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SINGULAR QUIVER VARIETIES OVER EXTENDED DYNKIN QUIVERS

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Abstract

We classify the singularities in the unframed Nakajima quiver varieties associated with extended Dynkin quivers and the corresponding minimal imaginary root with a small restriction on the parameters and use this to construct a number of hyper-Kähler cobordisms between binary polyhedral spaces.

1. Introduction

In [**19**], Nakajima introduced a family of spaces he called quiver varieties. A quiver is simply a finite directed graph (Q, I), where I is the set of vertices and Q is the set of edges. We typically denote the quiver by Q. Given a dimension vector $v \in \mathbb{Z}^I_{\geq 0}$, we form the vector space

$$\mathrm{Rep}(Q, v) := \bigoplus_{(h:\, i \to j) \in Q} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}),$$

which carries a natural linear action of the compact Lie group $G_v := \prod_{i \in I} U(v_i)$. The doubled quiver $\overline{Q}$ is obtained from Q by adjoining an opposite edge $\overline{h}\colon j \to i$ for each edge $h : i \to j$ in Q. In this situation, one may give the complex vector space $\mathrm{Rep}(\overline{Q}, v)$ a natural quaternionic structure preserved by the action of G_v. There is an associated hyper-Kähler moment map $\mu\colon \mathrm{Rep}(\overline{Q}, v) \to \mathbb{R}^3 \otimes \mathfrak{g}_v$, where $\mathfrak{g}_v = \mathrm{Lie}(G_v)$. The quiver varieties associated with Q and v are then defined to be the hyper-Kähler quotients

$$\mathcal{M}_\xi(Q, v) := \mu^{-1}(\xi)/G_v,$$

for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^I$. Here, ξ is regarded as an element of $\mathbb{R}^3 \otimes \mathfrak{g}_v$ using a canonical linear map from $\mathbb{R}^I$ onto the center of the Lie algebra. Given $w \in \mathbb{Z}^I$, let

$$D_w = \{\zeta \in \mathbb{R}^I : \zeta \cdot w = \sum_i \zeta_i w_i = 0\} \subset \mathbb{R}^I.$$

It is then necessary that $\xi \in \mathbb{R}^3 \otimes D_v$ for $\mathcal{M}_\xi(Q, v)$ to be non-empty; however, for almost all such parameters, the quiver variety $\mathcal{M}_\xi(Q, v)$ carries the structure of a smooth hyper-Kähler manifold.

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More generally, there is a decomposition

$$\mathcal{M}_\xi(Q,v) = \mathcal{M}_\xi^{\mathrm{reg}}(Q,v) \cup \mathcal{M}_\xi^{\mathrm{sing}}(Q,v),$$

where the regular set $\mathcal{M}_\xi^{\mathrm{reg}}(Q,v)$ is open and carries the structure of a smooth hyper-Kähler manifold, while the singular set $\mathcal{M}_\xi^{\mathrm{sing}}(Q,v)$ is its closed complement.

An extended Dynkin quiver Q is a quiver whose underlying unoriented graph is an extended Dynkin diagram of type $\widetilde{ADE}$, that is, type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. In this situation, there is a distinguished dimension vector $\delta \in \mathbb{Z}^I_{\geq 0}$, the minimal positive imaginary root in the associated root system. The purpose of this paper is to study the singular members of the family of quiver varieties $\mathcal{M}_\xi(Q,\delta)$ when Q is an extended Dynkin quiver. This family of spaces, whose non-singular members are the asymptotically locally euclidean (ALE) spaces, was first constructed and studied by Kronheimer [15] in a slightly different form. The fact that Kronheimer's construction can be expressed in the above form is explained in [19, p. 372–373].

The McKay correspondence [18] sets up a bijection between the isomorphism classes of finite subgroups $\Gamma \subset \mathrm{SU}(2)$ and the extended Dynkin diagrams of type $\widetilde{ADE}$. Kronheimer exploited this correspondence to show that the (non-empty) non-singular members of the family $\mathcal{M}_\xi(Q,\delta)$ for $\xi \in \mathbb{R}^3 \otimes D_\delta$ are smooth four-dimensional hyper-Kähler manifolds diffeomorphic to the minimal resolution of the quotient singularity $\mathbb{C}^2/\Gamma$, where $\Gamma \subset \mathrm{SU}(2)$ is the finite subgroup associated with the underlying graph of Q under the McKay correspondence.

To state our first main result, let Q be an extended Dynkin quiver with vertex set I and minimal positive imaginary root $\delta \in \mathbb{Z}^I$. By deleting any vertex $i \in I$ with $\delta_i = 1$ from Q, one recovers the associated (non-extended) Dynkin graph of type ADE. Identify the set of vertices with $\{0,1,\cdots,n\}$ for some $n \in \mathbb{N}$ such that $\delta_0 = 1$. One may then realize the root system associated with the underlying Dynkin graph as a subset $\Phi \subset \mathbb{Z}^n \subset \mathbb{R}^n$ with the coordinate vectors as a set of simple roots. Furthermore, there is a natural way to identify $\mathbb{R}^n \cong D_\delta \subset \mathbb{R}^{n+1}$, thereby identifying the set of parameters $\mathbb{R}^3 \otimes D_\delta \cong \mathbb{R}^3 \otimes \mathbb{R}^n$. With this in mind, our first main result can be stated as follows:

Theorem 1.1 *Let $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^n$ satisfy $\xi_1 = 0$. Then if $\Phi \cap \xi^\perp = \{\alpha \in \Phi : \alpha \cdot \xi_2 = \alpha \cdot \xi_3 = 0\}$ is non-empty, it is a root system in the subspace it spans and admits a decomposition into root systems of type ADE:*

$$\Phi \cap \xi^\perp = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r. \tag{1.1}$$

Furthermore, there is a natural bijection $\rho\colon \mathcal{M}_\xi^{\mathrm{sing}}(Q,\delta) \cong \{\Phi_1,\Phi_2,\cdots,\Phi_r\}$ and the local structure around the singularities can be described as follows: Let $x \in \mathcal{M}_\xi^{\mathrm{sing}}(Q,\delta)$ and let $\Gamma_x \subset \mathrm{SU}(2)$ be the finite group associated with $\rho(x)$ under the McKay correspondence. Then, there is an open neighborhood $x \in U_x \subset \mathcal{M}_\xi(Q,\delta)$ and a homeomorphism $\phi_x\colon U_x \to B_r(0)/\Gamma_x$, where $B_r(0) \subset \mathbb{C}^2$ is the open ball of radius r, that restricts to a diffeomorphism

$$\mathcal{M}_\xi^{\mathrm{reg}}(Q,\delta) \supset (U_x - \{x\}) \cong (B_r(0) - \{0\})/\Gamma_x.$$

The fact that $\mathcal{M}_\xi(Q,\delta)$ is non-singular when ξ avoids all the root walls D_θ for $\theta \in \Phi$ is the content of [15, Corollary 2.10].

We give a brief outline of the proof of Theorem 1.1 and, in particular, explain why we make the restriction $\xi_1 = 0$. The action of the compact group G_v on $\mathrm{Rep}(\overline{Q}, v)$ extends to a linear action of the complexification $G^c_\delta = \prod_{i=0}^{n} \mathrm{GL}(\delta_i, \mathbb{C})$. Moreover, the hyper-Kähler moment map splits

$$\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) \colon \mathrm{Rep}(\overline{Q}, \delta) \to \mathbb{R}^3 \otimes \mathfrak{g}_\delta \cong \mathfrak{g}_\delta \oplus \mathfrak{g}^c_\delta,$$

where $\mathfrak{g}^c_\delta = \mathrm{Lie}(G^c_\delta)$, and the second component is a moment for the action of G^c_δ with respect to a complex symplectic form on $\mathrm{Rep}(\overline{Q}, \delta)$. In the situation where the parameter $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^n$ satisfies $\xi_1 = 0$, there is a homeomorphism between the hyper-Kähler quotient $\mathcal{M}_\xi(Q, \delta)$ and the geometric invariant theory (GIT) quotient $\mu_{\mathbb{C}}^{-1}((\xi_2, \xi_3))//G^c_\delta$. The elements of the latter quotient have a representation theoretic interpretation. Indeed, if we write $\lambda = \xi_2 + i\xi_3 \in \mathbb{C}^I$, the points of $\mu_{\mathbb{C}}^{-1}(\lambda)//G^c_\delta$ are in natural bijection with the isomorphism classes of semi-simple modules of dimension δ over the deformed preprojective algebra $\Pi^\lambda = \Pi^\lambda(Q)$ introduced in [7]. Under these bijections, the singularities in $\mathcal{M}_\xi(Q, \delta)$ correspond precisely to the non-simple, semi-simple modules. Using the result by Crawley-Boevey [5] on the existence and uniqueness of simple Π^λ-modules, we are able to set up a bijection between the latter set and the root systems in the statement of the theorem.

To establish the homeomorphisms $\phi_x \colon U_x \to B_r(0)/\Gamma_x$, we employ the result of [17] that essentially reduces the statement to the determination of the complex symplectic slice (see Definition 7.1) at a point $\tilde{x} \in \mu^{-1}(0, \lambda)$ above x. We should note that a result along these lines is given in [15, Lemma 3.3]; however, the proof given there seems to contain a gap that we have been unable to close. For this reason, we have chosen to rely on the above-mentioned result instead.

The finite subgroups $\Gamma \subset \mathrm{SU}(2)$ are called the binary polyhedral groups. By restricting the canonical action of Γ to the three-sphere $S^3 \subset \mathbb{C}^2$, we obtain the binary polyhedral spaces S^3/Γ. In Proposition 8.3, we determine what kind of root space decomposition

$$\Phi \cap \xi = \Phi_1 \cup \cdots \cup \Phi_r,$$

one can obtain by varying the parameter ξ. Combining this with the above theorem, we obtain the following constructive procedure for hyper-Kähler manifolds with a number of ends modeled on $(0, \infty) \times S^3/\Gamma$ for finite subgroups $\Gamma \subset \mathrm{SU}(2)$. In the following statement, we say that a subgraph H of G is a full subgraph if every edge in G connecting a pair of vertices in H belongs to H.

THEOREM 1.2 *Let $\Gamma_0, \Gamma_1, \cdots, \Gamma_r \subset \mathrm{SU}(2)$ be finite subgroups and let $K_0, K_1, \cdots, K_r$ denote the corresponding (non-extended) Dynkin graphs. Let Q be an extended Dynkin quiver with vertex set I, whose underlying unoriented graph is the extended version of K_0. Then if $K_1 \sqcup K_2 \sqcup \cdots \sqcup K_r$ can be realized as a full subgraph of K_0, there exists a parameter $\xi \in \mathbb{R}^3 \otimes \mathbb{R}^I$ such that $X = \mathcal{M}^{\mathrm{reg}}_\xi(Q, \delta)$ satisfies the following properties:*

(1) *X is a connected hyper-Kähler manifold of dimension* 4.
(2) *There are disjoint open subsets $U_0, U_1, \cdots, U_r \subset X$ and for each $0 \leq i \leq r$ a diffeomorphism*

$$\phi_i \colon U_i \to (0, \infty) \times S^3/\Gamma_i.$$

(3) *The complement $Y = X - \bigcup_{i=0}^{r} U_i$ is a compact four-manifold with boundary components S^3/Γ_i for $0 \leq i \leq r$.*

Note that the diffeomorphism ϕ_i, $0 \leq i \leq r$, will generally not preserve the hyper-Kähler structure.

We wish to briefly describe the gauge theoretic motivation for pursuing the above result. In [11], the author calculates the equivariant instanton Floer homology in the sense of [10] for the binary polyhedral spaces. The key geometric input needed for the calculations is a close understanding of the appropriate moduli spaces of $\mathrm{SU}(2)$-instantons over the cylinders $\mathbb{R}\times S^3/\Gamma$ for finite $\Gamma\subset\mathrm{SU}(2)$. In [2], Austin tackled this problem using an equivariant version of the classical ADHM correspondence (see [9, Section 3.3] or [1]). This work inspired the generalized ADHM correspondence of Kronheimer and Nakajima [16] that describes instanton moduli spaces associated with unitary bundles over the ALE spaces as Nakajima quiver varieties (for this later reformulation, see [19]). To elaborate, if Q is an extended Dynkin quiver whose underlying graph corresponds to Γ under the McKay correspondence, then by [15, Corollary 3.2] one has

$$\mathbb{R}\times S^3/\Gamma\cong(\mathbb{C}^2-\{0\})/\Gamma\cong\mathcal{M}_0^{\mathrm{reg}}(Q,\delta),$$

where δ is the minimal imaginary root as before. With this in mind, the equivariant ADHM correspondence of [2] can be regarded as a degenerate case of the generalized ADHM correspondence of [16]. These two cases suggest that it should be possible to extend the ADHM correspondence to the singular situation considered in this paper as well. This conjectural leap would open up the possibility of studying cobordism maps in equivariant Floer homology associated with the many explicit cobordisms obtained from the above theorem.

The paper is organized as follows: In Section 2, we give the basic definitions concerning hyper-Kähler manifolds and hyper-Kähler reduction. In Section 3, we introduce quivers and quiver varieties and state the key results that will be needed concerning these. In Section 4, we recall the basic elements of the complex representation theory of quivers. Afterwards, we give the definition of the deformed preprojective algebras $\Pi^\lambda(Q)$ and spell out the relation between the quiver variety $\mathcal{M}_{(0,\lambda)}(Q,v)$ and the isomorphism classes of semi-simple $\Pi^\lambda(Q)$-modules. Finally, we recall the key result of [5] that eventually allows us to classify the singularities in $\mathcal{M}_{(0,\lambda)}(Q,v)$. In Section 5, we give the construction of the extended Dynkin diagrams from the underlying Dynkin diagram and review the necessary root space theory of the associated root systems.

Our original work starts in Section 6, where we establish the bijection between the singularities in the (extended Dynkin) quiver varieties and the components in the corresponding root space decomposition as in (1.1). In Section 7, we establish the local models around the singularities using a result of [17] and give the proof of Theorem 1.1. In the final section, we determine the possible configurations of singularities in the various quiver varieties and complete the proof of Theorem 1.2.

2. Hyper-Kähler reduction

A hyper-Kähler manifold is a tuple (M,g,I,J,K) consisting of a smooth manifold M, a Riemannian metric g and three almost complex structure maps $I,J,K\colon TM\to TM$ subject to the following conditions:

(1) I, J and K are orthogonal with respect to g,
(2) $IJK=-1_{TM}$ and
(3) $\nabla^g I=\nabla^g J=\nabla^g K=0$, where ∇^g is the Levi-Civita connection.

In particular, for each $S\in\{I,J,K\}$, the triple (M,g,S) is a Kähler manifold with Kähler form ω_S given by $(\omega_S)_p(v,w)=g_p(Sv,w)$ for each $p\in M$ and $v,w\in T_pM$.

Following the terminology of [17], a tri-Hamiltonian hyper-Kähler manifold is a triple (M,K,μ) consisting of a hyper-Kähler manifold M, a compact Lie group K acting on M preserving the hyper-Kähler structure and a hyper-Kähler moment map $\mu=(\mu_I,\mu_J,\mu_K)\colon M\to\mathbb{R}^3\otimes\mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K. Note that by definition μ is a hyper-Kähler moment map if and only if the components μ_I,μ_J,μ_K are moment maps for the corresponding symplectic forms $\omega_I,\omega_J,\omega_K$, respectively, in the sense familiar from symplectic geometry (see, for instance, [4]).

The group K acts on $\mathfrak{k}^*$ through the coadjoint action, and we denote the set of fixed points by $(\mathfrak{k}^*)^K$. For each $\xi\in\mathbb{R}^3\otimes(\mathfrak{k}^*)^K$, the fiber $\mu^{-1}(\xi)$ is K-invariant and the quotient space $\mu^{-1}(\xi)/K$ is called a hyper-Kähler quotient.

THEOREM 2.1 ([12]). *Let (M,K,μ) be a tri-Hamiltonian hyper-Kähler manifold and let $\xi\in\mathbb{R}^3\otimes(\mathfrak{k}^*)^K$. If K acts freely on $\mu^{-1}(\xi)$, then the following holds true.*

(1) *ξ is a regular value for μ so that $\mu^{-1}(\xi)$ is a smooth submanifold of M.*
(2) *The quotient $\mu^{-1}(\xi)/K$ is a smooth manifold of dimension* $\dim M-4\dim K$ *and the projection $\pi\colon\mu^{-1}(\xi)\to\mu^{-1}(\xi)/K$ is a principal K-bundle.*
(3) *There is a unique hyper-Kähler structure on $\mu^{-1}(\xi)/K$ with Kähler forms $\omega'_I,\omega'_J,\omega'_K$ such that $\pi^*(\omega'_S)=\omega_S|_{\mu^{-1}(\xi)}$ for each $S\in\{I,J,K\}$.*

The passage from (M,K,μ) to $\mu^{-1}(\xi)/K$ for $\xi\in\mathbb{R}^3\otimes(\mathfrak{k}^*)^K$ is called hyper-Kähler reduction. Even if the action of K on $\mu^{-1}(\xi)$ fails to be free, the hyper-Kähler quotient $X:=\mu^{-1}(\xi)/K$ admits a decomposition into smooth hyper-Kähler manifolds of various dimensions (see [17, Theorem 1.1]). For our purpose, it will be sufficient to note that if $U\subset M$ denotes the open (possibly empty) set consisting of the free K-orbits, then $\mu|_U\colon U\to\mathbb{R}^3\otimes\mathfrak{k}^*$ is a moment map for the action of K on U, and therefore, $(\mu^{-1}(\xi)\cap U)/K=:X^{\mathrm{reg}}\subset X$ carries the structure of a smooth hyper-Kähler manifold by the above theorem. The open subset X^{reg} is called the regular set, and its closed complement $X^{\mathrm{sing}}:=X-X^{\mathrm{reg}}$ is called the singular set.

We will only be interested in a very simple instance of the above procedure. Let V be a quaternionic vector space equipped with a compatible real inner product $g\colon V\times V\to\mathbb{R}$, that is, V is a real vector space equipped with three orthogonal endomorphisms $I,J,K\colon V\to V$ satisfying the relations of the quaternion algebra:

$$I^2=J^2=K^2=IJK=-1_V.$$

Using the standard identification $T_pV\cong V$ for each $p\in V$, we may regard (V,g,I,J,K) as a flat hyper-Kähler manifold. Let K be a compact Lie group acting linearly on V preserving (g,I,J,K). In this situation, the unique hyper-Kähler moment map vanishing at $0\in V$, $\mu=(\mu_I,\mu_J,\mu_K)\colon V\to\mathbb{R}^3\otimes\mathfrak{k}^*$, is given by

$$\mu_I(x)(\xi)=\frac{1}{2}\omega_I(\xi\cdot x,x)=\frac{1}{2}g(\xi\cdot Ix,x),$$

for $x\in V$, $\xi\in\mathfrak{k}$ and similarly for μ_J and μ_K. We call the triple (V,K,μ) a linear tri-Hamiltonian hyper-Kähler manifold.

3. Quiver varieties

A quiver is a finite directed graph (Q, I, s, t), where I is the set of vertices, Q is the set of edges and $s, t\colon Q \to I$ are the source and target maps. Given an edge $h \in Q$ with $s(h) = i \in I$ and $t(h) = j \in I$, we write $h\colon i \to j$. We will abuse notation slightly and refer to the quiver simply as Q or (Q, I) letting s and t be implicit. The purpose of this section is to fix our notation, define the quiver varieties of interest and state a few results needed for our later work. We will later restrict our attention to the quivers specified in the following definition.

Definition 3.1 An extended Dynkin quiver is a quiver Q whose underlying unoriented graph is an extended Dynkin diagram of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. Similarly, a Dynkin quiver is a quiver whose underlying unoriented graph is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.

Let (Q, I) be a quiver. For each $v = (v_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$, called a dimension vector, define

$$\begin{aligned} \mathrm{Rep}(Q, v) &:= \bigoplus_{h\in Q} \mathrm{Hom}(\mathbb{C}^{v_{s(h)}}, \mathbb{C}^{v_{t(h)}}) \\ G_v &:= \prod_{i\in I} U(v_i) \\ G_v^c &:= \prod_{i\in I} \mathrm{GL}(v_i, \mathbb{C}), \end{aligned}$$

where $U(v_i) \subset \mathrm{GL}(v_i, \mathbb{C})$ denotes the group of unitary matrices for each $i \in I$. There is an evident inclusion $G_v \subset G_v^c$ witnessing the fact that G_v^c is the complexification of G_v. The Lie algebras $\mathfrak{g}_v := \mathrm{Lie}(G_v)$ and $\mathfrak{g}_v^c := \mathrm{Lie}(G_v^c)$ are given by

$$\mathfrak{g}_v = \bigoplus_{i\in I} \mathfrak{u}(v_i) \quad \text{and} \quad \mathfrak{g}_v^c = \bigoplus_{i\in I} \mathrm{End}(\mathbb{C}^{v_i}).$$

The group G_v^c acts linearly on $\mathrm{Rep}(Q, v)$ by the formula

$$g \cdot x = (g_{t(h)} x_h g_{s(h)}^{-1})_{h\in Q} \text{ for } g = (g_i)_{i\in I} \in G_v^c \text{ and } x = (x_h)_{h\in Q} \in \mathrm{Rep}(Q, v).$$

The subgroup G_v acts by restriction along the inclusion $G_v \subset G_v^c$. The space $\mathrm{Rep}(Q, v)$ carries a Hermitian inner product preserved by the action of G_v. Explicitly,

$$(x, y) = \sum_{h\in Q} \mathrm{tr}(x_h y_h^*),$$

where tr is the trace and y_h^* is the adjoint of y_h with respect to the standard Hermitian inner product on $\mathbb{C}^{v_i}$ for $i \in I$.

Definition 3.2 Let Q be a quiver. The opposite quiver Q^{op} is defined by taking the same set of vertices and reverse the orientation of each edge. For an edge $h \in Q$, the opposite edge is denoted by $\overline{h} \in Q^{op}$. The doubled quiver $\overline{Q}$ is defined by taking the same set of vertices and let the set of edges be $Q \cup Q^{op}$. The orientation map $\epsilon\colon \overline{Q} \to \{\pm 1\}$ is defined by $\epsilon(h) = +1$ if $h \in Q$ and $\epsilon(h) = -1$ if $h \in Q^{op}$.

We extend the bijection $Q \to Q^{op}$, $h \mapsto \overline{h}$, to an involution of $\overline{Q}$ by setting $\overline{h_2} = h_1$ if and only if $\overline{h_1} = h_2$ for $h_1 \in Q$ and $h_2 \in Q^{op}$.

Given a quiver Q with vertex set I and a dimension vector $v \in \mathbb{Z}^I_{\geq 0}$, there is a natural decomposition

$$\operatorname{Rep}(\overline{Q}, v) = \operatorname{Rep}(Q, v) \oplus \operatorname{Rep}(Q^{op}, v).$$

This gives rise to a quaternionic structure $J\colon \operatorname{Rep}(\overline{Q}, v) \to \operatorname{Rep}(\overline{Q}, v)$. In terms of the above decomposition, J is given by $J(x,y) = (-y^*, x^*)$, where $(x^*)_h := (x_{\overline{h}})^*$ and similarly for y. The action of G_v commutes with this quaternionic structure, and we may therefore regard $\operatorname{Rep}(\overline{Q}, v)$ as a quaternionic representation of the compact group G_v. The components of the unique hyper-Kähler moment map $\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) := \operatorname{Rep}(\overline{Q}, v) \to \mathfrak{g}_v \oplus \mathfrak{g}_v^c$ vanishing at zero, where the Lie algebras are identified with their duals using the trace pairing, have the explicit forms [19, p. 370]

$$\begin{aligned} \mu_{\mathbb{R}}(x) &= \frac{\sqrt{-1}}{2}\left(\sum_{h \in t^{-1}(i)} x_h x_h^* - x_{\overline{h}}^* x_{\overline{h}}\right)_{i \in I} \\ \mu_{\mathbb{C}}(x) &= \left(\sum_{h \in t^{-1}(i)} \epsilon(h) x_h x_{\overline{h}}\right)_{i \in I}. \end{aligned} \tag{3.1}$$

In the terminology of the previous section, $(\operatorname{Rep}(\overline{Q}, v), G_v, \mu)$ is a linear tri-Hamiltonian hyper-Kähler manifold.

Under the identifications of $\mathfrak{g}_v$ and $\mathfrak{g}_v^c$ with their dual spaces, the subspaces fixed under the coadjoint action are identified with the centers $Z(\mathfrak{g}_v)$ and $Z(\mathfrak{g}_v^c)$. There are natural maps $\mathbb{R}^I \to Z(\mathfrak{g}_v)$ and $\mathbb{C}^I \to Z(\mathfrak{g}_v^c)$ given by

$$\begin{aligned} (\xi_i)_{i \in I} \in \mathbb{R}^I &\mapsto (\sqrt{-1}\xi_i \operatorname{Id}_{\mathbb{C}^{v_i}})_{i \in I} \in \bigoplus_{i \in I} Z(\mathfrak{u}(v_i)) \\ (\lambda_i)_{i \in I} \in \mathbb{C}^I &\mapsto (\lambda_i \operatorname{Id}_{\mathbb{C}^{v_i}})_{i \in I} \in \bigoplus_{i \in I} Z(\operatorname{End}(\mathbb{C}^{v_i}). \end{aligned}$$

If $v_i \neq 0$ for each $i \in I$, then both of these are isomorphisms. Otherwise, they restrict to isomorphisms from $\mathbb{R}^{\operatorname{supp} v}$ and $\mathbb{C}^{\operatorname{supp} v}$, respectively, where $\operatorname{supp} v = \{i \in I : v_i \neq 0\}$. For any dimension vector $v \in \mathbb{Z}^I_{\geq 0}$, we will tacitly regard elements $\xi \in \mathbb{R}^I$ and $\lambda \in \mathbb{C}^I$ as elements of $Z(\mathfrak{g}_v)$ and $Z(\mathfrak{g}_v^c)$, respectively, using the above maps.

DEFINITION 3.3 Let Q be a quiver with vertex set I. For any dimension vector $v \in \mathbb{Z}^I_{\geq 0}$ and parameter $\xi = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}}) \in \mathbb{R}^I \oplus \mathbb{C}^I$ define

$$\mathcal{M}_\xi(Q, v) := \mu^{-1}(\xi)/G_v.$$

These hyper-Kähler quotients are called (unframed) quiver varieties.

REMARK 3.4 In [19], Nakajima defines what one may call framed quiver varieties $\mathcal{M}_\xi(v, w)$ associated with a quiver Q with vertex set I and two dimension vectors $v, w \in \mathbb{Z}^I$. The above-defined spaces

$\mathcal{M}_\xi(Q,v)$ correspond to his $\mathcal{M}_\xi(v,0)$. According to [**5**, p. 261], the spaces $\mathcal{M}_\xi(v,w)$ can be expressed as $\mathcal{M}_{\xi'}(\overline{Q}_1,v')$, where Q_1 is a quiver obtained from Q by adjoining a single vertex and a number of arrows depending on w. There is therefore no loss in generality in only considering these (unframed) quivers.

The subgroup T of scalars, that is, $U(1) \cong T \subset G_v$, acts trivially on $\mathrm{Rep}(\overline{Q},v)$ so the action factors through $G_v \to G_v/T =: G_v'$. As explained in the previous section, we obtain a decomposition

$$\mathcal{M}_\xi(Q,v) = \mathcal{M}_\xi^{\mathrm{reg}}(Q,v) \cup \mathcal{M}_\xi^{\mathrm{sing}}(Q,v),$$

where the regular set $\mathcal{M}_\xi^{\mathrm{reg}}(Q,v)$ is the image of the free G_v'-orbits in $\mu^{-1}(\xi)$ or equivalently the points $x \in \mu^{-1}(\xi)$ with stabilizer T in G_v. The regular set is open in $\mathcal{M}_\xi(Q,v)$ and carries the structure of a smooth hyper-Kähler manifold. The singular set $\mathcal{M}_\xi^{\mathrm{sing}}(Q,v)$ is the closed complement of the regular set.

The fact that the action of G_v factors through G_v' has another important implication, namely, that the moment map $\mu\colon \mathrm{Rep}(\overline{Q},v) \to \mathbb{R}^3 \otimes \mathfrak{g}_v$ takes values in the subspace $\mathfrak{g}_{v,0} \subset \mathfrak{g}_v$ corresponding to $(\mathfrak{g}_v')^* = \mathrm{Lie}(G_v')^*$ under the isomorphism $\mathfrak{g}_v^* \cong \mathfrak{g}_v$. This subspace consists precisely of the $(a_i)_{i\in I} \in \mathfrak{g}_v$ satisfying $\sum_{i\in I} \mathrm{tr}(a_i) = 0$. A parameter $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 \otimes \mathbb{R}^I$ corresponds to an element satisfying this condition precisely when

$$v\cdot\xi_k = \sum_{i\in I} \mathrm{tr}((\xi_k)_i \,\mathrm{Id}_{\mathbb{C}^{v_i}}) = 0 \text{ for } k = 1,2,3,$$

where $\cdot$ denotes the usual scalar product. For each $\theta \in \mathbb{Z}^I$, define

$$D_\theta = \{u \in \mathbb{R}^I : u\cdot\theta = 0\} \subset \mathbb{R}^I.$$

The above then amounts to the fact that $\mu^{-1}(\xi) = \varnothing$ whenever $\xi_k \notin D_v$ for some $1 \le k \le 3$. However, for most parameters $\xi \in \mathbb{R}^3 \otimes D_v$, the space $\mathcal{M}_\xi(Q,v)$ will be a smooth hyper-Kähler manifold. To state the relevant result, we have to recall the definition of the symmetric bilinear form associated with a quiver (see, for instance, [**5**, Section 2]).

Definition 3.5 Let Q be a quiver with vertex set I. The symmetric bilinear form $(\cdot,\cdot)\colon \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ associated with the quiver is defined by

$$(v,w) := 2\sum_{i\in I} v_i w_i - \sum_{h\in\overline{Q}} v_{s(h)} w_{t(h)} \quad \text{for} \quad v,w \in \mathbb{Z}^I.$$

If we identify the set of vertices $I \cong \{1,2,\cdots,n\}$ for some $n \in \mathbb{N}$ and let $A = (a_{ij})$ be the adjacency matrix of the unoriented graph underlying Q, that is, $a_{ij} = a_{ji}$ is the number of edges connecting i and j, then $(v,w) = 2v\cdot w - v\cdot Aw$. Alternatively, $(v,w) = v\cdot Cw$, where $C = 2\,\mathrm{id} - A$. The symmetric bilinear form therefore only depends on the underlying unoriented graph. If Q is a (extended) Dynkin quiver, then C is the Cartan matrix associated with the corresponding (extended) Dynkin diagram.

The following theorem is the unframed analogue of [**19**, Theorem 2.8]. Note that even though Nakajima's result at first sight seems to cover the unframed case as well (in his notation, $w=0$), this is not really true as one can show that the condition [**19**, Equation (2.9)] forces the relevant quiver variety to be empty and therefore excludes the parameters of interest. In the case of extended Dynkin quivers, the result is due to Kronheimer (see [**15**, Proposition 2.8, Corollary 2.10]). In the language of quivers, the corresponding statement is given in [**19**, Proposition 2.12].

Let $\mathbb{Z}^I$ be partially ordered by $v \le w$ if and only if $v_i \le w_i$ for each $i \in I$.

THEOREM 3.6 *Let Q be a quiver with vertex set I. Given a dimension vector $v \in \mathbb{Z}^I_{\ge 0}$ define*

$$R_+(v) = \{\theta \in \mathbb{Z}^I : 0 < \theta < v \text{ and } (\theta,\theta) \le 2\}.$$

Then if

$$\xi \in \mathbb{R}^3 \otimes D_v - \left(\bigcup_{\theta \in R_+(v)} \mathbb{R}^3 \otimes (D_v \cap D_\theta) \right),$$

the group G'_v acts freely on $\mu^{-1}(\xi) \subset \operatorname{Rep}(\overline{Q}, v)$, and the quiver variety $\mathcal{M}_\xi(Q,v)$ is a (possibly empty) smooth hyper-Kähler manifold of dimension $4-2(v,v)$.

Let (Q,I) be a quiver and fix a dimension vector $v \in \mathbb{Z}^I_{\ge 0}$. The complex Lie group G^c_v acts on $\operatorname{Rep}(\overline{Q},v)$ preserving the complex symplectic form $\omega_{\mathbb{C}}$ given by the formula

$$\omega_{\mathbb{C}}(x,y) = \sum_{h \in \overline{Q}} \epsilon(h) \operatorname{tr}(x_h y_{\overline{h}}) \quad \text{for} \quad x,y \in \operatorname{Rep}(\overline{Q},v). \tag{3.2}$$

The corresponding moment map is precisely the component $\mu_{\mathbb{C}}\colon \operatorname{Rep}(\overline{Q},v) \to \mathfrak{g}^c_v$ in (3.1). From the given formula, it is clear that $\mu_{\mathbb{C}}$ is algebraic and therefore $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$ carries the structure of an affine variety for each $\xi_{\mathbb{C}} \in \mathbb{C}^I$. The action of the reductive group G^c_v is algebraic so there is a complex analytic quotient $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \to \mu^{-1}(\xi_{\mathbb{C}})//G^c_v$. This is the analytification of the affine GIT quotient

$$\operatorname{Spec} \mathbb{C}[\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})] \to \operatorname{Spec}(\mathbb{C}[\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})]^{G^c_v}).$$

We will need a few standard facts concerning this construction (see, for instance, [**8**, Chapter 6] for the algebraic side of the story and [**17**, Section 2.4.1] and the references contained therein for the analytical perspective).

LEMMA 3.7 *As a topological space $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G^c_v$ is homeomorphic to the quotient space $\mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})/\sim$ where $x \sim y$ if and only if $\overline{G^c_v \cdot x} \cap \overline{G^c_v \cdot y} \neq \varnothing$. Let $q\colon \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \to \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G^c_v$ denote the quotient map. Then, each fiber $q^{-1}(x)$ contains a unique closed orbit $G^c_v \cdot \tilde{x}$, and if $y \in q^{-1}(x)$, then $G^c_v \cdot x \subset \overline{G^c_v \cdot y}$.*

In this setting, we have the following result comparing the analytic quotient and the hyper-Kähler quotient.

THEOREM 3.8 ([**19**, Theorem 3.1]) *Let Q be a quiver with vertex set I and let $v \in \mathbb{Z}^I$ be a dimension vector. Then, for each $\xi_{\mathbb{C}} \in \mathbb{C}^I$, the inclusion $\mu^{-1}(0,\xi_{\mathbb{C}}) = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \hookrightarrow \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$ descends to* *a homeomorphism*

$$\mathcal{M}_{(0,\xi_{\mathbb{C}})}(Q,v) = (\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}))/G_v \cong \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G_v^c.$$

Moreover, each closed orbit $G_v^c \cdot x \subset \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})$ intersects $\mu_{\mathbb{R}}^{-1}(0)$ in a unique G_v-orbit.

REMARK 3.9 The final statement is not explicitly stated in [19] but seems to be well known. See, for instance, [20, Proposition 2.4].

The above result implies that $\mathcal{M}_{(0,\xi_{\mathbb{C}})}(Q,v)$ carries the structure of a complex analytic space. We will have use for one final result. Let $v \in \mathbb{Z}_{\geq 0}^I$ be a fixed-dimension vector and let $\xi_{\mathbb{C}} \in \mathbb{C}^I$ such that $\operatorname{Re}\xi_{\mathbb{C}}, \operatorname{Im}\xi_{\mathbb{C}} \in D_v$. Choose $\xi_{\mathbb{R}} \in D_v - \bigcup_{\theta \in R_+(v)} D_\theta$ and set $\xi = (0,\xi_{\mathbb{C}})$ and $\tilde{\xi} = (\xi_{\mathbb{R}}, \xi_{\mathbb{C}})$. The space $\mathcal{M}_{\tilde{\xi}}(Q,v)$ is a smooth hyper-Kähler manifold by Theorem 3.6. The inclusion

$$\mu^{-1}(\tilde{\xi}) = \mu_{\mathbb{R}}^{-1}(\xi_{\mathbb{R}}) \cap \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}) \hookrightarrow \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}}),$$

induces a map $\pi\colon \mathcal{M}_{\tilde{\xi}}(Q,v) \to \mu_{\mathbb{C}}^{-1}(\xi_{\mathbb{C}})//G_v^c \cong \mathcal{M}_{\xi}(Q,v)$. In the following result, we regard $\mathcal{M}_{\tilde{\xi}}(Q,v)$ as a complex manifold by fixing the complex structure induced by the standard complex vector space structure of $\operatorname{Rep}(\overline{Q},v)$.

THEOREM 3.10 ([19, Theorem 4.1]). *The map π is holomorphic and provided $\mathcal{M}_{\xi}^{\mathrm{reg}}(Q,v)$ is non-empty, it is a resolution of singularities, that is,*

(1) *$\pi\colon \mathcal{M}_{\tilde{\xi}}(Q,v) \to \mathcal{M}_{\xi}(Q,v)$ is proper,*
(2) *π induces an isomorphism $\pi^{-1}(\mathcal{M}_{\xi}^{\mathrm{reg}}(Q,v)) \cong \mathcal{M}_{\xi}^{\mathrm{reg}}(Q,v)$ and*
(3) *$\pi^{-1}(\mathcal{M}_{\xi}^{\mathrm{reg}}(Q,v))$ is a dense subset of $\mathcal{M}_{\tilde{\xi}}(Q,v)$.*

4. Representations of quivers

We briefly recall a few basic notions concerning the representation theory of quivers. An excellent reference for this material is [3]. Afterwards, we give the definition of the deformed preprojective algebras $\Pi^\lambda = \Pi^\lambda(Q)$ of [7] and spell out the correspondence between $\mathcal{M}_{(0,\lambda)}(Q,v)$ and the isomorphism classes of semi-simple Π^λ-modules. Finally, we recall the construction of the root system associated with a quiver and state the key result of [5] relevant for our purpose.

A (complex) representation of a quiver Q is a pair (V,f) where $V = (V_i)_{i\in I}$ is a family of complex vector spaces and $f = (f_h\colon V_{s(h)} \to V_{t(h)})_{h\in Q}$ is a family of linear maps. We will only be concerned with finite dimensional representations; that is, V_i is finite dimensional for each $i \in I$. The dimension of a representation (V,f) is $\dim V := (\dim(V_i))_{i\in I} \in \mathbb{Z}_{\geq 0}^I$. A homomorphism $u\colon (V,f) \to (W,g)$ of representations is a collection of linear maps $u_i\colon V_i \to W_i$ for $i \in I$ such that $f_h u_{s(h)} = u_{t(h)} g_h$ for each $h \in Q$. We therefore have a category of complex representations of Q. This category is equivalent to the category of left modules over the quiver algebra $\mathbb{C}Q$: the complex algebra generated by $\{e_i : i \in I\}$ and $\{h : h \in Q\}$ subject to the relations

$$e_i e_j = \delta_{ij} e_i, \;\; e_i h = \delta_{it(h)} h \;\text{ and }\; h e_j = \delta_{s(h)j} h,$$

for all $i,j \in I$ and $h \in Q$, where $\delta_{ij} = 1$ if $i=j$ and $\delta_{ij} = 0$ otherwise. The $\{e_i\}_{i\in I}$ is a complete set of mutually orthogonal idempotents, in particular $1_{\mathbb{C}Q} = \sum_{i\in I} e_i$.

We briefly recall the equivalence between representations of Q and left $\mathbb{C}Q$-modules. Let (V,f) be a representation of Q and put $X = \bigoplus_{i\in I} V_i$. For each $i \in I$, let $\iota_i\colon V_i \to X$ and $\pi_i\colon X \to V_i$ denote the inclusion and projection, respectively. Define $\rho\colon \mathbb{C}Q \to \mathrm{End}_{\mathbb{C}}(X)$ by $\rho(e_i) = \iota_i \circ \pi_i$ for each $i \in I$ and $\rho(h) = \iota_{t(h)} \circ f_h \circ \pi_{s(h)}$ for each $h \in Q$. One may then verify that ρ is a well-defined homomorphism of $\mathbb{C}$-algebras and therefore endows X with a $\mathbb{C}Q$-module structure. One may recover (V,f) from (X,ρ) by setting $V_i = e_i X$ for $i \in I$ and $f_h = \pi_{t(h)} \circ \rho(h) \circ \iota_{s(h)}$ for $h \in Q$. With this in mind, we will pass freely between the notion of a Q representation and a $\mathbb{C}Q$-module.

A $\mathbb{C}Q$-module X of dimension $v \in \mathbb{Z}^I$ defines a unique G_v^c-orbit $\mathcal{O}_X \subset \mathrm{Rep}(Q,v)$. A representative x for the orbit is obtained by choosing a basis for $V_i = e_i X$, thereby identifying $V_i \cong \mathbb{C}^{v_i}$, for each $i \in I$ and then letting $x_h\colon \mathbb{C}^{v_{s(h)}} \to \mathbb{C}^{v_{t(h)}}$ be the corresponding linear maps. The correspondence $X \mapsto \mathcal{O}_X$ sets up a bijection between the isomorphism classes of $\mathbb{C}Q$-modules of dimension v and the set of G_v^c-orbits in $\mathrm{Rep}(Q,v)$. Given a parameter $\lambda \in \mathbb{C}^I$, the G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q},v)$ have a representation theoretic interpretation as well.

Definition 4.1 ([7, p. 611]). Let Q be a quiver with vertex set I. The deformed preprojective algebra $\Pi^\lambda = \Pi^\lambda(Q)$ of weight $\lambda \in \mathbb{C}^I$ is defined to be the quotient of the quiver algebra $\mathbb{C}\overline{Q}$ by the two-sided ideal generated by

$$c := \sum_{i\in I} \lambda_i e_i - \sum_{h\in Q} (h\overline{h} - \overline{h}h).$$

Observe that there is a decomposition $c = \sum_i c_i$ where

$$c_i = e_i \left(\lambda_i 1_{\mathbb{C}\overline{Q}} - \sum_{h\in t^{-1}(i)} \epsilon(h) h\overline{h} \right).$$

In view of formula (3.1) for $\mu_{\mathbb{C}}$, it is not hard to see that the G_v^c-orbit of a $\mathbb{C}\overline{Q}$-module X is contained in $\mu^{-1}(\lambda)$ precisely when X descends to a Π^λ-module along the projection $\mathbb{C}\overline{Q} \to \Pi^\lambda$. Therefore, the G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q},v)$ are in natural bijection with the isomorphism classes of Π^λ-modules of dimension v.

We have the following result describing the closed G_v^c-orbits in $\mathrm{Rep}(Q,v)$ (see, for instance, [3, Section 2] for a proof). Note that a G_v^c-orbit is closed in the Zariski topology if and only if it is closed in the analytic topology.

Proposition 4.2 *Let Q be a quiver with vertex set I and let X be a finite dimensional $\mathbb{C}Q$-module of dimension $v \in \mathbb{Z}^I_{\geq 0}$. Let $\mathcal{O}_X$ denote the orbit corresponding to the isomorphism class of X in* $\mathrm{Rep}(Q,v)$. *Then $\mathcal{O}_X$ is closed if and only if X is semi-simple. Moreover, let*

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X,$$

be a composition series for X; that is, each quotient X_k/X_{k-1}, $1 \leq k \leq n$, is a simple module, and let $X_{ss} = \bigoplus_{i=1}^n X_i/X_{i-1}$ be the semi-simplification of X. Then, $\mathcal{O}_{X_{ss}}$ is the unique closed orbit contained in the closure of $\mathcal{O}_X$.

Let $\mathcal{SS}(\Pi^\lambda, v)$ denote the set of isomorphism classes of semi-simple Π^λ-modules of dimension v. For a semi-simple Π^λ-module X, we let $[X]$ denote its isomorphism class in $\mathcal{SS}(\Pi^\lambda, v)$.

Proposition 4.3 *Let Q be a quiver with vertex set I and let Π^λ be the associated deformed preprojective algebra of weight $\lambda \in \mathbb{C}^I$. Then, for each dimension vector $v \in \mathbb{Z}^I$, the map*

$$\rho\colon \mathcal{M}_{(0,\lambda)}(Q,v) \to \mathcal{SS}(\Pi^\lambda, v),$$

that assigns to a point $x \in \mathcal{M}_{(0,\lambda)}(Q,v)$, the isomorphism class of the Π^λ-module corresponding to any point $\tilde{x} \in \mu^{-1}(0,\lambda)$ in the fiber over x, is a well-defined bijection.

Moreover, if $\rho(x) = [X]$ and $X = \bigoplus_{j=1}^k n_j X_j$ with the X_j simple and $n_j \in \mathbb{N}$, then for any point $\tilde{x} \in \mu^{-1}(0,\lambda)$ above x, there are isomorphisms

$$(G_v)_{\tilde{x}} \cong \prod_{j=1}^k U(n_j) \text{ and } (G_v^c)_{\tilde{x}} \cong \prod_{j=1}^k \mathrm{GL}(n_j, \mathbb{C}).$$

In particular, $x \in \mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q,v)$ if and only if X is simple.

Proof. We divide the proof into four steps. The first sentence in each step is a claim that we then go on to verify.

Step 1: The rule $[X] \mapsto \mathcal{O}_X \subset \mu_{\mathbb{C}}^{-1}(\lambda)$ defines a bijection between $\mathcal{SS}(\Pi^\lambda, v)$ and the set of closed G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda)$. We have seen that the given rule sets up a bijection between the set of isomorphism classes of Π^λ-modules of dimension v and the G_v^c-orbits contained in $\mu_{\mathbb{C}}^{-1}(\lambda)$. Since a Π^λ-module X is semi-simple if and only if it is semi-simple as a $\mathbb{C}\overline{Q}$-module, Proposition 4.2 ensures that this bijection restricts to a bijection between the isomorphism classes of the semi-simple Π^λ-modules and the closed G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda)$.

Step 2: The rule $(G_v \cdot x) \mapsto (G_v^c \cdot x)$ for $x \in \mu^{-1}(0,\lambda)$ defines a bijection between the G_v-orbits in $\mu^{-1}(0,\lambda)$ and the closed G_v^c-orbits in $\mu_{\mathbb{C}}^{-1}(\lambda)$. For any dimension vector $v \in \mathbb{Z}^I$ we have a commutative diagram

$$\begin{tikzcd}
\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda) \arrow[r, "i"] \arrow[d, "p"] & \mu_{\mathbb{C}}^{-1}(\lambda) \arrow[d, "q"] \\
\mathcal{M}_{(0,\lambda)}(Q,v) \arrow[r, "j"] & \mu_{\mathbb{C}}^{-1}(\lambda)//G_v^c,
\end{tikzcd}$$

where p and q are the quotient maps, i is the inclusion and j is the induced map between the quotients. According to Theorem 3.8, the map j is a homeomorphism and in particular a bijection. Therefore, the only thing we need to prove is that for each $x \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda)$, the orbit $G_v^c \cdot x \subset \mu_{\mathbb{C}}^{-1}(\lambda)$ is closed. By Lemma 3.7, there is a unique closed orbit $G_v^c \cdot y \subset q^{-1}q(i(x))$. Moreover, by the second statement in Theorem 3.8, we may assume that $y = i(z)$ for some $z \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda)$. Then, as $jp(x) = qi(x) = qi(z) = jp(z)$ and j is injective, we conclude that $p(x) = p(z)$ and hence $G_v \cdot x = G_v \cdot z$. This implies that $G_v^c \cdot x = G_v^c \cdot z$, and as the latter orbit is closed by construction, the claim has been verified.

Step 3: The map $\rho\colon \mathcal{M}_{(0,\lambda)}(Q,v) \to \mathcal{SS}(\Pi^\lambda, v)$ is a well-defined bijection. Let $p\colon \mu^{-1}(0,\lambda) \to \mathcal{M}_{(0,\lambda)}(Q,v)$ denote the quotient map as in the above diagram. The map sending $x \in \mathcal{M}_{(0,\lambda)}(Q,v)$ to the G_v-orbit $p^{-1}(x) \subset \mu^{-1}(0,\lambda)$ is clearly a bijection. The map ρ sending a point $x \in \mathcal{M}_{(0,\lambda)}(Q,v)$ to the isomorphism class of the Π^λ-module associated with any choice of $\tilde{x} \in p^{-1}(x)$ is then precisely the composition of the bijection $x \mapsto p^{-1}(x) = G_v \cdot \tilde{x}$, the bijection of step 2 and the inverse of the bijection of step 1. It is then clear that ρ is a well-defined bijection.

Step 4: If $\rho(x) = [X]$ and $X = \sum_{j=1}^k n_j X_j$ is a decomposition of X into simple modules, then for any $\tilde{x} \in p^{-1}(x)$ it holds true that

$$(G_v)_{\tilde{x}} \cong \prod_{j=1}^k U(n_j) \text{ and } (G_v^c)_{\tilde{x}} \cong \prod_{j=1}^k \mathrm{GL}(n_j, \mathbb{C}).$$

Let $y \in \mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q}, v)$ and denote the corresponding Π^λ-module by Y. It is then easy to see that the stabilizer $(G_v^c)_y$ coincides with the module theoretic automorphism group $\mathrm{Aut}_{\Pi^\lambda}(Y)$. If Y is semi-simple and $Y = \bigoplus_{j=1}^k n_j Y_j$ is a decomposition into simple modules, it follows by Schur's lemma that

$$\mathrm{Aut}_{\Pi^\lambda}(Y) \cong \prod_{j=1}^k \mathrm{GL}(n_j, \mathbb{C}).$$

Let $x \in \mathcal{M}_{(0,\lambda)}$, let $\tilde{x} \in \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(\lambda)$ be a point above x and let $X = \sum_{j=1}^k n_j X_k$ be the corresponding semi-simple Π^λ-module decomposed into simple summands. From the above considerations, we may deduce that there is an isomorphism $(G_v^c)_{\tilde{x}} \cong \prod_{j=1}^k \mathrm{GL}(n_j, \mathbb{C})$. For any point $y \in \mu_{\mathbb{R}}^{-1}(0)$, it holds true that the inclusion of stabilizers $\iota\colon (G_v)_y \hookrightarrow (G_v^c)_y$ induces an isomorphism between the complexification of $(G_v)_y$ and $(G_v^c)_y$ (see [20, Proposition 1.6]). Applying this in the situation above, we deduce that $\prod_{j=1}^k \mathrm{GL}(n_j, \mathbb{C})$ is isomorphic to the complexification of $(G_v)_{\tilde{x}}$. In particular, $(G_v)_{\tilde{x}}$ is isomorphic to a maximal compact subgroup of $\prod_{j=1}^k \mathrm{GL}_{n_j}(\mathbb{C})$, and as all such subgroups are conjugate, we deduce that there is an isomorphism

$$(G_v)_{\tilde{x}} \cong \prod_{j=1}^k U(n_j).$$

This completes the final step and hence the proof. □

In [5], Crawley-Boevey gives a strong result on the existence and uniqueness of simple Π^λ-modules. To state the result, we need to recall the construction of the root system associated with a quiver. Here, we follow [5, Section 2].

Let Q be a quiver with vertex set I and let $(\cdot,\cdot)\colon \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$ be the associated symmetric bilinear form of Definition 3.5. Let $\{\epsilon_i \in \mathbb{Z}^I : i \in I\}$ denote the standard basis of $\mathbb{Z}^I$, that is, $(\epsilon_i)_j = \delta_{ij}$ for $i,j \in I$. To simplify the exposition slightly, we will assume that Q contains no edge loops; that is, there is no $h \in Q$ with $s(h) = t(h)$. This is valid in the case of (extended) Dynkin quivers. Note that this condition implies that $(\epsilon_i, \epsilon_i) = 2$ for each $i \in I$.

For each $i \in I$, there is a reflection $s_i\colon \mathbb{Z}^I \to \mathbb{Z}^I$ defined by $s_i(v) = v - (v, \epsilon_i)\epsilon_i$. These reflections generate a finite subgroup $W \subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^I)$ called the Weyl group. The action of the Weyl group on $\mathbb{Z}^I$ preserves the symmetric bilinear form associated with the quiver. The support of $\alpha \in \mathbb{Z}^I$ is the full subquiver of Q with vertex set $\{i \in I : \alpha_i \neq 0\}$. The fundamental domain $F \subset \mathbb{Z}_{\geq 0}^I - \{0\}$ is then defined to be the set of $\alpha \in \mathbb{Z}^I_{\geq 0}$ with connected support satisfying $(\alpha, \epsilon_i) \leq 0$ for each $i \in I$. The root system associated with the quiver Q is defined to be $\Phi := \Phi^{\mathrm{re}} \cup \Phi^{\mathrm{im}} \subset \mathbb{Z}^I$ where

$$\Phi^{\mathrm{re}} = \bigcup_{i\in I} W \cdot \epsilon_i \quad \text{and} \quad \Phi^{\mathrm{im}} = W \cdot (F \cup -F).$$

The elements of Φ^{re} are called real roots and the elements of Φ^{im} are called imaginary roots. One may show that there is a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots, where a root α is positive (respectively negative) if $\alpha \in \mathbb{Z}^I_{\geq 0}$ (respectively $\alpha \in \mathbb{Z}^I_{\leq 0}$). We record the following elementary fact.

LEMMA 4.4 *For each $\alpha \in \Phi^{\mathrm{re}}$, it holds true that $(\alpha,\alpha) = 2$. For each $\beta \in \Phi^{\mathrm{im}}$, it holds true that $(\beta,\beta) \leq 0$.*

Proof. As already noted, $(\epsilon_i, \epsilon_i) = 2$ for each $i \in I$. The first assertion now follows from the fact that each $\alpha \in \Phi^{\mathrm{re}}$ may be expressed in the form $w \cdot \epsilon_i$ for some $w \in W$ and $i \in I$. For the second assertion, we may assume without loss of generality that $\beta \in F$. Writing $\beta = \sum_{i\in I} b_i \epsilon_i$ with $b_i \geq 0$, we find

$$(\beta,\beta) = \sum_{i\in I} b_i(\beta, \epsilon_i) \leq 0,$$

since by definition $(\beta, \epsilon_i) \leq 0$ for each $i \in I$. □

We may now state the key result on the existence and uniqueness of simple Π^λ-modules. In the following result, the function $p\colon \mathbb{Z}^I \to \mathbb{Z}$ is defined by the formula $p(\alpha) = 1 - \frac{1}{2}(\alpha,\alpha)$.

THEOREM 4.5 ([5, Theorem 1.2]). *Let Q be a quiver with vertex set I. Let Π^λ be the associated deformed preprojective algebra of weight $\lambda \in \mathbb{C}^I$. Then, for each $\alpha \in \mathbb{Z}^I_{\geq 0}$, the following is equivalent*

(1) *There exists a simple Π^λ-module of dimension α.*
(2) *α is a positive root with $\lambda \cdot \alpha = 0$, and for every decomposition $\alpha = \sum_t \beta^{(t)}$ into positive roots satisfying $\lambda \cdot \beta^{(t)} = 0$, one has*

$$p(\alpha) > \sum_t p(\beta^{(t)}).$$

In that situation, $\mu_{\mathbb{C}}^{-1}(\lambda) \subset \mathrm{Rep}(\overline{Q}, \alpha)$ is a reduced and irreducible complete intersection of dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$, and the general element is a simple representation.

5. Extended Dynkin quivers and their root systems

In our later work, it will be important to have a firm grip on the relation between the Dynkin diagrams and root systems of type ADE and their extended counterparts of type $\widetilde{ADE}$. In this section, we briefly review the necessary root space theory, establish our notation and prove two basic lemmas needed to effectively apply Theorem 4.5.

Let K be a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8, for short type ADE. Fix an identification of the set of vertices with $\{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$. The Cartan matrix $C = (c_{ij})_{ij} \in M_n(\mathbb{Z})$ of K

is then defined by $c_{ij} = 2\delta_{ij} - a_{ij}$, where $a_{ij} = a_{ji} = 1$ precisely when there is an edge connecting i to j in K and otherwise 0. The associated root system $\Phi \subset \mathbb{Z}^n$ is then constructed just as in the previous section using the pairing $(v,w)_C := v \cdot Cw$. Note that this pairing is positive definite so that $\Phi = \Phi^{\mathrm{re}}$. In particular, the coordinate vectors $\{\epsilon_i : 1 \leq i \leq n\}$ are a set of simple roots for Φ. There is a unique maximal root $d \in \Phi^+ \subset \mathbb{Z}^n$ with respect to the partial ordering $\leq$ on $\mathbb{Z}^n$ (see [**13**, Section 10.4]). The extended Dynkin diagram $\widetilde{K}$ is constructed from K by adjoining a single vertex 0 and one edge connecting 0 to i if $(d,\epsilon_i)_C = 1$ for each $1 \leq i \leq n$. The extended Cartan matrix $\tilde{C}$ is constructed from $\widetilde{K}$ in the same way C was constructed from K. Explicitly, if we identify $\mathbb{Z}^{n+1} = \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n$,

$$\widetilde{C} = \begin{pmatrix} 2 & -d^tC \\ -Cd & C \end{pmatrix}.$$

The associated root system $\widetilde{\Phi} \subset \mathbb{Z}^{n+1}$ is then constructed using the pairing $(v,w)_{\widetilde{C}} := v^t\widetilde{C}w$. We have the following useful description of the real roots in Φ and $\widetilde{\Phi}$ (see [**14**, Proposition 5.10])

$$\Phi = \{\alpha \in \mathbb{Z}^n : (\alpha,\alpha)_C = 2\} \quad \text{and} \quad \widetilde{\Phi}^{re} = \{\beta \in \mathbb{Z}^{n+1} : (\beta,\beta)_{\widetilde{C}} = 2\}. \tag{5.1}$$

To understand the imaginary roots in $\widetilde{\Phi}$, define a linear map $\psi\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ by $\psi(\epsilon_0) = -d$ and $\psi(\epsilon_i) = \epsilon_i$ for $1 \leq i \leq n$. Then, using the above explicit description of $\widetilde{C}$, one obtains the following identity

$$(v,w)_{\widetilde{C}} = (\psi(v),\psi(w))_C.$$

As the latter pairing is positive definite, one deduces that $(\cdot,\cdot)_{\widetilde{C}}$ is positive semi-definite. It follows by Lemma 4.4 that the set of imaginary roots must coincide with the non-zero elements of $\mathrm{Ker}(\psi)$, that is,

$$\widetilde{\Phi}^{\mathrm{im}} = \{r\delta : r \in \mathbb{Z} - \{0\}\},$$

where $\delta = (1,d)^t \in \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n = \mathbb{Z}^{n+1}$ is the minimal positive imaginary root. We will need two lemmas concerning these root systems.

Lemma 5.1 *Define $\Sigma := \{\beta \in \widetilde{\Phi} : 0 < \beta < \delta\}$. Then, the map $\psi\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ restricts to a bijection $\psi\colon \Sigma \to \Phi$ with inverse given by*

$$\psi^{-1}(\alpha) = \begin{cases} (0,\alpha) & \text{if } \alpha \in \Phi^+ \\ (1,d+\alpha) & \text{if } \alpha \in \Phi^- \end{cases},$$

with respect to the decomposition $\mathbb{Z}^{n+1} = \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n$. Furthermore, the adjoint $\psi^*: \mathbb{R}^n \to \mathbb{R}^{n+1}$, determined by $\psi(\theta)\cdot\tau = \theta\cdot\psi^*(\tau)$ for $\theta \in \mathbb{Z}^{n+1}$ and $\tau \in \mathbb{R}^n$, is given by $\psi^*(\tau) = (-d\cdot\tau,\tau)$ and corestricts to an isomorphism $\mathbb{R}^n \cong \delta^\perp \subset \mathbb{R}^{n+1}$.

Proof. Note first $\Sigma \subset \widetilde{\Phi}^{\mathrm{re}}$ since δ is the minimal positive imaginary root. As $(\alpha,\beta)_{\widetilde{C}} = (\psi(\alpha),\psi(\beta))_C$ for all $\alpha,\beta \in \mathbb{Z}^{n+1}$, it follows from the description of the real roots in (5.1) that $\psi(\Sigma) \subset \Phi$. The same result shows that the map $\kappa\colon \Phi \to \Sigma$ given by $\kappa(\alpha) = (0,\alpha)$ if $\alpha \in \Phi^+$ and $\kappa(\beta) = (1,d+\beta)$ if $\beta \in \Phi^-$ is well defined. Using the definition of ψ, one easily verifies that $\psi\kappa = \mathrm{id}_\Phi$ and $\kappa\psi = \mathrm{id}_\Sigma$. Hence, ψ is a bijection with inverse $\psi^{-1} = \kappa$.

For the second part, note that ψ extends uniquely to a linear map $\psi\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ and hence has an adjoint $\psi^* : \mathbb{R}^n \to \mathbb{R}^{n+1}$ uniquely determined by the formula given in the statement. For each $1 \le i \le n$, we find $\psi^*(\tau)_i = \psi^*(\tau)\cdot\epsilon_i = \tau\cdot\psi(\epsilon_i) = \tau_i$, while $\psi^*(\tau_0) = \tau\cdot\psi(\epsilon_0) = -\tau\cdot d$. Thus, $\psi^*(\tau) = (-d\cdot\tau, \tau)$. Finally, since $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is surjective, it follows that ψ^* corestricts to an isomorphism onto $\mathrm{Ker}(\psi)^\perp = \delta^\perp$. □

In the following lemma, we regard $\Phi \subset \mathbb{Z}^n \subset \mathbb{R}^n$ as above, and we write $(\cdot,\cdot)\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ for the Cartan pairing.

Lemma 5.2 *For $\tau \in \mathbb{C}^n$, define $\tau^\perp := \mathrm{Span}_{\mathbb{R}}(\mathrm{Re}\,\tau, \mathrm{Im}\,\tau)^\perp \subset \mathbb{R}^n$ with respect to the standard scalar product on $\mathbb{R}^n$. Then, if $\tau^\perp \cap \Phi$ is non-empty, it is a root system in the subspace it spans and decomposes into a disjoint union of root systems of type ADE*

$$\tau^\perp \cap \Phi = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r.$$

Furthermore, Φ_j admits a unique base contained in Φ^+ for each $1 \le j \le r$.

Proof. Write $\Phi_\tau = \tau^\perp \cap \Phi$. The fact that $\tau^\perp \cap \Phi$ is a root system in the subspace it spans follows from [13, Exercise III.9.7]. To see that Φ_τ admits a base contained in $\Phi^+ \subset \mathbb{Z}^n_{\ge 0}$, we mimic the proof for the existence of bases in a root system in [13, p. 48]. We may write $\Phi_\tau = \Phi_\tau^+ \cup \Phi_\tau^-$ where $\Phi_\tau^\pm = \tau^\perp \cap \Phi^\pm$. As $\Phi_\tau^- = -\Phi_\tau^+$, it follows that Φ_τ is non-empty if and only if Φ_τ^+ is non-empty. We may therefore define $S \subset \Phi_\tau^+$ to be the subset of $\alpha \in \Phi_\tau^+$ that admits no decomposition $\alpha = \beta + \gamma$ for $\beta, \gamma \in \Phi_\tau^+$. This set is non-empty since any $\alpha = \sum_i a_i\epsilon_i \in \Phi_\tau^+$, $a_i \ge 0$, with $\sum_i a_i$ minimal must belong to S. For any pair $\alpha \ne \beta \in S$, we have have $(\alpha,\beta) \le 0$. Indeed, if $(\alpha,\beta) = 1$, then either $\alpha - \beta$ or $\beta - \alpha$ will belong to Φ_τ^+, contradicting either $\alpha \in S$ or $\beta \in S$. To see that the set S is linearly independent, suppose that $\sum_{s\in S} a_s s = 0$. Put $S_1 = \{s \in S : a_s > 0\}$, $S_2 = S - S_1$ and write $u = \sum_{s\in S_1} a_s s = \sum_{t \in S_2} b_t t$ where $a_s > 0$ and $b_t = -a_t \ge 0$. Then

$$(u,u) = \sum_{s,t} a_s b_t (s,t) \le 0,$$

which is only possible if $u = 0$. Hence, as each $s \in S$ is non-zero and has non-negative coefficients with respect to the standard basis ϵ_i, $1 \le i \le n$, it follows that $a_s = 0$ for all $s \in S$ as required. It is clear that every root $\alpha \in \Phi_\tau^+$ can be written as a positive integral linear combination of the elements of S, and we have thus verified that S is a base for Φ_τ. At this point, we may decompose $S = S_1 \cup S_2 \cup \cdots \cup S_r$ into pairwise orthogonal sets in such a way that each S_i is indecomposable, that is, admits no further decomposition into pairwise orthogonal sets. This yields a corresponding decomposition into irreducible root systems (see [13, Section 10.4]) $\Phi_\tau = \Phi_1 \cup \Phi_2 \cdots \cup \Phi_r$, where S_i is a base for Φ_i for each $1 \le i \le r$. As each Φ_j is contained in Φ, all the roots have the same length, and this implies that Φ_j must be of type *ADE* for each j. □

The graphs K and $\widetilde{K}$ are transformed into quivers by giving the edges arbitrary orientations. As already mentioned, the corresponding symmetric bilinear forms and root systems are independent of the choice of orientations. In particular, if Q is an extended Dynkin quiver, we may identify the set of vertices with $\{0, 1, \cdots, n\}$ for some $n \in \mathbb{N}$ and assume that we have root systems $\widetilde{\Phi} \subset \mathbb{Z}^{n+1}$, $\Phi \subset \mathbb{Z}^n$

such that the minimal positive imaginary root δ takes the form $(1,d) \in \mathbb{Z}\epsilon_0 \oplus \mathbb{Z}^n$, where $d \in \Phi$ is the maximal positive root. Furthermore, by Lemma 5.1, we have the map $\psi\colon \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ relating them and the adjoint $\psi^* : \mathbb{R}^n \to \mathbb{R}^{n+1}$ that allows us to identify $\mathbb{R}^n \cong \delta^\perp$. We will work under these assumptions whenever convenient in the rest of the paper.

6. Classification of singularities

Let Q be an extended Dynkin quiver with vertex set I and minimal imaginary root $\delta \in \mathbb{Z}^I$. In this section, we will give a description of the singular set in the quiver variety $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ for $\lambda \in \mathbb{C}^I$. According to Proposition 4.3, the singular set $\mathcal{M}^{\mathrm{sing}}_{(0,\lambda)}(Q,\delta)$ is in natural bijection with the isomorphism classes of semi-simple, non-simple Π^λ-modules of dimension δ, so it suffices to determine the latter set.

For this purpose, let $\widetilde{\Phi}$ denote the root system associated with Q and let $\Sigma = \{\alpha \in \widetilde{\Phi} : 0 < \alpha < \delta\}$ as in Lemma 5.1. For $\lambda \in \mathbb{C}^I$, define $\Sigma_\lambda = \{\alpha \in \Sigma : \alpha \cdot \lambda = 0\}$ and let this set be partially ordered by $\alpha \prec \beta$ if and only if $\beta - \alpha = \sum_t \gamma^{(t)}$ for some $\gamma^{(t)} \in \Sigma_\lambda$. Finally, let $\Sigma_\lambda^{\min} \subset \Sigma_\lambda$ denote the subset of minimal elements with respect to this partial ordering.

Lemma 6.1 *There exists a simple Π^λ-module of dimension δ if and only if $\delta \cdot \lambda = 0$. Moreover, there exists a simple Π^λ-module of dimension α satisfying $0 < \alpha < \delta$ if and only if $\alpha \in \Sigma_\lambda^{\min}$ and in that case the simple module is unique up to isomorphism.*

Proof. According to Theorem 4.5, there exists a simple Π^λ-module of dimension $\alpha \in \mathbb{Z}^I_{\geq 0}$ if and only if α is a root satisfying $\alpha \cdot \lambda = 0$, and for every decomposition $\alpha = \sum_t \beta^{(t)}$ into positive roots satisfying $\beta^{(t)} \cdot \lambda = 0$, it holds true that $p(\alpha) > \sum_t p(\beta^{(t)})$, where we recall that $p(\alpha) = 1 - \frac{1}{2}(\alpha,\alpha)$. In our case of an extended Dynkin quiver, we have $p(\delta) = 1 - \frac{1}{2}(\delta,\delta) = 1$ and $p(\alpha) = 1 - \frac{1}{2}(\alpha,\alpha) = 0$ for every real root $\alpha \in \widetilde{\Phi}^{\mathrm{re}}$. In any decomposition $\delta = \sum_t \beta^{(t)}$ into positive roots with at least two summands, the roots $\beta^{(t)}$ must be real because δ is the minimal positive imaginary root. Therefore, the condition $p(\delta) = 1 > 0 = \sum_t p(\beta^{(t)})$ is trivially satisfied. We conclude that there exists a simple Π^λ-module of dimension δ if and only if $\delta \cdot \lambda = 0$.

If α satisfies $0 < \alpha < \delta$, there exists a simple Π^λ-module of dimension α if and only if $\alpha \in \Sigma_\lambda$, and for every decomposition $\alpha = \sum_t \beta^{(t)}$ with $\beta^{(t)} \in \Sigma_\lambda$, it holds true that $p(\alpha) > \sum_t p(\beta^{(t)})$. This inequality is never satisfied since both sides reduce to zero. Consequently, the above condition can only be satisfied if α does not admit such a decomposition at all, and this is equivalent to $\alpha \in \Sigma_\lambda^{\min}$. The fact that the simple Π^λ-module is unique up to isomorphism in this case follows from the final part of Theorem 4.5 as explained in [**5**, p. 260]. □

Before we proceed, we record the following consequence.

Lemma 6.2 *Let Q be an extended Dynkin quiver with vertex set I and minimal imaginary root δ. Let $\lambda \in \mathbb{C}^I$ satisfy $\lambda \cdot \delta = 0$. Then, the quiver variety $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ is connected and $\mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q,\delta)$ is non-empty.*

Proof. By the above lemma, there exists a simple Π^λ-module of dimension δ in this situation. By Proposition 4.3, this implies that $\mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q,\delta)$ is non-empty. Furthermore, by the final part of Theorem 4.5, the variety $\mu_{\mathbb{C}}^{-1}(\lambda)$ is irreducible in the Zariski topology. It is therefore connected in the analytic topology, and it follows that the quotient $\mathcal{M}_{(0,\lambda)}(Q,\delta) \cong \mu_{\mathbb{C}}^{-1}(\lambda)//G^c_\delta$ is connected as well. □

In the following theorem, we make the assumptions on the extended Dynkin quiver Q as explained in the end of the previous section.

THEOREM 6.3 *Let Q be an extended Dynkin quiver with vertex set $\{0,1,\cdots,n\}$ and let Π^λ be the associated deformed preprojective algebra of weight $\lambda \in \mathbb{C}^{n+1}$ satisfying $\lambda\cdot\delta = 0$. Let $\Phi \subset \mathbb{Z}^n$ be the root system of type ADE associated with Q. Write $\lambda = (\lambda_1,\tau)$ where $\lambda_1 \in \mathbb{C}$ and $\tau \in \mathbb{C}^n$ and let*

$$\tau^\perp \cap \Phi = \Phi_1 \cup \cdots \cup \Phi_r,$$

be a decomposition into (irreducible) subsystems of type ADE as in Lemma 5.2. Then, there is a bijection between $\{\Phi_1,\cdots,\Phi_r\}$ and the isomorphism classes of semi-simple, non-simple Π^λ-modules of dimension δ.

Proof. Let $\widetilde{\Phi} \subset \mathbb{Z}^{n+1}$ be the root system associated with Q. Let $\Sigma^{\min}_\lambda \subset \Sigma_\lambda \subset \Sigma \subset \widetilde{\Phi}$ be defined as in the beginning of the section. The content of Lemma 6.1 is then that there exists a simple Π^λ-module of dimension α, $0<\alpha<\delta$, if and only if $\alpha \in \Sigma^{\min}_\lambda$, and in that case, the module is unique up to isomorphism. This implies that a semi-simple, non-simple Π^λ-module $X = \sum_{t=0}^k n_t X_t$ of dimension δ is uniquely determined up to isomorphism by the roots $\gamma_t := \dim X_t \in \Sigma^{\min}_\lambda$ and the multiplicities $n_t \in \mathbb{N}$. We therefore have a bijective correspondence between the isomorphism classes of semi-simple, non-simple Π^λ-modules of dimension δ and sets $\{(n_t,\gamma_t)\}_{t=0}^k$ for which $n_t \in \mathbb{N}$, $\gamma_t \in \Sigma^{\min}_\lambda$ for each t, $\delta = \sum_t n_t\gamma_t$ and either $k \geq 1$ or $n_0 > 1$.

Our task is to relate the collection of such sets with the root systems in the decomposition

$$\tau^\perp \cap \Phi = \Phi_1 \cup \cdots \cup \Phi_r,$$

given in the statement of the theorem. Suppose that $\{(n_t,\gamma_t)\}_{t=0}^k$ is such a set. As $\delta = (1,d) \in \mathbb{Z}\oplus\mathbb{Z}^n$, where $d \in \Phi$ is the maximal root, the condition $\delta = \sum_t n_t\gamma_t$ implies that there is a distinguished root γ_t with non-zero first component and thus necessarily $n_t = 1$. After possibly rearranging the roots, we may take this root to be γ_0. By Lemma 5.1, there are unique positive roots $\beta,\alpha_t \in \Phi^+$, $1 \leq t \leq k$, such that $\gamma_0 = \psi^{-1}(-\beta) = (1,d-\beta)$ and $\gamma_t = \psi^{-1}(\alpha_t) = (0,\alpha_t)$ for $1 \leq t \leq k$. Moreover, since $\lambda\cdot\delta = 0$, there is a unique $\tau \in \mathbb{C}^n$ such that $\lambda = (-d\cdot\tau,\tau) = \psi^*(\tau)$. The relation $\theta\cdot\lambda = \psi(\theta)\cdot\tau$ for each $\theta \in \mathbb{Z}^{n+1}$ ensures that the bijection $\psi\colon \Sigma \cong \Phi$ restricts to a bijection $\Sigma_\lambda \cong \Phi\cap\tau^\perp$. In particular, $\beta,\alpha_1,\cdots,\alpha_k \in \Phi\cap\tau^\perp$. Moreover, the minimality of $\gamma_t = (0,\alpha_t)$, $1\leq t\leq k$, translates to the fact that each α_t is minimal among the roots in $\Phi^+\cap\tau^\perp$, while the minimality of $\gamma_0 = (1,d-\beta)$ translates to the fact that $\beta \in \Phi^+\cap\tau^\perp$ is maximal. This means that β must be the unique maximal positive root in precisely one of the systems Φ_j occurring in the decomposition of $\Phi\cap\tau^\perp$. Furthermore, since the equality $\delta = \sum_t n_t\gamma_t$ is equivalent to the equality $\beta = \sum_t n_t\alpha_t$, we also deduce that $\{\alpha_t : 1 \leq t \leq k\}$ must be the unique positive base in the same system.

This procedure is clearly reversible. Given a system Φ_j, let α_t, $1\leq t\leq k$, be the unique positive base and let $\beta = \sum_t n_t\alpha_t$ be the maximal root. We may then define $\gamma_0 = (1,d-\beta) \in \Sigma^{\min}_\lambda$, $n_0 = 1$ and

$\gamma_t = (0, \alpha_t) \in \Sigma_\lambda^{\min}$ for $1 \le t \le k$. It then follows from our previous arguments that the set $\{(n_t, \gamma_t\}_{t=1}^k$ satisfies the required conditions: $n_t \in \mathbb{N}$, $\gamma_t \in \Sigma_\lambda^{\min}$ for all t and $\sum_t n_t \gamma_t = \delta$. This completes the proof of the theorem. □

7. Local structure and the proof of Theorem 1.1

The combination of Proposition 4.3 and Theorem 6.3 gives full control over the singularities in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ for an extended Dynkin quiver Q. In this section, we establish the final results needed to complete the proof of Theorem 1.1.

Let Q be a quiver with vertex set I and let $\lambda \in \mathbb{C}^I$ be a parameter. Given a point $x \in \mu_{\mathbb{C}}^{-1}(\lambda)$, consider the sequence

$$G_v^c \xrightarrow{b_x} \operatorname{Rep}(\overline{Q}, v) \xrightarrow{\mu_{\mathbb{C}}} \mathfrak{g}_v^c,$$

where $b_x(g) = g \cdot x$ is the orbit map at x. As $\mu_{\mathbb{C}}$ is G_v^c-equivariant and $\lambda \in \mathbb{C}^I$ is identified with an element of $Z(\mathfrak{g}_v^c)$, the composition $\mu_{\mathbb{C}} \circ b_x$ is the constant map at λ. Hence, by differentiating this sequence at $1 \in G_v^c$, we obtain a three-term complex

$$0 \longrightarrow \mathfrak{g}_v^c \xrightarrow{\sigma_x} \operatorname{Rep}(\overline{Q}, v) \xrightarrow{\nu_x} \mathfrak{g}_v^c \longrightarrow 0, \tag{7.1}$$

where $\sigma_x = d(b_x)_1$ and $\nu_x = d(\mu_{\mathbb{C}})_x$. By general properties of the moment map, it holds true that $\operatorname{Ker}(\nu_x) = \operatorname{Im}(\sigma_x)^{\omega_{\mathbb{C}}}$, where the uppercase $\omega_{\mathbb{C}}$ denotes the complex symplectic complement. The tangent space $T_x(G_v^c \cdot x) = \operatorname{Im}(\sigma_x)$ is therefore isotropic with respect to $\omega_{\mathbb{C}}$. Moreover, the stabilizer $H := (G_v^c)_x$ acts naturally on each space in (7.1) making σ_x and ν_x H-equivariant. From these facts, it follows that $T_x(G_v^c \cdot x)^{\omega_{\mathbb{C}}} / T_x(G_v^c \cdot x) = \operatorname{Ker}(\nu_x)/\operatorname{Im}(\sigma_x)$ inherits a complex symplectic form preserved by the induced action of H.

Definition 7.1 Let $x \in \mu_{\mathbb{C}}^{-1}(\lambda)$. Then, the complex symplectic slice at x is the complex symplectic $(G_v^c)_x$-representation

$$T_x(G_v^c \cdot x)^{\omega_{\mathbb{C}}} / T_x(G_v^c \cdot x) = \operatorname{Ker}(\nu_x)/\operatorname{Im}(\sigma_x).$$

The following result is a consequence of [**17**, Theorem 1.4(iv)]. Here, we regard $\mathcal{M}_{(0,\lambda)}(Q,v)$ as a complex analytic space using Theorem 3.8.

Lemma 7.2 *Let Q be a quiver with vertex set I, let $v \in \mathbb{Z}_{\ge 0}^I$ be a dimension vector and let $\lambda \in \mathbb{C}^I$ be a parameter. Let $y \in \mathcal{M}_{(0,\lambda)}(Q,v)$ and let $x \in \mu^{-1}(0,\lambda) \subset \operatorname{Rep}(\overline{Q}, v)$ be a point above y. Set*

$$H := (G_v^c)_x \quad \text{and} \quad W := T_p(G_v^c \cdot x)^{\omega_{\mathbb{C}}} / T_x(G_v^c \cdot x).$$

Let $\mu_W \colon W \to \mathfrak{h}^*$ be the unique complex symplectic moment map vanishing at 0, where $\mathfrak{h} = \operatorname{Lie}(H)$. Then, a neighborhood of $y \in \mathcal{M}_{(0,\lambda)}(Q,v)$ is biholomorphic with a neighborhood of 0 in (the analytification of) the GIT quotient $\mu_W^{-1}(0)//H$.

In view of this result, our task is to determine the complex symplectic slices at the points above the singular points in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$. It will be useful to introduce the following notation.

DEFINITION 7.3 Let Q be a quiver with vertex set I. For a pair of dimension vectors $v,w \in \mathbb{Z}^I_{\geq 0}$, define

$$\mathrm{Hom}(v,w) := \bigoplus_{i\in I} \mathrm{Hom}(V_i, W_i) \ \text{ and } \ \mathrm{Rep}(Q;v,w) := \bigoplus_{h\in \overline{Q}} \mathrm{Hom}(V_{s(h)}, W_{t(h)}),$$

where $V_i = \mathbb{C}^{v_i}$ and $W_i = \mathbb{C}^{w_i}$ for each $i \in I$.

Note that $\mathrm{Rep}(Q;v,v) = \mathrm{Rep}(Q,v)$ and that $\mathrm{End}(v) := \mathrm{Hom}(v,v) = \mathfrak{g}^c_v$. The complex in (7.1) also has a relative analogue. Let $v,w \in \mathbb{Z}^I$ be a pair of dimension vectors and let $x \in \mathrm{Rep}(\overline{Q},v)$ and $y \in \mathrm{Rep}(\overline{Q},w)$ satisfy $\mu_{\mathbb{C}}(x) = \mu_{\mathbb{C}}(y) = \lambda$ for some $\lambda \in \mathbb{C}^I$. Define $C_Q(x,y)$ to be the sequence given by

$$0 \longrightarrow \mathrm{Hom}(v,w) \xrightarrow{\sigma_{x,y}} \mathrm{Rep}(Q;v,w) \xrightarrow{\nu_{x,y}} \mathrm{Hom}(v,w) \longrightarrow 0,$$

where

$$\begin{aligned} \sigma_{x,y}((u_i)_{i\in I}) &= (u_{t(h)}x_h - y_h u_{s(h)})_{h\in\overline{Q}} \\ \nu_{x,y}((v_h)_{h\in\overline{Q}}) &= \left(\sum_{h\in t^{-1}(i)} \epsilon(h)(u_h x_{\overline{h}} + y_h u_{\overline{h}}) \right)_{i\in I} . \end{aligned}$$

Note that $C_Q(x,x)$ is the complex of (7.1).

LEMMA 7.4 *Let X and Y denote the Π^λ-modules corresponding to $x \in \mathrm{Rep}(\overline{Q},v)$ and $y \in \mathrm{Rep}(\overline{Q},w)$. Then, $C_Q(x,y)$ is a chain complex, that is, $\nu_{x,y}\circ\sigma_{x,y} = 0$, and if we denote the cohomology groups from left to right by $H^i_Q(x,y)$ for $0 \leq i \leq 2$, we have*

(1) $H^0_Q(x,y) \cong \mathrm{Hom}_{\Pi^\lambda}(X,Y)$,
(2) $H^2_Q(x,y) \cong \mathrm{Hom}_{\Pi^\lambda}(Y,X)^*$,
(3) $\dim_{\mathbb{C}} H^1_Q(x,y) = \dim_{\mathbb{C}} H^0_Q(x,y) + \dim_{\mathbb{C}} H^2_Q(x,y) - (v,w)$.

Proof. To simplify the notation, we will write $V_i = \mathbb{C}^{v_i}$ and $W_i = \mathbb{C}^{w_i}$ for $i \in I$. Let $u = (u_i\colon V_i \to W_i)_{i\in I} \in \mathrm{Hom}(v,w)$. Then, using the definitions of $\sigma_{x,y}$ and $\nu_{x,y}$, we see that $\nu_{x,y}\circ\sigma_{x,y}(u)$ equals

$$\begin{aligned} &= \left(\sum_{h\in t^{-1}(i)} \epsilon(h)(u_{t(h)}x_h x_{\overline{h}} - y_h u_{s(h)} x_{\overline{h}} + y_h u_{t(\overline{h})} x_{\overline{h}} - y_h y_{\overline{h}} u_{s(\overline{h})}) \right)_{i\in I} \\ &= \left(\sum_{h\in t^{-1}(h)} u_i(\epsilon(h)x_h x_{\overline{h}}) - (\epsilon(h) y_h y_{\overline{h}})u_i \right)_{i\in I} = (u_i\lambda_i - \lambda_i u_i)_{i\in I} = 0. \end{aligned}$$

Here, we have used that $s(\overline{h}) = t(h)$, $t(\overline{h}) = s(h)$ and that $\mu_{\mathbb{C}}(x) = \mu_{\mathbb{C}}(y) = \lambda$. This shows that $C_Q(x,y)$ is a chain complex.

Recall that Π^λ was defined to be a quotient of the quiver algebra $\mathbb{C}\overline{Q}$. Therefore, we may also regard X and Y as $\mathbb{C}\overline{Q}$-modules and clearly $\mathrm{Hom}_{\mathbb{C}\overline{Q}}(X,Y) = \mathrm{Hom}_{\Pi^\lambda}(X,Y)$. From the definition of a homomorphism of representations, it is clear that $\mathrm{Hom}_{\mathbb{C}\overline{Q}}(X,Y) = \mathrm{Ker}(\sigma_{x,y}) = H^0_Q(x,y)$ proving part (1).

For the second part, we use an idea from the proof of [6, Lemma 3.1] (this lemma and its proof implies our result for $\lambda = 0$). Let $\phi\colon \mathrm{Hom}(w,v) \to \mathrm{Hom}(v,w)^*$ be the isomorphism given by $\phi(u)(v) = \sum_{i\in I} \mathrm{tr}(u_i v_i)$ and let $\psi\colon \mathrm{Rep}(Q;w,v) \to \mathrm{Rep}(Q;v,w)^*$ be the isomorphism given by $\psi(f)(g) = \sum_{h\in\overline{Q}} \epsilon(h)\,\mathrm{tr}(f_h g_{\overline{h}})$. Then, a rather tedious calculation shows that the following diagram commutes.

$$\begin{tikzcd} \mathrm{Hom}(w,v) \arrow[r, "\sigma_{y,x}"] \arrow[d, "\phi"] & \mathrm{Rep}(Q;w,v) \arrow[d, "\psi"] \\ \mathrm{Hom}(v,w)^* \arrow[r, "(\nu_{x,y})^*"] & \mathrm{Rep}(Q;v,w)^*. \end{tikzcd}$$

Since both the vertical maps are isomorphisms, we conclude that

$$\mathrm{Coker}(\nu_{x,y})^* \cong \mathrm{Ker}((\nu_{x,y})^*) \cong \mathrm{Ker}(\sigma_{y,x}) = \mathrm{Hom}_{\Pi^\lambda}(Y,X),$$

where the final equality follows from the first part. Hence, $H^2_Q(x,y) \cong \mathrm{Hom}_{\Pi^\lambda}(Y,X)^*$.

For the final part observe that

$$(v,w) = 2\sum_{i\in I} v_i w_i - \sum_{h\in\overline{Q}} v_{s(h)} w_{t(h)} = 2\dim_{\mathbb{C}} \mathrm{Hom}(v,w) - \dim_{\mathbb{C}} \mathrm{Rep}(Q;v,w),$$

is the Euler characteristic of the complex $C_Q(v,w)$. Since the Euler characteristic is preserved upon passage to cohomology, we obtain $(v,w) = \dim_{\mathbb{C}} H^0_Q(v,w) - \dim_{\mathbb{C}} H^1_Q(v,w) + \dim_{\mathbb{C}} H^2_Q(v,w)$, and this is equivalent to the formula stated in part (3). □

Remark 7.5 It is in fact also true that $H^1_Q(x,y) \cong \mathrm{Ext}^1_{\Pi^\lambda}(X,Y)$. We give a sketch of the proof. By [3, Corollary 1.4.2], it holds true that $\mathrm{Coker}(\sigma_{x,y}) = \mathrm{Ext}^1_{\mathbb{C}\overline{Q}}(X,Y)$. Moreover, there is an explicit way to relate this group to the set of isomorphism classes of extensions $0 \to Y \to Z \to X \to 0$. Given an element $[z] \in \mathrm{Ext}^1_{\mathbb{C}\overline{Q}}(X,Y)$ represented by $z = (z_u\colon V_{s(h)} \to W_{t(h)})$, one may construct the extension Z by setting $e_i Z = U_i = V_i \oplus W_i$ for each $i \in I$ and letting $z_h\colon U_{s(h)} \to U_{t(h)}$ for $h \in \overline{Q}$ be given by the matrix

$$z_h = \begin{pmatrix} x_h & 0 \\ z_h & y_h \end{pmatrix}.$$

The exact sequence $0 \to Y \to Z \to X \to 0$ is given componentwise by the canonical exact sequence $0 \to W_i \to V_i \oplus W_i \to V_i \to 0$. This is then an extension of Π^λ-modules if and only if $\mu_{\mathbb{C}}(Z) = \lambda$. It is then a matter of calculation to check that this is the case if and only if $z \in \mathrm{Ker}(\nu_{x,y})$.

Let Q be an extended Dynkin quiver with vertex set identified with $\{0,1,\cdots,n\}$ and minimal imaginary root $\delta \in \mathbb{Z}^{n+1}$. Let $\lambda = (\lambda_1, \tau) \in \mathbb{C} \oplus \mathbb{C}^n = \mathbb{C}^{n+1}$ satisfy $\delta \cdot \lambda = 0$. Denote the root systems by $\Phi \subset \mathbb{Z}^{n+1}$ and $\Phi \subset \mathbb{Z}^n$ as usual. By Proposition 4.3 and Theorem 6.3, the singular points in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ are in bijection with the components in the root space decomposition

$$\Phi \cap \tau^\perp = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r.$$

Write $\mathcal{M}^{\text{sing}}_{(0,\lambda)}(Q,\delta) = \{y_1, y_2, \cdots, y_r\}$, where y_i corresponds to Φ_i for each $1 \le i \le r$.

PROPOSITION 7.6 *In the above situation fix i, $1 \le i \le r$, and let $x \in \mu^{-1}(0,\lambda) \subset \operatorname{Rep}(Q,\delta)$ be a point above y_i. Let Q' be the extended Dynkin quiver associated with the root system Φ_i and let δ' denote its minimal imaginary root. Then, there is an isomorphism $(G^c_\delta)_x \cong G^c_{\delta'}$ and there is a complex symplectic isomorphism*

$$T_x(G^c_\delta \cdot x)^{\omega_\mathbb{C}} / T_x(G^c_\delta \cdot x) \cong \operatorname{Rep}(\overline{Q'}, \delta'),$$

equivariant along the above isomorphism of groups.

Proof. First note that the complex symplectic slice at x is precisely the cohomology group $H^1_Q(x,x)$. We will determine this complex symplectic space as an $H := (G^c_\delta)_x$ representation. Let $X = \bigoplus_{t=0}^k n_t Z_t$ denote the semi-simple Π^λ-module corresponding to x decomposed into simple summands. Then, according to Proposition 4.3, we have $H = \prod_{t=0}^k \operatorname{GL}(n_t, \mathbb{C})$. Recall from the proof of Theorem 6.3 that if we write $\gamma_t = \dim Z_t \in \mathbb{Z}^{n+1}$ for $0 \le t \le k$, then after possibly rearranging the indices, we have $\gamma_0 = (1, d-\beta) \in \mathbb{Z}^{n+1}$ and $\gamma_t = (0,\alpha_t) \in \mathbb{Z}^{n+1}$, $1 \le t \le k$, where $\alpha_1, \alpha_2, \cdots, \alpha_t \in \Phi_i \subset \Phi \cap \tau^\perp$ is a base and $\beta = \sum_{t=1}^n n_t \alpha_t$ is the maximal root.

Let $z_j \in \mu_\mathbb{C}^{-1}(\lambda) \subset \operatorname{Rep}(Q,\gamma_j)$ be the point corresponding to Z_j. Then, the complex $C_Q(x,x)$ decomposes according to the decomposition $X = \sum_{t=0}^k n_t Z_t$, namely,

$$C_Q(x,x) \cong \bigoplus_{t,s} \operatorname{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}) \otimes C_Q(z_s, z_t).$$

The stabilizer $H = \prod_{t=0}^k \operatorname{GL}(n_t, \mathbb{C})$ only acts on the first factors, that is,

$$(u_j)_j \cdot (f_{t,s} \otimes B_{t,s})_{t,s} = (u_t f_{t,s} u_s^{-1} \otimes B_{t,s})_{t,s},$$

for $(u_j)_j \in H$ and $f_{t,s} \otimes B_{t,s} \in \operatorname{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}) \otimes C_Q(z_t, z_s)$. Passing to cohomology, we obtain

$$H^1_Q(x,x) \cong \bigoplus_{s,t} \operatorname{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}) \otimes H^1_Q(z_s, z_t), \tag{7.2}$$

and the action of H is the same as described above. By Lemma 7.4 part (3) and the fact that each Z_t is a simple module, we find

$$\begin{aligned} \dim_\mathbb{C} H^1_Q(z_s, z_t) &= \dim_C \operatorname{Hom}_{\Pi^\lambda}(Z_s, Z_t) + \dim_\mathbb{C} \operatorname{Hom}_{\Pi^\lambda}(Z_t, Z_s)^* - (\gamma_s, \gamma_t) \\ &= 2\delta_{st} - (\gamma_s, \gamma_t). \end{aligned} \tag{7.3}$$

Let $\widetilde{K}$ be the extended Dynkin graph associated with the root system Φ_i. Specifically, the vertex set is $I = \{0, 1, \cdots, k\}$ corresponding to the roots $\alpha_0 = -\beta, \alpha_1, \cdots, \alpha_t$ and a single edge connecting s to t if and only if $(\alpha_s, \alpha_t) = -1$. As $(\gamma_s, \gamma_t) = (\alpha_s, \alpha_t)$ for all s, t, we conclude by the dimension formula (7.3) that $H^1_Q(z_s, z_t) \cong \mathbb{C}$ precisely when $s \neq t$ and s and t are adjacent in $\widetilde{K}$ and $H^1_Q(z_s, z_t) = 0$ otherwise. The expression in (7.2) then takes the form

$$H^1_Q(x,x) \cong \bigoplus_{s \to t \text{ in } \widetilde{K}} \mathrm{Hom}(\mathbb{C}^{n_s}, \mathbb{C}^{n_t}),$$

where each edge is repeated twice once with each orientation. If the identifications $H^1_Q(z_s, z_t) \cong \mathbb{C}$ for s and t adjacent in $\widetilde{K}$ are chosen appropriately, the induced symplectic form is given by

$$\omega((f_{s,t})_{s,t}, (g_{s,t})_{s,t})) = \sum_{s<t} \epsilon(s,t)(\mathrm{tr}(f_{s,t} g_{t,s}) - \mathrm{tr}(f_{t,s} g_{s,t})),$$

for some $\epsilon(s,t) = \pm 1$. If $s < t$ and there is an edge connecting s to t, we specify the orientation of the edge by $s \to t$ if $\epsilon(s,t) = 1$ and $t \to s$ if $\epsilon(t,s) = -1$. This gives rise to an extended Dynkin quiver Q' with minimal imaginary root $\delta' = (n_0 = 1, n_1, \cdots, n_k)$. It is now clear from the above work that $H^1_Q(x,x) \cong \mathrm{Rep}(\overline{Q'}, \delta')$ as complex symplectic $H \cong \prod_{t=0}^{k} \mathrm{GL}(n_t, \mathbb{C}) = G^c_{\delta'}$ representations. □

To complete the proof of Theorem 1.1, we will need the following result.

LEMMA 7.7 ([**15**, Corollary 3.2]). *Let Q be an extended Dynkin quiver with minimal imaginary root δ. Let $\Gamma \subset \mathrm{SU}(2)$ be the finite subgroup associated with the underlying unoriented graph of Q under the McKay correspondence. Then, there is a homeomorphism*

$$\mathcal{M}_0(Q, \delta) \cong \mathbb{C}^2/\Gamma,$$

that restricts to an isometry away from the singular point. In particular, $\mathcal{M}_0^{\mathrm{reg}}(Q, \delta) = \mathcal{M}_0(Q, \delta) - \{0\}$.

Proof of Theorem 1.1. Let Q be an extended Dynkin quiver with vertex set $\{0, 1, \cdots, n\}$ and minimal imaginary root $\delta = (1, d) \in \mathbb{Z}^{n+1}$, where d is the maximal positive root in the associated root system $\Phi \subset \mathbb{Z}^n$ of type *ADE*. Let $\lambda \in \mathbb{C}^{n+1}$ be a parameter satisfying $\lambda \cdot \delta = 0$ and write $\lambda = (\lambda_1, \tau) \in \mathbb{C} \oplus \mathbb{C}^n$. Then, by Theorem 6.3, there is a bijection between $\mathcal{M}^{\mathrm{sing}}_{(0,\lambda)}(Q, \delta)$ and the components in the root space decomposition

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_q.$$

Write $\mathcal{M}^{\mathrm{sing}}_{(0,\lambda)}(Q, \delta) = \{x_1, \cdots, x_q\}$, where x_i corresponds to Φ_i for $1 \leq i \leq q$. For each $1 \leq i \leq q$, let $Q^{(i)}$ denote the extended Dynkin quiver associated with the root system Φ_i and let $\delta^{(i)}$ be the associated minimal positive imaginary root. Then, according to Proposition 7.6 and Lemma 7.2, there is for each $1 \leq i \leq q$ an open neighborhood U_i of $x_i \in \mathcal{M}_{(0,\lambda)}(Q, \delta)$, an open neighborhood V_i of $0 \in \mathcal{M}_0(Q^{(i)}, \delta^{(i)})$ and a biholomorphism $\rho_i \colon U_i \to V_i$. Importantly, since the category of complex manifolds is a full subcategory of the category of complex analytic spaces, this biholomorphism restricts to a biholomorphism $\rho_i \colon U_i^{\mathrm{reg}} \cong V_i^{\mathrm{reg}}$ of complex manifolds.

Let $\Gamma_i \subset \mathrm{SU}(2)$ be the finite subgroup associated with $Q^{(i)}$ under the McKay correspondence. By the above lemma, there is for each i, $1 \leq i \leq q$, a homeomorphism $\mathcal{M}_0(Q^{(i)}, \delta^{(i)}) \cong \mathbb{C}^2/\Gamma_i$ that restricts to an isometry away from the singular point. This map restricts to a homeomorphism $\kappa_i\colon V_i \cong W_i \subset \mathbb{C}^2/\Gamma_i$ for some open neighborhood W_i around 0. By shrinking the U_i and V_i if necessary, we may assume that $W_i = B_r(0)/\Gamma_i$ for some $r > 0$ for each $1 \le i \le q$. The compositions $\phi_i := \kappa_i \circ \rho_i\colon U_i \to B_r(0)/\Gamma_i$ are then the required homeomorphisms. Indeed, for each i, both ρ_i and κ_i restrict to diffeomorphisms away from the singular point, so we deduce that the restriction

$$\phi_i = \kappa_i \circ \phi_i\colon \mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q,\delta) \cap U_i = U_i - \{x_i\} \cong (B_r(0) - \{0\})/\Gamma_i,$$

is a diffeomorphism. This completes the proof. □

8. Configurations of singularities and the proof of Theorem 1.2

Let Q be an extended Dynkin quiver with vertex set $I = \{0,1,\cdots,n\}$ and minimal imaginary root $\delta \in \mathbb{Z}^{n+1}$. In this section, we take up the question of what kind of configurations of singularities that can occur in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ by varying the parameter λ. Assume that $\lambda \cdot \delta = 0$ and write $\lambda = (\lambda_1, \tau) \in \mathbb{C} \oplus \mathbb{C}^n$. Then, according to Theorem 6.3 and the local structure result in the previous section, the configuration of singularities is uniquely determined by the root space decomposition

$$\Phi \cap \tau^\perp = \Phi_1 \cup \cdots \cup \Phi_r,$$

where $\Phi \subset \mathbb{Z}^n$ is the root system of type *ADE* associated with Q. The problem therefore reduces to determining the number and types of root systems that can occur in the above root space decomposition.

Give $\mathbb{C}$ the total ordering determined by $z \le w$ if and only if either $\mathrm{Re}(z) \le \mathrm{Re}(w)$ or $\mathrm{Re}\,z = \mathrm{Re}\,w$ and $\mathrm{Im}\,z \le \mathrm{Im}\,w$. Note that this ordering is additive, that is, $z \le w \implies z + c \le w + c$ for each $c \in \mathbb{C}$. We say that an element $\tau \in \mathbb{C}^n$ is dominant if $\tau_i \ge 0$ for each i. The value of this notion comes from the simple observation that if $\tau \in \mathbb{C}^n$ is dominant and $\theta \in \mathbb{Z}^n$, then $\tau \cdot \theta = 0$ if and only if $\mathrm{supp}(\theta) \cap \mathrm{supp}(\tau) = \varnothing$.

LEMMA 8.1 *Let K denote the Dynkin diagram associated with the root system $\Phi \subset \mathbb{Z}^n$. Suppose $\tau \in \mathbb{C}^n$ is dominant and let J be the complement of* $\mathrm{supp}(\tau)$ *in* $\{1,2,\cdots,n\}$. *Let $K_J \subset K$ be the full subgraph of K with vertex set $J \subset \{1,2,\cdots,n\}$. Let*

$$K_J = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_r,$$

be the decomposition of K_J into connected components. Then,

$$\Phi \cap \tau^\perp = \Phi_1 \cup \Phi_2 \cup \cdots \cup \Phi_r,$$

where Φ_i is the ADE root system associated with K_i for each $1 \le i \le r$.

Proof. Note first that every connected subgraph of a Dynkin graph of type *ADE* is again a Dynkin graph of type *ADE*. Let J_i be the set of vertices for K_i in the decomposition in the statement and put $S_i = \{\epsilon_j : j \in J_i\}$. We claim that $S = \cup_i S_i$ is a base for $\Phi \cap \tau^\perp$. Indeed, S clearly consists of linearly independent elements, and every element $\alpha \in \Phi^+ \cap \tau^\perp$ satisfies $\mathrm{supp}(\alpha) \subset J$ so it can be written as a positive linear integral combination of the elements of S. Then, as in the proof of Lemma 5.2, the root space decomposition

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \cdots \cup \Phi_r,$$

is obtained by decomposing S into minimal pairwise orthogonal sets $S = \cup_i S_i$ and letting Φ_i be the subsystem generated by S_i. Importantly, this decomposition $S = \cup_i S_i$ is precisely the decomposition introduced in the beginning. We conclude that Φ_i is the root system associated with the Dynkin graph K_i for each $1 \leq i \leq r$. □

For completeness, we also show that the decomposition for an arbitrary parameter τ can in fact be put in the above standard form. Recall that the Weyl group associated with Φ is the finite group $W \subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^n)$ generated by the simple reflections $s_i\colon \mathbb{Z}^n \to \mathbb{Z}^n$ in the coordinate vectors ϵ_i for $1 \leq i \leq n$. There is a unique action of W on $\mathbb{C}^n$ such that $(w\alpha)\cdot\tau = \alpha\cdot(w^{-1}\tau)$ for all $\alpha \in \mathbb{Z}^n$ and $\tau \in \mathbb{C}^n$. This is the complexification of the dual action, where we identify $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ using the standard scalar product.

The following lemma follows essentially from the proof in [13, p. 51], see also [7, Lemma 7.2].

Lemma 8.2 *For every $\tau \in \mathbb{C}^n$, there exists $w \in W$ such that $w\tau$ is dominant.*

Proof. Write $\Phi = \Phi^+ \cup \Phi^-$ and define $\gamma = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. By [13, p. 50], one has $s_i(\gamma) = \gamma - \epsilon_i$ for each $1 \leq i \leq n$. Choose $w \in W$ such that $\gamma\cdot w\tau \geq \gamma\cdot w'\tau$ for every $w' \in W$ with respect to the total ordering on $\mathbb{C}$. We claim that $\tau' := w\cdot\tau$ is dominant. Indeed, for each $1 \leq i \leq n$, it holds true that

$$\gamma\cdot\tau' \geq \gamma\cdot s_i\tau' = s_i\gamma\cdot\tau' = \gamma\cdot\tau' - \epsilon_i\cdot\tau',$$

or equivalently $\tau_i' = \epsilon_i\cdot\tau' \geq 0$. This shows that $w\tau = \tau'$ is dominant. □

Proposition 8.3 *Let K denote the Dynkin diagram associated with the root system $\Phi \subset \mathbb{Z}^n$. Given $\tau \in \mathbb{C}^n$, let*

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \cdots \cup \Phi_r,$$

be the corresponding decomposition into ADE root systems. Then, there exists a full subgraph $K' \subset K$ and a decomposition $K' = K_1 \sqcup \cdots \sqcup K_r$ into connected components such that Φ_i is isomorphic to the root system associated with K_i for each i.

Proof. By the previous lemma, there exists a Weyl transformation $w \in W$ such that $w\tau \in \mathbb{C}^n$ is dominant. From the relation $\tau\cdot\alpha = w\tau\cdot w\alpha$, we deduce that the isomorphism $w\colon \Phi \to \Phi$ restricts to an isomorphism $\tau^{\perp}\cap\Phi \to (w\tau)^{\perp}\cap\Phi$. As this is an isomorphism of root systems, it preserves the decomposition into irreducible components. The result therefore follows from Lemma 8.1 as $w\tau$ is dominant. □

The final ingredient needed to complete the proof of Theorem 1.2 is contained in the following proposition. We use the notation $B_r(x) \subset \mathbb{C}^2$ and $\overline{B}_r(x) \subset \mathbb{C}^2$ for the open and closed ball, respectively, with center $x \in \mathbb{C}^2$ and radius r.

Proposition 8.4 *Let Q be an extended Dynkin quiver with minimal imaginary root δ. Let $\Gamma \subset \mathrm{SU}(2)$ be the finite subgroup associated with the underlying extended Dynkin graph under the McKay correspondence. Let $\lambda \in \mathbb{C}^{n+1}$ be a parameter with $\lambda \cdot \delta = 0$. Then, there is an open subset $U \subset \mathcal{M}^{\mathrm{reg}}_{(0,\lambda)}(Q,\delta)$ with compact complement in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ and a diffeomorphism $\phi\colon U \to (\mathbb{C}^2 - \overline{B}_R(0))/\Gamma$. Moreover, $\phi^{-1}((\mathbb{C}^2 - B_{R'}(0))/\Gamma)$ is closed in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ for each $R' > R$.*

Remark 8.5 The final assertion is included to explicitly state that there are no limit points in $\mathcal{M}_{(0,\lambda)}(Q,\delta)$ as $x \in (\mathbb{C}^2 - \overline{B}_R(0))/\Gamma$ tends to ∞.

Proof. Choose a parameter $\zeta \in \mathbb{R}^{n+1}$ satisfying $\zeta \cdot \delta = 0$ and $\zeta \cdot \theta \neq 0$ for each $\theta \in R_+(\delta)$ (defined in Theorem 3.6) and put $\xi = (0,\lambda)$ and $\tilde{\xi} = (\zeta,\lambda)$. To simplify the notation, write

$$\widetilde{X} = \mathcal{M}_{\tilde{\xi}}(Q,\delta) \quad \text{and} \quad X = \mathcal{M}_{\xi}(Q,\delta).$$

Then, according to Theorem 3.10, there is a holomorphic map $\pi\colon \widetilde{X} \to X$ which is a resolution of singularities. Furthermore, by Kronheimer's result mentioned in the introduction [15, Corollary 3.12], the smooth four-dimensional hyper-Kähler manifold $\widetilde{X}$ is diffeomorphic to the minimal resolution of the quotient singularity $\mathbb{C}^2/\Gamma$. We may therefore assume that there is a continuous proper map $\hat{\pi}\colon \widetilde{X} \to \mathbb{C}^2/\Gamma$ that restricts to a diffeomorphism $\hat{\pi}^{-1}((\mathbb{C}^2 - \{0\})/\Gamma) \cong (\mathbb{C}^2 - \{0\})/\Gamma$. The situation is summarized in the following diagram

$$X \xleftarrow{\pi} \widetilde{X} \xrightarrow{\hat{\pi}} \mathbb{C}^2/\Gamma.$$

Since the open sets $\hat{\pi}^{-1}(B_R(0)/\Gamma)$ for $1 < R < \infty$ cover $\widetilde{X}$ and $\pi^{-1}(X^{\mathrm{sing}})$ are compact, there exists an R such that $\pi^{-1}(X^{\mathrm{sing}}) \subset \hat{\pi}^{-1}(B_R(0)/\Gamma)$. Hence,

$$V := \hat{\pi}^{-1}((\mathbb{C}^2 - \overline{B}_R(0))/\Gamma) \subset \pi^{-1}(X^{\mathrm{reg}}),$$

and as $\hat{\pi}$ is proper, $X - V = \hat{\pi}^{-1}(\overline{B}_R(0)/\Gamma)$ is compact. The biholomorphism $\pi\colon \pi^{-1}(X^{\mathrm{reg}}) \cong X^{\mathrm{reg}}$ therefore maps V onto an open subset $U \subset X^{\mathrm{reg}}$. The composition of the restrictions $\pi^{-1}\colon U \to V$ and $\hat{\pi}\colon V \to (\mathbb{C}^2 - \overline{B}_R(0))/\Gamma$ gives the required diffeomorphism $\phi\colon U \cong (\mathbb{C}^2 - \overline{B}_R(0))/\Gamma$. Finally,

$$\phi^{-1}(\mathbb{C}^2 - B_{R'}(0)))/\Gamma = \pi(\hat{\pi}^{-1}(\mathbb{C}^2 - B_{R'}(0))/\Gamma),$$

is closed in X for each $R' > R$ because $\hat{\pi}$ is continuous and π is a closed map (as it is proper and X is locally compact Hausdorff). □

Proof of Theorem 1.2. Let $\Gamma_0, \Gamma_1, \cdots, \Gamma_q \subset \mathrm{SU}(2)$ be finite subgroups and let K_i denote the Dynkin diagram associated with K_i for each $0 \le i \le q$. Assume that $K' := K_1 \sqcup K_2 \sqcup \cdots \sqcup K_q$ can be realized as a full subgraph of K_0. Identify the vertex set of K_0 with $\{1,2,\cdots,n\}$ for some $n \in \mathbb{N}$ and let $J \subset \{1,\cdots,n\}$ be the vertices of the subgraph K'. Let $\Phi \subset \mathbb{Z}^n$ be the root system associated with K and specify $\tau \in \mathbb{C}^n$ by $\tau_j = 1$ if $j \notin J$ and $\tau_j = 0$ otherwise. Then, τ is dominant and $\operatorname{supp}\tau$ is complementary to J. By Lemma 8.1, we have a root space decomposition

$$\Phi \cap \tau^{\perp} = \Phi_1 \cup \cdots \cup \Phi_q, \tag{8.1}$$

where Φ_i is the *ADE* root system associated with the Dynkin graph K_i for each $1 \le i \le q$.

Let Q be an extended Dynkin quiver with the underlying extended Dynkin graph corresponding to Γ_0 under the McKay correspondence (that is, $\widetilde{K_0}$). We identify the set of vertices with $\{0,1,\cdots,n\}$ such that the minimal imaginary root is given by $(1,d)\in\mathbb{Z}^{n+1}$ where $d\in\Phi\subset\mathbb{Z}^n$ is the maximal positive root. Then, $\lambda:=(-d\cdot\tau,\tau)\in\mathbb{C}^{n+1}$ satisfies $\lambda\cdot\delta=0$. Set $X:=\mathcal{M}_{(0,\lambda)}(Q,\delta)$. Then, according to Theorem 1.1, we may write $X^{\mathrm{sing}}=\{x_1,x_2,\cdots,x_q\}$, and for each $1\le i\le q$, there is an open neighborhood $x_i\subset V_i\subset X$ and a homeomorphism $\phi_i\colon V_i\to B_r(0)/\Gamma_i$, for some fixed r independent of i. Furthermore, each ϕ_i restricts to a diffeomorphism away from the singular point. Next, by Proposition 8.4, there is an open subset $U'\subset X^{\mathrm{reg}}$ with $X-U'$ compact and a diffeomorphism $\phi_0\colon U'\cong(\mathbb{C}^2-\overline{B}_{R'}(0))/\Gamma_0$ for some $R'>0$. In addition, $\phi_0^{-1}((\mathbb{C}^2-B_R(0))/\Gamma)$ is closed in X for each $R>R'$.

For part (1), we already know that X^{reg} is a smooth hyper-Kähler four-manifold. The space X is connected by Lemma 6.2, and in view of the above local models around the singularities, it is clear that $X^{\mathrm{reg}}=X-\{x_1,\cdots,x_q\}$ is connected as well.

For parts (2) and (3), fix $R>R'$ and let $C\subset X$ be the closed subset $\phi^{-1}((\mathbb{C}^2-B_R(0))/\Gamma)$. Since $C\subset X^{\mathrm{reg}}$ and X is Hausdorff, we may assume after possibly shrinking the V_i (and hence $r>0$) that the open sets $V_1,V_2,\cdots,V_q$ are pairwise disjoint and that $V_i\cap C=\varnothing$ for each i. Put

$$U_0:=\phi^{-1}((\mathbb{C}^2-\overline{B}_R(0))/\Gamma)\subset X^{\mathrm{reg}}\text{ and }\ U_i:=V_i-\{x_i\}\subset X^{\mathrm{reg}},\ 1\le i\le q.$$

Then, the open subsets $U_0,U_1,U_2,\cdots,U_q$ are pairwise disjoint, the complement of their union is compact in X^{reg} and we have diffeomorphisms $\phi_0\colon U_0\cong(\mathbb{C}^2-\overline{B}_R(0))/\Gamma$ and $\phi_i\colon U_i\cong(B_r(0)-\{0\})/\Gamma$ for $1\le i\le q$. We now decrease r and increase R slightly to ensure that each ϕ_i extends over a slightly bigger open set for each $0\le i\le q$. The proof of part (2) is completed by composing ϕ_0 with the evident diffeomorphism $(\mathbb{C}^2-\overline{B}_R(0))/\Gamma\cong(R,\infty)\times S^3/\Gamma\cong(0,\infty)\times S^3/\Gamma$ and by composing ϕ_i with the diffeomorphism

$$(B_r(0)-\{0\})/\Gamma_i\cong(0,r)\times S^3/\Gamma_i\cong(0,\infty)\times S^3/\Gamma_i,$$

where the final diffeomorphism includes a time reversal, for each $1\le i\le q$. Finally, $Y=X^{\mathrm{reg}}-\cup_{i=0}^{q}U_i$ is a compact manifold with boundary components S^3/Γ_i, $0\le i\le q$, because we arranged that ϕ_i actually extends to a diffeomorphism $\phi_i'\colon U_i'\cong(-t_0,\infty)\times S^3/\Gamma_i$ for some $t_0>0$ for each $0\le i\le q$. This completes the verification of part (3) and hence the proof. $\square$

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