi(Y^{\pi,\theta}(t)), \theta(Y^{\pi,\theta}(t)))dt + \varphi(Y^{\pi,\theta}(\tau \wedge m)] \\
&\geq E^y[\int_0^{\tau} -A^{\pi,\theta}\varphi((Y^{\pi,\theta}(t), \pi(Y^{\pi,\theta}(t)), \theta(Y^{\pi,\theta}(t)))dt + g(Y^{\pi,\theta}(\tau)\chi_{\tau<\infty}].
\end{aligned}$$

From (vi) we have that

$$\varphi(y) \geq E^y[\int_0^\tau f((Y^{\pi,\hat{\theta}}(t), \pi(Y^{\pi,\hat{\theta}}(t)), \hat{\theta}(Y^{\pi,\hat{\theta}}(t)))dt + g(Y^{\pi,\hat{\theta}}(\tau)\chi_{\tau<\infty}]$$
$$= J^{\tau,\pi,\hat{\theta}}(y).$$

We now have

$$\varphi(y) \geq J^{\tau,\pi,\hat{\theta}}(y).$$

So since this holds for all $\pi \in \Pi$ and $\tau \leq \mathcal{T}$ we can conclude that

$$\varphi(y) \geq \sup_{\pi\in\Pi}\left[\sup_{\tau\in\mathcal{T}} J^{\tau,\pi,\theta}(y)\right] \geq \inf_{\theta\in\Theta}\left[\sup_{\tau\in\mathcal{T}}\left(\sup_{\pi\in\Pi} J^{\tau,\pi,\theta}(y)\right)\right] = \Phi(y). \tag{7.2.3}$$

(b) Using Dynkin to $\tau = \tau_D, \hat{\pi}, \hat{\theta} \in \mathcal{T}, \Pi, \Theta$ and (x) we get that

$$\varphi(y) = J^{\tau_D,\hat{\pi},\hat{\theta}},$$

and $\hat{\tau} = \tau_D$.

(c) Using Dynkin to $\tau = \tau_D = \hat{\tau}, \hat{\pi}, \theta \in \mathcal{T}, \Pi, \Theta$ and (iii) we get that

$$\varphi(y) = E^y[\int_0^{\hat{\tau}} -A^{\pi,\theta}\varphi((Y^{\pi,\theta}(t), \pi(Y^{\pi,\theta}(t)), \theta(Y^{\pi,\theta}(t)))dt + g(Y^{\pi,\theta}(\hat{\tau})\chi_{\hat{\tau}<\infty}].$$

So by using (vii)

$$\varphi(y) \leq E^y[\int_0^{\hat{\tau}} f((Y^{\hat{\pi},\theta}(t), \hat{\pi}, \hat{\theta})dt + g(Y^{\hat{\pi},\theta})(\hat{\tau})\chi_{\tau<\infty}]$$
$$= J^{\hat{\tau},\hat{\pi},\theta} \leq \sup_{\pi\in\Pi}\left[\sup_{\tau\in\mathcal{T}} J^{\tau,\pi,\theta}(y)\right].$$

Since this holds for all θ we get that

$$\varphi(y) \leq \inf_{\theta\in\Theta}\left[\sup_{\tau\in\mathcal{T}}\left(\sup_{\pi\in\Pi} J^{\tau,\pi,\theta}(y)\right)\right] = \Phi(y).$$

Combining this with (7.2.3) we get

$$\varphi(y) = E^y[\int_0^{\hat{\tau}} f(Y^{\hat{\pi},\hat{\theta}}(t))dt + g(Y^{\hat{\pi},\hat{\theta}}(\hat{\tau})\chi_{\hat{\tau}<\infty}]$$
$$= J^{\hat{\tau},\hat{\pi},\hat{\theta}} = \inf_{\theta\in\Theta}\left[\sup_{\tau\in\mathcal{T}}\left(\sup_{\pi\in\Pi} J^{\tau,\pi,\theta}(y)\right)\right] = \Phi(y).$$

Step2. Next let us prove that

$$\varphi(y) = \sup_{\pi}\left[\sup_{\tau}\left(\inf_{\theta\in\Theta} J^{\tau,\pi,\theta}\right)\right].$$

By Dynkin's formula applied to $\tau_m := \min(\tau, m), m = 1, 2, \ldots$ we have

$$E^y[\varphi(Y(\tau \wedge m))] = \varphi(y) + E^y[\int_0^{\tau \wedge m} A^{\phi,\theta}\varphi(Y^{\pi,\theta}(t))dt],$$

Hence by (ii) and Fatous's lemma

$$\begin{aligned}\varphi(y) &= \varliminf_{m\to\infty} E^y[\int_0^{\tau_s^N} -A\varphi((Y(t), \pi(Y(t)), \theta(Y(t)))dt + \varphi(Y(\tau \wedge m)] \\ &\geq E^y[\int_0^{\tau} -A\varphi((Y(t), \pi(Y(t)), \theta(Y(t)))dt + g(Y(\tau)\chi_{\tau<\infty}].\end{aligned}$$

From (vi) we have that

$$\varphi(y) \geq J^{\tau,\pi,\hat{\theta}}(y) \geq \inf_{\theta\in\Theta} J^{\tau,\pi,\theta}.$$

Since this holds for all $\pi \in \Pi$ and $\tau \in \mathcal{T}$

$$\varphi(y) \geq \sup_{\pi}\left[\sup_{\tau}\left(\inf_{\theta\in\Theta} J^{\tau,\pi,\theta}\right)\right].$$

On the other hand we have that

$$\inf_{\theta\in\Theta} J^{\tau,\pi,\theta} \leq J^{\tau,\pi,\theta'} \text{ for all } \theta' \in \Theta, \pi \in \Pi \text{ and } \tau \in \mathcal{T},$$

so

$$\inf_{\theta\in\Theta} J^{\tau,\pi,\theta} \leq \sup_{\pi} J^{\tau,\pi,\theta'} \text{ for all } \theta' \in \Theta, \pi \in \Pi \text{ and } \tau \in \mathcal{T},$$

We thus have that, by taking supremum on both sides

$$\sup_{\tau}\left(\inf_{\theta\in\Theta} J^{\tau,\pi,\theta}\right) \leq \sup_{\tau}\left(\sup_{\pi} J^{\tau,\pi,\theta'}\right) \text{ for all } \theta' \in \Theta \text{ and } \pi \in \Pi,$$

Since this holds for all $\theta \in \Theta$

$$\sup_{\tau}\left(\inf_{\theta\in\Theta} J^{\tau,\pi,\theta}\right) \leq \inf_{\theta}\left[\sup_{\tau}\left(\sup_{\pi} J^{\tau,\pi,\theta}(y)\right)\right].$$

Again, Since this holds for all $\pi \in \Pi$

$$\sup_{\pi}\left[\sup_{\tau}\left(\inf_{\theta\in\Theta} J^{\tau,\pi,\theta}\right)\right] \leq \inf_{\theta}\left[\sup_{\tau}\left(\sup_{\pi} J^{\tau,\pi,\theta}(y)\right)\right].$$

so we conclude that

$$\Phi(y) = \varphi(y) = \sup_{\pi}\left[\sup_{\tau}\left(\inf_{\theta\in\Theta} J^{\tau,\pi,\theta}\right)\right].$$

Step3. By applying the same approach we get the other equalities.

$\square$

7.3. Examples

Let us look at some control problem where we include stopping times as one of the controls. We then apply the result of the previous section to find a solution. We will look at both a jump and a non-jump market.

As usual, we consider the marked given by (2.3.2) and (2.3.3).

EXAMPLE 7.3.1 (Optimal Stopping and Control in a Classic Black-Scholes Marked). Let us apply the above theorem to example 4.1 in Øksendal and Sulem [2007]. Let

$$dP(t) = P(t)[\alpha(t)dt + \beta(t)dB(t)]; P(0) = y_1 > 0.$$

Let Q_t be the amount of remaining resource at time t, and let the dynamics be described by

$$dQ_t = -u_t Q_t dt; Q(0) = y_2 \geq 0.$$

where u_t controls the consumption of resource Q, and m is the maximum extraction rate. We let

$$\begin{cases} dY_0(t) & = dt \\ dY_1(t) & = dP(t); P(0) = y_1 > 0, \\ dY_2(t) & = dQ_t; Q(0) = y_2 \geq 0, \\ dY_3(t) & = -\theta(t)Y_3(t)dB(t); Y_3(0) = y_3 > 0. \end{cases}$$

Let the running cost be given by $K_0+K_1u_t$ ($K_0, K_1 \geq 0$, constants). Then we let our performance functional be given by

$$\begin{aligned} &J^{\tau,u,\theta}(s, y_1, y_2, y_3) \\ &= E^y\left[\int_0^\tau e^{-\delta(s+t)}(u(t)P(t)Q(t) - K_1) - K_0)Y_3(t)dt + e^{-\delta(s+\tau)}(MP(\tau)Q(\tau) - a)Y_3(\tau)\right], \end{aligned}$$

where $\delta > 0$ is the discounting rate,$\delta \geq 1$, and $M > 0, a > 0$ are constant (a can be seen as a transaction cost). Our problem is to find $(\hat{\tau}, \hat{u}, \hat{\theta})$ in $\mathcal{T} \times U \times \Theta$ such that

$$\Phi(y) = \Phi(s, y_1, y_2) = \sup_u \left[\inf_\theta \left(\sup_\tau J^{\tau,u,\theta}(y)\right)\right] = J^{\hat{\tau},\hat{u},\hat{\theta}}(y).$$

Then the generator of $Y^{u,\theta}$ is given by;

$$A^{u,\theta}\varphi(y) = A^{u,\theta}\varphi(s,y_1,y_2,y_3) = \frac{\partial\varphi}{\partial s} + y_1\alpha\frac{\partial\varphi}{\partial y_1} - uy_2\frac{\partial\varphi}{\partial y_2} + \frac{1}{2}y_1^2\beta^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_3^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_3} - y_1y_3\beta\theta\frac{\partial^2\varphi}{\partial y_1\partial y_3}.$$

We need to find a subset D of $S = \mathbb{R}_+^4 = [0,\infty)^4$ and $\varphi(s,y_1,y_2,y_3)$ such that

$$\varphi(s,y_1,y_2,y_3) = e^{-\delta s}(My_1y_2 - a)y_3,\ \forall\ (s,y_1,y_2,y_3) \notin D,$$

$$\varphi(s,y_1,y_2,y_3) \geq e^{-\delta s}(My_1y_2 - a)y_3,\ \forall\ (s,y_1,y_2,y_3) \in S,$$

$$A^{u,\theta}\varphi(s,y_1,y_2,y_3) + e^{-\delta s}(u(y_1y_2 - K_1) - K_0)y_3 \leq 0,\ \forall\ (s,y_1,y_2,y_3) \in S^0\backslash\bar{D},\ \forall\ u \in [0,m],$$

$$A^{u,\theta}\varphi(s,y_1,y_2,y_3) + e^{-\delta s}(u(y_1y_2 - K_1) - K_0)y_3 \geq 0,\ \forall\ (s,y_1,y_2,y_3) \in S^0\backslash D^0,\ \forall\ u \in [0,m],$$

$$\sup_u\left[\inf_\theta\{A^{u,\theta}\varphi(s,y_1,y_2,y_3) + e^{-\delta s}(u(y_1y_2 - K_1) - K_0)y_3\}\right] = 0,\ \forall\ (s,y_1,y_2,y_3) \in D.$$

$$\hat{\theta} = \frac{y_1}{y_3}\beta\frac{\varphi_{13}}{\varphi_3},$$

and $\hat{u}$ is the solution of

$$\sup_u\{e^{-\delta s}uy_3(y_1y_2 - K_1) - uy_2\varphi_2\}.$$

Let us try a function on the form

$$\varphi(s,y_1,y_2,y_3) = e^{-\delta s}F(\omega),\ \text{where}\ \omega = y_1y_2y_3.$$

Then

$$\hat{u} = \begin{cases} m, & \text{if } F'(\omega) < 1 - \frac{K_1}{y_1y_2}. \\ 0. & \text{otherwise.} \end{cases}$$

and

$$\hat{\theta} = \frac{y_1}{y_2}\beta(\frac{F'(\omega) + F''(\omega)y_1y_2y_3}{F'(\omega)y_1}).$$

so for $F'(\omega) < 1 - \frac{K_1}{y_1y_2}$ we have

$$\begin{aligned} A^{\hat{u},\hat{\theta}}F(s,y_1,y_2,y_3) &= -\delta e^{-\delta s}F(\omega) + \omega e^{-\delta s}\alpha F'(\omega) - m\omega e^{-\delta s}F'(\omega) + \frac{1}{2}\omega^2\beta^2F''(\omega)e^{-\delta s} \\ &+ \frac{1}{2}\omega^2F''(\omega)e^{-\delta s}\left(\frac{F''^2(\omega + 2F'(\omega)F''(\omega)y_1y_3y_3 + F''^2(\omega)y_1^2y_2^2y_3^2}{F'^2(\omega)y_1^2}\right) \\ &- y_1y_3\beta\frac{(e^{-\delta s}F'(\omega)y_2 + e^{-\delta s}F''(\omega)y_1y_2^2y_3)^2}{e^{-\delta s}F'(\omega)y_1y_2}. \end{aligned}$$

Solving this equation provides us with the function, φ, that we then verify satisfy the requirements so that we ensure that φ is our solution, i.e. $\varphi = \Phi$.

EXAMPLE 7.3.2 (Optimal control and stopping in a Lévy -market). Let our dynamics be given by

$$\begin{aligned}
dY_0(t) &= dt; && Y_0(0) = s \in \mathbb{R}.\\
dY_1(t) &= dV^{\pi}(t) = (Y_1(t)\alpha(t) - u(t))dt + Y_1(t)\beta\pi(t)dB(t)\\
&\quad + Y_1(t^-)\pi(t^-)\int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz); && Y_1(0) = y_1 > 0.\\
dY_2(t) &= -Y_2(t)\theta_0(t)dB(t) - Y_2(t)\int_{\mathbb{R}} \theta_1(s,z)\tilde{N}(ds,dz); && Y_2(0) = y_2 > 0.
\end{aligned}$$

Solve

$$\Phi(s,x) = \sup_u \left[\sup_\tau \left(\inf_{\theta_0,\theta_1} J^{\theta,\tau}\right]\right)$$

where

$$J^{\theta,\tau}(s,x) = E^x\left[\int_0^{\tau_S} e^{-\delta(s+t)}\frac{u^\lambda}{\lambda}Y_2(t)dt\right]$$

Then our generator becomes

$$\begin{aligned}
&A^\theta\varphi(s,y_1,y_2) = \frac{\partial\varphi}{\partial s} + (y_1\alpha - u)\frac{\partial\varphi}{\partial y_1} + y_2\theta_0\frac{\partial\varphi}{\partial y_2} + \frac{1}{2}y_1^2\beta^2\frac{\partial^2\varphi}{\partial^2 y_1}\\
&+ \frac{1}{2}y_2^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_2}\\
&+ \int_{\mathbb{R}}\left[\varphi(s,y_1+y_1\gamma(t,z),y_2-y_2\theta_1(t,z)) - \varphi(s,y_1,y_2) - y_1\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2}\right]v(dz).
\end{aligned}$$

and

$$\begin{aligned}
&A^\theta\varphi(s,y_1,y_2) + f(s,y_1,y_2) = \frac{\partial\varphi}{\partial s} + (y_1\alpha - u)\frac{\partial\varphi}{\partial y_1} + y_2\theta_0\frac{\partial\varphi}{\partial y_2} + \frac{1}{2}y_1^2\beta^2\frac{\partial^2\varphi}{\partial^2 y_1}\\
&+ \frac{1}{2}y_2^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_2}\\
&+ \int_{\mathbb{R}}\left[\varphi(s,y_1+y_1\gamma(t,z),y_2-y_2\theta_1(t,z)) - \varphi(s,y_1,y_2) - y_1\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2}\right]v(dz)\\
&+ e^{-\delta s}\frac{u^\lambda}{\lambda}y_2.
\end{aligned}$$

Imposing the first-order condition we get

$$\varphi_2 = \varphi(s,y_1+y_1\gamma(t,z),y_2-y_2\theta_1(t,z))_2,$$

and

$$\theta_0 = \frac{y_1}{y_2}\beta\frac{\varphi_{12}}{\varphi_{22}} - \frac{1}{y_2}\frac{\varphi_2}{\varphi_{22}}.$$

Then we get that

$$\hat{u} = (\frac{e^{\delta s}\varphi_1}{y_2})^{\frac{1}{\lambda-1}}.$$

so

$$\begin{aligned}
A^\theta\varphi(s,y_1,y_2) + f(s,y_1,y_2) &= \frac{\partial\varphi}{\partial s} + y_1(\alpha - (\frac{y_1 e^{-\delta s}}{y_2\varphi_1})^{\frac{1}{\lambda-1}})\frac{\partial\varphi}{\partial y_1} + y_1\beta\frac{\varphi_{12}\varphi_2}{\varphi_{22}} \\
&- \frac{\varphi_2^2}{\varphi_{22}} + \frac{1}{2}\frac{y_1^2}{y_2^2}\beta^2\frac{\varphi_{12}^2}{\varphi_2 2} - \frac{1}{2}\frac{\varphi_2^2}{y_2^2} + \frac{y_1}{y_2^2}\beta\varphi_2\varphi_{12} + \frac{1}{2}y_1^2\beta^2\frac{\partial^2\varphi}{\partial^2 y_1} - y_1^2\beta^2\frac{\varphi_{12}}{\varphi_{22}} - y_1\beta\frac{\varphi_2\varphi_{12}}{\varphi_{22}} \\
&+ \int_{\mathbb{R}}\left[\varphi(s,y_1+y_1\gamma(t,z),y_2-y_2\theta_1(t,z)) - \varphi(s,y_1,y_2) - y_1\gamma(t,z)\frac{\partial\varphi}{\partial y_1} + y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2}\right]v(dz) \\
&+ e^{-\delta s}\frac{(\frac{y_1 e^{-\delta s}}{y_2\varphi_1})^{\frac{\lambda}{\lambda-1}}}{\lambda}y_2.
\end{aligned}$$

Let us try a function

$$\varphi(s,y_1,y_2) = e^{-\delta s}y_1^\lambda F(y_2).$$

Then

$$\hat{u} = (F(y_2)\lambda)^{\frac{1}{\lambda-1}}y_1.$$

and

$$\theta_0 = \beta\frac{\lambda}{\lambda-1} - \frac{1}{\lambda-1}.$$

So

$$\begin{aligned}
A^\theta\varphi(s,y_1,y_2) + f(s,y_1,y_2) &= -\delta e^{-\delta s}y_1^\lambda F(y_2) + (y_1\alpha - (F(y_2)\lambda)^{\frac{1}{\lambda-1}}y_1)\lambda e^{-\delta s}y_1^{\lambda-1}F(y_2) \\
&+ y_2(\beta\frac{\lambda}{\lambda-1} - \frac{1}{\lambda-1})e^{-\delta s}y_1^\lambda F'(y_2) + \frac{1}{2}y_1^2\beta^2\lambda(\lambda-1)e^{-\delta s}y_1^{\lambda-2}F''(y_2) \\
&+ \frac{1}{2}y_2^2(\beta\frac{\lambda}{\lambda-1} - \frac{1}{\lambda-1})^2 e^{-\delta s}y_1^\lambda F(y_2) - y_1y_2\beta(\beta\frac{\lambda}{\lambda-1} - \frac{1}{\lambda-1})\lambda e^{-\delta s}y_1^{\lambda-1}F'(y_2) \\
&+ \int_{\mathbb{R}}\left[\lambda e^{-\delta s}y_1^\lambda F'(y_2) - e^{-\delta s}y_1^\lambda F(y_2) - y_1\gamma(t,z)\lambda e^{-\delta s}y_1^{\lambda-1}F(y_2) + y_2\theta_1(z)e^{-\delta s}y_1^\lambda F'(y_2)\right]v(dz) \\
&+ e^{-\delta s}\frac{(F(y_2)\lambda)^{\frac{\lambda}{\lambda-1}}}{\lambda}y_1^\lambda y_2.
\end{aligned}$$

so

$$
\begin{aligned}
&A^{\theta}\varphi(s,y_1,y_2)+f(s,y_1,y_2)=-\delta e^{-\delta s}F(y_2)+(\alpha-(F(y_2)\lambda)^{\frac{1}{\lambda-1}})\lambda e^{-\delta s}F(y_2)\\
&+y_2(\beta\frac{\lambda}{\lambda-1}-\frac{1}{\lambda-1})e^{-\delta s}F'(y_2)+\frac{1}{2}\beta^2\lambda(\lambda-1)e^{-\delta s}F''(y_2)\\
&+\frac{1}{2}y_2^2(\beta\frac{\lambda}{\lambda-1}-\frac{1}{\lambda-1})^2e^{-\delta s}F(y_2)-y_2\beta(\beta\frac{\lambda}{\lambda-1}-\frac{1}{\lambda-1})\lambda e^{-\delta s}F'(y_2)\\
&+\int_{\mathbb{R}}\left[\lambda e^{-\delta s}F'(y_2)-e^{-\delta s}F(y_2)-\gamma(t,z)\lambda e^{-\delta s}F(y_2)+y_2\theta_1(z)e^{-\delta s}F'(y_2)\right]v(dz)\\
&+e^{-\delta s}\frac{(F(y_2)\lambda)^{\frac{\lambda}{\lambda-1}}}{\lambda}y_2.
\end{aligned}
$$

Solving this integro-differential equation and requiring that $F > 0$we see that the requirements of the theorem are satisfied and we conclude that we have

$$\varphi(s,y_1,y_2)=\Phi(s,y_1,y_2).$$

EXAMPLE 7.3.3 (Optimal control and stopping in a Lévy -market). In this scenario, let us look at an investor who has invested in a risky-asset,Y_1 and wants to find the optimal time to sell where he would stress test over several scenarios by allowing the market to control a diffusion, Y_2 where the dynamics are given by

$$
\begin{aligned}
dY_0(t)&=dt; && Y_0(0)=s\in\mathbb{R}.\\
dY_1(t)&=dV^{\pi}(t)=Y_1(t)[r(t)dt+\beta dB(t)]\\
&\quad+Y_1(t^-)\int_{\mathbb{R}}\gamma(s,z)\tilde{N}(ds,dz); && Y_1(0)=y_1>0.\\
dY_2(t)&=-Y_2(t)\theta_0(t)dB(t)-Y_2(t)\int_{\mathbb{R}}\theta_1(s,z)\tilde{N}(ds,dz); && Y_2(0)=y_2>0.
\end{aligned}
$$

Solve

$$\Phi(s,x)=\sup_{\tau}\left[\inf_{\theta_0,\theta_1}J^{\theta,\tau}\right]$$

where

$$J^{\theta,\tau}(s,x)=E^x\left[e^{-\delta\tau}\lambda Y_1(\tau)Y_2(\tau)\right],$$

where $0<\lambda\leq 1$ and $(1-\lambda)$ is a percentage transaction cost. Then our generator becomes

$$
\begin{aligned}
&A^{\theta}\varphi(s,y_1,y_2)+f(s,y_1,y_2)=\frac{\partial\varphi}{\partial s}+y_1r\frac{\partial\varphi}{\partial y_1}+y_2\theta_0\frac{\partial\varphi}{\partial y_2}+\frac{1}{2}y_1^2\beta^2\frac{\partial^2\varphi}{\partial^2 y_1}\\
&+\frac{1}{2}y_2^2\theta_0^2\frac{\partial^2\varphi}{\partial^2 y_2}-y_1y_2\beta\theta_0\frac{\partial^2\varphi}{\partial y_1\partial y_2}\\
&+\int_{\mathbb{R}}\left[\varphi(s,y_1+y_1\gamma(t,z),y_2-y_2\theta_1(t,z))-\varphi(s,y_1,y_2)-y_1\gamma(t,z)\frac{\partial\varphi}{\partial y_1}+y_2\theta_1(z)\frac{\partial\varphi}{\partial y_2}\right]v(dz).
\end{aligned}
$$

Imposing the first-order condition we get

$$\varphi_2 = \varphi(s, y_1 + y_1\gamma(t,z), y_2 - y_2\theta_1(t,z))_2,$$

and

$$\theta_0 = \frac{y_1}{y_2}\beta\frac{\varphi_{12}}{\varphi_{22}} - \frac{1}{y_2}\frac{\varphi_2}{\varphi_{22}}.$$

We then have that

$$Ag = e^{-\delta s}y_1y_2\lambda(-\delta + r + \theta_0 - \beta\theta_0 + 1 - \int \gamma\theta_1\nu(dz)).$$

So we see that if

$$r + 1 + \theta_0(1-\beta) - \delta < \int \theta_1\gamma\nu(dz),$$

it is best to stop immediately and $\varphi = g$. Otherwise $U = \{(s, y_1, y_2)|Ag(s, y_1, y_2) > 0\} = [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \subset D$ and its never optimal to stop.

Example 7.3.4 (Optimal control and stopping in a Lévy -market). In this scenario, let us look at an investor who has invested in a risky-asset,Y_1 and wants to find the optimal time to sell where he would stress test over several scenarios by allowing the market to control a diffusion, Y_2 where the dynamics are given by

$$\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}.\\
&dY_1(t) = dV^\pi(t) = Y_1(t)[(r(t) + (\alpha(t) - r(t))\pi(t))dt + \beta\pi(t)dB(t)]; && Y_1(0) = y_1 > 0.\\
&dY_2(t) = -Y_2(t)\theta(t)dB(t); && Y_2(0) = y_2 > 0.
\end{aligned}$$

Solve

$$\Phi(s,x) = \sup_\pi\left[\sup_\tau\left(\inf_\theta J^{\pi,\theta,\tau}\right)\right]$$

where

$$J^{\pi,\theta,\tau}(s,x) = E^x\left[e^{-\delta\tau}\lambda Y_1(\tau)Y_2(\tau)\right],$$

where $0 < \lambda \leq 1$ and $(1-\lambda)$ is a percentage transaction cost. The generator is

$$\begin{aligned}
A^\theta\varphi(s,y_1,y_2) + f(s,y_1,y_2) = {} & \frac{\partial\varphi}{\partial s} + y_1(r + (\alpha - r)\pi)\frac{\partial\varphi}{\partial y_1} + y_2\theta\frac{\partial\varphi}{\partial y_2} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1}\\
&+ \frac{1}{2}y_2^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2}.
\end{aligned}$$

From the first order condition we get that

$$\hat{\pi} = \frac{(\alpha - r)\varphi_1 + y_1 y_2 \beta\theta\varphi_{12}}{y_1^2 \beta^2 \varphi_{11}},$$

and

$$\hat{\theta} = \frac{\varphi_{11}}{\varphi_{12}}(\varphi_{12} + y_2\varphi_{22}) + \frac{\alpha - r}{y_1 y_2 \beta}\varphi_1.$$

From Ag we set that we get

$$\hat{\pi} = 0,$$

and

$$\hat{\theta} = 0.$$

So if

$$r - \delta \leq 0,$$

the best thing is to stop immediately and $\varphi = g$. If

$$r - \delta > 0,$$

then

$$D = [0, T] \times \mathbb{R}^k \times \mathbb{R}^k,$$

so $\hat{\tau} = \infty$.

EXAMPLE 7.3.5. Now, let us look at an investor who has invested in a risky-asset,Y_1 and wants to find the optimal time to sell when he consumes, let the dynamics are given by

$$\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}.\\
&dY_1(t) = dV^\pi(t) = Y_1(t)[(\alpha(t) - c(t))dt + \beta dB(t)]; && Y_1(0) = y_1 > 0.\\
&dY_2(t) = -Y_2(t)\theta(t)dB(t); && Y_2(0) = y_2 > 0.
\end{aligned}$$

Solve

$$\Phi(s, x) = \sup_c \left[\sup_\tau \left(\inf_\theta J^{c,\theta,\tau}\right)\right]$$

where

$$J^{c,\theta,\tau}(s, x) = E^x\left[\int_0^\tau \lambda e^{-\delta t} c(t)dt + e^{-\delta\tau}\lambda Y_1(\tau)Y_2(\tau)\right],$$

where $0 < \lambda \le 1$ and $(1-\lambda)$ is a percentage transaction cost. The generator is

$$A^{\theta}\varphi(s,y_1,y_2) + f(s,y_1,y_2) = \frac{\partial\varphi}{\partial s} + y_1(\alpha-c)\frac{\partial\varphi}{\partial y_1} + y_2\theta\frac{\partial\varphi}{\partial y_2} + \frac{1}{2}y_1^2\beta^2\frac{\partial^2\varphi}{\partial^2 y_1}$$
$$+\frac{1}{2}y_2^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta\frac{\partial^2\varphi}{\partial y_1\partial y_2} + \lambda e^{-\delta s}c.$$

and we have that

$$\hat{\theta} = \frac{y_1y_2\beta\varphi_{12} - y_2\varphi_2}{y_2^2\varphi_{22}}.$$

and if $y_1\varphi_1 < \lambda e^{-\delta s}$, then $\hat{c} = y_1$ otherwise $\hat{c} = 0$.

From

$$Ag + f = y_1y_2\delta + y_1y_2(\alpha-c) + y_1^2\beta - y_1^2 + \frac{1}{2}y_1^2y_2^2\beta^2 - y_1^2\beta^2 + y_1^2\beta + c.$$

We see that

$$U = \{(s,y_1,y_2)|Ag+f>0\} = \begin{cases} [0,T]\times\mathbb{R}_+\times\mathbb{R}_+; & \delta + (\alpha-c) + \theta(1-\beta)) \ge 0. \\ 0 < y_1y_2 < \frac{-c}{(\delta+(\alpha-c)+\theta(1-\beta))}; & \text{otherwise.} \end{cases}$$

This gives us an indication about the our set D and φ. So we try a function on the form

$$\varphi(s,y_1,y_2) = e^{-\delta s}F(\omega);\ \text{where } \omega = y_1y_2.$$

Then we have that

$$A\varphi + f = -\delta F(\omega)e^{-\delta s} + \omega F'(\omega)e^{-\delta s}(\alpha - c + \theta) + \omega^2 e^{-\delta s}F''(\omega)(\frac{1}{2}(\beta^2+\theta^2) - \beta\theta) + \lambda e^{-\delta s}c.$$

Solving this (Euler)differential equation $AF = 0$ we get that

$$F(\omega) = C_1\omega^{\lambda_1} + C_2\omega^{\lambda_2} - \frac{\lambda c}{\delta},$$

where C_1, C_2 are constants and λ_1, λ_2 are solution to the equation

$$h(\lambda) = -\delta + \lambda(\alpha - c + \theta) + \lambda(\lambda-1)(\frac{1}{2}(\beta^2+\theta^2) - \beta\theta) = 0.$$

So we get that

$$F(\omega)\begin{cases} \lambda\omega & \text{if } \omega_0 < \omega, \\ C_1\omega^{\lambda_1} + C_2\omega^{\lambda_2} - \frac{\lambda c}{\delta}; & \text{otherwise.} \end{cases}$$

Let us assume that $C_2 = 0$. From the differentiability and continuity requirement we get

$$C_1 \omega_0^{\lambda_1} - \frac{\lambda c}{\delta} = \lambda \omega_0,$$

and

$$C_1 \lambda_1 \omega_0^{\lambda_1 - 1} = \lambda.$$

We find that

$$\omega_0 = \frac{c}{\delta} \frac{1}{1-\lambda},$$

and

$$C_1 = \lambda (\frac{c}{\delta} \frac{1}{1-\lambda})^{1-\lambda_1} + \frac{\lambda}{\delta} c (\frac{c}{\delta} \frac{1}{1-\lambda})^{-\lambda_1}.$$

It remains to verify that F satisfy our requirements.

Example 7.3.6 (Another example of Optimal Stopping and Control in a Black-Scholes Marked). Now we look at a non-jump dynamics given by

$$dY_0(t) = dt; \qquad Y_0(0) = s \in \mathbb{R}.$$

$$dY_1(t) = dV^{\pi}(t) = (Y_1(t)\alpha(t) - u(t))dt + \beta Y_1(t) dB(t)]; \qquad Y_1(0) = y_1 > 0.$$

Solve

$$\Phi(s, x) = \sup_{u,\tau} \left[\inf_{\alpha} J^{u,\theta,\tau} \right]$$

where

$$J^{u,\theta,\tau}(s, x) = E^x \left[\int_0^{\tau_S} e^{-\delta(s+t)} \frac{u^\gamma}{\gamma} dt + \lambda e^{-\delta(s+\tau)} Y_1^\gamma(\tau) \right].$$

Then the generator becomes

$$A^{u,\theta} \varphi(s, y_1) + f(s, y_1) = \frac{\partial \varphi}{\partial s} + (y_1 \alpha - u) \frac{\partial \varphi}{\partial y_1} + \frac{1}{2} y_1^2 \beta^2 \frac{\partial^2 \varphi}{\partial^2 y_1}$$

Imposing the first-order condition we get

$$\hat{u} = (\varphi_1 e^{\delta s})^{\frac{1}{\delta - 1}},$$

and

$$\hat{\alpha} = 0.$$

Let us try

$$\varphi(s, y_1) = Ke^{-\delta s} y_1^{\gamma}.$$

Then we get that

$$\hat{u} = (\gamma K)^{\frac{1}{\delta - 1}} y_1$$

So

$$-\delta K e^{-\delta s} y_1^{\gamma} - y_1 (\gamma K)^{\frac{1}{\gamma-1}} \gamma K e^{-\delta s} y_1^{\gamma-1} + \frac{1}{2}\beta^2 y_1^2 \gamma(\gamma-1) K e^{-\delta s} y_1^{\gamma-2} + (\gamma K)^{\frac{\gamma}{\gamma-1}} y_1^{\gamma} \frac{1}{\gamma}.$$

or

$$-\delta K e^{-\delta s} - (\gamma K)^{\frac{1}{\gamma-1}} \gamma K e^{-\delta s} + \frac{1}{2}\beta^2 \gamma(\gamma-1) K e^{-\delta s} + (\gamma K)^{\frac{\gamma}{\gamma-1}} \frac{1}{\gamma}.$$

We require that

$$-\delta K e^{-\delta s} - (\gamma K)^{\frac{1}{\gamma-1}} \gamma K e^{-\delta s} + \frac{1}{2}\beta^2 \gamma(\gamma-1) K e^{-\delta s} + e^{-\delta s} (\gamma K)^{\frac{\gamma}{\gamma-1}} \frac{1}{\gamma} = 0.$$

or

$$-\delta - \gamma^{\frac{\gamma}{\gamma-1}} K^{\frac{1}{\gamma-1}} + \frac{1}{2}\beta^2 \gamma(\gamma-1) + \gamma^{\frac{1}{\gamma-1}} K^{\frac{1}{\gamma-1}} = 0.$$

So

$$K = \frac{1}{\gamma}[\frac{1}{1-\gamma}(\delta - \frac{1}{2}\beta^2\gamma(\gamma-1))]^{\frac{1}{\gamma-1}}.$$

Assume $\delta - \frac{1}{2}\beta^2\gamma(\gamma-1)) > 0$, then $K > 0$. Now if we assume $\lambda \geq K$ we let

$$\varphi(s, y_1,) = \lambda e^{-\delta s} y_1^{\gamma}.$$

Then it is clear from (ii) that

$$\varphi(s, y_1) \geq \Phi(s, y_1).$$

But by choosing $\tau = 0$, we get $\varphi(s, y_1)$ so

$$\varphi(s, y_1) \leq \Phi(s, y_1, y_2).$$

and it follows that

$$\varphi(s, y_1) = \Phi(s, y_1),$$

$\hat{\tau} = 0$ and $D = \emptyset$.

If we now assume $\lambda < K$, let again

$$\varphi(s, y_1) = Ke^{-\delta s} y_1^{\gamma}.$$

then we always have that

$$\varphi(s, y_1) > \lambda e^{-\delta s} y_1^{\gamma}.$$

So $D = \mathbb{R} \times (0, \infty) \times (0, \infty)$ so we conclude that

$$\Phi(s, y_1) \leq \lambda e^{-\delta s} y_1^{\gamma}.$$

If we apply the control

$$\hat{u} = (\gamma K)^{\frac{1}{\delta - 1}} y_1,$$

and

$$\hat{\alpha} = 0.$$

then we get that $J^{\hat{u}, \hat{\alpha}} = K e^{-\delta s} y_1^{\gamma}$.

So

$$\Phi(s, y_1) = K e^{-\delta s} y_1^{\gamma},$$

and

$$\hat{\tau} = \infty.$$

CHAPTER 8

VISCOSITY SOLUTIONS FOR THE HJBI EQUATIONS

In this chapter we will investigate the cases where Φ is not smooth, i.e. not C^1. The assumption that Φ should be smooth is restrictive, so to find a rigorous assertion without the restrictive assumptions, Crandall and Lions introduced the viscosity solution. If this is the case Φ still satisfy the corresponding verification theorems if we consider this weak solution. In the cases of linear partial differential operators the given viscosity solution is the same as the classical solution.

8.1. Viscosity Solutions

We will now investigate the idea of viscosity solutions in our HJBI equations. Let our model be described by (4.1.1) where we try to solve problem 4.2.1. Let us define a viscosity solution as follows;

DEFINITION 8.1.1 (Modification of definition 5.1 in Yong and Zhou [1999]). A function $v \in C([0,T] \times \mathbb{R}^n)$ is called a viscosity subsolution of the HJB equation if

$$v(T,x) \leq g(x), \forall x \in \mathbb{R}^n, \tag{8.1.1}$$

and for any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, whenever $v-\varphi$ attains a local maximum at $(t,x) \in [0,T)\times\mathbb{R}^n$, we have

$$A^{\delta,\pi,\theta}\varphi(x) + f(x,\delta,\pi,\theta) \leq 0, \tag{8.1.2}$$

where A is as usual the generator of X. A function $v \in C([0,T]\times\mathbb{R}^n)$ is called a viscosity supersolution if

$$v(T,x) \geq g(x), \forall x \in \mathbb{R}^n, \tag{8.1.3}$$

and for any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, whenever $v-\varphi$ attains a local minimum at $(t,x) \in [0,T)\times\mathbb{R}^n$, we have

$$A^{\delta,\pi,\theta}\varphi(x) + f(x,\delta,\pi,\theta) \geq 0. \tag{8.1.4}$$

Further, if $v \in C([0,T]\times\mathbb{R}^n)$ is both a viscosity subsolution and a viscosity supersolution it is called a viscosity solution.

We now show that under some conditions Φ is a viscosity solution to the equation in Theorem 4.3.1;

THEOREM 8.1.1 (A viscosity solution theorem for HJBI). *If the set of admissible controls are compact and such that* (4.1.1) *admits a strong solution, the the value function* Φ *is a viscosity solution of problem 4.2.1.*

PROOF. To prove this we need a supporting theorem that gives us the following representation of the value function;

THEOREM 8.1.2 (Theorem 3.3 in Yong and Zhou [1999]).

$$\Phi(s,y) = \inf_{\theta,\delta}\left[\sup_{\pi} E[\int_s^{\hat{s}} f(t,X(t),u)dt + \Phi(\hat{s},X(\hat{s}))]\right] \textit{ for all } 0 \leq s \leq \hat{s} \leq T.$$

For any $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, assume $V-\varphi$ attains a local maximum at $(s,y) \in [0,T]\times\mathbb{R}^n$. Since U is closed, there is $\hat{u} = (\hat{\delta},\hat{\pi},\hat{\theta}) \in U$ such that $J^{\hat{\delta},\hat{\pi},\hat{\theta}}(s,y) = \Phi(s,y)$.

Let $\hat{s} > s$ and $\hat{u} \in U$. From theorem 8.1.2 we have that

$$\begin{aligned}
0 &\leq \frac{E\left[V(s,y) - \varphi(s,y) - (V(\hat{s}, X^{\hat{u}}(\hat{s})) - \varphi(\hat{s}, X^{\hat{u}}(\hat{s})))\right]}{\hat{s} - s} \\
&= \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} f(t, X^{\hat{u}}(t), \hat{u})dt + \varphi(\hat{s}, X(\hat{s})) - \varphi(s,y)\right] \\
&= \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} f(t, X^{\hat{u}}(t), \hat{u}) + \frac{d\varphi}{dt}(t, X^{\hat{u}}(t))dt\right].
\end{aligned}$$

We have from Itô

$$\begin{aligned}
d\varphi(t,x) &= \varphi_t(t,x)dt + \varphi_x(t,x)dX + \frac{1}{2}\varphi_{x^2}(t,x)(dX)^2 \\
&= \varphi_t(t,x)dt + \varphi_x(t,x)bdt + \varphi_x(t,x)\sigma dB_t + \frac{1}{2}\varphi_{x^2}(t,x)\sigma^2 dt \\
&= (\varphi_t(t,x)dt + \varphi_x(t,x)b + \frac{1}{2}\varphi_{x^2}(t,x)\sigma^2)dt + \varphi_x(t,x)\sigma dB_t
\end{aligned}$$

so

$$\begin{aligned}
&\frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} f(t, X^{\hat{u}}(t), \hat{u}) + \frac{d\varphi}{dt}(t, X^{\hat{u}}(t))dt\right] \\
&= \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} f(t, X^{\hat{u}}(t), \hat{u}) + A\varphi(t, X^{\hat{u}}(t))dt\right] \\
&\underset{\hat{s}\to s}{\longrightarrow} A\varphi(s, X^{\hat{u}}) + f(s, X^{\hat{u}}, \hat{u}).
\end{aligned}$$

Since thes holds for all $u \in U$, we conclude that

$$A\varphi(s, X^{\hat{u}}) + f(s, X^{\hat{u}}, \hat{u}) \geq 0. \tag{8.1.5}$$

To prove the opposite inequality, we assume $V - \varphi$ attains a local minimum at $(s,y) \in [0,T] \times \mathbb{R}^n$. For $\hat{s} > s$ we have

$$\begin{aligned}
0 &\geq \frac{1}{\hat{s}-s} E\left[V(s,y) - \varphi(s,y) - (V(\hat{s}, X(\hat{s})) - \varphi(\hat{s}, X(\hat{s})))\right] \\
&= \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} f(t, X^{\hat{u}}(t), \hat{u}) + A\varphi(t, X^{\hat{u}}(t))dt\right] \\
&\underset{\hat{s}\to s}{\longrightarrow} A\varphi(s, X^{\hat{u}}(s)) + f(s, X^{\hat{u}}(s), \hat{u}) \text{ for all } u \in U.
\end{aligned}$$

This leaves us with

$$A\varphi(s, X^{\hat{u}})(s) + f(s, X^{\hat{u}}, \hat{u}) \leq 0. \tag{8.1.6}$$

Combined with (8.1.5) we have that

$$A\varphi(s, X^{\hat{u}}) + f(s, X^{\hat{u}}, \hat{u}) = 0.$$

So Φ is a viscosity solution of theorem 4.3.1. $\square$

Remark 8.1.1. Notice that we have not shown uniqueness of the viscosity solution. This is vital as it is often used as a verification theorem. This uniqueness for the HJBI can be shown similar to Øksendal and Reikvam [1998].

8.2. Examples

We will now apply the result of the previous chapter to give a example of an optimal control problem where the value function Φ, is not everywhere C^1 so that we can not use the theorem of chapter 4. (Note that as we mentioned above we have not shown uniqueness of the viscosity solution.)

We consider the marked given by (2.3.2) and (2.3.3).

EXAMPLE 8.2.1 (A viscosity solution for a non-C^1 function). Let

$$f(x) = \begin{cases} -x, \text{ for } x \leq 0. \\ x, \text{ for } x > 0. \end{cases}$$

where $0 < \lambda < 1$, and

$$\begin{aligned}
&dY_0(t) = dt; && Y_0(0) = s \in \mathbb{R}. \\
&dY_1(t) = dV^\pi(t) = Y_1(t)[(r(t) + (\alpha(t) - r(t))\pi(t))dt + \beta\pi(t)dB(t)]; && Y_1(0) = y_1 > 0. \\
&dY_2(t) = -Y_2(t)\theta(t)dB(t); && Y_2(0) = y_2 > 0.
\end{aligned}$$

Then, let our control problem be

$$\Phi(y) = \Phi(s, y_1, y_2) = \inf_\alpha \left[\sup_\pi \left(\inf_\theta J^{\alpha,\pi,\theta}(s, y) \right) \right],$$

where

$$J^{\alpha,\pi,\theta}(s, y) = E^y \left[e^{-r(T-s)} f(Y_1(T)) Y_2(T) \right].$$

So, our generator becomes

(8.2.7)

$$A^{\alpha,\pi,\theta}\varphi(y) = \frac{\partial\varphi}{\partial s} + y_1(r + (\alpha - r)\pi)\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\theta^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1 y_2 \beta\theta\pi\frac{\partial^2\varphi}{\partial y_1 \partial y_2}.$$

Then we have that

$$\varphi(T, y_1, y_2) = \begin{cases} -y_1 y_2, \text{ for } y_1 \leq 0, \\ y_1 y_2, \text{ for } y_1 > 0. \end{cases}$$

Let π and α be fixed and minimize

$$f(\theta) := \frac{1}{2}y_2^2\theta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\theta\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2},$$

for functions $\theta(t,z)$. We minimize and find minimum

$$\hat{\theta} = \frac{y_1}{y_2}\frac{\beta}{\pi}\frac{\varphi_{12}}{\varphi_{22}}.$$

when $\varphi_{22} \neq 0$. We then use

$$g(\pi) := y_1(\alpha - r)\pi\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\hat{\theta}^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\hat{\theta}\pi\frac{\partial^2\varphi}{\partial y_1\partial y_2},$$

to maximize over π. So we get

$$\hat{\pi} = \frac{r-\alpha}{y_1\beta^2}\frac{\varphi_1}{\varphi_{11}}.$$

Further, for α, we let

$$l(\alpha) = \frac{1}{\beta^2}y_1^2(r-\alpha)^2(\frac{1}{2}y_1^2 - 1)\frac{\varphi_1^2}{\varphi_{11}}$$

and find

$$\hat{\alpha} = r.$$

So

$$\hat{\pi} = 0.$$

Then we have an optimal triple $(\hat{\alpha}, \hat{\pi}, \hat{\theta})$ which is substituted into (8.2.7) to give

$$\frac{\partial\varphi}{\partial s} + y_1(r + (\hat{\alpha} - r)\hat{\pi})\frac{\partial\varphi}{\partial y_1} + \frac{1}{2}y_1^2\beta^2\hat{\pi}^2\frac{\partial^2\varphi}{\partial^2 y_1} + \frac{1}{2}y_2^2\hat{\theta}^2\pi^2\frac{\partial^2\varphi}{\partial^2 y_2} - y_1y_2\beta\hat{\theta}\hat{\pi}\frac{\partial^2\varphi}{\partial y_1\partial y_2}$$

This gives

$$\frac{\partial\varphi}{\partial s} + y_1 r\frac{\partial\varphi}{\partial y_1} = 0. \tag{8.2.8}$$

Lets try a function

$$\varphi(s, y_1, y_2) = e^{r(T-s)}y_1y_2, \text{ for } 0 < y_1.$$

Then we have that

$$-re^{-r(T-s)}\phi + y_1re^{-r(T-s)}\frac{\partial\phi}{\partial y_1} = 0.$$

We then let

$$\varphi(s, y_1, y_2) = \begin{cases} e^{-r(T-s)} y_1 y_2, \text{ for } y_1 \leq 0. \\ e^{r(T-s)} y_1 y_2, \text{ for } 1 > y_1 > 0. \end{cases}$$

Then $\varphi(y)$ satisfy equation (8.2.8). This makes sense considering we only invest in the risk free asset. We then verify that φ is a viscosity solution. Let $h \in C^2(\mathbb{R})$, $h \geq \varphi$ and $h(y_0) = \varphi(y_0)$. Then φ is C^2 between 0 and 1, so it has a local minimum, so

$$D(h - \varphi)(y_0) = 0,$$

where D is the differential operator. Because of the linearity of D we get that

$$Dh(y_0) - D\varphi(y_0) = 0,$$

so

$$-rh(y_0) + y_1 r \frac{\partial h}{\partial y_1}(y_0) \geq -r\phi(y_0) + y_1 r \frac{\partial \varphi}{\partial y_1}(y_0) = 0. \tag{8.2.9}$$

and

$$A^{\hat{\alpha}, \hat{\pi}, \hat{\theta}} h(y_0) \geq A^{\hat{\alpha}, \hat{\pi}, \hat{\theta}} \varphi(y_0) = 0.$$

so φ is a viscosity subsolution. The same approach applies to proving that φ is a viscosity supersolution and hence a viscosity solution.

Part 3

CONCLUSION, DISCUSSION AND FURTHER REASEARCH

CHAPTER 9

DISCUSSION

9.1. Summary and Conclusions

The main purpose of this paper was to develop HJBI equations for multidimensional optimization problems that cohere to the theory of risk measures. We went through the pros and cons of the widely used risk measure VaR. As a consequence of the unstructured approach to most risk measures we concluded that there is a need for a solid theory on risk measures. It was made clear that it was and still is, need for more research on the subject. We then went through the existing work on risk measures in chapter 3. The extension from coherent risk measures to convex risk measures were discussed. These risk measures were used as a starting point for our model. In consummation we arrived at the problem of finding a value functional, $\Phi(y)$, and controls, $(\delta^*, \pi^*, \theta^*) \in \Delta \times \Pi \times \Theta$, such that

$$\Phi(y) = \inf_{\delta} \left[\sup_{\pi} \left(\inf_{\theta} J^{\delta,\pi,\theta}(y) \right) \right],$$

where our performance functional is defined as;

$$J^{\delta,\pi,\theta}(y) = E^y \left[\int_0^{\tau_s} f(X(t), u_0(t)) + g(X(\tau_s)) \right].$$

The theory of risk measures provided us with a foundation for our model; it needed to be valid for risk measures. Incorporating risk measures gives a solid and rigorous foundation that is in accordance with established financial theory. This enabled us to construct and prove several HJBI equations, both for convex risk measures and the extension to dynamic risk measures. We proved a general HJBI for a 3 dimensional game which included the convex risk measure case. One can interpret this by thinking of an investor as player 1 and the market as player 2. The market controls both the scenarios and a marked variable. This is proved to be equal to a 2 dimensional game with a 2 dimensional controller as is the case in Mataramvura and Øksendal [2008]. Next, we found and proved a HJBI equation for a Nash-equilibrium. We extended on Mataramvura and Øksendal [2008] by allowing the market to play a role through scenarios. As we explained, most risk management decisions spans over a several time periods. So we provided a model for dynamic risk measures. This gave us a HJBI equation that is linked to a FBSDE.

Choosing a time to maximize an expected reward or minimize an expected cost is known as an optimal stopping problem. To incorporate this problem into our model, we did provide an optimal control and stopping theorem. We showed that the optimal control and stopping problem could be written in the form of a Bellman equation, and is therefore solved using dynamic programming. This was not generalized to dynamic risk measures but adheres to both coherent and convex risk measures.

Last, as the value function Φ is not always smooth, we introduced viscosity solutions where our function need not to be everywhere differentiable. As the differential may not exist at some points, the superdifferential and the subdifferential were defined. We then proved that the value function was a viscosity solution of the HJBI equation in theorem 4.3.1.

9.2. Discussion

In our dynamic setting we constructed a performance functional by taking the expectation of an integral over a dynamic risk measure. This provided us with the following performance functional:

$$J^{\pi,\theta}(x) := E_Q^x\left[\int_0^T -\rho_t(X^{\hat{\pi},\hat{\theta}})dt\right] := E_Q^x\left[\int_0^T \epsilon_g(X^{\pi}(T)|\mathcal{F}_t]dt\right] := E_Q^x\left[\int_0^T y_t dt\right].$$

By constructing the performance functional this way we remove some of the dynamics and loose some information on the risks involved. Also, our theorem requires us to find a function, v, for each controller. This is an unpractical theorem in applications and its usefulness is questionable.

9.3. Topics for Further Research

1. [**Alternative representation for the dynamic risk measure model**] Our performance functional removes some of the dynamics of the risk measure. One possibility for further study is to construct a performance functional as

   $$\int_0^T \rho_t(X(T))dt.$$

   This gives us a stochastic function whose interpretation is not so obvious. It also requires a stochastic version of the HJBI equation.

2. [**A simpler verification theorem for dynamic risk measures**] We developed a HJBI which would require us to find a function, v, for each controller. This gives a complex system to solve and a simpler model is a possibility for further research.
3. [**Including optimal stopping in the Nash-equilibrium model**] Optimal stopping is a variant of stochastic games where two players may stop a randomly moving process. The consequence of their actions is that, whoever stops first, player 1 will receive a predefined function of the random process at the time of stopping which is to be paid for by player 2. Extending our zero-sum game model for optimal stopping to Nash-equilibrium games could be a valuable extension that could justify further study.
4. [**A model with random jump fields**] We could try to extend our model to the case where the dynamics is depended not on only on time but also some other space variable, i.e. we get a partial differential equation $dY(t,x)$. While it may be possible to solve using dynamic programming, an approach could be to formulate a maximum principle, see below.
5. [**The zero-sum game with maximum principle**] A useful alternative to the dynamic programming verification techniques we have studied in this paper is the maximum principle. In the article by Framstad et al. [2004] it is remarked that the HJB-equation in the jump diffusion case involves complicated integro-differential equations. Therefore they provide a maximum principle alternative. It seems to be attainable to prove a similar result for the games we have studied in this paper. In theorem 2.1 in Framstad et al. [2004] we can exchange the requirement

$$H(t,\hat{X(t)},\hat{u(t)},\hat{p(t)},\hat{q(t)},\hat{r(t)}) = \sup_u H(t,\hat{X(t)},u(t),\hat{p(t)},\hat{q(t)},\hat{r(t)}),$$

with

$$H(t,\hat{X(t)},\hat{\alpha},\hat{\pi},\hat{\theta},\hat{p(t)},\hat{q(t)},\hat{r(t)}) = \inf_\alpha \left[\sup_\pi \left(\inf_\theta H(t,\hat{X(t)},u(t),\hat{p(t)},\hat{q(t)},\hat{r(t)})\right)\right].$$

This would still lead to a similar proof.
6. [**Uniqueness of the viscosity solution**] As noted above we have not shown an uniqueness theorem for the viscosity solution. This is an important concept in the case where we want to verify that a function is a viscosity solution of the corresponding variational inequalities. It seems provable from similar theorems that

THEOREM 9.3.1 (Uniqueness). *Suppose that*

$$\tau_S^0 < \infty \text{ a.s. for all } y \in S^0.$$

Let $\varphi \in C(\bar{S})$ be a viscosity solution of the HJBI with the property that

$$\text{the family } \{\varphi(Y(\tau)) | \tau \leq \tau_S^0\} \text{ is uniformly integrable for all } y \in S^0.$$

Then

$$\varphi(y) = \Phi(y) \text{ for all } y \in \bar{S}.$$

This would require further studies.

7. [**A model for singular control**] To incorporate transaction costs to our model we could look at singular control. If we let $\kappa = [\kappa_{ij}] : \mathbb{R}^k \to \mathbb{R}^{k\times p}$ and the state described by

$$\begin{aligned} dY^u(t) &= b(Y^{u(t)}, u(t))dt + \sigma(Y^{u(t)}, u(t))dB(t) \\ &\quad + \int_{\mathbb{R}^k} \gamma(Y^{u(t^-)}, u(t^-), z)\tilde{N}(dt, dz) + \kappa(Y(t^-))d\xi(t). \\ Y^{u(0^-)} &= y \in \mathbb{R} \end{aligned}$$

Further, we let the performance functional be given by

$$J^{\pi,\xi,\theta} = E^y[\int_0^{T_R} f(Y(t), u(t))dt + g(Y(T_R))\chi_{\{\tau_S<\infty\}} + \sum_{j=1}^{p}\int_0^{T_R} \vartheta_j^T(Y(t^-))d\xi(t)]$$

Then we want to find a value function $\Phi(y)$ and an optimal control $(\pi, u, \xi) \in \mathcal{A}$ such that

$$\Phi(y) = \sup_u \left[\sup_\pi \left(\inf_\theta J^{\pi,\xi,\theta}\right)\right] = J^{\hat{\pi},\hat{\xi},\hat{\theta}}.$$

This control is called singular since the investment control measure $d\xi(t)$ is allowed to be singular with respect to the Lebegue measure dt. We then want a verification theorem similar to the ones we have developed above. From Øksendal and Sulem [2007] we derive that our theorem should be constructed in a sounding similar to: Suppose we can find a function $\varphi \in C^2(\mathcal{S}) \cap C(\mathbb{R}^k)$ such that

(i) $A^{\pi,\hat{\theta}}\varphi(y) + f(y,u) = 0$ for all $y \in \mathcal{S}$ and all $\pi \in \Pi$.

(ii) $A^{\hat{\pi},\hat{\theta}}\varphi(y) + f(y,u) = 0$ for all $y \in \mathcal{S}$.

(iii) $\sum_{i=1}^k \kappa_{ij}(y)\frac{\partial\varphi}{\partial y_i}(y) + \vartheta_j(y) \leq 0$, ll $y \in \mathcal{S}, j+1,\ldots,p$.

(iv) $E^y[\int_0^{\tau_S}\{|\sigma^T(Y(t),u(t))\nabla\varphi(Y(t))|^2$
$+\sum_{k=1}^l \int_{\mathbb{R}} |\varphi(Y(t) + \gamma^{(k)}(Y(t),u(t),z)) - \varphi(Y(t))|^2\nu_k(dz)\}dt] < \infty$ for all $(u,\xi) \in \mathcal{A}$.

(v) $\lim_{t\to\tau_s^-} \varphi(X^{\delta,\pi,\theta}(t)) = g(X^{\delta,\pi,\theta}(\tau_S))\chi_{\{\tau_s<\infty\}}$ a.s. for all $(u,\xi) \in \mathcal{A}$.

(vi) The family $\{\varphi(X^{\delta,\pi,\theta}(\tau))\}_{\tau\in\mathcal{T}}$ is uniformly integrable, for all $(u,\xi) \in \mathcal{A}$ and all $y \in \mathcal{S}$.

Define the nonintervention region D by

$$D = \{y \in \mathcal{S} | \max_{1\leq j\leq p}\{\sum_{i=1}^{k} \kappa_{ij}(y)\frac{\partial\varphi}{\partial y_i}(y) + \vartheta_j(y)\} < 0\}.$$

(vii) $Y^{\hat{u},\hat{\xi}}(t) \in \bar{D}$ for all t.

(viii) $\sum_{j=1}^p\{\sum_{i=1}^k \kappa_{ij}(y)\frac{\partial\varphi}{\partial y_i}(y) + \vartheta_j(y)\}d\bar{\xi}_j^(c) = 0$, for all $1 \leq j \leq p$, where $\xi^(c)$ is the continuous part of ξ.

(ix) $\Delta_{\hat{\xi}}\varphi(Y(t_n)) + \sum_{j=0}^{p} \vartheta_j(Y(t_n^-))\Delta\hat{\xi}_j(t_n) = 0$ for all jumping times t_n of $\hat{\xi}(t)$.

(x) $\lim_{R\to\infty} E^y[\varphi(Y^{\hat{u},\hat{\xi}}(T_R))] = E^y[g(Y^{\hat{u},\hat{\xi}}(T_R))\chi_{\tau_S<\infty}]$ where $T_R = \min(\tau_S, R)$ for $R < \infty$.

Then

$$\varphi(y) = \Phi(y),$$

and

$$(\hat{u}, \hat{\xi}) \text{ is an optimal control}$$

This would seem to be a simple extension to the case in chapter 5 of Øksendal and Sulem [2007], and the proof would be a combination of our above methods and the proof in Øksendal and Sulem [2007].

In the the proof in Øksendal and Sulem [2007], we have that by using Itô for semi-martingales we get

$$\begin{aligned}
E^y[\varphi(Y(T_R))] = \varphi(y) + E^y\Big[&\int_0^{T_R} A^u\varphi(Y(t))dt \\
&+ \int_0^{T_R} \sum_{i=1}^{k} \frac{\partial\varphi}{\partial y_i}(Y(t^-)) \sum_{j=1}^{p} \kappa_{ij}(Y(t^-))d\xi_j^{(c)}(t) \\
&+ \sum_{0<t_n\le T_R} \Delta_\xi\varphi(Y(t_n))\Big].
\end{aligned}$$

By the mean value theorem

$$\Delta_\xi\varphi(Y(t_n)) = \nabla\varphi(\hat{Y}^{(n)})^T\Delta_\xi Y(t_n) = \sum_{i=1}^{k}\sum_{j=1}^{p} \frac{\partial\varphi}{\partial y_i}(Y(t^-))\kappa_{ij}(Y(t^-))\Delta\xi_j(t_n),$$

So we have

$$\begin{aligned}
\varphi(y) \ge E^y[&\int_0^{T_R} f(Y(t), u(t))dt + \varphi(Y(T_R)) \\
&- \sum_{i=1}^{k}\sum_{j=1}^{p}\{\int_0^{T_R} \frac{\partial\varphi}{\partial y_i}(Y(t^-))\kappa_{ij}(Y(t^-))d\xi_j^{(c)}(t) \\
&+ \frac{\partial\varphi}{\partial y_i}(Y(t^-))\kappa_{ij}(Y(t^-))\Delta\xi_j(t_n)\}] \\
\ge E^y[&\int_0^{T_R} f(Y(t), u(t))dt + \varphi(Y(T_R)) + \sum_{j=1}^{p}\int_0^{T_R} \vartheta_j(Y(t^-))d\xi_j(t)].
\end{aligned}$$

Letting $R \to \infty$

$$\varphi(y) \ge J^{u,\xi}(y) \ge \inf_\theta J^{u,\xi}(y).$$

Since this holds for all $\pi \in \Pi$

$$\varphi(y) \ge \sup_\pi \left[\inf_\theta J^{u,\xi}(y)\right].$$

Proving the opposite inequality seems to be similar to the proofs we have shown before, but further research is required.

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