Pricing of spread options in energy markets

by

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THESIS

for the degree of

MASTER OF SCIENCE

(Master i Modellering og Dataanalyse)

Faculty of Mathematics and Natural Sciences
University of Oslo

May 2012

Det Matematisk-naturvitenskapelige fakultet
Universitetet i Oslo

Abstract

The objective of this thesis is to value spread options with payoff function on the form $\max(S_1(T) - hS_2(T) - K, 0)$, where $S_1(T)$ and $S_2(T)$ are the spot prices of two energy commodities at maturity time T, h is the heatrate and K is the strike price. We model $(X_1(t), X_2(t)) = (\log(S_1(t)), \log(S_2(t))$ as a bivariate Ornstein-Uhlenbeck Lévy process. First, we consider an Ornstein-Uhlenbeck process driven by a bivariate Brownian motion, then we extend the model to an Ornstein-Uhlenbeck process driven by a bivariate Lévy process with jumps. We compute the characteristic function of $(X_1(t), X_2(t))$ in both models, and study the stationary properties of the distribution of $(X_1(t), X_2(t))$. Then we we derive a closed form formula for the option price in the continuous model for the case $K = 0$. In the model with jumps, we use a Fourier transform method to express the price as an integral of the characteristic function of $(X_1(t), X_2(t))$ times the Fourier transform of the payoff function. When $K \neq 0$, we use a first order Taylor-expansion to approximate the option price. We find a closed form formula for the approximated price in the continuous model, and use simulations to check how good the approximation is for different values of K and for different values of the correlation between the two underlying price processes.

Acknowledgements

First of all, I would like to thank my supervisor Fred Espen Benth, for always having time for me, for his good advice and quick feedback, and for keeping my motivation up by always beeing positive to my work.

Thanks to my all fellow students at room B802 for their company during the writing process. A special thanks to Vu Duy Pham, for taking the time to discuss certain problems.

Finally, I would like to thank Ole-Jørgen, my family and friends for all support and encouragement during this semester.

Contents

Chapter 1

Introduction

Since the liberalization of energy markets started two decades ago, these markets have been steadily growing, and many new financial contracts have been introduced. This has led to new risk management tools for the traditional participants in these markets, like e.g. plant owners, but it has also opened up a new investment area for speculators as investment banks and pension funds. We will first give a short introduction to energy markets and some contracts which are traded in these markets. To value energy derivatives, we need to find appropriate stochastic models describing the underlying commodity prices. We will mainly consider electricity, since this is the energy commodity that we face most challenges when modelling. In Section 1.2 we discuss the special attributes of electricity prices. An important difference between electricity and other energy commodities is that electricity is not storable, so hedging against undesirable price movements by buying and keeping the underlying is not possible. Electricity prices are also depending on prices in other energy markets, e.g. the temperature, natural gas and coal markets. One example of a class of energy derivatives is spread options, which we will study more closely in this thesis. We introduce some commonly traded spread options in Section 1.3.

1.1 Energy markets

The liberalization of the energy markets started in the early 1990´s. In Norway, the energy law of 1991 led to the establishment of Statnett Marked (later Nord Pool ASA) in 1993, an exchange for trading electricity contracts. Earlier, the local electricity producers had been responsible for delivering electricity in their own area, and the price differences across the country could be large. The other Nordic countries joined Nord Pool in the following years. In 2002, Nord Pool Spot AS was separated from Nord Pool ASA. Nord Pool ASA was bought by Nasdaq OMX in 2008.

There are two markets for physical delivery of electricity, the real-time market and the day-ahead market. The real-time market is organized by the TSO (Transmission System Operator), which is responsible for the power grids. In Norway, Statnett is the TSO. They continuously buy and sell electricity from the producers to maintain a stable frequency on the power grid. See [9] for more details about this.

On the day-ahead market, Elspot, which is organized by Nord Pool Spot, the producers and consumers submit their bids on how much electricity they want to buy or sell during each hour the next day. Prices for each hour the following day are then decided at noon. This price is called the system price, or the spot price. This is the price we will use as our underlying price in the spread options we consider.

There is also a market called Elbas, where contracts from the Elspot market are traded after the prices are fixed and until the delivery hour the next day.

In the financial electricity market, futures and options on the system price are traded. The electricity future contracts traded at Nord Pool are written on the average of the hourly day-ahead prices over a specified delivery period. So these contracts are in fact swap contracts. Options on the future contracts are also traded. The financial market is used to hedge against a decrease or increase in the electricity price, or by speculators as mentioned earlier.

Another energy market is the gas market, which is linked to the electricity market through gas fired power plants. Both the spot market for gas, which is the market for short-term delivery of gas, and the market for gas futures are mostly traded OTC (over the counter). Other energy markets related to the electricity market are the markets for temperature, oil, coal and CO_2 emmisions. See [1] for further details on electricity and related markets.

1.2 Modelling the spot price

To price derivative contracts in energy markets, we first need to find appropriate models for the underlying spot prices. The models must describe the special features of these markets, but they also need to be analytically tractable in order to be able to compute the prices. The spot price is in reality a discrete process, since it is constant for each hour during the day. However, modelling it as a discrete process leads to a futures price that is also constant for each hour. Modelling the spot price as an underlying continuous process solves this problem (see [1, p.30]).

The price of electricity is dependeding on many different factors. An important attribute of electricity is that it can not be stored, so it needs to be instantly consumed. Both demand and supply are strongly dependent on weather. The demand is depending on temperature, with higher demand when the temperature is low. In Norway, where the majority of produced electricity comes from hydropower plants, the production is dependent on the precipitation. Weather can change suddenly, which in combination with the limited storage possibilities leads to imbalances between supply and demand. This leads to high volatility and sudden fluctuations or spikes in the prices. Temperatures are also more variable during the winter, so the volatility is seasonally dependent.

The gas market has many similarities with the electricity market. The prices are seasonal with sudden spikes. Contrary to electricity, gas can be stored, which makes hedging possible.

An important difference from stock prices is the mean-reverting property, because prices in energy markets are driven by the balance between production and demand. Ornstein-Uhlenbeck (OU) processes, which posess the mean-reverting property, are therefore natural processes for spot price modelling in energy markets. The Schwarz model, which is the classical spot price model in energy markets, models the prices as exponential OU processes (see [1, p.20]). By letting the OU processes be driven by Lévy processes instead of Brownian motions, we can model the spikes. The seasonally varying mean level can be modelled by multiplying the price process with a deterministic function. However, since Lévy processes are stationary, they can not be used to model a seasonally dependent variance. To include seasonality in the jump size and jump frequency, we can use independent increment (II) processes instead. II processes are a generalization of Lévy processes which are not necessarily stationary. We will not consider general II processes here. See [1] for more information about II processes and how they are used in energy market modelling.

Chapter 2 includes some background theory on Lévy processes and other topics that we will need in the following chapters. In Chapter 3, we take a closer look at both a continuous Schwarz model and a Lévy process model for the spot price. We calculate the characteristic functions and study the stationary distributions of the logarithmic spot prices. We also simulate a realization of the spot price process in the continuous model.

1.3 Spread options in energy markets

A spread option is an option written on the difference between two underlying assets. We will consider spread options of European type, i.e. call or put options on the price difference between the two assets. Such spread options have a payoff on the form $\max(S_1(T) - hS_2(T) - K, 0)$, where $S_1(t)$ and $S_2(t)$ are the price processes of the two underlying assets, K is the strike price, T is the maturity time of the option and h is called the heat rate. The heat rate is also called the efficiency rate, since it can be interpreted as the efficiency of a specific power plant. In energy markets, the maturity time is usually a period rather than a point in time.

Examples of spread options in energy markets are the spark spread option, the dark spread option and the crack spread option. The spark spread is written on the price difference between electricity and gas. The value of such an option reflects the cost of producing electricity from natural gas at a specific power plant. It can be used by plant owners to hedge against an increase in the gas price or a decrease in the electricity price. A dark spread option is written on the difference between the electricity price and the price of coal, the crack spread on the difference between crude and refined petroleum products. The last ones are used for risk management by oil refiners.

There also exist spread options written on only one underlying commodity, like locations spread, calendar spreads and quality spreads. Processing spreads are written on the price difference between the input and the output of a production process. Spread options depending on two underlying assets are harder to price because one has to take into account the dependency between the price processes. There also exist spread options written on several underlying commodities. See [2] for more details about the different types of spread options.

In Chapter 4 we will calculate the price of a spread option with strike price $K = 0$ for both of the spot price models in Chapter 3. In Chapter 5 we will find an approximation for the option price in the continuous model for the case when $K \neq 0$, and we use simulations to see how good the approximation is.

Chapter 2

Theoretical background

We will in this chapter introduce some topics that we will use in the later chapters. First, we give the definition of a Lévy process, and state some basic properties of these processes.

Further, we study characteristic functions, which are useful a useful tool to describe the distributional properties of the spot price processes we will consider in Chapter 3. It is often very difficult or even impossible to find a closed form formula for the distribution of a Lévy process, but finding its characteristic function is much easier. The characteristic function gives a complete and unique characterization of the distribution, and many properties of the distribution can be derived from the characteristic function. In Chapter 4.2, we are going to express the spread option price in the model with jumps, in terms of a characteristic function.

Then we introduce the Esscher transform, through which we can define a new equivalent probability measure. This measure change will also be helpful when we compute the spread option price in the Lévy process model in Chapter 4.

In the last section we define the Fourier transform and give an example that we will use in Chapter 4.

2.1 Lévy processes

Definition 2.1. [8, p. 3] *Let* $(\Omega, \mathscr{F}, P)$ *be a complete probability space equipped with a filtration* $\{\mathscr{F}_t,\, t \geq 0\}$*. A* Lévy process $\{L(t)\}_{t\geq 0}$ *is an* $\mathscr{F}_t$*-adapted process that possesses the following properties:*

(i) $L(0) = 0$ *a.s.*

(ii) $L(t)$ *has independent increments, i.e., for any* $n \geq 1$ *and* $0 \leq t_0 <$

$t_1 < t_2 < ... < t_n$, the random variables $L(t_0)$, $L(t_1) - L(t_0)$,...,$L(t_n) - L(t_{n-1})$ are independent.

(*iii*) *$L(t)$ has stationary increments, i.e., the distribution of $L(t+s) - L(t)$ does not depend on t.*

(*iv*) *$L(t)$ is stochastically continuous, i.e., for any $t \geq 0$ and $\epsilon > 0$, $\lim_{s \to 0} P(|L(t+s) - L(t)| > \epsilon) = 0$.*

A stochastic process is *càdlàg* if its paths are right-continuous for $t \geq 0$ and have limits from the left for $t > 0$ a.s. It can be shown that every Lévy process has a càdlàg version, so we can assume that the Lévy processes we use are càdlàg.

The *Poisson random measure* or *jump measure* $N(U,t)$ of $L(t)$ is defined as

$$N(U,t) = \sum_{0<s\leq t} 1_U(\Delta L(s)), \quad \forall \quad U \in \mathbf{B_0},$$

where $\mathbf{B_0}$ is the family of Borel sets $U \in \mathbb{R}$ such that $0 \notin \bar{U}$, and $\Delta L(t) = L(t) - L(t^-)$ is the jump of $L(t)$ at time $t \geq 0$. $N(U,t)$ is a random measure which measures the number of jumps between time 0 and time t where $\Delta L(s) \in U, 0 < s \leq t$. The Lévy process can have infinitely many small jumps, so the jump measure can be infinite. It is although finite for all $U \in \mathbf{B_0}$ because $L(t)$ has the càdlàg property (see [7]). The *Lévy measure* ν of $L(t)$ is defined as

$$\nu(U) = E\left[N(U,1)\right].$$

The Lévy measure measures the expected number of jumps within a time interval. We also define the *compensated Poisson random measure* $\tilde{N}(U,t)$ of $L(t)$:

$$\tilde{N}(U,t) = N(U,t) - \nu(U)t.$$

$\tilde{N}(U,t)$ is a local martingale. Every Lévy process can be decomposed in the following way:

Theorem 2.2 Itô-Lévy decomposition. [7, p. 3] *Let $L(t)$ be a Lévy process. Then it has the following decomposition:*

$$L(t) = \alpha t + \sigma B(t) + \int_0^t \int_{|z|<1} z\tilde{N}(dz,dt) + \int_0^t \int_{|z|\geq 1} zN(dz,dt), \qquad (2.1)$$

where $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and $B(t)$ is a Brownian motion independent of N.

The triplet (α, σ, ν) is called the *Lévy triplet* of $L(t)$, and it is unique. The reason why we need to use the measure $\tilde{N}$ instead of N for the small jumps, is to assure convergence of the integral, because the sum of the small jumps does not necessarily converge.

The *compound Poisson process* is an example of a Lévy process with finite Lévy measure. A compound Poisson process $X(t)$ with intensity λ and jump size distribution μ is defined as

$$X(t) = \sum_{j=1}^{N(t)} Y(j),\ t > 0,$$

where $Y(j)$, $j \in \mathbb{N}$, is a is a sequence of i.i.d. random variables taking values in $\mathbb{R}$ with common distribution μ, and $N(t)$ is a Poisson process with intensity λ, independent of $Y(j)$ for all j. $X(t)$ has Lévy measure $\lambda\mu$, and since it has no continuous component, the Lévy triplet is $(0, 0, \lambda\mu)$.

An *Itô-Lévy process* $X(t)$ is a process on the form:

$$\begin{aligned} X(t) = X(0) &+ \int_0^t \alpha(s)\, ds + \int_0^t \beta(s)\, dB(s) \\ &+ \int_0^t \int_{|z|<1} \gamma(s,z)\tilde{N}(dz, dt) + \int_0^t \int_{|z|\geq 1} \gamma(s,z) N(dz, dt), \end{aligned}$$

where

$$\int_0^t \left(|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}} \gamma^2(s,z)\, \nu(dz) \right) ds < \infty \quad \text{a.s.} \quad \forall t > 0.$$

We are going to use an Ornstein-Uhlenbeck Itô-Lévy process as a model for the logarithmic spot price process.

2.2 Characteristic functions

Definition 2.3 Characteristic function. [3, p. 30] *Let $X \in \mathbb{R}^n$ be a random variable with distribution μ_X. Then its* characteristic function *$\Phi_X : \mathbb{R}^n \to \mathbb{R}$ is defined as:*

$$\Phi_X(u) = E[e^{i\langle u, X\rangle}] = \int_{\mathbb{R}^n} e^{i\langle u, x\rangle}\, \mu_X(dx), \quad \forall u \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidian inner product in $\mathbb{R}^n$.

To show that this integral converges for all $u \in \mathbb{R}^n$ and all probability distributions μ_X, we use that $|e^{i\langle u,x\rangle}| = |\cos(\langle u,x\rangle) + i\sin(\langle u,x\rangle)| = 1$ and that $\mu_X(A) \geq 0$ for all $A \subseteq \mathbb{R}^n$.

$$\int_{\mathbb{R}^n} e^{iux}\, \mu_X(dx) \leq \int_{\mathbb{R}^n} |e^{iux}|\, \mu_X(dx) = \int_{\mathbb{R}^n} \mu_X(dx) = 1.$$

Hence, the characteristic function of a random variable is always well-defined. It is also continuous [8, p. 8], and we see that $\Phi_X(0) = 1$.

Theorem 2.4. [8, p. 8] *Let $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^n$ be two random variables with distributions μ_X and μ_Y. If $\Phi_X(u) = \Phi_Y(u)$ for all $u \in \mathbb{R}^n$, then $\mu_X = \mu_Y$.*

So the characteristic function of a random variable uniquely determines its distribution. The moments of a random variable X, if they exist, can be found through differentiation of its characteristic function.

Proposition 2.5 Characteristic function and moments. [3, p. 30] *Let $X \in \mathbb{R}^n$ be a random variable with characteristic function Φ_X.*

1. *If $E[X^n]$ is finite for some $n \in \mathbb{N}$, then Φ_X has n continuous derivatives at $u = 0$, and*

$$E[X^k] = (-i)^k \frac{\partial^k \Phi_X}{\partial u^k}(0), \quad \forall k = 1, \ldots, n.$$

2. *$E[X^n]$ is finite for all $n \in \mathbb{N}$ if and only if Φ_X has infinitely many continuous derivatives at $u = 0$. Then we have that*

$$E[X^n] = (-i)^n \frac{\partial^n \Phi_X}{\partial u^n}(0).$$

We state the characteristic function of a multivariate normal random variable, since we will use it in Chapter 3.

Proposition 2.6 Characteristic function of a multivariate normal random variable. [8, p. 11] *Let $X = (X_1, \ldots, X_n)$ be a random variable in $\mathbb{R}^n$ following a multivariate normal distribution with expectation μ and covariance matrix σ. Then the characteristic function of X is*

$$\Phi_X(u) = \exp\left\{ i\langle u, \mu\rangle - \frac{1}{2}\langle u, \sigma u\rangle \right\}, \quad \forall u \in \mathbb{R}^n.$$

As an example of a characteristic function of a Lévy process with jumps, we calculate the characteristic function of a compound Poisson process.

Proposition 2.7 Characteristic function of a compound Poisson random variable. *Let $X(t)$ be a compound Poisson process with intensity λ and jump size distribution μ. Then the characteristic function $\Phi_{X(t)}$ of $X(t)$ has the following representation:*

$$\Phi_{X(t)}(u) = \exp\left\{\lambda t \int_{-\infty}^{\infty} (e^{iuy} - 1)\mu(dy)\right\}, \quad \forall u \in \mathbb{R}.$$

Proof. We first calculate the characteristic function of the jump size distribution. The characteristic function of $Y(1)$, which equals the characteristic function of $Y(j)$ for all $j \in \mathbb{N}$, is:

$$\Phi_{Y(1)}(u) = E[e^{iuY(1)}] = \int_{-\infty}^{\infty} e^{iuy}\mu(dy), \quad \forall u \in \mathbb{R}. \tag{2.2}$$

The characteristic function $\Phi_{X(t)}$ of $X(t)$ becomes:

$$\begin{aligned}\Phi_{X(t)}(u) &= E[e^{iuX(t)}] \\ &= E\left[E\left(e^{iuX(t)}|N(t)\right)\right] \\ &= E\left[E\left(e^{iu\sum_{j=1}^{N(t)} Y(j)}|N(t)\right)\right] \\ &= E\left[E\left(\prod_{j=1}^{N(t)} e^{iuY(j)}|N(t)\right)\right].\end{aligned}$$

Since the random variable $E[Y|X]$ is a function of X, the outer expectation can be taken with respect to the distribution of X. Hence we get

$$\begin{aligned}&\sum_{n=0}^{\infty} E\left(\prod_{j=1}^{N(t)} e^{iuY(j)}|N(t) = n\right) P(N(t) = n). \\ =&\sum_{n=0}^{\infty} E\left(\prod_{j=1}^{n} e^{iuY(j)}\right) P(N(t) = n).\end{aligned}$$

Since the jumps $Y(j)$ are independent and identically distributed, we see that this is equal to

$$\sum_{n=0}^{\infty} \left(\prod_{j=1}^{n} E(e^{iuY(j)})\right) P(N(t) = n).$$

$$= \sum_{n=0}^{\infty} \Phi_{Y(1)}(u)^n \, P(N(t) = n).$$

Using that the number of jumps $N(t)$ is Poisson distributed with parameter λt and $\{N(t);\, t \geq 0\}$ is independent of $\{Y(j);\, j \in \mathbb{N}\}$ for all t, this is equal to

$$e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\left(\lambda t \Phi_{Y(1)}(u)\right)^n}{n!}.$$

From the Taylor expansion of the exponential function, we see that this is equal to

$$\begin{aligned} &\exp\left\{\lambda t \left(\Phi_{Y(1)}(u) - 1\right)\right\} \\ \overset{(2.2)}{=} &\exp\left\{\lambda t \left(\int_{-\infty}^{\infty} e^{iuy} \mu(dy) - 1\right)\right\} \\ = &\exp\left\{\lambda t \int_{-\infty}^{\infty} (e^{iuy} - 1)\mu(dy)\right\}. \end{aligned}$$

□

The Lévy-Khinchin formula gives us the characteristic function of a general Lévy process. Since we will mostly work with two-dimensional Lévy processes, we state it for an n-dimensional Lévy process $L(t) = (L_1(t), \ldots, L_n(t))$.

Theorem 2.8 Lévy-Khintchine formula. [8, p. 37] *Let $L(t)$ be a Lévy process with Lévy measure ν. Then $\int_{\mathbb{R}} min(1, |z|^2)\nu(dz) < \infty$ and*

$$E[e^{i\langle u, L(t)\rangle}] = e^{t\psi(u)}, \quad u \in \mathbb{R}^n, \tag{2.3}$$

where

$$\psi(u) = i\langle u, \alpha\rangle - \frac{1}{2}\langle u, \sigma u\rangle + \int_{\mathbb{R}^n} (e^{i\langle u, z\rangle} - 1 - i\langle u, z\rangle 1_{|z|<1})\nu(dz), \tag{2.4}$$

where $\alpha \in \mathbb{R}^n$ and σ is a symmetric non-negative definite $n \times n$-matrix.

Conversely, given α and σ satisfying the conditions above, and a measure ν on $\mathbb{R}^n$ such that $\int_{\mathbb{R}} min(1, |z|^2)\nu(dz) < \infty$, there exists a Lévy process $L(t)$ such that (2.3) and (2.4) hold.

The last term in the integral in equation (2.4) corresponds to the compensation of the small jumps. We use the Lévy-Khintchine formula to derive a formula for the characteristic function of a bivariate Itô-Levy process.

Proposition 2.9 Characteristic function of a bivariate Itô-Lévy process. *Let $X(t) = (X_1(t), X_2(t))$ be an Itô-Levy process on the form*

$$X(t) = \eta(t) + \int_0^t \gamma(t,s)dL(s)$$

where $\eta(t) = (\eta_1(t), \eta_2(t))$, $\gamma(t,s) = (\gamma_1(t,s), \gamma_2(t,s))$ and $L(t) = (L_1(t), L_2(t))$ is a bivariate Lévy process with dynamics:

$$dL_i(t) = \sigma_i dB_i(t) + \int_{|z|<1} z_i \tilde{N}(dz_1, dz_2, ds) + \int_{|z|\geq 1} z_i N(dz_1, dz_2, ds), \quad i = 1, 2,$$

where B_1 and B_2 are Brownian motions with correlation ρ, N is a Poisson random measure and $\tilde{N}$ is the corresponding compensated Poisson random measure. Then the characteristic function of $X(t)$ is:

$$\Phi_{X(t)}(u) = \exp\left\{ i\langle u, \eta(t)\rangle + \int_0^t \Psi(v(t,s))\, ds \right\},$$

where

$$\begin{aligned}
u &= (u_1, u_2) \\
\Psi(\cdot) &= -\frac{1}{2}\langle \cdot, \sigma \cdot\rangle + \int_{\mathbb{R}_0^2} \left(e^{i\langle \cdot, z\rangle} - 1 - i\langle \cdot, z\rangle 1_{\{|z|<1\}}\right) \nu(dz) \\
v(t,s) &= (u_1\gamma(t,s), u_2\gamma(t,s)) \\
\sigma &= \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.
\end{aligned}$$

Proof.

$$\begin{aligned}
\Phi_{X(t)}(u) &= E\left[e^{i\langle u, X(t)\rangle}\right] \\
&= E\left[e^{i(u_1,u_2)\begin{pmatrix} X_1(t) \\ X_2(t)\end{pmatrix}}\right] \\
&= E\left[e^{iu_1X_1(t) + iu_2X_2(t)}\right] \\
&= E\left[e^{iu_1\left(\eta_1(t) + \int_0^t \gamma_1(t,s)dL_1(s)\right) + iu_2\left(\eta_2(t) + \int_0^t \gamma_2(t,s)dL_2(s)\right)}\right] \\
&= e^{i\langle u, \eta(t)\rangle} E\left[e^{i\left(u_1\int_0^t \gamma_1(t,s)dL_1(s) + u_2\int_0^t \gamma_2(t,s)dL_2(s)\right)}\right].
\end{aligned}$$

Let Π_n be a partition of the interval $[0, t]$ which divides it into $J_n - 1$ intervals $[s_j, s_{j+1}]$, $1 \le j < J_n$, with

$$|\Pi_n| = \max_{1 \le j < J_n} |s_{j+1}^n - s_j^n| \underset{n\to\infty}{\longrightarrow} 0.$$

We can write the integrals in the expectation above as limits of integrals of elementary functions. Then we use the dominated convergence theorem to put the limit outside of the expectation. We also use the independent increment property of Lévy processes.

$$\begin{aligned}
&E\left[e^{i\left(u_1 \int_0^t \gamma_1(t,s)dL_1(s) + u_2 \int_0^t \gamma_2(t,s)dL_2(s)\right)}\right] \\
&= E\left[e^{i\left(u_1 \lim\limits_{n\to\infty} \sum_{j=1}^{J_n} \gamma_1(t,s_j^n)\Delta L_1(s_j^n) + u_2 \lim\limits_{n\to\infty} \sum_{j=1}^{J_n} \gamma_2(t,s_j^n)\Delta L_2(s_j^n)\right)}\right] \\
&= E\left[e^{i \lim\limits_{n\to\infty} \sum_{j=1}^{J_n}\left(u_1\gamma_1(t,s_j^n)\Delta L_1(s_j^n) + u_2\gamma_2(t,s_j^n)\Delta L_2(s_j^n)\right)}\right] \\
&= E\left[\lim_{n\to\infty} e^{i \sum_{j=1}^{J_n}\left(u_1\gamma_1(t,s_j^n)\Delta L_1(s_j^n) + u_2\gamma_2(t,s_j^n)\Delta L_2(s_j^n)\right)}\right] \\
&= \lim_{n\to\infty} E\left[e^{i \sum_{j=1}^{J_n}\left(u_1\gamma_1(t,s_j^n)\Delta L_1(s_j^n) + u_2\gamma_2(t,s_j^n)\Delta L_2(s_j^n)\right)}\right] \\
&= \lim_{n\to\infty} \prod_{j=1}^{J_n} E\left[e^{i\left(u_1\gamma_1(t,s_j^n)\Delta L_1(s_j^n) + u_2\gamma_2(t,s_j^n)\Delta L_2(s_j^n)\right)}\right] \\
&= \lim_{n\to\infty} \prod_{j=1}^{J_n} E\left[e^{i\left(u_1\gamma_1(t,s_j^n), u_2\gamma_2(t,s_j^n)\right)\left(\begin{smallmatrix}\Delta L_1(s_j^n)\\ \Delta L_2(s_j^n)\end{smallmatrix}\right)}\right] \\
&= \lim_{n\to\infty} \prod_{j=1}^{J_n} E\left[e^{i\langle v(t,s_j^n), \Delta L(s_j^n)\rangle}\right],
\end{aligned}$$

where

$$v(t, s_j^n) = \left(u_1\gamma_1(t, s_j^n), u_2\gamma_2(t, s_j^n)\right)$$

and

$$\Delta L(s_j^n) = \left(\begin{smallmatrix}\Delta L_1(s_j^n)\\ \Delta L_2(s_j^n)\end{smallmatrix}\right).$$

$\Delta L(t)$ is a two-dimensional Lévy process, and its characteristic function is given by the Lévy-Khinchin formula:

$$E\left[e^{i\langle u, \Delta L(t)\rangle}\right] = e^{\Psi(u)\Delta t},$$

where

$$\Psi(u) = -\frac{1}{2}\langle u, \sigma u\rangle + \int_{\mathbb{R}_0^2} \left(e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle 1_{\{|z|<1\}}\right) \nu(dz)$$

and

$$\sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

So we get that:

$$\begin{aligned}
&\lim_{n\to\infty} \prod_{j=1}^{J_n} E\left[e^{i\langle v(t,s_j^n),\Delta L(s_j^n)\rangle}\right] \\
&= \lim_{n\to\infty} \prod_{j=1}^{J_n} e^{\Psi(v(t,s_j^n))\Delta s_j^n} \\
&= \lim_{n\to\infty} e^{\sum_{j=1}^{J_n} \Psi(v(t,s_j^n))\Delta s_j^n} \\
&= e^{\lim_{n\to\infty} \sum_{j=1}^{J_n} \Psi(v(t,s_j^n))\Delta s_j^n} \\
&= e^{\int_0^t \Psi(v(t,s))ds}.
\end{aligned}$$

Putting everything together, we have that

$$\Phi_{X(t)}(u) = \exp\left\{ i\langle u, \eta(t)\rangle + \int_0^t \Psi(v(t,s))\, ds \right\}.$$

□

We also give the definition of the moment-generating function, which is closely related to the characteristic function.

Definition 2.10. *Let $X \in \mathbb{R}^n$ be a random variable with distribution μ_X. Then its* moment generating function $\Psi_X : \mathbb{R}^n \to \mathbb{R}$ *is defined as:*

$$\Psi_X(u) = E[e^{\langle u,X\rangle}] = \int_{\mathbb{R}^n} e^{\langle u,x\rangle}\, d\mu_X(x), \quad \forall u \in \mathbb{R}^n.$$

We see that $\Psi_X(u) = \Phi_X(-iu)$. As opposed to the characteristic function, the moment generating function is not always well-defined.

2.3 Esscher transform

The *Esscher transform* of a stochastic variable with density f, is defined as

$$f(x,\theta) = \frac{e^{\theta x} f(x)}{\int_{\mathbb{R}} e^{\theta y} f(y) dy},$$

where $\theta \in \mathbb{R}$, and it was introduced by Esscher in 1932 (see [4]). Gerber and Shiu extended it to stochastic processes in 1994 (see [5]), in which case the Esscher transform defines a new equivalent probability measure. For a stochastic process $X(t)$, the *Esscher measure* Q^θ is defined as:

$$\left.\frac{dQ^\theta}{dP}\right|_{\mathscr{F}_t} = \frac{e^{\theta X(t)}}{E_P[e^{\theta X(t)}]},$$

where $E_P[e^{\theta X(t)}]$ is the moment-generating function of $X(t)$.

In the continuous case, the Esscher transform is the same as the Girsanov transform. But the Esscher transform can also be applied to stochastic processes with jumps.

In [1], the Esscher measure is defined for n-dimensional Lévy processes and with a time-dependent parameter $\theta(t)$. Let $L_i(t)$ be a Lévy process with Lévy triplet $(\alpha_i, \sigma_i, \nu_i)$ for $i = 1, \dots, n$:

$$L_i(t) = \alpha_i t + \sigma_i B_i(t) + J_i(t), \quad i = 1, \dots, n,$$

where $B_1, \dots, B_n$ are correlated Brownian motions and $J_i(t)$ denotes the jump term of $L_i(t)$:

$$J_i(t) = \int_0^t \int_{|z|<1} z_i \tilde{N}(dz, dt) + \int_0^t \int_{|z|\geq 1} z_i N(dz, dt),$$

where $N(dz, dt)$ is a Poisson random measure with compensator measure $\nu(dz, dt) = (\nu_1(dz, dt), \dots, \nu_n(dz, dt))$. Let

$$\hat{\theta}(t) = (\hat{\theta}_1(t), \dots, \hat{\theta}_n(t)),$$
$$\tilde{\theta}(t) = (\tilde{\theta}_1(t), \dots, \tilde{\theta}_n(t)),$$

and let $\theta(t)$ be a 2n-dimensional vector:

$$\theta(t) = (\hat{\theta}(t), \tilde{\theta}(t)).$$

We define for $i = 1, \dots n$,

$$\hat{Z}_i^\theta(t) = \exp\left\{ \int_0^t \hat{\theta}_i(s) dB(s) - \frac{1}{2} \int_0^t \hat{\theta}_i^2(s) ds \right\},$$

$$\tilde{Z}_i^\theta(t) = \exp\left\{\int_0^t \tilde{\theta}_i(s)dJ(s) - \phi(0,t,\tilde{\theta}_i(\cdot))\right\},$$

where $\exp\left\{\phi(0,t,\tilde{\theta}_i(\cdot))\right\}$ is the moment-generating function of $J(t)$, and we let

$$Z^\theta(t) = \prod_{i=1}^n \left(\hat{Z}_i^\theta(t) \times \tilde{Z}_i^\theta(t)\right).$$

Assuming that $\sup_{0\le t\le T} |\tilde{\theta}_i(t)| \le c_i$, where $c_i > 0$ is a constant ensuring that

$$\int_0^T \int_1^\infty (e^{c_i z_i} - 1)\nu_i(dz,ds) < \infty,$$

it follows that $\tilde{Z}(t)$ is a martingale process (see [1]). $\hat{Z}(t)$ is also a martingale since the Novikov condition holds. Therefore, we can define an equivalent probability measure Q^θ in the following way:

$$\left.\frac{dQ^\theta}{dP}\right|_{\mathscr{F}_t} = Z^\theta(t),$$

for $0 \le t \le T$. The following proposition states how the drift and the Lévy measure of the process changes when we apply the Esscher transform.

Proposition 2.11. [1, p. 97] *With respect to the probability measure Q^θ, the processes*

$$B_i^\theta(t) = B_i(t) - \int_0^t \hat{\theta}_i(s)ds$$

are Brownian motions for $i = 1,\dots,n$ and $0 \le t \le T$. Furthermore, for each $i = 1,\dots,n$, $L_i(t)$ is a Lévy process on $0 \le t \le T$ with drift

$$\alpha_i t + \int_0^t \int_{|z|<1} z_i(e^{\tilde{\theta}(s)z} - 1)\nu_i(dz,ds),$$

and compensator measure $e^{\tilde{\theta}(t)z}\nu_i(dz,dt)$. Under Q^θ, we denote the random jump measure associated with $J_i(t)$ by $N_i^\theta(dz,dt)$, and its compensator measure by $\tilde{N}_i^\theta(dz,dt)$.

We remark that the Brownian motions $B_1,\dots,B_n$ are still correlated after the measure change.

2.4 Fourier transform

If f is a bounded and continuous function in $L^1(\mathbb{R})$, the Fourier transform $\hat{f}$ of f is defined as:

$$\hat{f}(y) = \int_{\mathbb{R}} e^{-ixy} f(x) dx.$$

We see that the characteristic function of a random variable is the Fourier transform of its density. f is uniquely determined by its Fourier transform, and if $\hat{f} \in L^1(\mathbb{R})$, we can find f by the following inversion formula (see [1, p. 247]):

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixy} \hat{f}(y) dy.$$

This is not the most common definition of a Fourier transform, but it is the one that is used in [1]. The reason for having a minus sign in the exponent, is to get the following relation between the characteristic function of a random variable X, the Fourier transform of f and $E[f(X)]$.

$$\begin{aligned} E[f(X)] &= E\left[\frac{1}{2\pi} \int_{\mathbb{R}} e^{iXy} \hat{f}(y) dy\right] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) E\left[e^{iXy}\right] dy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) \phi_X(y) dy, \end{aligned}$$

where we used the Fubini theorem. So if we can find the Fourier transform of f and the characteristic function of X, we can find the expectation of $f(X)$. We will use this approach to find the spread option price in the model with jumps in Chapter 4.

Example 2.12. *We let X be a random variable, and $f(x) = \max(e^x - 1)$. We want to compute $E[f(X)]$, but since $f \in L^1\mathbb{R}$, we can not compute the Fourier transform directly. We therefore define a function f_α by*

$$f_\alpha(x) = e^{-\alpha x} f(x).$$

Since $f_\alpha \in L^1(\mathbb{R})$, we can compute its Fourier transform $\hat{f}_\alpha$.

$$\begin{aligned} \hat{f}_\alpha(y) &= \int_{\mathbb{R}} e^{-ixy} f_\alpha(x) dx \\ &= \int_{\mathbb{R}} e^{-ixy} e^{-\alpha x} \max(e^x - 1, 0) dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-ixy}e^{-\alpha x}(e^x - 1)dx \\
&= \int_0^\infty \left(e^{-(iy+\alpha-1)x} - e^{-(iy+\alpha)x}\right) dx \\
&= \frac{1}{iy+\alpha-1} - \frac{1}{iy+\alpha} \\
&= \frac{1}{\alpha^2 - \alpha - y^2 + i(2\alpha-1)y}.
\end{aligned}
$$

By the inversion formula, we have that

$$f_\alpha(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_\alpha(y)e^{ixy}dy.$$

So

$$
\begin{aligned}
f(x) &= e^{\alpha x} f_\alpha(x) \\
&= \frac{e^{\alpha x}}{2\pi}\int_{\mathbb{R}} \hat{f}_\alpha(y)e^{ixy}dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_\alpha(y)e^{(iy+\alpha)x}dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_\alpha(y)e^{i(y-i\alpha)x}dy.
\end{aligned}
$$

Hence, we see that

$$
\begin{aligned}
E[f(X)] &= E\left[\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_\alpha(y)e^{i(y-i\alpha)X}dy\right] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_\alpha(y)E\left[e^{i(y-i\alpha)X}\right] dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}_\alpha(y)\Phi_X\left(y - i\alpha\right) dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\Phi_X\left(y-i\alpha\right)}{\alpha^2-\alpha-y^2+i(2\alpha-1)y}dy.
\end{aligned}
$$

This result will be used in Chapter 4 to write the payoff function of a spread option in terms of a characteristic function.

Chapter 3

Two spot price models

As we argued in Chapter 1, Ornstein-Uhlenbeck processes are a natural class of processes when modelling spot prices in energy markets. OU processes are mean-reverting, which is an important feature distinguishing energy commodity prices from stock prices. We will model the logarithm of the price process (log-price) as an OU process, to assure positive prices.

The first model we will consider in this chapter, is an OU process driven by a bivariate Brownian motion. In this continuous model we can easily calculate the distributional properties of the prices, which will lead us to a closed form formula for the spread option price in the next chapter.

The drawback of the continuous model is that we are not able to explain the spikes in the observed prices. In the second model we consider, we use an OU process driven by a bivariate Lévy process with jumps instead of a Brownian motion. This model captures the spikes, but since Lévy processes are stationary, the seasonality is still not captured. A further improvement that we will not exploit here, is therefore to use Independent Increment (II) processes instead, as we mentioned in the introduction.

We will in the following let $S_1(t)$ denote the electricity price and $S_2(t)$ the gas price at time t. The prices are on the form $(S_1(t), S_2(t)) = (e^{X_1(t)}, e^{X_2(t)})$, where

$$\begin{cases} dX_i(t) = (\mu_i - \alpha_i X_i(t))dt + dL_i(t) \\ X_i(0) = x_i \end{cases} , \quad i = 1, 2,$$

and $(L_1(t), L_2(t))$ is a bivariate Lévy process. We see that the process fluctuates around a mean level $\left(\frac{\mu_i}{\alpha_i}, \frac{\mu_i}{\alpha_i}\right)$. When the process is below this mean level, the drift is positive, and when it is above the mean level, the drift is negative.

After solving the SDE´s describing the dynamics of $(X_1(t), X_2(t))$, we compute the characteristic function of $(X_1(t), X_2(t))$ in both models. The process converges to a stationary distribution because it is mean-reverting with stationary increments. We use the characteristic function to find the stationary distribution in the continuous model. We also simulate a realization of $(X_1(t), X_2(t))$ and $(S_1(t), S_2(t))$ for the continuous model.

3.1 Continuous model

We will start by looking at the continuous model. In this model, the increments of the logarithmic spot price process are normally distributed. As mentioned earlier, this model is less realistic than the model with jumps, bevause it does not capture the spikes in the prices. However, the model is easy to analyse, and it is therefore a natural start. We calculate the characteristic function of the spot price process, and use it to find the stationary distribution of the log-prices. Let

$$\begin{cases} dX_i(t) = (\mu_i - \alpha_i X_i(t))dt + \sigma_i dB_i(t) \\ X_i(0) = x_i \end{cases}, \quad i = 1, 2, \tag{3.1}$$

where B_1 and B_2 are Brownian motions with correlation ρ. We first solve the equation for $X_i(t)$.

Proposition 3.1. *Let $X_i(t)$ be given by the dynamics* (3.1) *above. Then*

$$X_i(t) = \eta_i(t) + \sigma_i \int_0^t e^{-\alpha_i(t-s)} dB_i(s), \tag{3.2}$$

where

$$\eta_i(t) = x_i e^{-\alpha_i t} + \frac{\mu_i}{\alpha_i}(1 - e^{-\alpha_i t}), \ i = 1, 2.$$

Proof. We let $Y_i(t) = e^{at} X_i(t)$, and calculate $dY_i(t)$ using Itô´s formula (see [6, p. 44]).

$$\begin{aligned} dY_i(t) &= \alpha_i e^{\alpha_i t} X_i(t)dt + e^{\alpha_i t} dX_i(t) \\ &= \alpha_i e^{\alpha_i t} X_i(t)dt + e^{\alpha_i t}\left[(\mu - \alpha_i X_i(t))dt + \sigma_i dB_i(t)\right] \\ &= e^{\alpha_i t}(\mu_i dt + \sigma_i dB_i(t)). \end{aligned}$$

It follows that

$$Y_i(t) = Y_i(0) + \mu_i \int_0^t e^{\alpha_i s} ds + \sigma_i \int_0^t e^{\alpha_i s} dB_i(s)$$

$$= Y_i(0) + \frac{\mu_i}{\alpha_i}(e^{\alpha_i t} - 1) + \sigma_i \int_0^t e^{\alpha_i s} dB_i(s).$$

Observing that $Y_i(0) = x_i$, we get that

$$\begin{aligned} X_i(t) &= e^{-\alpha_i t} Y_i(t) \\ &= x_i e^{\alpha_i t} + \frac{\mu_i}{\alpha_i}(1 - e^{\alpha_i t}) + \sigma_i \int_0^t e^{\alpha_i (s-t)} dB_i(s). \end{aligned}$$

□

The dynamics of the spot price process becomes:

$$\begin{aligned} dS_i(t) &= d(e^{X_i(t)}) \\ &= e^{X_i(t)} dX_i(t) + \frac{1}{2} e^{X_i(t)} (dX_i(t))^2 \\ &= S_i(t) \left[(\mu_i - \alpha_i X_i(t))\, dt + \sigma_i dB_i(t) \right] + \frac{1}{2} S_i(t) \sigma_i^2 dt \\ &= S_i(t) \left[\left(\mu_i - \alpha_i X_i(t) + \frac{1}{2}\sigma_i^2 \right) dt + \sigma_i dB_i(t) \right]. \end{aligned}$$

We will simulate a realization of the processes $X_1(t)$ and $X_2(t)$ from $t = 0$ to $t = 365$, where t is measured in days. The parameters we use are shown in Table 3.1.

Table 3.1: Parameters in the simulation

Parameter	Value
μ_1	0.4
μ_2	0.6
α_1	0.1
α_2	0.15
σ_1	0.1
σ_2	0.1
ρ	0.5

We have chosen the parameters such that $\frac{\mu_1}{\alpha_1} = \frac{\mu_2}{\alpha_2} = 4$, so that both price processes fluctuate around the same mean level $e^4 \approx 54.6$. We let both processes start at the mean level. σ_1 and σ_2 are equal. Since α_2 is larger

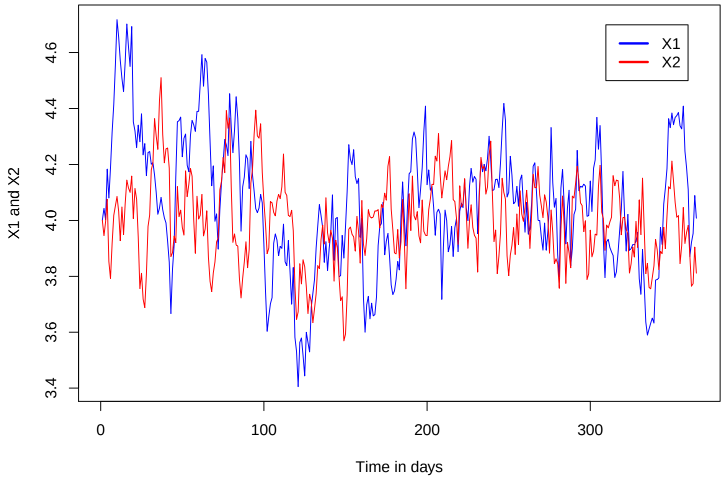

Figure 3.1: Simulation of $X_1(t)$ and $X_2(t)$ for $t = 1, \ldots, 365$.

than α_1, $X_2(t)$ has a stronger mean-reversion than $X_1(t)$. This leads to larger spikes in $X_1(t)$, which we clearly see in Figure 3.1.

To study the distribution of $X(t) = (X_1(t), X_2(t))$ and analyse its stationary properties, we calculate its characteristic function. We use the result from Proposition 2.9.

Proposition 3.2 Characteristic function of $X(t)$. *Let $X(t) = (X_1(t), X_2(t))$ be a bivariate gaussian Ornstein-Uhlenbeck process on the form* (3.2)*. Then the characteristic function of $X(t)$ is*

$$\Phi_{X(t)}(u_1, u_2) = \exp\left\{ iu_1\eta_1(t) + iu_2\eta_2(t) - \frac{1}{2}\left(u_1^2\beta_1(t) + u_2^2\beta_2(t) - \rho u_1 u_2 \zeta(t)\right)\right\},$$

where

$$\beta_i(t) = \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i t}),\ i = 1, 2,$$
$$\zeta(t) = \frac{\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(1 - e^{-(\alpha_1+\alpha_2)t}).$$

Proof. We use Proposition 2.9 with $\eta(t) = (\eta_1(t), \eta_2(t))$ and $\gamma(t, s) = (e^{-\alpha_1(t-s)}, e^{-\alpha_2(t-s)})$. It follows that

$$\Phi_{X(t)}(u) = \exp\left\{ iu\eta(t) - \frac{1}{2}\int_0^t v(t, s)\sigma v(t, s)^T\, ds\right\},$$

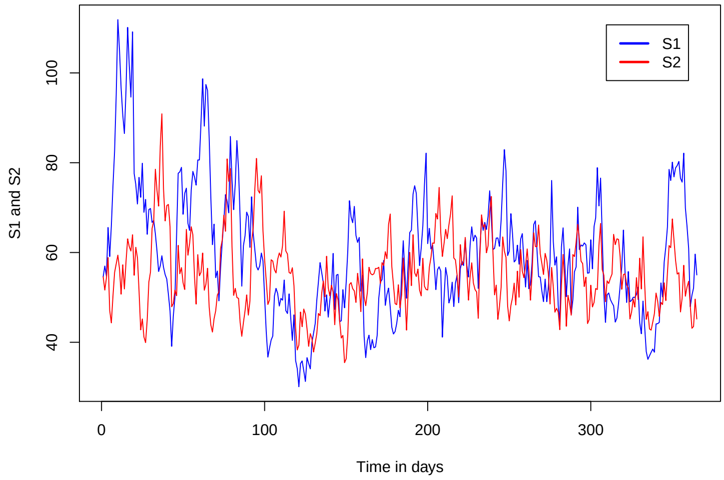

Figure 3.2: Simulation of $S_1(t)$ and $S_2(t)$ for $t = 1, \ldots, 365$.

where

$$v(t,s) = (u_1 e^{-\alpha_1(t-s)}, u_2 e^{-\alpha_2(t-s)}).$$

We see that

$$v(t,s)\sigma v(t,s)^T = u_1^2\sigma_1^2 e^{-2\alpha_1(t-s)} + u_2^2\sigma_2^2 e^{-2\alpha_2(t-s)} + 2\rho u_1 u_2 \sigma_1\sigma_2 e^{-(\alpha_1+\alpha_2)(t-s)},$$

and hence

$$\begin{aligned}\int_0^t v(t,s)\sigma v(t,s)^T ds = {} & \frac{u_1^2\sigma_1^2}{2\alpha_1}(1-e^{-2\alpha_1 t}) + \frac{u_2^2\sigma_2^2}{2\alpha_2}(1-e^{-2\alpha_2 t}) \\ & + \frac{\rho u_1 u_2 \sigma_1 \sigma_2}{\alpha_1+\alpha_2}(1-e^{-(\alpha_1+\alpha_2)t}).\end{aligned}$$

This gives us the desired result. □

From the characteristic function we see that $(X_1(t), X_2(t))$ follows a bivariate normal distribution with expectation $(\eta_1(t), \eta_2(t))$ and covariance matrix

$$\sigma(t) = \begin{pmatrix} \beta_1(t) & \rho\zeta(t) \\ \rho\zeta(t) & \beta_2(t) \end{pmatrix}.$$

So

$$X_i(t) \sim N\left(\eta_i(t), \beta_i(t)\right),\ i = 1,2$$

and

$$Cov(X_1(t), X_2(t)) = \rho\zeta(t).$$

The correlation between $X_1(t)$ and $X_2(t)$ is

$$\begin{aligned}\rho_{X_1,X_2}(t) &= \frac{\rho\zeta(t)}{\sqrt{\beta_1(t)\beta_2(t)}}\\ &= \frac{2\rho(1-e^{-(\alpha_1+\alpha_2)t})}{\alpha_1+\alpha_2}\sqrt{\frac{\alpha_1\alpha_2}{(1-e^{-2\alpha_1 t})(1-e^{-2\alpha_2 t})}}.\end{aligned}$$

To find the stationary distribution, we let $t \to \infty$. Then we see that $\eta_i(t) \to \frac{\mu_i}{\alpha_i}$, $\beta_i(t) \to \frac{\sigma_i^2}{2\alpha_i}$ and $\zeta(t) \to \frac{\sigma_1\sigma_2}{\alpha_1+\alpha_2}$. So the stationary distribution of $(X_1(t), X_2(t))$ is a bivariate normal distribution with expectation $(\frac{\mu_1}{\alpha_1}, \frac{\mu_2}{\alpha_2})$ and covariance matrix

$$\sigma = \begin{pmatrix} \frac{\sigma_1^2}{2\alpha_1} & \frac{\rho\sigma_1\sigma_2}{\alpha_1+\alpha_2} \\ \frac{\rho\sigma_1\sigma_2}{\alpha_1+\alpha_2} & \frac{\sigma_2^2}{2\alpha_2} \end{pmatrix}.$$

The correlation between $X_1(t)$ and $X_2(t)$ converges to

$$\rho_{X_1,X_2} = \frac{2\rho\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}.$$

3.2 Model with jumps

Now we move on to the model where the log-price process $X(t) = (X_1(t), X_2(t))$ is described by an OU Lévy process with jumps. Let

$$\begin{cases} dX_i(t) = (\mu_i - \alpha_i X_i(t))dt + dL_i(t) \\ X_i(0) = x_i \end{cases}, \quad i = 1, 2, \tag{3.3}$$

where $(L_1(t), L_2(t))$ is a bivariate Lévy process on the form:

$$dL_i(t) = \sigma_i dB_i(t) + dJ_i(t), \quad i = 1, 2,$$

where B_1 and B_2 are Brownian motions with correlation ρ as before, and $J_i(t)$ denotes the jump term:

$$dJ_i(t) = \int_{|z|<1} z_i\tilde{N}(dz_1, dz_2, dt) + \int_{|z|\geq 1} z_i N(dz_1, dz_2, dt), \quad i = 1, 2,$$

where N is a Poisson random measure and $\tilde{N}$ is the corresponding compensated Poisson random measure. The correlations between the jumps sizes and jumps times are now contained in the measure N. We will not specify this measure any further in this thesis.

We solve the equation for $X_i(t)$ using Itô´s formula for Lévy processes.

PROPOSITION 3.3. *Let $X_i(t)$, $i = 1, 2$, be given by the dynamics* (3.3) *above. Then*

$$X_i(t) = \eta_i(t) + \int_0^t e^{-\alpha_i(t-s)} dL_i(s), \tag{3.4}$$

where

$$\eta_i(t) = x_i e^{-\alpha t} + \frac{\mu_i}{\alpha_i}(1 - e^{-\alpha_i t}).$$

Proof. We let $Y_i(t) = e^{\alpha_i t} X_i(t)$, and use Itô´s formula for Lévy processes (see [7, p. 6]) to find $dY_i(t)$.

$$\begin{aligned} dY_i(t) =& \alpha_i e^{\alpha_i t} X_i(t)dt + e^{\alpha_i t}\left((\mu_i - \alpha_i X_i(t))dt + \sigma_i dB_i(t)\right) \\ &+ \int_{|z|<1} \left(e^{\alpha_i t}(X_i(t) + z) - e^{\alpha_i t} X_i(t) - e^{\alpha_i t} z\right) \nu(dz_1, dz_2)ds \\ &+ \int_{|z|<1} \left(e^{\alpha_i t}(X_i(t) + z) - e^{\alpha_i t} X_i(t)\right) \tilde{N}(dz_1, dz_2, dt) \\ &+ \int_{|z|\geq 1} \left(e^{\alpha_i t}(X_i(t) + z) - e^{\alpha_i t} X_i(t)\right) N(dz_1, dz_2, dt) \\ =& e^{\alpha_i t}\left(\mu_i dt + \sigma_i dB_i(t)\right) + \int_{|z|<1} z_i e^{\alpha_i t} \tilde{N}(dz_1, dz_2, dt) \\ &+ \int_{|z|\geq 1} z_i e^{\alpha_i t} N(dz_1, dz_2, dt). \end{aligned}$$

It follows that

$$\begin{aligned} Y_i(t) =& Y_i(0) + \mu_i \int_0^t e^{\alpha_i s} ds + \sigma \int_0^t e^{\alpha_i s} dB_i(s) + \int_0^t \int_{|z|<1} z_i e^{\alpha_i s} \tilde{N}(dz_1, dz_2, ds) \\ &+ \int_0^t \int_{|z|\geq 1} z_i e^{\alpha_i s} N(dz_1, dz_2, ds). \end{aligned}$$

Observing that $X_i(0) = Y_i(0)$, it follows that

$$\begin{aligned} X_i(t) =& e^{-\alpha_i t} Y_i(t) \\ =& e^{-\alpha_i t} x_i + \frac{\mu_i}{\alpha_i}\left(1 - e^{-\alpha t}\right) + \sigma_i \int_0^t e^{-\alpha_i(t-s)} dB(s) \\ &+ \int_0^t \int_{|z|<1} z_i e^{\alpha_i(t-s)} \tilde{N}(dz_1, dz_2, dt) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{|z|\geq 1} z_i e^{\alpha_i(t-s)} N(dz_1, dz_2, dt) \\
=&x_i e^{-\alpha_i t} + \frac{\mu_i}{\alpha_i}\left(1-e^{-\alpha t}\right) + \int_0^t e^{-\alpha_i(t-s)} dL(s).
\end{aligned}
$$

□

To find the spot price dynamics, we also use Itô´s formula for Lévy processes.

$$
\begin{aligned}
dS_i(t) =& d(e^{X_i(t)}) \\
=& e^{X_i(t)}\left((\mu_i - \alpha_i X_i(t))dt + \sigma_i dB_i(t)\right) + \frac{1}{2}\sigma_i^2 e^{X_i(t)} dt \\
&+ \int_{|z|<1} \left(e^{X_i(t)+z} - e^{X_i(t)} - e^{X_i(t)} z\right) \nu(dz_1, dz_2) dt \\
&+ \int_{|z|<1} \left(e^{X_i(t)+z} - e^{X_i(t)}\right) \tilde{N}(dz_1, dz_2, dt) \\
&+ \int_{|z|\geq 1} \left(e^{X_i(t)+z} - e^{X_i(t)}\right) N(dz_1, dz_2, dt) \\
=& e^{X_i(t)}\Bigg[\left(\mu_i - \alpha_i X_i(t) + \frac{1}{2}\sigma_i^2\right) dt + \sigma_i dB_i(t) + \int_{|z|<1} \left(e^z - 1 - z\right) \nu(dz_1, dz_2) dt \\
&+ \int_{|z|<1} \left(e^z - 1\right) \tilde{N}(dz_1, dz_2, dt) + \int_{|z|\geq 1} \left(e^z - 1\right) N(dz_1, dz_2, dt)\Bigg] \\
=& S_i(t)\Bigg[\left(\mu_i - \alpha_i X_i(t) + \frac{1}{2}\sigma_i^2 + \int_{|z|<1} \left(e^z - 1 - z\right) \nu(dz_1, dz_2)\right) dt + \sigma_i dB_i(t) \\
&+ \int_{|z|<1} \left(e^z - 1\right) \tilde{N}(dz_1, dz_2, dt) + \int_{|z|\geq 1} \left(e^z - 1\right) N(dz_1, dz_2, dt)\Bigg].
\end{aligned}
$$

To find the characteristic function of $(X_1(t), X_2(t))$, we use Proposition 2.9 where we found the characteristic function of a bivariate Itô-Lévy process.

Proposition 3.4 Characteristic function of $X(t)$. *Let $X(t) = (X_1(t), X_2(t))$ be a bivariate Ito-Lévy process on the form (3.4). Then the characteristic function of $X(t)$ is:*

$$
\Phi_{X(t)}(u_1, u_2) = \exp\Bigg\{ iu_1\eta_1(t) + iu_2\eta_2(t) - \frac{1}{2}\left(u_1^2\beta_1(t) + u_2^2\beta_2(t) - \rho u_1 u_2 \zeta(t)\right)
$$

$$+\int_0^t \int_{\mathbb{R}_0^2} \left(e^{i\langle v(s),z\rangle} - 1 - i\langle v(s), z\rangle 1_{\{|z|<1\}}\right) \nu(dz)ds \Bigg\},$$

where

$$\begin{aligned}
\eta_i(t) &= x_i e^{-\alpha_i t} + \frac{\mu_i}{\alpha_1}(1 - e^{-\alpha_i t}),\ i = 1, 2,\\
\beta_i(t) &= \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i t}),\ i = 1, 2,\\
\zeta(t) &= \frac{\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(1 - e^{-(\alpha_1+\alpha_2)t}),\\
v(s) &= (u_1\, e^{-\alpha_1 s}, u_2\, e^{-\alpha_2 s}).
\end{aligned}$$

Proof. We use Proposition 2.9 with

$$\eta(t) = \left(x_1 e^{-\alpha_1 t} + \frac{\mu_1}{\alpha_1}(1 - e^{-\alpha_1 t}), x_2 e^{-\alpha_2 t} + \frac{\mu_2}{\alpha_2}(1 - e^{-\alpha_2 t})\right)$$

and

$$\gamma(t, s) = (e^{-\alpha_1(t-s)}, e^{-\alpha_2(t-s)}).$$

It follows that

$$\Phi_{X(t)}(u) = \exp\left\{i\langle u, \eta(t)\rangle + \int_0^t \Psi(v(t, s))\, ds\right\},$$

where

$$v(t, s) = (u_1 e^{-\alpha_1(t-s)}, u_2 e^{-\alpha_2(t-s)}).$$

and

$$\Psi(\cdot) = -\frac{1}{2}\langle\cdot, \sigma\cdot\rangle + \int_{\mathbb{R}_0^2} \left(e^{i\langle\cdot,z\rangle} - 1 - i\langle\cdot, z\rangle 1_{\{|z|<1\}}\right) \nu(dz).$$

We observe that $v(t, s)$ is a function of $t - s$, so we do a variable change and let $w = t - s$. Let

$$v(w) = (u_1 e^{-\alpha_1 w}, u_2 e^{-\alpha_2 w}).$$

We see that the limits in the integral are interchanged when we do the variable change, so we get a minus sign in front of the integral. However, since $dw = -ds$, the minus sign is cancelled. Denoting w by s, the result follows. □

We can´t say anything about the properties of the stationary distribution in this case without specifying the Lévy measure.

Chapter 4

Pricing spread options with strike price $K = 0$

We recall from Section 1.3 that a European spread option has payoff function

$$\max(S_1(T) - hS_2(T) - K, 0), \tag{4.1}$$

at time T, where $S_1(T)$ and $S_2(T)$ are the prices of the two underlying commodities at maturity time T, K is the strike price and h is the heatrate. In this chapter, we will let $K = 0$. In the continuous case, this will lead to a closed form formula for the option price, corresponding to the *Margrabe formula* for spread options on stock prices (see [2]). The prices are modelled as exponential Ornstein-Uhlenbeck processes as in the previous chapter.

$$S_i(t) = e^{X_i(t)}, \quad i = 1, 2,$$

where

$$X_i(t) = \eta_i(t) + \int_0^t e^{-\alpha_i(t-s)} dL_i(s),\ i = 1, 2,$$
$$\eta_i(t) = e^{-\alpha_i t} x_i + \frac{\mu_i}{\alpha_i}(1 - e^{-\alpha_i t}),\ i = 1, 2$$

and (L_1, L_2) is a bivariate Lévy process.

The price of an option is given as the expected value of its payoff under an equivalent martingale measure, discounted by the risk-free interest rate. This follows from arbitrage arguments (see, for instance, [6]). The equivalent martingale measure is a probability measure equivalent to the market probability P under which the discounted price process is a martingale. But since electricity and temperature are not storable, hedging is not possible in these markets, and hence any probability measure equivalent to P is an equivalent martingale measure (see [1]). We will therefore use the market probability

P when computing the prices. Since it does not exist a unique equivalent martingale measure, these markets are not complete.

We assume that the risk free interest rate r is constant. The price of the option at time $0 \leq t \leq T$, is therefore given as:

$$p(t,T) = e^{-r(T-t)}E[\max(S_1(T) - hS_2(T), 0)|\mathscr{F}_t].$$

We can rewrite this as

$$p(t,T) = e^{-r(T-t)}E\left[S_2(T)\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\bigg|\mathscr{F}_t\right].$$

We will perform a measure change to get the maximum function alone inside the conditional expectation. We define a probability measure Q equivalent to P by

$$\left.\frac{dQ}{dP}\right|_{\mathscr{F}_t} = Z(t), \quad 0 \leq t \leq T.,$$

where $Z(t)$ is a martingale. For the continuous model, Q will be defined by the Girsanov theorem. For the model with jumps, it will be defined by the Esscher transform. We get that:

$$\begin{aligned} p(t,T) &= e^{-r(T-t)}\frac{E_Q\left[\frac{S_2(T)}{Z(T)}\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\Big|\mathscr{F}_t\right]}{E_Q\left[\frac{1}{Z(T)}\Big|\mathscr{F}_t\right]} \\ &= e^{-r(T-t)}Z(t)E_Q\left[\frac{S_2(T)}{Z(T)}\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\bigg|\mathscr{F}_t\right] \end{aligned}$$

where we used Bayes formula (see [6]) and the fact that $Z(t)$ is a martingale. $\frac{S_2(T)}{Z(T)}$ is a deterministic function and can be put outside of the conditional expectation. Hence we get

$$p(t,T) = e^{-r(T-t)}Z(t)\frac{S_2(T)}{Z(T)}E_Q\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\bigg|\mathscr{F}_t\right]. \tag{4.2}$$

If we can find the dynamics of $\frac{S_1(t)}{S_2(t)}$ under Q, we have only one underlying price process, and our problem is reduced to the problem of pricing a call option. In the continuous case, we can then apply the Black and Scholes formula. In the jump case, we will use a Fourier technique to find an expression for the price in terms of the characteristic function of $X_1(T) - X_2(T)$. Note that this approach can only be used when $K = 0$.

4.1 Continuous model

We let

$$X_i(t) = \eta_i(t) + \sigma_i \int_0^t e^{-\alpha_i(t-s)} dB_i(s), \quad i = 1, 2, \tag{4.3}$$

where

$$\eta_i(t) = e^{-\alpha_i t} x_i + \frac{\mu_i}{\alpha_i}(1 - e^{-\alpha_i t}),\ i = 1, 2,$$

and $B_1(t)$ and $B_2(t)$ are Brownian motions with correlation ρ.

Proposition 4.1. *Let $X_i(t) = (X_1(t), X_2(t))$ be given by the dynamics (4.3). Consider a spread option with payoff (4.1) at time T, and let the interest rate r be constant. Let $p_0(0,T)$ denote the price of the spread option at time $t = 0$. Then,*

$$\begin{aligned} p(0,T) &= \exp\left\{-rT + \eta_1(T) + \frac{1}{2}\beta_1(T)\right\} \Phi(k + \sigma(T)) \\ &\quad - \exp\left\{-rT + \eta_2(T) + \frac{1}{2}\beta_2(T)\right\} h\Phi(k), \end{aligned}$$

where

$$k = \frac{\eta_1(T) - \eta_2(T) + \frac{1}{2}(\beta_1(T) - \beta_2(T)) - \log h}{\sigma(T)} - \frac{1}{2}\sigma(T)$$

and

$$\begin{aligned} \beta_i(T) &= \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i T}), \quad i = 1, 2, \\ \zeta(T) &= \frac{\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(1 - e^{-(\alpha_1+\alpha_2)T}) \text{ and} \\ \sigma(T) &= \sqrt{\beta_1(T) + \beta_2(T) - 2\rho\zeta(T)}. \end{aligned}$$

Proof. We let

$$\theta_T(t) = \sigma_2 e^{-\alpha_2(T-t)}$$

and

$$\begin{aligned} Z_T(t) &= \exp\left\{\int_0^t \theta_T(s)\, dB_2(s) - \frac{1}{2}\int_0^t \theta_T^2(s)\, ds\right\} \\ &= \exp\left\{\sigma_2 \int_0^t e^{-\alpha_2(T-s)}\, dB_2(s) - \frac{\beta_2}{2}\int_0^t e^{-2\alpha_2(t-s)}\, ds\right\} \end{aligned}$$

$$= \exp\left\{\sigma_2 \int_0^t e^{-\alpha_2(T-s)}\, dB_2(s) - \frac{\beta_2}{4\alpha_2}(e^{-2\alpha_2(T-t)} - e^{-2\alpha_2 T})\right\}.$$

From Girsanovs theorem we know that $Z_T(t)$ is a martingale, and we can define a new probability measure Q equivalent to P by $\frac{dQ}{dP}\big|_{\mathscr{F}_t} = Z_T(t)$. From (4.2) we know that

$$\begin{aligned} p(0,T) &= e^{-rT} Z_T(t) \frac{S_2(T)}{Z_T(T)} E\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\right] \\ &= e^{-rT + \frac{1}{2}\beta_2(T)} Z_T(t) E\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\right]. \end{aligned}$$

Since $\frac{S_1(T)}{S_2(T)} = e^{X_1(T)-X_2(T)}$, we want to find the distribution of $X_1(T)-X_2(T)$ under Q. From Girsanovs theorem we know that

$$W_2(t) = B_2(t) - \int_0^t \theta_T(s)ds \tag{4.4}$$

is a Brownian motion under Q. Since $B_1(t)$ and $B_2(t)$ have correlation ρ, $B_1(t)$ can be written as

$$B_1(t) = \rho B_2(t) + \sqrt{1-\rho^2} B_3(t),$$

where B_3 is a Brownian motion under P which is independent of B_2. Under Q, B_3 is still a Brownian motion independent of B_2. We define W_1 as

$$W_1(t) = \rho W_2(t) + \sqrt{1-\rho^2} B_3(t).$$

W_1 is a Brownian motion under Q which has correlation ρ with B_2. Using (4.4) we get that

$$\begin{aligned} W_1(t) &= \rho B_2(t) - \rho \int_0^t \theta_T(s)ds + \sqrt{1-\rho^2} B_3(t) \\ &= B_1(t) - \rho \int_0^t \theta_T(s)ds. \end{aligned}$$

So we get that:

$$\begin{aligned} X_1(t) &= \eta_1(t) + \sigma_1 \int_0^t e^{-\alpha_1(t-s)} dB_1(s) \\ &= \eta_1(t) + \sigma_1 \int_0^t e^{-\alpha_1(t-s)}\, (dW_1(s) + \rho\theta_T(s)ds) \end{aligned}$$

$$
\begin{aligned}
&= \eta_1(t) + \rho\sigma_1\sigma_2 \int_0^t e^{-\alpha_1(t-s)} e^{-\alpha_2(T-s)} ds + \sigma_1 \int_0^t e^{-\alpha_1(t-s)} dW_1(s) \\
&= \eta_1(t) + \frac{\rho\sigma_1\sigma_2}{\alpha_1+\alpha_2}\left(e^{-\alpha_2(T-t)} - e^{-\alpha_1 t-\alpha_2 T}\right) + \sigma_1 \int_0^t e^{-\alpha_1(t-s)} dW_1(s)
\end{aligned}
$$

and

$$
\begin{aligned}
X_2(t) &= \eta_2(t) + \sigma_2 \int_0^t e^{-\alpha_2(t-s)} dB_2(s) \\
&= \eta_2(t) + \sigma_1 \int_0^t e^{-\alpha_2(t-s)} \left(dW_2(s) + \theta_T(s)ds\right) \\
&= \eta_2(t) + \sigma_2^2 \int_0^t e^{-\alpha_2(t-s)} e^{-\alpha_2(T-s)} ds + \sigma_2 \int_0^t e^{-\alpha_2(t-s)} dW_2(s) \\
&= \eta_2(t) + \frac{\sigma_2^2}{2\alpha_2}\left(e^{-\alpha_2(T-t)} - e^{-\alpha_2(t+T)}\right) + \sigma_2 \int_0^t e^{-\alpha_2(t-s)} dW_2(s).
\end{aligned}
$$

So $X_i(t)$ can still be written on the form (3.2), but the drift $\eta_i(t)$ has changed. Since the correlation between $W_1(t)$ and $W_2(t)$ is still ρ, it follows from Proposition 2.9 that the characteristic function of $(X_1(t), X_2(t))$ is

$$
\Phi_{X(t)}(u_1,u_2) = \exp\left\{ iu_1\eta_1'(t) + iu_2\eta_2'(t) - \frac{1}{2}\left(u_1^2\beta_1(t) + u_2^2\beta_2(t) - \rho u_1 u_2 \zeta(t)\right)\right\},
$$

where

$$
\begin{aligned}
\eta_1'(t) &= \eta_1(t) + \frac{\rho\sigma_1\sigma_2}{\alpha_1+\alpha_2}\left(e^{-\alpha_2(T-t)} - e^{-\alpha_1 t-\alpha_2 T}\right), \\
\eta_2'(t) &= \eta_2(t) + \frac{\sigma_2^2}{2\alpha_2}\left(e^{-\alpha_2(T-t)} - e^{-\alpha_2(t+T)}\right), \\
\beta_i(t) &= \frac{\sigma_i^2}{2\alpha_i}(1-e^{-2\alpha_i t}),\ i=1,2, \\
\zeta(t) &= \frac{\sigma_1\sigma_2}{\alpha_1+\alpha_2}(1-e^{-(\alpha_1+\alpha_2)t}).
\end{aligned}
$$

So from Chapter 3.1 we know that $(X_1(t), X_2(t))$ follows a bivariate normal distribution with expectation $(\eta_1'(t), \eta_2'(t))$ and covariance matrix

$$
\sigma(t) = \begin{pmatrix} \beta_1(t) & \rho\zeta(t) \\ \rho\zeta(t) & \beta_2(t) \end{pmatrix}.
$$

So $X_1(t) - X_2(t)$ is normally distributed with

$$
E\left[X_1(t) - X_2(t)\right] = \eta_1'(t) - \eta_2'(t)
$$

and

$$Var\left(X_1(t) - X_2(t)\right) = \beta_1(t) + \beta_2(t) - 2\rho\zeta(t).$$

To ease the notation, we let

$$\sigma(t)^2 = \beta_1(t) + \beta_2(t) - 2\rho\zeta(t).$$

We also observe that $\eta_1'(t) = \beta_1(T)$ and $\eta_2'(t) = \rho\zeta(T)$, and we let

$$g(T) = \eta_1(T) - \eta_2(T) + \rho\zeta(T) - \beta(T).$$

So $\frac{S_1(T)}{S_2(T)}$ is lognormally distributed, and we can write it as

$$\frac{S_1(T)}{S_2(T)} = e^{X_1(T)-X_2(T)} = e^{g(T)+\sigma(T)\cdot\epsilon},$$

where $\epsilon \sim N(0,1)$. We can now calculate

$$E_Q\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\right]$$

by integrating $\frac{S_1(T)}{S_2(T)} - h$ times the density of the standard normal distribution over all values of ϵ such that $\frac{S_1(T)}{S_2(T)} - h > 0$, i.e. for all $\epsilon > -k$, where

$$k = \frac{g(T) - \log h}{\sigma(T)}.$$

We compute the integral:

$$\begin{aligned} E_Q\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\right] &= \int_{-k}^{\infty} \left(e^{g(T)+\sigma(T)x} - h\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &= e^{g(T)} \int_{-k}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\sigma(T)x - \frac{x^2}{2}}\, dx - h \int_{-k}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx. \end{aligned}$$

We see that

$$h \int_{-k}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = h(1 - \Phi(-k)) = h\Phi(k),$$

where Φ is the distribution function of the standard normal distribution. To solve the other integral, we observe that

$$\sigma(T)x - \frac{x^2}{2} = -\frac{1}{2}(x - \sigma(T))^2 + \frac{\sigma^2(T)}{2}.$$

Letting $v = x - \sigma(T)$, we get

$$\begin{aligned}\int_{-k}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\sigma(T)x - \frac{x^2}{2}} dx &= e^{\frac{\sigma^2(T)}{2}} \int_{-k-\sigma(T)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{v^2}{2}} dv \\ &= e^{\frac{\sigma^2(T)}{2}} (1 - \Phi(-k - \sigma(T))) \\ &= e^{\frac{\sigma^2(T)}{2}} \Phi(k + \sigma(T)).\end{aligned}$$

So

$$\begin{aligned}&E_Q\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\right] \\ &= \exp\left\{g(T) + \frac{1}{2}\sigma^2(T)\right\}\Phi(k + \sigma(T)) - h\Phi(k) \\ &= \exp\left\{\eta_1(T) - \eta_2(T) + \frac{1}{2}(\beta_1(T) - \beta_2(T))\right\}\Phi(k + \sigma(T)) - h\Phi(k).\end{aligned}$$

Finally we have

$$\begin{aligned}&E[\max(S_1(T) - hS_2(T), 0)] \\ &= \exp\left\{\eta_2(T) + \frac{1}{2}\beta_2(T)\right\} E_Q\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\right] \\ &= \exp\left\{\eta_2(T) + \frac{1}{2}\beta_2(T) + \eta_1(T) - \eta_2(T) + \frac{1}{2}(\beta_1(T) - \beta_2(T))\right\}\Phi(k + \sigma(T)) \\ &\quad - \exp\left\{\eta_2(T) + \frac{1}{2}\beta_2(T)\right\} h\Phi(k) \\ &= \exp\left\{\eta_1(T) + \frac{1}{2}\beta_1(T)\right\}\Phi(k + \sigma(T)) - \exp\left\{\eta_2(T) + \frac{1}{2}\beta_2(T)\right\} h\Phi(k).\end{aligned}$$

We observe that

$$\begin{aligned}k &= \frac{g(T) - \log h}{\sigma(T)} \\ &= \frac{\eta_1(T) - \eta_2(T) + \rho\zeta(T) - \beta_2(T) - \log h}{\sigma(T)} \\ &= \frac{\eta_1(T) - \eta_2(T) + \rho\zeta(T) - \beta_2(T) - \log h + \frac{1}{2}\sigma^2(T)}{\sigma(T)} - \frac{1}{2}\sigma(T)\end{aligned}$$

$$= \frac{\eta_1(T) - \eta_2(T) + \frac{1}{2}(\beta_1(T) - \beta_2(T)) - \log h}{\sigma(T)} - \frac{1}{2}\sigma(T).$$

□

To see how this spread option price is depending on ρ, we plot in Figure 4.1 $p(0,T)$ as a function of ρ. We let the other parameters be as in Chapter 3 (see Table 3.1).

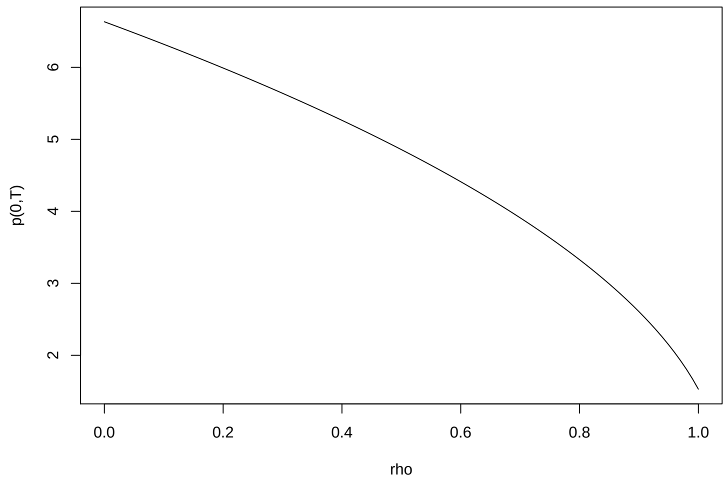

Figure 4.1: $p(0,T)$ as a function of ρ.

We see that the price is a decreasing function of ρ. When the correlation is high, the probability of a large deviation between the underlying prices, and hence a large payoff, is small.

4.2 Model with jumps

We now let

$$X_i(t) = \eta_i(t) + \int_0^t e^{-\alpha_i(t-s)} dL_i(s), \quad i = 1, 2, \tag{4.5}$$

where

$$\eta_i(T) = x_i e^{-\alpha_i T} + \frac{\mu_i}{\alpha_1}(1 - e^{-\alpha_i T}),\ i = 1, 2,$$

and $L(t) = (L_1(t), L_2(t))$ is a bivariate Lévy process with jumps on the form

$$L_i(t) = \sigma_i B_i(t) + J_i(t), \quad i = 1, 2,$$

where $B_1(t)$ and $B_2(t)$ are Brownian motions with correlation ρ, and $J_i(t)$ denotes the jump terms:

$$J_i(t) = \int_0^t \int_{|z|<1} z_i \tilde{N}(dz_1, dz_2, ds) + \int_0^t \int_{|z|\geq 1} z_i N(dz_1, dz_2, ds), \quad i = 1, 2,$$

where N is a Poisson random measure and $\tilde{N}$ is the corresponding compensated Poisson random measure.

We recall from equation (4.2) that the spread option price is given by the formula

$$p(t,T) = e^{-r(T-t)} Z(t) \frac{S_2(T)}{Z(T)} E_Q \left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0 \right) \middle| \mathscr{F}_t \right].$$

We will in this model use the Esscher transform to change the measure. Then we find the dynamics of $X_1(T) - X_2(T)$ under the new measure, and compute its characteristic function of $X_1(T) - X_2(T)$. Finally, we find an expression for the price in terms of this characteristic function.

Proposition 4.2. *Let $X_i(t) = (X_1(t), X_2(t))$ be given by the dynamics (4.5). Consider a spread option with payoff (4.1) at time T, and let the interest rate r be constant. Let $p_0(0,T)$ denote the price of the spread option at time $t = 0$. Then,*

$$p(0,T) = \frac{1}{2\pi} \exp\left\{ -rT + \frac{1}{2}\beta_2(T) - h(T) \right\} \int_{\mathbb{R}} \frac{\Phi_{X_1(T)-X_2(T)}(y - i\alpha)}{\alpha^2 - \alpha - y^2 + i(2\alpha - 1)y} dy,$$

where $\alpha \in \mathbb{R}$,

$$\beta_2(T) = \frac{\sigma_2^2}{2\alpha_2}(1 - e^{-2\alpha_2 T}),$$

$$h(T) = \int_0^T \int_{\mathbb{R}_0^2} \left(e^{z_2 e^{-\alpha_2(T-s)}} - 1 - z_2 e^{-\alpha_2(T-s)} 1_{\{|z|<1\}} \nu(dz_1, dz_2) \right) ds,$$

and $\Phi_{X_1(T)-X_2(T)}$ is the characteristic function of $X_1(T) - X_2(T)$;

$$\Phi_{X_1(T)-X_2(T)}(u)$$
$$= \exp\left\{ iu \left(\eta_1(T) - \eta_2(T) + \rho\zeta(T) - \beta(T) + \int_0^T \int_{|z|<1} (z_1 - z_2)(e^{z_2 e^{-\alpha_2(T-s)}} - 1)\nu(dz_1, dz_2) ds \right) \right.$$

$$- \frac{1}{2}u^2 \left(\beta_1(T) + \beta_2(T) - \rho\zeta(T)\right) + \int_0^T \int_{\mathbb{R}_0^2} \left(e^{i\langle v(s),z\rangle} - 1 - i\langle v(s), z\rangle 1_{\{|z|<1\}}\right) \nu^{\theta_T}(dz)ds \Bigg\},$$

where

$$\beta_i(T) = \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i T}),\ i = 1, 2,$$
$$\zeta(T) = \frac{\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(1 - e^{-(\alpha_1+\alpha_2)T}),$$
$$v(s) = (e^{-\alpha_1 s}, -e^{-\alpha_2 s}).$$

Proof. We define the Esscher transform for $L(t)$ as in Chapter 1. Let

$$\theta_T(t) = (\hat{\theta}_{1,T}(t), \hat{\theta}_{2,T}(t), \tilde{\theta}_{1,T}(t), \tilde{\theta}_{2,T}(t)) = (0, \sigma_2 e^{-\alpha_2(T-t)}, 0, e^{-\alpha_2(T-t)}).$$

Then we let

$$\hat{Z}^{\theta_T}(t) = \exp\left\{\int_0^t \hat{\theta}_T(s)dB(s) - \frac{1}{2}\int_0^t \hat{\theta}_T^2(s)ds\right\} = \exp\left\{\sigma_2 \int_0^t e^{-\alpha_2(T-s)}dB_2(s) - \frac{\beta_2}{4\alpha_2}(e^{-2\alpha_2(T-t)} - e^{-2\alpha_2 t})\right\}$$

and

$$\tilde{Z}^{\theta_T}(t) = \exp\left\{\int_0^t \tilde{\theta}_T(s)dI(s) - \phi(0, t, \tilde{\theta}_T(\cdot))\right\}$$
$$= \exp\left\{\int_0^t \int_{|z|<1} z_2 e^{-\alpha_2(T-s)}\tilde{N}(dz_1, dz_2, ds) + \int_0^t \int_{|z|\geq 1} z_2 e^{-\alpha_2(T-s)}N(dz_1, dz_2, ds) + \int_0^t \int_{\mathbb{R}_0^2} \left(e^{z_2 e^{-\alpha_2(T-s)}} - 1 - z_2 e^{-\alpha_2(T-s)} 1_{\{|z|<1\}}\nu(dz_1, dz_2)\right) ds\right\}.$$

We let $Z^{\theta_T}(t) = \hat{Z}^{\theta_T}(t) \times \tilde{Z}^{\theta_T}(t)$, and define the equivalent probability measure Q^{θ_T} as in Chapter 1:

$$\left.\frac{dQ^{\theta_T}}{dP}\right|_{\mathscr{F}_t} = Z^{\theta_T}(t), \quad 0 \leq t \leq T.$$

Under Q^{θ_T}, we know from the previous section that

$$B_1^{\hat{\theta}_T}(t) = B_1(t) - \rho \int_0^t \hat{\theta}_{2,T}(s)ds$$

and

$$B_2^{\hat{\theta}_T}(t) = B_2(t) - \int_0^t \hat{\theta}_{2,T}(s)ds$$

are Brownian motions with correlation ρ. From Proposition 2.11, we see that $J_i(t)$ gets a drift

$$\int_0^t \int_{|z|<1} z_i(e^{\hat{\theta}_{2,T}(s)z_2} - 1)\nu(dz_1, dz_2)ds,$$

and the Lévy measure under Q^{θ_T} becomes

$$\nu^{\theta_T}(dz_1, dz_2)dt = e^{\hat{\theta}_{2,T}(t)z_2}\nu(dz_1, dz_2)dt.$$

From equation (4.2) we have that

$$p(0,T) = e^{-r(T-t)}\frac{S_2(T)}{Z^{\theta_T}(T)}E_Q\left[\max\left(\frac{S_1(T)}{S_2(T)} - h, 0\right)\right],$$

since $Z^{\theta_T}(0) = 1$. We see that

$$\frac{S_2(T)}{Z^{\theta_T}(T)} = \exp\left\{\frac{1}{2}\beta_2(T) - h(T)\right\},$$

where

$$\beta_2(T) = \frac{\beta_2}{4\alpha_2}(1 - e^{-2\alpha_2 T})$$

and

$$h(T) = \int_0^T \int_{\mathbb{R}_0^2} \left(e^{z_2 e^{-\alpha_2(T-s)}} - 1 - z_2 e^{-\alpha_2(T-s)} 1_{\{|z|<1\}}\nu(dz_1, dz_2)\right) ds.$$

We compute the dynamics of $X_1(t) - X_2(t)$ under Q^{θ_T}.

$$\begin{aligned}
&X_1(t) - X_2(t) \\
=&\eta_1(t) - \eta_2(t) + \int_0^t e^{-\alpha_1(t-s)}dL_1(s) - \int_0^t e^{-\alpha_2(t-s)}dL_2(s) \\
=&\eta_1(t) - \eta_2(t) + \sigma_1 \int_0^t e^{-\alpha_1(t-s)}dB_1(s) - \sigma_2 \int_0^t e^{-\alpha_2(t-s)}dB_2(s)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{|z|<1} (z_1 e^{-\alpha_1(t-s)} - z_2 e^{-\alpha_2(t-s)}) \tilde{N}(dz_1, dz_2, ds) \\
&+\int_0^t \int_{|z|\geq 1} (z_1 e^{-\alpha_1(t-s)} - z_2 e^{-\alpha_2(t-s)}) N(dz_1, dz_2, ds) \\
=&\eta_1(t) - \eta_2(t) + \int_0^t \int_{|z|<1} (z_1 - z_2)(e^{z_2 e^{-\alpha_2(T-s)}} - 1)\nu(dz_1, dz_2) ds \\
&+\sigma_1 \int_0^t e^{-\alpha_1(t-s)} \left(dB_1^{\hat{\theta}_T}(s) + \rho\sigma_2 e^{-\alpha_2(T-s)} ds \right) \\
&-\sigma_2 \int_0^t e^{-\alpha_2(t-s)} \left(dB_2^{\hat{\theta}_T}(s) + \sigma_2 e^{-\alpha_2(T-s)} ds \right) \\
&+\int_0^t \int_{|z|<1} (z_1 e^{-\alpha_1(t-s)} - z_2 e^{-\alpha_2(t-s)}) \tilde{N}_{\theta_T}(dz_1, dz_2, ds) \\
&+\int_0^t \int_{|z|\geq 1} (z_1 e^{-\alpha_1(t-s)} - z_2 e^{-\alpha_2(t-s)}) N_{\theta_T}(dz_1, dz_2, ds) \\
=&\eta_1(t) - \eta_2(t) + \frac{\rho\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(e^{-\alpha_2(T-t)} - e^{(\alpha_1 t - \alpha_2 T)}) - \frac{\sigma_2^2}{2\alpha_2}(e^{-\alpha_2(T-t)} - e^{-\alpha_2(T+t)}) \\
&+\int_0^t \int_{|z|<1} (z_1 - z_2)(e^{z_2 e^{-\alpha_2(T-s)}} - 1)\nu(dz_1, dz_2) ds \\
&+\sigma_1 \int_0^t e^{-\alpha_1(t-s)} dB_1^{\hat{\theta}_T}(s) - \sigma_2 \int_0^t e^{-\alpha_2(t-s)} dB_2^{\hat{\theta}_T}(s) \\
&+\int_0^t \int_{|z|<1} (z_1 e^{-\alpha_1(t-s)} - z_2 e^{-\alpha_2(t-s)}) \tilde{N}_{\theta_T}(dz_1, dz_2, ds) \\
&+\int_0^t \int_{|z|\geq 1} (z_1 e^{-\alpha_1(t-s)} - z_2 e^{-\alpha_2(t-s)}) N_{\theta_T}(dz_1, dz_2, ds).
\end{aligned}
$$

So at time $t = T$ we have

$$
\begin{aligned}
X_1(T) - X_2(T) =&\eta_1(T) - \eta_2(T) + \rho\zeta(T) - \beta(T) \\
&+\int_0^T \int_{|z|<1} (z_1 - z_2)(e^{z_2 e^{-\alpha_2(T-s)}} - 1)\nu(dz_1, dz_2) ds \\
&+\sigma_1 \int_0^T e^{-\alpha_1(T-s)} dB_1^{\hat{\theta}_T}(s) - \sigma_2 \int_0^T e^{-\alpha_2(T-s)} dB_2^{\hat{\theta}_T}(s) \\
&+\int_0^T \int_{|z|<1} (z_1 e^{-\alpha_1(T-s)} - z_2 e^{-\alpha_2(T-s)}) \tilde{N}_{\theta_T}(dz_1, dz_2, ds)
\end{aligned}
$$

$$+ \int_0^T \int_{|z|\geq 1} (z_1 e^{-\alpha_1(T-s)} - z_2 e^{-\alpha_2(T-s)}) N_{\theta_T}(dz_1, dz_2, ds),$$

where

$$\beta_i(T) = \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i T}),\ i = 1, 2,$$
$$\zeta(T) = \frac{\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(1 - e^{-(\alpha_1+\alpha_2)T}).$$

The characteristic function of $X_1(T) - X_2(T)$, $\Phi_{X_1(T)-X_2(T)}$ follows from 3.4 with $(u_1, u_2) = (u, -u)$. Using the result from 2.12, we see that

$$E_{Q^{\theta_T}}[f(X_1(T) - X_2(T))] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\Phi_{X_1(T)-X_2(T)}(y - i\alpha)}{\alpha^2 - \alpha - y^2 + i(2\alpha - 1)y} dy.$$

So the spread option price at time $t = 0$ is then:

$$p(0, T) = \frac{1}{2\pi}\exp\left\{-rT + \frac{1}{2}\beta_2(T) - h(T)\right\}\int_{\mathbb{R}} \frac{\Phi_{X_1(T)-X_2(T)}(y - i\alpha)}{\alpha^2 - \alpha - y^2 + i(2\alpha - 1)y} dy,$$

□

The integral can be computed numerically, using a FFT (Fast Fourier Transform), but we will not do this here.

Chapter 5

Approximation of the spread option price when $K \neq 0$

When $K \neq 0$, it is no longer possible to reduce the spread option pricing problem to the problem of pricing a call option on a single asset. We can´t find an explicit formula for the price, but we will find an approximation by using a Taylor expansion. The payoff function of a spread option with maturity time T and strike price K is

$$\max(S_1(T) - hS_2(T) - K, 0). \tag{5.1}$$

We denote by $p_K(t, T)$ the price of the option at time t. We are interested in finding the price at time 0.

$$p_K(0, T) = e^{-rT} E\left[\max(S_1(T) - hS_2(T) - K, 0)\right].$$

We write the expectation of the payoff function as a function of K:

$$f(K) = E\left[\max(S_1(T) - hS_2(T) - K, 0)\right].$$

Then we use a first order Taylor expansion to approximate f(K).

$$f(K) \approx f(0) + f'(0)K.$$

The approximated price is denoted by $\hat{p}_K(0, T)$, and it is

$$\begin{aligned} \hat{p}_K(0, T) &= e^{-rT} f(K) \\ &= e^{-rT} f(0) + e^{-rT} f'(0)K \\ &= p_0(0, T) + Ke^{-rT} \frac{\partial}{\partial K} \left(E\left[\max(S_1(T) - hS_2(T) - K, 0)\right]\right)\Big|_{K=0}. \end{aligned} \tag{5.2}$$

This approximation can be used when K is small, and we will see in the next section how good it is for different values of K. We have already computed $p_0(0,T)$ in the previous chapter, so we need to find $f'(K)$. We let $Y = S_1(T) - hS_2(T)$, and we let $p_Y(y)$ denote the density of Y.

$$\begin{aligned} f(K) &= E\left[\max(S_1(T) - hS_2(T) - K, 0)\right] \\ &= E\left[\max(Y - K, 0)\right] \\ &= \int_{\mathbb{R}} \max(y - K, 0) p_Y(y) dy \\ &= \int_K^\infty (y - K) p_Y(y) dy \\ &= \int_K^\infty y p_Y(y) dy + K \int_K^\infty p_Y(y) dy. \end{aligned}$$

We differentiate this expression with respect to K.

$$\begin{aligned} f'(K) &= -Kp_Y(K) - \left(K(-p_Y(K) + \int_K^\infty p_Y(y)dy)\right) \\ &= -Kp_Y(K) + Kp_Y(K) - \int_K^\infty p_Y(y)dy \\ &= -P(S_1(T) - hS_2(T) > K). \end{aligned}$$

Putting $K = 0$, we have that

$$\begin{aligned} f'(0) &= -P(S_1(T) > hS_2(T)) \\ &= -P(\log(S_1(T) > \log(hS_2(T)))) \\ &= -P(X_1(T) > \log(h) + X_2(T)) \\ &= -P(X_1(T) - X_2(T) > \log(h)). \end{aligned}$$

5.1 Continuous model

In the continuous model, $X_1(T) - X_2(T)$ is normally distributed, so $P(X_1(T) - X_2(T) > \log(h))$ can be easily computed. We get the following result:

Proposition 5.1. *Let $X_i(t) = (X_1(t), X_2(t))$ be given by the dynamics* (4.3). *Consider a spread option with payoff* (5.1) *at time T, and let the* *interest rate r be constant. Let $\hat{p}_K(0,T)$ denote the approximated price of the spread option at time $t = 0$. Then,*

$$\hat{p}_K(0,T) = p(0,T) - Ke^{-rT}\Phi\left(\frac{\eta_1(T) - \eta_2(T) - \log(h)}{\sqrt{\beta_1(T) + \beta_2(T) - 2\rho\zeta(T)}}\right),$$

where $p(0,T)$ is given by Proposition 4.1.

Proof. We know from Chapter 3 that

$$X_i(T) \sim N\left(\eta_i(T), \beta_i(T)\right),$$

where

$$\eta_i(T) = x_i e^{-\alpha_i T} + \frac{\mu_i}{\alpha_i}(1 - e^{-\alpha_i T})$$

and

$$\beta_i(T) = \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i T}).$$

The covariance between $X_1(T)$ and $X_2(T)$ is $\rho\zeta(T)$, where

$$\zeta(T) = \frac{\sigma_1\sigma_2}{\alpha_1 + \alpha_2}(1 - e^{-(\alpha_1+\alpha_2)T}).$$

So

$$X_1(T) - X_2(T) \sim N\left(\eta_1(T) - \eta_2(T), \beta_1(T) + \beta_2(T) - 2\rho\zeta(T)\right).$$

We see that

$$\begin{aligned} P(X_1(T) - X_2(T) > \log(h)) &= \Phi\left(-\frac{\log(h) - (\eta_1(T) - \eta_2(T))}{\sqrt{\beta_1(T) + \beta_2(T) - 2\rho\zeta(T)}}\right) \\ &= \Phi\left(\frac{\eta_1(T) - \eta_2(T) - \log(h)}{\sqrt{\beta_1(T) + \beta_2(T) - 2\rho\zeta(T)}}\right), \end{aligned}$$

where Φ is here the cumulative distribution function of the standard normal distribution. Since $f'(0) = -P(X_1(T) - X_2(T) > \log(h))$, we have that

$$f'(0) = -\Phi\left(\frac{\eta_1(T) - \eta_2(T) - \log(h)}{\sqrt{\beta_1(T) + \beta_2(T) - 2\rho\zeta(T)}}\right).$$

The approximated price then follows from equation (5.2) and Proposition 4.1.

$$\hat{p}_K(0,T) = p_0(0,T) + Ke^{-rT}f'(0)$$

$$= \exp\left\{-rT + \eta_1(T) + \frac{1}{2}\beta_1(T)\right\}\Phi(k + \sigma(T))$$
$$- \exp\left\{-rT + \eta_2(T) + \frac{1}{2}\beta_2(T)\right\} h\Phi(k)$$
$$- Ke^{-rT}\Phi\left(\frac{\eta_1(T) - \eta_2(T) - \log(h)}{\sqrt{\beta_1(T) + \beta_2(T) - 2\rho\zeta(T)}}\right).$$

□

In the model with jumps, we don´t know the distribution of $X_1(T) - X_2(T)$ for a general Lévy measure, and we can therefore not find a closed form formula for $f'(0)$.

5.2 Simulation of the approximation error

To estimate how good this approximation is for different values of K, we will simulate the spread option price for different values of ρ and K, and compare it to the approximated price. We let $r = 0$, so the price equals the expected payoff of the option. We also let $h = 1$. To find the simulated price, we simulate the payoff of the option 1000000 times, and compute the mean of the simulated payoffs. We denote by $\tilde{p}_{\rho,K}$ the simulated price as a function of ρ and K, and by $\hat{p}_{\rho,K}$ the approximated price $\hat{p}_K(0,T)$ as a function of ρ and K. We use the same parameters as in Chapter 3 (see Table 3.1), and we let $T = 365$ days.

We first plot a histogram of the simulated values of $S_1(T) - S_2(T)$ for three different values of ρ; $\rho = 0.2$, $\rho = 0.5$ and $\rho = 0.8$. See Figures 5.1, 5.2 and 5.3. We see that the variance of the spread $S_1(T) - S_2(T)$ is larger for smaller ρ. This is natural, since a smaller ρ gives a smaller correlation between $S_1(T)$ and $S_2(T)$, and hence the probability of a high absolute value of the spread gets larger. Then we plot the approximated and the simulated price as a function of K for the same three values of ρ, where K varies between 0 and 4. See Figures 5.4, 5.5 and 5.6.

We compute the error $\epsilon_{\rho,K}$ of the approximation by the following formula:

$$\epsilon_{\rho,K} = \left|\frac{\hat{p}_{\rho,K} - \tilde{p}_{\rho,K}}{\tilde{p}_{\rho,K}}\right|.$$

In Figure 5.7, we plot $\epsilon_{\rho,K}$ as a function of K for the three different values of ρ. We see that the larger ρ is, the faster $\epsilon_{\rho,K}$ increases as a function of K. When $\rho = 0.8$, the error is larger than 5% for $K > 2$. For $\rho = 0.5$, we must increase the strike price to $K = 3$ to get an error of the same size. When $\rho = 0.2$, we can can have a K just below 4 and still get an error under 5%.

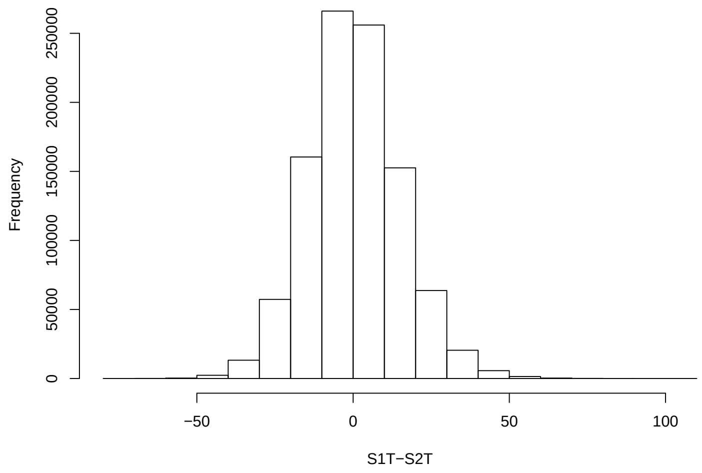

Figure 5.1: Histogram of $S_1(T) - S_2(T)$ when $\rho = 0.2$.

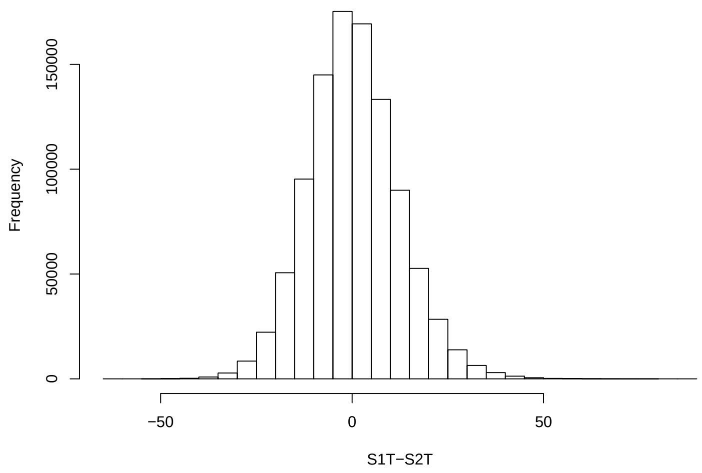

Figure 5.2: Histogram of $S_1(T) - S_2(T)$ when $\rho = 0.5$.

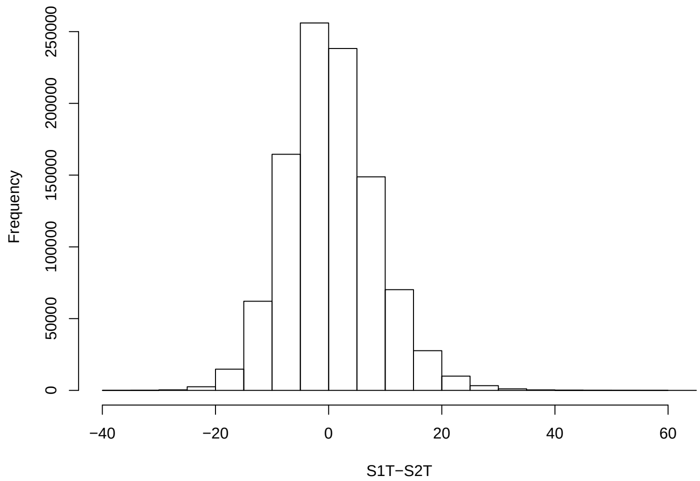

Figure 5.3: Histogram of $S_1(T) - S_2(T)$ when $\rho = 0.8$.

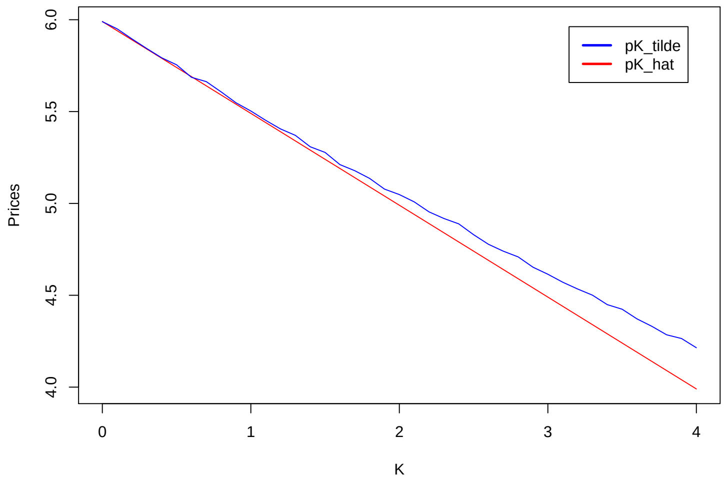

Figure 5.4: Approximated and simulated price as a function of K when $\rho = 0.2$.

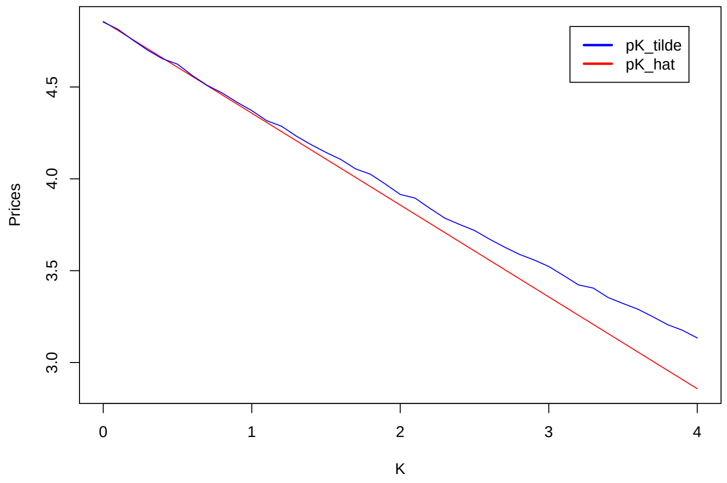

Figure 5.5: Approximated and simulated price as a function of K when $\rho = 0.5$.

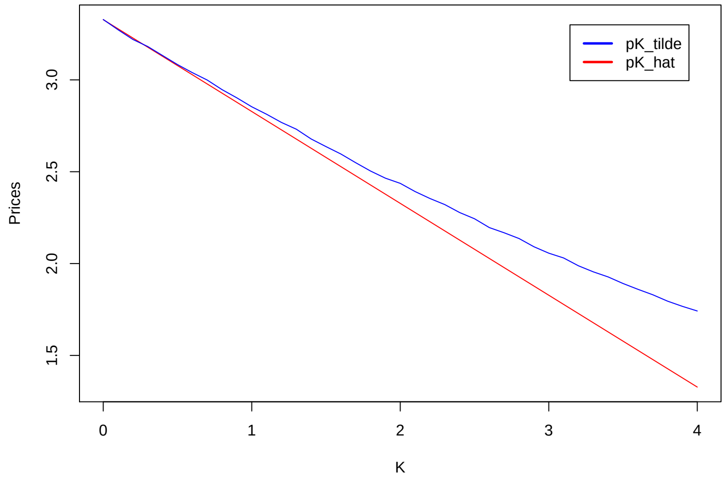

Figure 5.6: Approximated and simulated price as a function of K when $\rho = 0.8$.

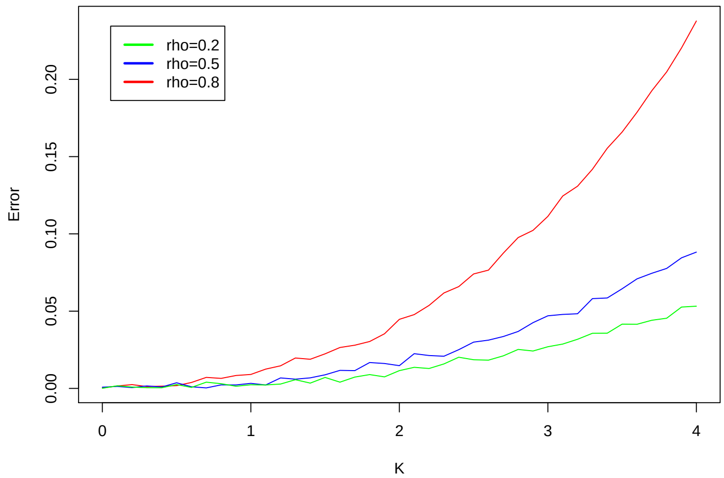

Figure 5.7: $\epsilon_{\rho,K}$ as a function of K for three different values of ρ.

Appendix A

R code

This appendix contains the R code for the simulations.

```
#Simulation of the Ornstein−Uhlenbeck processes
#========================================

T=365
mu1=0.4
mu2=0.6
alpha1=0.1
alpha2=0.15
sigma1=0.1
sigma2=0.1
rho=0.5

X1=rep(0,T)
X2=rep(0,T)

X1[1]=mu1/alpha1
X2[1]=mu2/alpha2

Z1=rnorm(T)
Z2=rho*Z1+sqrt(1−rho^2)*rnorm(T)

for (t in 2:T){
        X1[t]=X1[t−1]+(mu1−alpha1*X1[t−1])+sigma1*Z1[t]
        X2[t]=X2[t−1]+(mu2−alpha2*X2[t−1])+sigma2*Z2[t]
}

S1=exp(X1)
S2=exp(X2)

plot(X1,type="l",xlab="Time in days",ylab="X1 and X2",col="blue")
lines(X2,type="l",col="red")
legend("topright",inset=0.05,c("X1","X2"),
        lty=c(1,1),lwd=c(2.5,2.5),col=c("blue","red"))
savePlot(filename="X1X2",type="pdf",device=dev.cur(),restoreConsole=TRUE)

plot(S1,type="l",xlab="Time in days",ylab="S1 and S2",col="blue")
lines(S2,type="l",col="red")
legend("topright",inset=0.05,c("S1","S2"),
        lty=c(1,1),lwd=c(2.5,2.5),col=c("blue","red"))
savePlot(filename="S1S2",type="pdf",device=dev.cur(),restoreConsole=TRUE)
```

```
#The option price when K=0 as a function of rho
#==============================================

eta1T=exp(-alpha1*T)+mu1/alpha1*(1-exp(-alpha1*T))
eta2T=exp(-alpha2*T)+mu2/alpha2*(1-exp(-alpha2*T))
beta1T=(sigma1^2)/(2*alpha1)*(1-exp(-2*alpha1*T))
beta2T=(sigma2^2)/(2*alpha2)*(1-exp(-2*alpha2*T))
zetaT=sigma1*sigma2/(alpha1+alpha2)*(1-exp(-(alpha1+alpha2)*T))

p0=function(rho){
        sigmaT=sqrt(beta1T+beta2T-2*rho*zetaT)
        k=(eta1T-eta2T+0.5*(beta1T-beta2T))/sigmaT-0.5*sigmaT
        p0=exp(eta1T+0.5*beta1T)*pnorm(k+sigmaT)-exp(eta2T+0.5*beta2T)*pnorm(k)
        return(p0)
}

rhos=seq(0,1,by=0.01)
prices=p0(rhos)
plot(rhos,prices,type="l",xlab="rho",ylab="p(0,T)")
savePlot(filename="p0",type="pdf",device=dev.cur(),restoreConsole=TRUE)

#The approximated price as a function of rho and K
#================================================

pK_hat=function(rho,K){
        sigmaT=sqrt(beta1T+beta2T-2*rho*zetaT)
        k=(eta1T-eta2T+0.5*(beta1T-beta2T))/sigmaT-0.5*sigmaT
        pK_hat=p0(rho)-pnorm((eta1T-eta2T)/sigmaT)*K
        return(pK_hat)
}

#The simulated price as a function of rho and K
#==============================================
sim=1000000

pK_tilde=function(rho,K){
        corr=rho*zetaT/sqrt(beta1T*beta2T)
        Z1=rnorm(sim)
        Z=rnorm(sim)
        Z2=corr*Z1+sqrt(1-corr^2)*Z
        X1T=rep(eta1T,sim)+Z1*rep(sqrt(beta1T),sim)
        X2T=rep(eta2T,sim)+Z2*rep(sqrt(beta2T),sim)
        S1T=exp(X1T)
        S2T=exp(X2T)
        spread=S1T-S2T-rep(K,sim)
        payoff=spread[spread>0]
        pK_tilde=sum(payoff)/sim
        return(c(pK_tilde,spread))
}

#Plot a histogram of S1T-S2T when rho=0.2
#========================================
spread=pK_tilde(0.2,0)[2:(sim+1)]
hist(spread,xlab="S1T-S2T",main="")
savePlot(filename="S1S2rho1",type="pdf",device=dev.cur(),restoreConsole=TRUE)

#Plot a histogram of S1T-S2T when rho=0.5
#========================================
spread=pK_tilde(0.5,0)[2:(sim+1)]
hist(spread,xlab="S1T-S2T",main="")
savePlot(filename="S1S2rho2",type="pdf",device=dev.cur(),restoreConsole=TRUE)
```

```
#Plot a histogram of S1T-S2T when rho=0.8
#=====================================
spread=pK_tilde(0.8,0)[2:(sim+1)]
hist(spread,xlab="S1T-S2T",main="")
savePlot(filename="S1S2rho3",type="pdf",device=dev.cur(),restoreConsole=TRUE)

#The error as a function of rho and K
#=================================

error=function(rho,K){
        pK_hat=pK_hat(rho,K)
        pK_tilde=pK_tilde(rho,K)[1]
        error=abs((pK_hat-pK_tilde)/pK_tilde)
        return(error)
}

#Calculate the prices and errors for different values of rho and K
#==========================================================

rhos=cbind(0.2,0.5,0.8)
Ks=seq(0,4,by=0.1)
n=length(rhos)
m=length(Ks)
pK_tildes=matrix(rep(0,n*m),n,m)
pK_hats=matrix(rep(0,n*m),n,m)
errors=matrix(rep(0,n*m),n,m)

for (i in 1:n){
        for (j in 1:m){
                pK_tildes[i,j]=pK_tilde(rhos[i],Ks[j])[1]
                pK_hats[i,j]=pK_hat(rhos[i],Ks[j])
                errors[i,j]=abs((pK_tildes[i,j]-pK_hats[i,j])/pK_tildes[i,j])
        }
}

#Plot the simulated and approximated price as a function of K when rho=0.2
#==================================================================

plot(Ks,pK_hats[1,],type="l",col="red",xlab="K",ylab="Prices")
lines(Ks,pK_tildes[1,],type="l",col="blue")
legend("topright",inset=0.05,c("pK_tilde","pK_hat"),
        lty=c(1,1),lwd=c(2.5,2.5),col=c("blue","red"))
savePlot(filename="rho1",type="pdf",device=dev.cur(),restoreConsole=TRUE)

#Plot the simulated and approximated price as a function of K when rho=0.5
#==================================================================

plot(Ks,pK_hats[2,],type="l",col="red",xlab="K",ylab="Prices")
lines(Ks,pK_tildes[2,],type="l",col="blue")
legend("topright",inset=0.05,c("pK_tilde","pK_hat"),
        lty=c(1,1),lwd=c(2.5,2.5),col=c("blue","red"))
savePlot(filename="rho2",type="pdf",device=dev.cur(),restoreConsole=TRUE)

#Plot the simulated and approximated price as a function of K when rho=0.8
#==================================================================

plot(Ks,pK_hats[3,],type="l",col="red",xlab="K",ylab="Prices")
lines(Ks,pK_tildes[3,],type="l",col="blue")
legend("topright",inset=0.05,c("pK_tilde","pK_hat"),
        lty=c(1,1),lwd=c(2.5,2.5),col=c("blue","red"))
savePlot(filename="rho3",type="pdf",device=dev.cur(),restoreConsole=TRUE)
```

```
#Plot the error as a function of K for different rho´s
#==========================================================

plot(Ks,errors[3,],type="l",col="red",xlab="K",ylab="Error")
lines(Ks,errors[2,],type="l",col="blue")
lines(Ks,errors[1,],type="l",col="green")
legend("topleft",inset=0.05,c("rho=0.2","rho=0.5","rho=0.8"),
        lty=c(1,1,1),lwd=c(2.5,2.5,2.5),col=c("green","blue","red"))
savePlot(filename="error",type="pdf",device=dev.cur(),restoreConsole=TRUE)
```

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