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MULTI-ALTERNATING SIGN MATRICES*

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Abstract. We introduce a generalization of alternating sign matrices (ASMs) called multiASMs and develop some of their properties. Classes of multiASMs with specified row and column sum vectors R and S extend the classes of $(0,1)$-matrices with specified R and S. The special case when $R = S$ is a constant vector, in particular all 2's, is treated in more detail. We also investigate the polytope spanned by a class of multiASMs. Finally, we discuss the possibility of defining a Bruhat order on a class of multiASMs.

Key words. Multipermutation (matrix), Multi-alternating sign matrix (multiASM), k-regular ASM.

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1. Introduction. Let n and $m_1, m_2, \ldots, m_n$ be positive integers. A *multipermutation* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ is a vector of length $N = m_1 + m_2 + \cdots + m_n$ whose components are $1, 2, \ldots, n$ where each j occurs exactly m_j times $(j \leq n)$. The set of such multipermutations is denoted by $\mathcal{S}_n^{\times(m_1,m_2,\ldots,m_n)}$. Associated with σ is its $N \times n$ *incidence matrix* A_σ which is the $(0,1)$-matrix whose row i has a 1 in column σ_i and all other entries zero $(i \leq N)$. We call A_σ a *multipermutation matrix.* The row sums of this matrix all equal 1 while the column sums are $m_1, m_2, \ldots, m_n$, respectively. By a *partial row sum* in a matrix, we mean the sum of the entries in a row from the first entry up to some column. Partial column sums are defined similarly. When we display matrices, we sometimes replaces 0's with empty cells and ± 1 by simply the sign $+$ or $-$.

Alternating sign matrices, abbreviated to ASMs, are generalizations of $n \times n$ permutation matrices. We define an $(m_1, m_2, \ldots, m_n)$-*ASM* to be a generalization of a multipermutation matrix as follows:

A is an $N \times n$ $(0, \pm 1)$ such that the following hold:

(i) the row sums of A all equal 1,
(ii) the column sums of A are $m_1, m_2, \ldots, m_n$, respectively,
(iii) the partial row sums are 0 or 1, and
(iv) the partial column sums are between 0 and m_j, respectively.

From this, it follows that the matrix A does not have any -1's in the first row or the first column. An ordinary ASM is an $n \times n$ $(1, 1, \ldots, 1)$-ASM. Thus, $(m_1, m_2, \ldots, m_n)$-ASMs are to multipermutation matrices as ASMs are to permutation matrices. A multipermutation matrix is an $(m_1, m_2, \ldots, m_n)$-ASM without any -1's. We use the generic term *multiASM* for an $(m_1, m_2, \ldots, m_n)$-ASM corresponding to any $(m_1, m_2, \ldots, m_n)$.

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Example 1.1. Consider

$$A = \left[\begin{array}{c|c|c} & + & \\ \hline + & - & + \\ \hline & + & \\ \hline + & - & + \\ \hline & + & \\ \hline & + & \end{array}\right].$$

Then, A is a $(2,2,2)$-ASM as is easily checked. □

More generally, let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integral vectors. Then, we define an (R, S)-*ASM* to be an $m \times n$ $(0, \pm 1)$-matrix such that

(i) R is row sum vector of A,
(ii) S is the column sum vector of A,
(iii) the partial row sums of row i are between 0 and r_i, $(1 \le i \le m)$, and
(iv) the partial column sums of column j are between 0 and s_j, $(1 \le j \le n)$.

Thus, an $(m_1, m_2, \ldots, m_n)$-ASM is an (R, S)-ASM with $R = (1, 1, \ldots, 1)$ and $S = (m_1, m_2, \ldots, m_n)$.

The much-studied class $\mathcal{A}(R, S)$ of $(0, 1)$-matrices with row sum vector R and column sum vector S consists of the $(0, 1)$ matrices in the (R, S)-*ASM class* $\mathcal{A}^{\pm}(R, S)$ of (R, S)-ASMs, and thus, $\mathcal{A}(R, S) \subseteq \mathcal{A}^{\pm}(R, S)$. If $m = n$ and $R = S = (1, 1, \ldots, 1)$, then $\mathcal{A}^{\pm}(R, S)$ is the class $\mathcal{A}^{\pm}_n$ of $n \times n$ ASMs.

Example 1.2. Let $R = (5, 2, 5)$ and $S = (2, 3, 1, 1, 3, 2)$. Then, the following matrix A is an (R, S)-ASM.

$$A = \left[\begin{array}{c|c|c|c|c|c} + & + & + & + & + & \\ \hline + & + & - & - & + & + \\ \hline & + & + & + & + & + \end{array}\right].$$

As this example shows, unlike ordinary ASMs, an (R, S)-ASM may have consecutive 1's and consecutive -1's. □

Lemma 1.3. *Let $A = [a_{ij}]$ be an (R, S)-ASM. Then*

$$0 \le \textstyle\sum_{k=j}^{n} a_{ik} \le r_i \ \ (i \le m,\ j \le n),$$
$$0 \le \textstyle\sum_{k=i}^{m} a_{kj} \le s_j \ \ (i \le m,\ j \le n).$$

Moreover, the first and last nonzero in a row or column must be a 1. *In particular, A does not have any -1's in the first and last row and column. Finally, if some $r_i = 0$ (resp. some $s_j = 0$), then row i (resp. column j) is all* 0*'s.*

Proof. Consider a row $i \le m$, and let $j \le n$. Since $0 \le \sum_{k=1}^{j-1} a_{ik} \le r_i$, we have $0 \le \sum_{k=j}^{n} a_{ik} = r_i - \sum_{k=1}^{j-1} a_{ik} \le r_i$. The inequalities for columns are obtained similarly. If A has a -1 in the first or last nonzero in a row or column, then some partial row or column sum taken from left or right, or from top or bottom, would be negative, a contradiction. □

Next we describe some ways to construct classes of (R, S)-ASMs. First, let A be an $n \times n$ ASM and let J_k be the $k \times k$ matrix of all 1's. Then the $kn \times kn$ matrix $J_k \otimes A$ (the tensor product) obtained by

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replacing each 1 of J_k with A is an $kn \times kn$ (R,S)-ASM with $R = S = (k, k, \ldots k)$. Note also we can choose a different $n \times n$ ASM for each 1 of J_k; e.g., if $k = 3$,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{1.1}$$

where the A_{ij} are $n \times n$ ASMs.

Another (R,S)-ASM is obtained when A has the form in (1.1) where $A_{pq} = J_k$ for each $(p,q) \neq (2,2)$ and A_{22} is an arbitrary $(0,\pm1)$-matrix. This example also shows that *any* $(0,\pm1)$-matrix is a submatrix of some (R,S)-ASM, for suitable R and S, and the submatrix may even consist of consecutive rows and columns. We generalize this example in the next section.

To conclude this section, we mention an important basic property of ASMs that is not satisfied by multiASMs in general. If A is an $n \times n$ ASM, then the alternating property of the rows and columns implies that A is uniquely determined by its 0's. The following example shows that this property does not hold for multiASMs even when R and S are constant.

Example 1.4. Let $R = S = (2,2,2,2,2,2,2,2)$. Then, the following two (R,S)-ASMs have the same zero positions:

$$\left[\begin{array}{c|c|c|c|c|c|c|c} & & & + & & + & & \\ \hline & & + & & + & & & \\ \hline + & + & & & & & & \\ \hline + & & & x & y & & + & \\ \hline & + & & u & v & & & + \\ \hline & & & & & + & + & \\ \hline & & + & + & & & & \\ \hline & & & & + & + & & \end{array}\right]$$

where

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} + & - \\ - & + \end{bmatrix} \text{ or } \begin{bmatrix} - & + \\ + & - \end{bmatrix}.$$

□

The remaining paper is organized as follows. In Section 2, we discuss some elementary results about the classes $\mathcal{A}^{\pm}(R,S)$. Section 3 is concerned in more detail with the case where $R = S$ is a constant vector. In Section 4, we consider the polytope of convex combinations of multiASMs. Section 5 is concerning with decompositions of multiASMs. Finally, in Section 6 we discuss the possible extension of the Bruhat order on ASMs to multiASMs.

Notation: We identify n-tuples and corresponding column vectors. The row sum vector and column sum vectors of a matrix A are denoted respectively by $R(A)$ and $S(A)$. For a matrix, A $\nu^+(A)$ and $\nu^-(A)$ denote, respectively, the number of positive and negative entries in A. We set $\nu(A) = \nu^+(A) + \nu^-(A)$, the total number of nonzeros in A, the positions of which constitute the *pattern* of A. $\mathcal{P}_n$ denotes the set of permutation matrices of order n. J_{mn} is the all ones matrix of size $m \times n$.

2. Various results. Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be nonnegative integer vectors. We now study the class $\mathcal{A}^{\pm}(R,S)$, motivated by the following fundamental questions:

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(i) When is $\mathcal{A}^{\pm}(R,S)$ nonempty? Is there a characterization that resembles the majorization condition $S \preceq R^*$, with R^* equal to the conjugate vector of R, in the Gale-Ryser theorem for the nonemptiness of $\mathcal{A}(R,S)$? What are the relationships between $\mathcal{A}^{\pm}(R,S)$ and $\mathcal{A}(R,S)$ in general?

(ii) For any pair of matrices in $\mathcal{A}^{\pm}(R,S)$, can one find a sequence of certain elementary transformations that takes one matrix into the other where all of the intermediate matrices are also in $\mathcal{A}^{\pm}(R,S)$?

Concerning (ii), let A_1, A_2 be distinct matrices in a class $\mathcal{A}^{\pm}(R,S)$. Then, the matrix $\Delta = A_2 - A_1$ has all line sums zero and its entries lie in $\{0, \pm 1, \pm 2\}$. As for the class $\mathcal{A}(R,S)$, one may try to find a decomposition of Δ as

$$\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_k,$$

where each Δ_i has all line sums equal to 0 but its entries are in $\{0, \pm 1\}$, and the intermediary matrices between A_1 and A_2 satisfy

$$A_1 + \Delta_1 + \cdots + \Delta_s \in \mathcal{A}^{\pm}(R,S) \quad (s \le k),$$

so that for $s = k$ we obtain A_2. We refer to such a collection $\mathcal{E}_{m,n}$ of matrices Δ_i as *elementary* matrices. Consider the set of elementary matrices whose nonzeros form one of the two matrices

$$\left[\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right] \text{ or } \left[\begin{array}{c|c} - & + \\ \hline + & - \end{array}\right].$$

Then, adding one of these two elementary matrices to $A \in \mathcal{A}^{\pm}(R,S)$ is called an *interchange* on A, provided that the resulting matrix is also in $\mathcal{A}^{\pm}(R,S)$. As is well known, this set $\mathcal{I}_{m,n}$ of such elementary matrices allows one to transform any matrix in $\mathcal{A}(R,S)$ to any other such matrix. (We shall later also consider "interchanges" based on more complicated elementary matrices, such as those containing entries that are ± 2.)

Example 2.1. Let $R = S = (2,1,1)$ and consider the matrices $A_1, A_2 \in \mathcal{A}^{\pm}(R,S)$

$$A_1 = \left[\begin{array}{c|c|c} + & + & \\ \hline + & - & + \\ \hline & + & \end{array}\right], \quad A_2 = \left[\begin{array}{c|c|c} + & + & \\ \hline + & & \\ \hline & & + \end{array}\right].$$

Then, A_2 is obtained from A_1 (and vice versa) by an interchange. This example illustrates that $\mathcal{A}(R,S)$ will, in general, be a nonempty, strict subset of $\mathcal{A}^{\pm}(R,S)$.

Now consider $R = S = (3,1,3)$ and

$$A = \left[\begin{array}{c|c|c} + & + & + \\ \hline + & - & + \\ \hline + & + & + \end{array}\right].$$

Then, $A \in \mathcal{A}^{\pm}(R,S)$ and it is easy to see that this is the unique matrix in this class. Here ,$S \not\preceq R^* = (3,2,2)$, so $\mathcal{A}(R,S)$ is empty. □

This second example shows that a class $\mathcal{A}(R,S)$ may be empty and the class $\mathcal{A}^{\pm}(R,S)$ not empty. (A more general similar example is when A is a matrix obtained from the all ones matrix $J_{m,n}$ by changing a single entry to -1, and where this entry is not in the first or last row or column. These two examples emphasize that we may have

$$\emptyset \neq \mathcal{A}(R,S) \subset \mathcal{A}^{\pm}(R,S), \text{ and } \emptyset = \mathcal{A}(R,S),\ \mathcal{A}^{\pm}(R,S) \neq \emptyset.$$

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Observe that for $A \in \mathcal{A}^{\pm}(R,S)$, if A' is obtained from A by changing some of its 0's into 1's, then $A' \in \mathcal{A}^{\pm}(R',S')$ where $R' = R(A')$ and $S' = S(A')$.

A distinguishing property of classes $\mathcal{A}(R,S)$ and $\mathcal{A}^{\pm}(R,S)$ is the following: if $\mathcal{A}(R,S) \neq \emptyset$, then $\mathcal{A}(R',S') \neq \emptyset$ whenever R' and S' are permutations of R and S respectively, but we may have $\mathcal{A}^{\pm}(R,S) \neq \emptyset$ but $\mathcal{A}^{\pm}(R',S') = \emptyset$.

For the class $\mathcal{A}(R,S)$, the corresponding *maximal matrix* plays a special role. It is the $(0,1)$-matrix $\bar{A} = \bar{A}(R)$ with r_i leading ones on row i followed by $n - r_i$ zeros ($i \leq m$). Thus, its ones form a Ferrers diagram. (We assume R and S are monotone here; if not, we have a permuted form of a Ferrers diagram.) Thus, the row sum vector of $\bar{A}$ is R and its column sum vector is R^*, the conjugate vector of R.

Example 2.2. Let $m = n = 6$ and $R = (6,4,4,2,2,1)$. Then, the maximal matrix $\bar{A}$ is

$$\bar{A} = \left[\begin{array}{c|c|c|c|c|c} + & + & + & + & + & + \\ \hline + & + & + & + & & \\ \hline + & + & + & + & & \\ \hline + & + & & & & \\ \hline + & + & & & & \\ \hline + & & & & & \end{array}\right].$$

□

The maximal matrix $\bar{A}$ is the unique matrix in $\mathcal{A}(R,R^*)$. We now show that it is also the unique matrix in $\mathcal{A}^{\pm}(R,R^*)$.

Lemma 2.3. *The class $\mathcal{A}^{\pm}(R,R^*)$ contains a unique matrix, which is the maximal matrix $\bar{A}(R)$. So, $\mathcal{A}^{\pm}(R,R^*) = \mathcal{A}(R,R^*)$.*

Proof. We prove this by induction on the number $m+n$ of lines. If the last row of the maximal matrix has only zeros, so $r_m = 0$, then any $A \in \mathcal{A}^{\pm}(R,S)$ will also have zeros in the last row. Thus, we can delete the last row, and we are done, by induction. Otherwise, the last row of $\bar{A}$ is nonzero, so it contains a 1 in its first column. This means, by the Ferrers property, that the first column only contains 1's, so $s_1 = m$. Therefore, any $A \in \mathcal{A}^{\pm}(R,S)$ must also only have 1's in the first column (otherwise its column sum is less than m). Thus, we can delete the first column, and we are done, by induction. □

Next, we consider a concept of minimality. Let $A \in \mathcal{A}^{\pm}(R,S)$. Then, A is called $(R,S)^{\pm}$-*minimal* (shortened to just *minimal* when R and S are understood) if no interchange exists that results in a matrix in $\mathcal{A}^{\pm}(R,S)$ with fewer negative entries. In particular, if A is a $(0,1)$-matrix it is $(R,S)^{\pm}$-minimal. But not all minimal matrices are $(0,1)$-matrices. In the next proposition, minimal matrices with three rows having at least one -1 are characterized.

Proposition 2.4. *Let R and S be given where $m = 3$ and n is arbitrary. Let $A = [a_{ij}] \in \mathcal{A}^{\pm}(R,S)$. Then A is $(R,S)^{\pm}$-minimal if and only if for each j with $a_{2j} = 1$ the j'th column has only 1's.*

Proof. Assume first that A is $(R,S)^{\pm}$-minimal with at least one negative entry. By Lemma 1.3, these -1's can only be in the second row. Consider the first -1 in row 2, say it is in column j. Let $K = \{k < j : a_{2k} = 1\}$. Then, K must be nonempty. Moreover, for each $k \in K$, column k of A must only contain 1's. Otherwise we make an interchange in columns k and j and eliminate the -1 in position $(2,j)$, and the new matrix lies in $\mathcal{A}^{\pm}(R,S)$ and has fewer negative entries than A, contradicting minimality. Thus, each column before j

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has the form $(1,0,0)$, $(0,0,1)$, $(1,0,1)$ or $(1,1,1)$ (not the zero vector as we assume each $s_i > 0$). Next, we consider the columns after column j until (possible) another column $j' > j$ with $a_{2j'} = -1$. Repeating the interchange argument for columns $j+1, j+2, \ldots, j'$, we assure that these columns (apart from column j') are of the form $(1,0,0)$, $(0,0,1)$, $(1,0,1)$ or $(1,1,1)$. We continue doing this until the last column j^* with a negative entry. Then, the columns after j^* has the desired property; otherwise, we would do the same operation of an interchange as above. Thus, A has the property described in the theorem. Conversely, if A satisfies the condition of the theorem, it is easy to see that no interchange can remove a -1 and give a matrix in $\mathcal{A}^{\pm}(R,S)$. So, then A is minimal. $\square$

By Proposition 2.4, a minimal (R,S)-ASM with $m = 3$ has columns of the form

$$a_1 = (1,1,1), a_2 = (1,-1,1), a_3 = (1,0,1), a_4 = (1,0,0), a_5 = (0,0,1).$$

Moreover, if there are columns of type a_2, then there must be more columns of type a_1. From this, one can describe explicitly the possible, realizable R and S in case $m = 3$.

Example 2.5. Let $R = (5,1,5)$ and $S = (2,1,3,1,1,3)$. For the matrix $A \in \mathcal{A}^{\pm}(R,S)$ below, we show one interchange as described in the proof.

$$A = \left[\begin{array}{c|c|c|c|c|c} & + & + & + & + & + \\ \hline + & - & + & - & & + \\ \hline + & + & + & + & & + \end{array}\right] \to A_1 = \left[\begin{array}{c|c|c|c|c|c} + & & + & + & + & + \\ \hline & & + & - & & + \\ \hline + & + & + & + & & + \end{array}\right].$$

Then, no further interchange can remove the -1 in A_1 (while giving a matrix in $\mathcal{A}^{\pm}(R,S)$) so A_1 is minimal and therefore it satisfies the conditions in Proposition 2.4. Note that $R^* = (3,2,2,2,2,0)$ and $S \not\preceq R^*$, so $\mathcal{A}(R,S)$ is empty. $\square$

The minimal matrices in the previous result may be generalized as follows. Consider a matrix of the form

$$A = \begin{bmatrix} J_{m_1 n_1} & J_{m_1 n_2} & J_{m_1 n_3} \\ J_{m_2 n_1} & -J_{m_2 n_2} & J_{m_2 n_3} \\ J_{m_3 n_1} & J_{m_3 n_2} & J_{m_3 n_3} \end{bmatrix}, \tag{2.2}$$

where the sizes of the block matrices satisfy $m_2 \le \min\{m_1, m_3\}$ and $n_2 \le \min\{n_1, n_3\}$. These assumptions assure that A lies in $\mathcal{A}^{\pm}(R,S)$ for suitable R and S. A possible interchange to reduce the number of -1's involves a 2×2 submatrix. Any such submatrix has either a row or a column with equal (nonzero) entries, and then the interchange leads to a matrix outside $\mathcal{A}^{\pm}(R,S)$. Thus, these matrices are minimal (R,S)-ASMs.

Remark 2.6. Let $A = [a_{ij}] \in \mathcal{A}^{\pm}(R,S)$. Choose a position (i,j) with $a_{ij} = -1$ and where $i + j$ is minimal with this property. Then, by minimality, $a_{il} \in \{0,1\}$ for each $l < j$, and $a_{kj} \in \{0,1\}$ for each $k < i$. Let

$$I(i,j) = \{l < j : a_{il} = 1\}, \text{ and } J(i,j) = \{k < i : a_{kj} = 1\}.$$

By the partial line sum constraints both these sets are nonempty. Assume there exists $k \in I(i,j)$ and $l \in J(i,j)$ with $a_{kl} = 0$. Then, we can perform an interchange with rows k and i, and columns l and j, and then reduce the number of -1's by one. For instance, this is always possible for an ASM (then each of the sets $I(i,j)$ and $J(i,j)$ has cardinality one). Alternatively, and this can happen for (R,S)-ASMs, we have that

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$$a_{kl} = 1 \text{ for all } k \in I(i,j) \text{ and } l \in J(i,j).$$

This situation can be investigated further (when for all such "extreme" positions (i,j) this property holds). □

3. Regular (R,S)-ASMs. Let $m, n,$ and k be positive integers. Let R be the m-vector $\mathbf{k}_m = (k, k, \ldots, k)$ and S be the n-vector $\ell_n = (l, l, \ldots, l)$ where $mk = nl$. Then we call an (R,S)-ASM a (k,l)*-regular ASM*. If $k = l$ (so $m = n$) we abbreviate this to k*-regular ASM*. The 1-regular ASMs are the usual ASMs. The 2-regular ASMs are of some interest being "1-step up" from ASMs. If an ASM A has a permutation P set of places occupied by 0's, then $A + P$ is a 2-regular ASM.

Example 3.1. An $m \times n$ $(0,1)$-matrix with row sums equal to k and column sums equal to l is also a (k,l)-regular ASM. The following matrix is a $(2,3)$-regular ASM:

$$\begin{bmatrix} & + & & + \\ + & - & + & + \\ + & & + & \\ + & + & - & + \\ & + & + & \\ & + & + & \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} & & + & & + & \\ & + & - & + & + & \\ + & & + & - & & + \\ & + & + & & & \\ + & - & & + & & + \\ & + & & + & & \end{bmatrix}$$

is a 2-regular ASM. □

(We note that in [1] and [3] a different notion of a k-ASM is studied.)

We now confine our attention to 2-regular ASMs which already constitute an intriguing class of $(0,\pm 1)$-matrices.

For ASMs of order n, the maximum number of -1's is known, and it is attained for the diamond ASMs, for example, with $n = 5$ and $n = 6$:

$$\begin{bmatrix} & & + & & \\ & + & - & + & \\ + & - & + & - & + \\ & + & - & + & \\ & & + & & \end{bmatrix} \text{ and } \begin{bmatrix} & & + & & & \\ & + & - & + & & \\ + & - & + & - & + & \\ & + & - & + & - & + \\ & & + & - & + & \\ & & & + & & \end{bmatrix}.$$

In case n is even, the vertical reflection is also a diamond ASM. We consider a similar question for 2-regular ASMs.

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Let A be a 2-regular ASM. Then, each line (i.e., row or column) in A contains an even number of nonzeros. In fact, if there are p (-1)'s, there must be $(p+2)$ 1's, so the total number of nonzeros is $2p+2$. Define

$$N_n = \max\{\nu^-(A) : A \text{ is a 2-regular ASM of order } n\},$$

as the maximum number of -1's in this class.

Lemma 3.2. *For each n*

$$N_{n+2} \geq N_n.$$

Proof. If A is a 2-regular ASM of order n, then the direct sum $A \oplus J_2$ is a 2-regular ASM of order $n+2$, which gives the inequality. ∎

For $n = 2$ and $n = 3$, it is clear that 2-regular ASM cannot have any -1's. So, $N_2 = N_3 = 0$.

Example 3.3. The following are 2-regular ASMs of orders 4, 5 and 6, respectively:

$$A_4^* = \begin{bmatrix} & + & + & \\ + & + & - & + \\ + & - & + & + \\ & + & + & \end{bmatrix}, \quad A_5^* = \begin{bmatrix} & & + & + & \\ & + & + & - & + \\ + & + & - & + & \\ + & - & + & & + \\ & + & & + & \end{bmatrix},$$

$$A_6^* = \begin{bmatrix} & + & & & + & \\ + & - & + & + & - & + \\ & + & + & - & + & \\ & + & - & + & + & \\ + & - & + & + & - & + \\ & + & & & + & \end{bmatrix}. \tag{3.3}$$

□

For small n, we have the following.

Proposition 3.4. *The values of N_n for $n \leq 6$ are*

$$N_2 = N_3 = 0,\ N_4 = 2,\ N_5 = 3,\ N_6 = 6.$$

Moreover, the three 2-regular ASMs in (3.3) satisfy $\nu^-(A_n^*) = N_n$ for $n = 4, 5, 6$.

Proof. Each line (row or column) which is not the first or last will be called an *internal* line.

(i) $n = 4$: Then, each internal line has at most one -1, so $N_4 \leq 2 \cdot 1 = 2$. The matrix A_4^* satisfies $\nu^-(A_4^*) = 2$, so $N_4 = 2$.

(ii) $n = 5$: Then, each internal line has at most one -1, so $N_5 \leq 3 \cdot 1 = 2$. The matrix A_5^* satisfies $\nu^-(A_5^*) = 3$, so $N_5 = 3$.

(iii) $n = 6$: Then, each internal row has at most two -1's. An internal row with two -1's has four 1's, so it has a 1 in the first and last column. Since the first column has exactly two nonzeros, both 1, there can be at most two rows with two -1's in each. The remaining internal rows each have at most one -1. Thus, we obtain the upper bound

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$$N_6 \leq 2 \cdot 2 + 2 \cdot 1 = 6.$$

The matrix A_6^* satisfies $\nu^-(A_6^*) = 6$, so $N_6 = 6$. □

Determining the maximum number of nonzeros in an (R,S)-ASM, and those (R,S)-ASMs attaining it, is a difficult question in general. We consider this question for $n \times n$ 2-regular ASMs with n even.

Let $n \geq 4$ be even, and let $1 \leq k \leq n/2$. Let

$$D_i^+ = \{(i, k-i+1), (i+1, k-i+2), \ldots, (n-k+i, n-i+1)\} \ \ (1 \leq i \leq k).$$

This set consists of $n-k+1$ positions joining $(i, k-i+1)$ and $(n-k+i, n-i+1)$ in a line segment parallel to the main diagonal. Next, define

$$D_i^- = \{(i, k-i+2), (i+1, k-i+3), \ldots, (n-k+i-1, n-i+1)\} \ \ (2 \leq i \leq k),$$

as $(n-k)$ positions, also parallel to the main diagonal. Let (initially) $A = A^{(k,n)} = [a_{ij}]$ be the $(0, \pm 1)$-matrix where (i) $a_{rs} = 1$ for each $(r,s) \in \bigcup_i D_i^+$, (ii) $a_{rs} = -1$ for each $(r,s) \in \bigcup_i D_i^-$. Let N be the set of nonzero positions just defined. The remaining nonzeros of A are defined by reflection with respect to the middle vertical line in the matrix: for each $(i,j) \in N$ where the reflection (i,j') is undefined let $a_{ij'} = a_{ij}$. All other entries are zero. The nonzeros in $A^{(k,n)}$ form four similar "rectangles," obtained by rotation, plus a central "square." Also, in every line parallel to the main diagonal all entries are equal, except for leading and trailing zeros. The next example illustrates this, where the four similar rectangles are shaded.

Example 3.5. Let $n = 16$ and $k = 6$. Then,

$$A^{(6,16)} = \left[\begin{array}{cccccccccccccccc}
 & & & & & + & & & & & + & & & & & \\
 & & & & + & - & + & & & + & - & + & & & & \\
 & & & + & - & + & - & + & + & - & + & - & + & & & \\
 & & + & - & + & - & + & - & + & + & - & + & - & + & & \\
 & + & - & + & - & + & - & + & - & + & + & - & + & - & + & \\
+ & - & + & - & + & - & + & - & + & - & + & + & - & + & - & + \\
 & + & - & + & - & + & - & + & - & + & - & + & + & - & + & \\
 & & + & - & + & - & + & - & + & - & + & - & + & + & & \\
 & & + & + & - & + & - & + & - & + & - & + & - & + & & \\
 & + & - & + & + & - & + & - & + & - & + & - & + & - & + & \\
+ & - & + & - & + & + & - & + & - & + & - & + & - & + & - & + \\
 & + & - & + & - & + & + & - & + & - & + & - & + & - & + & \\
 & & + & - & + & - & + & + & - & + & - & + & - & + & & \\
 & & & + & - & + & - & + & + & - & + & - & + & & & \\
 & & & & + & - & + & & & + & - & + & & & & \\
 & & & & & + & & & & & + & & & & &
\end{array}\right].$$

It will follow from the next theorem that this matrix has the largest number of nonzeros among all 2-regular ASMs of order 16. □

Let $\sigma(A)$ be the number of nonzeros in a matrix A, and define

$$\hat{\sigma}_n^{(2)} = \max\{\sigma(A) : A \text{ is a 2-regular ASM of order } n\}.$$

Theorem 3.6. *Let $n \geq 4$ be even, and let $1 \leq k \leq n/2$. Then, the matrix $A^{(k,n)}$ is a symmetric 2-regular ASM, and*

$$\sigma(A^{(1,n)}) < \sigma(A^{(2,n)}) < \cdots < \sigma(A^{(k^*,n)}),$$

where $k^* = k^*(n) = \lceil n/3 \rceil$. Moreover,

$$\sigma(A^{(k^*,n)}) = \hat{\sigma}_n^{(2)} = n^2 - 2k^*(k^*+1) - 4(n/2 - k^*)(n/2 - k^* + 1), \tag{3.4}$$

so $A^{(k^*,n)}$ has the largest number of nonzeros in the class of $n \times n$ 2-regular ASMs.

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Proof. The general structure is illustrated in Example 3.5. Consider a row in the matrix. Its nonzeros consists of three consecutive parts: (i) an alternating sequence of 1's and -1's, starting and ending with a 1, (ii) an alternating sequence of 1's and -1's, starting with a 1 and ending with a -1, or vice versa, (iii) an alternating sequence of 1's and -1's, starting and ending with a 1. It follows that the row sum is 2, and that all partial row sums are 0, 1 or 2. It is easy to check that $A^{(k,n)}$ is symmetric. From these facts, we conclude that $A^{(k,n)}$ is a symmetric 2-regular ASM.

Next, observe that the support of $A^{(k,n)}$ is rotation symmetric in the four blocks obtained by the first and last $n/2$ rows and columns, so each block has the same number of nonzeros. We therefore count the number of zeros in the leading submatrix of $n/2$ rows and columns. This is

$$\sum_{i=1}^{k-1} i + 2\sum_{i=1}^{n/2-k} i = (3/2)k^2 - (n+3/2)k + (n/2)(n/2+1).$$

For fixed n, this gives a convex quadratic polynomial in k with (real) minimum for $k = n/3 + 1/2$. Thus, the number of nonzeros in $A^{(k,n)}$ is a strictly decreasing function of k, from $k = 1$ up to $k = k^*(n) = \lceil n/3 \rceil$.

It remains to prove (3.4). Observe, for a general 2-regular ASM, the maximum number of nonzeros in row i is $4i-2$; this is a consequence of the partial line sum constraints. A similar bound holds for the last rows, and the first and last columns. Let $A^* = A^{(k^*,n)}$ where $k^* = \lceil n/3 \rceil$. The first row in A^* has a 1 in positions k^* and $n-k^*+1$ and otherwise zeros. Then, row i has $4i-2$ nonzeros for $i = 1, 2, \ldots, n/2-k^*$. So, in each of these rows we have the mentioned maximum number of nonzeros. Let α denote the total number of nonzeros in these $n/2-k^*$ rows. In row $n/2-k^*+1$, the nonzeros are consecutive. Moreover, the submatrix A' obtained from A^* be deleting the first and last $n/2-k^*$ rows and columns has *only* nonzeros. Let V denote the set of positions in A^* that are not in the submatrix A'. It is clear that every 2-regular ASM has at most 4α nonzeros in the positions in V, and A^* has exactly 4α nonzeros in these positions. Moreover, in the remaining positions, those of the submatrix A', A^* has only nonzeros. It follows that $\sigma(A^{(k^*,n)}) = \hat{\sigma}_n^{(2)}$, as desired. Moreover, by counting the nonzeros in A^*, we compute the indicated expression for $\hat{\sigma}_n^{(2)}$. ∎

Example 3.7. Let $n = 10$. Below are the matrices $A^{(k,10)}$ for $k = 1, 2, 3, 4, 5$.

$$\begin{bmatrix}
+ & & & & & & & & & + \\
 & + & & & & & & & + & \\
 & & + & & & & & + & & \\
 & & & + & & & + & & & \\
 & & & & + & + & & & & \\
 & & & & + & + & & & & \\
 & & & + & & & + & & & \\
 & & + & & & & & + & & \\
 & + & & & & & & & + & \\
+ & & & & & & & & & +
\end{bmatrix}, \quad
\begin{bmatrix}
 & + & & & & & & & + & \\
+ & - & + & & & & & + & - & + \\
 & + & - & + & & & + & - & + & \\
 & & + & - & + & + & - & + & & \\
 & & & + & - & + & + & & & \\
 & & & + & + & - & + & & & \\
 & & + & - & + & + & - & + & & \\
 & + & - & + & & & + & - & + & \\
+ & - & + & & & & & + & - & + \\
 & + & & & & & & & + &
\end{bmatrix},$$

$$\begin{bmatrix}
 & & + & & & & & + & & \\
 & + & - & + & & & + & - & + & \\
+ & - & + & - & + & + & - & + & - & + \\
 & + & - & + & - & + & + & - & + & \\
 & & + & - & + & - & + & + & & \\
 & & + & + & - & + & - & + & & \\
 & + & - & + & + & - & + & - & + & \\
+ & - & + & - & + & + & - & + & - & + \\
 & + & - & + & & & + & - & + & \\
 & & + & & & & & + & &
\end{bmatrix}, \quad
\begin{bmatrix}
 & & & + & & & + & & & \\
 & & + & - & + & + & - & + & & \\
 & + & - & + & - & + & + & - & + & \\
+ & - & + & - & + & - & + & + & - & + \\
 & + & - & + & - & + & - & + & + & \\
 & + & + & - & + & - & + & - & + & \\
+ & - & + & + & - & + & - & + & - & + \\
 & + & - & + & + & - & + & - & + & \\
 & & + & - & + & + & - & + & & \\
 & & & + & & & + & & &
\end{bmatrix},$$

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$$\begin{bmatrix} & & & & + & + & & & & \\ & & & + & - & + & + & & & \\ & & + & - & + & - & + & + & & \\ & + & - & + & - & + & - & + & + & \\ + & - & + & - & + & - & + & - & + & + \\ + & + & - & + & - & + & - & + & - & + \\ & + & + & - & + & - & + & - & + & \\ & & + & + & - & + & - & + & & \\ & & & + & + & - & + & & & \\ & & & & + & + & & & & \end{bmatrix}.$$

The following table shows the number of zeros in the leading 5×5 submatrix as a function of k:

k	1	2	3	4	5
#	20	13	9	8	10

Here, $k^* = k^*(10) = \lceil 10/3 \rceil = 4$, so the matrix $A^{(4,10)}$ above has the maximum number of nonzeros among 2-regular ASMs of order 10. Additional information concerning dense ASMs is in [11]. □

Theorem 3.6 determines the maximum number of nonzeros in a 2-regular ASM when n is even. A similar construction should be possible in the case when n is odd but we do not pursue it here. For instance, let $n = 7$. Then, the following 2-regular ASM has the maximum number of nonzeros

$$A = \begin{bmatrix} & & + & + & & & \\ & + & - & + & + & & \\ + & - & + & - & + & + & \\ & + & - & + & - & + & + \\ & + & + & - & + & - & + \\ + & - & + & + & - & + & \\ & + & & & + & & \end{bmatrix}$$

among 2-regular ASMs of this order. To see this, note that the four 1's in the boundary (first and last row and column) is the maximum. In the remaining positions there is only one 0, and it is not possible to avoid zeros in this internal block, as every line has an even number of nonzeros.

Remark 3.8. The construction for n even can be generalized to k-regular ASMs in general. In fact, in the construction of the maximizing matrix $A^{(k^*,n)}$ the first rows contains the top triangle of a diamond ASM, repeated twice. In general, we can repeat this k times and otherwise use the same construction. Similar arguments should prove that this is a maximizer, but again we omit going into the details.

Finally, note that, in contrast to the fact that the pattern of an ASM determines the ASM uniquely, the pattern of a 2-regular ASM does not, in general, determine the 2-regular ASM. For example, for $m = n = 4$, the following two 2-regular ASMs have the same pattern:

$$\begin{bmatrix} & + & + & \\ + & - & + & + \\ + & + & - & + \\ & + & + & \end{bmatrix} \text{ and } \begin{bmatrix} & + & + & \\ + & + & - & + \\ + & - & + & + \\ & + & + & \end{bmatrix}.$$

So we simply negate a 2×2 submatrix which performs an interchange. But this cannot be done arbitrarily. So one could ask which patterns of 2-regular ASMs determine the 2-regular ASM uniquely? □

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4. The (R,S)-ASM polytope. Recall that the class of (R,S)-ASMs is denoted by $\mathcal{A}^{\pm}(R,S)$. Define

$$\mathcal{P}_{\mathcal{A}^{\pm}(R,S)} = \mathrm{conv}\,(\mathcal{A}^{\pm}(R,S)),$$

as the convex hull of the matrices in $\mathcal{A}^{\pm}(R,S)$. Since $\mathcal{A}^{\pm}(R,S)$ is a finite set, $\mathcal{P}_{\mathcal{A}^{\pm}(R,S)}$ is a polytope within the space $M_{m,n}$ of $m \times n$ real matrices, and we call it the *(R,S)-ASM polytope*. Clearly, this polytope is contained in the polyhedron described by the partial line sum constraints of (R,S)-ASMs and the trivial inequalities $-1 \leq a_{ij} \leq 1$ for each i,j. A surprising result is that these two sets are actually equal, as the next theorem says.

THEOREM 4.1. *$\mathcal{P}_{\mathcal{A}^{\pm}(R,S)}$ equals the set of real matrices $A = [a_{ij}] \in M_{m,n}$ satisfying*

$$\begin{aligned} 0 \leq \textstyle\sum_{k=1}^{j} a_{ik} \leq r_i & \quad (i \leq m,\ j \leq n), \\ 0 \leq \textstyle\sum_{k=1}^{i} a_{kj} \leq s_j & \quad (i \leq m,\ j \leq n), \\ -1 \leq a_{ij} \leq 1 \quad\quad & \quad (i \leq m,\ j \leq n). \end{aligned} \tag{4.5}$$

Proof. Let P^* be the bounded polyhedron consisting of every *real* matrix $A = [a_{ij}] \in M_{m,n}$ satisfying the inequalities in (4.5). As remarked above, it suffices to show that $P^* \subseteq \mathcal{P}_{\mathcal{A}^{\pm}(R,S)}$, and we do this by showing that every extreme point of P^* is integral, and therefore, it must be an (R,S)-ASM. To do this, we introduce some concepts.

Let $A = [a_{ij}] \in M_{m,n}$ be an extreme point of P^*. Let $i \leq m$ and let $J^{=}(i)$ be the set of those $j \leq n$ for which $\sum_{k=1}^{j} a_{ik}$ equals either 0 or r_i. Thus, $J^{=}(i)$ indicates the active partial row sum constraints in row i and consists of integers

$$1 \leq j_1 < j_2 < \cdots < j_t = n.$$

This defines consecutive, disjoint intervals

$$J(i,s) = \{j : j_{s-1} + 1 \leq j \leq j_s\} \ \ (1 \leq s \leq t),$$

where we let $j_0 = 0$. We call the corresponding sets of positions $\{(i,j) : j \in J(i,s)\}$ in row i an *A-interval* in row i. These give a partition of the positions in row i. For each s with $1 \leq s \leq t$, we have

$$\sum_{j \in J(i,s)} a_{ij} \in \{0, \pm r_i\}.$$

This follows easily from the definition of $J^{=}(i)$. In particular, this sum is an integer. A real number $-1 \leq a \leq 1$ will be called *fractional* if it is not an integer.

Claim 1*: If an A-interval in a row contains a fractional entry, then this interval contains at least two fractional entries.*

This holds because the sum of the entries in an A-interval is an integer. In a similar way, we can consider the active partial column sum constraints, and define A-intervals in columns. Therefore, we also have the following.

Claim 2*: If an A-interval in a column contains a fractional entry, then this interval contains at least two fractional entries.*

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We shall prove that the extreme point A is integral, i.e., that all entries are integers. So, assume, to the contrary, that some entry $a_{i_1j_1}$ is fractional. By Claim 1, there is another fractional entry $a_{i_1j_2}$ such that (i_1, j_1) and (i_1, j_2) are in the same A-interval in row i_1. Then, by Claim 2 there is another fractional entry $a_{i_2j_2}$ such that (i_1, j_2) and (i_2, j_2) are in the same A-interval in column j_2. By repeating this argument, we conclude that there is a cycle C of positions with non-integral entries. Let now A_1 be obtained from A by perturbing the entries in C as follows: along C add $\pm\epsilon$ in an alternating way where ϵ is a suitably small positive number. The alternating property assures that A_1 has the same line sums as A. Moreover, all changes are done within A-intervals; this means that for all active partial line sum constraints the sum is unchanged, so these constraints still hold. It follows that A_1 is an (R, S)-ASM. Let A_2 be obtained from A_1 by changing the sign before each ϵ. The A_2 is also an (R, S)-ASM. Moreover,

$$A = (1/2)A_1 + (1/2)A_2.$$

This contradicts that A is an extreme point, and it proves that every extreme point must be integral, and the proof is complete. □

This theorem opens up the possibility of studying faces, in particular facets, of the (R, S)-ASM polytope $\mathcal{P}_{\mathcal{A}^\pm(R,S)}$. This can be done by replacing some subset of the inequalities in (4.5) by corresponding equalities.

In the ASM case, i.e., when R and S are all ones vectors of length n, then every ASM of order n is an extreme point of the ASM polytope $\operatorname{conv}(\mathcal{A}_n)$ [3]. For more general R and S, this may not be the case, as the following example shows.

Example 4.2. Let $m = n = 4$ and $R = S = (2, 2, 2, 2)$. Then the following three matrices are (R, S)-ASMs

$$A = \left[\begin{array}{c|c|c|c} & + & + & \\ \hline + & & & + \\ \hline + & & & + \\ \hline & + & + & \end{array}\right], \quad A_1 = \left[\begin{array}{c|c|c|c} & + & + & \\ \hline + & - & + & + \\ \hline + & + & - & + \\ \hline & + & + & \end{array}\right], \quad A_2 = \left[\begin{array}{c|c|c|c} & + & + & \\ \hline + & + & - & + \\ \hline + & - & + & + \\ \hline & + & + & \end{array}\right].$$

Moreover, $A = (1/2)A_1 + (1/2)A_2$, which proves that A is not an extreme point of $\mathcal{P}_{\mathcal{A}^\pm(R,S)}$. □

A $(0, 1)$-matrix which is a 2-regular ASM will be called a 2-*permutation matrix*. Any such matrix of order n is the sum of two disjoint $n \times n$ permutation matrices. In light of the previous example, where A is a 2-permutation matrix, we now consider which 2-permutation matrices are extreme points of the polytope $\mathcal{P}_{\mathcal{A}^\pm(R,S)}$ equal to the convex hull of the 2-regular ASMs. Let A be a 2-permutation matrix of order n. Define an associated graph G_A with vertex set consisting of the positions (i, j) between the two 1's in *both* row i and column j $(i, j \leq n)$. The edges correspond to adjacent positions vertical or horizontal in the usual way. This is a bipartite graph. The following theorem characterizes those 2-permutation matrices which are extreme points of $\mathcal{P}_{\mathcal{A}^\pm}(2, 2)$.

Theorem 4.3. *Let A be a 2-permutation matrix of order n. Then A is an extreme point of $\mathcal{P}_{\mathcal{A}^\pm}(2, 2)$ if and only if G_A is a forest.*

Proof. Assume first that G_A is not a forest, so it contains a cycle C. The consecutive vertices in C are in the same line (row or column), and they are both between the two 1's of that line. Therefore the matrix A' obtained from A by replacing entries in C by alternating ± 1 is a 2-regular ASM. The same is true for the matrix A'' obtained from A' by changing signs on entries in C. As a result $A = (1/2)A' + (1/2)A''$, so A is not an extreme point.

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To prove the opposite implication, assume that G_A is a (nontrivial) forest. Moreover, let $A = \sum_{j=1}^{t} \lambda_j A_j$ be a convex combination of 2-regular ASMs A_j $(j \le t)$, so $\lambda_j > 0$ and $\sum_j \lambda_j = 1$. It suffices to show that $A_j = A$ $(j \le t)$. First observe that in any line each A_j must have only zeros outside the interval between the two 1's of A (otherwise one A_j would have a leading -1). Moreover, in each A_j the entry corresponding to a 1 in A must also be 1 (as a positive convex combination of entries ≤ 1 is 1 only if all those entries are 1). Next, consider the entries corresponding to the vertices of G_A. Since the graph is a nontrivial forest there is a pendent vertex, so a line with only one vertex in G_A, so A contains (say) a 1 in positions (i,j) and $(i,j+2)$ (or similar for a column). Then, each A_j has a zero in $(i,j+1)$; otherwise, its line sum would not be 2. We can delete the vertex $(i,j+1)$ from the graph G_A and the new graph is also a forest, so we can repeat this argument. Eventually, we can conclude that every entry corresponding to a vertex in G_A contains a zero in every A_j $(j \le t)$. This shows that $A_j = A$ $(j \le t)$, and therefore, A is an extreme point of the polytope. $\square$

Example 4.4. Let $n = 5$ and consider the 2-permutation matrix

$$A = \left[\begin{array}{c|c|c|c|c} & + & & & + \\ \hline & & + & + & \\ \hline + & * & * & + & \\ \hline & + & * & & + \\ \hline + & & + & & \end{array}\right].$$

The stars indicate the vertices in G_A, and we see that G_A does not have any cycles; in fact, G_A is a path of three vertices. Therefore, A is an extreme point of $\mathcal{P}_{\mathcal{A}^{\pm}}(2,2)$.

Below is a 2-permutation matrix B which is not an extreme point of $\mathcal{P}_{\mathcal{A}^{\pm}}(2,2)$. Its graph is a cycle of 4 vertices. B is a convex combination of the two matrices to the right (1/2 times each).

$$B = \left[\begin{array}{c|c|c|c|c} & + & & + & \\ \hline + & & & & + \\ \hline + & & & & + \\ \hline & + & + & & \\ \hline & & + & + & \end{array}\right] \to \left[\begin{array}{c|c|c|c|c} & + & & + & \\ \hline + & - & & + & + \\ \hline + & + & & - & + \\ \hline & + & + & & \\ \hline & & + & + & \end{array}\right], \left[\begin{array}{c|c|c|c|c} & + & & + & \\ \hline + & + & & - & + \\ \hline + & - & & + & + \\ \hline & + & + & & \\ \hline & & + & + & \end{array}\right].$$

$\square$

More generally, one might try to determine which (R,S)-ASMs are extreme points of the convex polytope $\mathcal{P}_{\mathcal{A}^{\pm}(R,S)}$. As the above example shows they may be a proper subset of $\mathcal{A}^{\pm}(R,S)$. This, however, seems more difficult to answer than in the special case above. The reason is that finding positions for a cycle such that the addition gives 2-regular ASMs is more complicated in general.

5. Decomposition. In this section, we study decompositions of 2-regular ASMs and, more generally, decompositions of k-regular ASMs. The following example illustrates the question considered.

Example 5.1. Below is a 2-regular ASM A which can be decomposed as sum of two ASMs, indicated by the light and dark shading:

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$$A = \begin{bmatrix} & & & + & + & \\ & + & & + & & \\ + & - & + & - & + & + \\ & + & & + & - & + \\ + & & + & & & \\ & + & & & + & \end{bmatrix}.$$

In this case, the two matrices are disjoint, i.e., they have disjoint supports. $\square$

We now state our main result on decompositions.

THEOREM 5.2. *Let A be an $n \times n$ k-regular ASM. Then, A may be written as a sum of k ASMs, i.e.,*

$$A = A_1 + A_2 + \cdots + A_k,$$

for suitable $n \times n$ ASMs $A_1, A_2, \ldots, A_k$.

Note that, in this theorem, the supports of the ASMs may intersect. The proof of the theorem is now given, and it basically consists in applying a result in our previous paper [5] which we now explain.

Let $B = [b_{ij}]$ be an $n \times n$ nonnegative matrix. Then, an $n \times n$ matrix $A = [a_{ij}]$ is *sum-majorized* by B if

$$\begin{array}{rl} 0 \le \sum_{j'=1}^{j} a_{ij'} \le b_{ij} & (1 \le i, j \le n), \\ 0 \le \sum_{i'=1}^{i} a_{i'j} \le b_{ij} & (1 \le i, j \le n), \\ \sum_{j=1}^{n} a_{ij} = b_{in} & (1 \le i \le n), \\ \sum_{i=1}^{n} a_{ij} = b_{nj} & (1 \le j \le n). \end{array} \tag{5.6}$$

Then, the following lemma clearly holds.

LEMMA 5.3. *If A is an k-regular ASM of order n, then A is sum-majorized by $B = kJ$, where J is the $n \times n$ all ones matrix.*

Proof. This follows from the defining inequalities of a k-ASM, because for $B = kJ$ all entries b_{ij} in (5.6) are equal to k. $\square$

Note that in the notion of sum-majorization for $B = kJ$, the entries of the matrix A are not required to be between -1 and 1; it only follows from the constraints (5.6) that entries are between $-k$ and k. (We remark that in the paper [5], we used the term k-ASM with a different meaning than k-regular ASM as used here.)

The following two theorems are proved in [5].

THEOREM 5.4. [5] *Let $B = [b_{ij}]$ be an $n \times n$ nonnegative matrix. The convex hull of all integral matrices that are sum-majorized by B equals the set of real matrices $A = [a_{ij}]$ satisfying the linear system in (5.6).*

THEOREM 5.5. [5] *Let A be a square matrix and k a positive integer. Then, A is sum-majorized by $B = kJ$ if and only if there exist ASMs $A_1, A_2, \ldots, A_k$ such that*

$$A = A_1 + A_2 + \cdots + A_k.$$

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We remark that the proof of Theorem 5.4 uses a certain linear isomorphism to obtain a linear system with totally unimodular (TU) coefficient matrix, and then applying different results from the theory of TU-matrices, see [5] for details.

Proof of Theorem 5.2. This follows from Lemma 5.3 and Theorem 5.5. ∎

Note that a converse of Theorem 5.2 is not true. The sum of ASMs can easily have entries different from $0, \pm 1$. If, however, one sums ASMs with (pairwise) disjoint support, then the resulting matrix has all partial line sums between 0 and k, so it is a k-regular ASM. This leads to the question of determining which k-regular ASMs can be decomposed as sums of k ASMs with pairwise disjoint supports.

The next example shows that that there are 2-regular ASMs that cannot be decomposed as a sum of two ASMs with disjoint supports if one of the ASMs is assumed to be the simplest possible ASM, that is, a permutation matrix.

Example 5.6. Consider the following 2-regular ASM

$$A = \begin{bmatrix} & & & + & & + & & \\ & + & & & & + & & \\ & + & & + & & - & + & \\ + & - & + & - & + & + & - & + \\ & + & & + & & - & + & \\ & & & & & + & & + \\ + & & & & + & & & \\ & & + & & & & + & \end{bmatrix}.$$

Then, it is impossible to decompose A into $A = B + P$ where A is an ASM and P is a permutation matrix where B and P have disjoint supports. To see this, consider row 5 in A. Say that P has a 1 in position $(5, 2)$ (where A has a 1). Then the ASM B must have nonzeros as the remaining nonzeros in column 2 (as P is a permutation matrix). This is impossible as the last nonzero in column 2 would be negative. A similar argument shows that P cannot have a 1 in any of the positions of a 1 in row 5 in A. So, P cannot have a 1 in row 5, a contradiction. This completes the argument. However, the matrix A may be written is a sum of two pattern disjoint ASMs as follows

$$A = \begin{bmatrix} & & & + & & & & \\ & & & & & + & & \\ & + & & & & - & + & \\ + & - & & & & + & - & + \\ & + & & & & - & + & \\ & & & & & + & & \\ & & & & + & & & \\ & & + & & & & & \end{bmatrix} + \begin{bmatrix} & & & & & + & & \\ & + & & & & & & \\ & & & + & & & & \\ & & + & - & + & & & \\ & & & + & & & & \\ & & & & & & & + \\ + & & & & & & & \\ & & & & & & + & \end{bmatrix}.$$

This decomposition is found by starting from the -1 is row 5 which "belongs" to one matrix; many other entries follow from this, until the final selections can be done by inspection. ∎

A related question concerns extending an ASM to a 2-regular ASM by adding a pattern disjoint ASM. In the following result, we consider this in the case when we add a permutation matrix.

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Theorem 5.7. *Let A be an $n \times n$ ASM. Then, there exists a permutation matrix P whose support is disjoint from that of A such that $A + P$ is a 2-regular ASM if and only if A does not have a row or column with only nonzeros. This condition is always satisfied when n is even.*

Proof. Clearly, if A has a line (row or column) with only nonzeros, then it is impossible to extend with a permutation matrix as described. To prove the converse, assume that $A = [a_{ij}]$ does not have a line with only nonzeros. In our proof, we repeatedly use the following elementary fact, which is Lemma 2.1 in [8]: row $i \leq \lfloor n/2 \rfloor$ contains at most $(2i-1)$ nonzeros $(i \leq k)$; row i is *full* if it contains $(2i-1)$ nonzeros. The same bound holds when we count rows from the bottom of the matrix. A row with the number of nonzeros equal to this upper bound will be called *full*.

First suppose that n is even, say $n = 2k$. We construct the positions of the ones in a permutation matrix successively by, starting from the middle and considering pairs of rows, namely row $k - p$ and $k + 1 + p$ for $p = 0, 1, \ldots, k - 1$. Define $I(p) = \{k - p, k + 1 + p\}$. Let $0 \leq p \leq k - 1$ and assume that we have selected positions in rows $I(0), I(1), \ldots, I(p-1)$ so that they no two are in the same row or column and they avoid all nonzeros of A. As noted in Section 1, an ASM is uniquely determined by the set of the positions of its zeros. Hence if $p = 0$, there are positions (k, j_k) and $(k+1, j_{k+1})$ with $j_k \neq j_{k+1}$ and where A has a zero in these two positions; otherwise the rows would have to be equal, and this contradicts the ASM property. Next, assume that $p > 0$. Each of the two rows in $I(p)$ (in A) contains at most $2(k-p) - 1 = n - 2p - 1$ nonzeros. There are $2p$ columns that must be avoided due to previous selection of positions. If all the previously considered rows and the rows in $I(p)$ are full, then the two rows in $I(p)$ must be different (otherwise the ASM property would be violated). If the two rows in $I(p)$ are full, but some previously selected row is not full, then we can again select two distinct columns, possibly after an interchange of selected positions. Finally, if one of the rows in $I(p)$ is not full, there are two available columns, as desired.

Now suppose that n is odd, say $n = 2k + 1$. By assumption row $k + 1$ of A has a zero, and we choose such a position $(k + 1, j_{k+1})$. The rest of the construction is almost identical to the even case above, so it is omitted. □

Example 5.8. We illustrate the proof of Theorem 5.7 with an 8×8 ASM. First, we note that if rows 4 and 5 are full, then they must be:

$$\left[\begin{array}{c|c|c|c|c|c|c|c} + & - & + & - & + & - & + & 0 \\ \hline 0 & + & - & + & - & + & - & + \end{array}\right] \text{ or its horizontal reflection.}$$

Then, the 6×6 submatrix of A obtained by deleting rows 4 and 5 and columns 1 and 8 is a 6×6 ASM, and in this case, we could proceed by induction, choosing the two 0's in columns 1 and 8. (In fact, if we only know that columns 1 and 8 have 0's in rows 4 and 5, we could proceed by choosing the two 0's.) If only one of row 4 and 5 is not full, then we have something like:

$$\left[\begin{array}{c|c|c|c|c|c|c|c} + & - & + & - & + & - & + & 0 \\ \hline 0 & + & 0 & 0 & - & + & 0 & 0 \end{array}\right] \text{ or its horizontal reflection.}$$

We could again choose the two 0's in columns 1 and 8 and proceed. □

The following general matrix decomposition is well-known and follows, for example, from Theorem 4.4.3 in [9].

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Theorem 5.9. *Let A be a an $n \times n$ nonnegative integral matrix with every row and column sum equal to k. Then there are permutation matrices $P_1, P_2, \ldots, P_k \in \mathcal{P}_n$ such that*

$$A = P_1 + P_2 + \cdots + P_k.$$

Corollary 5.10. *Let $1 \leq k \leq n$ and let A be a k-regular ASM of order n. Then there are permutations matrices $P_1, P_2, \ldots, P_k \in \mathcal{P}_n$ such that the following decomposition holds*

$$A + J_n = P_1 + P_2 + \cdots + P_{k+n}. \tag{5.7}$$

Proof. The matrix $A + J_n$ is nonnegative and integral. Moreover, it has all row and column sums equal to $k + n$. The result now follows from Theorem 5.9. ☐

A converse of Corollary 5.10 is clearly not true as the partial line sum constrains may be violated for a matrix of the form $P_1 + P_2 + \cdots + P_{k+n} - J_n$.

Example 5.11. Consider the 2-regular ASM

$$A = \left[\begin{array}{c|c|c|c} & 1 & 1 & \\ \hline 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline & 1 & 1 & \end{array}\right].$$

Then, a decomposition of $A + J_4$ is

$$A + J_4 = \left[\begin{array}{c|c|c|c} 1 & 2 & 2 & 1 \\ \hline 2 & 2 & 0 & 2 \\ \hline 2 & 0 & 2 & 2 \\ \hline 1 & 2 & 2 & 1 \end{array}\right] = \left[\begin{array}{c|c|c|c} 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right] + \left[\begin{array}{c|c|c|c} & 1 & & \\ \hline 1 & & & \\ \hline & & & 1 \\ \hline & & 1 & \end{array}\right] + \left[\begin{array}{c|c|c|c} & 1 & & \\ \hline 1 & & & \\ \hline & & & 1 \\ \hline & & 1 & \end{array}\right] +$$

$$+ \left[\begin{array}{c|c|c|c} & & 1 & \\ \hline & & & 1 \\ \hline 1 & & & \\ \hline & 1 & & \end{array}\right] + \left[\begin{array}{c|c|c|c} & & 1 & \\ \hline & & & 1 \\ \hline 1 & & & \\ \hline & 1 & & \end{array}\right] + \left[\begin{array}{c|c|c|c} & & & 1 \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline 1 & & & \end{array}\right].$$

☐

Conjecture 5.12. Let A be a k-regular ASM. Then, there are k ASMs $A_1, A_2, \ldots, A_k$ such that

$$A = A_1 + A_2 + \cdots + A_k,$$

where the matrices $A_1, A_2, \ldots, A_k$ are pairwise, pattern disjoint, thus extending Theorem 5.9 from permutation matrices to ASMs. If this is not true as stated, maybe it holds with some overlap in positions. The special case of 2-regular ASMs already seems difficult but worthy of investigation. A less-restrictive conjecture is as follows: Every 2-regular ASM A contains an ASM, that is, an ASM can be obtained by changing some ± 1's of A to 0's. ☐

Another possibility for a relationship between ASMs and 2-regular ASMs is in the following question.

Question 5.13. Let A be an $n \times n$ ASM. So all row and column sums of A equal 1 (and total sum equals n), as opposed to a 2-regular ASM where all row and column sums equal 2 (and total sum is $2n$). Is

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it possible to replace $0 \leq x \leq n$ 0's of A by 1's and $(n-x)$ -1's of A with 0's so that the resulting matrix A' is a 2-regular ASM? The total sum of A' would be $2n$ as it would have to be for a 2-regular ASM. Note that the difference between A' and A would have to be a *signed permutation matrix*, i.e., a permutation matrix in which some of the 1's have been replaced by -1's. Not every 2-regular ASM results in this way from an ASM. The example of a 2-regular ASM with the maximum number of nonzeros when $n = 10$, the matrix $A^{(4,10)}$ given in Example 3.5, has 68 nonzeros while the maximum number of nonzeros of a ASM with $n = 10$ is 50 (the diamond ASM); but $68 - 50 = 18 \neq 10$. For example,

$$\left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline & 1 & -1 & 1 & \\ \hline 1 & -1 & 1 & -1 & 1 \\ \hline & 1 & -1 & 1 & \\ \hline & & 1 & & \end{array}\right] \rightarrow \left[\begin{array}{c|c|c|c|c} & 1 & 1 & & \\ \hline & 1 & 0 & 1 & \\ \hline 1 & -1 & 1 & 0 & 1 \\ \hline 1 & 1 & -1 & 1 & \\ \hline & & 1 & & 1 \end{array}\right].$$

□

6. Coda - the Bruhat order. Let $A = [a_{ij}]$ be an arbitrary $m \times n$ matrix, and let

$$\sigma_{ij} = \sum_{k=1}^{i} \sum_{l=1}^{j} a_{ij} \quad (1 \leq i \leq m, 1 \leq j \leq n),$$

the sum of the entries in the leading $i \times j$ submatrix of A. The $m \times n$ matrix

$$\Sigma(A) = [\sigma_{ij}],$$

is the *sum matrix* of A. The matrix A is recoverable from $\Sigma(A)$ since

$$a_{ij} = \sigma_{ij} + \sigma_{i-1,j-1} - \sigma_{i-1,j} - \sigma_{i,j-1} = (\sigma_{ij} - \sigma_{i-1,j}) - (\sigma_{i,j-1} - \sigma_{i-1,j-1}), \tag{6.8}$$

where $\sigma_{i0} = \sigma_{0j} = 0$. Note that if A is an $n \times n$ ASM then by (6.8), a_{ij} is the difference of two numbers equal to 0 or 1, and so $a_{ij} = 0, \pm 1$. Thus ASMs are recoverable from (6.8).

The *Bruhat order* on $\mathcal{A}(R, S)$ (see e.g., [9]) has been defined by

$$A_1 \preceq_b A_2 \text{ if and only if } \Sigma(A_1) \geq \Sigma(A_2) \ (\textit{entrywise}).$$

In the set $\mathcal{P}_n$ of $n \times n$ permutation matrices, this is equivalent to the usual definition of its Bruhat order (a sequence of inversions each of which reduces the number of inversions by 1) and gives a partially ordered set with the identity matrix I_n as the unique minimal matrix in the Bruhat order and the Hankel diagonal (anti-diagonal) permutation matrix as the unique maximal element. As proved in [7], in the set $\mathcal{A}(n, 2)$ of $n \times n$ (0,1)-matrices with row and column sums all equal to 2 (so the set of $n \times n$ 2-permutation matrices), a matrix is a minimal element if and only if it is a direct sum, in any order, of matrices equal to the 2×2 matrix J_2 of all 1's and the matrix

$$F_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

In an analogous way, a matrix in $\mathcal{A}(n, 2)$ is a maximal matrix provided it is a *skew-direct sum* (in any order) of matrices equal to J_2 and

$$F_3' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

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There is also a known characterization of the minimal matrices in $\mathcal{A}(n,3)$ [9]. In $\mathcal{A}(n,2)$, it is also known that minimal matrices are those that do not contain as a submatrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In fact (see [7, 2]), the Bruhat order on $\mathcal{A}(R,S)$ where $R = S = (2,2,\ldots,2)$ is equivalent to: $A_1 \preceq_b A_2$ if and only if A_1 can be gotten from A_2 by the *one-sided interchanges* on 2×2 submatrices of the form:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

But note that this is not true for $\mathcal{A}(R,S)$ in general. See [12] and some of its references for further information on this issue.

We extend the *Bruhat order* on $\mathcal{A}(R,S)$ to $\mathcal{A}^{\pm}(R,S)$ by

$$A_1 \preceq_b A_2 \text{ if and only if } \Sigma(A_1) \geq \Sigma(A_2) \ (\textit{entrywise}).$$

LEMMA 6.1. *The minimal matrices in the Bruhat order on $\mathcal{A}^{\pm}(n,2)$ are the minimal matrices in the Bruhat order on $\mathcal{A}(n,2)$.*

Proof. The sum matrix of J_2 begins with

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

and that of F_3 begins with

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}.$$

Hence if A is a minimal matrix in $\mathcal{A}^{\pm}(n,2)$, it must be of the form

$$\left[\begin{array}{c|c|c|c|c} 1 & 1 & 0 & \cdots & \\ \hline 1 & a & b & & \\ \hline 0 & c & d & & \\ \hline \vdots & & & & \\ \hline & & & & \end{array}\right].$$

If $a = 1$, then the lemma follows by induction where $A = J_2 \oplus A'$. We cannot have $a = -1$, because then $J_2 \oplus I_{n-2}$ would be strictly smaller than A in the Bruhat order. Finally, consider $a = 0$, in which case we see that $b = c = d = 1$. $\square$

A theorem of MacNeille (see e.g., [14]) asserts that every finite poset has a unique, up to isomorphism, minimal completion to a finite lattice, now called its *Dedekind-MacNeille completion*. The Dedekind-MacNeille completion of the partially ordered set $(\mathcal{P}_n, \preceq_b)$ of the $n \times n$ permutation matrices with the Bruhat order is the partially ordered set $(\mathcal{A}_n, \preceq_b)$ of the $n \times n$ ASMs with the Bruhat order [13].

QUESTION 6.2. Is $(\mathcal{A}^{\pm}(R,S), \preceq_b)$ the Dedekind-MacNeille completion of the partially ordered set $(\mathcal{A}(R,S), \preceq_b)$? In particular, is the Dedekind-MacNeille completion of the partially ordered set $(\mathcal{A}(n,2), \preceq_b)$ of $n \times n$ $(0,1)$-matrices with two 1's in each row and column (the 2-permutation matrices) partially ordered by the Bruhat order, equal to $(\mathcal{A}^{\pm}_n(n,2), \preceq_b)$ of $n \times n$ of 2-regular ASMs partially ordered by the Bruhat order? $\square$

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To consider Question 6.2 for 2-regular ASMs, we give a characterization of the sum matrices of 2-regular ASMs.

For ASMs, we know that $X = [x_{ij}]$ is the sum matrix of an $n \times n$ ASM $A = [a_{ij}]$ if and only if X satisfies

1. Integers in each row and column i are taken from the set $\{0, 1, 2, \ldots, i\}$ beginning with 0 or 1 and ending with i, and are nondecreasing.
2. Two consecutive entries in a row or column are either equal or there is an increase of 1.

By (6.8), these properties determine A to be a $(0, \pm 1)$-matrix. So the $n \times n$ ASMs can be regarded as the $n \times n$ nonnegative integral matrices satisfying the above properties. What are the corresponding properties for (R, S)-ASMs, in particular for 2-regular ASMs? If A is a 2-regular ASM, then the following are satisfied by $\Sigma(A)$:

(i) Integers in each row and column i are taken from the set $\{0, 1, 2, \ldots, 2i\}$, beginning with 0 or 1 and ending with $2i$, and are nondecreasing,

(ii) Two consecutive entries in a row or column are either equal or there is an increase of 1 or 2 (see the following example which shows that an increase of 2 is possible).

Recall that

$$a_{ij} = x_{ij} + x_{i-1,j-1} - x_{i-1,j} - x_{i,j-1} = (x_{ij} - x_{i-1,j}) - (x_{i,j-1} - x_{i-1,j-1}), \tag{6.9}$$

where $x_{i0} = x_{0j} = 0$, the difference of two integers each of which could equal 0, 1, or 2, and hence the difference has to be guaranteed to be $\neq \pm 2$ that is, the following must hold:

$$(x_{ij} - x_{i-1,j}) = 2 \text{ implies } (x_{i,j-1} - x_{i-1,j-1}) \neq 0, \text{ and}$$

$$(x_{ij} - x_{i-1,j}) = 0 \text{ implies } (x_{i,j-1} - x_{i-1,j-1}) \neq 2.$$

Otherwise we get an entry of A equal ± 2 which is not allowed for a 2-regular ASM. Thus we also need the property:

(iii) $x_{ij} + x_{i-1,j-1} - x_{i-1,j} - x_{i,j-1} = ((x_{ij} - x_{i-1,j}) - (x_{i,j-1} - x_{i-1,j-1})) \in \{0, \pm 1\}$.

Defining a_{ij} as in (6.9) we see by (iii) that the a_{ij} equal 0, 1, or -1, and hence, by (i) the row and column sums equal 2. Thus, the properties (i), (ii), and (iii) characterize sum matrices of 2-regular ASMs. It follows from the relation (6.9) between A and its sum matrix X that, whenever (i) and (ii) both hold, condition (iii) is equivalent to A being a $(0, \pm 1)$-matrix. Therefore, for a $(0, \pm 1)$-matrix A, it will be a 2-regular ASM if and only it its sum matrix satisfies conditions (i) and (ii).

Example 6.3. Consider the two 2-permutation matrices

$$P_1 = \left[\begin{array}{c|c|c|c} 1 & 1 & & \\ \hline & 1 & 1 & \\ \hline & & 1 & 1 \\ \hline 1 & & & 1 \end{array}\right] \to \Sigma(P_1) = \left[\begin{array}{c|c|c|c} 1 & 2 & 2 & 2 \\ \hline 1 & 3 & 4 & 4 \\ \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & 6 & 8 \end{array}\right],$$

and

$$P_2 = \left[\begin{array}{c|c|c|c} 1 & & & 1 \\ \hline 1 & 1 & & \\ \hline & 1 & 1 & \\ \hline & & 1 & 1 \end{array}\right] \to \Sigma(P_2) = \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & 4 \\ \hline 2 & 4 & 5 & 6 \\ \hline 2 & 4 & 6 & 8 \end{array}\right].$$

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We have

$$\min\{\Sigma(P_1),\Sigma(P_2)\} = \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 2 \\ \hline 1 & 3 & 3 & 4 \\ \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & 6 & 8 \end{array}\right].$$

Inverting this minimum, we get

$$\left[\begin{array}{c|c|c|c} 1 & & & 1 \\ \hline & 2 & & \\ \hline & & 2 & \\ \hline 1 & & & 1 \end{array}\right],$$

which is not a 2-regular ASM! Thus, the Bruhat order on 2-regular ASMs does not give a lattice. □

Finally, we observe that if P is a permutation matrix, then P can be recovered from $\Sigma(P)$ by the following algorithm applied to $\Sigma(P)$: start with row 1 and put a 1 where the first nonzero or increase occurs. Delete row 1 and that column, and proceed by induction.

Example 6.4.

$$\left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline 1 & & & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline & 1 & & & \end{array}\right] \rightarrow \left[\begin{array}{c|c|c|c|c} 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 & 2 \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline 1 & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 & 5 \end{array}\right] \rightarrow \left[\begin{array}{c|c|c|c|c} 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 & 2 \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline 1 & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 & 5 \end{array}\right].$$

□

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