The Price and Risk of Guaranteed Interest Rate Products

by

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Abstract

This paper examines the utility indifference price of interest rate products and the risk associated with these. Such products can be compared with put options and are here considered to be written on a non-tradeable asset which can be hedged with a correlated asset. Initially, we look at the case where both the tradeable and non-tradeable assets can be modeled by two geometric Brownian motions. This model is later extended to the case where it is assumed that the tradeable asset follows a Lévy process.

The paper is based on the article 'Utility indifference pricing of interest-rate guarantees' by Fred Espen Benth and Frank Proske, but is meant to be an independent paper. The definitions of the utility indifference price and the residual risk remaining after hedging are the same as in their paper.

The residual risk is measured with several different risk measures such as Value at Risk, Conditional Value at Risk and Expected Shortfall. These measures, with others, are closely examined and evaluated.

Numerical examples are included showing that the utility indifference price is lower for negative correlation than for positive and that the price can be even lower if the tradeable asset follows a Lévy process. Thus, if e.g. life companies can hedge in assets allowing jumps, and that are negatively correlated with their pension fund, they may offer lower prices with practically unaltered measures of risk.

Analysis of the pricing and hedging of interest rate guarantees are not only relevant for life companies, but also for other financial institutions offering investment products where there is a guaranteed least rate of return.

Preface

This thesis has been prepared in partial fulfillment of the requirements for my master degree at the Department of Mathematics, Faculty of Mathematics and Natural Science at the University of Oslo.

Professor Fred Espen Benth, University of Oslo, has been thesis supervisor while Professor Frank Proske, University of Oslo, has been co-supervisor.
The thesis was written over three months in the spring term of 2009 and was mainly produced independently.

There is a certain number of people I wish to thank in relation to my thesis. In particular, I would like to thank my thesis supervisor Professor Fred Espen Benth for an interesting and relatively applicable subject and for useful comments. I would also like to thank co-supervisor Professor Frank Proske for always showing a sincere interest in my work and for his help with various issues. Their theoretical as well as applied knowledge has generated inspiring and helpful discussions.

Lastly, I would like to thank my fellow students and friends for good spirit and friendly atmosphere.

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Ørjan Mekkalgården Ressem

Contents

Introduction

I will in this paper look at the risk associated with interest rate products when priced with the indifference utility pricing method. This method is a tool to be used if it is dubious to assume that a perfect hedge can be achieved and we need to utilize sub-optimal replication strategies. In other words, this price can be viewed as a substitute for the standard Black & Scholes framework. Reasons for why the Black & Scholes framework can be inadequate is e.g. illiquidity or that contracts simply are not long enough compared to the time perspective of a pension fund.

Life insurance companies often offer pension saving deals with a guaranteed least rate of return to their clients. Of course, the companies aim for a higher rate of return, while the customers are protected against low returns. This offer is equivalent to issuing a put option with strike dependent on the guaranteed rate of return and the time perspective. By issuing such a put option, the companies undertake a risk of having to cover the loss should their investments fail to achieve the guaranteed rate of return.

To take this risk, the the issuer needs to be compensated. This compensation is here decided by the utility indifference price. The utility indifference price is defined at the level where the issuer of an option is indifferent between entering the market by its own or issuing the option and entering the market with the collected premium. These two optimal investment problems are solved using stochastic control theory and the difference between them gives the utility based hedging strategy. The construction and notation will be recognizable with the one in Benth and Proske[6].

Choosing the 'best' model is always a difficult choice. The optimal model is a model describing the reality close to perfect and, at the same time, has few and easy-to-get parameters. This being said, we will in the earlier chapters use a model with as few parameters as possible in this context, and then try to expand it to better fit the reality in the later chapters. In detail the paper will progress as follows.

The first chapter will introduce a model for the financial market of which we are operating in and then use it to define the utility indifference price of a put option and the risk of issuing these. It is all dependent on the risk aversion of the issuer. The lowest price of which a life company is willing to issue such guarantees is obtained when the issuer is indifferent to risk.

In the second chapter, I will look at different risk measures, including Value at Risk, Conditional Value at Risk and Expected Shortfall. I find it difficult discussing the topic of risk without mentioning the highly interesting changes being made in the Basel (worldwide) and Solvency (EU) frameworks, namely the newly implemented Basel II and the soon-to-be implemented Solvency II directives. These will be mentioned briefly.

Chapter three will be a continuation of both chapter one and chapter two and will contain a risk analysis of the option defined in chapter one.

In chapter four the model in chapter one will extended to include the possibility of investing in a stock behaving like a Lévy process. The implied changes in price and risk will be studied.

The fifth chapter will contain a conclusion of the findings in the paper together with a survey of some of the limitations of the model and some possible improvements and extensions.

Assumptions

We are assuming an incomplete market, meaning that we cannot hedge any claims perfectly. In other words, there exist no explicit arbitrage-free price, but rather a continuum of such. With the indifference utility pricing method we will find the price within this continuum that is most profitable for a certain agent, i.e. has the highest utility for a given company or investor.

Further, this is a situation where the life company can only use a part of the fund for hedging. Therefor, the fund is interpreted as a non-tradeable asset, whereas the part that can be traded is modeled as a separate correlated asset.

1 The Model

1.1 Defining the model

In this paper, as in Benth and Proske[6], we are assuming a complete probability space

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$$

supporting two independent Brownian motions B and W. Our algebra of information $\mathcal{F}_t$ is assumed to fulfill 'the usual conditions', namely that $\mathcal{F}_t$ is right continuous and that $\mathcal{F}_0$ contains all the null sets[1].

We consider a market model consisting of a bank account with deterministic return (a risk free asset), a stock driven by B (a risky asset) and a fund driven by the stock and W (an untradeable asset). The extent of which the fund is driven by both the stock and the Brownian motion W is determined by the correlation parameter, $\rho \in (-1, 1)$. Note that in order to use the utility indifference pricing method, we have to assume that our market is incomplete, meaning that all claims can not be perfectly hedged, i.e. $|\rho| \neq 1$. Why this has to hold will be revealed shortly, when we define the utility indifference price.

We assume that the value of the bank account at time t, S_t^0, the value of the stock at time t, S_t, and the value of the fund at time t, Y_t[2], have the following dynamics.

$$\begin{aligned}
\frac{dS_t^0}{S_t^0} &= rdt \\
\frac{dS_t}{S_t} &= \mu dt + \sigma dB_t \\
\frac{dY_t}{Y_t} &= \nu dt + \eta\left(\rho dB_t + \sqrt{1-\rho^2}dW_t\right)
\end{aligned}$$

The parameters, r, μ, ν, σ and η are real constants and the two last ones are positive, by definition since they are volatilities. The three first ones will also be positive in any normal market, but there are situations in which at least μ and ν could be negative; for instance the financial crisis we just witnessed. It is also possible for r to become negative, but this is highly unlikely and belongs only in the field of crisis modeling. This paper will at all times use a positive set of parameters to fit a healthy market.

On the upside, this choice of model assures us that both the stock and the fund is positive at all times. The downside, of course, is that not only will the stock and the fund be positive at all times, they will also be strictly positive at all times. While the former is a property corresponding to the real world, the latter is not; Companies go bankrupt all the time, causing their stocks to loose all

[1] That is if $B \subset A \in \mathcal{F}_0$ with $P(A) = 0$, then $B \in \mathcal{F}_0$

[2] It is shown in Benth and Proske that Y_t is a Markov process

value and never recover. This could be modeled by introducing a low threshold b saying that if $S_t < b$, then $S_t = 0$ and the company would be bankrupt. The threshold should be chosen such that the probability of S_t assuming a value below b, $P(S_t < b)$, corresponds to the real market. However, this is not within the topic of this paper, and will not be treated further.

As mentioned in the introduction, the pricing of the fund will be equivalent with the pricing of an European put option with exercise time T and strike $K = Y_0 e^{gT}$, where g is the guaranteed rate of return of the portfolio. We consider the case when an investor is short λ such put options and we let θ_t be the cash amount invested in S_t while the remaining cash or wealth is invested in the risk free asset, S_t^0. This gives us the wealth portfolio dynamic:

$$dX_t^{\lambda,\theta_t} = \theta_t \frac{dS_t}{S_t} + r(X_t^{\lambda,\theta_t} - \theta_t)dt \tag{1}$$

where the trading strategy θ_t is admissible when the equation above has a unique strong solution[3], X_t^{λ,θ_t}, for $t \in [0,T]$ and

$$\mathbb{E}[-U(X_T^{\lambda,\theta_t})] < \infty \tag{2}$$

Here U is the utility function our indifference pricing is based on, and is of the form $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, where $\gamma > 0$ is the risk aversion of the company. Small γ implies low risk aversion, meaning that the company is willing to take on more risk than if they had a higher γ. At first, one might think that this risk aversion parameter is hard to determine, but, as found in Benth and Proske[6], this parameter can easily be determined from past investments of the company. This is also mentioned in Benth et al.[7]

As shown in Benth and Proske[6] and the references therein, we can express the utility indifference price as

Definition 1.1.

$$p_\lambda^\gamma(t,y) = e^{-r(T-t)} \frac{\ln w(t,y)}{\gamma(1-\rho^2)} \tag{3}$$

Here the function w is of the form

$$w(t,y) = \mathbb{E}_{Q^0}\left[\exp\left\{\lambda\gamma(1-\rho^2)(K-Y_T)^+\right\}|Y_t = y\right] \tag{4}$$

By using that Y_t is a Markov process under Q^0, we can redefine equation (4) to the following equation

$$w(t,y) = \mathbb{E}^0\left[\exp\left\{\lambda\gamma(1-\rho^2)(K-Y_T^{t,y}))^+\right\}\right] \tag{5}$$

[3]As defined in page 72 of Øksendal[2]

where

$$Y_T^{t,y} \quad = \quad y \cdot \exp\left\{(\delta - \frac{1}{2}\eta^2)(T-t) + \eta\left(\rho(B_T^0 - B_t^0) + \sqrt{1-\rho^2}(W_T - W_t)\right)\right\}$$

This expectation is under the minimal martingale measure Q^0, under which (B^0, W) are two independent Brownian motions, with $dB_t^0 = dB_t + \frac{\mu - r}{\sigma}dt$. The fact that B^0 is a Brownian motion under Q^0 follows from the Girsanov Theorem. Further, the Q^0 dynamics of S_t and Y_t becomes

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma dW_t^0 \\ \frac{dY_t}{Y_t} &= \delta dt + \eta\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right) \end{aligned}$$

for $\delta = \nu - \eta\rho\frac{\mu - r}{\sigma}$.
One should note that since the fund, Y_t, is not tradeable, our market is not complete. Hence the utility indifference price, when ρ is tending to ± 1, will in general not be equal to the Black and Scholes price.

1.1.1 The utility indifference price under a general martingale measure

Since companies often have a risk aversion not tending to zero, we have to consider the price under some Equivalent Martingale Measure (EMM) Q_λ^γ. We note that this measure is dependent both on the number of put options issued, λ, and the risk aversion factor γ, as it should since the indifference price is nonlinear in both. It is shown in Theorem 3.2 in Benth and Proske[6] that such an EMM exists, making the utility indifference price p_λ^γ arbitrage free and stating the explicit form of the EMM yielding the utility indifference price. The theorem states the following:

Theorem 1.2. *There exists an equivalent martingale measure Q_λ^γ such that*

$$p_\lambda^\gamma(t,y) \quad = \quad e^{-r(T-t)}E_{Q_\lambda^\gamma}\left[\lambda(K - Y(T))^+ | Y(t) = y\right] \tag{6}$$

Moreover, the Q_λ^γ-dynamics of Y_t and S_t are given by

$$\frac{dS_t}{S_t} \quad = \quad rdt + \sigma dB^0 \tag{7}$$

$$\frac{dY_t}{Y_t} \quad = \quad \delta^\gamma(t, Y(t))dt + \eta\left(\rho dB^0 + \sqrt{1-\rho^2}dW_\lambda^\gamma\right) \tag{8}$$

where (B^0, W_λ^γ) are two independent Brownian motions under Q_λ^γ, with

$$dW_\lambda^\gamma(t) \quad = \quad dW(t) - \frac{1}{2}\eta\gamma\sqrt{1-\rho^2}e^{r(T-t)}Y(t)\partial_y p_\lambda^\gamma(t, Y(t))dt \tag{9}$$

Finally,

$$\delta^\gamma(t,y) \quad = \quad \delta + \frac{1}{2}\eta^2\gamma(1-\rho^2)e^{r(T-t)}y\partial_y p_\lambda^\gamma(t,y) \tag{10}$$

Proof. The proof is carried out in full by Benth and Proske showing that Q_λ^γ is a probability measure and then showing that this implies that the representation of the price as a conditional expectation holds. $\square$

The indifference pricing measure defined above can be said to be risk neutral in some sense, since the discounted stock price process $e^{-rt}S_t$ is a martingale under it. It is also important to note that even if the utility indifference price is arbitrage free, the market need not be. This is because the price is dependent on the risk aversion, thus making a price that is fair for one trader with risk aversion γ_1 and that is an arbitrage opportunity for another with risk aversion γ_2.

Benth and Proske[6] also notes that $Q_\lambda^\gamma \to Q^0$ when $\gamma \to 0$ and that

$$\lim_{\gamma \to 0} p_\lambda^\gamma = p_\lambda^0$$

which are important points saying that the lowest indifference price is obtained when the trader has zero risk aversion. This is perfectly intuitive and easy to verify.

1.1.2 The residual risk

The residual risk is the risk the company is left with after hedging. When we, as in Benth and Proske[6], define the hedge, H_λ^γ, as

$$H_\lambda^\gamma = X_\lambda^{\theta^\lambda} - X_0^{\theta^0} \tag{11}$$

the residual risk, $\mathcal{R}_\lambda^\gamma$, becomes

$$\mathcal{R}_\lambda^\gamma = X_\lambda^{\theta^\lambda}(T) - X_0^{\theta^0}(T) - \lambda(K - Y_T)^+$$

that is the payoff of the put option(s) subtracted from the hedge at terminal time. Since $X_\lambda^{\theta^\lambda}$ and $X_0^{\theta^0}$ is the wealth portfolio when issuing λ and 0 put options, respectively, the risk can be construed as the difference between the wealth portfolio when issuing λ put options subtracted the value of λ put options and the value of the wealth portfolio if we did not issue any options. It is shown in section four of Benth and Proske that the residual risk can also be interpreted as the cumulative value of the perfect hedge[4] with respect to a residual risk process. In other words, it can be represented by

$$\mathcal{R}_\lambda^\gamma = \int_0^T e^{r(T-t)} Y_t \partial_y p_\lambda^\gamma(t, Y_t) dR_t \tag{12}$$

Here, dR_t is a residual risk process given by

$$dR_t = \frac{\eta\rho}{\sigma}\left(\frac{dS_t}{S_t} - rdt\right) - \left(\frac{dY_t}{Y_t} - \delta^\gamma(t, Y_t)dt\right) \tag{13}$$

[4] A perfect hedge, a hedge that would eliminate all risk, is only possible in our risk neutral world Q_λ^γ of a complete market

where dS_t and dY_t is the Q_λ^γ-dynamics given by equations (7) and (8).

Further, we note that the integrand in equation (12) is the value of the cash amount invested in the fund at time T for a perfect hedge in the risk neutral world Q_λ^γ. That is, if the risk free rate of return was given by δ^γ and the fund was tradeable, the perfect hedging strategy of λ put options would be given by the integrand in (12). Thus we have a quantification of the residual risk as the accumulated value of the perfect hedge with respect to the risk process R_t.

1.2 Simulating the model

I have used Matlab to describe the model to my computer. To do this in an as effective way as possible, some rewritings, simplifications and assumptions are needed.

First off, we need to descretify the model presented above. For instance, we have that

$$\begin{aligned}\int_0^T \phi_t dB_t &= \lim_{n\to\infty}\sum_{i=1}^n \phi_{t_i}\Delta B_{t_i} \\ &\approx \sum_{i=1}^n \phi_{t_i}\Delta B_{t_i}\end{aligned}$$

where $0 \leq t_1 < \ldots < t_n \leq T$ and the approximation is better for larger n. The new and discrete model is updated daily in stead of continuously, which means that $n = T$ and $\Delta B_{t_i} \sim N(0,1)$. This gives an error with respect to the continuous model, but it can be argued that this error is insignificant compared to other errors such as parameter insecurity. Also, a portfolio can not be continuously rebalanced since that would cause transaction costs to be enormous. One could also say that the market is not continuous, and therefore our model must be descretified to better fit the way that the market is behaving. Even daily updating may be a bit too often, but that is what I have decided to use. One could find an estimate of the error implied by descretifying the model in such a way by adjusting the time steps. This will not be included in this paper.

Secondly, we need to compute $\partial_y p_\lambda^\gamma(t, Y(t))$ for $p_\lambda^\gamma(t, Y(t))$ as defined in equation (3) on page 4. Here ∂_y is a shorthand notation for $\frac{\partial}{\partial y}$ and will be used consequently.

$$\begin{aligned}\partial_y p_\lambda^\gamma(t, Y(t)) &= \partial_y\left(e^{-r(T-t)}\frac{ln(w(t,y))}{\gamma(1-\rho^2)}\right) \\ &= e^{-r(T-t)}\frac{\partial_y ln(w(t,y))}{\gamma(1-\rho^2)} \\ &= e^{-r(T-t)}\frac{\partial_y w(t,y)}{w(t,y)\gamma(1-\rho^2)}\end{aligned}$$

Using the above rewritings, one can find a direct derivation with respect to y

of $w(t,y)$ as

$$\begin{aligned}
\partial_y w(t,y) &= \partial_y E_Q\left[\exp\left\{\lambda\gamma(1-\rho^2)(K-Y_T^{t,y})^+\right\}\right] \\
&= E_Q\left[\partial_y \exp\left\{\lambda\gamma(1-\rho^2)(K-Y_T^{t,y})^+\right\}\right] \\
&= E_Q\left[\exp\left\{\lambda\gamma(1-\rho^2)(K-Y_T^{t,y})^+\right\}\lambda\gamma(1-\rho^2)\partial_y\left(K-Y_T^{t,y}\right)^+\right]
\end{aligned}$$

where

$$\partial_y\left(K-Y_T^{t,y}\right)^+ = \begin{cases} -Y_t^{t,1} & ,K > Y_T^{t,y} \\ 0 & ,\ \text{else} \end{cases}$$

Although this method gives seemingly nice results, the above derivation of w might be a bit dodgy. This is because a function needs to be smooth in all points to be differentiated, which w is not. Actually, w will have a breakpoint almost surely since $Y_T^{t,y} = K$ with probability 1 for some t. Hence, the derivation should rather be done using Malliavin calculus[5].

Do also note that for modeling purposes our process $Y_T^{t,y}$ as defined in equation (6) on page 5 may be replaced by

$$\begin{aligned}
X_T^{t,x} &= x\cdot\exp\left\{(\delta-\frac{\eta^2}{2})(T-t)+\eta(W_T-W_t)\right\} \qquad (14) \\
&= x\cdot\exp\left\{(\delta-\frac{\eta^2}{2})(T-t)+\eta\sqrt{(T-t)}\epsilon\right\}\ \text{for } \epsilon\sim N(0,1)
\end{aligned}$$

without any loss of generality. This is because $\rho W_t^1+\sqrt{1-\rho^2}W_t^2 \sim N(0,t)$ which we recognize as the distribution of a single Brownian motion.

With this in mind, it can be shown that $\partial_x w(t,x)$ can be expressed as

$$\frac{\partial}{\partial x}w(t,x) = E\left[\exp\left\{\lambda\gamma(1-\rho^2)(K-X_T^{t,x})^+\right\}\frac{B_T}{x\eta T}\right] \qquad (15)$$

using the techniques presented in page 57 in di Nunno et al.[8] and the simplification $X_T^{t,x}$ of $Y_T^{t,y}$ as stated in equation (14) above.

1.3 Example

Now, we can look at some examples on how the price behaves for different inputs. Especially, we will look at how the risk aversion affect the price in time, $t\in[0,252]$, and correlation, $\rho\in[-0.99,0.99]$.

The figures showing the price for each set of parameters will contain a total of 6 subplots each. Subplot 1 is a surface plot of the price over both time and correlation while the others are borders of the plot, except subplot 5 which is

[5] I thank Frank Proske for pointing this out for me

a cross section of the price over time with correlation fixed at zero. Hence, the five smaller plots are just to help read the surface better. The dots above and below the price lines in the last five plots of each figure indicates the standard deviation of the price when it is calculated 20 times. As we can see from all the figures below, the simulation error is insignificant for small values of γ. If, on the other hand, γ is as high as 0.5, uncertainty is larger, and hence our simulation error is bigger, as seen in figure 3.

Lets start with the market given in table 1 and suppose the rest of the parameters are as given in the same table.

$r = 0.02$	$g = 0.035$	$Y_0 = 100$
$\mu = 0.06$	$\nu = 0.05$	$\lambda = 1$
$\sigma = 0.10$	$\eta = 0.07$	$\gamma = 10^{-11}$

Table 1: Parameters for the first example

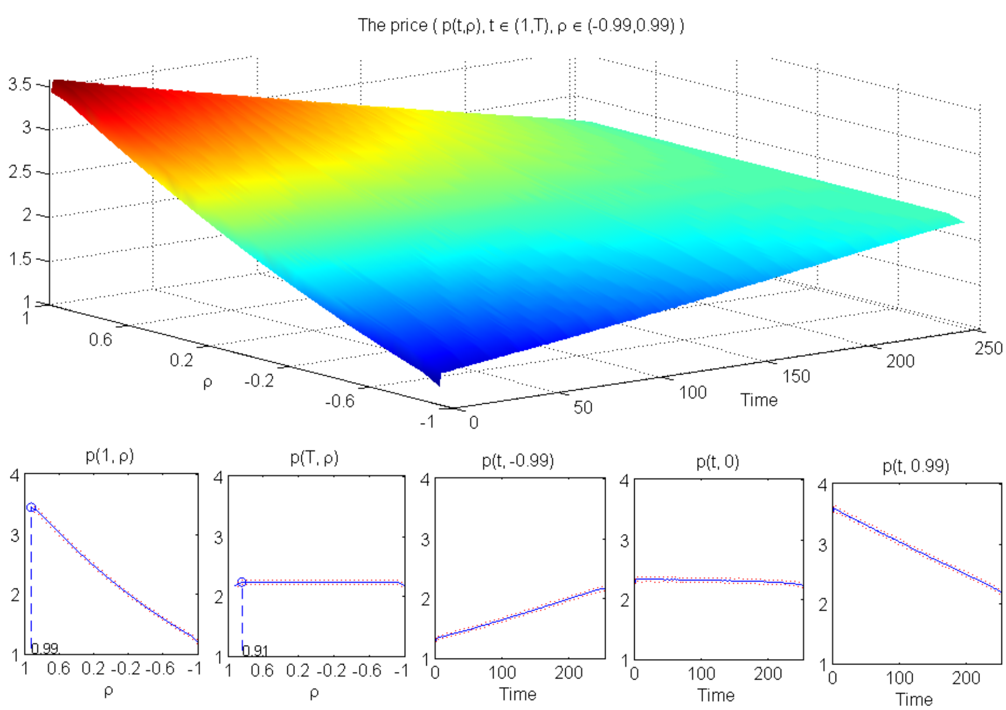

Figure 1: The price with $\gamma \approx 0$

As we can see from figure 1, the minimum price when risk aversion is close to zero is obtained if it is possible to invest in an asset having the opposite behavior compared to fund, in the meaning of correlation being equal to -1. It seems as if correlation plays no role at maturity as we can see from subplot 3. Further, the price is approximately constant over time if the risky asset is independent of the fund, i.e. $\rho = 0$, as seen from subplot 5. Since the price is approximately linear in time, one can easily calculate the approximate daily change in price

$\frac{p(t+\Delta_t)-p(t)}{\Delta_t}$, which is seen to be an increase by 0.33% for $\rho \approx -1$ and a decrease of 0.56% for $\rho \approx 1$. It is very close to zero for $\rho = 0$. The correlation giving the highest price is $\rho \approx 1$ at initial time, and $\rho = 0.91$ at maturity. While the observation for the initial time is 'spot on', the observation at maturity is not. The top point at maturity is decided by chance, and the 'real' price at this time is a flat line.

To conclude one can say that for a low risk aversion, the correlation parameter is of great impact at the initial time. This impact, however, decays as time tends to maturity, as seen in subplots 4 and 6 of figure 1, and is of no relevance at maturity as seen in subplot 5 of the same figure. The dots, representing the standard deviation of the price, are very close to the line, representing the mean of the price, indicating that the price is accurate at 10,000 simulations.

If we increase the risk aversion to $\gamma = 0.1$, we can first note that the ρ giving the largest price has changed from close to 1 to about 0.75 for the initial time, while it has stabilized around zero for maturity time, as seen in figure 2.

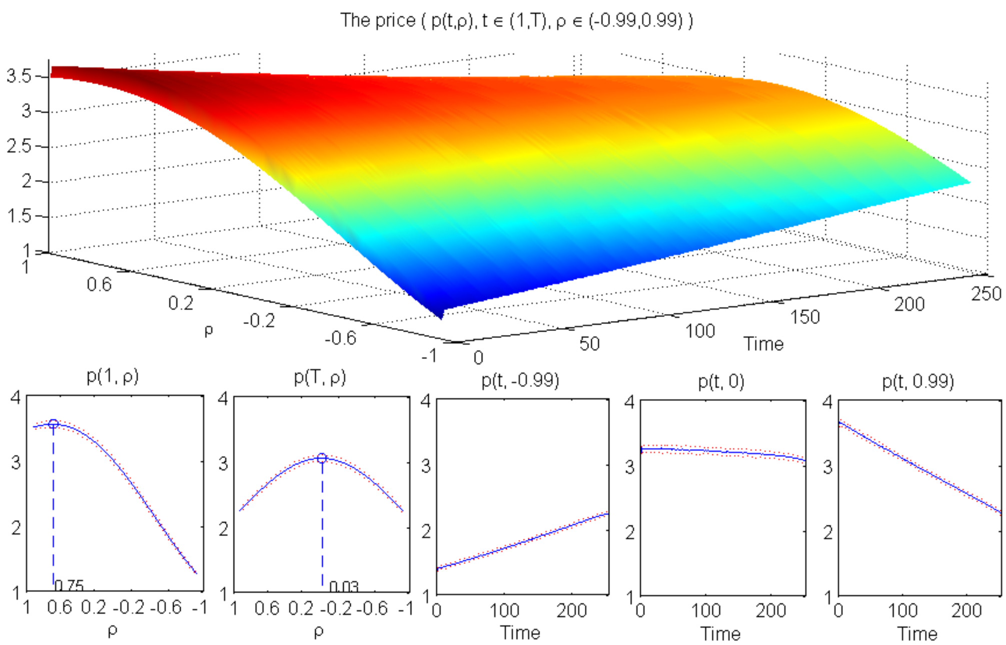

Figure 2: The price with $\gamma = 0.1$

Subplots 4 and 6 are quite alike their corresponding subplots in figure 1, while subplot 5 has noticeable higher values. This means that in order to maintain a low price with higher risk aversion, it is crucial to procure a negative correlated asset to hedge in. At least at initial time. When one get closer to maturity, it is also possible to hedge in a positively correlated asset. The main point is that it is not independent. The form of subplot 5 is still approximately the same as it was in figure 1, allthough it is decreasing slightly more here. Accuracy is still high in all subplots agreed by the low standard deviation. Subplots 2 and 3 are the ones that have changed the most, Subplot 3 has become more concave than it was when γ was close to zero and subplot 2 has gone from being convex to being mostly[6] concave.

Further increasing the risk aversion to $\gamma = 0.5$, imply a new shape of the price as seen in figure 3. The price is now much more symmetric in the sense that the top point of the price has moved further against zero. The maximum price is also much larger and continues to be over time, as we can see from subplot 5 of figure 3. By increasing the risk aversion from 0.1 to 0.5, the maximum price is increased from about 3.5 to about 10. The price in the extreme cases of ρ being near -1 or 1, have not changed all that much from when γ was close to zero. It is also important to notice that the uncertainty of the price is starting to show in the interval where ρ is small, that is for $\rho \in [-0.25, 0.25]$ approximately. The uncertainty is largest for $\rho = 0$, where the standard deviation is about 0.8.

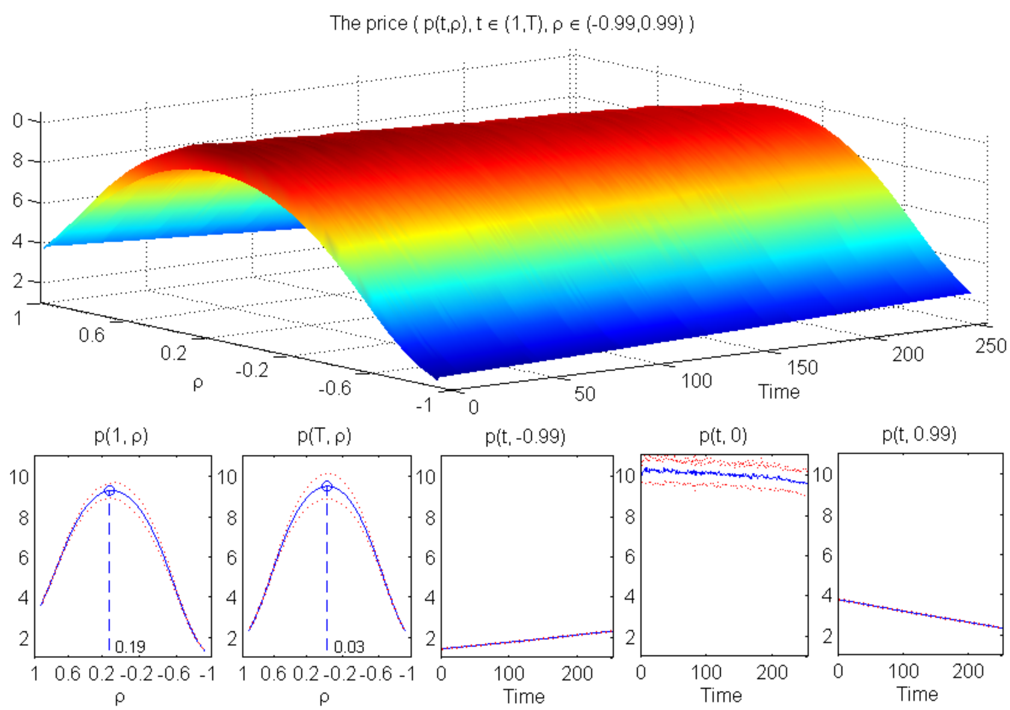

Figure 3: The price with $\gamma = 0.5$

If one were to raise risk aversion even further, say to $\gamma = 1$, one would see exactly the same changes as from raising it from 0.1 to 0.5, except one. It seems as if the ρ giving the maximum price stabilizes at about 0.20-0.30. It makes perfect sense that the correlation giving the largest price is positive because one should benefit from diversification, which is done by hedging in a stock with negative correlation. Hence the price should always be larger when hedging in positively correlated stocks. This might be interesting for someone worrying that the correlation might change over time, which it easily could. If, for

[6] It seems as if it turns convex for values of ρ close to -1.

instance, they knew that they had the ρ giving the highest price, they would know that any change in that correlation, would give a lower price. This could be worth knowing. First I wanted to find this ρ analytical, but I found it to be, well, rather troublesome, so instead I 'solved' it using Monte Carlo.

$r = 0.035$	$g = 0.00$	$Y_0 = 100$
$\mu = 0.07$	$\nu = 0.08$	$\lambda = 1$
$\sigma = 0.12$	$\eta = 0.15$	$\gamma \in (0, 0.5]$

Table 2: Parameters for the second example

In figure 4, the program has been run 150 times, each time simulating the price with 10,000 paths. The risk aversion γ ranges from close to zero to zero point five with increments of 0.01. This gave me 150 simulations of the ρ giving the largest price for each point in γ, of which I calculated the mean and standard deviation as plotted for both initial time and capital time. Capital time is 1 year, or 252 days, from initial time, while the other parameters are as given in table 2.

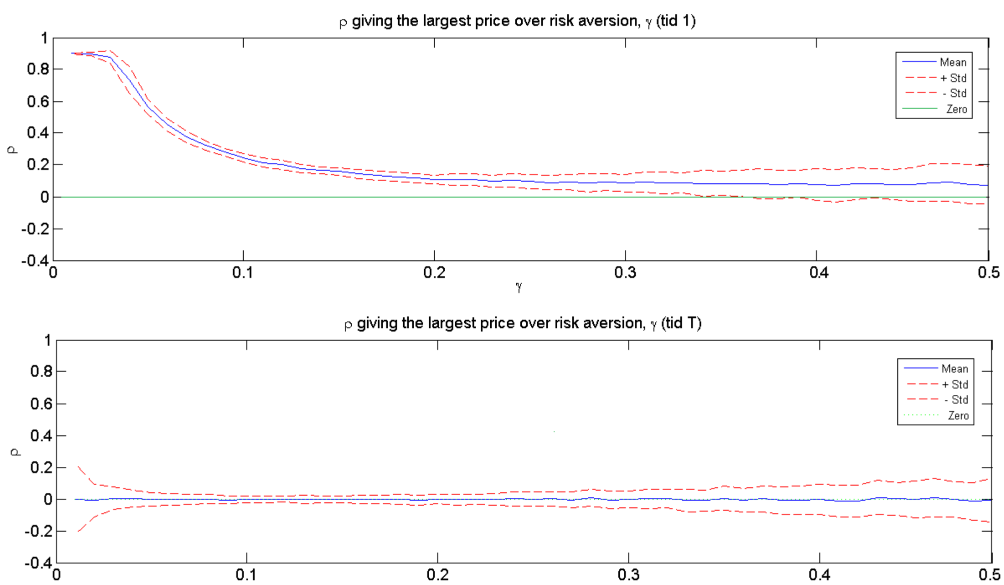

Figure 4: The ρ giving the maximum price against risk aversion $\gamma \in (0, 0.5]$

As we can see from figure 4, the ρ giving the largest price is close to 1 for small γ's and decrease as γ increase. The decrease, however, does seem to stagger at about $\rho = 0.1$. The function has decreased to $\rho = 0.1$ at $\gamma = 0.22$, and at $\gamma = 0.5$ it is still not significantly lower than 0.1 if one take uncertainty into account. The uncertainty, here represented by the standard deviation of 150 results, on the other hand seem to increase as γ increase. As we can see from the second plot on figure 4, the ρ giving the largest price at maturity, is $\rho = 0$. The standard deviation is at this time quite large for small γ's and decreases up to $\gamma = 0.14$ or so, where it starts to increase to about the same level as for initial time. The reason for the large uncertainty for smaller γ's at capital time was seen in subplot 3 of figure 1, where the price for a small γ at maturity is flat, implying that a maximum price could occur over the whole scale due to randomness.

One may further be interested in how this plot looks like for larger risk aversions. The market in figure 5 is the same as the one behind figure 4 and is given in table 2 except that here we let γ range from close to zero to 5. As we saw in figure 4, $\hat{\rho}$, the ρ giving the maximum price, decreased as γ increased from zero to zero point five. It might look as if $\hat{\rho}$ stabilized at some positive level, or one could think that $\hat{\rho}$ converged to zero. As we can see from figure 5, neither of those is what is really happening. In this figure we can clearly see that $\hat{\rho}$ actually start to increase at $\gamma \approx 0.6$. It is important to note that the uncertainty in this region is quite massive, meaning that the point of where $\hat{\rho}$ starts to increase may well be a bit higher or lower in reality.

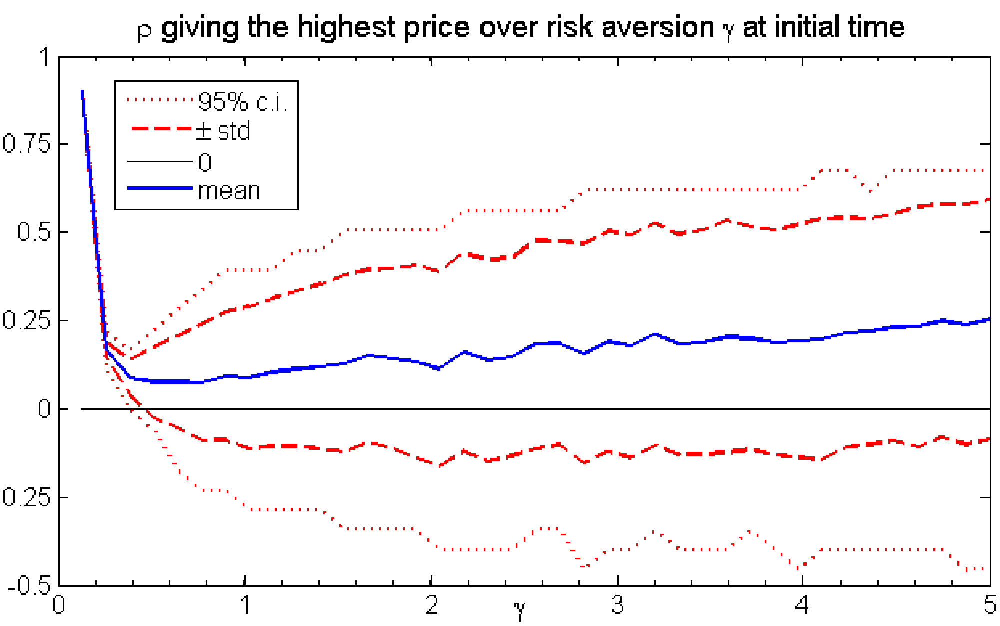

Figure 5: The ρ giving the maximum price against risk aversion $\gamma \in (0, 5]$

To put some numbers with this, I denoted the mean and standard deviation of $\hat{\rho}$. These numbers are noted in table 3. As we can see in this table as well as in figure 5, the standard deviation has a steady increase with respect to γ, while the mean has a large decrease at first and then starts to increase again.
It is also worth noticing that the distribution of this $\hat{\rho}$ is quite skew. We can see this by looking at the standard deviation and the 95% confidence interval of $\hat{\rho}$ also plotted in figure 5. Here we can see that the upper 95% quantile is quite close to the standard deviation added to the mean while the 5% quantile is a good deal lower than the standard deviation subtracted from the mean.

γ	0	0.2	0.4	0.6	0.8	1
E[$\hat{\rho}$], std($\hat{\rho}$)	0.99, 0	0.14, 0.04	0.08, 0.09	0.08, 0.13	0.09, 0.18	0.10, 0.21

Table 3: The values of figure 5 denoted ($\gamma \in (0, 1]$)

In other words, $\hat{\rho}$ can vary from -0.5 to 0.5, but it is more common that it is on the positive side, hence the mean is positive. Theoretically, it could also be of interest to look at the limit of $\hat{\rho}$ when γ tends to infinity. However, one could argue that no company will have a risk aversion that large, so it would be of limited practical interest. It might also be interesting to note that $\hat{\rho}$'s distribution is more heavytailed than a normal distribution since its confidence interval is larger than its standard deviation.

Another property worth noticing and which is visualized in figures 1, 2 and 3 is that for all times, all correlations and all risk aversions, the price is growing with respect to $\rho \in (-1, 0)$. In fact, it can be proved[7] that $p_{\lambda}^{\gamma}(t, y)$ is increasing with respect to $\rho \in (-1, 0)$ if $\frac{\mu - r}{\delta} > 0$ and that it is decreasing with respect to $\rho \in (0, 1)$ if $\frac{\mu - r}{\delta} < 0$. This is proved in appendix A.1, and it tells us that if the expected growth of the stock is larger than a banks interest rate, which is fairly common, one can obtain a lower price by investing in a lower correlated stock. If the bank rate should exceed the expected return of the stock, the lowest price would be obtained by having a correlation as close to one as possible.

As we can see from some of the plots in this section, $p_{\lambda}^{\gamma}(t, y)$ is increasing when $\mu \geq r$ for $\rho \in (-1, 1)$ for low γ's and decreasing when $\mu \geq r$ for $\rho \in [0, 1)$ for higher γ's. This is somewhat harder to prove analytically and has not been emphasized.

1.4 Summary of the model

In this chapter, the model of which we will work with in the rest of this paper has been defined and tested. The main definitions would be the formulation of the utility indifference price of a interest rate guarantee, or put option, and the residual risk of issuing such a guarantee or option. Further, this model was descretified in order to be simulated and lastly, we looked at some examples of the price, showing how the minimum price was obtained for minimal γ's and how the price increased with γ.

In particular we looked at which ρ gave the maximum price as γ was increased and we found that this ρ was positive in expectation, had its maximum at γ close to zero and its local, and probably also global, expected minimum at $\gamma \approx 0.6$.

[7] I thank Frank Proske for his help here.

2 Financial Risk and how to Measure it

Risk is a term most people are familiar with. However, there are several types of risk including *market risk*, *model risk*, *credit risk* and *operational risk*. The latter is the most recent notion.

2.1 Financial risk

Operational risk is defined by the Basel Committee as the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events, as stated in Embrechts et al.[10]. Examples of this risk are technological failure, errors in data processing, fraud, environmental risks, etc.

Market risk is the risk that the value of an investment will change due to changes in the market risk factors, such as interest rates, exchange rates, volatility, correlation, etc.

Credit risk is the risk of financial losses due to the counter party defaulting on a contract, typically a bond-holder being concerned that the bond-issuer will default. The horror example here is the LTCM scandal in 1998. LTCM, or Long Term Capital Management, was founded in 1994 and had amazing returns the first years, but folded in 2000 due to Russia defaulting on a rather large contract in 1998[8]. A good paper for modeling credit risk is Duffie[5].

Model risk can be defined as the risk that a financial institution incurs losses because its risk-management models are misspecified or because some of the assumptions required are not met. For instance, we might work with a normal distribution to model losses, whereas the real distribution is heavy-tailed, or we might fail to recognize the presence of volatility clustering or tail dependences. Since any financial model is a simplification and therefore an imperfect representation of the economic world, it is fair to say that every risk-management model is subject to model risk of some extent.

Until recent years, the banking and insurance industry only focused on market and credit risk, not spending too much thought on operational risk and the potential losses it could cause. Before, operational risk was included in credit risk, making credit risk more difficult to model than it really was. Now, we want to divide credit and operational risk by implementing the Basel II and Solvency II frameworks. Basel II and Solvency II also requires formal risk modeling for banks and insurance companies, respectively. Model risk has always been a problem and might have increased due to several new and sophisticated but not necessarily consistent models

[8] I am not saying that Russia is the one to blame for the collapse of LTCM, it would probably have happened sooner or later despite what happened with Russia.

Basel II[9] requires that banks are to set aside at least 8% of the total capital invested in risky assets. The aim for Basel II is to ensure that capital allocations are more risk sensitive than they have been in the recent past, separating operational risk from credit risk, quantifying both and preventing regulatory arbitrage[10] by aligning economic and regulatory capital.

It is believed that Basel II, by being an international standard, can help protect the international financial system from the types of problems that might arise should a major bank or even a series of banks collapse, as happened this last autumn. In practice, Basel II attempts to accomplish this by setting up rigorous risk and capital management requirements designed to ensure that a bank holds capital reserves appropriate to the risk the bank exposes itself to through its lending and investment practices.

As mentioned above, the insurance business have a similar framework under development, namely Solvency II which is expected to be implemented within 2012. The aim of Solvency II is to implement more deliberated solvency requirements with respect to the risks that companies face, and to deliver a consistent supervisory system that will be implemented all over the European Union.

Solvency II will reduce the probability that insurance companies get into trouble by introducing a far more comprehensive framework for risk management for defining required capital levels and to implement procedures to identify, measure, and manage risk levels than Solvency I could offer. As a side effect, Solvency II will most likely improve the confidence among policyholders (both current and potential) that the financial part of the insurance companies are steady by reducing the chances of policyholders losing if insurance companies get into difficulties. Longevity will also be considered in a greater extent then under Solvency I.

It is in my opinion that neither the Basel II nor the Solvency II framework would have prevented the current financial crisis from evolving even if it had been implemented many years ago. However, I do believe that it could have greatly reduced some of the severe negative effects we have seen lately and probably will see more of in 2009.

The field of risk analysis is a rather large one, and I will not attempt to cover it all. Instead, I will from now on focus on market and model risk and how to measure and control it. The primary aim, when modeling risk, is to quantify likely losses of a portfolio. To do this quantification, we need to know what a risk measure is.

[9] Recommendations on banking laws and regulations issued by the Basel Committee on Banking Supervision

[10] Arbitrage that can arise from regulated institutions taking advantage of the difference between its economic risk and the regulatory position it has

2.2 Risk measures

Risk can be thought of as a random variable, say X_t, which is defined on the same filtered probability space as in chapter 1, namely $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P\right)$. A risk measure ϑ is then a relation between the set of random variables, X_t, and the real line, that is $\vartheta(X_t) \in \mathbb{R}$. If, for instance, X_t is the expected loss of a portfolio given some security level, $\vartheta(X_t)$ can be the additional amount of money the company need to set aside to survive with that security level. The definition of risk measures is quite general, meaning that not all risk measures necessarily are *good* risk measures. Therefor, one might want risk measures to fulfill some additional conditions. For instance, one would probably want a risk measure to not exceed the largest possible loss.

For this reason, one has divided risk measures into several sets and subsets. The largest set is the set of monotary risk measures. This is the set of risk measures that satisfy monotonicity and translational invariance. A subset of these are the convex risk measures, recognized by the above set of economically desirable properties, as well as convexity. In 1999, Artzner et al.[13] performed the first systematic study of risk measures properties within finance and defined the class of *coherent* risk measures. This set is a subset of the convex risk measures satisfy the following axiom:

Axiom

A coherent risk measure ϑ is a risk measure, which for all risks X_t and Y_t and all constants $c \geq 0$ satisfy

(a) Translational invariance: $\vartheta(X_t + c) = c + \vartheta(X_t)$

(b) Positive homogeneity: $\vartheta(c \cdot X_t) = c \cdot \vartheta(X_t)$

(c) Monotonicity: if $X_t \leq Y_t$, then $\vartheta(X_t) \leq \vartheta(Y_t)$

(d) Subadditivity: $\vartheta(X_t + Y_t) \leq \vartheta(X_t) + \vartheta(Y_t)$

In words, one could explain these axioms in the following way: *Translational invariance:* Adding an amount of cash to the portfolio decreases its risk by the same amount. *Positive homogeneity:* If we increase the size of all risky positions in a portfolio, the risk of the portfolio will be increased by the same size. *Monotonicity:* If losses in one portfolio are larger than losses in another portfolio for all possible risk scenarios, then the risk of the first portfolio is higher than the risk of the second. *Subadditivity:* The risk of a portfolio is smaller than or equal to the sum of risks of its sub portfolios, or in other words; Risk in general should be reduced by diversification.

There exists numerous versions of this axiom depending on whether, for instance, one define losses as negative or positive values. In our case, X is the value of a risky position at the end of the holding period, hence it is a random variable at one point in time, namely the terminal time. The risk in this paper is denoted by $\mathcal{R}_{\lambda}^{\gamma}$ and is defined on page 6.

Now that we have defined what the risk is, lets have a look at some of the ways to measure it.

2.2.1 Standard deviation

In 1952, Harry Markowitz wrote a paper about modern portfolio theory where he defined the standard deviation of the value of a portfolio as a risk measure. This was the very first widely used risk measure. While standard deviation is quite easy to calculate, it penalizes losses as well as profits, and is therefor, in general, not a very good risk measure. It is subadditive, but still not coherent since it violates the monotonicity property.

2.2.2 Value at Risk

Value at Risk (VaR) has been the most used risk measure since J. P. Morgan released their RiskMetrics system in 1994. VaR gives us the answer to how much the value of a portfolio can drop given some probability level α. We say that VaR is a downside risk measure, since it typically describes the probability boundary of potential losses. One can define Value at Risk formally in the following way:

Definition 2.1. *Given a risk X with cumulative distribution function F_X and a probability level of $\alpha \in (0,1)$, then*

$$VaR_{\alpha}(X) \quad = \quad F_X^{-1}(\alpha) = inf\,\{x \in \mathbb{R} : F_X(x) \geq \alpha\}$$

is the Value at Risk of X at α level.

α is often chosen to be among 0.95, 0.99 or 0.999. From a statisticians point of view, VaR is nothing but the α quantile of some sorted distribution, and says that with a confidence of $\alpha \cdot 100\%$, you will not loose more than VaR_{α}. In Jorion[14] page 27, VaR is represented as $VaR_{\alpha}(X) = E[F(X)] - Q_{\alpha}[F(X)]$ where $Q_{\alpha}[F(X)]$ is the quantile of the qumulative distribution of X, matching the confidence level α. This representation is nothing but VaR viewed from a different angle and is consistent with the definition above.

Even though VaR has become the benchmark risk measure in the financial world, it is not perfect. As almost everything else in this world it has some flaws that one should know before using it. Being a one-period risk measure, VaR does not care what happens with the portfolio value in the holding period, it only cares about the level at capital time. It also assumes that the market position is the same at all times, which is highly unlikely in practice. However, one can argue that these flaws are minor and will drown in comparison to other uncertainties.

More serious is the fact that VaR does not measure the potential size of a loss, given that it exceeds VaR. The most serious of VaR's flaws, is the fact that it

Bonds	Initial	Defaults					Risk measure	
		No	Soft A	Hard A	Soft B	Hard B	$\mathrm{VaR}_{0.95}$	$\mathrm{ES}_{0.95}$
A	104.6	108	100	0	108	108	4.6	64.6
B	104.6	108	108	108	100	0	4.6	64.6
A+B	209.2	216	208	108	208	108	101.2	101.2
Probabilities:		0.9	0.02	0.03	0.02	0.03		

Table 4: Table showing how Value at Risk is not subadditive, while Expected Shortfall is.

is, in general, not subadditive, and hence not a coherent risk measure. For the normal distribution and other light-tailed distributions, VaR might well be subadditive. It is when one is dealing with either extremely skewed distributions (e.g. exponential), heavy tailed distributions (e.g. Pareto) or distributions with a special dependence structure (e.g. copulas) that one has to pay attention. The implications of not having a subadditive risk measure are severe when modeling financial or economic values, as such a risk measure dissuade diversification. One can then create severe aggregation problems when adding risk.
VaR has been popularized as the risk measure of choice among investment banks wishing to measure their portfolio risk for the benefit of banking regulators. However, due to the lack of subadditivity, VaR appears to be unfit for such calculations.

We can now take a look at a some examples of using VaR, beginning with a practical example of how subadditivity may fail to work.

Example 2.2. Suppose we have a bond **A** and that, at maturity, there are three possible outcomes, i.e. $\Omega = \{\omega_1, \omega_2, \omega_3\}$

ω_1 **No default:** The bond redeems both its face value of 100 Euro and the coupon of 8 Euro with probability $P(\omega_1) = 0.95$

ω_1 **Soft default:** The bond redeems only its face value of 100 Euro with probability $P(\omega_2) = 0.02$

ω_1 **Hard default:** The bond redeems nothing with probability $P(\omega_3) = 0.03$

Note: If we ignore the second possible outcome, and say that $P(\omega_1) = 0.97$ and $P(\omega_3) = 0.03$, we get that $\mathrm{VaR}_{0.95}(A) = 0$, even though the risk of such a bond is greater than zero.

Further, suppose there is another bond **B** that is identical to **A**, but issued by another agent. The risks of these two bonds are each others opposites, in the meaning that if bond **A** defaults, bond **B** will not, and vice versa.
As we can clearly see from the second last line of table 4, VaR is not subadditive since $4.6 + 4.6 < 101.2$, hence VaR clearly disfavors diversification.

To end my discussion of VaR, I would like to take a look at an example found in Embrechts et al.[10] showing that VaR *is* subadditive for Gaussian distributed risks.

Example 2.3. Suppose X_1 and X_2 are jointly Gaussian distributed with mean μ and covariance matrix Σ

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where $\rho \in [-1, 1]$ and $\sigma_i > 0, i \in \{1, 2\}$. If $\alpha \in [0.5, 1)$, then

$$\mathrm{VaR}_\alpha(X_1 + X_2) \leq \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) \tag{16}$$

Since (X_1, X_2) is bivariate Gaussian distributed, X_1, X_2 and $X_1 + X_2$ are all univariate Gaussian distributed. It follows that

$$\begin{aligned} \mathrm{VaR}_\alpha(X_i) &= \mu_i + \sigma_i q_\alpha(\epsilon), \qquad i \in \{1, 2\} \\ \mathrm{VaR}_\alpha(X_1 + X_2) &= \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}\, q_\alpha(\epsilon) \end{aligned}$$

where $q_\alpha(\epsilon)$ is the α-quantile of a standard Gaussian distributed random variable, ϵ. Inserting this into (16) gives us that $\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2 \leq (\sigma_1 + \sigma_2)^2$ since $\rho \leq 1$. Hence the subadditivity property is kept.

It is shown in McNeil et al.[1], Theorem 6.8, that VaR is subadditive for the set of linear combinations of components of a multivariate elliptical distribution.

2.2.3 Expected Shortfall

Since VaR does not tell us anything about the size of our potential loss, Artzner et al.[13] considered the notion of Expected Shortfall (ES) also known as conditional tail expectation. In words, one can define Expected Shortfall at α-level as the expected risk in the worst $\alpha \cdot 100\%$ of the cases. The meaning of this level is that ES ignores the most profitable but unlikely possibilities for high α's while it focuses on the worst losses for low α's. In more formal terms, one can define ES as follows:

Definition 2.4. *Let X be a risk and $\alpha \in (0, 1)$. Expected Shortfall is then defined as the conditional expected risk given that the risk exceeds $VaR_\alpha(X)$:*

$$\mathrm{ES}_\alpha(X) = E[X | X > \mathrm{VaR}_\alpha(X)]$$

Note: The direction of the inequality sign in the definition above is decided by how you define VaR. The point is that the expectation is to be taken over the variables that exceed VaR in the tail. A more precise representation is made by McNeil et al. in [1] page 44 by observing that for a continuous random variable X, one has that $\forall\ \alpha \in (0, 1)$

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(X) du \tag{17}$$

For such continuous X, Expected Shortfall is subadditive and therefor a coherent risk measure, see Artzner et al.[13]. For discrete random variables X, equation (17) needs to be slightly modified to achieve the same properties as in the continuous case. For a discrete random variable X, one has $\forall\ \alpha \in (0,1)$

$$\begin{aligned} \mathrm{ES}_\alpha(X) &= \frac{1}{1-\alpha}\left(E[X|X \geq \mathrm{VaR}_\alpha(X)]\right. \\ &\quad \left. + \mathrm{VaR}_\alpha(1-\alpha-P(X \geq \mathrm{VaR}_\alpha(X)))\right) && (18) \\ &= \lim_{n\to\infty} \frac{\sum_{i=1}^{[n(1-\alpha)]} X_{i,n}}{n(1-\alpha)} \text{ a.s.} && (19) \end{aligned}$$

Here, in equation (19), $X_{1,n} \geq \ldots \geq X_{n,n}$ is the ordered statistics of the sequence $(X_i)_{i\in\mathbb{N}}$ of independent identical distributed (i.i.d.) random variables and $[n(1-\alpha)]$ denotes the largest integer not exceeding $n(1-\alpha)$. A proof of this latter equation is found in Proposition 4.1 of Acerbi and Tasche [4].

Among ES's properties we find that ES_α increases as α increases and that ES_α is worse than, or equal to, the Value at Risk (VaR_α) at α level, i.e. ES is more conservative. See example 2.3 for a case where ES is subadditive while VaR is not.

2.2.4 Conditional Value at Risk

A third highly popular risk measure is Conditional Value at Risk (CVaR) also known as 'Mean Excess Loss', 'Mean Shortfall' or 'Tail VaR'. CVaR is meant to be an extension of VaR in the sense that it coincides with VaR for elliptical distributed risks, but remains coherent for general distributed risks. Conditional Value at Risk is computed by assessing the likelihood (at a given confidence level) that a specific loss will exceed the Value at Risk. Mathematically speaking, CVaR is derived by taking a weighted average between the Value at Risk and losses exceeding the Value at Risk, as we shall see soon. The following definition of CVaR for *continuous* X is found in Rockafellar and Uryasev[15]:

Definition 2.5. *Conditional Value at Risk at level $\alpha \in (0,1]$ of a continuous random variable X is:*

$$\mathrm{CVaR}_\alpha(X) = E\left[\Psi_\alpha(X,\zeta)\right]$$

where $\Psi_\alpha(X,\cdot)$ is the α-tail distribution of X defined as,

$$\Psi_\alpha(X,\zeta) = \begin{cases} \left(\Psi(X,\zeta)-\alpha\right)/(1-\alpha) & ,\zeta \geq \mathrm{VaR}_\alpha(X) \\ 0 & ,\ \textit{else} \end{cases}$$

and $\Psi(X,\zeta) = P(X|X \leq \zeta)$.

Note that both Ψ and Ψ_α are non-decreasing and right-continuous, and that $\Psi_\alpha(X,\zeta) \to 1$ as $\zeta \to \infty$. Hence, the α-tail distribution referred to above is well defined. As a side-note, ζ can be interpreted as VaR and in fact, for continuous distributed risks, CVaR and ES coincide.

However, since I am interested in evaluating simulated distributions, i.e. not continuous ones, I will need a representation of CVaR that is more appropriate.

Luckily, Rockafellar and Uryasev[15], in their Proposition 6, has proved the following.

Proposition 2.6. *Let $\lambda_\alpha(X)$ be the probability assigned to the loss amount $VaR_\alpha(X)$ by the α-tail distribution, that is*

$$\lambda_\alpha(X) = \frac{\Psi(X, VaR_\alpha(X)) - \alpha}{1-\alpha} \in [0,1]$$

when$\Psi(X, VaR_\alpha(X)) < 1$,

$$CVaR_\alpha(X) = \lambda_\alpha(X) \cdot VaR_\alpha(X) + (1 - \lambda_\alpha(X)) \cdot ES_\alpha(X)$$

while $\Psi(X, VaR_\alpha(X)) = 1$ *gives*

$$CVaR_\alpha(X) = VaR_\alpha(X)$$

The proof of this and the reason behind the choice of λ_α is included in Rockafellar and Uryasev[15].

While the former proposition is best for understanding, the next one is probably better for computations.

Proposition 2.7. *Suppose the probability measure P is concentrated in finitely many points y_k of Y such that for each $x \in X$, the distribution of loss $z = f(x,y)$ is likewise concentrated in finitely many points, and $\Psi(x,\cdot)$ is a step function with jumps at those points. Fixing x, let those corresponding loss points be ordered as $z_1 < z_2 < \ldots < z_N$, with $P(z_k) = p_k > 0$. Further let k_α be the unique index such that*

$$\sum_{k=1}^{k_\alpha} p_k \geq \alpha > \sum_{k=1}^{k_\alpha - 1} p_k$$

The VaR_α of the loss is then given by

$$VaR_\alpha(x) = z_{k_\alpha}$$

while the $CVaR_\alpha$ is given by

$$CVaR_\alpha = \frac{1}{1-\alpha}\left(\left(\sum_{k=1}^{k_\alpha} p_k - \alpha\right) z_{k_\alpha} + \sum_{k=k_\alpha+1}^{N} p_k z_k\right)$$

The proof of this is also carried out in Rockafellar and Uryasev[15] under their Proposition 8. I suggest taking a look at this paper for further reading about this risk measure.

It is this last proposition I have used to calculate CVaR in my programs.

2.2.5 Entropic risk measure

The scientific meaning of an entropy is a measure of the disorder, or uncertainty, in a given system. An entropy is a measure on how great the unexpected changes in a system is. The entropic risk measure is said to be the most famous example of a convex risk measure, introduced by Föllmer and Schied[9]. The definition is quite straight-forward and is given by the following

Definition 2.8. *The entropic risk measure, ENT, of a non-negative random variable X is given by*

$$ENT(X) = \frac{1}{\gamma} \log E[e^{-\gamma X}]$$

where γ is the risk aversion introduced in the first chapter of this paper.

The entropic risk measure is in general not coherent but it is convex. By conditioning the expectation on a filtration, $\mathcal{F}_t$ generated by the Brownian motion, the entropic risk measure can be extended to a dynamic risk measure as defined in section 3 of Barrieu and Karoui[11]. I am confident that the field of dynamic risk measures is a very interesting and useful one, but far to extensive to be treated in this short paper. I refer to Barrieu and Karoui[11] for a more comprehensive talk about the subject. As it turns out, the risks in this paper are not non-negative, so this measure should not be used here. It is still an interesting measure having several areas of use and for this, I will include it in my calculations.

2.2.6 Other risk measures

There are of course many other risk measures than the ones I have looked at here including the *Incremental risk measures* (s.a. Incremental VaR (IVaR), Incremental Expected Shortfall (IES), and Incremental Standard Deviation (ISD)), *Marginal Value at Risk* and *Conditional Drawdown at Risk*. This paper, however, is not meant to be a total run-through over all existing risk measures. It is rather meant to be an examination of how some risk measures behaves when used on utility based pricing of interest guarantees and the risk of such. I will for these reasons not go into close examination of rare and elusive risk measures, albeit I would say it is needed.

2.2.7 Important purposes of risk measures

As I mentioned earlier in this chapter, the primary aim of a risk measure is to quantify likely losses or the likeliness of a given loss. This is a quite general statement, and I would like to mention some of the most important purposes of a risk measure.

Determination of risk capital and capital adequacy. The principal function of risk management within finance, included insurance, is to determine the amount of capital a financial institution needs to hold as a buffer against unexpected

losses in its portfolio in order to satisfy a regulator concerned with the solvency of the institution.
Insurance premiums compensate an insurance company for bearing the risk of the insured claims. The size of this compensation can be viewed as a measure of the size of these risks.
Management tool. Risk measures are often used as a tool for limiting the amount of risk a unit within a firm may take. For instance, traders in a bank are often constrained by the rule that the daily 95% Value at Risk of their position should not exceed a given bound.

2.3 Summary of risk measures

In this chapter we have looked at several different risk measures and their pros and cons. After working with them all, I would like to emphasize Expected Shortfall for its properties and its easy-to-understand calculations and meaning. In the end of the day, what we really want from such a measure, is for it to be easy to use and easy to explain for others.

I would also like to note that even though the entropic risk measure need positive inputs by definition, and the residual risk defined in chapter 1 might well be negative, this risk measure will still be evaluated and used later in this paper. Strictly out of curiosity.

The difference of CVaR and ES. When doing research for this part of my paper, I found that there exist some inconsistent opinions on CVaR and ES. This inconsistency consist mainly of some people considering ES and CVaR to coincide, while others do not. After careful reading of selected opinions on the subject, I decided to agree with the ones saying there is a difference even if it is rather small. In this paper, the difference of ES and CVaR is that CVaR is a weighted average of VaR and ES, as defined earlier in this chapter. The difference in value might not be significant, but the two ideas are quite inequal.

As a rule of thumb, one can say that risk measures represented by an expectation are in general coherent. The entropic risk measure is not included in this rule.

3 The Relationship of Price and Risk

In this chapter, I will give a close examination of the relationship between the utility indifference price, p_λ^γ, and the residual risk, $\mathcal{R}_\lambda^\gamma$, as defined in chapter 1. We remember $\mathcal{R}_\lambda^\gamma$ to be the risk a company is left with after hedging. This examination will be done by looking at the price and risk of different sets of parameters given in tables below. I will also check for which sets of parameters the coherence property of the different risk measures is retained. The funds initial value, Y_0, will be equal to 100 in all the examples below.
All calculations in this chapter are done by double Monte Carlo with 10,000 simulations repeated 20 times.

3.1 A 'money back' guarantee

This example is fetched from Benth and Proske[6]. Consider a contract agreed at time $t \in [0, T]$ which guarantees that an investor gets his money back at a predefined time, T. In other words, the guaranteed rate of return is zero percent. This is a relevant situation for Norwegian pension funds, where the investor may have buffers to cover possible deficits in the fund. This buffer is built up in years with surplus returns over the guaranteed, which is usually around 3.5%. However, new legislations enforce the manager to cover a possible negative return on the fund irrespective to the amount of buffer capital at hand. Since most investors have large buffers, the manager is basically issuing an at-the-money put option, i.e. a guarantee against a negative return.

Suppose that the risk free rate of return, r is equal to 3.5%, the expected return of the pension fund, $\nu = 8\%$ and its volatility, $\eta = 15\%$. The hedging asset S has a slightly lower expected return, $\mu = 7\%$ and its volatility, $\sigma = 0.12$ indicates less risk. These are all yearly values, and are summarized in table 5 with the number of options issued, λ and the risk aversion γ.

$r = 0.035$	$g = 0.0$	$p^{B\&S} = 4.32$
$\mu = 0.07$	$\nu = 0.08$	$\lambda = 1$
$\sigma = 0.12$	$\eta = 0.15$	$\gamma = 10^{-11}$

Table 5: Money back guarantee with quite high drift and volatility

Lets first look at the minimum price obtained whenever the insurance company is tolerant to risk, i.e. they have a low risk aversion. As we can see in figure 6, the lowest price for this market is given at $(t, \rho) = (0, -1)$ and is 1.87. This is significantly lower than the price found at $(t, \rho) = (0, 1)$, which is 4.27. Note that both are lower than the B&S price. As time goes by, the price will increase or decrease depending on whether ρ is negative or positive, respectively. For $\rho = 0$, the price seems to remain roughly the same for all times and is approximately equal to the price at time T for all ρ. This is a little deceptive since if one were to zoom in on it, one would see that this is not the case. It starts at 3.2 at initial time and then fluctuates around that value for about 100 days before it starts to rapidly decrease to reach its final value of 3.0 at time T. From the figure, the price at maturity seems quite flat over ρ. In both these cases, the uncertainty of the price, here in form of the standard deviation, is 0.08, which is not so high compared to the error occurring from a prospective parameter estimation and model error. Taking such errors into account, makes a detailed discussion of no direct practical value and will be omitted hereafter.

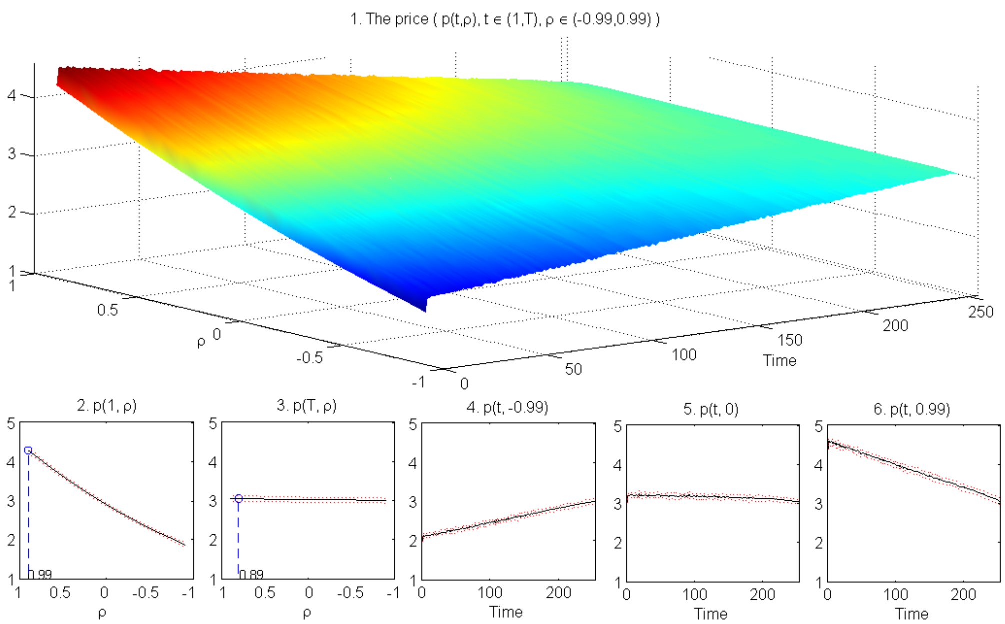

Figure 6: The minimum price

Now, lets take a look at the residual risk process for this zero percent interest rate guarantee. To get an overview of how it behaves, I have included various plots of it in figure 7. Subplot 1 is an overview of the sorted realizations of 10,000 simulations done 20 times. As we can see, it takes both positive and negative values, and its value is most volatile at $|\rho|$ close to ± 1. The first thing we should notice, is the scale of the y-axis, which has a value of 10^{11}, or 100 billions. This very high number is a direct consequence of the very low value of the chosen risk aversion, γ. It is interesting to note that $\gamma = 10^{-11} = \frac{1}{10^{11}}$.

Further, the second subplot is the mean of the residual risk over correlation. Allthough it is significantly lower than its highest value, it is still quite high with values up to 20 billions. Its minimum point is at $\rho = 0$ and has the value of approximately 0.5 billions or 500 millions. Subplot three shows the proportion of positive residual risks, which is just above 50% for all ρ's and a bit larger for $|\rho| = \pm 1$, and has the same form as the mean. The fourth and final subplot shows histograms of the residual risk when ρ is chosen to be -1, 0 and 1. As we can see, the distribution of the risk is roughly the same for $\rho = \pm 1$, while it has significantly lower volatility for $\rho = 0$. This is also seen in subplot 1.

What we have learned here is that the minimum price comes with a very high risk. The company, of course, is not bothered by this, since their risk aversion is practically zero.

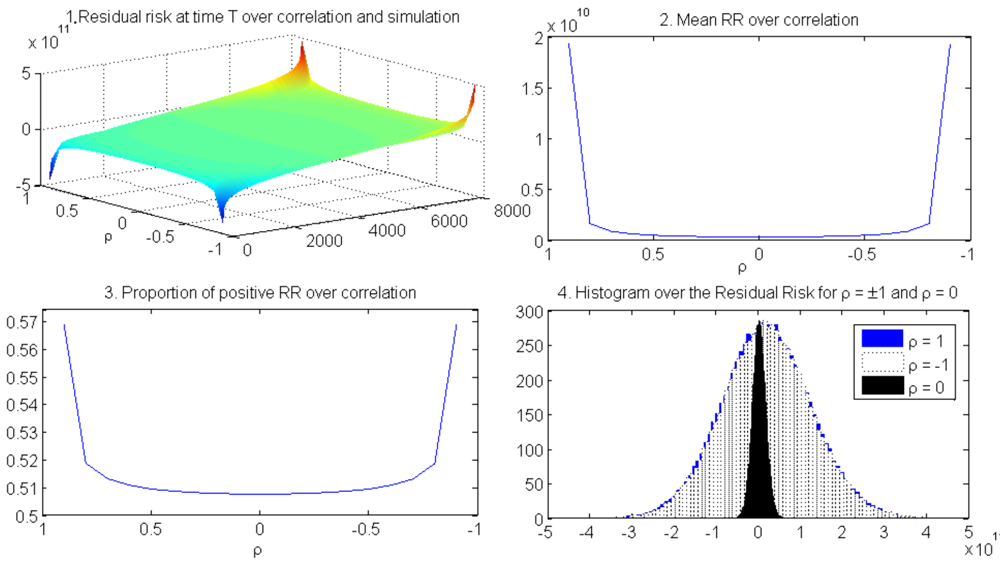

Figure 7: The residual risk

Now lets take a look at the different measures of this residual risk. The measure giving the lowest risk is the Entropical risk measure. This is probably because it is dependent on the risk aversion parameter, γ, scaling it down. The standard deviation is almost as low, while VaR, CVaR and ES agree that the risk is higher. They all agree that the largest risk is when $\rho = \pm 1$ and that the least risk is obtained for lower values of $|\rho|$. This is shown in figure 8.

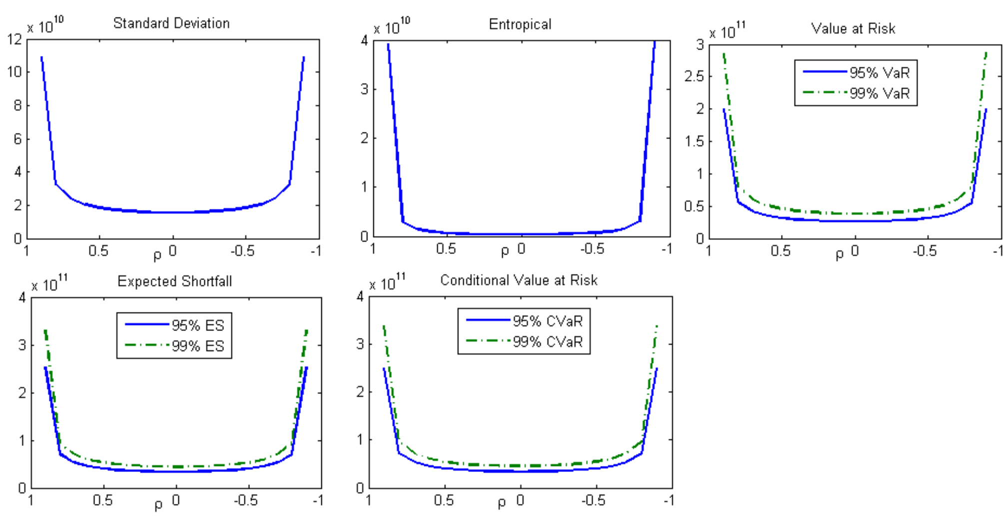

Figure 8: The residual risk measured

3.1.1 Raising γ

Lets now look at what happens when we increase the company's risk aversion. As we saw in the example in chapter 1, this made the price increase, and it should make the risk decrease. As we can see from figure 9, the prices for

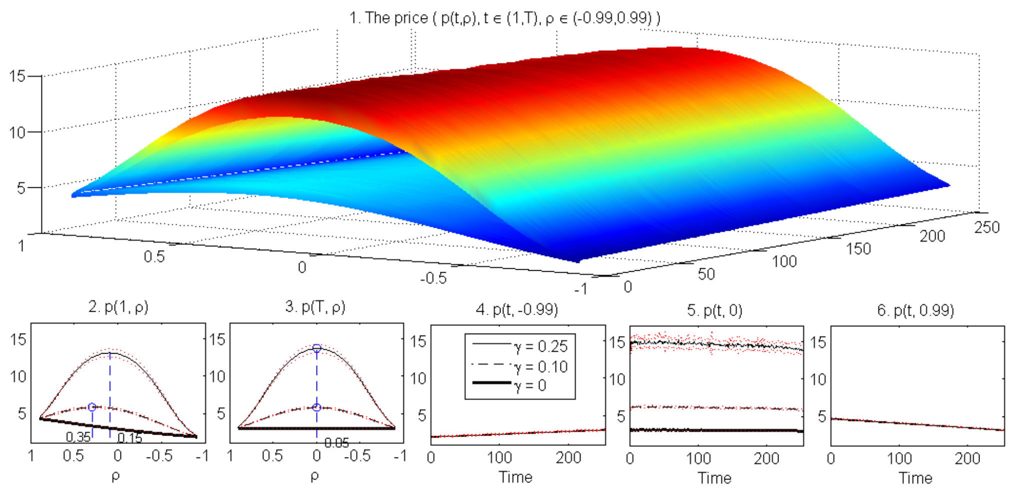

Figure 9: The price with $\gamma \in \{0.10, 0.25\}$

$\gamma = 0.1$ and $\gamma = 0.25$ does not differ much from the minimum price in the extreme points of $\rho \approx \pm 1$. The difference is significantly larger for smaller $|\rho|$ where $p_1^{0.10}$ is up to twice as high as the minimum price while $p_1^{0.25}$ is up to 5 times higher than the minimum price. It is obvious that the price does not depend on γ for values of ρ close to one since we then have an almost complete market where the utility price would be the same as the Black&Scholes price, i.e. unique for all γ's.

Whilst the price behaves as expected when we adjust the risk aversion, the risk measures do not. As far as I was concerned, the risk should go down as gamma was increasing, but this is not always true. As we can see from figure 10, the risk when γ is 0.1 and 0.25 has decreased considerably compared to the case where γ were approximately zero. However, the interesting point here is that the highest risk among these two is achieved with the highest risk aversion, given that the correlation is not too high. In other words we have that $\vartheta(\mathcal{R}^{0.1}) \leq \vartheta(\mathcal{R}^{0.25}) \leq \vartheta(\mathcal{R}^{0})$ for $\rho \in [-0.8, 0.8]$ approximately. This means that there must exist some $\hat{\gamma}$ for which the residual risk is at its minimum.
Suppose we have a risk measure ϑ and a risk $\mathcal{R}^{\gamma}$ representing the residual risk process defined in chapter 1. My proposition is then that there exist some $\hat{\gamma} > 0$ such that

$$\vartheta(\mathcal{R}^{\hat{\gamma}}) \leq \vartheta(\mathcal{R}^{\gamma}) \quad \forall \quad \gamma \in \mathbb{R}^{+}$$

The proof of this is not included since it is not done as of yet, hence the above statement is merely an idea or thought of how it needs to be.

If the above turns out to be true, it would be possible for traders in the market to issue the interest rate guarantees discussed here priced as a result of minimizing the traders risk independent of his risk measure.

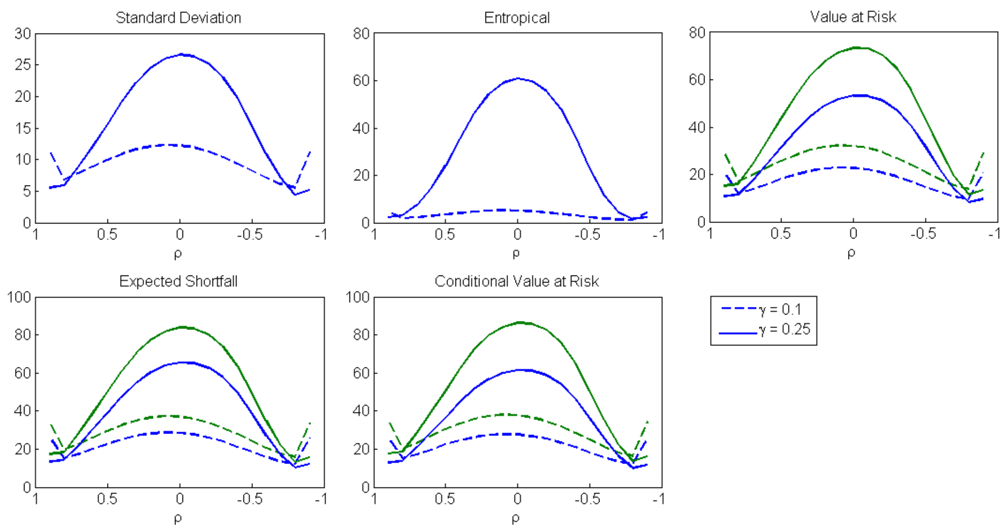

Figure 10: The different risk measures with $\gamma \in \{0.10, 0.25\}$

3.1.2 Raising λ

As a final investigation of the relationship between price and risk in this market, lets see what will happen if we raise λ. This also gives us an opportunity to check the coherence property, i.e. subadditivity, of Var, CVaR and ES. We remember from chapter 2 that a risk measure ϑ is said to be subadditive if

$$\vartheta(X+Y) \leq \vartheta(X) + \vartheta(Y)$$

holds for two risks X and Y. This represents that merging two portfolios should not cause more risk, and that diversification in general should be favored. In this paper, the above inequality translates into the following:

$$\vartheta(\mathcal{R}_\lambda) \leq \lambda \cdot \vartheta(\mathcal{R}_1) \Rightarrow \frac{\vartheta(\mathcal{R}_\lambda)}{\lambda} \leq \vartheta(\mathcal{R}_1)$$

Introducing λ_1 and λ_2 where $\lambda_1 \geq \lambda_2$, we get the more general inequality

$$\frac{\vartheta(\mathcal{R}_{\lambda_1})}{\lambda_1} \leq \frac{\vartheta(\mathcal{R}_{\lambda_2})}{\lambda_2}$$

Figure 11 shows that the above equation is intact with $\lambda = \{1, 3, 10\}$ even for standard deviation and Value at Risk which in general are not subadditive when $\gamma = 10^{-11}$.

In figure 12, λ is as above but γ has been raised to 0.1. As we can see, the subadditivity property has now been destroyed when $\rho \in [-0.95, 0.95]$. This

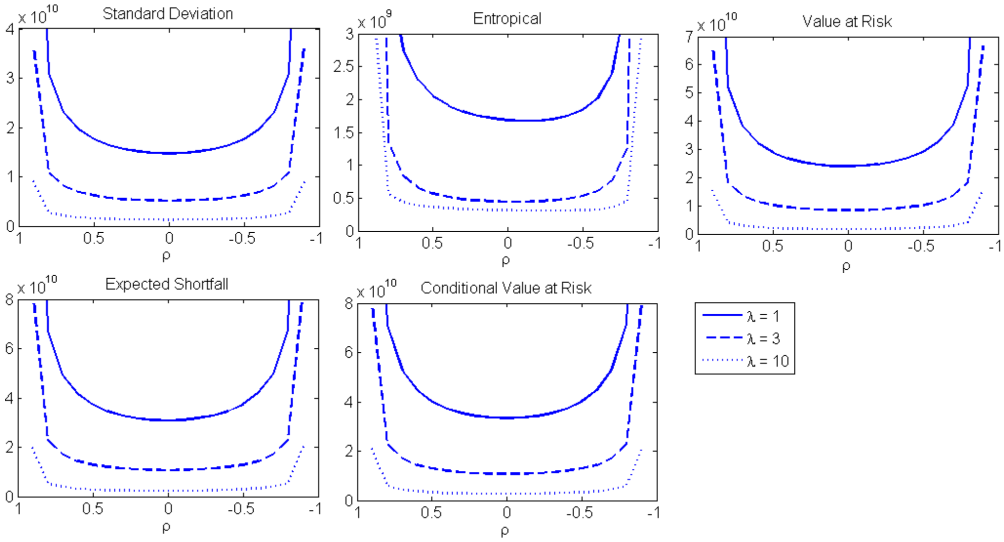

Figure 11: The different risk measures with $\lambda \in \{1, 3, 10\}1$ for $\gamma = 10^{-11}$

can be explained by the risk aversion factor in the meaning that an issuer with a given risk aversion might not want to take on more risk than he already has due to his aversion to risk. Since the subadditivity property holds for low γ's but not for higher ones, there must exist some $\tilde{\gamma}$ such that

$$\vartheta(\mathcal{R}_{\lambda}^{\tilde{\gamma}}) = \lambda \cdot \vartheta(\mathcal{R}_{1}^{\tilde{\gamma}})$$

This $\tilde{\gamma}$ would most likely be dependent on λ, but might not be dependent on the other market parameters such as drift or volatility.

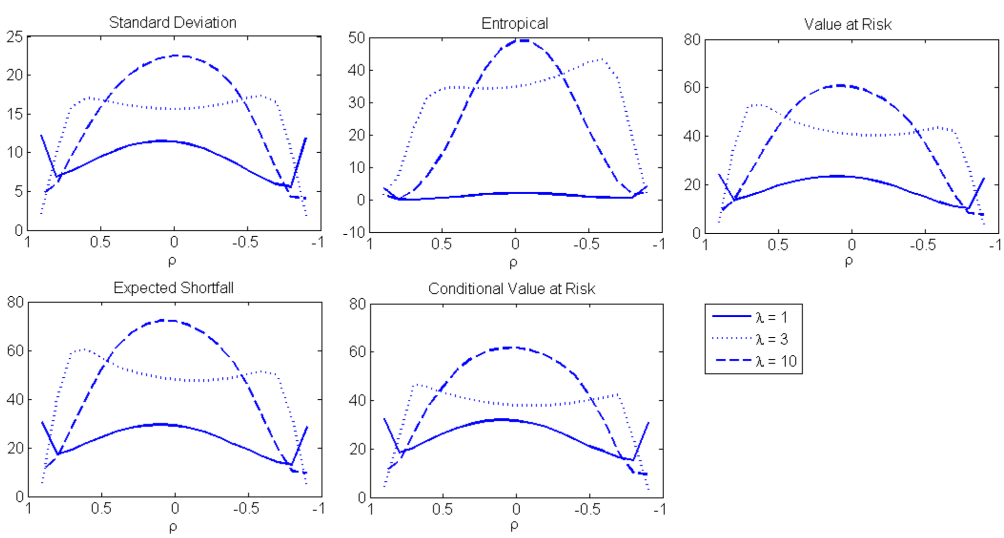

Figure 12: The different risk measures with $\lambda \in \{1, 3, 10\}1$ for $\gamma = 0.1$

3.2 A minimum interest rate guarantee

Let us now consider a market where the issuer have to repay the money with an interest rate of 3.5% while the riskfree interest rate is only 2%. Both the stock and the fund have lower expected returns than in the market considered above. However, the uncertainties are also significantly lower as they usually are for lower returns. This is a market situation experienced in Norway in 2004/2005 and is given in detail in table 6.

$r = 0.02$	$g = 0.035$	$p^{B\&S} = 3.60$
$\mu = 0.06$	$\nu = 0.05$	$\lambda = 1$
$\sigma = 0.10$	$\eta = 0.07$	$\gamma = 10^{-11}$

Table 6: Minimum interest rate guarantee with lower returns and volatility

The fact that all returns are lowered compared to the market in section 3.1 and that the issuer needs to repay the invested money with a guarantee should imply the price to be high. As we can see from the B&S prices in table 6 and 5, this is not the case. By comparing these two B&S prices, we can see that the one in this example is 20% lower than the one in the former example. This can be explained by the fact that the volatilities are significantly lowered. In particular the volatility of the fund, which is only about half here than what it was in the former section.

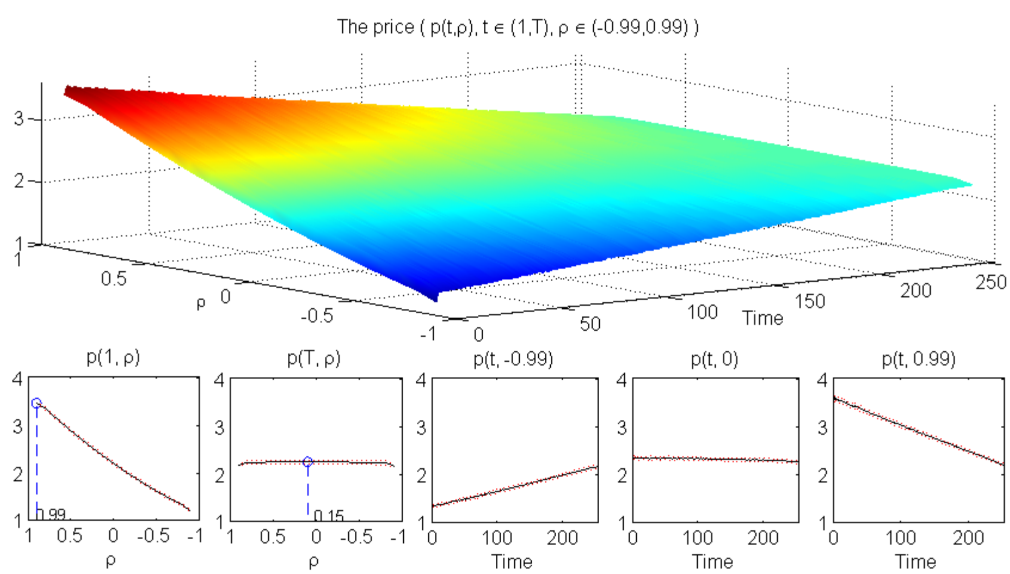

Figure 13: The utility indifference price of investments in the market described by table 6

The form of the price curve in figure 13 is remarkably similar to the one in figure 6. The only real difference is the level of which it lies on. While the

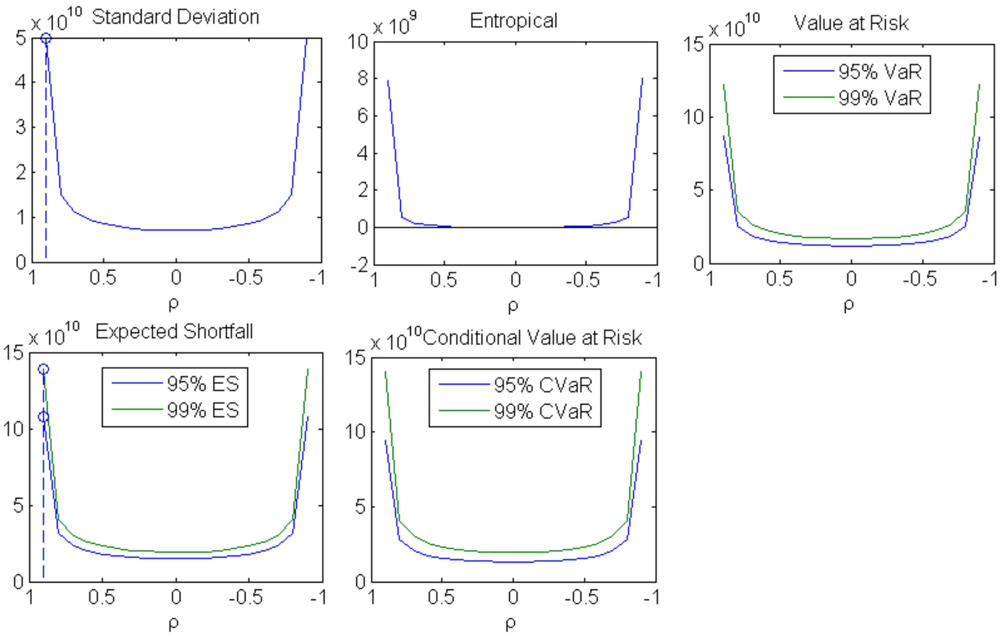

Figure 14: The residual risk of investments in the market described by table 6

highest and lowest price in figure 6 was approximate 4.3 and 1.9, respectively, the same numbers here are 3.45 and 1.2. Hence, the minimum price for this market is found on a significantly lower level than for the former market. As we saw above, the B&S price was 20% lower here than what it was under the first market. The differences between the largest and the lowest indifference prices in the two markeds are 25% and 58%, respectively. Hence the decrease is rather skew with a larger increase for positive correlation than for negative. This may well be a result of diversification.

In figure 14 I have plotted some measures of the risk of issuing the given guarantee in the given marked. As we can see, the risk is still high, but compared to the risk in the former section, its VaR, CVaR and ES values has decreased by about 60% while its standard deviation has decreased by nearly 50%. So despite the fact that the guaranteed return has increased from zero to 3.5 percent, and the expected returns of all asstes has been lowered, the minimum price and its risk have decreased quite drastically.

3.2.1 Raising γ again

As we did for the first market we examined, let us now try to raise the risk aversion of the life company wanting to issue an interest rate guarantee. We just looked at the case where the life company had close to no risk aversion.

Even when we raise the risk aversion, the price in the current market is lower than in the former. In fact, the difference has increased dramatically. This is seen when figure 15 is compared to figure 9 on page 28. Here we can see that

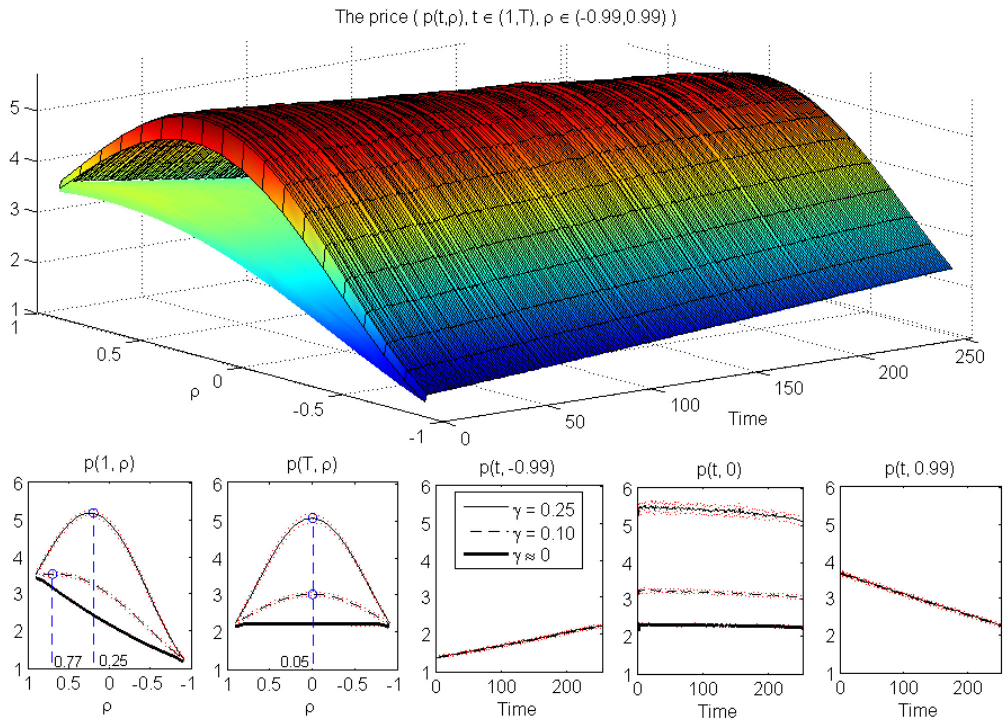

Figure 15: The indifference price of an interest rate guarantee in the market described by table 6 with $\gamma \in \{0.10, 0.25\}$

the highest price in this market is about one third of the equivalent price in the first market for $\gamma = 0.25$ and a little less than two thirds when $\gamma = 0.10$.

Compared to the minimum price, as seen in figure 13 and the thick line of figure 15, we have the same changes as in the former market. That is for instance that the ρ giving the maximum price is decreased while the price is increased for all ρ when γ is increased. Also for this market, we see that the price for $\rho \approx \pm 1$ is unaltered and thereby constant over γ. The uncertainty of the price does not differ by much between the markets.

As for the risk, the first thing to notice is the scale. By increasing the risk aversion parameter from close to zero to perhaps a more common level, the risk declines rapidly as displayed in figure 16. Here we can see that the measures of the residual risk have decreased to a two digit number at the most.

Compared to the other market we looked at, the measures of the residual risk has approximately the same form, but as with the price, at a much lower level. For instance, if $\rho = 0$, the 95% ES of the former market had a value of about 66 while the 95% ES of the current market has a value of about 16. The values for greater values of $|\rho|$, has a slight smaller difference, but it is still highly notable.

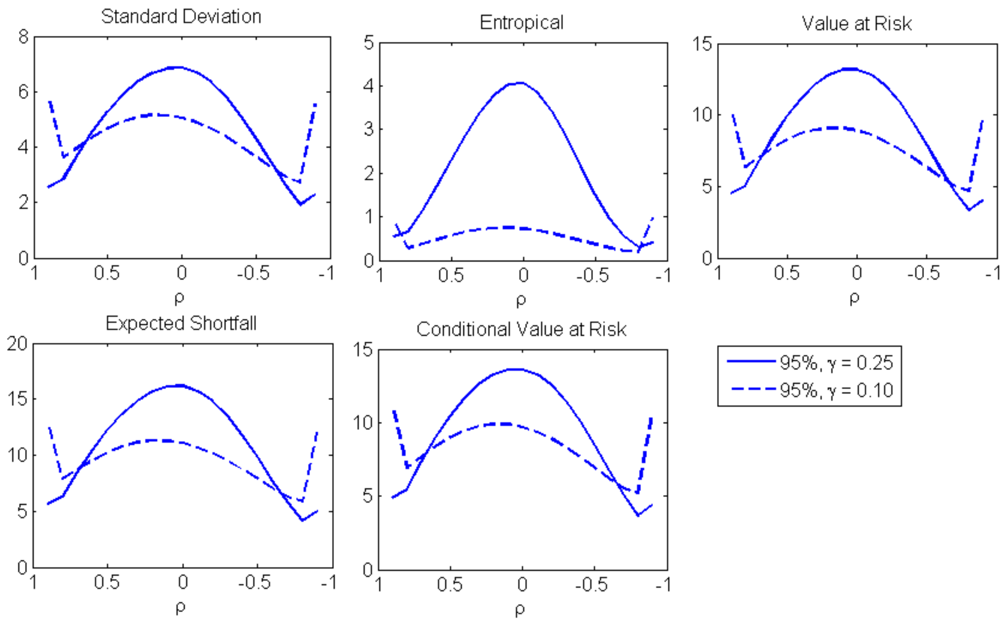

Figure 16: The residual risk of an interest rate guarantee in the market described by table 6 with $\gamma \in \{0.10, 0.25\}$

3.3 Summary of risk and price compared

What we have seen in this chapter is that the price and residual risk of an interest rate guarantee depends heavily on the risk aversion parameter γ. For low γ's, the price can be quite low, but at the cost of bearing a severe risk. For higher γ's the price is increased, but the risk is decreased.

We also saw that both the price and the residual risk depended on the the market with particularly emphasize on the volatility of the fund. For lower volatilities, the price was greatly reduced even when the other market parameters suggested the market to be harder to invest in, as we saw in section 3.2.1. Roughly speaking, only λ and γ changed the form of the price curve while the other parameters just changed the level of the price.

In section 3.1.1 we found that there must exist some $\hat{\gamma}$ such that $\vartheta(\mathcal{R}^{\hat{\gamma}}) \leq \vartheta(\mathcal{R}^{\gamma})$ $\forall\ \gamma \in \mathbb{R}^+$ and that this γ is quite low, that is around 0.1 or so.
Lastly, in section 3.1.2 we saw how the coherence property was present for all the risk measures used when γ was low and how it was destroyed for larger γ's, hence there must exist some $\tilde{\gamma}$ such that $\vartheta(\mathcal{R}_\lambda^{\tilde{\gamma}}) = \lambda \cdot \vartheta(\mathcal{R}_1^{\tilde{\gamma}})$. This point might be a bit confusing, since one would think that even if a trader had high risk risk aversion, he should like to lower it. However, the only way to lower the risk by diversification, is in our case to issue more options. A trader with high risk aversion would not want to do this since it means taking on more total risk, hence the coherence property should be destroyed.

4 Implementing a Lévy process

In this chapter I will investigate how the price would behave if the stock we were to invest in allowed jumps, i.e. were not continuous. A stock behaving like this is more intuitive since the economic market operates in discrete time, which could cause a stock to increase drastically, or 'jump', in no time. Investing in a stock behaving like this, would cause the fund Y_t to become discontinuous as well. This discontinuous new stock will be denoted by Z_t. I will both try a direct approach and an adjusted approach, where S_t and Z_t will have some common properties. The direct approach will be carried out by simply replacing S_t with Z_t, or in other words, just adding a compound Poisson process to S_t. But first, we will need some notion of such a jump or Lévy process.

4.1 Defining and fitting the Lévy process

Definition 4.1. *Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ be a filtered probability space. An $\mathcal{F}_t$-adapted process $\{L_t\}_{t\geq 0} \subset \mathbb{R}$ with $L_0 = 0$ a.s. is called a Lévy process if L_t is continuous in probability and has stationary and independent increments.*

Theorem 4.2. *Let $\{L_t\}$ be a Lévy process. Then L_t has a càdlàg[11] version which is also a Lévy process.*

Proof. The proof is carried out in Protter[12] Theorem 30. □

Protter also proves that a Lévy process is a strong Markov process and that it is a semi martingale. While the latter is interesting but not directly necessary to know in this case, the former tells us that implementing a Lévy process into our fund, Y_t, will not destroy the funds Markov-property. Without this information we might have had to find another way to calculate the conditional expectation in $w(t, y)$ given in equation (4) on page 4 in chapter 1.

Because of the result of Thorem 4.2, we can assume L_t to be càdlàg. The jump of L_t at time t is then defined by $\Delta L_t = L_t - L_{t^-}$ while the number of jumps of size $\Delta L_s \in U$ can be denoted by

$$N_t = \sum_{s\in(0,t]} \mathcal{X}_U(\Delta L_s)$$

for a fixed domain U.

Now, the most natural way to create a jump, would be by using a Poisson distributed variable for each point in time, to decide if the process should jump at that time or not. By specializing Theorem 1.35 of Protter[12] to our simplistic case, we can deduce the following theorem.

Theorem 4.3. *Let $\Delta L_t \in \mathbb{N}$. Then the process N_t is a Poisson process.*

[11] Right continuous with left limits

For a more rigorous and general treatment of the creation of a Lévy process, I refer to e.g. chapter 1 in Øksendal and Sulem[3] or Protter[12].
As we recall from chapter 1 of this paper, our stock, S_t, is a stochastic process driven by a Brownian Motion, W_t. From time s to time t, the stock has an expected profit of $\mu(t-s)$ and a volatility of $\sigma\sqrt{t-s}$, i.e. the value of the stock at time t is given by

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

To extend this stock to a stock allowing jumps, one 'simply' need to add a compound Poisson process

$$C_t = \sum_{i=1}^{N_t} D_{t,i}$$

where $N_t \sim \text{poiss}(\mathcal{I}_T)$, $t \in [O, T]$ and $D_{t,i}$ is some random variable. An increment of this process from time s to time t, i.e. $s > t$, is given by

$$C_s - C_t = \sum_{i=N_{t+1}}^{N_s} D_{s-t,i}$$

This is independent of $D_{t,1}, \ldots, D_{t,N_t}$ and depends only on the difference $s - t$. Thus C_t is a Lévy process.

To decide the size of the jumps, one will need to choose an appropriate probability distribution for the $D_{t,i}$'s. In this paper, $D_{t,i}$ is chosen to be standard normal. It would also be interesting to look at other distributions, but the normal distributions is very well known and easy to work with. Further, $\mathcal{I} = \mathcal{I}_T$ is the jump intensity, usually a quite low number defined by the number of expected jumps in a given time interval $[0, T]$. The dynamics of the stock allowing jumps, denoted Z_t, will then be the following

$$dZ_t \;=\; Z_t \left(\mu_z dt + \sigma_z dB_t + \xi dC_t\right) \tag{20}$$

To model this Lévy process, Euler's method method was used. The idea in this method is to simply discretify the diffusion as it is, to create a recursive formula to simulate from. In appendix A.2, I have proved that such a formula will be of the form

$$Z_t \;=\; Z_{t-1}\left(1 + \mu_z + \sigma_z \epsilon_t + \xi \sum_{i=1}^{\pi_t} D_{t,i}\right) \tag{21}$$

when time increments are equal to one. Here μ_z and σ_z is the drift and volatility of the process, respectively, while ξ decides the size of the jumps and $\pi_t \sim \text{poiss}(\mathcal{I})$.

To be able to compare these two models in a sensible way, one should make sure they have some equal properties, i.e. equal expectation and variance. In other words, one would like to calibrate the Lévy process to make it match the geometric Brownian motion driven process.

Brief discussion: Is this calibration natural to perform? One might argue that if one were to add the possibility of jumps in a process, that should imply that the process should become more volatile. I believe that argument is well addressed, but that is not the point here. The point here is to investigate the robustness of the utility indifference price if one were to allow jumps in the stock market, and to compare that price to the price in a continuous[12] market. In doing this, one is basically comparing two prices of derivatives having two quite significantly different behaviors. Therefore, at least their properties such as expectation and variance should be the same.

The most natural way of achieving this calibration would be to find and fix a suiting value for our two new parameters $\mathcal{I}$ and ξ and then to adjust μ_z and σ_z. Since I have chosen $D_{t,i}$ to be standard normal for all $i \in [1, N_t]$, $t \in [0, T]$, my hypothesis is that $\mu_z = \mu$ will give $\mathrm{E}[Z_t] = \mathrm{E}[S_t]$. The reason for this is that since the Gaussian distribution is symmetric, I do not add or subtract anything to the drift of the stock by adding the compound Poisson process C_t.

On the other hand, the drift of a process is indirectly affected by its volatility, so since we have to adjust the volatility of Z_t, σ_z, we might also have to adjust its drift μ_z in a more or less significant way. Note that if we let any of the parameters controlling the compound Poisson process, $\mathcal{I}$ or ξ, tend to zero, $Z_t \to S_t \; \forall \, t$.

In appendix A.3, I have proved that my hypothesis concerning μ_z was correct and that

$$\sigma_z = \sqrt{\sigma^2 - \xi^2 \mathcal{I}} \tag{22}$$

Note that this means that $\sigma^2 > \xi^2 \mathcal{I}$ or $\sigma > \xi\sqrt{\mathcal{I}}$ since the volatility parameter is assumed to be positive. If there is equality here, Z_t will be a pure jump-process with no dependency of the Brownian motion. We know for sure that this is not how stocks fluctuates in general. Also note that since the above boundary is present, Z_t can either have few, but large, jumps, or it can have small, but many, jumps. The extreme case of the latter, that is letting $\xi \to 0$, would be the same as just adding another Brownian motion with ξ as volatility since my $D_{t,i}$'s are standard normal. Since ξ tends to zero in this case, this would be like adding nothing.

In figure 17, I have compared the expectation, standard deviation, 95% and 99% confidence intervals of the continuous stock

$$S_t = S_{t-1} \cdot (1 + \mu + \sigma \epsilon_t)$$

and the calibrated stock allowing jumps

$$Z_t = Z_{t-1} \cdot \left(1 + \mu + \sigma_z \epsilon_t + \xi \sum_{i=1}^{\pi_t} D_{t,i} \right)$$

[12] Continuous in the sense that no jumps are allowed, but we still consider a discrete market

Starting from the top of the plots figure 17 we have the 99% and 95% confidence interval, the standard deviation and then the mean. As we can see from the left subplot, S_t and Z_t behave quite alike both in expectation and standard deviation already at 500 simulations. The tail behavior is more uncertain, but we can clearly see signs of resemblance. In the right subplot, S_t and Z_t seem to have identical distributions. For $t \in [0, 50]$, there is a slight difference in the 99% confidence interval, but I would say it is insignificant. Figure 18 shows us five

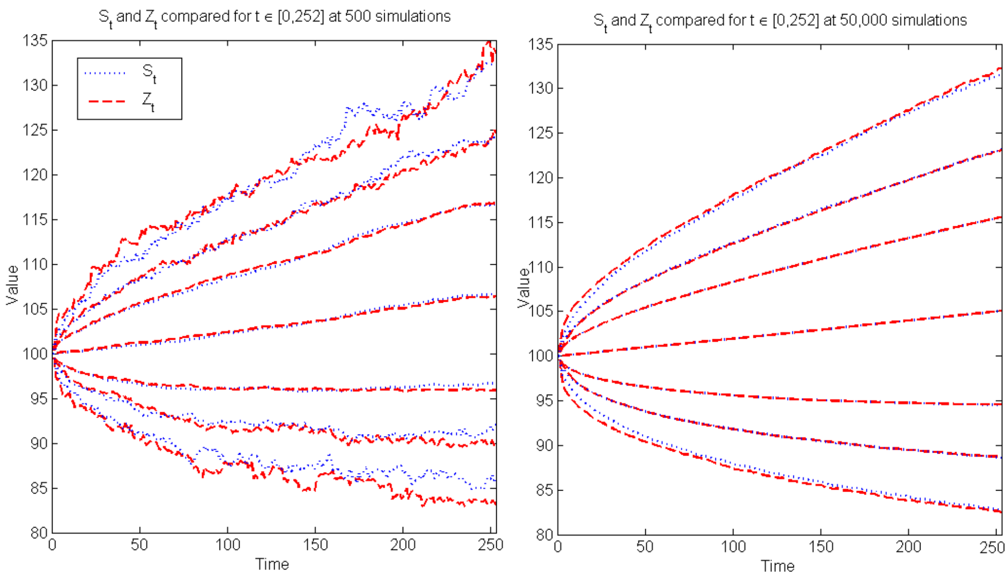

Figure 17: Comparison of S_t and Z_t

independent runs of S_t and Z_t. Parameters are $(\mu, \sigma, \xi, \mathcal{I}) = (0.05, 0.1, 10, 5)$, that is per year 5% expected growth, 10% standard deviation and 5 expected jumps which are normally distributed with expectation 0 and standard deviation equal to 10. σ_z is given by equation (22) above.

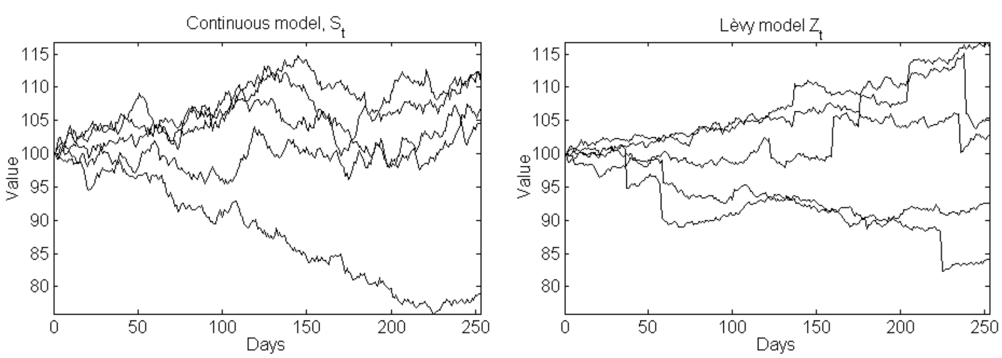

Figure 18: Five independent runs of S_t and Z_t

Further, as we can see from figure 19, the distribution of S_T and Z_T are practically identical when we have a high number of simulations, here 50,000.

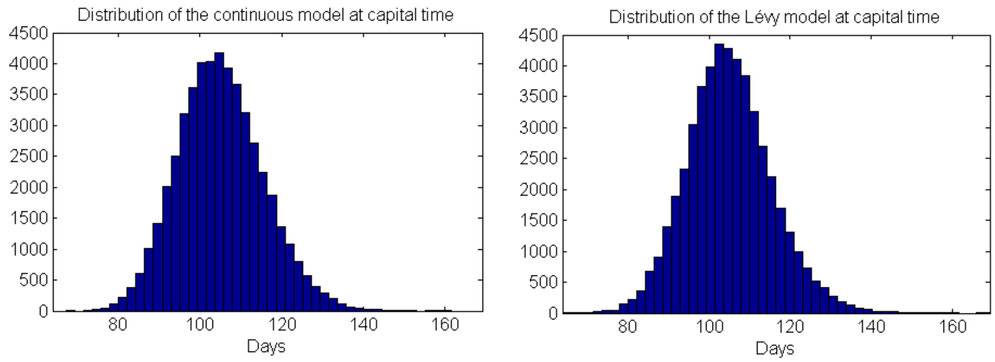

Figure 19: The distribution of 50,000 simulations of S_t and Z_t

4.2 The utility indifference price with an underlying allowing jumps

To simplify notation, let me first define $p(S_t)$ and $p(Z_t)$ as the utility indifference prices of issuing such interest rate guarantees where the fund of the issuer depends on a continuous stock and a stock allowing jumps, respectively. As introduced in the first chapter of this paper, the value dynamics of the fund, Y correlated by a tradeable asset S, is

$$\frac{dY_t}{Y_t} = \nu dt + \eta \left(\rho dB_t + \sqrt{1-\rho^2} dW_t \right)$$

where S has the dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

Now, we will replace S by our jump process Z, and the dynamics of the fund will then be

$$\frac{dY_t}{Y_t} = \nu dt + \eta \left(\rho (dB_t + \xi dC_t) + \sqrt{1-\rho^2} dW_t \right)$$

The latter is the Y I am referring to in rest of this chapter.

Basically, what one is doing when adding a compound Poisson process to the value of the fund, is adding more volatility, more risk. Therefore, and since our price is a compensation for the risk the company is taking by issuing such a product, one would think that the price would increase. For correlation $\rho > 0$ our expectation is met, but it appears that this is not the case if the correlation is negative. This is seen in figure 20 where the parameters used are as described in table 7.

As we can see from subplots 4 and 6, the difference is greatest at initial time, and then decrease as time goes by until it is about zero at capital time, as seen in subplot 3. Subplot 5 shows us that $p(S_t)$ and $p(Z_t)$ are about the same for all times if $\rho = 0$. The main point here, is displayed best in subplot 2, where

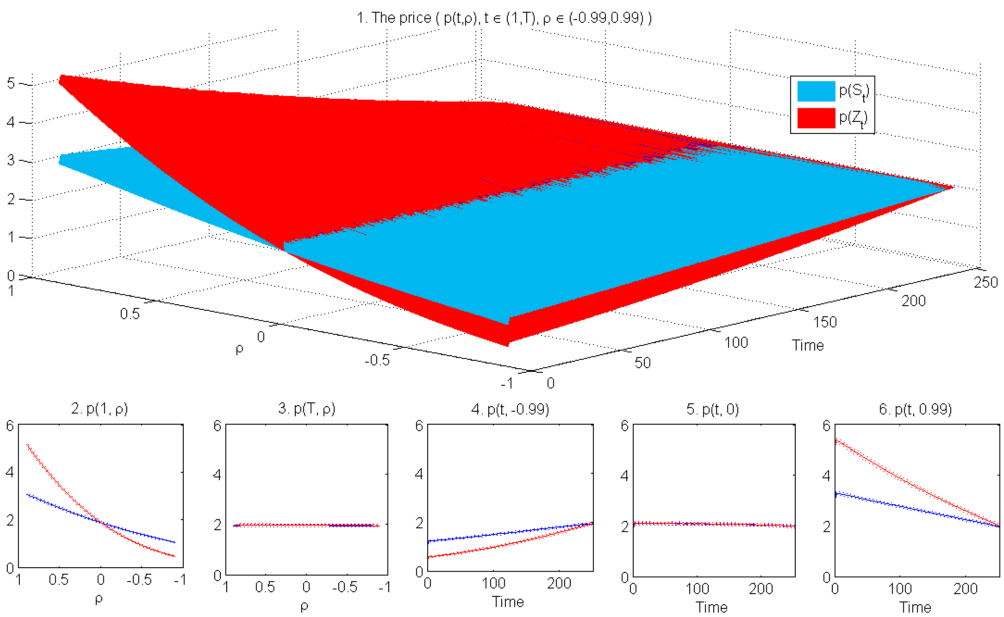

Figure 20: The prices $p(S_t)$ and $p(Z_t)$ compared

$r = 0.035$	$g = 0.0$	$Y_0 = 100$
$\mu = 0.07$	$\nu = 0.08$	$\lambda = 1$
$\sigma = 0.10$	$\eta = 0.12$	$\gamma = 10^{-11}$
$\mathcal{I} = 5$	$\xi = 0.04$	$\sigma_z = \sqrt{\sigma^2 - \xi^2 \mathcal{I}}$

Table 7: Parameters used to produce figure 20

the difference of $p(S_0)$ and $p(Z_0)$ is seen in detail. Here we can clearly see the intersection of $p(S_0)$ and $p(Z_0)$ at $\rho = 0$ and how $p(Z_0)$ is lower than $p(S_0)$ for negative ρ's and greater for positive ρ's. As time goes by, the intersection of $p(S_0)$ and $p(Z_0)$ fluctuates around zero with growing volatility and ends up a little below zero at capital time. This intersection is interesting since it tells us whether we are to invest in a stock allowing jumps or not at a given time if we, for instance, want a low price.

Even though this is a constructed scenario, one might think that it would be possible to find two stocks having the same (total) volatility, but only one of them are allowing jumps. The optimal investment to attain low prices, would then be to invest in the continuous one if it were positively correlated to the fund and to invest in the one allowing jumps if it were negatively correlated with the fund. Remember that we are still assuming a risk aversion, γ, close to zero.

While the price changed pretty much by introducing Lévy processes to the fund, the residual risk process did not change at all. Or at least so it seems when one compare them to each other as done in figure 21.

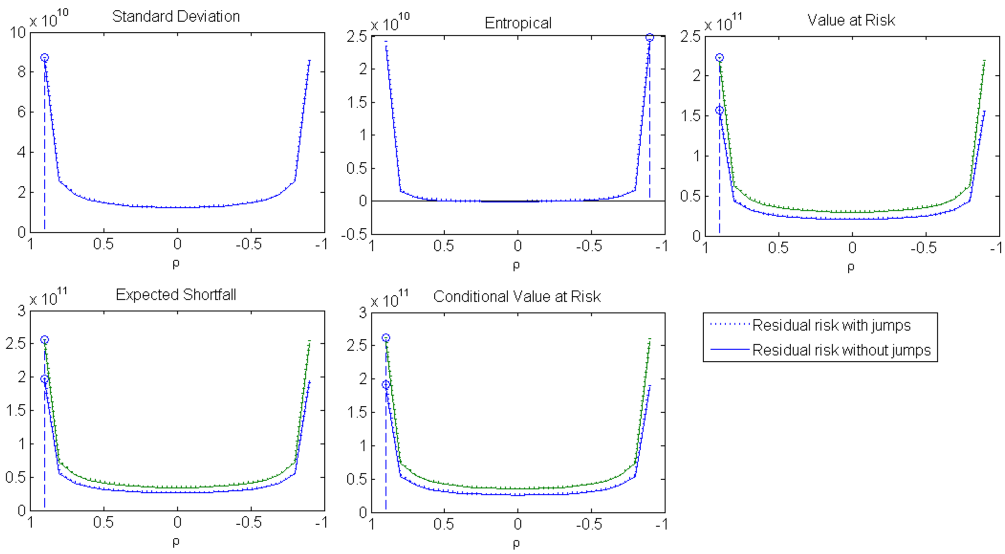

Figure 21: Residual risk with and without jumps measured and plotted

However, what we must take into account here, is the magnitude of the numbers. Therefore, one should rather look at the difference of the measures in stead of comparing them. This is done in figure 22 where we can see that there is a remarkable difference. Allthough the difference is of magnitude 10^9, one could argue that the difference is insignificant and may as well be a product of model uncertainty. However, the difference is consistent and supports the theory that adding risk should give higher risk measures. Still, the increase is quite small, roughly 2%, so I will conclude that it is insignificant.

It is interesting to note that the prices $p(S_t)$ and $p(Z_t)$ intersected and shifted at $\rho \approx 0$, but the measures of the residual risk is symmetric around zero.

Since it is possible to lower the price by investing in a stock that can allow jumps, but has the same volatility as before, one may ask; How low is it possible to get the price under such assumptions or circumstances? The answer is zero. Theoretically, at least. However, to achieve this, one have to choose the parameters σ, ξ and $\mathcal{I}$ in such a way that σ_z is close to zero. By doing this, Z_t will become approximately a pure jump diffusion. This case is highly unlikely to be experienced, but it is nevertheless interesting to examine to get a better view of what can happen in strange circumstances.

The results of such a scenario is shown in figure 23. Here, $\sigma = 0.1$ and $\mathcal{I} = 5$ as before, but ξ has been increased to 0.0445 which yields that $\sigma_z = 6.2599 \cdot 10^{-4}$ by equation (22). At first glance, the price looked as if it were zero for quite a few values of t and ρ. But a closer examination, revealed that the price was

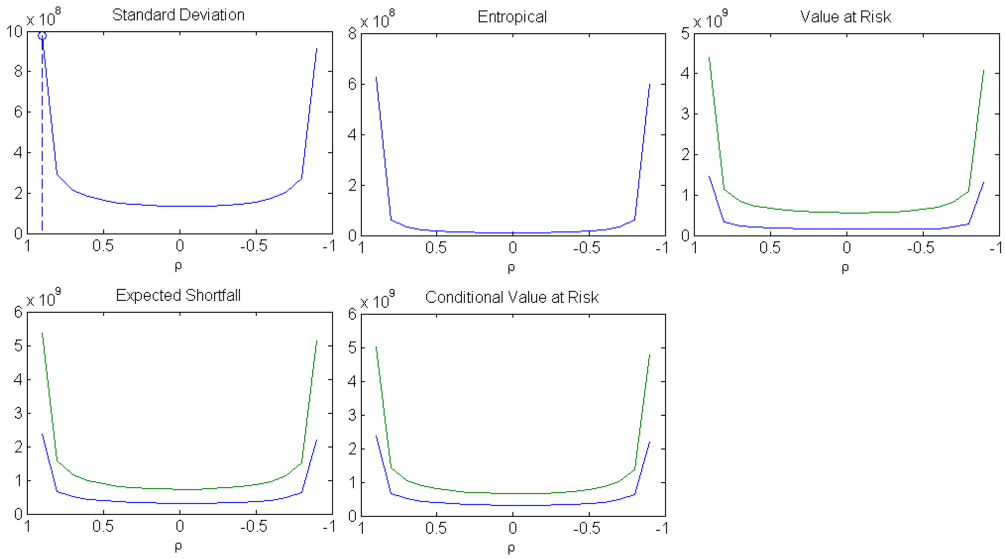

Figure 22: The difference of the measures of residual risks with and without jumps

equal to zero at only one point, namely $(t, \rho) = (0, -1)$. Therefore, I added a constant surface equal to 10^{-3} to better show us where the price has a notable value.

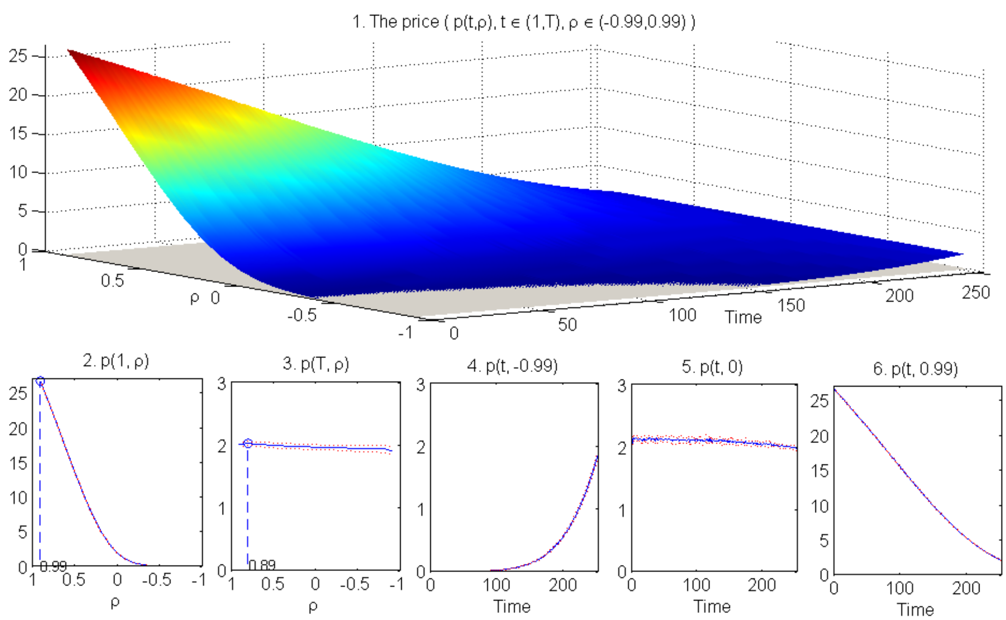

Figure 23: $p(Z_t) = 0$ when σ_z is close to zero

As we can see, still from figure 23, the price $p_\lambda^\gamma(t, y) \leq 10^{-3}$ for values $t \in [0, 150]$ and $\rho \in (-1, -0.5)$ approximate which form a triangle-looking shape. One should also notice that the price for ρ close to one is very high compared to the other figures we have looked at, for example figure 20. This effect decays over time and at maturity the prices are the same.

Even though the price is almost equal to zero, that does not mean that the company charging it has almost no risk, which one could think. It is rather that the company does not care about whether they take risk or not since their risk parameter is low. If one were able to hedge in such an asset that also is negatively correlated with our fund, one would be able to practically give the guarantee away. This does require that it is possible to find something to hedge in which behaves like a pure Lévy process. The residual risk process did not change much in this case either, but remained at a very high level, namely in a magnitude of 10^{11}. The same as we saw in chapter 3.

4.2.1 No calibration

Now, what would happen to the price if we kept all parameters as above, but did not calibrate, that is $\sigma_z = \sigma$? The answer is 'nothing'. Hence it may seem as if the price is independent of the choices of ξ and $\mathcal{I}$ and even Lévy or Brownian motion. In my opinion, the price should increase for all $\rho \in [-1, 1]$, $t \in [0, T]$ since we add more risk to the portfolio, and the price should reflect the risk the company takes on. As we can see in figure 24, this is not the case. In this figure, the price over time and correlation is plotted for a price where the fund is correlated with a continuous stock (black), and a stock allowing jumps (colored). We can see that there are very few differences, and the differences present is probably from simulation error.

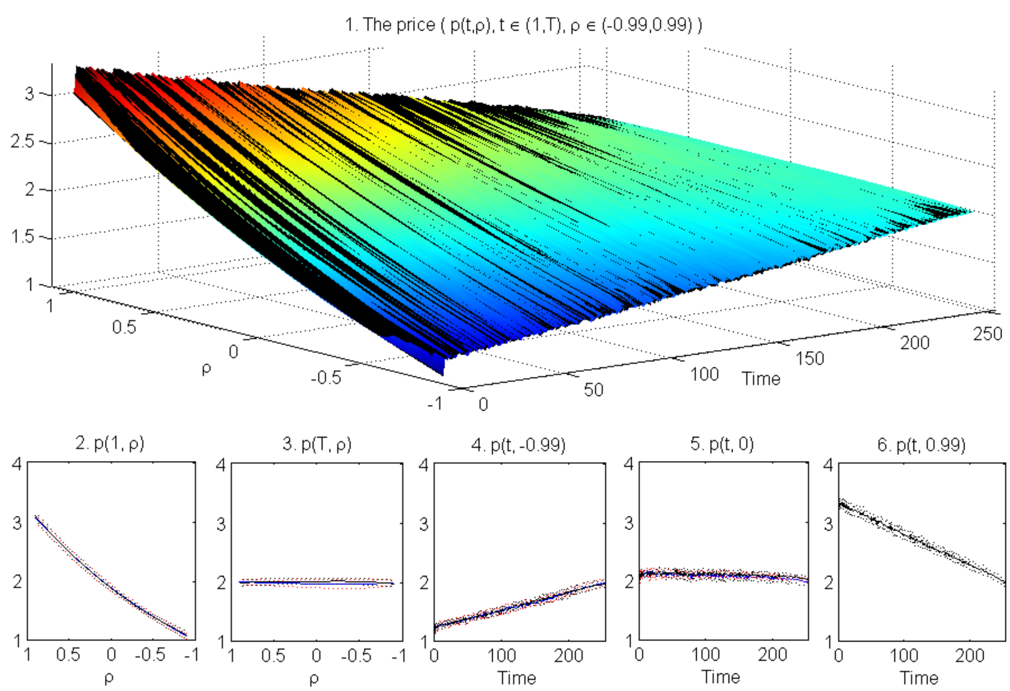

Figure 24: Figure of the price without calibrating. Parameters are as in table 7

At first, one might think that the risk aversion parameter, γ, is to blame since it is still held at a very low level. However, it turns out that even if we increase γ, the prices still do not differ significantly from each other. No difference is present in the residual risk process or the measure of it either.

4.2.2 Rising risk aversion

Now, lets go back to the case where we calibrate the Lévy process, as it is still my opinion that this is the 'correct' way to do it. But this time, lets see what happens if we raise the risk aversion. The price should be higher in both cases, and it should also change its form as we saw in chapter 3. In figure 25, we can see that the price curves here are both higher, at least in some cases, and have a different form compared with the ones in figure 20 on page 40. But, they are also similar to the ones in figure 20 in the way that their difference get their sign reversed at approximately $\rho = 0$ at time zero. This effect is only present if one choose ξ and $\mathcal{I}$ such that σ_z is quite small. In the figure discussed here, $\sigma_z = 0.02$. The larger σ_z, i.e. the lower ξ and/or $\mathcal{I}$ is, the more $p(S_t)$ and $p(Z_t)$

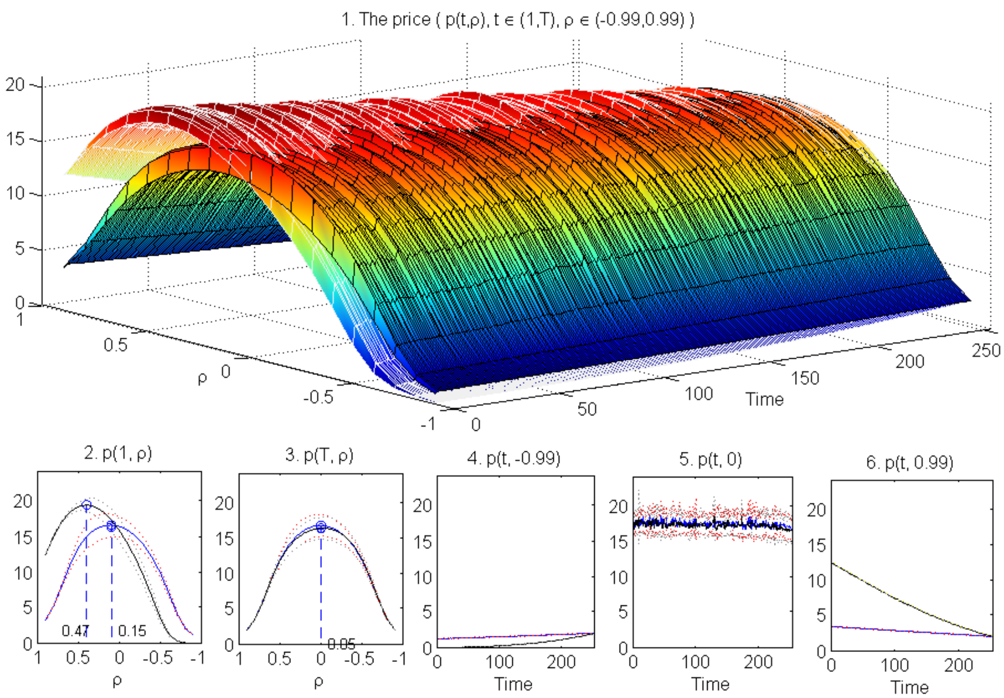

Figure 25: The price when risk aversion $\gamma = 0.5$

are alike which it should be as discussed before.

The most important subplot of figure 25 is subplot 2. Here the line to the left, with its peak at $\rho = 0.47$, is representing $p(Z_t)$ and the other is representing $p(S_t)$. In subplot 1, $p(Z_t)$ is the one with white stripes. Subplot 3 and 5 shows us that the two prices are almost equal at terminal time and for $\rho = 0$, respectively. The fourth and sixth subplot shows us that $p(Z_t)$ is lowest for $\rho = -0.99$

while $p(S_t)$ is lowest for $\rho = 1$. We also note that the insecurity is quite large for $\rho \in (-0.5, 0.5)$, approximately. The parameters here are as in table 8.

$r = 0.035$	$g = 0.0$	$Y_0 = 100$
$\mu = 0.07$	$\nu = 0.08$	$\lambda = 1$
$\sigma = 0.10$	$\eta = 0.12$	$\gamma = 0.5$
$\mathcal{I} = 6$	$\xi = 0.04$	$\sigma_z = \sqrt{\sigma^2 - \xi^2 \mathcal{I}} = 0.02$

Table 8: Parameters used to produce figure 25

So what we have found here, is that there exist a set of $(\xi, \mathcal{I})$ such that the utility indifference price is significantly lower if it depends on a Lévy process and $\rho < 0$, than it would be if it depended on a Brownian motion. If $\rho > 0$, the opposite holds. We also note that for $\rho \approx 0$, this effect fades away quite fast as time goes by. In the opposite case, when $\rho \approx \pm 1$, we can see from subplots 4 and 6 that the effect lasts almost until maturity, where it is lost for all ρ. This effect remains the same if one were to raise λ as well. It does also seem as if the result of the examination I did on $\hat{\rho}$[13] in chapter 1 would have held even stronger for $p(Z_t)$.

Now, what would be really interesting were if we were looking at parameters estimated from the market. Then we could see which sets of $(\xi, \mathcal{I})$ are common and which are not. However, jump intensity parameters can be hard to estimate, and it is not within the scope of this paper to do so.

4.3 Summary of the Implementation of Lévy processes

In this chapter I have developed a way to compare a Brownian motion driven fund to a fund that is driven by a Lévy process. What we found was that the price did not depend on the choice of model (Lévy or Brownian motion), unless the Lévy process was calibrated to have the same properties as the Brownian motion. I find this to be a bit odd, but I rely on my calculations.

Further, the price when the fund was driven by the Lévy process was lowest for negative correlation and highest for positive correlation compared to the price when the fund was driven by the Brownian motion. We also found that the price $p(Z_t)$ could be very close to zero for some appropriate parameters. Even though the price in the continuous case and the Lévy case had quite a few differences, the measures of the risk remained roughly the same independent of parameter choice.

[13]The correlation giving the largest price at initial time

5 Conclusions and Extensions

5.1 Conclusions

In this paper, we have looked at the price and risk of interest rate guarantees for different input parameters with special emphasize on the correlation coefficient, ρ, the risk aversion parameter, γ and the number of options issued, λ. Since I have not used estimated parameters, the inspection has been more of a stress-testing nature and it is not necessary to evaluate parameter insecurities. Had the parameters been estimated, it would have been natural and important to discuss the insecurity of these.

My calculations support the conclusion of Benth and Proske[6] where they find that the minimum price is obtained if one can hedge in negative correlated assets. This also follows from the diversification argument. By implementing the Lévy framework, I find that this minimum price can be further reduces without increasing risk substantially. In order to obtain this, it must be possible to hedge in an asset that is negatively correlated and that allows jumps. This effect is only observed when S_t and Z_t are adapted to each other, i.e. they have the same drift and total volatility. We also saw that the price when hedging in a stock allowing jumps could be almost zero for certain parameters in the Lévy process.

As I mentioned in the introduction, a good model should be as close to the real world as possible, but should also have as few parameters as possible. One could therefore argue that an extension to include Lévy processes is not necessary and does not give as much information as it adds uncertainty. Uncertainty around estimated Lévy parameters is quite high, however, some points could still be of interest to have in mind, such as the point mentioned above.

Further, we saw that for risk aversion close to zero, the residual risk was high with values in the region of 10^{11}. This number decreased as the risk aversion increased until a certain point. It seemed as if the residual risk had a minimum for $\gamma \approx 0.1$ for $\rho \in [-0.8, 0.8]$ approximately. Increasing γ further actually seemed to increase the residual risk. In other words, there exist an optimal γ to obtain the lowest risk measures as we saw in section 3.1.1. This means that it is possible to optimize the price-risk problem in two ways; Either one can 'choose' γ such that one obtain the lowest price, or one can 'choose' γ such that the residual risk is minimized. While the first choice means that the residual risk will be very high, the second choice does not imply that the price necessarily will increase drastically. A natural extension of this paper would, in addition to the other extensions below, be to prove this claim and perhaps find a formula for the price when the residual risk has been minimized.

By implementing, and working with, the different risk measures, I also found another property that needs to be satisfied if a risk measure is to be called a good risk measure. In conjunction to the properties listed in chapter two, a good risk measure also needs to be easy to work with and easy to understand for a third party. For these reasons, I would like to emphasize Expected Shortfall as the risk measure of choice. This is because it has the 'right' properties, read subadditivity and thereby coherence, it is easy to calculate and easy to understand. That being said, it turned out that all the risk measures I looked at were equally coherent in this context. In chapter 3 we saw that the coherence property held for small risk aversions, but was destroyed for all the measures when γ increased.

5.2 Possible extensions and known weaknesses

Both the field of option pricing and risk analysis are quite large and there are therefore numerous ways of extending this paper. I will in this section try to list and discuss which ones I think are the most obvious or interesting ones.

The most obvious extension in my opinion, would be to look at how the price behaves if some of the inputs were stochastic variables. Stochastic volatility, interest rate and correlation would most likely have been interesting to take a look at. The correlation between a pair of stocks, for instance, is not a constant relationship. It can change through several unexpected circumstances. Therefore, adding a noise part to the correlation would make sense. This would make the price more volatile, and companies wanting to sell a put option may then need to charge an even higher price if they want to be sure of not incurring a loss.

It is also reasonable to believe that the volatility of a given stock can change over time. Research has shown that even in the most liquid markets, volatility displays dynamics and it is normal to observe clustering effects, where it is observed that large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes. By modeling the volatility, one would catch these clustering phenomena, and see how they affect the price. The effect of this might not be as large as making the correlation stochastic, but would indeed have been noticeable. Both these parameters could have been modeled using GARCH or ARMA models.

Another adjustment in this case would have been to model the interest rate, for instance with Black-Karasinski, but it is my opinion that this extension would have been far less noticeable than the ones above.

How the price and risk would have changed under such circumstances would have been interesting to examine. But again, introducing these models would have required more estimated parameters and would have led to even larger uncertainty of the entire model, as I mentioned briefly in the introduction.

It could also be interesting to look at another utility function. Allthough the utility function presented here, is an intuitive one, it could be interesting to look at others. This would, however, change the form of the utility indifference price quite dramatically depending on which function is chosen.
Further, I would also have added rigorous proofs of more of the things I have

'discovered'. I would especially like to prove that $E[\hat{\rho}]^{14}$ is positive and perhaps derive a closed formula for it.

Also, as previously discussed, it would be interesting to fit the Lévy process to marked data, allthough it is my impression that the parameters needed are hard to estimate. I have basically stress tested the model, but it would be interesting to see how it behaves when fitted to a dataset. It could also be interesting to let the fund be dependent on, say, two Lévy processes.

Limitations of the model

The most serious limitation is that if one choose λ or γ too high, the values needed to calculate w gets too high for the computer to handle. I do not know if this is problem persists when solving $w(t, y)$ as a parabolic differential equation as done in Benth and Proske[6], but from looking at the formulas, I think it would. This is a serious problem since it in practice is interesting to look at arbitrary high values of the number of options issued, λ. Especially when analyzing the residual risk. As we saw, the risk measures were coherent for low risk aversions. In such cases the quantity of options issued would be of importance.

In hindsight, it would probably have been better to solve $w(t, y)$ by the parabolic differential equation (pde) in stead of using Monte Carlo on the Feynman-Kac representation, but since the pde was used in Benth and Proske [6], I thought it would be interesting to use Monte Carlo

[14] As introduced in section 1.3

Appendix

A Technical calculations and proofs

A.1 The increase and decrease of the utility indifference price

This is a proof of why p_λ^γ increases with respect to $\rho \in (-1, 0)$ if $\mu \geq r$ and decreases with respect to $\rho \in (0, 1)$ if $\mu \leq r$.

We have from the definitions in chapter one that

$$p_\lambda^\gamma(t, y) \quad = \quad e^{-r(T-t)} \frac{\ln w(t, y)}{\gamma(1-\rho^2)}$$

where

$$w(t, y) \quad = \quad E\left[\exp\left\{\lambda\gamma(1-\rho^2)(K - S_T)^+\right\}\right]$$

for $S_T = y \cdot \exp\left\{(\delta - \frac{1}{2}\eta^2)(T-t) + \eta(B_T - B_t)\right\}$ and $\delta = \nu - \eta\rho\frac{\mu - r}{\sigma}$.

To simplify calculations, lets split $w(t, y)$ into two parts, say A_1 and A_2:

$$\begin{aligned} w(t, y) \quad &= \quad P\left(K \leq S_T\right) + E\left[\exp\left\{\lambda\gamma(1-\rho^2)\right\} \mathbf{1}_{\{K \geq S_T\}}\right] \\ &= \quad A_1 + A_2 \end{aligned}$$

One can then differentiate this part by part, starting with A_1:

$$\begin{aligned} A_1 \quad &= \quad P\left(\left(\ln\frac{K}{y} + (\frac{1}{2}\eta^2 - \delta)(T-t)\right)\eta^{-1} \leq B_{T-t}\right) \\ &= \quad 1 - P\left(B_{T-t} < \left(\ln\frac{K}{y} + (\frac{1}{2}\eta^2 - \delta)(T-t)\right)\eta^{-1}\right) \\ &= \quad 1 - \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{u} \exp\left\{\frac{-y^2}{2(T-t)}\right\} dy \end{aligned}$$

for $u = \left(\ln\frac{K}{y} + (\frac{1}{2}\eta^2 - \delta)(T-t)\right)\eta^{-1}$ which gives that

$$\frac{\partial}{\partial\rho}A_1 \quad = \quad -\frac{\mu - r}{\sigma}(T-t)\frac{1}{\sqrt{2\pi(T-t)}}\exp\left\{\frac{-u^2}{2(T-t)}\right\}$$

And then A_2:

$$A_2 \quad = \quad \int_{-\infty}^{u} C_1(x)dx$$

for

$$C_1(x) = \frac{1}{\sqrt{2\pi(T-t)}}\exp\left\{\lambda\gamma(1-\rho^2)\left(K - y\cdot\exp\left\{\eta x + (\delta - \frac{1}{2}\eta^2)(T-t)\right\}\right) - \frac{x^2}{2(T-t)}\right\}$$

This gives that

$$\begin{aligned}\frac{\partial}{\partial\rho}A_2 &= \frac{\mu-r}{\sigma}(T-t)\cdot C_1(u)\\&\quad -2\lambda\gamma\rho\int_{-\infty}^{u}\left(K-y\cdot\exp\left\{\eta x+(\delta-\frac{1}{2}\eta^2)(T-t)\right\}\right)\cdot C_1(x)dx\\&\quad +\frac{\mu-r}{\sigma}\lambda\gamma(1-\rho^2)\int_{-\infty}^{u}y\cdot\exp\left\{\eta x+(\delta-\frac{1}{2}\eta^2)(T-t)\right\}\cdot C_1(x)dx\end{aligned}$$

Let us now rejoin our two parts and note that $C_1(u)=\frac{\exp\left\{-\frac{u^2}{2(T-t)}\right\}}{\sqrt{2\pi(T-t)}}$.
We then get

$$\begin{aligned}\frac{\partial}{\partial\rho}w(t,y) &= \frac{\partial}{\partial\rho}A_1+\frac{\partial}{\partial\rho}A_2 \qquad (23)\\&= -\frac{\mu-r}{\sigma}(T-t)C_1(u)+\frac{\mu-r}{\sigma}(T-t)C_1(u)\\&\quad -2\lambda\gamma\rho\int_{-\infty}^{u}\left(K-y\cdot\exp\left\{\eta x+(\delta-\frac{1}{2}\eta^2)(T-t)\right\}\right)\cdot C_1(x)dx\\&\quad +\frac{\mu-r}{\sigma}\lambda\gamma(1-\rho^2)\int_{-\infty}^{u}y\cdot\exp\left\{\eta x+(\delta-\frac{1}{2}\eta^2)(T-t)\right\}\cdot C_1(x)dx\end{aligned}$$

To get a better overview, we introduce the positive variables C_2 and C_3:

$$\begin{aligned}C_2 &= \int_{-\infty}^{u}\left(K-y\cdot\exp\left\{\eta x+(\delta-\frac{1}{2}\eta^2)(T-t)\right\}\right)\cdot C_1(x)dx>0\\C_3 &= \int_{-\infty}^{u}y\cdot\exp\left\{\eta x+(\delta-\frac{1}{2}\eta^2)(T-t)\right\}\cdot C_1(x)dx>0\end{aligned}$$

Using this abbreviation, we get that

$$\frac{\partial}{\partial\rho}w(t,y) = \frac{\mu-r}{\sigma}\lambda\gamma\eta(1-\rho^2)C_3-2\lambda\gamma\rho C_2$$

We can then easily see that

$\frac{\partial}{\partial\rho}w(t,y)\geq 0$ when $\mu\geq r$ and $\rho\in(-1,0]$ and

$\frac{\partial}{\partial\rho}w(t,y)\leq 0$ when $\mu\leq r$ and $\rho\in[0,1)$.

Now, since we know that $w(t,y)$ and even $\ln w(t,y)$ are positive for all input values and that

$$\frac{\partial}{\partial\rho}p_\lambda^\gamma(t,y)=\frac{e^{-r(T-t)}}{\gamma(1-\rho^2)}\left(\frac{\frac{\partial}{\partial\rho}w(t,y)}{w(t,y)}-\frac{-2\rho}{(1-\rho^2)}\ln w(t,y)\right)$$

we get that
p_λ^γ is increasing for $\mu\geq r$ and $\rho\in(-1,0]$ while
p_λ^γ is decreasing for $\mu\leq r$ and $\rho\in[0,1)$
as was what we wished to show.

A.2 Euler's method used on a jump diffusion

The diffusion is given by $dZ_t = Z_t\left(\mu_z dt + \sigma_z dB_t + \xi dC_t\right)$. Here B_t is a Brownian motion and C_t is given by

$$C_t = \sum_{i=1}^{N_t} D_{t,i}$$

where N_t is a Poisson process with intensity $\mathcal{I}$ and $D_{t,i}$ is i.i.d $N(0,1)$ $\forall\ i \in [1, N_t]$. Discretifying Z_t with time increments $\Delta t = 1$ gives

$$\begin{aligned}
Z_t - Z_{t-1} &= Z_{t-1}\cdot\left(\mu_z + \sigma_z(B_t - B_{t-1}) + \xi\Delta C_t\right)\\
&= Z_{t-1}\cdot\left(\mu_z + \sigma_z\epsilon_t + \xi\sum_{i=N_{t-1}}^{N_t} D_{t,i}\right)\\
Z_t &= Z_{t-1}\cdot\left(1 + \mu_z + \sigma_z\epsilon_t + \xi\sum_{i=1}^{N_t - N_{t-1}} D_{t,i}\right)\\
&= Z_{t-1}\cdot\left(1 + \mu_z + \sigma_z\epsilon_t + \xi\sum_{i=1}^{\pi_t} D_{t,i}\right)
\end{aligned}$$

Here, I have used that $\epsilon_t = B_t - B_{t-1} \sim N(0,1)\ \forall\ t \in [0,T]$, that

$$\Delta C_t = C_t - C_{t-1} = \sum_{i=1}^{N_t} D_{t,i} - \sum_{i=1}^{N_{t-1}} D_{t,i} = \sum_{i=N_{t-1}}^{N_t} D_{t,i} = \sum_{i=1}^{N_t - N_{t-1}} D_{t,i}$$

and that $\pi_t = N_t - N_{t-1} \sim poiss(\mathcal{I})$.

A.3 Calibrating Z_t to fit S_t

Using the assumptions and results from A.2 it is easy to show that Z_t and S_t has the same drift, and to calculate σ_z if $\mathcal{I}$ and ξ are given. First off I will show that $\mu_z = \mu$ gives $E[Z_t] = E[S_t]$.

$$\begin{aligned}
E\left[\frac{Z_t}{Z_{t-1}}\right] &= 1 + \mu + \sigma E[\epsilon_t] + \xi E[\pi_t]\cdot E[D]\\
&= 1 + \mu = E\left[\frac{S_t}{S_{t-1}}\right]
\end{aligned}$$

And now for the volatility

$$\begin{aligned}
\mathrm{var}\left(\frac{Z_t}{Z_{t-1}}\right) &= \mathrm{var}\left(1 + \mu + \sigma\epsilon_t + \xi\sum_{i=1}^{\pi_t} D_{t,i}\right)\\
&= \sigma^2\mathrm{var}\left(\epsilon_t\right) + \xi^2\mathrm{var}\left(\sum_{i=1}^{\pi_t} D_{t,i}\right)\\
&= \sigma^2 + \xi^2\mathcal{I}
\end{aligned}$$

since by the law of conditional variance we have that

$$\begin{aligned}
\operatorname{var}\left(\sum_{i=1}^{\pi_t} D_{t,i}\right) &= E\left[\operatorname{var}\left(\sum_{i=1}^{\pi_t} D_{t,i}|\pi_t\right)\right] + \operatorname{var}\left(E\left[\sum_{i=1}^{\pi_t} D_{t,i}|\pi_t\right]\right) \\
&= E[\pi_t \operatorname{var}(D)] + \operatorname{var}(\pi_t E[D]) \\
&= \operatorname{var}(D)E[\pi_t] + E[D]^2 \operatorname{var}(\pi_t) \\
&= E[\pi_t] \\
&= \mathcal{I}
\end{aligned}$$

which gives that for

$$\begin{aligned}
\operatorname{var}\left(\frac{Z_t}{Z_{t-1}}\right) &= \operatorname{var}\left(\frac{S_t}{S_{t-1}}\right) \\
\sigma_z^2 + \xi^2 \mathcal{I} &= \sigma^2 \\
\sigma_z &= \sqrt{\sigma^2 - \xi^2 \mathcal{I}}
\end{aligned}$$

Hence, the volatility of Z_t, σ_z, must have the above form if Z_t and S_t are to have the same total volatility.

B Source code for Matlab

B.1 Input file

This file is a main file controlling and calling the other functions such that they are run with the same parameters. Basically makes it easier to switch between parameter sets while being confident that the different files uses the same set.

```
priceON = 1;                    %1 - The price is calculated
riskON = 0;                     %1 - The risk is calculated
price_jumpON = 0;               %1 - The price with jumps is calculated
risk_jumpON = 0;                %1 - The risk with jumps is calculated
model = 1;                      %Chooses between the different models

%Timeperspective
years = 1;                      %Not necessarily an integer
days = 252;                     %Number of financial days pr year
T = round(years*days)+1;        %Timeperspective
L = 19;                         %partition of the vector \rho

sim = 10000;                    %Simulations
m = 100;                        %Number of times the program is run

%The different market models:
if model == 1           %5.1
    ry = 0.035;                 %Annual interestrate (riskless asset)
    muy = 0.07;                 %Annual expected profit (risky asset)
    nuy = 0.08;                 %Annual expected profit (fund)
    sigmay = 0.12;              %Annual volatility (asset)
    etay = 0.15;                %Annual volatility (fund)
    gamma = 0.1;                %Risk aversion
    g = 0.00;                   %Annual guaranteed profit of the fund
    jiy = 6;                    %Annual JumpIntensity
    xi = 0.04;                  %The std of the jumpsizes
    lambda = 1;                 %Number of putoptions the investor is
                                %short

elseif model == 2       %5.2
    ry = 0.02;                  %Annual interestrate (riskless asset)
    muy = 0.07;                 %Annual expected profit (risky asset)
    nuy = 0.08;                 %Annual expected profit (fund)
    sigmay = 0.12;              %Annual volatility (risky asset)
    etay = 0.15;                %Annual volatility (fund)
    gamma = 0.5;                %Risk aversion
    g = 0.00;                   %Annual guaranteed profit of the fund
    jiy = 5;                    %Annual JumpIntensity
    xi = 0.04;                  %The std of the jumpsizes
    lambda = 1;                 %Number of putoptions the investor is
                                %short

elseif model == 3       %5.3
    ry = 0.02;                  %Annual interestrate (riskless asset)
    muy = 0.06;                 %Annual expected profit (risky asset)
    nuy = 0.05;                 %Annual expected profit (fund)
    sigmay = 0.10;              %Annual volatility (risky asset)
    etay = 0.07;                %Annual volatility (fund)
    gamma = 0.5;                %Risk aversion
```

```
    g = 0.035;                   %Annual guaranteed profit of the fund
    jiy = 4;                     %Annual JumpIntensity
    xi = 0.06;                   %The std of the jumpsizes
    lambda = 1;                  %Number of putoptions the investor is
end                              %short

%Constant calculations:
Y0 = 100;                        %Initial value of the fund
g = log(1+g);
r = log(1+ry)/days;              %Daily interest rate (riskless asset)
mu = log(1+muy)/days;            %Daily expected profit (asset)
nu = log(1+nuy)/days;            %Daily expected profit (fond)
sigma = sigmay/sqrt(days);       %Daily volatility (asset)
eta = etay/sqrt(days);           %Daily volatility (fun)
K = Y0*exp(g*T/days);            %Strike at time T
ji = jiy/(T-1);                  %Daily JumpIntensity

if priceON
    disp('Calculating the price')
    price(T,sim,m,r,mu,nu,sigma,eta,Y0,lambda,gamma,L,K)
end
if riskON
    disp('Calculating the risk')
    risk(T,sim,m,r,mu,nu,sigma,eta,Y0,lambda,gamma,L,K)
end
if price_jumpON
    disp('Calculating the price p(Z_t)')
    price_jump(T,sim,m,r,mu,nu,sigma,eta,Y0,lambda,gamma,L,K,xi,ji)
end
if risk_jumpON
    disp('Calculating the risk R(Z_t)')
    risk_jump(T,sim,m,r,mu,nu,sigma,eta,Y0,lambda,gamma,L,K,xi,ji)
end
```

B.2 Utility indifference price

This function calculates the price using the Feynman-Kac representation and Monte Carlo technique. The price is normalized with respect to λ, i.e. output is the price pr option issued, not the total price of the investment. Input is the market variables along with technical variables such as the number of times the program should be run. Output is a (time X correlation) matrix containing the price for all times $t \in [0, T]$ and all $\rho \in (-1, 1)$.

```
function [] = price(T,sim,m,r,mu,nu,sigma,eta,Y0,lambda,gamma,L,K)

rho = linspace(-0.99,0.99,L);              %Correlation Coefficient
tid = zeros(1,m-1);                         %For timing purposes
P = NaN(m,T,L);                             %The price pr option
for l = 1:m
    tic;                                    %For timing purposes

    %################### - The Model - ###################
    %Independent normally distributed variables
    epsW1 = random('normal',0,1,T,sim);
    epsW2 = random('normal',0,1,T,sim);
```

```

    Y = zeros(T,sim,L);                    %The fund under Qo
    epsW1o = epsW1 + (mu-r)/sigma;         %Qo Brownian motion
    delta = nu - eta*rho*(mu-r)/sigma;

    Y(1,:,:) = Y0;
    for t = 2:T
        for j = 1:L
            Y(t,:,j) = Y(t-1,:,j).*(1 + delta(j)
                            + eta*(rho(j)*epsW1o(t,:)
                            + sqrt(1-rho(j)^2)*epsW2(t,:)));
        end
    end

    %############ - Utility Indifference Pricing - ############
    tempw = zeros(T,sim,L);                %To estimate E[w(t,y)]
    for t = 1:T
        for j = 1:L
            a = Y(t,:,j)*exp((delta(j)-0.5*eta^2)*(T-(t-1)));
            b = exp(eta*sqrt(T-(t-1))*(rho(j)*epsW1(t,:)
                + sqrt(1-rho(j)^2)*epsW2(t,:)));
            tempw(t,:,j) = exp(lambda*gamma*(1-rho(j)^2)
                              *(K-a.*b).*(K>a.*b));
        end
    end

    clear epsW1 epsW2 a b y;               %Saving the memory by
    w = mean(tempw,2);                     %deleting used matrices
    clear tempw;

    pp = zeros(T,L);                       %The price pr option
    for t = 1:T
        for j = 1:L
            pp(t,j) = exp(-r*(T-t))*log(w(t,1,j))
                        /(lambda*gamma*(1-rho(j)^2));
        end
    end

    clear int dpdy dwdy epsW1o w
    P(l,:,:) = pp;
    clear pp;
    tid(l) = toc;                          %For timing purposes
    if m > 1 && l < m
        disp([char('Done: ') num2str(l/m*100) ' %'])
        ETL(tid,l)                         %For timing purposes
    end
end

p = zeros(T,L);
error = zeros(T,L);
for t = 1:T
    for l = 1:L
        error(t,l) = std(P(:,t,l));
        p(t,l) = mean(P(:,t,l));
    end
end
clear P;
```

```
%######################## - Output - ########################
disp('- - - - - - - - - - - - - - - - - - - - - - - ')
disp([char('TIME ELAPSED:') num2str(cputime-sum(tid)) 'seconds'])

disp('    min(p)    max(p)    p(1,1)    p(1,L)')
disp([min(min(p)) max(max(p)) p(1,1) p(1,L)])

minplt = floor(max(min(min(p-error)), 0));
maxplt = ceil(max(max(p+error)));

L2 = 100;
rho2 = linspace(-0.99,0.99,L2);

figure;
hold on;
subplot(3,5,1:10), surf(p(:,:)'),shading interp
set(gca,'YTickLabel',{'-1','-0.5','0', '0.5', '1'})
axis([0 T 0 L+1 floor(min(min(p))) max(max(p))])
title('The price (p(t,\rho), t\in(1,T),\rho\in(-0.99,0.99))')

p1 = 0*p;
error1 = 0*error;
for l = 1:L %Reversing to better fit the other plots
    p1(:,l) = p(:,L-l+1);
    error1(:,l) = error(:,L-l+1);
end

hold on;
subplot(3,5,11); plot(p1(1,:), 'k')
set(gca,'XTickLabel',{'1','0.5','0', '-0.5', '-1'})
axis([0 L+1 minplt maxplt])
xlabel('\rho')
title('p(1, \rho)')
hold on;
text(1.1*maxIndex(p1(1,:)),minplt+0.1,
num2str(rho2(round(maxIndex(p(1,:))/L*L2))))
stem(maxIndex(p1(1,:)),max(p1(1,:)),'--');
plot(p1(1,:)+error1(1,:),':r')
plot(p1(1,:)-error1(1,:),':r')
hold off;

hold on;
subplot(3,5,12); plot(p1(T,:), 'k')
set(gca,'XTickLabel',{'1','0.5','0', '-0.5', '-1'})
axis([0 L+1 minplt maxplt])
xlabel('\rho')
title('p(T, \rho)')
hold on;
text(1.1*maxIndex(p1(T,:)),minplt+0.1,
num2str(rho2(round(maxIndex(p(T,:))/L*L2))))
stem(maxIndex(p1(T,:)),max(p1(T,:)),'--');
plot(p1(T,:)+error1(T,:),':r')
plot(p1(T,:)-error1(T,:),':r')
hold off;
clear p1;
clear error1;
```

```
hold on;
subplot(3,5,13); plot(p(:,1), 'k')
hold on;
plot(p(:,1)+error(:,1),':r')
plot(p(:,1)-error(:,1),':r')
hold off;
axis([0 T minplt maxplt])
xlabel('Time')
title('p(t, -0.99)')

hold on;
subplot(3,5,14); plot(p(:,round(L/2)), 'k')
hold on;
plot(p(:,round(L/2))+error(:,round(L/2)),':r')
plot(p(:,round(L/2))-error(:,round(L/2)),':r')
hold off;
axis([0 T minplt maxplt])
xlabel('Time')
title('p(t, 0)')

hold on;
subplot(3,5,15); plot(p(:,L), 'k')
hold on;
plot(p(:,L)+error(:,L),':r')
plot(p(:,L)-error(:,L),':r')
hold off;
axis([0 T minplt maxplt])
xlabel('Time')
title('p(t, 0.99)')
```

B.3 Residual risk

This program models the residual risk a company takes on by issuing such guaranties as modeled in the first program. The input is market data while the output is a (number of simulations X correlation) matrix containing the residual risk at terminal time T. At the end of the program, the measure function (B.5) is called with the risk matrix as input.

```
function [] = risk(T,sim,m,r,mu,nu,sigma,eta,Y0,lambda,gamma,L,K)

rho = linspace(-0.99,0.99,L);              %Correlation Coefficient
delta = nu - eta*rho*(mu-r)/sigma;
RRF = zeros(L,sim);                         %Residual Risk
tid = zeros(1,m-1);
for l = 1:m
    tic;

    %################### - The Model - ###################
    epsW = random('normal',0,1,T,sim);   %i.i.d normal distributed
    X = zeros(T,sim,L);              %Supportprocess for the fund
    X(1,:,:) = Y0;
    WT = 0;                          %Brownain motion
    for t = 2:T
        WT = WT + epsW(t,:);
        for j = 1:L
```

```
                X(t,:,j) = X(t-1,:,j).*exp(delta(j)-0.5*eta^2
                                    + eta*epsW(t,:));
            end
        end

        %################### - The Hedge - ###################
        tempw = zeros(T,sim,L);
        tempdw = zeros(T,sim,L);

        for t = 1:T
            for j = 1:L
                e = X(t,:,j).*exp((delta(j)-0.5*eta^2)*(T-t)
                            + eta*sqrt(T-t)*epsW(t,:));
                tempw(t,:,j) = exp(lambda*gamma*(1-rho(j)^2)
                            *(K-e).*(K>e));
                tempdw(t,:,j) = tempw(t,:,j).*WT./(X(t,:,j)*eta*T);
            end
        end
        clear e WT;
        w = mean(tempw,2);
        clear tempw;
        dw = mean(tempdw,2);
        clear tempdw;

        int = zeros(1,sim);
        R = zeros(T,sim,L);
        RR = zeros(L,sim);        %Residual risk at terminal time

        for j = 1:L
            for t = 1:T
                dp = dw(t,1,j)/(exp(r*(T-t))
                     *w(t,1,j)*gamma*(1-rho(j)^2));
                int = int + exp(r*(T-t))*X(t,:,j).*dp;
                R(t,:,j) = 0.5*eta^2*gamma*(1-rho(j)^2)*int
                        - eta*sqrt(1-rho(j)^2)*epsW(t,:)*sqrt(t);
                if t == 1
                    RR(j,:) = RR(j,:) + exp(r*(T-t))
                            *X(t,:,j).*dp.*(R(t,:,j) - 0);
                else
                    RR(j,:) = RR(j,:) + exp(r*(T-t))
                            *X(t,:,j).*dp.*(R(t,:,j) - R(t-1,:,j));
                end
            end
        end

        clear epsW X R;
        RRF = RRF + sort(RR,2)/m;
        clear RR;

        if m > 1 && m > l
            disp([char('Done: ') num2str(l/m*100) ' %'])
            tid(l) = toc;
            ETL(tid,l)        %Prints estimated remaining runtime.
        end
    end

    meanF1 = mean(RRF,2);
```

```
meanPosF1 = mean(RRF > 0,2);
meanF = 0*meanF1;
meanPosF = 0*meanPosF1;
for l = 1:L
    meanF(l) = meanF1(L-l+1);
    meanPosF(l) = meanPosF1(L-l+1);
end

%######################## - Output - ########################
display('-------------------------------------------------')
display([char('TIME ELAPSED: ') num2str(cputime-sum(tid)) 'seconds'])

measure(RRF,lambda,gamma)

figure; subplot(2,2,1), surf(RRF),shading interp
set(gca,'YTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
set(gca,'YTickLabel',{'-1', '-0.5','0', '0.5', '1'})
title('Residual risk at time T over correlation and simulation')

subplot(2,2,2), plot(meanF')
set(gca,'XTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
set(gca,'XTickLabel',{'1', '0.5','0', '-0.5', '-1'})
title('Mean RR over correlation')
hold on;
text(1.1*maxIndex(meanF),0,num2str(rho(round(maxIndex(meanF)))))
stem(maxIndex(meanF),max(meanF),'--');
hold off;

subplot(2,2,3), plot(meanPosF')
ylim([min([min(meanPosF) max(meanPosF)*0.99]) max(meanPosF)*1.01])
set(gca,'XTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
set(gca,'XTickLabel',{'1', '0.5','0', '-0.5', '-1'})
title('Andel av positive RR over corr')
hold on;
stem(maxIndex(meanPosF),max(meanPosF),'--');
hold off;

subplot(2,2,4), hist(RRF(1,:),100)
title('Histogram over the Residual Risk (for \rho=0?)')
hold on;
hist(RRF(L,:),100,'FaceColor','r')
hist(RRF(round(L/2),:),100,'FaceColor','k')
hold off;
```

B.4 Lévy process

Substitute these lines with the ones in the lines 18-25 of the price file and lines 13-22 of the risk file to implement a Lévy process. Also, the functions should be extended such that they can receive the two additional parameters needed for the Lévy process.

```
%############ - Substitue into the price function - ############
sigmaZ = sqrt(sigma^2-xi^2*ji);          %Calibration
sigma = sigmaZ;

N = cumsum(random('poiss',ji,T,sim)); %Poisson Process
```

```
Y(1,:,:) = Y0;
for t = 2:T
    dC = zeros(1,sim);                    %Compound Poisson dynamics
    for i = 1:sim
        if N(t,i) > N(t-1,i)
            dC(i) = sum(random('normal',0,1,N(t,i) - N(t-1,i),1));
        end
    end
    for j = 1:L
        a1 = eta*(rho(j)*(epsW1o(t,:) + xi*dC));
        b1 = eta*(sqrt(1-rho(j)^2)*epsW2(t,:));
        Y(t,:,j) = Y(t-1,:,j).*(1 + delta(j) + a1 + b1);
    end
end

%############ - Substitue into the risk function - ############
sigmaZ = sqrt(sigma^2-xi^2*ji);         %Calibration
sigma = sigmaZ;

N = cumsum(random('poiss',ji,T,sim));
X = zeros(T,sim,L);      %Support prosess for the fund  Y
X(1,:,:) = Y0;
WT = 0;
for t = 2:T
    WT = WT + epsW(t,:);
    dC = zeros(1,sim);
    for i = 1:sim
        if N(t,i) > N(t-1,i)
            dC(i) = sum(random('normal', 0,1,N(t,i) - N(t-1,i),1));
        end
    end
    for j = 1:L
        a = eta*(epsW(t,:)+xi*dC);
        X(t,:,j) = X(t-1,:,j).*(1 + delta(j) + a);
    end
end
```

B.5 Measure function

Function that measures the residual risk with sd, VaR, CVaR, ES and ENT. The function receives a matrix and prints the measures discussed in this paper. Measures are normalized with respect to λ, in the meaning that output is risk pr option issued.

```
function [] = measure(risk,lam, gam)

risk = sort(risk,2,'descend');
%(ASCEND OR DECEND) = (alpha or 1-alpha)
[L,sim] = size(risk1); %Assume sim>500

sd = zeros(L,1);
VaR = zeros(L,2);   %(:,1) = 95, (:,2) = 99
ES = zeros(L,2);    %(:,1) = 95, (:,2) = 99
CVaR = zeros(L,2); %(:,1) = 95, (:,2) = 99
ENT = zeros(L,1);

```

```
simhi = 15;          %Initial partition of p
if sim >= 2000
    simhi = 20;
end
if sim >= 10000
    simhi = 25;
end
p = zeros(L,simhi);
check = 0;
for b = 1:L
    sd(b) = std(risk(b,:));

    VaR(b,1) = risk(b,round(0.05*sim));
    VaR(b,2) = risk(b,round(0.01*sim));

    ES(b,1) = mean(risk(b,1:(round(0.05*sim)-1)));
    ES(b,2) = mean(risk(b,1:(round(0.01*sim)-1)));

    ENT(b,1) = 1/gam*log(mean(exp(-gam*risk1(b,:))));

    p(b,:) = sort(hist(risk(b,:),simhi))/sim;
    if min(p(b,:)) <= 0  %Checking if p is valid, that is p>0
        if check == 0
            figure;
            title('Decrease "simhi" in measure.m (min(p)=0)')
            check = 1;
        end
        hold on
        plot(p(b,:))
    end

    % 95% CVaR:
    sumA = 0;
    kalphaA = 0;
    while sumA < 0.05
        kalphaA = kalphaA + 1;
        sumA = sumA + p(b,kalphaA);
    end
    A1 = sum(p(b,1:kalphaA) - 0.05)*VaR(b,1);
    A2 = sum(p(b,kalphaA+1:simhi).*risk(b,kalphaA+1:simhi));
    CVaR(b,1) = 1/(1-0.05)*(A1 + A2);

    % 99% CVaR:
    sumB = 0;
    kalphaB = 0;
    while sumB < 0.01
        kalphaB = kalphaB + 1;
        sumB = sumB + p(b,kalphaB);
    end
    B1 = sum(p(b,1:kalphaB) - 0.01)*VaR(b,2);
    B2 = sum(p(b,kalphaB+1:simhi).*risk(b,kalphaB+1:simhi));
    CVaR(b,2) = 1/(1-0.01)*(B1 + B2);
end
clear risk riskent p a1 a2 b1 b2

sd = sd/lam;
VaR = VaR/lam;
```

```
ES = ES/lam;
CVaR = CVaR/lam;
ENT = ENT/lam;

%Reversing the vectors to better fit the other plots:
sdtmp = sd;
VaRtmp = VaR;
EStmp = ES;
CVaRtmp = CVaR;
ENTtmp = ENT;
for l = 1:L
    sd(l) = sdtmp(L-l+1,:);
    VaR(l,:) = VaRtmp(L-l+1,:);
    ES(l,:) = EStmp(L-l+1,:);
    CVaR(l,:) = CVaRtmp(L-l+1,:);
    ENT(l,:) = ENTtmp(L-l+1,:);
end

if L == 1
    disp('Standard deviation:')
    disp(sd)
    disp('Value-at-Risk')
    disp(VaR)
    disp('Expected shortfall')
    disp(ES)
    disp('Conditional Value-at-Risk')
    disp(CVaR)
    disp('Entropic risk')
    disp(ENT)
else
    figure;
    subplot(2,3,1), hold on; plot(sd);
    title('Standard Deviation'); xlabel('\rho')
    set(gca,'XTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
    set(gca,'XTickLabel',{'1', '0.5','0', '-0.5', '-1'})
    stem(maxIndex(sd),max(sd),'--');
    hold off;

    subplot(2,3,3), hold on; plot(VaR);
    title('Value at Risk'); xlabel('\rho')
    set(gca,'XTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
    set(gca,'XTickLabel',{'1', '0.5','0', '-0.5', '-1'})
    stem(maxIndex(VaR(:,1)),max(VaR(:,1)),'--');
    stem(maxIndex(VaR(:,2)),max(VaR(:,2)),'--');
    hold off;

    subplot(2,3,4), hold on; plot(ES);
    title('Expected Shortfall'); xlabel('\rho')
    set(gca,'XTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
    set(gca,'XTickLabel',{'1', '0.5','0', '-0.5', '-1'})
    stem(maxIndex(ES(:,1)),max(ES(:,1)),'--');
    stem(maxIndex(ES(:,2)),max(ES(:,2)),'--');
    hold off;

    subplot(2,3,5), hold on; plot(CVaR);
    title('Conditional Value at Risk'); xlabel('\rho')
    set(gca,'XTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
```

```
    set(gca,'XTickLabel',{'1', '0.5','0', '-0.5', '-1'})
    stem(maxIndex(CVaR(:,1)),max(CVaR(:,1)),'--');
    stem(maxIndex(CVaR(:,2)),max(CVaR(:,2)),'--');
    hold off;

    subplot(2,3,2), hold on; plot(ENT);
    title('Entropical'); xlabel('\rho')
    set(gca,'XTick',[0 (L+1)/4 (L+1)/2 3*(L+1)/4 L+1])
    set(gca,'XTickLabel',{'1', '0.5','0', '-0.5', '-1'})
    stem(maxIndex(ENT),max(ENT),'--');
    hold off;

    disp('Max Standard deviation:')
    disp(max(sd))
    disp('Max Value-at-Risk')
    disp(max(VaR))
    disp('Max Expected shortfall')
    disp(max(ES))
    disp('Max Conditional Value-at-Risk');
    disp(max(CVaR))
end
```

B.6 Maximum index

Receives a vector and gives the index where the maximum value of the vector is found. Assuming the vector to be concave, which I know they are in this case.

```
function [mI] = maxIndex(vector)
m = max(vector);
mI = 0;
while mI == 0
    if vector(i) == m
        mI = i;
    end
end
%The index of a vector maximizing the vector.
```

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