



Cancellation theorems for reciprocity sheaves

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ABSTRACT

We prove cancellation theorems for reciprocity sheaves and cube-invariant modulus sheaves with transfers of Kahn–Miyazaki–Saito–Yamazaki. This generalizes a cancellation theorem for \mathbf{A}^1 -invariant sheaves with transfers, which was proved by Voevodsky. As an application, we get some new formulas for internal homs of the sheaves Ω^i of absolute Kähler differentials.

1. Introduction

We fix once and for all a perfect field k . Let \mathbf{Sm} be the category of separated smooth schemes of finite type over k . Let \mathbf{Cor} be the category of finite correspondences: \mathbf{Cor} has the same objects as \mathbf{Sm} , and morphisms in \mathbf{Cor} are finite correspondences. Let \mathbf{PST} be the category of additive presheaves of abelian groups on \mathbf{Cor} , called presheaves with transfers. Let $\mathbf{NST} \subset \mathbf{PST}$ be the full subcategory of Nisnevich sheaves, that is, those objects $F \in \mathbf{PST}$ whose restrictions F_X to the small étale site $X_{\text{ét}}$ over X are Nisnevich sheaves for all $X \in \mathbf{Sm}$. By a fundamental result of Voevodsky, the inclusion $\mathbf{NST} \rightarrow \mathbf{PST}$ has an exact left adjoint $a_{\mathbf{Nis}}^V$ such that for any $F \in \mathbf{PST}$ and $X \in \mathbf{Sm}$, the restriction $(a_{\mathbf{Nis}}^V F)_X$ is the Nisnevich sheafification of F_X as a presheaf on $X_{\mathbf{Nis}}$. In Voevodsky’s theory of motives, a fundamental role is played by \mathbf{A}^1 -invariant objects $F \in \mathbf{NST}$, namely such F that the maps $F(X) \rightarrow F(X \times \mathbf{A}^1)$ induced by the projection $X \times \mathbf{A}^1 \rightarrow X$ are isomorphisms for all $X \in \mathbf{Sm}$. The \mathbf{A}^1 -invariant objects form a full abelian subcategory $\mathbf{HI}_{\mathbf{Nis}} \subset \mathbf{NST}$ that carries a symmetric monoidal structure $\otimes_{\mathbf{HI}}^{\mathbf{Nis}}$ such that

$$F \otimes_{\mathbf{HI}}^{\mathbf{Nis}} G = h_0^{\mathbf{A}^1, \mathbf{Nis}} a_{\mathbf{Nis}}^V(F \otimes_{\mathbf{PST}} G) \quad \text{for } F, G \in \mathbf{HI}_{\mathbf{Nis}},$$

where $\otimes_{\mathbf{PST}}$ is the symmetric monoidal structure on \mathbf{PST} induced formally from that on \mathbf{Cor} and $h_0^{\mathbf{A}^1, \mathbf{Nis}}$ is a left adjoint to the inclusion functor $\mathbf{HI}_{\mathbf{Nis}} \rightarrow \mathbf{NST}$, which sends an object of \mathbf{NST} to its maximal \mathbf{A}^1 -invariant quotient in \mathbf{NST} . For integers $n > 0$, the twists of $F \in \mathbf{HI}_{\mathbf{Nis}}$ are then defined as

$$F(1) = F \otimes_{\mathbf{HI}}^{\mathbf{Nis}} \mathbf{G}_m, \quad F(n) := F(n-1) \otimes_{\mathbf{HI}}^{\mathbf{Nis}} \mathbf{G}_m,$$

where $\mathbf{G}_m \in \mathbf{NST}$ is given by $X \rightarrow \Gamma(X, \mathcal{O}^\times)$ for $X \in \mathbf{Sm}$.

Noting that $- \otimes_{\mathbf{HI}}^{\mathbf{Nis}} \mathbf{G}_m$ is an endofunctor on $\mathbf{HI}_{\mathbf{Nis}}$, we get a natural map

$$\iota_{F,G}: \text{Hom}_{\mathbf{PST}}(F, G) \rightarrow \text{Hom}_{\mathbf{PST}}(F(1), G(1)) \quad \text{for } F, G \in \mathbf{HI}_{\mathbf{Nis}}. \quad (1.1)$$

Received 31 October 2020, accepted in final form 21 July 2022.

2020 Mathematics Subject Classification 19E15, 14F42, 19D45, 19F15.

Keywords: motives, algebraic cycles, cohomology theories.

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The first author is supported by the Swiss National Science Foundation (SNF), project 200020_178729. The second author is supported by JSPS KAKENHI Grant (15H03606).

One key ingredient in Voevodsky's theory is the cancellation theorem [Voe10, Corollary 4.10], which implies the following theorem.

THEOREM 1.1. *For $F, G \in \mathbf{HI}_{\text{Nis}}$, the map $\iota_{F,G}$ is an isomorphism.*

The purpose of this paper is to generalize Theorem 1.1 to reciprocity sheaves. The category $\mathbf{RSC}_{\text{Nis}}$ of reciprocity sheaves was introduced in [KSY16, KSY22] as a full subcategory of \mathbf{NST} that contains \mathbf{HI}_{Nis} as well as interesting non- \mathbf{A}^1 -invariant objects such as the additive group scheme \mathbf{G}_a , the sheaf of absolute Kähler differentials Ω^i and the de Rham–Witt sheaves $W_n\Omega^i$. In [RSY22], a lax monoidal structure $(-, -)_{\mathbf{RSC}_{\text{Nis}}}$ on $\mathbf{RSC}_{\text{Nis}}$ is defined in such a way that

$$(F, G)_{\mathbf{RSC}_{\text{Nis}}} = F \otimes_{\mathbf{HI}}^{\text{Nis}} G \quad \text{for } F, G \in \mathbf{HI}_{\text{Nis}} .$$

It allows us to define the twists for $F \in \mathbf{RSC}_{\text{Nis}}$ recursively as

$$F\langle 1 \rangle := (F, \mathbf{G}_m)_{\mathbf{RSC}_{\text{Nis}}}, \quad F\langle n \rangle := (F\langle n-1 \rangle, \mathbf{G}_m)_{\mathbf{RSC}_{\text{Nis}}} .$$

Some examples of twists were computed in [RSY22]: if $F \in \mathbf{HI}_{\text{Nis}}$, then $F\langle n \rangle = F(n)$; in particular, $\mathbb{Z}\langle n \rangle \cong \mathcal{K}_n^M$ (the Milnor K -sheaf), and $\mathbf{G}_a\langle n \rangle \cong \Omega^n$ if $\text{ch}(k) = 0$.

By the fact that $(-, \mathbf{G}_m)_{\mathbf{RSC}_{\text{Nis}}}$ is an endofunctor on $\mathbf{RSC}_{\text{Nis}}$, we get a natural map (cf. (6.14))

$$\iota_{F,G}: \text{Hom}_{\mathbf{PST}}(F, G) \rightarrow \text{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) \quad \text{for } F, G \in \mathbf{RSC}_{\text{Nis}}, \quad (1.2)$$

which coincides with (1.1) if $F, G \in \mathbf{HI}_{\text{Nis}}$. We will also get a natural map in \mathbf{NST} :

$$\lambda_F: F \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle) \quad \text{for } F \in \mathbf{RSC}_{\text{Nis}}, \quad (1.3)$$

using the functoriality of $(-, \mathbf{G}_m)_{\mathbf{RSC}_{\text{Nis}}}$, where $\underline{\text{Hom}}_{\mathbf{PST}}$ denotes the internal hom in \mathbf{PST} .

The main result of this paper is the following.

THEOREM 1.2 (Theorems 6.4 and 6.2). *The maps $\iota_{F,G}$ and λ_F are isomorphisms.*

As an application of the above theorem, we prove the following.

COROLLARY 1.3 (Theorem 7.2). *Assume $\text{ch}(k) = 0$. For integers $m, n \geq 0$, there are natural isomorphisms in \mathbf{NST} :*

$$\begin{aligned} \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m) &\cong \Omega^{m-n} \oplus \Omega^{m-n-1}, \\ \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m) &\cong \Omega^{m-n}, \end{aligned}$$

where $\Omega^i = 0$ for $i < 0$ by convention.

Let \mathbf{PS} be the category of additive presheaves of abelian groups on \mathbf{Sm} (without transfers). Note that \mathbf{PST} is viewed as a subcategory of \mathbf{PS} . By a lemma due to Kay Rülling (see Lemma 2.1), we have a natural isomorphism in \mathbf{PS} :

$$\underline{\text{Hom}}_{\mathbf{PST}}(G, \Omega^m) \cong \underline{\text{Hom}}_{\mathbf{PS}}(G, \Omega^m) \quad \text{for any } G \in \mathbf{PST}, \quad (1.4)$$

where $\underline{\text{Hom}}_{\mathbf{PS}}$ is the internal hom in \mathbf{PS} . Thanks to (1.4), the isomorphisms of Corollary 1.3 and their explicit descriptions (7.1) and (7.3) imply that

$$\begin{aligned} \text{Hom}_{\mathbf{PS}}(\Omega^n, \Omega^m) &= \{\omega_1 \wedge (-) + \omega_2 \wedge d(-) \mid \omega_1 \in \Omega_k^{m-n}, \omega_2 \in \Omega_k^{m-n-1}\}, \\ \text{Hom}_{\mathbf{PS}}(\mathcal{K}_n^M, \Omega^m) &= \{\omega \wedge \text{dlog}(-) \mid \omega \in \Omega_k^{m-n}\}, \end{aligned}$$

where $\text{dlog}: \mathcal{K}_n^M \rightarrow \Omega^m$ is the map $\{x_1, \dots, x_n\} \rightarrow \text{dlog } x_1 \wedge \dots \wedge \text{dlog } x_n$. It would be an interesting question whether there is a direct proof of these formulas which does not use the machinery of modulus sheaves with transfers explained below.

Reciprocity sheaves are closely related to *modulus sheaves with transfers*, introduced in [KMSY21a, KMSY21b]: Voevodsky's category \mathbf{Cor} of finite correspondences is enlarged to a new category \mathbf{MCor} of *modulus pairs*: Its objects are pairs $\mathcal{X} = (X, D)$, where X is a separated scheme of finite type over k and D is an effective Cartier divisor on X such that $\mathcal{X}^\circ := X - |D| \in \mathbf{Sm}$ (here \mathcal{X}° is called the interior of \mathcal{X}). The morphisms are finite correspondences on interiors satisfying some admissibility and properness conditions. Let $\mathbf{MCor} \subset \mathbf{MCor}$ be the full subcategory of such objects (X, D) with X proper over k . There is a symmetric monoidal structure \otimes on \mathbf{MCor} , which also induces one on \mathbf{MCor} by restriction (cf. § 2.19).

We then define \mathbf{MPST} (respectively, \mathbf{MPST}) as the category of additive presheaves of abelian groups on \mathbf{MCor} (respectively, \mathbf{MCor}). We have a functor

$$\underline{\omega}: \mathbf{MCor} \rightarrow \mathbf{Cor}, \quad (\overline{X}, X_\infty) \mapsto \overline{X} - |X_\infty|$$

and two adjunctions

$$\mathbf{MPST} \begin{array}{c} \xleftarrow{\tau^*} \\ \xrightarrow{\tau_!} \end{array} \mathbf{MPST}, \quad \mathbf{MPST} \begin{array}{c} \xleftarrow{\underline{\omega}^*} \\ \xrightarrow{\underline{\omega}_!} \end{array} \mathbf{PST},$$

where $\underline{\omega}^*$ is induced by $\underline{\omega}$ and $\underline{\omega}_!$ is its left Kan extension, and τ^* is induced by the inclusion $\tau: \mathbf{MCor} \rightarrow \mathbf{MCor}$ and $\tau_!$ is its left Kan extension, which turns out to be exact and fully faithful.

For $F \in \mathbf{MPST}$ and $\mathcal{X} = (X, D) \in \mathbf{MCor}$, write $F_{\mathcal{X}}$ for the presheaf on the small étale site $X_{\text{ét}}$ over X given by $U \rightarrow F(\mathcal{X}_U)$ for $U \rightarrow X$ étale, where $\mathcal{X}_U = (U, D \times_X U) \in \mathbf{MCor}$. We say that F is a Nisnevich sheaf if $F_{\mathcal{X}}$ is one for all $\mathcal{X} \in \mathbf{MCor}$. We write $\mathbf{MNST} \subset \mathbf{MPST}$ for the full subcategory of Nisnevich sheaves.

The replacement of the \mathbf{A}^1 -invariance in this new framework is the $\overline{\square}$ -invariance, where $\overline{\square} := (\mathbf{P}^1, \infty) \in \mathbf{MCor}$: Let $\mathbf{CI} \subset \mathbf{MPST}$ be the full subcategory of those objects F such that the maps $F(\mathcal{X}) \rightarrow F(\mathcal{X} \otimes \overline{\square})$ induced by the projection $\mathcal{X} \otimes \overline{\square} \rightarrow \mathcal{X}$ are isomorphisms for all $\mathcal{X} \in \mathbf{MCor}$. Let $\mathbf{CI}^\tau \subset \mathbf{MPST}$ be the essential image of \mathbf{CI} under $\tau_!$ and $\mathbf{CI}^{\tau, \text{sp}} \subset \mathbf{CI}^\tau$ be the full subcategory of *semipure* objects F , namely such objects that the natural maps $F(X, D) \rightarrow F(X - D, \emptyset)$ are injective for all $(X, D) \in \mathbf{MCor}$. We also define $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}} = \mathbf{CI}^{\tau, \text{sp}} \cap \mathbf{MNST}$ as a full subcategory of \mathbf{MNST} . A symmetric monoidal structure $\otimes_{\mathbf{CI}}^{\text{sp}}$ (respectively, $\otimes_{\mathbf{CI}}^{\text{Nis, sp}}$) on $\mathbf{CI}^{\tau, \text{sp}}$ (respectively, on $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$) can be defined in the same spirit as $\otimes_{\mathbf{HI}}^{\text{Nis}}$ (see § 4).

The relationship between reciprocity (pre)sheaves and $\overline{\square}$ -invariant modulus (pre)sheaves with transfers is encoded in

$$\mathbf{RSC} = \underline{\omega}_!(\mathbf{CI}^{\tau, \text{sp}}) \quad \text{and} \quad \mathbf{RSC}_{\text{Nis}} = \underline{\omega}_!(\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}).$$

There is a pair of adjoint functors

$$\mathbf{CI}^{\tau, \text{sp}} \begin{array}{c} \xleftarrow{\underline{\omega}^{\mathbf{CI}}} \\ \xrightarrow{\underline{\omega}_!} \end{array} \mathbf{RSC} \quad \text{and} \quad \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}} \begin{array}{c} \xleftarrow{\underline{\omega}^{\mathbf{CI}}} \\ \xrightarrow{\underline{\omega}_!} \end{array} \mathbf{RSC}_{\text{Nis}}$$

such that $\underline{\omega}^{\mathbf{CI}} F = \underline{\omega}^* F$ for $F \in \mathbf{HI}$. Moreover, the lax monoidal structure on $\mathbf{RSC}_{\text{Nis}}$ is induced by the symmetric monoidal structure on $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ via the formula

$$(F, G)_{\mathbf{RSC}_{\text{Nis}}} := \underline{\omega}_!(\underline{\omega}^{\mathbf{CI}} F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} \underline{\omega}^{\mathbf{CI}} G) \quad \text{for } F, G \in \mathbf{RSC}_{\text{Nis}}.$$

The endofunctor $-\otimes_{\mathbf{CI}}^{\text{Nis,sp}} \underline{\omega}^* \mathbf{G}_m$ on $\mathbf{CI}_{\text{Nis}}^{\tau,\text{sp}}$ induces a natural map for $F \in \mathbf{CI}_{\text{Nis}}^{\tau,\text{sp}}$:

$$\iota_F: F \rightarrow \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, F \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \underline{\omega}^* \mathbf{G}_m), \quad (1.5)$$

where $\underline{\text{Hom}}_{\mathbf{MPST}}$ denotes the internal hom in \mathbf{MPST} . Now Theorem 1.2 will be a consequence of the following result.

THEOREM 1.4 (Corollary 4.6). *For $F \in \mathbf{RSC}_{\text{Nis}}$ and $\tilde{F} = \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}_{\text{Nis}}^{\tau,\text{sp}}$, the map $\iota_{\tilde{F}}$ is an isomorphism.*

We give an outline of the content of the paper: In Section 2, we first review basic definitions and results of the theory of modulus (pre)sheaves with transfers and reciprocity sheaves from [KMSY21a, KMSY21b] and [Sai20]. We also prove some technical lemmas which will be used in the later sections.

In Section 3, we define the contraction functors γ on $\mathbf{CI}^{\tau,\text{sp}}$ and $\mathbf{CI}_{\text{Nis}}^{\tau,\text{sp}}$, which generalize Voevodsky's contraction functors on \mathbf{HI} and \mathbf{HI}_{Nis} (cf. [MVW06, Lecture 23]) to the setting of modulus (pre)sheaves with transfers. We prove some technical lemmas which will be used in the later sections.

In Section 4, we define the symmetric monoidal structure $\otimes_{\mathbf{CI}}^{\text{sp}}$ (respectively, $\otimes_{\mathbf{CI}}^{\text{Nis,sp}}$) on $\mathbf{CI}^{\tau,\text{sp}}$ (respectively, on $\mathbf{CI}_{\text{Nis}}^{\tau,\text{sp}}$) using results from Section 2. The endofunctor $-\otimes_{\mathbf{CI}}^{\text{sp}} \underline{\omega}^* \mathbf{G}_m$ on $\mathbf{CI}^{\tau,\text{sp}}$ induces a natural map for $F \in \mathbf{CI}^{\tau,\text{sp}}$:

$$\iota_F: F \rightarrow \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, F \otimes_{\mathbf{CI}}^{\text{sp}} \underline{\omega}^* \mathbf{G}_m). \quad (1.6)$$

We state the main result, Theorem 4.4: ι_F is an isomorphism. Theorem 1.4 is deduced from it by using results from Section 3.

The last half of Section 4 is devoted to the proof of the split-injectivity of the map ι_F in (1.6). In order to construct a section of ι_F , we follow the same strategy as in [Voe10] by generalizing the techniques used therein.

In Section 5, we finish the proof of Theorem 4.4 by showing the surjectivity of ι_F . We again follow the same strategy as in [Voe10] by generalizing the results of [Gra05, § 2.7]: here a technical problem is that for $(X, D) \in \mathbf{MCor}$, the diagonal map $X \rightarrow X \times X$ does not induce a map $(X, D) \rightarrow (X, D) \otimes (X, D)$ in \mathbf{MCor} but only induces a map $(X, 2D) \rightarrow (X, D) \otimes (X, D)$, where $2D \hookrightarrow X$ is the thickening of $D \hookrightarrow X$ defined by the square of the ideal sheaf. This is the main reason why we need to work with $\mathbf{CI}^{\tau,\text{sp}}$ instead of \mathbf{CI}^{τ} , employing much more intricate arguments than those in [Voe10] and [Gra05, § 2.7], for which we need the technical results in Sections 2 and 3.

In Section 6, we deduce Theorem 1.2 from Theorem 1.4. In Section 7, we deduce Corollary 1.3 from Theorem 1.2.

Conventions 1.5. In the whole paper, we fix a perfect base field k . Let $\widetilde{\mathbf{Sm}}$ be the category of k -schemes X which are essentially smooth over k ; that is, X is a limit $\varprojlim_{i \in I} X_i$ over a filtered set I , where X_i is smooth over k and all transition maps are étale. Note that $\text{Spec } K \in \widetilde{\mathbf{Sm}}$ for a function field K over k thanks to the assumption that k is perfect. We frequently allow $F \in \mathbf{PST}$ to take values on objects of $\widetilde{\mathbf{Sm}}$ by setting $F(X) := \varinjlim_{i \in I} F(X_i)$ for X as above.

2. Background on modulus sheaves with transfers

In this section, we recall the definitions and basic properties of modulus sheaves with transfers from [KMSY21a] and [Sai20] (see also [KSY22] for a more detailed summary).

2.1. Denote by \mathbf{Sch} the category of separated schemes of finite type over k and by \mathbf{Sm} the full subcategory of smooth schemes. For $X, Y \in \mathbf{Sm}$, an integral closed subscheme of $X \times Y$ that is finite and surjective over a connected component of X is called a *prime correspondence from X to Y* . The category \mathbf{Cor} of finite correspondences has the same objects as \mathbf{Sm} , and for $X, Y \in \mathbf{Sm}$, the object $\mathbf{Cor}(X, Y)$ is the free abelian group on the set of all prime correspondences from X to Y (see [MVW06]). We consider \mathbf{Sm} as a subcategory of \mathbf{Cor} by regarding a morphism in \mathbf{Sm} as its graph in \mathbf{Cor} .

Let $\mathbf{PST} = \text{Fun}(\mathbf{Cor}, \mathbf{Ab})$ be the category of additive presheaves of abelian groups on \mathbf{Cor} whose objects are called *presheaves with transfers*. Let $\mathbf{NST} \subseteq \mathbf{PST}$ be the category of Nisnevich sheaves with transfers, and let

$$a_{\text{Nis}}^V : \mathbf{PST} \rightarrow \mathbf{NST} \quad (2.1)$$

be Voevodsky's Nisnevich sheafification functor, which is an exact left adjoint to the inclusion $\mathbf{NST} \rightarrow \mathbf{PST}$. Let $\mathbf{HI} \subseteq \mathbf{PST}$ be the category of \mathbf{A}^1 -invariant presheaves, and put $\mathbf{HI}_{\text{Nis}} = \mathbf{HI} \cap \mathbf{NST} \subseteq \mathbf{NST}$. The product \times on \mathbf{Sm} yields a symmetric monoidal structure on \mathbf{Cor} , which induces a symmetric monoidal structure on \mathbf{PST} in the usual way.

2.2. We recall the definition of the category \mathbf{MCor} from [KMSY21a, Definition 1.3.1]. A pair $\mathcal{X} = (X, D_X)$ of $X \in \mathbf{Sch}$ and an effective Cartier divisor D on X is called a *modulus pair* if $X - |D_X| \in \mathbf{Sm}$. Let $\mathcal{X} = (X, D_X)$ and $\mathcal{Y} = (Y, D_Y)$ be modulus pairs and $\Gamma \in \mathbf{Cor}(X - D_X, Y - D_Y)$ be a prime correspondence. Let $\bar{\Gamma} \subseteq X \times Y$ be the closure of Γ , and let $\bar{\Gamma}^N \rightarrow X \times Y$ be the normalization. We say that Γ is *admissible* (respectively, *left proper*) if $(D_X)_{\bar{\Gamma}^N} \geq (D_Y)_{\bar{\Gamma}^N}$ (respectively, if $\bar{\Gamma}$ is proper over X). Let $\mathbf{MCor}(\mathcal{X}, \mathcal{Y})$ be the subgroup of $\mathbf{Cor}(X - D_X, Y - D_Y)$ generated by all admissible left proper prime correspondences. The category \mathbf{MCor} has modulus pairs as objects and $\mathbf{MCor}(\mathcal{X}, \mathcal{Y})$ as the group of morphisms from \mathcal{X} to \mathcal{Y} .

2.3. Let $\mathbf{MCor}_{\text{ls}} \subset \mathbf{MCor}$ be the full subcategory of $(X, D) \in \mathbf{MCor}$ with $X \in \mathbf{Sm}$ and $|D|$ a simple normal crossing divisor on X . As observed in [Sai20, Remark 1.14], after assuming resolution of singularities, we can assume $\mathbf{MCor} \cong \mathbf{MCor}_{\text{ls}}$, as for every object $(X, D) \in \mathbf{MCor}$, there exists a proper birational map $p: X' \rightarrow X$ that is an isomorphism on $X - |D|$ and such that $|p^*D|$ is a simple normal crossing divisor. Hence the modulus correspondence $(X', D') \rightarrow (X, D)$ induced by the graph of p is invertible in \mathbf{MCor} .

2.4. There is a canonical pair of adjoint functors $\lambda \dashv \omega$:

$$\begin{aligned} \lambda : \mathbf{Cor} &\rightarrow \mathbf{MCor}, & X &\mapsto (X, \emptyset), \\ \omega : \mathbf{MCor} &\rightarrow \mathbf{Cor}, & (X, D) &\mapsto X - |D|. \end{aligned}$$

2.5. There is a full subcategory $\mathbf{MCor} \subset \mathbf{MCor}$ consisting of *proper modulus pairs*, where a modulus pair (X, D) is *proper* if X is proper. Let $\tau : \mathbf{MCor} \hookrightarrow \mathbf{MCor}$ be the inclusion functor and $\omega = \omega\tau$.

2.6. For all $n > 0$, there is an endofunctor $(-)^{(n)}$ on \mathbf{MCor} preserving \mathbf{MCor} , such that $(X, D)^{(n)} = (X, nD)$, where nD is the n th thickening of D .

2.7. We have two categories of *modulus presheaves with transfers*:

$$\mathbf{MPST} = \mathbf{Fun}(\mathbf{MCor}, \mathbf{Ab}) \quad \text{and} \quad \underline{\mathbf{MPST}} = \mathbf{Fun}(\underline{\mathbf{MCor}}, \mathbf{Ab}).$$

Let $\mathbb{Z}_{\text{tr}}(\mathcal{X}) = \underline{\mathbf{MCor}}(-, \mathcal{X}) \in \underline{\mathbf{MPST}}$ be the representable presheaf for $\mathcal{X} \in \mathbf{MCor}$. In this paper, we frequently write \mathcal{X} for $\mathbb{Z}_{\text{tr}}(\mathcal{X})$ for simplicity.

2.8. The adjunction $\lambda \dashv \underline{\omega}$ induce a string of four adjoint functors ($\lambda_! = \underline{\omega}^!$, $\lambda^* = \underline{\omega}_!$, $\lambda_* = \underline{\omega}^*$, $\underline{\omega}_*$) (cf. [KMSY21a, Proposition 2.3.1]):

$$\underline{\mathbf{MPST}} \begin{array}{c} \xleftarrow{\underline{\omega}^!} \\ \xrightarrow{\underline{\omega}_!} \\ \xleftarrow{\underline{\omega}^*} \\ \xrightarrow{\underline{\omega}_*} \end{array} \mathbf{PST},$$

where $\underline{\omega}_!$ and $\underline{\omega}_*$ are localisations and $\underline{\omega}^!$ and $\underline{\omega}^*$ are fully faithful.

2.9. The functor ω yields a string of three adjoint functors ($\omega_!, \omega^*, \omega_*$) (cf. [KMSY21a, Proposition 2.2.1]):

$$\mathbf{MPST} \begin{array}{c} \xrightarrow{\omega_!} \\ \xleftarrow{\omega^*} \\ \xrightarrow{\omega_*} \end{array} \mathbf{PST},$$

where $\omega_!$ and ω_* are localisations and ω^* is fully faithful.

2.10. The functor τ yields a string of 3 adjoint functors ($\tau_!, \tau^*, \tau_*$):

$$\mathbf{MPST} \begin{array}{c} \xrightarrow{\tau_!} \\ \xleftarrow{\tau^*} \\ \xrightarrow{\tau_*} \end{array} \underline{\mathbf{MPST}},$$

where $\tau_!$ and τ_* are fully faithful and τ^* is a localisation; $\tau_!$ has a pro-left adjoint $\tau^!$, hence is exact (cf. [KMSY21a, Proposition 2.4.1]). We will denote by $\underline{\mathbf{MPST}}^\tau$ the essential image of $\tau_!$ in $\underline{\mathbf{MPST}}$. Moreover, we have (cf. [KMSY21a, Lemma 2.4.2])

$$\omega_! = \underline{\omega}_! \tau_!, \quad \omega^* = \tau^* \underline{\omega}^*, \quad \tau_! \omega^* = \underline{\omega}^*. \quad (2.2)$$

2.11. For $F \in \underline{\mathbf{MPST}}$ and $\mathcal{X} = (X, D) \in \underline{\mathbf{MCor}}$, write $F_{\mathcal{X}}$ for the presheaf on the small étale site $X_{\text{ét}}$ over X given by $U \rightarrow F(\mathcal{X}_U)$ for $U \rightarrow X$ étale, where $\mathcal{X}_U = (U, D|_U) \in \underline{\mathbf{MCor}}$. We say that F is a Nisnevich sheaf if $F_{\mathcal{X}}$ is for all $\mathcal{X} \in \underline{\mathbf{MCor}}$ (see [KMSY21a, § 3]). We write $\underline{\mathbf{MNST}} \subset \underline{\mathbf{MPST}}$ for the full subcategory of Nisnevich sheaves. Let $\mathbf{MNST} \subset \mathbf{MPST}$ be the full subcategory of such objects F that $\tau_! F \in \underline{\mathbf{MNST}}$. By [KMSY21a, Proposition 3.5.3] and [KMSY21b, Theorem 2], the inclusion functors

$$\underline{i}_{\text{Nis}}: \underline{\mathbf{MNST}} \rightarrow \underline{\mathbf{MPST}} \quad \text{and} \quad i_{\text{Nis}}: \mathbf{MNST} \rightarrow \mathbf{MPST}$$

admit exact left adjoints $\underline{a}_{\text{Nis}}$ and a_{Nis} , respectively, and there are natural isomorphisms

$$\tau_! a_{\text{Nis}} \simeq \underline{a}_{\text{Nis}} \tau_! \quad \text{and} \quad a_{\text{Nis}} \tau^* \simeq \tau^* \underline{a}_{\text{Nis}}. \quad (2.3)$$

Moreover, the adjunction from § 2.10 induces an adjunction

$$\mathbf{MNST} \begin{array}{c} \xrightarrow{\tau_!} \\ \xleftarrow{\tau^*} \end{array} \underline{\mathbf{MNST}}.$$

The functor $\underline{a}_{\text{Nis}}$ has the following description: For $F \in \underline{\mathbf{MPST}}$ and $\mathcal{Y} \in \underline{\mathbf{MCor}}$, let $F_{\mathcal{Y}, \text{Nis}}$ be the usual Nisnevich sheafification of $F_{\mathcal{Y}}$. Then, for $(X, D) \in \underline{\mathbf{MCor}}$, we have

$$\underline{a}_{\text{Nis}}F(X, D) = \varinjlim_{f: Y \rightarrow X} F_{(Y, f^*D), \text{Nis}}(Y), \quad (2.4)$$

where the colimit is taken over all proper maps $f: Y \rightarrow X$ that induce isomorphisms $Y - |f^*D| \xrightarrow{\simeq} X - |D|$.

2.12. For $X \in \mathbf{Sch}$, let $\text{Sh}(X_{\text{Nis}}, \mathbf{Ab})$ be the abelian category of additive sheaves on X_{Nis} . By the definition of $\underline{\mathbf{MNST}}$, we have an additive functor for $\mathcal{X} = (X, D) \in \underline{\mathbf{MCor}}$,

$$\underline{\mathbf{MNST}} \rightarrow \text{Sh}(X_{\text{Nis}}, \mathbf{Ab}), \quad F \mapsto F_{\mathcal{X}}.$$

The functor is not exact in general, but it is left exact by (2.4).

2.13. By [KMSY21b, Proposition 6.2.1], the functors $\underline{\omega}^*$ and $\underline{\omega}_!$ respect $\underline{\mathbf{MNST}}$ and \mathbf{NST} and induce a pair of adjoint functors

$$\underline{\mathbf{MNST}} \begin{array}{c} \xrightarrow{\underline{\omega}_!} \\ \xleftarrow{\underline{\omega}^*} \end{array} \mathbf{NST},$$

which are both exact. Moreover, we have

$$\underline{\omega}_! \underline{a}_{\text{Nis}} = a_{\text{Nis}}^V \underline{\omega}_! \quad \text{and} \quad \underline{a}_{\text{Nis}} \underline{\omega}^* = \underline{\omega}^* a_{\text{Nis}}^V.$$

2.14. We say that $F \in \underline{\mathbf{MPST}}$ (respectively, \mathbf{MPST}) is *semipure* if the unit map

$$u: F \rightarrow \underline{\omega}^* \underline{\omega}_! F \quad (\text{respectively, } u: F \rightarrow \omega^* \omega_! F)$$

is injective. For $F \in \underline{\mathbf{MPST}}$ (respectively, $F \in \mathbf{MPST}$), let $F^{\text{sp}} \in \underline{\mathbf{MPST}}$ (respectively, $F^{\text{sp}} \in \mathbf{MPST}$) be the image of $F \rightarrow \underline{\omega}^* \underline{\omega}_! F$ (respectively, $F \rightarrow \omega^* \omega_! F$) (called the semipurification of F). One easily sees that the association $F \rightarrow F^{\text{sp}}$ gives a left adjoint to the inclusion of the full subcategories of semipure objects into $\underline{\mathbf{MPST}}$ and \mathbf{MPST} . For $F \in \mathbf{MPST}$, we have

$$\tau_1(F^{\text{sp}}) \simeq (\tau_1 F)^{\text{sp}}. \quad (2.5)$$

This follows from the fact that τ_1 is exact and commutes with $\omega^* \omega_!$ and $\underline{\omega}^* \underline{\omega}_!$ since $\tau_1 \omega^* = \underline{\omega}^*$ and $\underline{\omega}_! \tau_1 = \tau_1$ (cf. § 2.10). In particular, $F \in \mathbf{MPST}$ is semipure if and only if $\tau_1 F \in \underline{\mathbf{MPST}}$ is. For $F \in \underline{\mathbf{MPST}}$, we have

$$\underline{a}_{\text{Nis}}(F^{\text{sp}}) \simeq (\underline{a}_{\text{Nis}} F)^{\text{sp}}, \quad (2.6)$$

where the $(-)^{\text{sp}}$ on the right is defined for $F \in \underline{\mathbf{MNST}}$ in the same way as above. This follows from the fact that $\underline{a}_{\text{Nis}}$ is exact and commutes with $\omega^* \omega_!$ and $\underline{\omega}^* \underline{\omega}_!$ (cf. § 2.13).

2.15. Let $\overline{\square} := (\mathbf{P}^1, \infty) \in \mathbf{MCor}$. We say that $F \in \mathbf{MPST}$ is $\overline{\square}$ -invariant if $p^*: F(\mathcal{X}) \rightarrow F(\mathcal{X} \otimes \overline{\square})$ is an isomorphism for any $\mathcal{X} \in \mathbf{MCor}$, where $p: \mathcal{X} \otimes \overline{\square} \rightarrow \mathcal{X}$ is the projection. Let \mathbf{CI} be the full subcategory of \mathbf{MPST} consisting of all $\overline{\square}$ -invariant objects.

Recall from [KSY22, Theorem 2.1.8] that \mathbf{CI} is a Serre subcategory of \mathbf{MPST} and that the inclusion functor $i^{\overline{\square}}: \mathbf{CI} \rightarrow \mathbf{MPST}$ has a left adjoint $h_0^{\overline{\square}}$ and a right adjoint $h_{\overline{\square}}^0$ given for $F \in \mathbf{MPST}$ and $\mathcal{X} \in \mathbf{MCor}$ by

$$\begin{aligned} h_0^{\overline{\square}}(F)(\mathcal{X}) &= \text{Coker}(i_0^* - i_1^*: F(\mathcal{X} \otimes \overline{\square}) \rightarrow F(\mathcal{X})), \\ h_{\overline{\square}}^0(F)(\mathcal{X}) &= \text{Hom}(h_0^{\overline{\square}}(\mathcal{X}), F), \end{aligned}$$

where for $a \in k$, the section $i_a: \mathcal{X} \rightarrow \mathcal{X} \otimes \bar{\square}$ is induced by the map $k[t] \rightarrow k[t]/(t-a) \cong k$.

For $\mathcal{X} \in \mathbf{MCor}$, we write $h_0^{\bar{\square}}(\mathcal{X}) = h_0^{\bar{\square}}(\mathbb{Z}_{\text{tr}}(\mathcal{X})) \in \mathbf{CI}$.

2.16. Let $\mathbf{CI}^\tau = \tau_! \mathbf{CI} \subset \mathbf{MPST}$ be the essential image of \mathbf{CI} under $\tau_!$. In this paper, for $F \in \mathbf{CI}$, we also let F denote $\tau_! F \in \mathbf{CI}^\tau$ by abuse of notation. Let $\mathbf{CI}^{\text{SP}} \subset \mathbf{CI}$ (respectively, $\mathbf{CI}^{\tau, \text{SP}} \subset \mathbf{CI}^\tau$) be the full subcategory of semipure objects. By (2.5), we have

$$F^{\text{SP}} \in \mathbf{CI}^\tau \quad \text{for } F \in \mathbf{CI}^\tau, \quad (2.7)$$

and $\tau_!$ and τ^* induce an equivalence of categories

$$\tau_! : \mathbf{CI}^{\text{SP}} \simeq \mathbf{CI}^{\tau, \text{SP}} : \tau^* \quad (2.8)$$

with natural isomorphisms $\tau^* \tau_! \simeq \text{id}$ and $\tau_! \tau^* \simeq \text{id}$.

We also consider the full subcategories

$$\begin{aligned} \mathbf{CI}_{\text{Nis}}^{\text{SP}} &= \mathbf{CI}^{\text{SP}} \cap \mathbf{MNST} \subset \mathbf{MNST}, \\ \mathbf{CI}_{\text{Nis}}^\tau &= \mathbf{CI}^\tau \cap \mathbf{MNST} \subset \mathbf{MNST}, \\ \mathbf{CI}_{\text{Nis}}^{\tau, \text{SP}} &= \mathbf{CI}^{\tau, \text{SP}} \cap \mathbf{MNST} \subset \mathbf{MNST}. \end{aligned}$$

By [Sai20, Theorem 0.4], we have

$$\underline{a}_{\text{Nis}}(\mathbf{CI}^{\tau, \text{SP}}) \subset \mathbf{CI}_{\text{Nis}}^{\tau, \text{SP}}. \quad (2.9)$$

By [KMSY21b, Theorem 2(1)], the maps $\tau_!$ and τ^* induce an equivalence of categories

$$\tau_! : \mathbf{CI}_{\text{Nis}}^{\text{SP}} \simeq \mathbf{CI}_{\text{Nis}}^{\tau, \text{SP}} : \tau^* \quad (2.10)$$

with natural isomorphisms $\tau^* \tau_! \simeq \text{id}$ and $\tau_! \tau^* \simeq \text{id}$.

2.17. We write $\mathbf{RSC} \subseteq \mathbf{PST}$ for the essential image of \mathbf{CI} under $\omega_!$ (which is the same as the essential image of $\mathbf{CI}^{\tau, \text{SP}}$ under $\underline{\omega}_!$ since $\omega_! = \underline{\omega}_! \tau_!$ and $\omega_! F = \underline{\omega}_! F^{\text{SP}}$). Put $\mathbf{RSC}_{\text{Nis}} = \mathbf{RSC} \cap \mathbf{NST}$. The objects of \mathbf{RSC} (respectively, $\mathbf{RSC}_{\text{Nis}}$) are called reciprocity presheaves (respectively, sheaves). We have $\mathbf{HI} \subseteq \mathbf{RSC}$, and \mathbf{RSC} also contains smooth commutative group schemes (which may have non-trivial unipotent part), the sheaf Ω^i of Kähler differentials and the de Rham–Witt sheaves $W\Omega^i$ (see [KSY16, KSY22]).

2.18. By [KSY22, Proposition 2.3.7], we have a pair of adjoint functors

$$\mathbf{CI} \begin{array}{c} \xrightarrow{\omega_!} \\ \omega_{\mathbf{CI}} \\ \xleftarrow{\quad} \end{array} \mathbf{RSC}, \quad (2.11)$$

where $\omega^{\mathbf{CI}} = h_{\bar{\square}}^0 \omega^*$, which is fully faithful. It induces a pair of adjoint functors

$$\mathbf{CI}^\tau \begin{array}{c} \xrightarrow{\omega_!} \\ \omega_{\mathbf{CI}} \\ \xleftarrow{\quad} \end{array} \mathbf{RSC}, \quad (2.12)$$

where $\omega^{\mathbf{CI}} = \tau_! h_{\bar{\square}}^0 \omega^*$, which is fully faithful. Indeed, let $F = \tau_! \hat{F}$ for $\hat{F} \in \mathbf{CI}$ and $G \in \mathbf{RSC}$. In view of § 2.15 and the exactness and full faithfulness of $\tau_!$, we have

$$\begin{aligned} \text{Hom}_{\mathbf{CI}^\tau}(F, \tau_! h_{\bar{\square}}^0 \omega^* G) &\simeq \text{Hom}_{\mathbf{CI}}(\hat{F}, h_{\bar{\square}}^0 \omega^* G) \simeq \text{Hom}_{\mathbf{MPST}}(\hat{F}, \omega^* G) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(\tau_! \hat{F}, \omega^* G) \simeq \text{Hom}_{\mathbf{RSC}}(\omega_! F, G). \end{aligned}$$

By [KSY22, Theorem 2.4.1(2)], the pair (2.12) induce a pair of adjoint functors

$$\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}} \begin{array}{c} \xrightarrow{\omega_!} \\ \xleftarrow{\omega^{\text{CI}}} \end{array} \mathbf{RSC}_{\text{Nis}} . \quad (2.13)$$

If $F \in \mathbf{CI}^{\tau}$, the adjunction induces a canonical map $F \rightarrow \omega^{\text{CI}} \omega_! F$, which is injective if $F \in \mathbf{CI}^{\tau, \text{sp}}$.

2.19. The category $\underline{\mathbf{MCor}}$ is equipped with a symmetric monoidal structure given by

$$(X, D_X) \otimes (Y, D_Y) := (X \times Y, D_X \times Y + X \times D_Y),$$

and \mathbf{MCor} is clearly a \otimes -subcategory. Notice that the product is not a categorical product since the diagonal map is not admissible. It is admissible as a correspondence

$$(X, D_X)^{(n)} \rightarrow (X, D_X) \otimes (X, D_X) \quad \text{for } n \geq 2.$$

The symmetric monoidal structure \otimes on $\underline{\mathbf{MCor}}$ (respectively, \mathbf{MCor}) induces a symmetric monoidal structure on $\underline{\mathbf{MPST}}$ (respectively, \mathbf{MPST}) in the usual way, and $\tau_!$, $\omega_!$ and $\omega_!$ from §§ 2.10, 2.9 and 2.8 are all monoidal (see [RSY22, § 3]).

We end this section with some lemmas that will be needed in the rest of the paper.

The proof of the following lemma is due to Kay Rülling. We thank him for letting us include it in our paper.

LEMMA 2.1. *Let p be the exponential characteristic of the base field k . Let $F \in \mathbf{PST}$ be such that*

- (1) *for all dominant étale maps $U \rightarrow X$ in \mathbf{Sm} , the pullback $F(X) \rightarrow F(U)$ is injective;*
- (2) *F has no p -torsion.*

Then, for any $G \in \mathbf{PST}$, the natural map

$$\underline{\mathbf{Hom}}_{\mathbf{PST}}(G, F) \rightarrow \underline{\mathbf{Hom}}_{\mathbf{PS}}(G, F)$$

is an isomorphism.

Proof (by Kay Rülling). First, we prove $\mathbf{Hom}_{\mathbf{PST}}(G, F) = \mathbf{Hom}_{\mathbf{PS}}(G, F)$; that is, any morphism $\varphi: G \rightarrow F$ of presheaves on \mathbf{Sm} is also a morphism in \mathbf{PST} . We have to show that $\varphi(f^*a) = f^*\varphi(a)$ in $F(X)$ for $a \in G(Y)$ and $f \in \mathbf{Cor}(X, Y)$ a prime correspondence. By condition (1), we can reduce to the case $X = \text{Spec } K$, with K a function field over k . In this case, we can write $f^* = h_*g^*$, where $h: \text{Spec } L \rightarrow \text{Spec } K$ is induced by a finite field extension L/K and $g: \text{Spec } L \rightarrow Y$ is a morphism. Since φ is a morphism of presheaves on \mathbf{Sm} , we are reduced to showing that

$$h_*\varphi(a) = \varphi(h_*a), \quad a \in G(L). \quad (2.14)$$

It suffices to consider the following two cases:

Case 1: L/K is finite separable. Let E/K be a finite Galois extension containing L/K , and denote by $j: \text{Spec } E \rightarrow \text{Spec } K$ the induced morphism and by $\sigma_i: \text{Spec } E \rightarrow \text{Spec } L$ the morphism induced by all K -embeddings of L into E . Since $G \in \mathbf{PST}$, we obtain in $G(E)$

$$j^*h_*a = (h^t \circ j)^*a = \sum_i \sigma_i^*(a).$$

Thus

$$j^*\varphi(h_*a) = \varphi(j^*h_*a) = \varphi\left(\sum_i \sigma_i^*(a)\right) = \sum_i \sigma_i^*\varphi(a) = j^*h_*\varphi(a).$$

Since $j^*: F(L) \rightarrow F(E)$ is injective by condition (1), this shows (2.14) in this case.

Case 2: L/K is purely inseparable of degree p . In this case, $h^*h_* = (h^t \circ h): G(L) \rightarrow G(L)$ is multiplication by p , as is $h_*h^*: G(K) \rightarrow G(K)$. Thus

$$h^*\varphi(h_*a) = \varphi(h^*h_*a) = p\varphi(a) = h^*h_*\varphi(a).$$

Applying h_* yields $p\varphi(h_*a) = ph_*\varphi(a)$; thus (2.14) follows from condition (2).

Next we prove the analogous statement for internal homs. Indeed, note that for $X \in \mathbf{Sm}$, the internal hom $\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) \in \mathbf{PST}$ also satisfies conditions (1) and (2) above and that we have

$$\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) = F(X \times -) = \underline{\mathrm{Hom}}_{\mathbf{PS}}(h_X, F) \quad \text{in } \mathbf{PS}, \quad (2.15)$$

where $h_X = \mathbb{Z}(\mathrm{Hom}_{\mathbf{Sm}}(-, X))$. Thus for $G \in \mathbf{PST}$,

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathbf{PST}}(G, F)(X) &= \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \underline{\mathrm{Hom}}_{\mathbf{PST}}(G, F)) \\ &= \mathrm{Hom}_{\mathbf{PST}}(G \otimes^{\mathbf{PST}} \mathbb{Z}_{\mathrm{tr}}(X), F) \\ &= \mathrm{Hom}_{\mathbf{PST}}(G, \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)) \\ &= \mathrm{Hom}_{\mathbf{PS}}(G, \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)), && \text{by (2.14)} \\ &= \mathrm{Hom}_{\mathbf{PS}}(G, \underline{\mathrm{Hom}}_{\mathbf{PS}}(h_X, F)), && \text{by (2.15)} \\ &= \mathrm{Hom}_{\mathbf{PS}}(G \otimes^{\mathbf{PS}} h_X, F) \\ &= \mathrm{Hom}_{\mathbf{PS}}(h_X, \underline{\mathrm{Hom}}_{\mathbf{PS}}(G, F)) \\ &= \underline{\mathrm{Hom}}_{\mathbf{PS}}(G, F)(X). \end{aligned}$$

This completes the proof of Lemma 2.1. □

LEMMA 2.2. For $F \in \mathbf{PST}$ and $X \in \mathbf{Sm}$, we have a natural isomorphism

$$\underline{\omega}^* \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) \simeq \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), \underline{\omega}^* F).$$

Proof. For $\mathcal{Y} = (Y, E) \in \mathbf{MCor}$ with $V = Y - |E|$, we have natural isomorphisms

$$\begin{aligned} \underline{\omega}^* \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)(\mathcal{Y}) &\simeq \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F)(V) \simeq \mathrm{Hom}_{\mathbf{PST}}(X \times V, F) \\ &\simeq \mathrm{Hom}_{\mathbf{MPST}}((X, \emptyset) \otimes \mathcal{Y}, \underline{\omega}^* F) \simeq \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), \underline{\omega}^* F)(\mathcal{Y}). \end{aligned}$$

This proves the lemma. □

LEMMA 2.3. For $F \in \mathbf{MPST}$ and $X \in \mathbf{Sm}$, we have a natural isomorphism

$$\underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), F) \simeq \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \underline{\omega}_! F).$$

Proof. For $Y \in \mathbf{Sm}$, we have natural isomorphisms

$$\begin{aligned} \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), F)(Y) &\simeq \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), F)(Y, \emptyset) \\ &\simeq \mathrm{Hom}_{\mathbf{MPST}}(\mathbb{Z}_{\mathrm{tr}}(X \times Y, \emptyset), F) \simeq \mathrm{Hom}_{\mathbf{PST}}(X \times Y, \underline{\omega}_! F) \\ &\simeq \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \underline{\omega}_! F)(Y). \end{aligned}$$

This proves the lemma. □

LEMMA 2.4. A complex in C^\bullet in \mathbf{NST} such that $C^n \in \mathbf{RSC}$ for all $n \in \mathbb{Z}$ is exact if and only if $C^\bullet(K)$ is exact as a complex of abelian groups for any function field K .

Proof. The cohomology sheaves $H_{\mathrm{Nis}}^n(C^\bullet)$ are in $\mathbf{RSC}_{\mathrm{Nis}}$ by [Sai20, Theorem 0.1]. Hence for all $X \in \mathbf{Sm}$, by [Sai20, Theorem 0.2], there is an injective map $(H_{\mathrm{Nis}}^n C^\bullet)(X) \hookrightarrow (H_{\mathrm{Nis}}^n C^\bullet)(k(X))$.

Hence the lemma follows from the fact that $(H_{\text{Nis}}^n C^\bullet)(k(X)) \cong H^n(C^\bullet(k(X)))$ since $k(X)$ is henselian local. \square

LEMMA 2.5. For $G \in \mathbf{RSC}$ and $F \in \mathbf{PST}$ such that F is a quotient of a representable sheaf, we have $\underline{\text{Hom}}_{\mathbf{PST}}(F, G) \in \mathbf{RSC}$.

Proof. First assume $F = \mathbb{Z}_{\text{tr}}(X)$ with $X \in \mathbf{Sm}$. Put $\tilde{G} = \underline{\omega}^{\mathbf{CI}} G \in \mathbf{CI}^\tau$ (cf. § 2.18). The adjunction (2.12) implies $\underline{\omega}_! \tilde{G} \simeq G$. Lemma 2.3 implies a natural isomorphism

$$\underline{\text{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), G) \simeq \underline{\omega}_! \underline{\text{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(X, \emptyset), \tilde{G}).$$

Thus it suffices to show

$$\underline{\text{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(X, \emptyset), \tilde{G}) \in \mathbf{CI}^\tau.$$

The \square -invariance follows directly from that for \tilde{G} . The fact that it is in \mathbf{MPST}^τ follows from [Sai20, Lemma 1.27(2)].

Now assume that there is a surjection $\mathbb{Z}_{\text{tr}}(X) \rightarrow F$ in \mathbf{PST} , where $X \in \mathbf{Sm}$. It induces an injection

$$\underline{\text{Hom}}_{\mathbf{PST}}(F, G) \hookrightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), G).$$

Since $\underline{\text{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), G) \in \mathbf{RSC}$ as shown above and $\mathbf{RSC} \subset \mathbf{PST}$ is closed under finite products and subobjects, we get $\underline{\text{Hom}}_{\mathbf{PST}}(F, G) \in \mathbf{RSC}$, as desired. This completes the proof. \square

LEMMA 2.6. Let $F \in \mathbf{MNST}$ be such that $F^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^\tau$ (cf. § 2.16). For any function field K over k , we have

$$H^i(\mathbf{P}_K^1, F_{(\mathbf{P}_K^1, 0+\infty)}) = 0 \quad \text{for } i > 0.$$

Proof. If F is semipure, the assertion follows from [Sai20, Theorem 9.1]. In general, we use the exact sequence in \mathbf{MNST}

$$0 \rightarrow C \rightarrow F \rightarrow F^{\text{sp}} \rightarrow 0$$

to reduce to the above case, observing that $H^i(\mathbf{P}_K^1, C_{(\mathbf{P}_K^1, 0+\infty)}) = 0$ for $i > 0$ since $C_{(\mathbf{P}_K^1, 0+\infty)}$ is supported on $\{0, \infty\}$. \square

LEMMA 2.7. For $F \in \mathbf{CI}^\tau$ and a function field K over k , we have

$$\underline{a}_{\text{Nis}} F(K) \xrightarrow{\sim} \underline{a}_{\text{Nis}} F(\square \otimes K).$$

Proof. We consider the exact sequence $0 \rightarrow C \rightarrow F \rightarrow F^{\text{sp}} \rightarrow 0$ in \mathbf{MPST} with $\underline{\omega}_! C = 0$. Since $\underline{a}_{\text{Nis}}$ is exact, from this we get an exact sequence $0 \rightarrow \underline{a}_{\text{Nis}} C \rightarrow \underline{a}_{\text{Nis}} F \rightarrow \underline{a}_{\text{Nis}} F^{\text{sp}} \rightarrow 0$ in \mathbf{MNST} . Since $C_{(\mathbf{P}_K^1, 0+\infty)}$ is supported on $\{0_K, \infty_K\}$, we have $(\underline{a}_{\text{Nis}} C)_{(\mathbf{P}_K^1, 0+\infty)} = C_{(\mathbf{P}_K^1, 0+\infty)}$ by (2.4). Hence the exact sequence gives rise to a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(K) & \longrightarrow & F(K) & \longrightarrow & F^{\text{sp}}(K) \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ 0 & \longrightarrow & C(\square \otimes K) & \longrightarrow & \underline{a}_{\text{Nis}} F(\square \otimes K) & \longrightarrow & \underline{a}_{\text{Nis}} F^{\text{sp}}(\square \otimes K). \end{array}$$

The left (respectively, right) vertical map is an isomorphism since $C \in \mathbf{CI}^\tau$ (respectively, thanks to [Sai20, Theorem 10.1]). This completes the proof. \square

Let $\mathbf{A}_t^1 = \text{Spec } k[t]$ be the affine line with the coordinate t . Consider the map in \mathbf{PST}

$$\lambda_{\mathbf{G}_m} : \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1 - \{0\}) \rightarrow \mathbf{G}_m$$

given by $t \in \mathbf{G}_m(\mathbf{A}_t^1 - \{0\}) = k[t, t^{-1}]^\times$ and the map in **PST**

$$\lambda_{\mathbf{G}_a} : \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1) \rightarrow \mathbf{G}_a$$

given by $t \in \mathbf{G}_a(\mathbf{A}_t^1) = k[t]$. Note that $\lambda_{\mathbf{G}_m}$ and $\lambda_{\mathbf{G}_a}$ factor through

$$\text{Coker}(\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1 - \{0\})) \quad \text{and} \quad \text{Coker}(\mathbb{Z} \xrightarrow{i_0} \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1)),$$

with i_1 and i_0 induced by the points $1 \in \mathbf{A}_t^1 - \{0\}$ and $0 \in \mathbf{A}_t^1$, respectively.

LEMMA 2.8. (1) *The composite map*

$$\omega_! \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty) \simeq \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1 - \{0\}) \xrightarrow{\lambda_{\mathbf{G}_m}} \mathbf{G}_m$$

induces an isomorphism

$$a_{\text{Nis}}^V \omega_! h_0^{\square}(\square_{\mathbf{G}_m}) \xrightarrow{\sim} \mathbf{G}_m, \quad (2.16)$$

where $\square_{\mathbf{G}_m} = \text{Coker}(\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty)) \in \mathbf{MPST}$.

(2) *The composite map*

$$\omega_! \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 2\infty) \simeq \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1) \xrightarrow{\lambda_{\mathbf{G}_a}} \mathbf{G}_a$$

induces an isomorphism

$$a_{\text{Nis}}^V \omega_! h_0^{\square}(\square_{\mathbf{G}_a}) \xrightarrow{\sim} \mathbf{G}_a, \quad (2.17)$$

where $\square_{\mathbf{G}_a} = \text{Coker}(\mathbb{Z} \xrightarrow{i_0} \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 2\infty)) \in \mathbf{MPST}$.

Proof. We prove only part (2). The proof of part (1) is similar. By [Sai20, Lemma 1.36 and Theorem 0.1], we have $a_{\text{Nis}}^V \omega_! h_0^{\square}(\square_{\mathbf{G}_a}) \in \mathbf{RSC}_{\text{Nis}}$. Hence, by Lemma 2.4, it suffices to show that the map $\lambda_{\mathbf{G}_a} : \mathbb{Z}_{\text{tr}}(\mathbf{A}^1)(K) \rightarrow \mathbf{G}_a(K) = K$ for a function field K over k induces an isomorphism $\omega_! h_0^{\square}(\square_{\mathbf{G}_a})(K) \simeq K$. We know that $\mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1)(K)$ is identified with the group of 0-cycles on $\mathbf{A}_K^1 = \mathbf{A}^1 \otimes_k K$. Then, by [KSY22, Theorem 3.2.1], the kernel of $\mathbb{Z}_{\text{tr}}(\mathbf{A}^1)(K) \rightarrow \omega_! h_0^{\square}(\square_{\mathbf{G}_a})(K)$ is generated by the class of $0 \in \mathbf{A}_K^1$ and $\text{div}_{\mathbf{A}_K^1}(f)$ for $f \in K(t)^\times$ such that $f \in 1 + \mathfrak{m}_\infty^2 \mathcal{O}_{\mathbf{P}_K^1, \infty}$, where \mathfrak{m}_∞ is the maximal ideal of the local ring $\mathcal{O}_{\mathbf{P}_K^1, \infty}$ of \mathbf{P}_K^1 at ∞ . Now part (2) follows by an elementary computation. \square

LEMMA 2.9. *We have*

$$\underline{\text{Hom}}_{\mathbf{MPST}}(G, F) \in \mathbf{MNST} \quad \text{for } G \in \mathbf{MPST}, F \in \mathbf{MNST}.$$

Proof. Put $H = \underline{\text{Hom}}_{\mathbf{MPST}}(G, F)$. Let $\mathcal{X} \in \mathbf{MCor}$ and

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathcal{X} \end{array}$$

be an \mathbf{MV}^{fin} -square as defined in [KMSY21a, Definition 3.2.1]. By [KMSY21a, Definition 4.5.2 and Lemma 4.2.3], it suffices to show the exactness of $0 \rightarrow H(\mathcal{X}) \rightarrow H(\mathcal{U}) \oplus H(\mathcal{Y}) \rightarrow H(\mathcal{W})$. By adjunction, we have $H(\mathcal{X}) = \underline{\text{Hom}}_{\mathbf{MPST}}(G, F^{\mathcal{X}})$ with $F^{\mathcal{X}} = \underline{\text{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(\mathcal{X}), F)$. Hence it suffices to show the exactness of the sequence $0 \rightarrow F^{\mathcal{X}} \rightarrow F^{\mathcal{U}} \oplus F^{\mathcal{Y}} \rightarrow F^{\mathcal{W}}$ in \mathbf{MPST} . Taking $\mathcal{Y} \in \mathbf{MCor}$, this is reduced to showing the exactness of

$$0 \rightarrow F(\mathcal{X} \otimes \mathcal{Y}) \rightarrow F(\mathcal{U} \otimes \mathcal{Y}) \oplus F(\mathcal{Y} \otimes \mathcal{Y}) \rightarrow F(\mathcal{W} \otimes \mathcal{Y}).$$

This follows from the fact that \mathbf{MV}^{fin} -squares are preserved by the product \otimes in \mathbf{MCor} . \square

PROPOSITION 2.10. (1) For $F, G \in \underline{\mathbf{MPST}}$, we have a natural isomorphism

$$\underline{a}_{\text{Nis}}(F \otimes_{\underline{\mathbf{MPST}}} G) \simeq \underline{a}_{\text{Nis}}(\underline{a}_{\text{Nis}}F \otimes_{\underline{\mathbf{MPST}}} \underline{a}_{\text{Nis}}G)$$

induced by the natural maps $F \rightarrow \underline{a}_{\text{Nis}}F$ and $G \rightarrow \underline{a}_{\text{Nis}}G$.

(2) For $F, G \in \mathbf{MPST}$, we have a natural isomorphism

$$a_{\text{Nis}}(F \otimes_{\mathbf{MPST}} G) \simeq a_{\text{Nis}}(a_{\text{Nis}}F \otimes_{\mathbf{MPST}} a_{\text{Nis}}G)$$

induced by the natural maps $F \rightarrow a_{\text{Nis}}F$ and $G \rightarrow a_{\text{Nis}}G$.

Proof. For $H \in \underline{\mathbf{MNST}}$, we have isomorphisms

$$\begin{aligned} \text{Hom}_{\underline{\mathbf{MNST}}}(\underline{a}_{\text{Nis}}(F \otimes_{\underline{\mathbf{MPST}}} G), H) &\simeq \text{Hom}_{\underline{\mathbf{MPST}}}(F \otimes_{\underline{\mathbf{MPST}}} G, H) \\ &\simeq \text{Hom}_{\underline{\mathbf{MPST}}}(F, \underline{\text{Hom}}_{\underline{\mathbf{MPST}}}(G, H)) \\ &\stackrel{(*1)}{\simeq} \text{Hom}_{\underline{\mathbf{MPST}}}(\underline{a}_{\text{Nis}}F, \underline{\text{Hom}}_{\underline{\mathbf{MPST}}}(G, H)) \\ &\simeq \text{Hom}_{\underline{\mathbf{MPST}}}(\underline{a}_{\text{Nis}}F \otimes_{\underline{\mathbf{MPST}}} G, H) \\ &\simeq \text{Hom}_{\underline{\mathbf{MPST}}}(G, \underline{\text{Hom}}_{\underline{\mathbf{MPST}}}(\underline{a}_{\text{Nis}}F, H)) \\ &\stackrel{(*2)}{\simeq} \text{Hom}_{\underline{\mathbf{MPST}}}(\underline{a}_{\text{Nis}}G, \underline{\text{Hom}}_{\underline{\mathbf{MPST}}}(\underline{a}_{\text{Nis}}F, H)) \\ &\simeq \text{Hom}_{\underline{\mathbf{MPST}}}(\underline{a}_{\text{Nis}}F \otimes_{\underline{\mathbf{MPST}}} \underline{a}_{\text{Nis}}G, H) \\ &\simeq \text{Hom}_{\underline{\mathbf{MNST}}}(\underline{a}_{\text{Nis}}(\underline{a}_{\text{Nis}}F \otimes_{\underline{\mathbf{MPST}}} \underline{a}_{\text{Nis}}G), H), \end{aligned}$$

where (*1) and (*2) follow from the fact that $\underline{\text{Hom}}_{\underline{\mathbf{MPST}}}(A, H) \in \underline{\mathbf{MNST}}$ for $A \in \underline{\mathbf{MPST}}$ by Lemma 2.9. This proves part (1).

For $F, G \in \mathbf{MPST}$, we have isomorphisms

$$\begin{aligned} \tau_! a_{\text{Nis}}(F \otimes_{\mathbf{MPST}} G) &\stackrel{(*1)}{\simeq} \underline{a}_{\text{Nis}} \tau_!(F \otimes_{\mathbf{MPST}} G) \\ &\stackrel{(*2)}{\simeq} \underline{a}_{\text{Nis}}(\tau_! F \otimes_{\underline{\mathbf{MPST}}} \tau_! G) \\ &\stackrel{(*3)}{\simeq} \underline{a}_{\text{Nis}}(\underline{a}_{\text{Nis}} \tau_! F \otimes_{\underline{\mathbf{MPST}}} \underline{a}_{\text{Nis}} \tau_! G) \\ &\stackrel{(*4)}{\simeq} \underline{a}_{\text{Nis}}(\tau_! a_{\text{Nis}} F \otimes_{\underline{\mathbf{MPST}}} \tau_! a_{\text{Nis}} G) \\ &\stackrel{(*5)}{\simeq} \underline{a}_{\text{Nis}} \tau_!(a_{\text{Nis}} F \otimes_{\underline{\mathbf{MPST}}} a_{\text{Nis}} G) \\ &\stackrel{(*6)}{\simeq} \tau_! a_{\text{Nis}}(a_{\text{Nis}} F \otimes_{\underline{\mathbf{MPST}}} a_{\text{Nis}} G), \end{aligned}$$

where (*1), (*4) and (*6) follow from (2.3), (*2) and (*5) follow from the monoidality of $\tau_!$ (see [RSY22, §3.8]), and (*3) follows from part (1). Since $\tau_!$ is fully faithful, this implies part (2). This completes the proof of the lemma. \square

LEMMA 2.11. There are natural isomorphisms for $F, G \in \mathbf{MPST}$

$$(F \otimes_{\mathbf{MPST}} G)^{\text{sp}} \simeq (F^{\text{sp}} \otimes_{\mathbf{MPST}} G)^{\text{sp}} \simeq (F^{\text{sp}} \otimes_{\mathbf{MPST}} G^{\text{sp}})^{\text{sp}}. \quad (2.18)$$

Proof. We have an exact sequence $0 \rightarrow C \rightarrow F \rightarrow F^{\text{sp}} \rightarrow 0$ in \mathbf{MPST} with $\omega_! C = 0$. Since $(-)\otimes_{\mathbf{MPST}} G: \mathbf{MPST} \rightarrow \mathbf{MPST}$ is right exact, we get an exact sequence

$$C \otimes_{\mathbf{MPST}} G \rightarrow F \otimes_{\mathbf{MPST}} G \rightarrow F^{\text{sp}} \otimes_{\mathbf{MPST}} G \rightarrow 0.$$

We have $\omega_!(C \otimes_{\mathbf{MPST}} G) = 0$ since $\omega_!: \mathbf{MPST} \rightarrow \mathbf{PST}$ is monoidal by [RSY22, §3.6]. Hence we get an isomorphism $(F \otimes_{\mathbf{MPST}} G)^{\text{sp}} \simeq (F^{\text{sp}} \otimes_{\mathbf{MPST}} G)^{\text{sp}}$. This implies (2.18). \square

LEMMA 2.12. *There are natural isomorphisms for $F, G, H \in \mathbf{MPST}$*

$$h_0^\square(F^{\text{sp}})^{\text{sp}} \simeq h_0^\square(F)^{\text{sp}}, \quad (2.19)$$

$$h_0^\square(F \otimes_{\mathbf{MPST}} G) \simeq h_0^\square(h_0^\square(F) \otimes_{\mathbf{MPST}} h_0^\square(G)). \quad (2.20)$$

Proof. We have an exact sequence $0 \rightarrow C \rightarrow F \rightarrow F^{\text{sp}} \rightarrow 0$ in \mathbf{MPST} with $\omega_! C = 0$. From this, we get an exact sequence $h_0^\square(C) \rightarrow h_0^\square(F) \rightarrow h_0^\square(F^{\text{sp}}) \rightarrow 0$ in \mathbf{MPST} since $h_0^\square: \mathbf{MPST} \rightarrow \mathbf{MPST}$ is right exact. We have $\omega_! h_0^\square(C) = 0$ since $\omega_!: \mathbf{MPST} \rightarrow \mathbf{PST}$ is exact and $h_0^\square(C)$ is a quotient of C . Hence we get an isomorphism $\omega_! h_0^\square(F) \simeq \omega_! h_0^\square(F^{\text{sp}})$. This implies (2.19).

For $H \in \mathbf{CI}$, we have isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbf{CI}}(h_0^\square(F \otimes_{\mathbf{MPST}} G), H) &\simeq \text{Hom}_{\mathbf{MPST}}(F \otimes_{\mathbf{MPST}} G, H) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(F, \underline{\text{Hom}}_{\mathbf{MPST}}(G, H)) \\ &\stackrel{(*)}{\simeq} \text{Hom}_{\mathbf{MPST}}(h_0^\square(F), \underline{\text{Hom}}_{\mathbf{MPST}}(G, H)) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(h_0^\square(F) \otimes_{\mathbf{MPST}} G, H) \\ &\simeq \text{Hom}_{\mathbf{CI}}(h_0^\square(h_0^\square(F) \otimes_{\mathbf{MPST}} G), H), \end{aligned}$$

where (*) follows from the fact that $\underline{\text{Hom}}_{\mathbf{MPST}}(G, H) \in \mathbf{CI}$ for $H \in \mathbf{CI}$, which follows easily from the definition. This shows

$$h_0^\square(F \otimes_{\mathbf{MPST}} G) \simeq h_0^\square(h_0^\square(F) \otimes_{\mathbf{MPST}} G),$$

which implies (2.20). \square

From (2.9), we have $\underline{a}_{\text{Nis}}(\mathbf{CI}^{\tau, \text{sp}}) \subset \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, which implies

$$a_{\text{Nis}}(\mathbf{CI}^{\text{sp}}) \subset \mathbf{CI}_{\text{Nis}}^{\text{sp}}.$$

Indeed, for $F \in \mathbf{CI}^{\text{sp}}$, we have $\tau_! a_{\text{Nis}} F \simeq \underline{a}_{\text{Nis}} \tau_! F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ by (2.3), which implies $a_{\text{Nis}} F \in \mathbf{CI}_{\text{Nis}}^{\text{sp}}$ by definition (cf. § 2.11 and [KMSY21b, Definition 3]). Thus we get an induced functor

$$a_{\text{Nis}}^{\mathbf{CI}}: \mathbf{CI}^{\text{sp}} \rightarrow \mathbf{CI}_{\text{Nis}}^{\text{sp}}. \quad (2.21)$$

By definition, we have

$$a_{\text{Nis}}^{\mathbf{CI}}(F) = a_{\text{Nis}} j(F) \quad \text{for } F \in \mathbf{CI}^{\text{sp}}, \quad (2.22)$$

where $j: \mathbf{CI}^{\text{sp}} \rightarrow \mathbf{MPST}$ is the inclusion.

LEMMA 2.13. *The functor $a_{\text{Nis}}^{\mathbf{CI}}$ is left adjoint to the inclusion $\mathbf{CI}_{\text{Nis}}^{\text{sp}} \rightarrow \mathbf{CI}^{\text{sp}}$.*

Proof. This follows easily from the facts that a_{Nis} is left adjoint to the inclusion $\mathbf{MNST} \rightarrow \mathbf{MPST}$ and that the inclusions $\mathbf{CI}^{\text{sp}} \rightarrow \mathbf{MPST}$ and $\mathbf{CI}_{\text{Nis}}^{\text{sp}} \rightarrow \mathbf{MNST}$ are fully faithful. \square

LEMMA 2.14. *Consider the functors*

$$\begin{aligned} h_0^{\square, \text{sp}}: \mathbf{MPST} &\rightarrow \mathbf{CI}^{\text{sp}}, & F &\mapsto h_0^\square(F)^{\text{sp}}, \\ h_{0, \text{Nis}}^{\square, \text{sp}}: \mathbf{MPST} &\rightarrow \mathbf{CI}_{\text{Nis}}^{\text{sp}}, & F &\mapsto a_{\text{Nis}}^{\mathbf{CI}} h_0^{\square, \text{sp}}(F). \end{aligned}$$

(1) *The functor $h_0^{\square, \text{sp}}$ (respectively, $h_{0, \text{Nis}}^{\square, \text{sp}}$) is a left adjoint to the inclusion $\mathbf{CI}^{\text{sp}} \rightarrow \mathbf{MPST}$ (respectively, $\mathbf{CI}_{\text{Nis}}^{\text{sp}} \rightarrow \mathbf{MPST}$). For $F \in \mathbf{MPST}$, we have natural isomorphisms*

$$h_0^{\square, \text{sp}}(F) \simeq h_0^{\square, \text{sp}} h_{0, \text{Nis}}^{\square, \text{sp}}(F) \quad \text{and} \quad h_{0, \text{Nis}}^{\square, \text{sp}}(F) \simeq h_{0, \text{Nis}}^{\square, \text{sp}} h_{0, \text{Nis}}^{\square, \text{sp}}(F).$$

(2) For $F \in \mathbf{MPST}$, the natural map $F \rightarrow a_{\mathbf{Nis}}F$ induces an isomorphism

$$h_{0, \mathbf{Nis}}^{\square, \text{sp}}(F) \simeq h_{0, \mathbf{Nis}}^{\square, \text{sp}}(a_{\mathbf{Nis}}F).$$

(3) For $F \in \mathbf{MPST}$, we have natural isomorphisms

$$\begin{aligned} h_0^{\square, \text{sp}}(F \otimes_{\mathbf{MPST}} G) &\simeq h_0^{\square, \text{sp}}(h_0^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_0^{\square, \text{sp}}(G)), \\ h_{0, \mathbf{Nis}}^{\square, \text{sp}}(F \otimes_{\mathbf{MPST}} G) &\simeq h_{0, \mathbf{Nis}}^{\square, \text{sp}}(h_{0, \mathbf{Nis}}^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_{0, \mathbf{Nis}}^{\square, \text{sp}}(G)). \end{aligned}$$

Proof. The first statement of item (1) follows from the left-adjointness of $h_0^{\square, \text{sp}}$, $(-)^{\text{sp}}$ and $a_{\mathbf{Nis}}$. The second statement of item (1) is a formal consequence of the first since the inclusions are fully faithful.

To show item (2), consider the commutative diagram

$$\begin{array}{ccc} \mathbf{CI}_{\mathbf{Nis}}^{\text{sp}} & \xrightarrow{i_{\mathbf{CI}}} & \mathbf{CI}^{\text{sp}} \\ \downarrow j_{\mathbf{Nis}} & & \downarrow j \\ \mathbf{MNST} & \xrightarrow{i} & \mathbf{MPST}, \end{array}$$

where the functors are inclusions. For $F \in \mathbf{MPST}$ and $G \in \mathbf{CI}_{\mathbf{Nis}}^{\text{sp}}$, we have isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbf{CI}_{\mathbf{Nis}}^{\text{sp}}}(h_{0, \mathbf{Nis}}^{\square, \text{sp}} ia_{\mathbf{Nis}}F, G) &\stackrel{(*1)}{\simeq} \text{Hom}_{\mathbf{CI}^{\text{sp}}}(h_0^{\square, \text{sp}} ia_{\mathbf{Nis}}F, i_{\mathbf{CI}}G) \\ &\stackrel{(*2)}{\simeq} \text{Hom}_{\mathbf{MPST}}(ia_{\mathbf{Nis}}F, ji_{\mathbf{CI}}G) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(ia_{\mathbf{Nis}}F, ij_{\mathbf{Nis}}G) \\ &\stackrel{(*3)}{\simeq} \text{Hom}_{\mathbf{MNST}}(a_{\mathbf{Nis}}F, j_{\mathbf{Nis}}G) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(F, ij_{\mathbf{Nis}}G) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(F, ji_{\mathbf{CI}}G) \\ &\stackrel{(*4)}{\simeq} \text{Hom}_{\mathbf{MPST}}(h_0^{\square, \text{sp}}F, i_{\mathbf{CI}}G) \\ &\stackrel{(*5)}{\simeq} \text{Hom}_{\mathbf{MPST}}(a_{\mathbf{Nis}}^{\mathbf{CI}} h_0^{\square, \text{sp}}F, G), \end{aligned}$$

where (*1) and (*5) (respectively, (*2) and (*4), (*3)) follow from Lemma 2.13 (respectively, item (1), the full faithfulness of i). This proves item (2).

For $F, G \in \mathbf{MPST}$, we have natural isomorphisms

$$\begin{aligned} h_0^{\square, \text{sp}}(F \otimes_{\mathbf{MPST}} G) &\stackrel{(2.19)}{\simeq} h_0^{\square}((F \otimes_{\mathbf{MPST}} G)^{\text{sp}})^{\text{sp}} \\ &\stackrel{(2.18)}{\simeq} h_0^{\square}((F^{\text{sp}} \otimes_{\mathbf{MPST}} G^{\text{sp}})^{\text{sp}})^{\text{sp}} \\ &\stackrel{(2.19)}{\simeq} h_0^{\square}(F^{\text{sp}} \otimes_{\mathbf{MPST}} G^{\text{sp}})^{\text{sp}} \\ &\stackrel{(2.20)}{\simeq} h_0^{\square}(h_0^{\square}(F^{\text{sp}}) \otimes_{\mathbf{MPST}} h_0^{\square}(G^{\text{sp}}))^{\text{sp}} \\ &\stackrel{(2.19)}{\simeq} h_0^{\square}((h_0^{\square}(F^{\text{sp}}) \otimes_{\mathbf{MPST}} h_0^{\square}(G^{\text{sp}}))^{\text{sp}})^{\text{sp}} \\ &\stackrel{(2.18)}{\simeq} h_0^{\square}((h_0^{\square}(F^{\text{sp}})^{\text{sp}} \otimes_{\mathbf{MPST}} h_0^{\square}(G^{\text{sp}})^{\text{sp}})^{\text{sp}})^{\text{sp}} \\ &\stackrel{(2.19)}{\simeq} h_0^{\square}((h_0^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_0^{\square, \text{sp}}(G))^{\text{sp}})^{\text{sp}} \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(2.19)}{\simeq} h_0^{\square} (h_0^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_0^{\square, \text{sp}}(G))^{\text{sp}} \\
 & = h_0^{\square, \text{sp}} (h_0^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_0^{\square, \text{sp}}(G)) .
 \end{aligned}$$

This proves the first isomorphism of item (3). From this, we get natural isomorphisms

$$\begin{aligned}
 h_{0, \text{Nis}}^{\square, \text{sp}}(F \otimes_{\mathbf{MPST}} G) & \simeq h_{0, \text{Nis}}^{\square, \text{sp}}(h_0^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_0^{\square, \text{sp}}(G)) \\
 & \stackrel{(*1)}{\simeq} h_{0, \text{Nis}}^{\square, \text{sp}} a_{\text{Nis}}(h_0^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_0^{\square, \text{sp}}(G)) \\
 & \stackrel{(*2)}{\simeq} h_{0, \text{Nis}}^{\square, \text{sp}} a_{\text{Nis}}(h_{0, \text{Nis}}^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_{0, \text{Nis}}^{\square, \text{sp}}(G)) \\
 & \stackrel{(*3)}{\simeq} h_{0, \text{Nis}}^{\square, \text{sp}}(h_{0, \text{Nis}}^{\square, \text{sp}}(F) \otimes_{\mathbf{MPST}} h_{0, \text{Nis}}^{\square, \text{sp}}(G)) ,
 \end{aligned}$$

where (*1) and (*3) follow from item (2) and (*2) follows from Proposition 2.10 in view of (2.22). This completes the proof of the lemma. \square

3. Some lemmas on contractions

For an integer $a \geq 1$, put $\overline{\square}^{(a)} = (\mathbf{P}^1, a(0 + \infty)) \in \mathbf{MCor}$ and

$$\overline{\square}_{\text{red}}^{(a)} = \text{Ker}(\mathbb{Z}_{\text{tr}}(\overline{\square}^{(a)}) \rightarrow \mathbb{Z} = \mathbb{Z}_{\text{tr}}(\text{Spec } k, \emptyset)) .$$

The inclusion $\mathbf{A}^1 - \{0\} \hookrightarrow \mathbf{A}^1$ induces a map $\overline{\square}^{(a)} \rightarrow \overline{\square}$ in \mathbf{MCor} for all a . Note that the composite map

$$\overline{\square}_{\text{red}}^{(1)} \hookrightarrow \overline{\square}^{(1)} \rightarrow \overline{\square}_{\mathbf{G}_m} \tag{3.1}$$

is an isomorphism, where $\overline{\square}_{\mathbf{G}_m}$ is from (2.16).

For $F \in \mathbf{MPST}$, we write

$$\gamma F = \text{Coker}(\underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}, F) \rightarrow \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}^{(1)}, F)) \in \mathbf{MPST} ,$$

where the map is induced by $\overline{\square}^{(1)} \rightarrow \overline{\square}$ in \mathbf{MCor} . If $F \in \mathbf{CI}^r$, the projection $\overline{\square} \rightarrow \text{Spec } k$ induces an isomorphism

$$F = \underline{\text{Hom}}_{\mathbf{MPST}}(\text{Spec } k, F) \simeq \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}, F) .$$

Thus we get an isomorphism

$$\gamma F \simeq \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)}, F) \stackrel{(*)}{\cong} \underline{\text{Hom}}_{\mathbf{MPST}}(h_0^{\square}(\overline{\square}_{\text{red}}^{(1)}), F) \quad \text{for } F \in \mathbf{CI}^r , \tag{3.2}$$

where the equality (*) follows from the adjunction from §2.15. Note that $\gamma F \in \mathbf{CI}^{r, \text{sp}}$ for $F \in \mathbf{CI}^{r, \text{sp}}$. We also define

$$\gamma_{\text{Nis}} F = a_{\text{Nis}} \gamma F \in \mathbf{MNST} .$$

By (3.2) and Lemma 2.9, we have

$$\gamma_{\text{Nis}} F = \gamma F \quad \text{for } F \in \mathbf{CI}_{\text{Nis}}^r .$$

For an integer $n \geq 1$, we write (cf. §2.19)

$$\gamma^n F \cong \underline{\text{Hom}}_{\mathbf{MPST}}((\overline{\square}_{\text{red}}^{(1)})^{\otimes_{\mathbf{MPST}} n}, F) \cong \overbrace{\gamma \gamma \cdots \gamma}^{n \text{ times}} F . \tag{3.3}$$

The proof of the following lemma is due to Kay Rülling. We thank him for letting us include it in our paper.

LEMMA 3.1. *The unit map*

$$\underline{a}_{\text{Nis}} h_0^{\square}(\square^{(1)})^{\text{sp}} \xrightarrow{\simeq} \underline{\omega}^* \underline{\omega}_! \underline{a}_{\text{Nis}} h_0^{\square}(\square^{(1)}) \cong \underline{\omega}^*(\mathbf{G}_m \oplus \mathbb{Z}) \quad (3.4)$$

is an isomorphism, where the second isomorphism in (3.4) holds by Lemma 2.8 and (3.1).

Proof (by Kay Rülling). The unit map is injective by semipurity. It remains to show the surjectivity. By the definition of the sheafification functor, it suffices to show the surjectivity on $(\text{Spec } R, (f))$, where R is an integral local k -algebra and $f \in R \setminus \{0\}$ is such that R_f is regular. Denote by

$$\psi: \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty)(R, f) \rightarrow R_f^{\times} \oplus \mathbb{Z}$$

the precomposition of (3.4) evaluated at (R, f) with the quotient map $\mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 0 + \infty)(R, f) \rightarrow \underline{a}_{\text{Nis}} h_0^{\square}(\square^{(1)})^{\text{sp}}$.

We show that ψ is surjective. To this end, observe that for $a \in R_f^{\times}$, we can find an $N \geq 0$ and a $b \in R$ such that

$$ab = f^N \quad \text{and} \quad af^N \in R. \quad (3.5)$$

Set $W := V(t^N - a) \subset \text{Spec } R_f[t, 1/t]$ and $K := \text{Frac}(R)$.

The map $\mathbf{Cor}(K, \mathbf{A}^1 - \{0\}) \rightarrow \text{Pic}(\mathbf{P}_K^1, 0 + \infty) \cong K^{\times} \oplus \mathbb{Z}$, which induces the second isomorphism of (3.4), sends a prime correspondence $V(a_0 + a_1 t + \cdots + a_r t^r)$ to $((-1)^r a_0/a_r, r)$; hence we have

$$\psi(V(a_0 + a_1 t + \cdots + a_r t^r)) = ((-1)^r a_0/a_r, r) \quad (3.6)$$

provided that $V(a_0 + a_1 t + \cdots + a_r t^r) \in \underline{\mathbf{MCor}}((R, f), (\mathbf{P}^1, 0 + \infty))$.

For any $a \in R_f^{\times}$, consider $h = t^N - a$; let $h = \prod_i h_i$ be the decomposition into monic irreducible factors in $K[t, 1/t]$, and denote by $W_i \subset \text{Spec } R_f[t, 1/t]$ the closure of $V(h_i)$. (Note that $W_i = W_j$ for $i \neq j$ is allowed.)

The W_i correspond to the components of W which are dominant over R_f ; since W is finite and surjective over R_f , so are the W_i . We claim

$$W_i \in \underline{\mathbf{MCor}}((R, f), (\mathbf{P}^1, 0 + \infty)). \quad (3.7)$$

Indeed, let I_i (respectively, J_i) be the ideal of the closure of W_i in $\text{Spec } R[t]$ (respectively, $\text{Spec } R[z]$ with $z = 1/t$). By (3.5), we have $bt^N - f^N \in I_i$ and $f^N - f^N a z^N \in J_i$. Hence $(f/t)^N \in R[t]/I_i$ and $(f/z)^N \in R[z]/J_i$. It follows that f/t (respectively, f/z) is integral over $R[t]/I_i$ (respectively, $R[z]/J_i$); thus (3.7) holds. We claim

$$\psi\left(\sum_i W_i\right) = ((-1)^{N+1} a, N).$$

Indeed, it suffices to show this after restriction to the generic point of R , in which case it follows directly from the definition of the W_i and (3.6). Since $\psi(V(t \pm 1)) = (-(\pm 1), 1)$, this implies the surjectivity of ψ and proves the lemma. \square

COROLLARY 3.2. (1) *There is a natural isomorphism*

$$\underline{a}_{\text{Nis}} h_0^{\square}(\square_{\text{red}}^{(1)})^{\text{sp}} \simeq \underline{\omega}^* \mathbf{G}_m.$$

(2) *For $F \in \mathbf{CI}_{\text{Nis}}^{7, \text{sp}}$, we have a natural isomorphism*

$$\gamma F \simeq \underline{\text{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, F). \quad (3.8)$$

Proof. Item (1) is a direct consequence of Lemma 3.1. In view of (3.2), item (2) follows from item (1) and the adjunction of $\underline{a}_{\text{Nis}}$ and that from §2.14. \square

LEMMA 3.3. Consider an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\underline{\mathbf{MNST}}$.

(1) Assume $A, B, C \in \mathbf{CI}_{\text{Nis}}^\tau$. Then the following sequence in \mathbf{NST} is exact:

$$0 \rightarrow \underline{\omega}_1 \gamma A \rightarrow \underline{\omega}_1 \gamma B \rightarrow \underline{\omega}_1 \gamma C \rightarrow 0.$$

(2) Assume $\underline{\omega}_1 A = 0$ and $C \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$. Then the sequence

$$0 \rightarrow \gamma A(K) \rightarrow \gamma B(K) \rightarrow \gamma C(K) \rightarrow 0$$

is exact for any function field K over k .

Proof. First assume $A, B, C \in \mathbf{CI}_{\text{Nis}}^\tau$. Then all terms of the sequence in item (1) are in $\mathbf{RSC}_{\text{Nis}}$. By Lemma 2.4, it suffices to show the exactness of

$$0 \rightarrow \gamma A(K) \rightarrow \gamma B(K) \rightarrow \gamma C(K) \rightarrow 0$$

for a function field K over k .

By (3.2), we have $\gamma F(K) = \text{Hom}(\overline{\square}_{\text{red}, K}^{(1)}, F)$ for all $F \in \mathbf{CI}^\tau$, where $\overline{\square}_{\text{red}, K}^{(1)} = \overline{\square}_{\text{red}}^{(1)} \otimes \text{Spec } K$. Since $\overline{\square}_{\text{red}, K}^{(1)}$ is a direct summand of $\mathbb{Z}_{\text{tr}}(\mathbf{P}_K^1, 0 + \infty)$, it is enough to show that

$$\text{Ext}_{\underline{\mathbf{MNST}}}^1(\mathbb{Z}_{\text{tr}}(\mathbf{P}_K^1, 0 + \infty), A) = 0.$$

By using [KMSY21a, Theorem 2(2)], we can compute

$$\text{Ext}_{\underline{\mathbf{MNST}}}^1(\mathbb{Z}_{\text{tr}}(\mathbf{P}_K^1, 0 + \infty), A) \simeq H_{\text{Nis}}^1(\mathbf{P}_K^1, A_{(\mathbf{P}_K^1, 0 + \infty)}),$$

where we use the fact that any proper birational map $X \rightarrow \mathbf{P}_K^1$ is an isomorphism. Thus the vanishing follows from Lemma 2.6. This proves item (1).

Next we assume $\underline{\omega}_1 A = 0$ and $C \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$. For a function field K over k , we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A(\mathbf{P}_K^1, \infty) & \longrightarrow & B(\mathbf{P}_K^1, \infty) & \longrightarrow & C(\mathbf{P}_K^1, \infty) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow c \\ 0 & \longrightarrow & A(\mathbf{P}_K^1, 0 + \infty) & \longrightarrow & B(\mathbf{P}_K^1, 0 + \infty) & \longrightarrow & C(\mathbf{P}_K^1, 0 + \infty) \longrightarrow 0, \end{array}$$

where the sequences are exact since for every effective Cartier divisor D on \mathbf{P}_K^1 ,

$$\text{Ext}_{\underline{\mathbf{MNST}}}^1(\mathbb{Z}_{\text{tr}}(\mathbf{P}_K^1, D), A) \simeq H_{\text{Nis}}^1(\mathbf{P}_K^1, A_{(\mathbf{P}_K^1, D)}) = 0$$

by [KMSY21a, Theorem 2(2)] and the fact that $A_{(\mathbf{P}_K^1, D)}$ is supported on the zero-dimensional scheme $|D|$ by the assumption. Finally, $\text{Ker}(c) = 0$ by [Sai20, Theorem 3.1]; hence the snake lemma gives the exact sequence of item (2). \square

PROPOSITION 3.4. (1) Take $F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ (cf. §2.16). For $\mathcal{X} = (X, D_X) \in \underline{\mathbf{MCor}}_{\text{ls}}$ (cf. §2.3), there exists a map functorial in \mathcal{X}

$$\gamma F(\mathcal{X}) \rightarrow H^1(\mathbf{P}^1 \times X, F_{\mathbf{P}^1 \otimes \mathcal{X}}). \quad (3.9)$$

Moreover, if X is henselian local, it is an isomorphism.

(2) Let $F \in \underline{\mathbf{MNST}}$ be such that $F^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$. For $X \in \mathbf{Sm}$, there exists a map functorial in X

$$\gamma F(X) \rightarrow H^1(\mathbf{P}^1 \times X, F_{\mathbf{P}^1 \times X}). \quad (3.10)$$

Moreover, it is an isomorphism either if $F \in \mathbf{CI}_{\text{Nis}}^{\tau}$ and X is henselian local, or if $X = \text{Spec}(K)$ is the spectrum of a function field over k and the natural map $F(K) \rightarrow F(\overline{\square} \otimes K)$ is an isomorphism.

Proof. Let $L = (\mathbf{P}^1, 0)$. We prove item (1). By (3.2) and [Sai20, Lemma 7.1], there exists an exact sequence of sheaves on $(\mathbf{P}^1 \times X)_{\text{Nis}}$

$$0 \rightarrow F_{\mathbf{P}^1 \otimes \mathcal{X}} \rightarrow F_{L \otimes \mathcal{X}} \rightarrow i_* \gamma F_{\mathcal{X}} \rightarrow 0, \quad (3.11)$$

where $i: X \rightarrow \mathbf{P}^1 \times X$ is induced by $0 \in \mathbf{P}^1$. Taking cohomology, we get the map (3.9). If X is henselian local, we have

$$H^1(\mathbf{P}^1 \times X, F_{L \otimes \mathcal{X}}) \simeq H^1(X, F_{\mathcal{X}}) = 0 \quad (3.12)$$

thanks to [Sai20, Theorem 9.3]. Note that the map $F(\mathcal{X}) \rightarrow F(L \otimes \mathcal{X})$ induced by the projection $L \otimes \mathcal{X} \rightarrow \mathcal{X}$ is an isomorphism by the \square -invariance of F . Since the projection factors as $L \otimes \mathcal{X} \rightarrow \mathbf{P}^1 \otimes \mathcal{X} \rightarrow \mathcal{X}$, this implies that the map $F(\mathbf{P}^1 \otimes \mathcal{X}) \rightarrow F(L \otimes \mathcal{X})$ is surjective. This implies that the map (3.9) is an isomorphism.

We now prove item (2). Consider the exact sequence of sheaves on $(\mathbf{P}^1 \times X)_{\text{Nis}}$

$$0 \rightarrow F_{\mathbf{P}^1 \times X} \rightarrow F_{L \otimes X} \rightarrow i_* \lambda_X F \rightarrow 0, \quad (3.13)$$

where $\lambda_X F = i^*(F_{L \otimes X}/F_{\mathbf{P}^1 \times X})$. The injectivity of the first map follows from [Sai20, Theorem 3.1], observing that $F_{\mathbf{P}^1 \times X} = F_{\mathbf{P}^1 \times X}^{\text{sp}}$ (the point is that X has empty modulus) and $F^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ by the assumption. Taking cohomology over an étale $U \rightarrow X$, we get a natural map in U

$$\lambda_X F(U) \rightarrow H^1(\mathbf{P}^1 \times U, F_{\mathbf{P}^1 \times U}).$$

To define the map (3.10), it suffices to show the following.

CLAIM 3.5. *There exists a natural map $\varphi_{F,X}: (\gamma_{\text{Nis}} F)_X \rightarrow \lambda_X F$ of sheaves on X_{Nis} . It is an isomorphism if $F \in \mathbf{CI}_{\text{Nis}}^{\tau}$. If $F \in \mathbf{MNST}$ and $F^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^{\tau}$, then $\varphi_{F,K}: (\gamma_{\text{Nis}} F)_K = (\gamma F)_K \rightarrow \lambda_X F$ is an isomorphism for a function field K over k .*

Proof of Claim 3.5. By definition, $\lambda_X F$ is the sheaf on X_{Nis} associated with the presheaf

$$\widetilde{\lambda_X F}: U \rightarrow \varinjlim_V F(V, 0_V)/F(V, \emptyset), \quad (3.14)$$

where V ranges over étale neighborhoods of $0_U = i(U) \subset \mathbf{P}^1 \times U$. On the other hand, we have

$$(\gamma F)_X(U) = F(\mathbf{P}^1 \times U, 0 + \infty)/F(\mathbf{P}^1 \times U, \infty).$$

Since the colimit in (3.14) does not change when taken over étale neighborhood of $0_U \subset \mathbf{A}^1 \times U$, there is a natural map

$$(\gamma F)_X(U) \rightarrow F(\mathbf{A}^1 \times U, 0)/F(\mathbf{A}^1 \times U, \emptyset) \rightarrow \widetilde{\lambda_X F}(U),$$

which induces the desired map $\varphi_{F,X}$.

Next we show that $\varphi_{F,X}$ is an isomorphism if $F \in \mathbf{CI}_{\text{Nis}}^{\tau}$, or if $F \in \mathbf{MNST}$ with $F^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ and $X = K$ is a function field over k . If F is semipure, the assertion follows from [Sai20, Lemma 7.1]. In general, we consider the exact sequence in \mathbf{MNST}

$$0 \rightarrow C \rightarrow F \rightarrow F^{\text{sp}} \rightarrow 0 \quad \text{with } \underline{\omega}_1 C = 0. \quad (3.15)$$

It gives rise to a commutative diagram of sheaves on $(\mathbf{P}^1 \times X)_{\text{Nis}}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{\mathbf{P}^1 \times X} & \longrightarrow & F_{\mathbf{P}^1 \times X} & \longrightarrow & F_{\mathbf{P}^1 \times X}^{\text{sp}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_{L \otimes X} & \longrightarrow & F_{L \otimes X} & \longrightarrow & F_{L \otimes X}^{\text{sp}}, \end{array}$$

where the upper (respectively, lower) sequence is exact by the exactness of $\omega_1: \underline{\text{MNST}} \rightarrow \text{NST}$ from § 2.13 (respectively, by § 2.12). The right vertical map is injective by [Sai20, Theorem 3.1]. This implies the exactness of the lower sequence of the following commutative diagram on X_{Nis} :

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\gamma C)_X & \longrightarrow & (\gamma F)_X & \longrightarrow & (\gamma F^{\text{sp}})_X \longrightarrow 0 \\ & & \downarrow \varphi_{C,X} & & \downarrow \varphi_{F,X} & & \downarrow \varphi_{F^{\text{sp}},X} \\ 0 & \longrightarrow & \lambda_X C & \longrightarrow & \lambda_X F & \longrightarrow & \lambda_X F^{\text{sp}}. \end{array}$$

The upper sequence is exact by Lemma 3.3. Since we know that $\varphi_{F^{\text{sp}},X}$ is an isomorphism, it suffices to show that $\varphi_{C,X}$ is an isomorphism. Indeed, for an étale $U \rightarrow X$ with U henselian local, we have

$$\begin{aligned} (\gamma C)_X(U) &= C(\mathbf{P}^1 \times U, 0 + \infty) / C(\mathbf{P}^1 \times U, \infty) \\ &\simeq \varinjlim_V C(V, 0_V) / C(V, \emptyset) = \widetilde{\lambda_X C}(U), \end{aligned}$$

where the V are as in (3.14) and the isomorphism comes from the excision as $C_{(\mathbf{P}^1 \times U, 0 + \infty)}$ (respectively, $C_{(\mathbf{P}^1 \times U, \infty)}$) is supported on $\{0_U, \infty_U\}$ (respectively, ∞_U). This proves that $\varphi_{C,X}$ is an isomorphism and completes the proof of the claim. \square

To show the second assertion of item (2), we look at the cohomology exact sequence arising from (3.13). Note that $F(\mathbf{P}^1 \times X) \rightarrow F(L \otimes X)$ is surjective since $F(X) \xrightarrow{\sim} F(L \otimes X)$ by the assumption. Hence it suffices to show $H^1(\mathbf{P}^1 \times X, F_{L \otimes X}) = 0$. If F is semipure, this follows from (3.12). In general, it is reduced to the above case using (3.15) and noting that $H^1(\mathbf{P}^1 \times X, C_{L \otimes X}) = 0$ since $C_{L \otimes X}$ is supported on $0 \times X$. This completes the proof of the proposition. \square

COROLLARY 3.6. *Let $G \in \mathbf{CI}^\tau$, and let K be a function field K over k .*

- (1) *There is a natural isomorphism $\gamma_{\underline{a}_{\text{Nis}}} G(K) \simeq H^1(\mathbf{P}_K^1, \underline{a}_{\text{Nis}} G)$.*
- (2) *The natural map $\gamma_{\underline{a}_{\text{Nis}}} G(K) \rightarrow \gamma_{\underline{a}_{\text{Nis}}} G^{\text{sp}}(K)$ is an isomorphism.*

Proof. Letting $F = \underline{a}_{\text{Nis}} G$, we have $F^{\text{sp}} = \underline{a}_{\text{Nis}} G^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ by (2.9). By Lemma 2.7, the sheaf F satisfies the second assumption of Proposition 3.4(1). Hence item (1) follows from Proposition 3.4(2). Item (2) follows from the isomorphisms

$$\begin{aligned} \gamma_{\underline{a}_{\text{Nis}}} G(K) &\simeq H^1(\mathbf{P}_K^1, \underline{a}_{\text{Nis}} G) \simeq H^1(\mathbf{P}_K^1, \omega_! \underline{a}_{\text{Nis}} G) \simeq H^1(\mathbf{P}_K^1, a_{\text{Nis}}^V \omega_! G) \\ &\simeq H^1(\mathbf{P}_K^1, a_{\text{Nis}}^V \omega_! G^{\text{sp}}) \simeq H^1(\mathbf{P}_K^1, \underline{a}_{\text{Nis}} G^{\text{sp}}) \simeq \gamma_{\underline{a}_{\text{Nis}}} G^{\text{sp}}(K), \end{aligned}$$

where the third and last isomorphisms follow from, respectively, § 2.13 and Proposition 3.4. \square

LEMMA 3.7. *Let $F \in \mathbf{CI}^\tau$.*

- (1) *The natural map $\gamma F(K) \rightarrow \gamma_{\underline{a}_{\text{Nis}}} F(K)$ is an isomorphism for any function field K over k .*
- (2) *The natural map $\underline{a}_{\text{Nis}} \gamma F^{\text{sp}} \rightarrow \gamma_{\underline{a}_{\text{Nis}}} F^{\text{sp}}$ is injective.*
- (3) *The natural map $\omega_! \underline{a}_{\text{Nis}} \gamma F^{\text{sp}} \rightarrow \omega_! \gamma_{\underline{a}_{\text{Nis}}} F^{\text{sp}}$ is an isomorphism.*

Proof. Consider the exact sequence in **MPST**

$$0 \rightarrow C \rightarrow F \rightarrow F^{\text{sp}} \rightarrow 0 \quad \text{with } \omega_! C = 0. \quad (3.16)$$

By § 2.7, we have $C, F^{\text{sp}} \in \mathbf{CI}^\tau$. It gives rise to an exact sequence $0 \rightarrow \underline{a}_{\text{Nis}} C \rightarrow \underline{a}_{\text{Nis}} F \rightarrow \underline{a}_{\text{Nis}} F^{\text{sp}} \rightarrow 0$ in **MNST** and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \gamma C(K) & \longrightarrow & \gamma F(K) & \longrightarrow & \gamma F^{\text{sp}}(K) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \gamma \underline{a}_{\text{Nis}} C(K) & \longrightarrow & \gamma \underline{a}_{\text{Nis}} F(K) & \longrightarrow & \gamma \underline{a}_{\text{Nis}} F^{\text{sp}}(K) \longrightarrow 0. \end{array}$$

The upper sequence is exact thanks to (3.2). The lower sequence is exact by Lemma 3.3(2), noting that $\underline{a}_{\text{Nis}} F^{\text{sp}} \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ by [Sai20, Theorem 10.1] and $\omega_! \underline{a}_{\text{Nis}} C = a_{\text{Nis}}^V \omega_! C = 0$ (cf. § 2.13). Since $C_{(\mathbf{P}_K^1, 0+\infty)}$ is supported on $\{0_K, \infty_K\}$, we have by § 2.4

$$(\underline{a}_{\text{Nis}} C)_{(\mathbf{P}_K^1, 0+\infty)} = C_{(\mathbf{P}_K^1, 0+\infty)},$$

where we use the fact that any proper birational map between normal schemes of dimension 1 is an isomorphism. Hence the left vertical map is an isomorphism. We may therefore assume that F is semipure. By § 2.9, we have $\underline{a}_{\text{Nis}} F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$. By [Sai20, Lemma 5.9], we have natural isomorphisms

$$\begin{aligned} \gamma F(K) &\simeq F(\mathbf{A}_K^1, 0) / F(\mathbf{A}_K^1, \emptyset), \\ \gamma \underline{a}_{\text{Nis}} F(K) &\simeq \underline{a}_{\text{Nis}} F(\mathbf{A}_K^1, 0) / \underline{a}_{\text{Nis}} F(\mathbf{A}_K^1, \emptyset). \end{aligned}$$

Hence item (1) follows from [Sai20, Theorem 4.1].

To show items (2) and (3), first note that $F^{\text{sp}} \in \mathbf{CI}^{\tau, \text{sp}}$ by the assumption and § 2.7 and hence $\gamma F^{\text{sp}} \in \mathbf{CI}^{\tau, \text{sp}}$. By § 2.9, the sheaves $\underline{a}_{\text{Nis}} \gamma F^{\text{sp}}$ and $\gamma \underline{a}_{\text{Nis}} F^{\text{sp}}$ are in $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, and hence $\omega_! \underline{a}_{\text{Nis}} \gamma F^{\text{sp}}$ and $\omega_! \gamma \underline{a}_{\text{Nis}} F^{\text{sp}}$ are in $\mathbf{RSC}_{\text{Nis}}$. Hence item (2) (respectively, item (3)) follows from item (1) for $F = F^{\text{sp}}$ and [Sai20, Corollary 3.4] (respectively, Lemma 2.4). \square

LEMMA 3.8. Consider a sequence $A \rightarrow B \rightarrow C$ in \mathbf{CI}^τ such that

$$\omega_! \underline{a}_{\text{Nis}} A \rightarrow \omega_! \underline{a}_{\text{Nis}} B \rightarrow \omega_! \underline{a}_{\text{Nis}} C \rightarrow 0$$

is exact in **NST**. Then the sequence

$$\gamma \underline{a}_{\text{Nis}} A(K) \rightarrow \gamma \underline{a}_{\text{Nis}} B(K) \rightarrow \gamma \underline{a}_{\text{Nis}} C(K) \rightarrow 0$$

is exact for any function field K over k .

Proof. In view of the right exactness of the functor $H^1(\mathbf{P}_K, -): \mathbf{NST} \rightarrow \mathbf{Ab}$, the lemma follows from Corollary 3.6(1) by applying this functor to the first exact sequence. \square

COROLLARY 3.9. Let $F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$. Then for any function field K , we have an isomorphism $\gamma F(K) \cong \gamma \omega^{\mathbf{CI}} \omega_! F(K)$.

Proof. Let $q: \gamma(F)(K) \rightarrow \gamma(\omega^{\mathbf{CI}} \omega_! F)(K)$ be the map induced by the unit map $F \hookrightarrow \omega^{\mathbf{CI}} \omega_! F$ for the adjunction (2.13), which is injective since it factors the map $F \hookrightarrow \omega^* \omega_! F$. Notice that q is injective by (3.2) and the fact that $\text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}, K}^{(1)}, -)$ preserves injective maps; hence it is enough to show that it is surjective. Let Q be the presheaf cokernel of $F \rightarrow \omega^{\mathbf{CI}} \omega_! F$; then $Q \in \mathbf{CI}^\tau$ and $\omega_! Q = 0$. By Lemma 3.8, we have an exact sequence

$$\gamma F(K) \xrightarrow{q} \gamma \omega^{\mathbf{CI}} \omega_! F(K) \rightarrow \gamma \underline{a}_{\text{Nis}} Q(K) \rightarrow 0.$$

By Corollary 3.6(2), we have $\gamma \underline{a}_{\text{Nis}} Q(K) \cong \gamma \underline{a}_{\text{Nis}} Q^{\text{sp}}(K) = 0$; hence q is surjective. \square

PROPOSITION 3.10. For $F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, there is a natural isomorphism

$$\omega_! \gamma F \simeq \omega_! \underline{\text{Hom}}_{\underline{\text{MPST}}}(\omega^* \mathbf{G}_m, F) \simeq \underline{\text{Hom}}_{\text{PST}}(\mathbf{G}_m, \omega_! F).$$

Proof. The first isomorphism follows from (3.2) and Corollary 3.2. For $F \in \underline{\text{MPST}}$ and $X \in \mathbf{Sm}$, put $F^X = \underline{\text{Hom}}_{\underline{\text{MPST}}}(\mathbb{Z}_{\text{tr}}(X, \emptyset), F)$. Note that $F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$ implies $F^X \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$. We compute

$$\begin{aligned} \omega_! \gamma F(X) &= \underline{\text{Hom}}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(1)}, F)(X, \emptyset) \\ &\simeq \underline{\text{Hom}}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(1)}, F^X) = \gamma F^X(k), \\ \underline{\text{Hom}}_{\text{PST}}(\mathbf{G}_m, \omega_! F)(X) &= \underline{\text{Hom}}_{\text{PST}}(\mathbf{G}_m, \underline{\text{Hom}}_{\text{PST}}(X, \omega_! F)) \\ &\simeq \underline{\text{Hom}}_{\text{PST}}(\mathbf{G}_m, \omega_! F^X)(k), \end{aligned}$$

where the last isomorphism comes from Lemma 2.3. Hence it suffices to show that for any $F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, there exists a natural isomorphism $\gamma F(k) \simeq \underline{\text{Hom}}_{\text{PST}}(\mathbf{G}_m, \omega_! F)$. We have isomorphisms

$$\begin{aligned} \underline{\text{Hom}}_{\text{PST}}(\mathbf{G}_m, \omega_! F) &\stackrel{(*1)}{\simeq} \underline{\text{Hom}}_{\underline{\text{MPST}}}(\omega^* \mathbf{G}_m, \omega^* \omega_! F) \\ &\stackrel{(*2)}{\simeq} \underline{\text{Hom}}_{\underline{\text{MPST}}}(\omega^* \mathbf{G}_m, \omega^* \mathbf{CI}_{\omega_! F}) \\ &\stackrel{(*3)}{\simeq} \underline{\text{Hom}}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(1)}, \omega^* \mathbf{CI}_{\omega_! F}) \\ &\stackrel{(*4)}{\simeq} \gamma \omega^* \mathbf{CI}_{\omega_! F}(k) \stackrel{(*5)}{\simeq} \gamma F(k), \end{aligned}$$

where (*1) follows from the fact that ω^* is fully faithful (cf. § 2.8), (*2) follows from the adjunction from § 2.15 (see also (2.12)) in view of the fact that $\omega^* \mathbf{G}_m \in \mathbf{CI}^{\tau}$ by Lemma 3.1, (*3) from Lemma 3.1, (*4) from (3.2) and (*5) from Corollary 3.9. \square

4. Weak cancellation theorem

For $F, G \in \underline{\text{MPST}}$, we write (cf. §§ 2.16 and 2.19 and Lemma 2.14)

$$\begin{aligned} F \otimes_{\mathbf{CI}} G &= h_0^{\overline{\square}}(F \otimes_{\underline{\text{MPST}}} G) \in \mathbf{CI}, \\ F \otimes_{\mathbf{CI}}^{\text{sp}} G &= h_0^{\overline{\square}, \text{sp}}(F \otimes_{\underline{\text{MPST}}} G) \in \mathbf{CI}^{\text{sp}}, \\ F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} G &= h_0^{\overline{\square}, \text{sp}}(F \otimes_{\underline{\text{MPST}}} G) \in \mathbf{CI}_{\text{Nis}}^{\text{sp}}. \end{aligned}$$

PROPOSITION 4.1. The product $\otimes_{\mathbf{CI}}$ (respectively, $\otimes_{\mathbf{CI}}^{\text{sp}}$, $\otimes_{\mathbf{CI}}^{\text{Nis, sp}}$) defines a symmetric monoidal structure on \mathbf{CI} (respectively, \mathbf{CI}^{sp} , $\mathbf{CI}_{\text{Nis}}^{\text{sp}}$).

Proof. The assertion except for the associativity follows immediately from the fact that $\otimes_{\underline{\text{MPST}}}$ defines a symmetric monoidal structure on $\underline{\text{MPST}}$. We prove the associativity only for $\otimes_{\mathbf{CI}}^{\text{Nis, sp}}$ (the other cases are similar). We need to show a natural isomorphism for $F, G, H \in \mathbf{CI}_{\text{Nis}}^{\text{sp}}$:

$$(F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} G) \otimes_{\mathbf{CI}}^{\text{Nis, sp}} H \simeq F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} (G \otimes_{\mathbf{CI}}^{\text{Nis, sp}} H).$$

For simplicity, we write $\lambda = h_0^{\overline{\square}, \text{sp}}$. For $F, G, H \in \mathbf{CI}_{\text{Nis}}^{\text{sp}}$, we have isomorphisms

$$\begin{aligned} \lambda(\lambda(F \otimes_{\underline{\text{MPST}}} G) \otimes_{\underline{\text{MPST}}} H) &\stackrel{(*1)}{\simeq} \lambda(\lambda^2(F \otimes_{\underline{\text{MPST}}} G) \otimes_{\underline{\text{MPST}}} \lambda H) \\ &\stackrel{(*2)}{\simeq} \lambda(\lambda(F \otimes_{\underline{\text{MPST}}} G) \otimes_{\underline{\text{MPST}}} \lambda H) \\ &\stackrel{(*3)}{\simeq} \lambda((F \otimes_{\underline{\text{MPST}}} G) \otimes_{\underline{\text{MPST}}} H), \end{aligned}$$

where (*1) (respectively, (*2), (*3)) follows from item (3) (respectively, item (1), item (3)) of Lemma 2.14. The lemma follows from this and the associativity of $\otimes_{\mathbf{MPST}}$. \square

For $F, G \in \mathbf{CI}^\tau$, we write

$$\begin{aligned} F \otimes_{\mathbf{CI}} G &= \tau_! h_0^{\square}(\tau^* F \otimes_{\mathbf{MPST}} \tau^* G) \in \mathbf{CI}^\tau, \\ F \otimes_{\mathbf{CI}}^{\text{sp}} G &= \tau_! h_0^{\square, \text{sp}}(\tau^* F \otimes_{\mathbf{MPST}} \tau^* G) \in \mathbf{CI}^{\tau, \text{sp}}, \\ F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} G &= \tau_! h_{0, \text{Nis}}^{\square, \text{sp}}(\tau^* F \otimes_{\mathbf{MPST}} \tau^* G) \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}. \end{aligned}$$

By § 2.3, we have a natural isomorphism

$$\underline{a}_{\text{Nis}}(F \otimes_{\mathbf{CI}}^{\text{sp}} G) \simeq F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} G. \quad (4.1)$$

In view of the equivalences (2.8) and (2.10), Proposition 4.1 implies the following.

PROPOSITION 4.2. *The product $\otimes_{\mathbf{CI}}$ (respectively, $\otimes_{\mathbf{CI}}^{\text{sp}}$, $\otimes_{\mathbf{CI}}^{\text{Nis, sp}}$) defines a symmetric monoidal structure on \mathbf{CI}^τ (respectively, $\mathbf{CI}^{\tau, \text{sp}}$, $\mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$). For $F, G, H \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, There is a natural isomorphism*

$$(F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} G) \otimes_{\mathbf{CI}}^{\text{Nis, sp}} H \simeq F \otimes_{\mathbf{CI}}^{\text{Nis, sp}} (G \otimes_{\mathbf{CI}}^{\text{Nis, sp}} H). \quad (4.2)$$

For $F \in \mathbf{CI}_{\text{Nis}}^\tau$ and an integer $d \geq 0$, we put

$$F(d) = (\overline{\square}_{\text{red}}^{(1)})^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} d} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} F. \quad (4.3)$$

Note that $F(d) = F(m)(n)$ with $d = m + n$ by (4.2).

For $F \in \mathbf{CI}^\tau$ and $f \in F(\mathcal{X})$ with $\mathcal{X} \in \mathbf{MCor}$, consider the composite map

$$\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{MPST}} \mathbb{Z}_{\text{tr}}(\mathcal{X}) \xrightarrow{\text{id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes f} \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{MPST}} F \rightarrow \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F.$$

By the adjunction $(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{MPST}} -) \dashv \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)}, -)$, this gives rise to a natural map

$$\iota_F: F \rightarrow \gamma(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F), \quad (4.4)$$

which induces

$$\iota_F^{\text{sp}}: F^{\text{sp}} \rightarrow \gamma(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F), \quad (4.5)$$

noting the adjunction from § 2.14 and the fact that $\gamma: \mathbf{MPST} \rightarrow \mathbf{MPST}$ preserves semipure objects.

If $F \in \mathbf{CI}_{\text{Nis}}^\tau$, this induces a natural map

$$\iota_F: F^{\text{sp}} \rightarrow \gamma(F(1)), \quad (4.6)$$

which generalizes to a natural map for $n \in \mathbb{Z}_{\geq 1}$ (cf. (4.3) and (3.3))

$$\iota_F^n: F^{\text{sp}} \rightarrow \gamma^n(F(n)), \quad (4.7)$$

noting that

$$\gamma^n F = \underline{\text{Hom}}_{\mathbf{MPST}}((\overline{\square}_{\text{red}}^{(1)})^{\otimes_{\mathbf{CI}^n}, F}) \quad \text{for } F \in \mathbf{CI}^\tau$$

thanks to the adjunction from § 2.15.

Question 4.3. For $F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, is the map (4.6) an isomorphism?

We will prove the following variant.

THEOREM 4.4. For $F \in \mathbf{CI}^\tau$, the map (4.5) is an isomorphism.

Before going into its proof, we give some consequences.

COROLLARY 4.5. For $F \in \mathbf{CI}^\tau$, the map (4.5) gives an isomorphism

$$\omega_! \iota_F: \omega_! \underline{a}_{\text{Nis}} F \xrightarrow{\sim} \omega_! \gamma \underline{a}_{\text{Nis}} (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F).$$

For $F \in \mathbf{CI}_{\text{Nis}}^\tau$, the map (4.7) induces an isomorphism

$$\omega_! \iota_F^n: \omega_! F \xrightarrow{\sim} \omega_! \gamma^n F(n).$$

Proof. The functors $\omega_!$ and $\underline{a}_{\text{Nis}}$ are exact, and $\omega_! \underline{a}_{\text{Nis}} G \cong \omega_! \underline{a}_{\text{Nis}} G^{\text{sp}}$ for all $G \in \mathbf{MPST}$. Hence Theorem 4.4 gives a natural isomorphism

$$\omega_! \underline{a}_{\text{Nis}} \iota_F: \omega_! \underline{a}_{\text{Nis}} F \xrightarrow{\sim} \omega_! \underline{a}_{\text{Nis}} \gamma (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F).$$

This proves the first assertion since Lemma 3.7(3) implies

$$\omega_! \underline{a}_{\text{Nis}} \gamma (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F) \simeq \omega_! \gamma \underline{a}_{\text{Nis}} (\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F).$$

The second assertion for the case $n = 1$ follows directly from the first. For $n > 1$, we proceed by induction on n to assume

$$\omega_! \iota_F^{n-1}: \omega_! F \xrightarrow{\sim} \omega_! \gamma^{n-1} F(n-1). \quad (4.8)$$

Then we have isomorphisms

$$\begin{aligned} \omega_! \gamma^n F(n) &\stackrel{(*1)}{\simeq} \omega_! \gamma \gamma^{n-1} F(n) \stackrel{(*2)}{\simeq} \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \omega_! \gamma^{n-1} F(n)) \\ &= \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \omega_! \gamma^{n-1} F(1)(n-1)) \\ &\stackrel{(*3)}{\simeq} \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \omega_! F(1)) \stackrel{(*4)}{\simeq} \omega_! \gamma F(1) \stackrel{(*5)}{\simeq} \omega_! F, \end{aligned}$$

where (*1) follows from (3.3), (*2) follows from Proposition 3.10 noting that $\gamma^{n-1} F(n) \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, (*3) follows from (4.8), (*4) follows from Proposition 3.10 and (*5) follows from the case $n = 1$. This completes the proof. \square

COROLLARY 4.6. For $F \in \mathbf{RSC}_{\text{Nis}}$ and $\tilde{F} = \omega^{\mathbf{CI}} F \in \mathbf{CI}_{\text{Nis}}^\tau$ (cf. (2.13)), the map $\iota_{\tilde{F}}^n: \tilde{F} \rightarrow \gamma^n \tilde{F}(n)$ from (4.7) is an isomorphism.

Proof. We have a commutative diagram

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{\iota_{\tilde{F}}^n} & \gamma^n \tilde{F}(n) \\ \downarrow \cong & & \downarrow \hookrightarrow \\ \omega^{\mathbf{CI}} \omega_! \tilde{F} & \xrightarrow{\omega^{\mathbf{CI}} \omega_! \iota_{\tilde{F}}^n} & \omega^{\mathbf{CI}} \omega_! \gamma^n \tilde{F}(n), \end{array}$$

where the vertical arrows come from the adjunction (2.13). The left (respectively, right) vertical arrow is an isomorphism (respectively, is injective) since $\omega_! \omega^{\mathbf{CI}} \simeq \text{id}$ (respectively, by the semipurity of $\gamma^n \tilde{F}(n)$). Since $\omega^{\mathbf{CI}} \omega_! \iota_{\tilde{F}}^n$ is an isomorphism by Corollary 4.5, this implies that $\iota_{\tilde{F}}^n$ is an isomorphism by the snake lemma. \square

COROLLARY 4.7. For $F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$, there is a natural injective map

$$\tilde{\rho}_F: \gamma^n F(n) \rightarrow \omega^{\mathbf{CI}} \omega_! F$$

whose composite with the map $\iota_F^n: F \rightarrow \gamma^n F(n)$ from (4.7) coincides with the unit map $u: F \rightarrow \omega^{\mathbf{CI}} \omega_! F$ for the adjunction (2.13). In particular, (4.7) is injective.

Proof. Define $\tilde{\rho}_F$ as the composite

$$\gamma^n F(n) \xrightarrow{u} \gamma^n \underline{\omega}^{\mathbf{CI}_{\omega_1} F}(n) \xrightarrow{(\iota_{\underline{\omega}^{\mathbf{CI}_{\omega_1} F}}^n)^{-1}} \underline{\omega}^{\mathbf{CI}_{\omega_1} F},$$

where the second map is the inverse of the isomorphism $\iota_{\underline{\omega}^{\mathbf{CI}_{\omega_1} F}}^n: \underline{\omega}^{\mathbf{CI}_{\omega_1} F} \cong \gamma^n \underline{\omega}^{\mathbf{CI}_{\omega_1} F}(n)$ from Corollary 4.6. Clearly, we have $\tilde{\rho}_F \circ \iota_F^n = u$. We easily see that $\tilde{\rho}_F$ coincides with the composite

$$\gamma^n F(n) \xrightarrow{u} \underline{\omega}^{\mathbf{CI}_{\omega_1} \gamma^n F}(n) \xrightarrow{(\underline{\omega}_1 \iota_F^n)^{-1}} \underline{\omega}^{\mathbf{CI}_{\omega_1} F},$$

where the first map is injective by the semipurity of $\gamma^n F(n)$ and the second map is induced by the inverse of the isomorphism $\underline{\omega}_1 \iota_F^n: \underline{\omega}_1 F \cong \underline{\omega}_1 \gamma^n F(n)$ from Corollary 4.5. This completes the proof. \square

In the rest of this section, we prove the following.

PROPOSITION 4.8. *For $F \in \mathbf{CI}^T$, the map ι_F^{sp} from (4.5) is split injective.*

For the proof of Proposition 4.8, we first recall a construction from [Voe10]. Take $X, Y \in \mathbf{Sm}$. For an integer $n > 0$, consider the rational function $g_n = (x_1^{n+1} - 1)/(x_1^{n+1} - x_2)$ on $\mathbf{A}_{x_1}^1 \times \mathbf{A}_{x_2}^1$. Let $D_{XY}(g_n)$ be the divisor of the pullback of g_n to $(\mathbf{A}_{x_1}^1 - 0) \times X \times (\mathbf{A}_{x_2}^1 - 0) \times Y$. Take a prime correspondence

$$Z \in \mathbf{Cor}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y). \quad (4.9)$$

Let $\bar{Z} \subset \mathbf{P}_{x_1}^1 \times X \times \mathbf{P}_{x_2}^1 \times Y$ be the closure of Z and \bar{Z}^N be its normalization.

LEMMA 4.9. (1) *Let $N > 0$ be an integer such that*

$$N(0_1 + \infty_1)|_{\bar{Z}^N} \geq (0_2 + \infty_2)|_{\bar{Z}^N}. \quad (4.10)$$

Then, for any integer $n \geq N$, the correspondence Z intersects properly with $|D_{XY}(g_n)|$, and any component of the intersection $Z \cdot D_{XY}(g_n)$ is finite and surjective over X . Thus we get

$$\rho_n(Z) \in \mathbf{Cor}(X, Y)$$

as the pushforward of $Z \cdot D_{XY}(g_n)$ in $X \times Y$.

(2) *If $Z = \text{Id}_{(\mathbf{A}_{x_1}^1 - 0)} \otimes W$ for $W \in \mathbf{Cor}(X, Y)$, then one can take $N = 1$ in item (1) and $\rho_n(Z) = W$.*

(3) *For any Z as in (4.9) such that $\rho_n(Z)$ is defined and for any $f \in \mathbf{Cor}(X', Y')$ with $X', Y' \in \mathbf{Sm}$, for*

$$Z \otimes f \in \mathbf{Cor}((\mathbf{A}_{x_1}^1 - 0) \times (X \times X'), (\mathbf{A}_{x_2}^1 - 0) \times (Y \times Y')),$$

$\rho_n(Z \otimes f)$ is defined, and we have

$$\rho_n(Z \otimes f) = \rho_n(Z) \otimes f \in \mathbf{Cor}(X \times X', Y \times Y').$$

(4) *For an integer $N > 0$, let*

$$\mathbf{Cor}^{(N)}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y)$$

be the subgroup of $\mathbf{Cor}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y)$ generated by prime correspondences satisfying condition (4.10). Then the presheaf on \mathbf{Sm} given by

$$X \rightarrow \mathbf{Cor}^{(N)}((\mathbf{A}_{x_1}^1 - 0) \times X, (\mathbf{A}_{x_2}^1 - 0) \times Y)$$

is a Nisnevich sheaf.

Proof. The assertions are proved in [Voe10, Lemmas 4.1, 4.3 and 4.5], except that item (4) follows from the fact that the condition (4.10) is Nisnevich local on X . \square

For an integer $a \geq 1$, put $\overline{\square}^{(a)} = (\mathbf{P}^1, a(0+\infty)) \in \mathbf{MCor}$. Take $\mathcal{X} = (\overline{X}, X_\infty), \mathcal{Y} = (\overline{Y}, Y_\infty) \in \mathbf{MCor}$ with $X = \overline{X} - |X_\infty|$ and $Y = \overline{Y} - |Y_\infty|$. For $a \geq 1$, take a prime correspondence

$$Z \in \mathbf{MCor}(\overline{\square}^{(a)} \otimes \mathcal{X}, \overline{\square}^{(1)} \otimes \mathcal{Y}).$$

By definition, $Z \in \mathbf{Cor}(X, Y)$, and Z satisfies

$$(0_2 + \infty_2)|_{\overline{Z}^N} + (Y_\infty)|_{\overline{Z}^N} \leq a(0_1 + \infty_1)|_{\overline{Z}^N} + (X_\infty)|_{\overline{Z}^N}, \quad (4.11)$$

where \overline{Z}^N is the normalization of the closure \overline{Z} of Z in $\mathbf{P}_{x_1}^1 \times X \times \mathbf{P}_{x_2}^1 \times \overline{Y}$.

For integers $n, m \geq a$, we consider the rational function on $\mathbf{A}_{x_1}^1 \times \mathbf{A}_t^1 \times \mathbf{A}_{x_2}^1$

$$h = tg_n + (1-t)g_m.$$

Let $D_{X\mathbf{A}^1Y}(h)$ be the divisor of the pullback of h to $(\mathbf{A}_{x_1}^1 - 0) \times X \times \mathbf{A}_t^1 \times (\mathbf{A}_{x_2}^1 - 0) \times Y$. By [Voe10, Remark 4.2], the product $Z \times \mathbf{A}_t^1$ intersects properly with $|D_{X\mathbf{A}^1Y}(h)|$, and any component of the intersection $(Z \times \mathbf{A}_t^1) \cdot D_{X\mathbf{A}^1Y}(h)$ is finite and surjective over $X \times \mathbf{A}_t^1$. Thus we get

$$\rho_h(Z \times \mathbf{A}_t^1) \in \mathbf{Cor}(X \times \mathbf{A}_t^1, Y).$$

It is easy to see that

$$i_0^* \rho_h(Z \times \mathbf{A}_t^1) = \rho_m(Z) \quad \text{and} \quad i_1^* \rho_h(Z \times \mathbf{A}_t^1) = \rho_n(Z). \quad (4.12)$$

LEMMA 4.10. For $n, m \geq a$, we have $\rho_h(Z \times \mathbf{A}_t^1) \in \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}, \mathcal{Y})$.

Proof. Let V be any component of $(Z \times \mathbf{A}_t^1) \cdot D_{X\mathbf{A}^1Y}(h)$ and \overline{V} be its closure in the product $\mathbf{P}_{x_1}^1 \times \overline{X} \times \mathbf{P}_t^1 \times \mathbf{P}_{x_2}^1 \times \overline{Y}$. Let $W \subset X \times \mathbf{A}_t^1 \times Y$ be the image of V and \overline{W} be its closure in $\overline{X} \times \mathbf{P}_t^1 \times \overline{Y}$. Then we have $\overline{W} = \pi(\overline{V})$, where $\pi: \mathbf{P}_{x_1}^1 \times \overline{X} \times \mathbf{P}_t^1 \times \mathbf{P}_{x_2}^1 \times \overline{Y} \rightarrow \overline{X} \times \mathbf{P}_t^1 \times \overline{Y}$ is the projection. We want to show that

$$(Y_\infty)|_{\overline{W}^N} \leq (\overline{X} \times \infty)|_{\overline{W}^N} + (X_\infty \times \mathbf{P}_t^1)|_{\overline{W}^N}.$$

Since $\pi: \overline{V}^N \rightarrow \overline{W}^N$ is proper and surjective, this is reduced to showing that

$$(Y_\infty)|_{\overline{V}^N} \leq (\overline{X} \times \infty)|_{\overline{V}^N} + (X_\infty \times \mathbf{P}_t^1)|_{\overline{V}^N},$$

by [KP12, Lemma 2.2]. By (4.11) and the containment lemma [KP12, Proposition 2.4] (see also [BS19, Lemma 2.1]), we have

$$(Y_\infty)|_{\overline{V}^N} + (0_2 + \infty_2)|_{\overline{V}^N} \leq a(0_1 + \infty_1)|_{\overline{V}^N} + (X_\infty \times \mathbf{P}_t^1)|_{\overline{V}^N}.$$

Thus it suffices to show that

$$a(0_1 + \infty_1)|_{\overline{V}^N} \leq (0_2 + \infty_2)|_{\overline{V}^N} + \infty|_{\overline{V}^N}.$$

Using [KP12, Proposition 2.4] again, this follows from

$$a(0_1 + \infty_1)|_T \leq (0_2 + \infty_2)|_T + \infty|_T, \quad (4.13)$$

where $T \subset \mathbf{P}_{x_1}^1 \times \mathbf{P}_t^1 \times \mathbf{P}_{x_2}^1$ is any component of the closure of the divisor of h on $(\mathbf{A}_{x_1}^1 - 0) \times \mathbf{A}_t^1 \times (\mathbf{A}_{x_2}^1 - 0)$. By an easy computation, T is contained in one of the closures $\overline{D(H)}$, $\overline{D(J_n)}$,

$\overline{D(J_m)}$ of the divisors of

$$H = t(x_1^{n+1} - x_1^{m+1})(1 - x_2) + (x_1^{m+1} - 1)(x_1^{n+1} - x_2),$$

$$J_n = x_1^{n+1} - x_2, \quad J_m = x_1^{m+1} - x_2,$$

respectively. Letting $\mathbf{P}_{x_i}^1 - 0 = \text{Spec } k[\tau_i]$ with $\tau_i = x_i^{-1}$ for $i = 1, 2$, the closures $\overline{D(H)}$, $\overline{D(J_n)}$, $\overline{D(J_m)}$ are defined in $(\mathbf{P}_{x_1}^1 - 0) \times \mathbf{A}_t^1 \times (\mathbf{P}_{x_2}^1 - 0)$ by the ideals generated by, respectively,

$$H' = t(\tau_1^{m+1} - \tau_1^{n+1})(\tau_2 - 1) + (1 - \tau_1^{m+1})(\tau_2 - \tau_1^{n+1}),$$

$$J'_n = \tau_2 - \tau_1^{n+1}, \quad J'_m = \tau_2 - \tau_1^{m+1}.$$

Hence, $\overline{D(H)}$, $\overline{D(J_n)}$, $\overline{D(J_m)}$ do not intersect with $\infty_1 \times \mathbf{P}_t^1 \times \mathbf{A}_{x_2}^1$.

By the assumption $n, m \geq a$, the ideals $(J_n, x_1^a), (J_m, x_1^a) \subset k[x_1, x_2]$ contain x_2 , and the ideals $(J'_n, \tau_1^a), (J'_m, \tau_1^a) \subset k[\tau_1, \tau_2]$ contain τ_2 , which implies (4.13) (without the last term) if T is contained in $\overline{D(J_m)}$ or $\overline{D(J_n)}$.

On the other hand, the ideal $(H, x_1^a) \subset k[x_1, x_2, t]$ contains x_2 , and the ideal $(H', \tau_1^a) \subset k[\tau_1, \tau_2, t]$ contains τ_2 . Over $\mathbf{P}_t^1 - 0 = \text{Spec } k[u]$ with $u = t^{-1}$, $\overline{D(H)} \cdot (\mathbf{A}_{x_1}^1 \times (\mathbf{P}_t^1 - 0) \times \mathbf{A}_{x_2}^1)$ is the zero divisor of

$$\tilde{H} = (x_1^{n+1} - x_1^{m+1})(1 - x_2) + u(x_1^{m+1} - 1)(x_1^{n+1} - x_2),$$

and $\overline{D(H)} \cdot ((\mathbf{P}_{x_1}^1 - 0) \times (\mathbf{P}_t^1 - 0) \times (\mathbf{P}_{x_2}^1 - 0))$ is the zero divisor of

$$\tilde{H}' = (\tau_1^{m+1} - \tau_1^{n+1})(\tau_2 - 1) + u(1 - \tau_1^{m+1})(\tau_2 - \tau_1^{n+1}).$$

The ideal $(\tilde{H}, x_1^a) \subset k[x_1, x_2, u]$ contains ux_2 , and the ideal $(\tilde{H}', \tau_1^a) \subset k[\tau_1, \tau_2, u]$ contains $u\tau_2$. This shows (4.13) if $T \subset \overline{D(H)}$ and completes the proof of the claim. \square

LEMMA 4.11. For $n \geq a$, we have $\rho_n(Z) \in \mathbf{MCor}(\mathcal{X}, \mathcal{Y})$.

Proof. This follows from Lemma 4.10 and (4.12). \square

For an integer $N \geq a$, let

$$\mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) \subset \mathbf{MCor}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y})$$

be the subgroup generated by prime correspondences lying in $\mathbf{Cor}^{(N)}((\mathbf{A}^1 - 0) \times X, (\mathbf{A}^1 - 0) \times Y)$. By Lemma 4.11, for $n \geq N \geq a$, we get a map

$$\rho_n^{(a)}: \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X}, \mathcal{Y}). \quad (4.14)$$

The map (4.14) induces a map of cubical complexes

$$\rho_n^{(a)\bullet}: \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X} \otimes \overline{\square}^\bullet, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}^\bullet, \mathcal{Y}). \quad (4.15)$$

By construction, the following diagram is commutative if $n \geq N \geq b \geq a$:

$$\begin{array}{ccc} \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X} \otimes \overline{\square}^\bullet, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) & \xrightarrow{\rho_n^{(a)\bullet}} & \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}^\bullet, \mathcal{Y}) \\ \downarrow \beta^* & \nearrow \rho_n^{(b)\bullet} & \\ \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(b)} \otimes \mathcal{X} \otimes \overline{\square}^\bullet, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) & & \end{array} \quad (4.16)$$

where β^* is induced by the natural map $\beta: \overline{\square}_{\text{red}}^{(b)} \rightarrow \overline{\square}_{\text{red}}^{(a)}$.

COROLLARY 4.12. For $m, n \geq N \geq a$, the maps $\rho_n^{(a)\bullet}$ and $\rho_m^{(a)\bullet}$ are homotopic.

Proof. By Lemma 4.10, we get a map

$$s_{m,n} = \rho_h(- \times \mathbf{A}_t^1): \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}, \mathcal{Y}) \quad (4.17)$$

such that $\partial \circ s_{m,n} = \rho_m^{(a)} - \rho_n^{(a)}$, where $\partial = i_0^* - i_1^*: \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}, \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X}, \mathcal{Y})$. Let

$$s_{m,n}^i: \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X} \otimes \overline{\square}^i, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) \rightarrow \mathbf{MCor}(\mathcal{X} \otimes \overline{\square}^{i+1}, \mathcal{Y})$$

be the map (4.17) defined by replacing \mathcal{X} by $\mathcal{X} \otimes \overline{\square}^i$. Then we have that

$$\partial \circ ((-1)^i s_{m,n}^i) + (-1)^{i-1} s_{m,n}^{i-1} \circ \partial = \rho_n^{(a),i} - \rho_m^{(a),i};$$

hence $\{(-1)^i s_{m,n}^i\}_i$ gives the desired homotopy. \square

Let $Z \in \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y})$; then for all $W \in \mathbf{MCor}(\mathcal{X}', \mathcal{X})$, by [Voe10, Lemma 4.4],

$$Z \circ (\text{Id}_{\mathbf{A}^1 - \{0\}} \otimes W) \in \mathbf{Cor}^{(N)}((\mathbf{A}^1 - 0) \times X, (\mathbf{A}^1 - 0) \times Y).$$

Moreover, by [KMSY21a, Proposition 1.2.4(i)], we have

$$Z \circ (\text{Id}_{\mathbf{A}^1 - \{0\}} \otimes W) \in \mathbf{MCor}(\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}),$$

which implies that

$$L_a(\mathcal{Y})^{(N)} = \underline{\text{Hom}}_{\mathbf{MPST}}^{(N)}(\overline{\square}_{\text{red}}^{(a)}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathbb{Z}_{\text{tr}}(\mathcal{Y})) = \mathbf{MCor}^{(N)}(\overline{\square}_{\text{red}}^{(a)} \otimes (-), \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y})$$

is an object of \mathbf{MPST} , which is a subobject of

$$L_a(\mathcal{Y}) = \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(a)}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathbb{Z}_{\text{tr}}(\mathcal{Y})) \in \mathbf{MPST},$$

and we have

$$L_a(\mathcal{Y}) = \varinjlim_{N>0} L_a(\mathcal{Y})^{(N)}. \quad (4.18)$$

The above construction gives a map of complexes $\rho_N^{(a)\bullet}: C_\bullet L_a(\mathcal{Y})^{(N)} \rightarrow C_\bullet(\mathcal{Y})$ in \mathbf{MPST} , where $C_\bullet(-)$ is the cubical Suslin complex. Let $\rho_N^{(a)}: H_i(C_\bullet L_a(\mathcal{Y})^{(N)}) \rightarrow H_i(C_\bullet(\mathcal{Y}))$ be the map in \mathbf{MPST} induced on cohomology presheaves. Thanks to Corollary 4.12, the diagram

$$\begin{array}{ccc} H_i(C_\bullet L_a(\mathcal{Y})^{(N)}) & \xrightarrow{\rho_N^{(a)}} & h_i^{\overline{\square}}(\mathcal{Y}) \\ \downarrow & \nearrow \rho_{N'}^{(a)} & \\ H_i(C_\bullet L_a(\mathcal{Y})^{(N')}) & & \end{array}$$

commutes for integers $N' \geq N$. Hence, by (4.18), we get maps $\rho^{(a)}: H_i(C_\bullet L_a(\mathcal{Y})) \rightarrow h_i^{\overline{\square}}(\mathcal{Y})$. Putting $\Phi = \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}$, we have

$$C_\bullet(L_a(\mathcal{Y})) = \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(a)}, \underline{\text{Hom}}_{\mathbf{MPST}}(\overline{\square}^\bullet, \Phi)).$$

Recall that for $F \in \mathbf{MPST}$ and $\mathcal{X} \in \mathbf{MCor}$, we have, by the Hom-tensor adjunction, an isomorphism

$$h_0^{\overline{\square}} \underline{\text{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(\mathcal{X}), F) \cong \underline{\text{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(\mathcal{X}), h_0^{\overline{\square}}(F)).$$

Hence, we get an isomorphism $H_0(C_\bullet L_a(\mathcal{Y})) \simeq \underline{\text{Hom}}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(a)}, h_0^{\overline{\square}}(\Phi))$ for $h_i^{\overline{\square}}(\Phi) = H_i(C_\bullet(\Phi))$, and we have an isomorphism $h_0^{\overline{\square}}(\Phi) = h_0^{\overline{\square}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) = \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \in \mathbf{CI}$. Hence we get a natural map

$$\rho_{\mathcal{Y}}^{(a)} : \gamma_a(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}) \rightarrow h_0^{\overline{\square}}(\mathcal{Y}), \quad (4.19)$$

where $\gamma_a(F) := \underline{\text{Hom}}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(a)}, F)$ for $F \in \underline{\text{MPST}}$, and by abuse of notation, for $C \in \mathbf{CI}$, we let C also denote $\tau_1 C \in \mathbf{CI}^\tau$ (cf. § 2.16). In view of (4.16), the following diagram is commutative (recall that we assume $b \geq a$):

$$\begin{array}{ccc} \underline{\text{Hom}}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(a)}, h_0^{\overline{\square}}(\Phi)) & \xrightarrow{\rho_{\mathcal{Y}}^{(a)}} & h_0^{\overline{\square}}(\mathcal{Y}) \\ \downarrow \beta^* & \nearrow \rho_{\mathcal{Y}}^{(b)} & \\ \underline{\text{Hom}}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(b)}, h_0^{\overline{\square}}(\Phi)) & & \end{array}$$

Now take any $F \in \mathbf{CI}^\tau$, and consider a presentation $A \rightarrow B \rightarrow F \rightarrow 0$ in $\underline{\text{MPST}}$, where A and B are the direct sums of $h_0^{\overline{\square}}(\mathcal{Y})$ for varying $\mathcal{Y} \in \mathbf{MCor}$. We then get a commutative diagram

$$\begin{array}{ccccccc} \gamma_a(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} A) & \rightarrow & \gamma_a(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} B) & \rightarrow & \gamma_a(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F) & \rightarrow & 0 \\ \downarrow \rho_A^{(a)} & & \downarrow \rho_B^{(a)} & & & & \\ A & \longrightarrow & B & \longrightarrow & F & \longrightarrow & 0, \end{array}$$

where the vertical maps are induced by (4.19). The upper sequence is exact by the right exactness of $\otimes_{\mathbf{CI}}$ and the fact that $\overline{\square}_{\text{red}}^{(a)}$ is a projective object of $\underline{\text{MPST}}$. Thus we get the induced map in $\underline{\text{MPST}}$

$$\rho_F^{(a)} : \gamma_a(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F) \rightarrow F. \quad (4.20)$$

Write $\rho_F = \rho_F^{(1)}$.

CLAIM 4.13. *The map ρ_F splits ι_F .*

Proof. By the construction of ρ_F , the proof is reduced to the case $F = h_0^{\overline{\square}}(\mathcal{Y})$ for $\mathcal{Y} \in \mathbf{MCor}$, which follows from Lemma 4.9(2). \square

The following result concludes the proof of Proposition 4.8.

LEMMA 4.14. *For $F \in \mathbf{CI}^\tau$, the map ρ_F factors through*

$$\rho_F^{\text{sp}} : \gamma(\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F) \rightarrow F^{\text{sp}}.$$

Moreover, it splits the map ι_F^{sp} from (4.5).

Proof. Take $\mathcal{X} \in \mathbf{MCor}$, and let φ be in the kernel of

$$\text{Hom}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F) \rightarrow \text{Hom}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F).$$

Note that the map is surjective since $\overline{\square}_{\text{red}}^{(a)} \otimes \mathcal{X}$ is a projective object of $\underline{\text{MPST}}$ by Yoneda's lemma. By the definition of semipurification (cf. § 2.14), there exists an integer $m > 0$ such that $\beta_m^* \varphi = 0$ in $\text{Hom}_{\underline{\text{MPST}}}(\overline{\square}_{\text{red}}^{(m)} \otimes \mathcal{X}^{(m)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} F)$, where $\beta_m : \overline{\square}_{\text{red}}^{(m)} \otimes \mathcal{X}^{(m)} \rightarrow \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}$ (cf.

§ 2.6). Then the maps from (4.20) induce a commutative diagram

$$\begin{array}{ccc}
 \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\mathrm{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}} F) & \xrightarrow{\rho_F} & F(\mathcal{X}) \\
 \downarrow & & \downarrow \theta_m^* \\
 \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\mathrm{red}}^{(1)} \otimes \mathcal{X}^{(m)}, \overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}} F) & \xrightarrow{\rho_F} & F(\mathcal{X}^{(m)}) \\
 \downarrow & \nearrow \rho_F^{(m)} & \\
 \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\mathrm{red}}^{(m)} \otimes \mathcal{X}^{(m)}, \overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}} F) & &
 \end{array}$$

β_m^* (curved arrow from top-left to bottom-left)

where θ_m^* is induced by $\theta_m: \mathcal{X}^{(m)} \rightarrow \mathcal{X}$ and the triangle commutes by (4.16). We have $\theta_m^* \rho_F(\varphi) = \rho_F^{(m)} \beta_m^*(\varphi) = 0$. Hence $\rho_F(\varphi)$ lies in the kernel of θ_m^* , which is contained in the kernel of the map $\mathrm{sp}_{\mathcal{X}}: F(\mathcal{X}) \rightarrow F^{\mathrm{sp}}(\mathcal{X})$ by the definition of semipurification. Hence the composite map

$$\mathrm{sp}_{\mathcal{X}} \circ \rho_F: \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\mathrm{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}} F) \rightarrow F^{\mathrm{sp}}(\mathcal{X})$$

factors through $\mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\mathrm{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}}^{\mathrm{sp}} F)$, inducing the desired map ρ_F^{sp} . Finally, to show the last assertion, consider the commutative diagram

$$\begin{array}{ccccc}
 F & \xrightarrow{\iota_F} & \gamma(\overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}} F) & \xrightarrow{\rho_F} & F \\
 \downarrow & & \downarrow & & \downarrow \\
 F^{\mathrm{sp}} & \xrightarrow{\iota_F^{\mathrm{sp}}} & \gamma(\overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}}^{\mathrm{sp}} F) & \xrightarrow{\rho_F^{\mathrm{sp}}} & F^{\mathrm{sp}},
 \end{array}$$

where $\rho_F \iota_F = \mathrm{id}_F$ by Claim 4.13. This implies $\rho_F^{\mathrm{sp}} \iota_F^{\mathrm{sp}} = \mathrm{id}_{F^{\mathrm{sp}}}$ since $F \rightarrow F^{\mathrm{sp}}$ is surjective. This completes the proof of Lemma 4.14. \square

5. Completion of the proof of the main theorem

In this section, we prove the following result.

PROPOSITION 5.1. *For $\varphi \in \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\mathrm{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\mathrm{red}}^{(1)} \otimes \mathcal{Y})$ with $\mathcal{X}, \mathcal{Y} \in \mathbf{MCor}$, there exists an $f \in \mathbf{MCor}(\mathcal{X}, \mathcal{Y})$ such that φ and $\mathrm{id}_{\overline{\square}_{\mathrm{red}}^{(1)}} \otimes f$ have the same image in $\mathrm{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\mathrm{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}}^{\mathrm{sp}} \mathcal{Y})$.*

First, we deduce Theorem 4.4 from Proposition 5.1. By Proposition 4.8, it suffices to show the surjectivity of the map ι_F^{sp} from (4.5). Proposition 5.1 implies that the composition

$$h_0^{\overline{\square}}(\mathcal{Y}) \rightarrow \gamma(\overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}) \rightarrow \gamma(\overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}}^{\mathrm{sp}} \mathcal{Y}) \simeq \gamma(\overline{\square}_{\mathrm{red}}^{(1)} \otimes_{\mathbf{CI}}^{\mathrm{sp}} h_0^{\overline{\square}}(\mathcal{Y}))$$

is surjective. Since the last object is semipure, it factors through $h_0^{\overline{\square}}(\mathcal{Y})^{\mathrm{sp}}$, proving the desired surjectivity for $F = h_0^{\overline{\square}}(\mathcal{Y})$.

For a general $F \in \mathbf{CI}^{\tau}$, consider a surjection $q: \bigoplus_{\mathcal{Y} \rightarrow F} h_0^{\overline{\square}}(\mathcal{Y}) \rightarrow F$, which gives a commuta-

tive diagram

$$\begin{array}{ccc} \bigoplus h_0^{\square}(\mathcal{Y})^{\text{sp}} & \xrightarrow{\oplus \iota_{\mathcal{Y}}^{\text{sp}}} & \bigoplus \gamma(\square_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} \mathcal{Y}) \\ \downarrow q^{\text{sp}} & & \downarrow \\ F^{\text{sp}} & \xrightarrow{\iota_F^{\text{sp}}} & \gamma(\square_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} F), \end{array}$$

where the top arrow is surjective and the vertical arrows are surjective since representable presheaves are projective objects of $\underline{\mathbf{MPST}}$ by Yoneda's lemma and the functors $(-)^{\text{sp}}$ and $\square_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}$ commute with direct sums and preserves surjective maps. This proves the desired surjectivity of ι_F .

The proof of Proposition 5.1 requires a construction analogous to the one in [Gra05]. For a variable T over k and for $i \geq 1$, we put

$$\square_T^{(i)} = (\mathbf{P}_T^1, i(0 + \infty)),$$

where \mathbf{P}_T^1 is the compactification of $\mathbf{G}_{m,T} = \text{Spec } k[T, T^{-1}]$. We also put (cf. (3.1))

$$\square_{T,\text{red}}^{(i)} = \text{Ker}(\mathbb{Z}_{\text{tr}}(\square_T^{(i)}) \xrightarrow{\text{pr}} \mathbb{Z} = \mathbb{Z}_{\text{tr}}(\text{Spec } k, \emptyset)) \in \mathbf{MPST},$$

where $\text{pr}: \mathbf{P}_T^1 \rightarrow \text{Spec } k$ is the projection. Let e be the composite of pr and $i_1: \mathbb{Z} \rightarrow \mathbb{Z}_{\text{tr}}(\square_T^{(1)})$ induced by $1 \in \mathbf{P}_T^1$. Then e is an idempotent of $\text{End}_{\mathbf{MPST}}(\square_T^{(1)})$, and $\text{id} - e \in \text{End}_{\mathbf{MPST}}(\square_T^{(1)})$, with id denoting the identity on $\square_T^{(i)}$, is a splitting of $\square_{T,\text{red}}^{(i)} \rightarrow \square_T^{(i)}$. Thus, we get a direct sum decomposition in \mathbf{MPST} (cf. (3.1))

$$\square_T^{(i)} = \square_{T,\text{red}}^{(i)} \oplus \mathbb{Z} \quad \text{with} \quad \square_{T,\text{red}}^{(i)} = (\text{id} - e)\square_T^{(i)}.$$

For $F \in \underline{\mathbf{MPST}}$ and integers $i_1, \dots, i_n \geq 1$, let

$$\pi: \text{Hom}_{\underline{\mathbf{MPST}}}(\square_T^{(i_1)} \otimes \dots \otimes \square_T^{(i_n)}, F) \rightarrow \text{Hom}_{\underline{\mathbf{MPST}}}(\square_{T,\text{red}}^{(i_1)} \otimes \dots \otimes \square_{T,\text{red}}^{(i_n)}, F)$$

be the projection induced by the above decomposition.

For $X \in \mathbf{Sm}$ and $a \in \Gamma(X, \mathcal{O}^\times)$, let $[a] \in \mathbf{Cor}(X, \mathbf{A}^1 - \{0\})$ be the map given by $z \mapsto a$, where $\mathbf{A}^1 = \text{Spec } k[z]$.

LEMMA 5.2. (1) *The correspondences*

$$[T], [U], [TU], [1] \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}), (\mathbf{A}^1 - \{0\}))$$

lie in $\mathbf{MCor}(\square_T^{(1)} \otimes \square_U^{(1)}, \square^{(1)})$. Moreover, we have

$$[T] + [U] - [TU] - [1] = 0 \in \text{Hom}_{\mathbf{MPST}}(\square_T^{(1)} \otimes \square_U^{(1)}, h_0^{\square}(\square^{(1)})).$$

(2) *The correspondences*

$$[-T], [-U], [-TU], [-1] \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}), (\mathbf{A}^1 - \{0\}))$$

lie in $\mathbf{MCor}(\square_T^{(1)} \otimes \square_U^{(1)}, \square^{(1)})$. Moreover, we have

$$[-T] + [-U] - [-TU] - [-1] = 0 \in \text{Hom}_{\mathbf{MPST}}(\square_T^{(1)} \otimes \square_U^{(1)}, h_0^{\square}(\square^{(1)})).$$

Proof. The first assertion of item (1) follows from the equalities

$$[T] = \mu(\text{id} \otimes [1]), \quad [U] = \mu(\text{id} \otimes [1]), \quad [TU] = \mu,$$

where $\mu: (\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}) \rightarrow (\mathbf{A}_W^1 - \{0\})$ is the multiplication $W = TU$, which lies in $\mathbf{MCor}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \overline{\square}_W^{(1)})$ by [Sai20, Claim 1.21].

To show the second assertion of item (1), consider as in [SV00, Proposition 3.4.3] the finite correspondence Z given by the following algebraic subset:

$$\begin{aligned} & \{V^2 - (W(T+U) + (1-W)(TU+1))V + TU = 0\} \\ & \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}) \times \mathbf{A}_W^1, \mathbf{A}_V^1 - \{0\}). \end{aligned} \quad (5.1)$$

Let

$$i_0, i_1: (\mathbf{A}_T^1 - 0) \times (\mathbf{A}_U^1 - 0) \times (\mathbf{A}_V^1 - 0) \rightarrow (\mathbf{A}_T^1 - 0) \times (\mathbf{A}_U^1 - 0) \times \mathbf{A}_W^1 \times (\mathbf{A}_V^1 - 0)$$

be the maps induced by the inclusion of 0_W and 1_W in \mathbf{A}_W^1 . It is clear that $(i_0^* - i_1^*)(Z) = ([TU] + [1]) - ([T] + [U])$ since

$$\begin{aligned} V^2 - (TU+1)V + TU &= (V - TU)(V - 1), \\ V^2 - (T+U)V + TU &= (V - T)(V - U). \end{aligned}$$

We need to check that Z lies in $\mathbf{MCor}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)} \otimes \overline{\square}_W, \overline{\square}_V^{(1)})$. Consider the compactification $(\mathbf{P}^1)^{\times 4}$ of $\mathbf{A}_T^1 \times \mathbf{A}_U^1 \times \mathbf{A}_W^1 \times \mathbf{A}_V^1$ given coordinates with the usual convention $[0 : 1] = \infty$ and $[1 : 0] = 0$:

$$([T_0 : T_\infty], [U_0 : U_\infty], [W_0 : W_\infty], [V_0 : V_\infty]).$$

Then the closure of Z is the hypersurface given by the polyhomogeneous polynomial

$$T_0 U_0 W_0 V_\infty^2 - (W_\infty(T_0 U_\infty + T_\infty U_0) + (W_0 - W_\infty)(T_\infty U_\infty + T_0 U_0)) V_\infty V_0 + T_\infty U_\infty W_0 V_0^2.$$

We have to check that it satisfies the modulus condition: letting $\varphi: \overline{Z} \rightarrow (\mathbf{P}^1)^{\times 4}$ be the inclusion, and letting

$$\begin{aligned} D_1 &= (\{0\} + \{\infty\}) \times \mathbf{P}_U^1 \times \mathbf{P}_W^1 \times \mathbf{P}_V^1 + \mathbf{P}_T^1 \times (\{0\} + \{\infty\}) \times \mathbf{P}_W^1 \times \mathbf{P}_V^1 \\ &\quad + \mathbf{P}_T^1 \times \mathbf{P}_U^1 \times \{\infty\} \times \mathbf{P}_V^1, \end{aligned}$$

$$D_2 = \mathbf{P}_T^1 \times \mathbf{P}_U^1 \times \mathbf{P}_W^1 \times (\{0\} + \{\infty\}),$$

we have to check the inequality

$$\varphi^*(D_1) \geq \varphi^*(D_2). \quad (5.2)$$

Consider the Zariski cover of $(\mathbf{P}^1)^{\times 4}$ given by

$$\{\mathcal{U}_{\alpha,\beta,\gamma,\delta} = (\mathbf{P}^1 - \alpha) \times (\mathbf{P}^1 - \beta) \times (\mathbf{P}^1 - \gamma) \times (\mathbf{P}^1 - \delta), \alpha, \beta, \gamma, \delta \in \{0, \infty\}\}.$$

Define $t_\alpha = T_\infty/T_0$ if $\alpha = \infty$ and $t_\alpha = T_0/T_\infty$ if $\alpha = 0$ and $u_\beta, w_\gamma, v_\delta$ similarly. Then

$$\mathcal{U}_{\alpha,\beta,\gamma,\delta} = \text{Spec}(k[t_\alpha, u_\beta, w_\gamma, v_\delta]).$$

On this cover, the Cartier divisors D_1 and D_2 are given by the systems of local equations

$$D_1 = \{(\mathcal{U}_{\alpha,\beta,0,\delta}, t_\alpha u_\beta w_0), (\mathcal{U}_{\alpha,\beta,\infty,\delta}, t_\alpha u_\beta)\}, \quad D_2 = \{(\mathcal{U}_{\alpha,\beta,\gamma,\delta}, v_\delta)\}.$$

The equation of \overline{Z} on $(\mathbf{P}^1)^{\times 4} - \{0\}$ is of the form $T_0 U_0 W_0 - v_0 F$ for some $F \in k[v_0][U_0, U_\infty, \dots]$. Hence (5.2) is satisfied on $\mathcal{U}_{\alpha,\beta,\gamma,0}$ if $\alpha = 0$ or $\beta = 0$ or $\gamma = 0$. Furthermore, $\overline{Z} \cdot \mathcal{U}_{\infty,\infty,\infty,0} \cdot D_2 = \emptyset$. Similarly, the equation of \overline{Z} on $(\mathbf{P}^1)^{\times 4} - \{\infty\}$ is of the form $T_\infty U_\infty W_0 - v_\infty G$ for some $G \in k[v_\infty][U_0, U_\infty, \dots]$. Hence (5.2) is satisfied on $\mathcal{U}_{\alpha,\beta,\gamma,\infty}$ if $\alpha = \infty$ or $\beta = \infty$ or $\gamma = 0$. Furthermore, $\overline{Z} \cdot \mathcal{U}_{0,0,\infty,\infty} \cdot D_2 = \emptyset$.

Item (2) is proved by the same argument using the following correspondence instead of (5.1):

$$\begin{aligned} & \{V^2 + (W(T+U) + (1-W)(TU+1))V + TU = 0\} \\ & \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}) \times \mathbf{A}_W^1, \mathbf{A}_V^1 - \{0\}). \end{aligned} \quad \square$$

COROLLARY 5.3. We have $\pi([TU]) = 0 \in \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{T,\text{red}}^{(1)} \otimes \overline{\square}_{U,\text{red}}^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)}))$.

Proof. This follows from Lemma 5.2 since

$$\begin{aligned} [TU] \circ ((\text{id} - e) \otimes (\text{id} - e)) &= [TU] - [TU] \circ (1 \otimes e) - [TU] \circ (e \otimes 1) + [TU] \circ (e \otimes e) \\ &= [TU] - [T] - [U] + [1] \quad \text{in } \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \overline{\square}^{(1)}). \end{aligned} \quad \square$$

For $X \in \mathbf{Sm}$ and $a, b \in \Gamma(X, \mathcal{O}^\times)$, let

$$[a, b] \in \mathbf{Cor}(X, (\mathbf{A}^1 - \{0\}) \otimes (\mathbf{A}^1 - \{0\}))$$

be the map given by $z \mapsto a$, $w \mapsto b$, where z (respectively, w) is the standard coordinate of the first (respectively, second) \mathbf{A}^1 .

COROLLARY 5.4. In $\mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)} \otimes \overline{\square}_V^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)}))$, we have

$$[T, V] + [U, V] - [TU, V] - [1, V] = [-T, V] + [-U, V] - [-TU, V] - [-1, V] = 0.$$

Proof. This follows from Lemma 5.2, noting that the endofunctor $_{-} \otimes \overline{\square}^{(1)}$ on \mathbf{MPST} is additive and $h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})$ is a quotient of $h_0^{\overline{\square}}(\overline{\square}^{(1)}) \otimes \overline{\square}^{(1)}$. \square

PROPOSITION 5.5. The correspondences

$$[U, T], [T^{-1}, U] \in \mathbf{Cor}((\mathbf{A}_T^1 - \{0\}) \times (\mathbf{A}_U^1 - \{0\}), (\mathbf{A}^1 - \{0\}) \times (\mathbf{A}^1 - \{0\}))$$

lie in $\mathbf{MCor}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \overline{\square}^{(1)} \otimes \overline{\square}^{(1)})$. Moreover, the element

$$\pi([U, T]) - \pi([T^{-1}, U]) \in \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{T,\text{red}}^{(1)} \otimes \overline{\square}_{U,\text{red}}^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)}))$$

lies in the kernel of the map

$$\mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{T,\text{red}}^{(1)} \otimes \overline{\square}_{U,\text{red}}^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})) \rightarrow \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{T,\text{red}}^{(2)} \otimes \overline{\square}_{U,\text{red}}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})).$$

Proof. (See [Gra05, Corollary 9].) The first assertion is easily checked. To show the second, consider the map in \mathbf{MCor}

$$\overline{\square}_S^{(2)} \rightarrow \overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \quad T \mapsto S, \quad U \mapsto S^{-1}.$$

Composing this with the correspondences of Lemma 5.2(1), we get $[S] + [S^{-1}] - 2[1] = 0 \in \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{S,\text{red}}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)}))$. Noting that $\pi([1]) = (\text{id} - e) \circ [1] = 0$, we get $\pi([S] + [S^{-1}]) = 0 \in \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{S,\text{red}}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)}))$.

This implies

$$\pi([S, V] + [S^{-1}, V]) = 0 \in \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{S,\text{red}}^{(2)} \otimes \overline{\square}_{V,\text{red}}^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})), \quad (5.3)$$

again noting that the endofunctor $_{-} \otimes \overline{\square}_V^{(1)}$ on \mathbf{MCor} is additive and $h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})$ is a quotient of $h_0^{\overline{\square}}(\overline{\square}^{(1)}) \otimes \overline{\square}^{(1)}$.

On the other hand, by tensoring the correspondence of Corollary 5.3 with another copy of itself, we get

$$\pi([TU, VW]) = 0 \quad \text{in } \mathbf{Hom}_{\mathbf{MPST}}(\overline{\square}_{T,\text{red}}^{(1)} \otimes \overline{\square}_{U,\text{red}}^{(1)} \otimes \overline{\square}_{V,\text{red}}^{(1)} \otimes \overline{\square}_{W,\text{red}}^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})). \quad (5.4)$$

There is a map in **MCor**

$$\overline{\square}_{S_1}^{(2)} \otimes \overline{\square}_{S_2}^{(2)} \rightarrow \overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)} \otimes \overline{\square}_V^{(1)} \otimes \overline{\square}_W^{(1)}, \quad T \mapsto S_1, U \mapsto S_2, V \mapsto -S_1, W \mapsto S_2,$$

which induces an element of

$$\mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_{S_1, \mathrm{red}}^{(2)} \otimes \overline{\square}_{S_2, \mathrm{red}}^{(2)}, \overline{\square}_{T, \mathrm{red}}^{(1)} \otimes \overline{\square}_{U, \mathrm{red}}^{(1)} \otimes \overline{\square}_{V, \mathrm{red}}^{(1)} \otimes \overline{\square}_{W, \mathrm{red}}^{(1)}).$$

Composing this with (5.4) and changing the variables (S_1, S_2) to (T, U) , we get

$$\pi([TU, -TU]) = 0 \in \mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_{T, \mathrm{red}}^{(2)} \otimes \overline{\square}_{U, \mathrm{red}}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})). \quad (5.5)$$

We make the following claim.

CLAIM 5.6. *In $\mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_{T, \mathrm{red}}^{(1)} \otimes \overline{\square}_{U, \mathrm{red}}^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)}))$, we have*

$$\pi([TU, -TU]) = \pi([T, -TU]) + \pi([U, -TU]), \quad (5.6)$$

$$\pi([T, -TU]) = \pi([T, U]), \quad (5.7)$$

$$\pi([U, -TU]) = \pi([U, T]). \quad (5.8)$$

Proof of Claim 5.6. Indeed, composing the first correspondence of Corollary 5.4 with the map in **MCor**

$$\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)} \rightarrow \overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)} \otimes \overline{\square}_V^{(1)} \quad (5.9)$$

given by $V \rightarrow -TU$ which is admissible by [Sai20, Claim 1.21], we get

$$[TU, -TU] + [1, -TU] - [T, -TU] - [U, -TU] = 0$$

in $\mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)}))$. Then (5.6) follows from the equality

$$\pi([1, -TU]) = 0 \in \mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_{T, \mathrm{red}}^{(1)} \otimes \overline{\square}_{U, \mathrm{red}}^{(1)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})).$$

Indeed, we have

$$\begin{aligned} & [1, -TU] \circ ((\mathrm{id} - e) \otimes (\mathrm{id} - e)) \\ &= [1, -TU] - [1, -TU] \circ (\mathrm{id} \otimes e) - [1, -TU] \circ (e \otimes \mathrm{id}) + [1, -TU] \circ (e \otimes e) \\ &= [1, -TU] - [1, -T] - [1, -U] + [1, -1] \stackrel{(*)}{=} 0 \end{aligned}$$

in $\mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_T^{(1)} \otimes \overline{\square}_U^{(1)}, \overline{\square}^{(1)} \otimes \overline{\square}^{(1)})$, where the equality $(*)$ follows from Corollary 5.4. Then (5.7) and (5.8) follow from Corollary 5.4 by an analogous argument considering the maps (5.9) given by $V \mapsto T$, $T \mapsto -T$ and $V \mapsto U$, $U \mapsto -U$, respectively, and noticing that

$$\begin{aligned} & [T, -T] \circ ((\mathrm{id} - e) \otimes (\mathrm{id} - e)) \\ &= [T, -T] - [T, -T] \circ (\mathrm{id} \otimes e) - [T, -T] \circ (e \otimes \mathrm{id}) + [T, -T] \circ (e \otimes e) \\ &= [T, -T] - [T, -T] - [1, -1] + [1, -1] = 0, \end{aligned}$$

and similarly for $[U, -U]$. This completes the proof of the claim. \square

By Claim 5.6, equation (5.5) implies

$$\pi[T, U] + \pi[U, T] = 0 \quad \text{in } \mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_{T, \mathrm{red}}^{(2)} \otimes \overline{\square}_{U, \mathrm{red}}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})). \quad (5.10)$$

Putting (5.3) and (5.10) together, we conclude that

$$\pi[T, U] - \pi[U^{-1}, T] = 0 \quad \text{in } \mathrm{Hom}_{\mathbf{MPST}}(\overline{\square}_{T, \mathrm{red}}^{(2)} \otimes \overline{\square}_{U, \mathrm{red}}^{(2)}, h_0^{\overline{\square}}(\overline{\square}^{(1)} \otimes \overline{\square}^{(1)})).$$

This completes the proof of Proposition 5.5. \square

Take $\mathcal{X}, \mathcal{Y} \in \mathbf{MCor}$ and $\varphi \in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y})$. It induces

$$\varphi_{\overline{\square}} \in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}).$$

Let $\varphi^* \in \text{Hom}_{\underline{\mathbf{MPST}}}(\mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)})$ be obtained from φ by the obvious permutation. It induces $\varphi_{\overline{\square}}^* \in \text{Hom}_{\underline{\mathbf{MPST}}}(\mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)})$. We then put

$$\begin{aligned} \varphi \otimes \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} &\in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)}), \\ \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \varphi^* &\in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)}), \end{aligned}$$

which induce

$$\begin{aligned} \varphi_{\overline{\square}} \otimes \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} &\in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}), \\ \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \varphi_{\overline{\square}}^* &\in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}). \end{aligned}$$

For $\mathcal{M} \in \mathbf{MCor}$, let $\sigma_{\mathcal{M}}: \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{M} \otimes \overline{\square}_{\text{red}}^{(1)} \rightarrow \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{M} \otimes \overline{\square}_{\text{red}}^{(1)}$ be the permutation of the two copies of $\overline{\square}_{\text{red}}^{(1)}$. We have

$$\varphi \otimes \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} = (\sigma_{\mathcal{Y}}) \circ (\text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \varphi^*) \circ (\sigma_{\mathcal{X}}).$$

Let T be the standard coordinate on \mathbf{A}^1 , and let

$$\iota: \overline{\square}_{\text{red}}^{(1)} \rightarrow \overline{\square}_{\text{red}}^{(1)} \tag{5.11}$$

be the map given by $T \rightarrow T^{-1}$. For all $\mathcal{M} \in \mathbf{MCor}$, let

$$\sigma'_{\mathcal{M}} = \sigma_{\mathcal{M}} - \text{Id}_{\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{M}} \otimes \iota: \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{M} \otimes \overline{\square}_{\text{red}}^{(1)} \rightarrow \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{M} \otimes \overline{\square}_{\text{red}}^{(1)}.$$

We can write

$$\varphi \otimes \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} = \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \varphi^* + (\sigma'_{\mathcal{Y}}) \circ p + q \circ (\sigma'_{\mathcal{X}})$$

for some $p, q \in \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)})$. Put $\Gamma_{\mathcal{X}} = \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{X} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}$ and $\Gamma_{\mathcal{Y}} = \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}$. Hence we can write

$$\varphi_{\overline{\square}} \otimes \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} = \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \varphi_{\overline{\square}}^* + \sigma'_{\overline{\square}, \mathcal{Y}} \circ p + q_{\overline{\square}} \circ \sigma'_{\overline{\square}, \mathcal{X}}, \tag{5.12}$$

where

$$\sigma'_{\overline{\square}, \mathcal{Y}}: \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)} \rightarrow \Gamma_{\mathcal{Y}}, \quad \sigma'_{\overline{\square}, \mathcal{X}}: \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)} \rightarrow \Gamma_{\mathcal{X}}, \quad q_{\overline{\square}}: \Gamma_{\mathcal{X}} \rightarrow \Gamma_{\mathcal{Y}}$$

are induced by $\sigma'_{\mathcal{Y}}$, $\sigma'_{\mathcal{X}}$ and q , respectively. For an integer $n > 0$, let $\mathcal{X}^{(n)} := (X, nD)$ if $\mathcal{X} = (X, D)$. Then we consider the map

$$\text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \Gamma_{\mathcal{Y}}) \xrightarrow{\beta_n^*} \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{\text{red}}^{(n)}, \Gamma_{\mathcal{Y}})$$

induced by the natural map $\beta_n: \overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{\text{red}}^{(n)} \rightarrow \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}$.

CLAIM 5.7. *There is an $N \geq 2$ such that for all $n \geq N$, the maps $\sigma'_{\overline{\square}, \mathcal{Y}} \circ p$ and $q_{\overline{\square}} \circ \sigma'_{\overline{\square}, \mathcal{X}}$ lie in the kernel of*

$$\text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \Gamma_{\mathcal{Y}}) \xrightarrow{\beta_n^*} \text{Hom}_{\underline{\mathbf{MPST}}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{\text{red}}^{(n)}, \Gamma_{\mathcal{Y}}).$$

Proof. By Proposition 5.5, the composite map

$$\overline{\square}_{\text{red}}^{(2)} \otimes \overline{\square}_{\text{red}}^{(2)} \xrightarrow{\beta_2} \overline{\square}_{\text{red}}^{(1)} \otimes \overline{\square}_{\text{red}}^{(1)} \xrightarrow{\sigma'} \overline{\square}_{\text{red}}^{(1)} \otimes \overline{\square}_{\text{red}}^{(1)} \rightarrow h_0^{\overline{\square}}(\overline{\square}_{\text{red}}^{(1)}) \otimes_{\mathbf{CI}} h_0^{\overline{\square}}(\overline{\square}_{\text{red}}^{(1)})$$

is zero, where $\sigma' = \sigma - \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \iota$ with σ the permutation of the two copies of $\overline{\square}_{\text{red}}^{(1)}$ and ι from (5.11). This immediately implies the claim for $q_{\overline{\square}} \circ \sigma'_{\overline{\square}, \mathcal{X}}$. We now show the claim for $\sigma'_{\overline{\square}, \mathcal{Y}} \circ p$. Choose an integer N such that for all $n \geq N$, there is map

$$p^{(n)} \in \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{\text{red}}^{(n)}, \overline{\square}_{\text{red}}^{(2)} \otimes \mathcal{Y}^{(2)} \otimes \overline{\square}_{\text{red}}^{(2)})$$

induced by p . For $\mathcal{M}, \mathcal{N} \in \mathbf{MCor}$, write

$$\begin{aligned} \Lambda_{\mathcal{M}, \mathcal{N}} &= \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{M} \otimes \overline{\square}_{\text{red}}^{(1)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{N} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}), \\ \Lambda_{\mathcal{M}, \mathcal{N}}^{(n)} &= \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{M}^{(n)} \otimes \overline{\square}_{\text{red}}^{(n)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{N} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}). \end{aligned}$$

Then for $n \geq N$, we have a commutative diagram

$$\begin{array}{ccc} \Lambda_{\mathcal{Y}, \mathcal{Y}} & \xrightarrow{p^*} & \Lambda_{\mathcal{X}, \mathcal{Y}} \\ \downarrow \beta_2^* & & \downarrow \beta_n^* \\ \Lambda_{\mathcal{Y}, \mathcal{Y}}^{(2)} & \xrightarrow{(p^{(n)})^*} & \Lambda_{\mathcal{X}, \mathcal{Y}}^{(n)}. \end{array} \quad (5.13)$$

The claim for $\sigma'_{\overline{\square}, \mathcal{Y}} \circ p$ follows from this. \square

We now complete the proof of Proposition 5.1. We consider the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)}) & \xrightarrow{\rho_1} & \text{Hom}_{\mathbf{MPST}}(\mathcal{X} \otimes \overline{\square}_{\text{red}}^{(1)}, \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}) \\ \downarrow \beta_n^* & & \downarrow \beta_n^* \\ \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)} \otimes \overline{\square}_{\text{red}}^{(n)}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)}) & \xrightarrow{\rho_n} & \text{Hom}_{\mathbf{MPST}}(\mathcal{X}^{(n)} \otimes \overline{\square}_{\text{red}}^{(n)}, \mathcal{Y} \otimes_{\mathbf{CI}} \overline{\square}_{\text{red}}^{(1)}), \end{array}$$

where the horizontal maps come from (4.19), replacing \mathcal{Y} by $\mathcal{Y} \otimes \overline{\square}_{\text{red}}^{(1)}$. By Lemma 4.9(3) and (2), we have $\rho_1(\varphi \otimes \text{id}_{\overline{\square}_{\text{red}}^{(1)}}) = \rho(\varphi) \otimes \text{Id}_{\overline{\square}_{\text{red}}^{(1)}}$ and $\rho_1(\text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \varphi^*) = \varphi_{\overline{\square}}^*$, where

$$\rho: \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{Y}) \rightarrow \text{Hom}_{\mathbf{MPST}}(\mathcal{X}, h_0^{\overline{\square}}(\mathcal{Y})) \quad (5.14)$$

is the map from (4.19). In view of the diagram, (5.12) and Claim 5.7 imply that there is an $n \gg 0$ such that $\beta_n^*(\varphi_{\overline{\square}}^* - \rho(\varphi_{\overline{\square}}) \otimes \text{Id}_{\overline{\square}_{\text{red}}^{(1)}}) = 0$, so that

$$\beta_n^*(\varphi_{\overline{\square}} - \text{Id}_{\overline{\square}_{\text{red}}^{(1)}} \otimes \rho(\varphi_{\overline{\square}})) = 0 \in \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}). \quad (5.15)$$

Consider the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}) & \longrightarrow & \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(1)} \otimes \mathcal{X}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} \mathcal{Y}) \\ \downarrow \beta_n^* & & \downarrow \beta_n^* \\ \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y}) & \longrightarrow & \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\text{red}}^{(n)} \otimes \mathcal{X}^{(n)}, \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} \mathcal{Y}). \end{array}$$

The two horizontal maps are surjective since representable presheaves are projective objects of \mathbf{MPST} and $\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}} \mathcal{Y} \rightarrow \overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} \mathcal{Y}$ is surjective. The map β_n^* on the right-hand side is injective

since $\overline{\square}_{\text{red}}^{(1)} \otimes_{\mathbf{CI}}^{\text{sp}} \mathcal{Y}$ is semipure. Hence Proposition 5.1 follows from (5.15).

6. Implications on reciprocity sheaves

Let $\mathbf{RSC}_{\text{Nis}}$ be the category of reciprocity sheaves (see § 2.17). Recall that for simplicity, for all $F \in \mathbf{RSC}_{\text{Nis}}$, we write (cf. § 2.18)

$$\tilde{F} := \underline{\omega}^{\mathbf{CI}} F \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}.$$

By [RSY22], there is a *lax* monoidal structure on $\mathbf{RSC}_{\text{Nis}}$ given by (cf. Proposition 4.1)

$$(F, G)_{\mathbf{RSC}_{\text{Nis}}} := \underline{\omega}_! (\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} \tilde{G}).$$

Following [RSY22, § 5.21], we define

$$F\langle 0 \rangle := F, \quad F\langle n \rangle := (F\langle n-1 \rangle, \mathbf{G}_m)_{\mathbf{RSC}_{\text{Nis}}} \quad \text{for } n \geq 1. \quad (6.1)$$

By Corollary 3.2(1), we have (cf. (4.3))

$$F\langle n \rangle \cong \underline{\omega}_! (\widetilde{F\langle n-1 \rangle(1)}). \quad (6.2)$$

By the recursiveness of the definition, we have

$$(F\langle n \rangle)\langle m \rangle \cong F\langle n+m \rangle. \quad (6.3)$$

By [RSY22, Proposition 5.6 and Corollary 5.22], we have isomorphisms

$$\underline{\omega}_! ((\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} n}) \cong \mathbb{Z}\langle n \rangle \cong \mathcal{K}_n^M, \quad \mathbf{G}_a\langle n \rangle \cong \Omega^n \quad \text{if } \text{ch}(k) = 0, \quad (6.4)$$

where the second isomorphism is defined as follows: for an affine $X = \text{Spec } A \in \mathbf{Sm}$, the composite map

$$\mathbf{G}_a(A) \otimes_{\mathbb{Z}} \mathbf{G}_m(A)^{\otimes_{\mathbb{Z}} n} \rightarrow (\mathbf{G}_a \otimes_{\mathbf{NST}} \mathbf{G}_m^{\otimes_{\mathbf{NST}} n})(A) \rightarrow \mathbf{G}_a\langle n \rangle(A) \xrightarrow{(6.4)} \Omega_A^n \quad (6.5)$$

sends $a \otimes f_1 \otimes \cdots \otimes f_n$ with $a \in A$ and $f_i \in A^\times$ to $a \text{dlog } f_1 \wedge \cdots \wedge \text{dlog } f_n$.

By [RSY22, § 5.21(4)], there is a natural surjective map for $F \in \mathbf{RSC}_{\text{Nis}}$

$$F \otimes_{\mathbf{NST}} \mathcal{K}_n^M \rightarrow F\langle n \rangle. \quad (6.6)$$

LEMMA 6.1. *The map (6.6) factors through a natural surjective map*

$$\underline{\omega}_! (\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} (\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} n}) \rightarrow F\langle n \rangle. \quad (6.7)$$

Proof. By [RSY22, 5.21(1)], there is a natural surjective map

$$\underline{\omega}_! a_{\text{Nis}} h_0^{\square} (\tilde{F} \otimes_{\mathbf{MPST}} (\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{MPST}} n}) \rightarrow F\langle n \rangle. \quad (6.8)$$

By Lemma 2.14(2) and (3), we have a natural isomorphism

$$\underline{\omega}_! a_{\text{Nis}} h_0^{\square} (\tilde{F} \otimes_{\mathbf{MPST}} (\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{MPST}} n}) \simeq \underline{\omega}_! (\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} (\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} n}).$$

Hence (6.8) induces (6.7). We have a surjective map

$$\begin{aligned} F \otimes_{\mathbf{PST}} \mathcal{K}_n^M &\stackrel{(6.4)}{\simeq} \underline{\omega}_! \tilde{F} \otimes_{\mathbf{PST}} \underline{\omega}_! ((\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} n}) \\ &\simeq \underline{\omega}_! (\tilde{F} \otimes_{\mathbf{MPST}} ((\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} n})) \rightarrow \underline{\omega}_! (\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} (\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} n}), \end{aligned}$$

where the second isomorphism comes from the monoidality of $\underline{\omega}_!$ (cf. § 2.19). By the adjunction from (2.1), this induces a surjective map

$$F \otimes_{\mathbf{NST}} \mathcal{K}_n^M = a_{\text{Nis}}^V (F \otimes_{\mathbf{PST}} \mathcal{K}_n^M) \rightarrow \underline{\omega}_! (\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} (\underline{\omega}^* \mathbf{G}_m)^{\otimes_{\mathbf{CI}}^{\text{Nis, sp}} n}). \quad (6.9)$$

By the construction of (6.8), it is straightforward to check that (6.6) is the composite (6.7) and (6.9). This completes the proof of the lemma. \square

We have a natural map in $X \in \mathbf{Sm}$:

$$\begin{aligned} F(X) = \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) &\xrightarrow{-\otimes \mathrm{id}_{\mathcal{K}_n^M}} \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X) \otimes_{\mathbf{NST}} \mathcal{K}_n^M, F \otimes_{\mathbf{NST}} \mathcal{K}_n^M) \\ &\longrightarrow \mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X) \otimes_{\mathbf{NST}} \mathcal{K}_n^M, F\langle n \rangle), \end{aligned} \quad (6.10)$$

where the last map is induced by (6.6). Thus we get a map

$$\lambda_F^n: F \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle). \quad (6.11)$$

THEOREM 6.2. *For $F \in \mathbf{RSC}_{\mathrm{Nis}}$, the map λ_F^n is an isomorphism.*

The proof will be given later. First, we prove the following.

PROPOSITION 6.3. *The map λ_F^n is an isomorphism for $n = 1$.*

Proof. Note that $\mathcal{K}_1^M = \mathbf{G}_m$ and that for $F_1, G_1, F_2, G_2 \in \underline{\mathbf{MPST}}$ and maps $f: F_1 \rightarrow F_2$ and $g: G_1 \rightarrow G_2$, the diagram

$$\begin{array}{ccc} \omega_! F_1 \otimes_{\mathbf{PST}} \omega_! G_1 & \xrightarrow{\omega_! f \otimes \omega_! g} & \omega_! F_2 \otimes_{\mathbf{PST}} \omega_! G_2 \\ \downarrow \simeq & & \downarrow \simeq \\ \omega_!(F_1 \otimes_{\underline{\mathbf{MPST}}} G_1) & \xrightarrow{\omega_!(f \otimes g)} & \omega_!(F_2 \otimes_{\underline{\mathbf{MPST}}} G_2) \end{array}$$

commutes, where the vertical isomorphisms follow from the monoidality of $\omega_!$. Thus, by Lemma 6.1, (6.10) with $n = 1$ coincides with the composite map

$$\begin{aligned} F(X) &= \omega_! \tilde{F}(X) \xrightarrow{\omega_!(-\otimes \mathrm{id}_{\omega^* \mathbf{G}_m})(X)} \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \omega^* \mathbf{G}_m)(X) \\ &\simeq \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\mathrm{tr}}(X), \emptyset), \tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \omega^* \mathbf{G}_m) \\ &\stackrel{(*1)}{\simeq} \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \omega^{\mathbf{CI}} \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\mathrm{tr}}(X), \emptyset), \tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \omega^* \mathbf{G}_m) \\ &\stackrel{(*2)}{\simeq} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\mathrm{tr}}(X), \emptyset), \tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \omega^* \mathbf{G}_m) \\ &\stackrel{(*3)}{\simeq} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\mathrm{Hom}}(\mathbb{Z}_{\mathrm{tr}}(X), \omega_!(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \omega^* \mathbf{G}_m))) \\ &\stackrel{(*4)}{\simeq} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, F\langle 1 \rangle)(X). \end{aligned} \quad (6.12)$$

Here (*1) is induced by the injective unit map $G \rightarrow \omega^{\mathbf{CI}} \omega_! G$ ($G \in \mathbf{CI}_{\mathrm{Nis}}^{\tau, \mathrm{sp}}$) for the adjunction (2.13) and is an isomorphism by Corollary 3.9 and the fact that $\underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\mathrm{tr}}(X), \emptyset), \tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \omega^* \mathbf{G}_m \in \mathbf{CI}_{\mathrm{Nis}}^{\tau, \mathrm{sp}}$; (*2) is given by the fully faithfulness of $\omega^{\mathbf{CI}}$ and the equality $\omega^{\mathbf{CI}} \mathbf{G}_m = \omega^* \mathbf{G}_m$ by [KSY22, Lemma 2.3.1]; (*3) follows from Lemma 2.3; and (*4) holds by the definition (6.1).

This gives a commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{\lambda_F^1} & \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, F\langle 1 \rangle) \\ \downarrow \simeq & & \uparrow \simeq \\ \omega_! \tilde{F} & \xrightarrow{\omega_! \iota_F^1} & \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \square_{\mathrm{red}}^{(1)}), \end{array} \quad (6.13)$$

where $\iota_{\tilde{F}}^1 = (- \otimes \text{id}_{\underline{\omega}^* \mathbf{G}_m})$ is an isomorphism from Corollary 4.6 (using Corollary 3.2). This proves the proposition. \square

For $F, G \in \mathbf{RSC}_{\text{Nis}}$, let

$$\iota_{F,G}: \text{Hom}_{\mathbf{PST}}(F, G) \rightarrow \text{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) \quad (6.14)$$

be the composite map

$$\begin{aligned} \text{Hom}_{\mathbf{PST}}(F, G) &\xrightarrow{\underline{\omega}^{\mathbf{CI}}} \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \tilde{G}) \\ &\xrightarrow{- \otimes_{\mathbf{CI}}^{\text{Nis}} \underline{\omega}^* \mathbf{G}_m} \text{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis}} \underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis}} \underline{\omega}^* \mathbf{G}_m) \xrightarrow{\underline{\omega}_!} \text{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle). \end{aligned}$$

THEOREM 6.4. *For $F, G \in \mathbf{RSC}_{\text{Nis}}$, the map $\iota_{F,G}$ is an isomorphism.*

Proof. We have isomorphisms (cf. § 2.18)

$$\begin{aligned} \text{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) &= \text{Hom}_{\mathbf{PST}}(\underline{\omega}_!(\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \underline{\omega}^* \mathbf{G}_m), \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)})) \\ &\cong \text{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)})) \\ &\cong \text{Hom}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{MPST}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)})) \\ &\cong \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)}))), \quad (6.15) \end{aligned}$$

where the first (respectively, second) isomorphism follows from (2.12) (respectively, the fact that $\underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \tau_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)}) \in \mathbf{CI}_{\text{Nis}}^{\tau, \text{sp}}$). Note that for $H \in \mathbf{CI}^{\tau, \text{sp}}$, the natural map $H \rightarrow \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! H$ is injective.

Hence we get injective maps

$$\begin{aligned} &\text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)})) \\ &\hookrightarrow \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)}))) \\ &\hookrightarrow \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)}))) \\ &\stackrel{(*1)}{\simeq} \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\omega}^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)}))) \\ &\stackrel{(*2)}{\simeq} \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\omega}^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, G\langle 1 \rangle)), \quad (6.16) \end{aligned}$$

where the isomorphism (*1) comes from Proposition 3.10 and $\underline{\omega}_! \underline{\omega}^{\mathbf{CI}} \simeq \text{id}$ (cf. § 2.18) and (*2) follows from (6.2). These maps fit into a commutative diagram

$$\begin{array}{ccc} & & \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \tilde{G}) \\ & \swarrow \alpha \simeq & \uparrow \underline{\omega}^{\mathbf{CI}} \\ & \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)})) & \text{Hom}_{\mathbf{PST}}(F, G) \\ & \downarrow \hookrightarrow & \downarrow \iota_{F,G} \\ \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_!(\tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis,sp}} \overline{\square}_{\text{red}}^{(1)}))) & \xleftarrow{\simeq (6.15)} & \text{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) \simeq \underline{\omega}^{\mathbf{CI}} \\ & \downarrow \hookrightarrow & \uparrow \simeq \\ \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \underline{\omega}^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, G\langle 1 \rangle)) & \xleftarrow{\simeq \beta} & \text{Hom}_{\mathbf{MPST}}(\tilde{F}, \tilde{G}). \end{array}$$

The two right vertical isomorphisms follow from the full faithfulness of $\omega^{\mathbf{CI}}$. The isomorphism α (respectively, β) comes from ι_G^1 from Corollaries 4.6 and 3.2 (respectively, λ_G^1 from Proposition 6.3). The squares are commutative by (6.13), noting that the left vertical maps are viewed as inclusions under the identifications

$$\begin{aligned} \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \overline{\square}_{\mathrm{red}}^{(1)}) &\simeq \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, G\langle 1 \rangle) \\ &\simeq \omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\omega^* \mathbf{G}_m, \omega^{\mathbf{CI}} \omega_!(\tilde{G} \otimes_{\mathbf{CI}}^{\mathrm{Nis,sp}} \overline{\square}_{\mathrm{red}}^{(1)})) \end{aligned}$$

coming from Proposition 3.10. This proves that the map $\iota_{F,G}$ is an isomorphism, as desired. \square

COROLLARY 6.5. *For $F, G \in \mathbf{RSC}_{\mathrm{Nis}}$, there exists a natural injective map in \mathbf{NST} for internal hom*

$$\underline{\mathrm{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) \hookrightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(F, G), \quad (6.17)$$

which coincides with the inverse of (6.14) on the k -valued points.

Proof. The surjective map $F \otimes_{\mathbf{NST}} \mathbf{G}_m \rightarrow F\langle 1 \rangle$ in \mathbf{NST} from (6.6) induces an injective map

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle) &\hookrightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(F \otimes_{\mathbf{NST}} \mathbf{G}_m, G\langle 1 \rangle) \\ &\simeq \underline{\mathrm{Hom}}_{\mathbf{PST}}(F, \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathbf{G}_m, G\langle 1 \rangle)), \end{aligned}$$

and the latter is isomorphic to $\underline{\mathrm{Hom}}_{\mathbf{PST}}(F, G)$ by Proposition 6.3. This completes the proof. \square

Proof of Theorem 6.2. Consider the map induced by (6.6):

$$q: \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F \otimes_{\mathbf{NST}} \mathcal{K}_n^M) \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle).$$

The map (6.11) is then the composition of q and the map

$$F \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F \otimes_{\mathbf{NST}} \mathcal{K}_n^M), \quad s \mapsto s \otimes \mathrm{id}_{\mathcal{K}_n^M}. \quad (6.18)$$

On the other hand, we have isomorphisms $\mathcal{K}_{i-1}^M\langle 1 \rangle \cong \mathcal{K}_i^M$ for all $i \geq 1$ by (6.4). Hence the map (6.17) for $F = \mathcal{K}_{i-1}^M$ gives an injective map

$$\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_i^M, F\langle i \rangle) \rightarrow \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_{i-1}^M, F\langle i-1 \rangle). \quad (6.19)$$

Composing (6.19) for all $i \leq n$, we get an injective map

$$\underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle) \hookrightarrow F, \quad (6.20)$$

which by definition sends $q(s \otimes \mathrm{id}_{\mathcal{K}_n^M})$ to s for a section s of F . Hence the composition

$$F \xrightarrow{(6.11)} \underline{\mathrm{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, F\langle n \rangle) \xrightarrow{(6.20)} F$$

is the identity, so (6.11) is an isomorphism. This completes the proof of Theorem 6.2. \square

Let $G \in \mathbf{RSC}_{\mathrm{Nis}}$ and $X \in \mathbf{Sm}$. By Lemma 2.3, we have a natural isomorphism

$$\omega_! \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \omega^{\mathbf{CI}} G) \simeq \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G).$$

Hence the unit map $\mathrm{id} \rightarrow \omega^{\mathbf{CI}} \omega_!$ from (2.13) induces a natural map

$$\underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}((X, \emptyset), \omega^{\mathbf{CI}} G) \rightarrow \omega^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G). \quad (6.21)$$

It is injective by the semipurity of $\underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\mathbb{Z}_{\mathrm{tr}}(X, \emptyset), \omega^{\mathbf{CI}} G)$ and becomes an isomorphism

after taking $\underline{\omega}_!$. Moreover, the following diagram is commutative:

$$\begin{array}{ccc}
 \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\omega}^{\mathbf{CI}}G) & \xrightarrow{(6.21)} & \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G) \\
 \downarrow \hookrightarrow & & \downarrow \hookrightarrow \\
 \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\omega}^*G) & \xrightarrow{\cong} & \underline{\omega}^* \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G),
 \end{array} \tag{6.22}$$

where the isomorphism comes from Lemma 2.2.

For $G \in \mathbf{RSC}_{\mathrm{Nis}}$ and $X \in \mathbf{Sm}$, we define the following condition:

(♣) $_X$ The map (6.21) is an isomorphism.

THEOREM 6.6. *Let $F, G \in \mathbf{RSC}_{\mathrm{Nis}}$. Assume one of the following:*

- (1) *The reciprocity sheaf G satisfies (♣) $_X$ for any $X \in \mathbf{Sm}$.*
- (2) *The reciprocity sheaf G satisfies (♣) $_{\mathrm{Spec}(K)}$ for any function field K over k , and F is the quotient of a direct sum of representable objects.*

Then (6.17) is an isomorphism.

Proof. Assume condition (1). Letting $\tilde{G} = \underline{\omega}^{\mathbf{CI}}G$, we have isomorphisms for $X \in \mathbf{Sm}$

$$\begin{aligned}
 \underline{\mathrm{Hom}}_{\mathbf{PST}}(F, G)(X) &= \mathrm{Hom}_{\mathbf{PST}}(F, \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G)) \\
 &\stackrel{(*)1}{\cong} \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\omega}^{\mathbf{CI}} \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G)) \stackrel{(*)2}{\cong} \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G})), \tag{6.23}
 \end{aligned}$$

where the isomorphism (*1) (respectively, (*2)) comes from the full faithfulness of $\underline{\omega}^{\mathbf{CI}}$ (respectively, (♣) $_X$). Moreover, we have isomorphisms

$$\begin{aligned}
 \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}) &\stackrel{(*)3}{\cong} \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \tilde{G}(1))) \\
 &\cong \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))), \tag{6.24}
 \end{aligned}$$

where the isomorphism (*3) comes from Corollaries 4.6 and 3.2. We also have isomorphisms

$$\begin{aligned}
 \underline{\mathrm{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle)(X) &= \mathrm{Hom}_{\mathbf{PST}}(F\langle 1 \rangle, \underline{\mathrm{Hom}}_{\mathbf{PST}}(X, G\langle 1 \rangle)) \\
 &\stackrel{(*)4}{\cong} \mathrm{Hom}_{\mathbf{PST}}(\underline{\omega}_!(\tilde{F} \otimes_{\mathbf{CI}}^{\mathrm{Nis}} \underline{\omega}^* \mathbf{G}_m), \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))) \tag{6.25} \\
 &\stackrel{(*)5}{\cong} \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F} \otimes_{\mathbf{MPST}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))) \\
 &\cong \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1)))) ,
 \end{aligned}$$

where (*4) (respectively, (*5)) comes from Lemma 2.3 (respectively, the adjunction (2.12)). These maps fit into a commutative diagram

$$\begin{array}{ccc}
 \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G})) & & \\
 \downarrow \simeq (6.24) & \swarrow \simeq (6.23) & \\
 \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1)))) & & \underline{\mathrm{Hom}}_{\mathbf{PST}}(F, G)(X) \\
 \downarrow \hookrightarrow (\dagger) & & \uparrow \hookrightarrow (6.17) \\
 \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\tilde{F}, \underline{\mathrm{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \underline{\omega}_! \underline{\mathrm{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1)))) & \xleftarrow{\simeq (6.25)} & \underline{\mathrm{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle)(X),
 \end{array}$$

where the injective map (\dagger) comes from the counit map $\text{id} \rightarrow \underline{\omega}^{\mathbf{CI}} \omega_!$ from the adjunction (2.12). We see that the diagram commutes as follows: The map (6.24) is induced by the map

$$\begin{aligned} \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}) &\rightarrow \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G}(1))) \\ &\simeq \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset) \otimes \underline{\omega}^* \mathbf{G}_m, \tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} \underline{\omega}^* \mathbf{G}_m) \end{aligned}$$

given by $f \mapsto f \otimes \text{id}_{\underline{\omega}^* \mathbf{G}_m}$. The map (6.17) is induced by the surjection $F \otimes_{\mathbf{NST}} \mathbf{G}_m \rightarrow F\langle 1 \rangle$ from (6.6) and the isomorphism $\underline{\text{Hom}}_{\mathbf{PST}}(F \otimes \mathbf{G}_m, G\langle 1 \rangle) \xrightarrow{\sim} \underline{\text{Hom}}_{\mathbf{PST}}(F, G)$ inverse to (6.11) given by $f \otimes \text{id}_{\mathbf{G}_m} \mapsto f$. The maps (6.23) and (\dagger) are inclusions under the identifications

$$\begin{aligned} \omega_! \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\text{Hom}}_{\mathbf{MPST}}(X, \emptyset), \tilde{G}(1)) \\ \simeq \underline{\text{Hom}}_{\mathbf{PST}}(\mathbf{G}_m \otimes X, G\langle 1 \rangle) \\ \simeq \omega_! \underline{\text{Hom}}_{\mathbf{MPST}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \omega_! \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset), \tilde{G} \otimes_{\mathbf{CI}}^{\text{Nis, sp}} \square_{\text{red}}^{(1)})) \end{aligned}$$

coming from Lemma 2.3 and Proposition 3.10. This proves that (6.17) is an isomorphism.

Next assume condition (2). In view of Lemma 2.5, we have that $\underline{\text{Hom}}_{\mathbf{PST}}(F, G)$ and $\underline{\text{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle)$ are in $\mathbf{RSC}_{\text{Nis}}$. Hence, by Lemma 2.4, it is enough to prove that (6.17) induces an isomorphism $\underline{\text{Hom}}_{\mathbf{PST}}(F\langle 1 \rangle, G\langle 1 \rangle)(K) \cong \underline{\text{Hom}}_{\mathbf{PST}}(F, G)(K)$ for any function field K over k . This follows from the same computations as above. \square

LEMMA 6.7. *Any $F \in \mathbf{HI}_{\text{Nis}}$ satisfies $(\clubsuit)_X$ for all $X \in \mathbf{Sm}$.*

Proof. We have

$$\begin{aligned} \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\omega}^{\mathbf{CI}} F) &= \underline{\text{Hom}}_{\mathbf{MPST}}((X, \emptyset), \underline{\omega}^* F) \\ &\stackrel{(*)}{\cong} \underline{\omega}^* \underline{\text{Hom}}_{\mathbf{PST}}(X, F) \stackrel{(**)}{\cong} \underline{\omega}^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(X, F), \end{aligned}$$

where the isomorphism $(*)$ follows from Lemma 2.2 and $(**)$ from the fact that $\underline{\text{Hom}}_{\mathbf{PST}}(X, F) \in \mathbf{HI}$, so that $\underline{\omega}^* \underline{\text{Hom}}_{\mathbf{PST}}(X, F) \in \mathbf{CI}^\tau$ by [KSY22, Lemma 2.3.1]. This completes the proof. \square

LEMMA 6.8. *If $\text{ch}(k) = 0$, then Ω^i satisfies $(\clubsuit)_X$ for all $X \in \mathbf{Sm}$.*

Proof. Put $\Gamma = \underline{\text{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), \Omega^i)$ and

$$G = \underline{\text{Hom}}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(X, \emptyset), \underline{\omega}^{\mathbf{CI}} \Omega^i), \quad G^* = \underline{\omega}^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), \Omega^i).$$

Note that $\Gamma \in \mathbf{RSC}_{\text{Nis}}$ by Lemma 2.5. By [RS22, Corollary 6.8], for $\mathcal{Y} = (Y, D) \in \mathbf{MCor}$, where $Y \in \mathbf{Sm}$ and D_{red} is a simple normal crossing divisor, we have

$$G(\mathcal{Y}) = \Gamma(Y \times X, \Omega^i(\log D_{\text{red}} \times X)((D - D_{\text{red}}) \times X)). \quad (6.26)$$

Hence the conductor c^G associated with G in the sense of [RS22, Definition 4.14] is given as follows (note that Lemma 2.3 implies $G \in \mathbf{CI}(\Gamma)$ under the notation of loc. cit.): Let Φ be as in [RS22, Definition 4.1]. For $a \in G(L) = H^0(X \otimes_k L, \Omega^i)$ with $L \in \Phi$, put $c_L^G(a) = 0$ if $a \in H^0(X \otimes_k \mathcal{O}_L, \Omega^i)$. Otherwise, put

$$c_L^G(a) = \min \left\{ n \geq 1 \mid a \in H^0 \left(X \otimes_k \mathcal{O}_L, \frac{1}{t^{n-1}} \cdot \Omega_{X \otimes_k \mathcal{O}_L}^i(\log) \right) \right\},$$

where t is a local parameter of \mathcal{O}_L and $\Omega_{X \otimes_k \mathcal{O}_L}^\bullet(\log)$ is the differential graded subalgebra of $\Omega_{X \otimes_k L}^\bullet$ generated by $\Omega_{X \otimes_k \mathcal{O}_L}^\bullet$ and $d \log t$ (cf. [RS22, §§ 6.1 and 6.3]). Moreover, one easily sees that for $\mathcal{Y} = (Y, D) \in \mathbf{MCor}$ as (6.26),

$$G(\mathcal{Y}) = \{ a \in G(Y - D) \mid c_L^G(a) \leq v_L(D) \text{ for any } L \in \Phi \}$$

(see [RS22, Notation 4.2] for $v_L(D)$). Hence by [RS22, Theorem 4.15(4)], it suffices to show $c^{G^*} = c^G$. We know $c^{G^*} \leq c^G$ by loc. cit., so that it suffices to show the following: Let $L \in \Phi$ and $a \in G(L)$. For $r \in \mathbb{Z}_{\geq 0}$, we have

$$c_L^{G^*}(a) \leq r \Rightarrow c_L^G(a) \leq r.$$

We prove this implication by descending induction on r . By [RS22, Corollary 4.44], this is reduced to showing the following: Choose a ring homomorphism $K \hookrightarrow \mathcal{O}_L$ such that $K \rightarrow \mathcal{O}_L \rightarrow \mathcal{O}_L/(t)$ is the identity, and extend it in the canonical way to $\sigma: K(x) \hookrightarrow \mathcal{O}_{L_x}$, where x is a variable and $L_x = \text{Frac}(\mathcal{O}_L[x]_{(t)}^h)$. Assume $c_L^G(a) \leq r + 1$. Then the following implication holds:

$$(a, 1 - xt^r)_{L_x, \sigma} = 0 \in G(K(x)) \Rightarrow c_L^G(a) \leq r, \quad (6.27)$$

where $(-, -)_{L_x, \sigma}$ is the local symbol for $\Gamma^i = \underline{\text{Hom}}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(X), \Omega^i)$ from [RS22, § 4.37]. Since the local symbol is uniquely determined by the properties (LS1)–(LS4) from [RS22, Lemma 4.38], we see that it is given by $(a, 1 - xt^r)_{L_x, \sigma} = \text{Res}_t(a \text{dlog}(1 - xt^r))$, where

$$\text{Res}_t: \Gamma^{i+1}(L_x) = H^0(X \otimes_k L_x, \Omega^{i+1}) \rightarrow \Gamma^i(K(x)) = H^0(X \otimes_k K(x), \Omega^i)$$

is induced by the residue map $\Omega_{L_x}^{i+1} \rightarrow \Omega_{K(x)}^i$, which is defined using the isomorphism $L_x \simeq K(x)((t))$ induced by $\sigma: K(x) \hookrightarrow \mathcal{O}_{L_x}$. To prove the implication (6.27), we may assume after replacing a with $a - b$ for some $b \in \Gamma(L)$ with $c_L^G(b) \leq r$,

$$a = \frac{1}{t^r} \alpha + \beta \frac{dt}{t^{r+1}} \quad \text{for } \alpha \in H^0(X \otimes_k K, \Omega^i), \beta \in H^0(X \otimes_k K, \Omega^{i-1}).$$

Then we compute in $H^0(X \otimes_k K(x), \Omega^i)$: $\text{Res}_t(a \text{dlog}(1 - xt^r)) = -rx\alpha + \beta dx$. This shows (6.27) and completes the proof. \square

7. Internal homs for Ω^n

In this section, we assume $\text{ch}(k) = 0$. Note that a section of $\underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)$ over $X \in \mathbf{Sm}$ is given by a collection of maps $\varphi_Y: H^0(Y, \Omega^n) \rightarrow H^0(X \times Y, \Omega^m)$ for $Y \in \mathbf{Sm}$, which are natural in $Y \in \mathbf{Cor}$. For $(\alpha, \beta) \in H^0(X, \Omega^{m-n}) \oplus H^0(X, \Omega^{m-n-1})$, we define

$$\varphi_{Y, \alpha, \beta}^{n, m}: H^0(Y, \Omega^n) \rightarrow H^0(X \times Y, \Omega^m), \quad \omega \mapsto p_X^* \alpha \wedge p_Y^* \omega + p_X^* \beta \wedge p_Y^* d\omega,$$

where $p_X: X \times Y \rightarrow X$ and $p_Y: X \times Y \rightarrow Y$ are the projections. The naturalness of $\varphi_{Y, \alpha, \beta}^{n, m}$ in $Y \in \mathbf{Cor}$ follows from [CR11]. Thus we get a natural map in \mathbf{NST} :

$$\Omega^{m-n} \oplus \Omega^{m-n-1} \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m), \quad (\alpha, \beta) \mapsto \{\varphi_{Y, \alpha, \beta}^{n, m}\}_{Y \in \mathbf{Sm}}, \quad (7.1)$$

where $\Omega^i = 0$ for $i < 0$ by convention. Taking the sections over $\text{Spec } k$, we get a natural map

$$\Phi^{n, m}: \Omega_k^{m-n} \oplus \Omega_k^{m-n-1} \rightarrow \text{Hom}_{\mathbf{PST}}(\Omega^n, \Omega^m). \quad (7.2)$$

We also consider the composite map in \mathbf{NST}

$$\Omega^{m-n} \xrightarrow{(7.1)} \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m) \xrightarrow{\text{dlog}^*} \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m), \quad (7.3)$$

where the second map is induced by the map $\text{dlog}: \mathcal{K}_n^M \rightarrow \Omega^n$. Taking the sections over $\text{Spec } k$, we get a natural map

$$\Psi^{n, m}: \Omega_k^{m-n} \rightarrow \text{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m). \quad (7.4)$$

The main result of this subsection is the following.

THEOREM 7.1. *The maps (7.1) and (7.3) are isomorphisms.*

First, we prove the following.

PROPOSITION 7.2. *The maps (7.2) and (7.4) are isomorphisms.*

This follows from Lemmas 7.3, 7.4 and 7.5 below, in light of Theorem 6.4. For $i \geq 0$, let us fix the isomorphisms

$$\sigma^i: \Omega^{i-1}\langle 1 \rangle \xrightarrow{\sim} \Omega^i, \quad \zeta^i: \mathcal{K}_{i-1}^M\langle 1 \rangle \xrightarrow{\sim} \mathcal{K}_i^M \quad (7.5)$$

coming from (6.3) and (6.4)

LEMMA 7.3. (1) *The following diagram is commutative:*

$$\begin{array}{ccc} \Omega_k^{m-n} \oplus \Omega_k^{m-n-1} & \xrightarrow{\Phi^{n,m}} & \mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \Omega^m) \\ \downarrow \Phi^{n-1,m-1} & & \uparrow \\ \mathrm{Hom}_{\mathbf{PST}}(\Omega^{n-1}, \Omega^{m-1}) & \xrightarrow{(6.14)} & \mathrm{Hom}_{\mathbf{PST}}(\Omega^{n-1}\langle 1 \rangle, \Omega^{m-1}\langle 1 \rangle), \end{array}$$

where the right vertical map is induced by the isomorphisms σ^m and $(\sigma^n)^{-1}$ from (7.5).

(2) *The following diagram is commutative:*

$$\begin{array}{ccc} \Omega_k^{m-n} & \xrightarrow{\Psi^{n,m}} & \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m) \\ \downarrow \Psi^{n-1,m-1} & & \uparrow \\ \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_{n-1}^M, \Omega^{m-1}) & \xrightarrow{(6.14)} & \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_{n-1}^M\langle 1 \rangle, \Omega^{m-1}\langle 1 \rangle), \end{array}$$

where the right vertical map is induced by the isomorphisms σ^m and $(\zeta^n)^{-1}$ from (7.5).

Proof. By [RSY22, Corollary 5.22], for an affine $X = \mathrm{Spec} A \in \mathbf{Sm}$ and $i \geq 0$, the composite map

$$\theta^i: \Omega_A^{i-1} \otimes_{\mathbb{Z}} A^\times \rightarrow (\Omega^{i-1} \otimes_{\mathbf{NST}} \mathbf{G}_m)(A) \xrightarrow{(6.6)} \Omega^{i-1}\langle 1 \rangle(A) \xrightarrow{\sigma^i} \Omega_A^i$$

sends $\omega \otimes f$ with $\omega \in \Omega_A^{i-1}$ and $f \in A^\times$ to $\omega \wedge \mathrm{dlog} f$. Moreover, for $\varphi \in \mathrm{Hom}_{\mathbf{PST}}(\Omega^{n-1}, \Omega^{m-1})$ and $\varphi' = \sigma^m \circ \varphi\langle 1 \rangle \circ (\sigma^n)^{-1}$, the diagram

$$\begin{array}{ccc} \Omega_A^{n-1} \otimes_{\mathbb{Z}} A^\times & \xrightarrow{\theta^n} & \Omega_A^n \\ \downarrow \varphi \otimes \mathrm{id}_{A^\times} & & \downarrow \varphi' \\ \Omega_A^{m-1} \otimes_{\mathbb{Z}} A^\times & \xrightarrow{\theta^m} & \Omega_A^m \end{array}$$

is commutative. Hence item (1) follows from the equation

$$\alpha \wedge (\omega \wedge \mathrm{dlog} f) + \beta \wedge d(\omega \wedge \mathrm{dlog} f) = (\alpha \wedge \omega + \beta \wedge d\omega) \wedge \mathrm{dlog} f,$$

where $\alpha \in \Omega_k^{m-n}$ and $\beta \in \Omega_k^{m-n-1}$.

Item (2) follows from item (1) and the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{K}_{n-1}^M\langle 1 \rangle & \xrightarrow{\mathrm{dlog}\langle 1 \rangle} & \Omega^{n-1}\langle 1 \rangle \\ \downarrow \zeta^n & & \downarrow \sigma^n \\ \mathcal{K}_n^M & \xrightarrow{\mathrm{dlog}} & \Omega^n, \end{array}$$

which can be verified using (6.5). \square

LEMMA 7.4. *For an integer $n \geq 1$, we have*

$$\mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \mathbf{G}_a) = \mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \mathbf{G}_a) = 0. \quad (7.6)$$

Proof. We have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{PST}}(\Omega^n, \mathbf{G}_a) &\simeq \mathrm{Hom}_{\mathbf{PST}}(\underline{\omega}_! (\widetilde{\Omega}^{n-1} \otimes_{\mathbf{CI}} \underline{\omega}^* \mathbf{G}_m), \mathbf{G}_a) \\ &\simeq \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\widetilde{\Omega}^{n-1} \otimes_{\mathbf{CI}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a) \\ &\simeq \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\widetilde{\Omega}^{n-1} \otimes_{\underline{\mathbf{MPST}}} \underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a) \\ &\simeq \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\widetilde{\Omega}^{n-1}, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a)), \end{aligned}$$

where the first isomorphism is induced by $(\sigma^n)^{-1}$, the inverse of the isomorphism σ^n from (7.5) and the second follows from (2.12). Similarly, we have an isomorphism using $(\zeta^n)^{-1}$ instead of $(\sigma^n)^{-1}$:

$$\mathrm{Hom}_{\mathbf{PST}}(\mathcal{K}_n^M, \mathbf{G}_a) \simeq \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathcal{K}_{n-1}^M, \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a)).$$

We compute

$$\begin{aligned} \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a)(X) &\hookrightarrow \underline{\mathrm{Hom}}_{\underline{\mathbf{MPST}}}(\underline{\omega}^* \mathbf{G}_m, \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a)(K(X)) \\ &\simeq \mathrm{Coker}(\underline{\omega}^{\mathbf{CI}} \mathbf{G}_a(K(X)) \rightarrow \underline{\omega}^{\mathbf{CI}} \mathbf{G}_a(\mathbf{P}_{K(X)}^1, 0 + \infty)) \\ &\simeq \mathrm{Coker}(K(X) \rightarrow H^0(\mathbf{P}_{K(X)}^1, \mathcal{O})) = 0, \end{aligned}$$

where the first map is injective by [Sai20, Corollary 0.3], and the first (respectively, last) isomorphism follows from Corollary 3.2(1) (respectively, [RS22, Corollary 6.8]). This completes the proof of Lemma 7.4. \square

LEMMA 7.5. *The maps (7.2) and (7.4) are isomorphisms for $n = 0$.*

Proof. The assertion for (7.4) is obvious since $\mathcal{K}_n^M = \mathbb{Z}$ for $n = 0$. We prove it for (7.2). We have isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) &\simeq \mathrm{Hom}_{\mathbf{PST}}(a_{\mathrm{Nis}}^V \omega_! h_0^\square(\square_{\mathbf{G}_a}), \Omega^i) \\ &\simeq \mathrm{Hom}_{\underline{\mathbf{MPST}}}(h_0^\square(\square_{\mathbf{G}_a}), \omega^{\mathbf{CI}} \Omega^i) \\ &\simeq \mathrm{Hom}_{\underline{\mathbf{MPST}}}(\square_{\mathbf{G}_a}, \omega^{\mathbf{CI}} \Omega^i) \\ &\simeq \mathrm{Ker}(H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^i(\log \infty)(\infty)) \xrightarrow{i_0^*} \Omega_k^i), \end{aligned} \quad (7.7)$$

where the first (respectively, last) isomorphism follows from (2.17) (respectively, [RS22, Corollary 6.8]). Since $\Omega_{\mathbf{P}^1/k}^i(\log \infty) = 0$ for $i > 1$, $\mathcal{O}_{\mathbf{P}^1}(\log \infty) = \mathcal{O}_{\mathbf{P}^1}$ and $\Omega_{\mathbf{P}^1/k}^1(\log \infty) = \Omega_{\mathbf{P}^1/k}^1(\infty)$, the standard exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1} \otimes_k \Omega_k^1 \rightarrow \Omega_{\mathbf{P}^1}^1(\log \infty) \rightarrow \Omega_{\mathbf{P}^1/k}^1(\log \infty) \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^1}(\infty) \otimes_k \Omega_k^i \rightarrow \Omega_{\mathbf{P}^1}^i(\log \infty)(\infty) \rightarrow \Omega_{\mathbf{P}^1/k}^1(2\infty) \otimes_k \Omega_k^{i-1} \rightarrow 0,$$

where $\Omega_k^{i-1} = 0$ if $i = 0$ by convention. Letting t be the standard coordinate of $\mathbf{A}^1 \subset \mathbf{P}^1$, we have

$$H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(\infty)) = k \cdot 1 \oplus k \cdot t, \quad H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1/k}^1(2\infty)) = k \cdot dt,$$

and dt lifts canonically to a section $dt \in H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^1(\log \infty)(\infty))$. Hence we get an isomorphism

$$H^0(\mathbf{P}^1, \Omega_{\mathbf{P}^1}^i(\log \infty)(\infty)) \simeq (k \cdot 1 \oplus k \cdot t) \otimes_k \Omega_k^i \oplus (k \cdot dt) \otimes_k \Omega_k^{i-1}. \quad (7.8)$$

Thus the last group of (7.7) is isomorphic to $k \cdot t \otimes_k \Omega_k^i \oplus k \cdot dt \otimes_k \Omega_k^{i-1} \simeq \Omega_k^i \oplus \Omega_k^{i-1}$. Hence, from (7.7), we get a natural isomorphism

$$\Omega_k^{i-1} \oplus \Omega_k^i \xrightarrow{\sim} \text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i). \quad (7.9)$$

Next we claim that the map (7.9) coincides with (7.2) for $n = 0$. By Lemma 2.8(2), we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1) & \xrightarrow{\lambda_{\mathbf{G}_a}} & \mathbf{G}_a \\ \downarrow \simeq & & \uparrow (2.17) \\ \omega_! \mathbb{Z}_{\text{tr}}(\mathbf{P}^1, 2\infty) & \longrightarrow & \omega_! h_0^{\square}(\square_{\mathbf{G}_a}), \end{array} \quad (7.10)$$

where $\lambda_{\mathbf{G}_a}$ is given by $t \in \mathbf{G}_a(\mathbf{A}_t^1) = k[t]$. The standard isomorphism

$$\Omega^i(\mathbf{A}_t^1) \simeq (\Omega_k^i \otimes_k k[t]) \oplus (\Omega_k^{i-1} \otimes_k k[t]dt)$$

induces a natural isomorphism

$$\text{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1), \Omega^i) = \Omega^i(\mathbf{A}_t^1) \simeq \Omega_k^i[t] \oplus \Omega_k^{i-1}[t] \wedge dt, \quad (7.11)$$

where $\Omega_k^i[t] = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \Omega_k^i \cdot t^m$ and $\Omega_k^{i-1}[t] \wedge dt = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \Omega_k^{i-1} \wedge t^m dt$. The map $\lambda_{\mathbf{G}_a}$ induces the inclusion

$$\lambda_{\mathbf{G}_a}^* : \text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \hookrightarrow \text{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\text{tr}}(\mathbf{A}_t^1), \Omega^i) = \Omega^i(\mathbf{A}_t^1)$$

such that

$$\lambda_{\mathbf{G}_a}^*(\varphi) = \varphi_{\mathbf{A}_t^1}(t) \quad \text{for } \varphi \in \text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i), \quad (7.12)$$

where $\varphi_{\mathbf{A}_t^1} : \mathbf{G}_a(\mathbf{A}_t^1) = k[t] \rightarrow \Omega^i(\mathbf{A}_t^1)$ is induced by φ . The following claim follows from (7.7), (7.8) and (7.10).

CLAIM 7.6. *The image of $\lambda_{\mathbf{G}_a}^*$ is identified under (7.11) with*

$$\Omega_k^i \cdot t \oplus \Omega_k^{i-1} \wedge dt \subset \Omega_k^i[t] \oplus \Omega_k^{i-1}[t] \wedge dt,$$

and the composite map

$$\Omega_k^i \oplus \Omega_k^{i-1} \xrightarrow{(7.9)} \text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \xrightarrow{\lambda_{\mathbf{G}_a}^*} \Omega_k^i \cdot t \oplus \Omega_k^{i-1} \wedge dt$$

is given by the obvious identifications $\Omega_k^i = \Omega_k^i \cdot t$ and $\Omega_k^{i-1} = \Omega_k^{i-1} \wedge dt$.

Let

$$\text{Hom}_{\mathbf{G}_a}(\mathbf{G}_a, \Omega^i) \subset \text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \quad (7.13)$$

be the subgroup of \mathbf{G}_a -linear morphisms. There is a natural isomorphism

$$\xi : \Omega_k^i \cong \text{Hom}_{\mathbf{G}_a}(\mathbf{G}_a, \Omega^i), \quad \omega \mapsto \{\lambda \mapsto \lambda\omega\} \quad (\lambda \in \mathbf{G}_a).$$

The group (7.13) is a direct summand since we have a splitting given by

$$\text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \rightarrow \text{Hom}_{\mathbf{G}_a}(\mathbf{G}_a, \Omega^i), \quad \varphi \mapsto \{\lambda \mapsto \lambda\varphi(1)\}.$$

The other summand is $\text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i)^0 := \{\varphi \mid \varphi(1) = 0\}$. There is a natural map

$$\xi': \Omega_k^{i-1} \rightarrow \text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i)^0, \quad \omega \mapsto \{\alpha \mapsto \omega \wedge d\alpha\}.$$

By (7.12), under the identification (7.11), we have

$$\lambda_{\mathbf{G}_a}^*(\xi(\omega)) = \omega \cdot t, \quad \lambda_{\mathbf{G}_a}^*(\xi'(\eta)) = \eta \wedge dt \quad (\omega \in \Omega^i, \eta \in \Omega^{i-1}).$$

Hence the composite map

$$\Omega_k^i \oplus \Omega_k^{i-1} \xrightarrow{\xi \oplus \xi'} \text{Hom}_{\mathbf{PST}}(\mathbf{G}_a, \Omega^i) \xrightarrow{\lambda_{\mathbf{G}_a}^*} \Omega_k^i \cdot t \oplus \Omega_k^{i-1} \wedge dt$$

is given by the obvious identifications $\Omega_k^i = \Omega_k^i \cdot t$ and $\Omega_k^{i-1} = \Omega_k^{i-1} \wedge dt$. By Claim 7.6, this proves the desired claim and completes the proof of Lemma 7.5. \square

To deduce Theorem 7.1 from Proposition 7.2, we need some preliminaries.

Let K be the function field of $S \in \mathbf{Sm}$, and define \mathbf{Cor}_K , \mathbf{PST}_K , \mathbf{MCor}_K , \mathbf{MPST}_K , etc. as \mathbf{Cor} , \mathbf{PST} , \mathbf{MCor} , \mathbf{MPST} , etc., where the base field k is replaced by K . We then have a map

$$r_K: \text{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K), \quad \varphi \mapsto \{\psi_Y\}_{Y \in \mathbf{Sm}}, \quad (7.14)$$

where ψ_Y for $Y \in \mathbf{Sm}$ is the composite map

$$H^0(Y, \Omega^n) \rightarrow H^0(Y \times_k K, \Omega^n) \rightarrow H^0(Y \times_k K, \Omega^m),$$

where the second map is $\varphi_{Y \times_k K}$ (note that $Y \times_k K \in \mathbf{Sm}_K$) and the first is the pullback by the projection $p_Y: Y \times_k K \rightarrow Y$. Similarly, we can define a map

$$r_K: \text{Hom}_{\mathbf{PST}_K}(\mathcal{K}_n^M, \Omega^m) \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m)(K). \quad (7.15)$$

By the definitions, the following diagrams are commutative:

$$\begin{array}{ccc} \Omega_K^{m-n} \oplus \Omega_K^{m-n-1} & \xrightarrow{(7.2)} & \text{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) & \quad & \Omega_K^{m-n} & \xrightarrow{(7.4)} & \text{Hom}_{\mathbf{PST}_K}(\mathcal{K}_n^M, \Omega^m) \\ & \searrow (7.1) & \downarrow r_K & & & \searrow (7.3) & \downarrow r_K \\ & & \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) & & & & \underline{\text{Hom}}_{\mathbf{PST}}(\mathcal{K}_n^M, \Omega^m)(K) \end{array}$$

In view of Lemma 2.4, Theorem 7.1 follows from Proposition 7.2 and the following.

LEMMA 7.7. *The maps (7.14) and (7.15) are isomorphisms.*

For the proof, we need the following. Recall from Conventions 1.5 that for $U = \varprojlim_i U_i \in \widetilde{\mathbf{Sm}}$ and $F \in \mathbf{PST}$, we let $F(U) := \varinjlim_i F(U_i)$. In general, for $(Y, D_Y) \in \mathbf{MCor}$ and $F \in \mathbf{MPST}$, we let

$$\underline{\text{Hom}}_{\mathbf{MPST}}(U, F)(Y, D_Y) := \varinjlim_i F(U_i \times_k Y, U_i \times_k D_Y),$$

and for $G \in \mathbf{MPST}$, we have

$$\text{Hom}(G, \underline{\text{Hom}}(U, F)) = \varinjlim_i \text{Hom}(G, \underline{\text{Hom}}(U_i, F)).$$

LEMMA 7.8. *For $\mathcal{X} = (X, D) \in \mathbf{MCor}$ and $\mathcal{X}_K = (X_K, D_K) \in \mathbf{MCor}(K)$ with $X_K = X \times_k K$ and $D_K = D \times_k K$, we have a natural isomorphism*

$$\text{Hom}_{\mathbf{MPST}_K}(\mathbb{Z}_{\text{tr}}(\mathcal{X}_K), \underline{\omega}^{\text{CI}_K} \Omega^n) \cong \text{Hom}_{\mathbf{MPST}}(\mathbb{Z}_{\text{tr}}(\mathcal{X}), \underline{\text{Hom}}_{\mathbf{MPST}}(K, \underline{\omega}^{\text{CI}} \Omega^n)).$$

Proof. By [KMSY21a, Proposition 1.9.2(c)] and resolutions of singularities (recall that we are assuming $\text{ch}(k) = 0$), we may assume $X \in \mathbf{Sm}$ and that D_{red} is a simple normal crossing divisor. From the explicit computation of $\underline{\omega}^{\mathbf{CI}}\Omega^n$ in [RS22, Corollary 6.8], we have

$$\begin{aligned} (\underline{\omega}^{\mathbf{CI}_K}\Omega^n)(X_K, D_K) &= H^0(X_K, \Omega_{X_K}^n(\log(D_K))(D_K - D_{K,\text{red}})) \\ &= (\underline{\omega}^{\mathbf{CI}}\Omega^n)(X_K, D_K) := \varinjlim_{U \subset S} (\underline{\omega}^{\mathbf{CI}}\Omega^n)(X \times_k U, D \times_k U), \end{aligned}$$

where U ranges over the open subsets of S . This proves the lemma. \square

Proof of Lemma 7.7. We only prove the assertion for (7.14). The proof for (7.15) is similar. Put

$$\overline{\square}_{\Omega^n} = \overline{\square}_{\mathbf{G}_a} \otimes_{\mathbf{MPST}} \overline{\square}_{\mathbf{G}_m}^{\otimes n},$$

where $\overline{\square}_{\mathbf{G}_a}$ and $\overline{\square}_{\mathbf{G}_m}$ are from Lemma 2.8. By (2.16) and (2.17) and [RSY22, Theorem 5.20], we have an isomorphism in \mathbf{PST} :

$$a_{\text{Nis}}^V \omega_! h_0^{\overline{\square}}(\overline{\square}_{\Omega^n}) \xrightarrow{\sim} \Omega^n. \quad (7.16)$$

Let $\overline{\square}_K = (\mathbf{P}_K^1, \infty) \in \mathbf{MCor}_K$ and $\overline{\square}_{\Omega^n, K} \in \mathbf{MPST}_K$ be defined as $\overline{\square}_{\Omega^n}$. We have isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) &\simeq \text{Hom}_{\mathbf{PST}_K}(\omega_! h_0^{\overline{\square}_K}(\overline{\square}_{\Omega^n, K}), \Omega^m) \\ &\simeq \text{Hom}_{\mathbf{MPST}_K}(\overline{\square}_{\Omega^n, K}, \underline{\omega}^{\mathbf{CI}_K}\Omega^m) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\Omega^n}, \underline{\text{Hom}}_{\mathbf{MPST}}(K, \underline{\omega}^{\mathbf{CI}}\Omega^m)), \end{aligned} \quad (7.17)$$

where the last isomorphism comes from Lemma 7.8. On the other hand, by (7.16) and Lemma 2.5, we have $\underline{\text{Hom}}(\mathbb{Z}_{\text{tr}}(U), \Omega^m) \in \mathbf{RSC}_{\text{Nis}}$ for $U \in \mathbf{Sm}$. Hence, writing $\text{Spec}(K) = \varprojlim_i U_i$ with $U_i \in \mathbf{Sm}$, we have isomorphisms (see Conventions 1.5)

$$\begin{aligned} \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) &= \varinjlim_i \text{Hom}_{\mathbf{PST}}(U_i, \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)) \\ &\simeq \varinjlim_i \text{Hom}_{\mathbf{PST}}(\Omega^n, \underline{\text{Hom}}_{\mathbf{PST}}(U_i, \Omega^m)) \\ &\simeq \varinjlim_i \text{Hom}_{\mathbf{PST}}(\omega_! h_0^{\overline{\square}}(\overline{\square}_{\Omega^n}), \underline{\text{Hom}}_{\mathbf{PST}}(U_i, \Omega^m)) \\ &\simeq \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\Omega^n}, \underline{\omega}^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(K, \Omega^m)). \end{aligned} \quad (7.18)$$

Hence Lemma 7.7 follows from Lemma 6.8 and the following claim. \square

CLAIM 7.9. *The following diagram is commutative:*

$$\begin{array}{ccc} \text{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) & \xrightarrow{(7.17)} & \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\Omega^n}, \underline{\text{Hom}}_{\mathbf{MPST}}(K, \underline{\omega}^{\mathbf{CI}}\Omega^m)) \\ \downarrow r_K & & \downarrow \\ \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) & \xrightarrow{(7.18)} & \text{Hom}_{\mathbf{MPST}}(\overline{\square}_{\Omega^n}, \underline{\omega}^{\mathbf{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(K, \Omega^m)), \end{array} \quad (7.19)$$

where the right vertical map is induced by the map (6.21).

Proof. To show the above claim, write $\mathbf{A}_{\Omega^n} = \mathbf{A}^1 \times (\mathbf{A}^1 - \{0\})^n$ and $\mathbf{A}_{\Omega^n, K} = \mathbf{A}_{\Omega^n} \otimes_k K$. Take the standard coordinates y on \mathbf{A}^1 and (x_1, \dots, x_n) on $(\mathbf{A}^1 - \{0\})^n$ so that

$$\mathbf{A}_{\Omega^n} = \text{Spec } k[y, x_1, \dots, x_n][x_1^{-1}, \dots, x_n^{-1}].$$

By the definition of $\overline{\square}_{\Omega^n}$, we have natural maps in \mathbf{MPST}

$$\mathbb{Z}_{\text{tr}}(\mathbf{A}_{\Omega^n}, \emptyset) \rightarrow (\mathbf{P}^1, 2\infty) \otimes (\mathbf{P}^1, 0 + \infty)^{\otimes n} \rightarrow \overline{\square}_{\Omega^n} \quad (7.20)$$

which induces a map in **PST**

$$\lambda_{\Omega^n} : \mathbb{Z}_{\text{tr}}(\mathbf{A}_{\Omega^n}) \rightarrow \omega_! \bar{\square}_{\Omega^n} \rightarrow \Omega^n, \quad (7.21)$$

where the last map is induced by (7.16). Let

$$\lambda_{\Omega^n, K} : \mathbb{Z}_{\text{tr}}(\mathbf{A}_{\Omega^n, K}) \rightarrow \Omega^n \quad (7.22)$$

be defined as (7.21) in which k is replaced by K . By the definitions of $\lambda_{\mathbf{G}_m}$ and $\lambda_{\mathbf{G}_a}$ (cf. Lemma 2.8) and (6.5), the map λ_{Ω^n} corresponds to

$$\omega_0 := y \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \in \Omega^n(\mathbf{A}_{\Omega^n}). \quad (7.23)$$

The map (7.20) induces injective maps

$$\text{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\bar{\square}_{\Omega^n}, \underline{\text{Hom}}_{\underline{\mathbf{M}}\mathbf{PST}}(K, \omega^{\text{CI}} \Omega^m)) \hookrightarrow H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m), \quad (7.24)$$

$$\text{Hom}_{\underline{\mathbf{M}}\mathbf{PST}}(\bar{\square}_{\Omega^n}, \omega^{\text{CI}} \underline{\text{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \hookrightarrow H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m) \quad (7.25)$$

which are compatible with the right vertical map in (7.19) since applying $\omega_!$, the map (6.21) is identified with the identity on $\underline{\text{Hom}}_{\mathbf{PST}}(K, \Omega^m)$ via the isomorphism in Lemma 2.3. Hence it suffices to show the commutativity of the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m) & \xrightarrow{\alpha} & H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m), \\ \downarrow r_K & \nearrow \beta & \\ \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) & & \end{array} \quad (7.26)$$

where α (respectively, β) is the composite of (7.17) and (7.24) (respectively, (7.18) and (7.25)). By definition, α is induced by the map $\lambda_{\Omega^n, K}$ from (7.22). As $\lambda_{\Omega^n, K}$ is given by the image $\omega_{0, K}$ of ω_0 from (7.23) under the pullback map

$$p^* : \Omega^n(\mathbf{A}_{\Omega^n}) \rightarrow \Omega^n(\mathbf{A}_{\Omega^n, K}),$$

we have

$$\alpha(\varphi) = \varphi_{\mathbf{A}_{\Omega^n, K}}(\omega_{0, K}) \quad \text{for } \varphi \in \text{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m),$$

where

$$\varphi_{\mathbf{A}_{\Omega^n, K}} : \Omega^n(\mathbf{A}_{\Omega^n, K}) \rightarrow \Omega^m(\mathbf{A}_{\Omega^n, K})$$

is induced by φ . On the other hand, by the definition of β , we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{A}_{\Omega^n, K}, \Omega^m) & \xrightarrow{\simeq} & \text{Hom}_{\mathbf{PST}}(\mathbf{A}_{\Omega^n}, \underline{\text{Hom}}_{\mathbf{PST}}(K, \Omega^m)) \\ \beta \uparrow & & \uparrow \lambda_{\Omega^n}^* \\ \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K) & \xrightarrow{\simeq} & \text{Hom}_{\mathbf{PST}}(\Omega^n, \underline{\text{Hom}}_{\mathbf{PST}}(K, \Omega^m)), \end{array}$$

where $\lambda_{\Omega^n}^*$ is induced by λ_{Ω^n} from (7.21). Hence we have

$$\beta(\psi) = \psi_{\mathbf{A}_{\Omega^n}}(\omega_0) \quad \text{for } \psi \in \underline{\text{Hom}}_{\mathbf{PST}}(\Omega^n, \Omega^m)(K),$$

where

$$\psi_{\mathbf{A}_{\Omega^n}} : \Omega^n(\mathbf{A}_{\Omega^n}) \rightarrow \underline{\text{Hom}}_{\mathbf{PST}}(K, \Omega^m)(\mathbf{A}_{\Omega^n}) = \Omega^m(\mathbf{A}_{\Omega^n, K})$$

is induced by ψ . Then, for $\varphi \in \text{Hom}_{\mathbf{PST}_K}(\Omega^n, \Omega^m)$, we get

$$\beta(r_K(\varphi)) = r_K(\varphi)_{\mathbf{A}_{\Omega^n}}(\omega_0) = \varphi_{\mathbf{A}_{\Omega^n, K}}(p^* \omega_0) = \varphi_{\mathbf{A}_{\Omega^n, K}}(\omega_{0, K}) = \alpha(\varphi),$$

which proves the commutativity of (7.26). This concludes the proof of Claim 7.9, and hence of Lemma 7.7 and therefore also that of Theorem 7.1. \square

ACKNOWLEDGEMENTS

The authors would like to thank Kay Rülling for letting them include his proofs of Lemmas 2.1 and 3.1 and for pointing out a mistake in the first version of this paper and also for many valuable comments. The first author would like to thank his PhD supervisor Joseph Ayoub for suggesting the study of modulus sheaves and for many discussions that led to the formulation of the results in this paper. He would like to thank Lorenzo Mantovani and Federico Binda for many helpful discussions. The second author would like to thank Joseph Ayoub for the invitation to the university of Zürich where his collaboration with the first author started.

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