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On the stochastic Euler-Poincaré equations driven by pseudo-differential/multiplicative noise

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The stochastic Euler-Poincaré equations with pseudo-differential/multiplicative noise are considered in this work. We first establish two new cancellation properties on pseudo-differential operators, which considerably extend the previous results for transport type noise only involving gradient operator. Then, we obtain results on local solution, blow-up criterion, and global existence. The interplay between stability on exiting times and continuous dependence of solution on initial data is also studied for the multiplicative noise case.

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1. Introduction and main results

Let $\mathbf{I}$ denote the identity operator. Then the Euler-Poincaré (**EP**) equations are given by:

$$\partial_t m + (u \cdot \nabla) m + (\nabla u)^T m + (\text{div} u) m = 0, \quad m = (\mathbf{I} - \alpha \Delta) u, \tag{1.1}$$

where $u = (u_j)_{1 \leq j \leq d}$ denotes the velocity, $m = (m_j)_{1 \leq j \leq d}$ with $m_j = (\mathbf{I} - \alpha\Delta) u_j(t, x)$ represents the momentum, α corresponds to the square of the length scale and A^T denotes the transpose of a matrix A.

As a higher-dimensional Camassa-Holm (CH) system for modeling and analyzing nonlinear shallow water waves, the **EP** equations (1.1) were first introduced in Holm et al. [24]. In the case where $d \geq 2$ and $m > d/2 + 3$, local existence and uniqueness of a strong solution belonging to H^m were established in Chae and Liu [9]. The blow-up phenomenon for the case $\alpha = 0$ was also obtained in the same work. For the case $\alpha > 0$, the blow-up and global existence of solutions to (1.1) were studied in Li et al. [29].

In this paper, we focus on the case where $\alpha > 0$ in (1.1) and assume $\alpha = 1$ for convenience. We use $I_{d \times d}$ to represent the $d \times d$ identity matrix and let div be the divergence operator (i.e., div $= \nabla \cdot$). The notation $f = (\mathbf{I} - \Delta)^{-1} g$ denotes that $f = G * g$, where G is the Green function for the Helmholtz operator $\mathbf{I} - \Delta$. We can then reformulate (1.1) into the following form (see Chae and Liu [9], Yan and Yin [52], Zhao et al. [53]):

$$u_t + (u \cdot \nabla) u + F(u) = 0, \tag{1.2}$$

where

$$\begin{cases} F(u) = (\mathbf{I}-\Delta)^{-1}\mathrm{div}F_1(u) + (\mathbf{I}-\Delta)^{-1}F_2(u), \\ F_1(u) = \nabla u(\nabla u + (\nabla u)^T) - (\nabla u)^T\nabla u - \nabla u(\mathrm{div}u) + \dfrac{1}{2}I_{d\times d}|\nabla u|^2, \\ F_2(u) = u(\mathrm{div}u) + u\cdot(\nabla u)^T. \end{cases} \tag{1.3}$$

Over the past two decades, Stochastic Partial Differential Equations (SPDEs) have emerged as a powerful tool for studying complex phenomena. They can incorporate phenomena such as turbulence and random energy exchange that elude traditional deterministic methods. For instance, deterministic methods struggle to accurately model cloud formation in weather forecasting. In this context, we will explore a stochastic version of (1.2). More precisely, let $\{(\widetilde{W}_k(t), W_k(t))\}_{k\geq 1}$ be a family of mutually independent 1-dimensional Brownian motions and then we consider the following stochastic **EP** equations:

$$\mathrm{d}u + [(u\cdot\nabla)u + F(u)]\,\mathrm{d}t = \sum_{k=1}^{\infty}\Big(\mathcal{Q}_k u \circ \mathrm{d}\widetilde{W}_k(t) + h_k(t,u)\,\mathrm{d}W_k(t)\Big), \tag{1.4}$$

where $\{\mathcal{Q}_k\}_{k\geq 1}$ is a sequence of differential operators, $\{h_k\}_{k\geq 1}$ is a sequence of nonlinear functions, $\mathrm{d}W_k(t)$ is the Itô stochastic differential and $\circ\,\mathrm{d}\widetilde{W}_k(t)$ is the Stratonovich stochastic differential.

In Mikulevicius and Rozovskii [31,32], the authors introduced randomness at the Lagrangian level by imposing a stochastic forcing $\sigma \circ \mathrm{d}W(t)$ in the equation for the streamlines $X = X(t,x)$ with undetermined local velocity u:

$$\mathrm{d}X(t,x) = u(t,X(t,x))\,\mathrm{d}t + \sigma(t,X(t,x))\circ \mathrm{d}W(t), \quad X(0,x) = x, \tag{1.5}$$

where $W(t)$ is a standard 1-dimensional Brownian motion. Roughly speaking, rewriting the system in terms of $u(t,X)$ leads to the transport noise. Similar ideas were incorporated in the Stochastic Advection by Lie Transport (SALT) framework developed in Holm [23], which has been a topic of much interest in recent years. See Albeverio et al. [2], Alonso-Orán and Bethencourt de León [3], Alonso-Orán et al. [5], Crisan et al. [11], Crisan and Holm [12], Goodair and Crisan [17], Holden et al. [21,22] for recent developments. Let $\mathrm{diag}(\cdot,\cdots,\cdot)$ be the diagonal operator. Mathematically, we observe that the classical transport noise coefficient and the noise structure given by the SALT operator (SALT noise) can be unified by taking $\mathcal{Q}$ as follows:

$$\begin{cases} \mathcal{Q}_k = \mathrm{diag}\Big(\big(\psi_k^{(1)}(x)\cdot\nabla\big),\cdots,\big(\psi_k^{(d)}\cdot\nabla\big)\Big) + \Phi_k(x)\mathbf{I}, \\ \psi_k^{(i)}(x) : \mathbb{R}^d\text{-valued},\ 1\leq i\leq d, \\ \Phi_k(x) = \big(\phi_k^{(i,j)}(x)\big)_{1\leq i,j\leq d} : \mathbb{R}^{d\times d}\text{-valued}. \end{cases} \tag{1.6}$$

For example, we observe the following cases:

- In velocity equations, the SALT operator $\mathfrak{L}_k$ can be formulated as (cf. [17, Section 1.4])

$$\mathfrak{L}_k(f) = (\eta_k \cdot \nabla) f + \sum_{j=1}^{d} f_j \nabla \eta_{k,j}, \quad \eta_k = \left(\eta_{k,l}\right)_{1 \le l \le d}, \tag{1.7}$$

which can be covered by (1.6) with $\psi_k^{(i)} = \eta_k$ and $\phi_k^{(i,j)} = \partial_{x_i} \eta_{k,j}$ $(1 \le i, j \le d)$.
- In 3-dimensional vorticity equations, the Lie derivative operator $\widetilde{\mathfrak{L}}_k$ (see Crisan et al. [11], Flandoli and Luo [16] or [17, Equation (42)]) takes the form:

$$\widetilde{\mathfrak{L}}_k(f) = (\eta_k \cdot \nabla) f - (f \cdot \nabla) \eta_k, \quad \eta_k = \left(\eta_{k,l}\right)_{1 \le l \le 3}, \tag{1.8}$$

Similarly, (1.8) can be also covered by (1.6).

1.1. Pseudo-differential noise and cancellation properties

We note that only the classical gradient operator is involved (and only on the main diagonal) in the classical transport noise coefficient and the SALT operator. In this paper, roughly speaking, we study the case where

$$\mathcal{Q}_k \text{ is a matrix-valued pseudo-differential operator,} \tag{1.9}$$

which significantly extends (1.6). We refer to the noise structure $\mathcal{Q}_k u \circ \mathrm{d}\widetilde{W}_k$ as *pseudo-differential noise*. See Section 2.1 for a precise definition of pseudo-differential operators.

Although the physical interpretation of such pseudo-differential noise may be unclear, the noise structure introduced in (1.9) can facilitate the exploration of non-local random interactions. Since pseudo-differential operators extend classical differential operators in a non-local manner, studying pseudo-differential noise provides a more flexible framework for modeling complex phenomena that involve non-local random interactions. This can be particularly useful in turbulence models, where the behavior of fluid at one point is influenced by the behavior of fluid at distant points.

Usually, if u and σ are smooth functions, the random perturbation in the trajectories of the Lagrangian fluid particles (1.5) is local in $X(t, x)$. To gain more insight into the non-locality introduced by the noise structure in (1.9), which the classical transport noise or the SALT noise cannot capture, we consider the following simple but intriguing model. As before, $W(t)$ is a standard 1-dimensional Brownian motion and we consider the Burgers' equation with the noise structure $\sqrt{2\mu}(-\partial_x^2)^\alpha u \circ \mathrm{d}W(t)$:

$$\mathrm{d}u + u\partial_x u \, \mathrm{d}t = \sqrt{2\mu}(-\partial_x^2)^\alpha u \circ \mathrm{d}W(t), \quad \alpha \in (0, 1/2], \quad \mu > 0. \tag{1.10}$$

We will show below that the noise term $\sqrt{2\mu}(-\partial_x^2)^\alpha u \circ \mathrm{d}W$ introduces non-locality to the classical transport term uu_x. Indeed, by using the following relation for a semi-martingale $\Theta(t)$:

$$\Theta(t) \circ \mathrm{d}W(t) = \Theta(t)\,\mathrm{d}W(t) + \frac{1}{2}\left\langle \Theta, W \right\rangle(t),\ \ \langle \cdot, \cdot \rangle \text{ is the quadratic variation.} \tag{1.11}$$

we can rewrite (1.10) as

$$\mathrm{d}u + u\partial_x u\,\mathrm{d}t = \sqrt{2\mu}(-\partial_x^2)^\alpha u\,\mathrm{d}W(t) + \mu(-\partial_x^2)^{2\alpha} u\,\mathrm{d}t.$$

Let $\Phi_\alpha(t) := \exp\left\{-\sqrt{2\mu}W(t)(-\partial_x^2)^\alpha\right\}$ be an operator-valued process. Then we have the following operator-valued SDE (understood in the sense of Fourier multiplier):

$$\mathrm{d}\Phi_\alpha(t) = -\sqrt{2\mu}(-\partial_x^2)^\alpha \Phi_\alpha(t)\,\mathrm{d}W(t) + \mu(-\partial_x^2)^{2\alpha}\Phi_\alpha(t)\,\mathrm{d}t.$$

From the above SDE and $[\Phi_\alpha(t), (-\partial_x^2)^\alpha] = 0$, it follows that $\theta(t) := [\Phi_\alpha u](t)$ satisfies

$$\mathrm{d}\theta = [\mathrm{d}\Phi_\alpha](u) + \Phi_\alpha(\mathrm{d}u) - 2\mu(-\partial_x^2)^{2\alpha}\Phi_\alpha u\,\mathrm{d}t = -\Phi_\alpha\left(\Phi_\alpha^{-1}\theta \cdot \partial_x \Phi_\alpha^{-1}\theta\right)\mathrm{d}t. \tag{1.12}$$

Thus, the term $\Phi_\alpha\left(\Phi_\alpha^{-1}\theta \cdot \partial_x \Phi_\alpha^{-1}\theta\right)$ reflects the non-local effect arising from the pseudo-differential noise $\sqrt{2\mu}(-\partial_x^2)^\alpha u \circ \mathrm{d}W(t)$ on the level of θ. In the classical derivative case, we have:

$$\mathrm{d}u + u\partial_x u\,\mathrm{d}t = \sqrt{2\mu}\partial_x u \circ \mathrm{d}W(t), \quad \mu > 0,$$

which is equivalent to

$$\partial_t\theta + \Phi\left(\Phi^{-1}\theta \cdot \partial_x\Phi^{-1}\theta\right) = 0, \quad \theta(t) := [\Phi u](t), \quad \Phi(t) := \exp\left\{-\sqrt{2\mu}W(t)\partial_x\right\}.$$

Pathwisely, it appears that the kernels of Φ_α and Φ_α^{-1} cannot be explicitly written down, while the kernels of Φ and Φ^{-1} are delta functions, indicating that they are local-in-x operators.

The analysis of the non-local effects from pseudo-differential noise is quite challenging. Our focus will be limited to the existence and uniqueness of solutions. Even in this case, there is a challenge presented by pseudo-differential noise: closing the *a priori* estimate for (1.4) in H^s becomes *non-trivial*. Here, H^s denotes the Sobolev space with regularity index s (see Section 2.1). To see this, we can rewrite (1.4) in Itô's form (see (2.8)) and then apply Itô's formula to $\|u\|_{H^s}^2$. This will result in two terms:

$$\sum_{k=1}^{\infty} \left\langle \mathcal{Q}_k u, u \right\rangle_{H^s}\, \mathrm{d}\widetilde{W}_k \ \text{ and } \ \sum_{k=1}^{\infty}\left[\left\langle \mathcal{Q}_k^2 u, u\right\rangle_{H^s} + \left\langle \mathcal{Q}_k u, \mathcal{Q}_k u\right\rangle_{H^s}\right]\,\mathrm{d}t.$$

If the order of $\mathcal{Q}_k$ is greater than zero, these two terms are *a priori* singular in terms of H^s since derivatives of order higher than s are involved. However, to close the *a priori* estimate for $\|u\|_{H^s}^2$, one must control these two terms by the H^s-norm of u. This means that the singularities cancel out and such estimates are called cancellation of singularities. The first main result in this work is the following cancellation properties:

Main Result (A) *(See Theorems 3.1, 3.2 and 3.3 for the precise statement). Cancellation properties for certain differential operators* $\{\mathcal{Q}_k\}_{k\geq 1}$*:*

$$\sup_{\mathcal{P}\in\mathscr{O}}\sum_{k=1}^{\infty}\left\langle \mathcal{P}\mathcal{Q}_k f, \mathcal{P}f\right\rangle_{L^2}^2 \lesssim \|f\|_{H^s}^4 , \tag{1.13}$$

$$\sup_{\mathcal{P}\in\mathscr{O}}\sum_{k=1}^{\infty}\left|\left\langle \mathcal{P}\mathcal{Q}_k^2 f, \mathcal{P}f\right\rangle_{L^2} + \left\langle \mathcal{P}\mathcal{Q}_k f, \mathcal{P}\mathcal{Q}_k f\right\rangle_{L^2}\right| \lesssim \|f\|_{H^s}^2 , \tag{1.14}$$

where $\mathscr{O} \subset \mathrm{OP}\mathcal{S}^s$ *is a bounded set and* $f : \mathbb{K}^d \to \mathbb{R}^m$ ($\mathbb{K} = \mathbb{R}$ *or* $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$) *is sufficiently regular.*

1.2. Existence, uniqueness and initial-data dependence

With the above cancellation properties (1.13) and (1.14), we can obtain the second result in the paper:

Main Result (B) *(See Theorem 4.1 for the detailed statement). Local existence, uniqueness, blow-up criterion and global existence of solutions to the following Cauchy problem of the stochastic* ***EP*** *equations* (1.4) *on* $\mathbb{K}^d$*,*

$$\begin{cases} \mathrm{d}u + [(u\cdot\nabla)u + F(u)]\,\mathrm{d}t = \sum_{k=1}^{\infty}\Big(\mathcal{Q}_k u \circ \mathrm{d}\widetilde{W}_k(t) + h_k(t,u)\,\mathrm{d}W_k(t)\Big), \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.15}$$

where $\mathcal{Q}_k$ *and* h_k *satisfy certain conditions.*

The effect of noise is a key and interesting question for the study of SPDEs and noise has been observed to produce various regularization effects. We refer to Alonso-Orán et al. [4], Chen et al. [10], Flandoli et al. [15], Flandoli and Luo [16], Tang and Wang [45] and the references therein for some examples.

According to Hadamard, the concept of well-posedness for an abstract Cauchy problem requires the existence, uniqueness, and stability (continuous dependence of the solution on initial data). Here we highlight that in the case of nonlinear stochastic evolution equations, the dependence on initial conditions presents a much more complex problem than that in cases of linear growth or determinism. This is because solutions may only exist up to an interval $[0, \tau)$ with τ being a stopping time, and in general, there are no estimates available for this τ, i.e., there is a lack of lifespan estimate in the stochastic setting.

In contrast to many previous studies where noise effects were mainly explored in terms of regularity or uniqueness, we examine the impact of noise on the dependence of solutions on initial data. Comparing the noise and Laplacian provides an intriguing perspective on this issue. Specifically, on one hand, "regularization by noise" is formally related to the regularization produced by an additional Laplacian. On the other hand, the presence of an actual Laplacian in governing equations may improve the dependence on initial data in some cases. For example, whereas the deterministic Euler equations have at most continuous dependence on initial data (cf. Himonas and Misiołek [20]), the deterministic Navier-Stokes equations have at least Lipschitz continuity (cf. [18, pages 79-81]). So far, we have not been able to completely determine the impact of $\mathcal{Q}_k u \circ \mathrm{d}\widetilde{W}_k$. Therefore, we are considering the special case of (1.15):

$$\mathrm{d}u + [(u\cdot\nabla)\,u + F(u)]\;\mathrm{d}t = \sum_{k=1}^{\infty} h_k(t,u)\,\mathrm{d}W_k(t), \quad u\big|_{t=0} = u_0. \tag{1.16}$$

The third result in this paper can be roughly stated as follows:

Main Result (C) *(The detailed statement is in Theorem 5.1). The solution map $u_0 \mapsto u$ defined by (1.16) on $\mathbb{T}^d$ is weakly unstable in the sense that:*

(1) *Either the exiting time of solution $u \equiv 0$ is not strongly stable (see Definition 2.3);*
(2) *Or the dependence on initial data is not uniformly continuous.*

1.3. Plan on the paper and preliminary remarks

- In Section 2, we introduce the notations, review some related preliminary results, and provide definitions.
- In Section 3, we establish the cancellation properties (1.13) and (1.14) for an extensive family of $\mathcal{Q}_k$. These results are, to the best of our knowledge, novel in the analysis of pseudo-differential operators. Theorems 3.1 and 3.2 pertain to operators of order $\alpha \in [0,1]$, while Theorem 3.3 addresses operators that are independent of x with order $\beta \geq 0$. Section 3.3 elaborates on the assumptions and techniques (see Remark 3.1), provides examples (see Example 3.1), and discusses other potential extensions (see Theorem 3.4 and Remark 3.2). We list two preliminary remarks below before going into the details:

  (1) When $\mathscr{O} = \{\partial_x^n : n = 0,1,\cdots,m\} \subset \mathrm{OP}\mathcal{S}^m$ and $\mathcal{Q}_k$ is the SALT operator of the form (1.7) or (1.8), the estimates of type (1.13) and (1.14) have been established in Crisan et al. [11], Goodair and Crisan [17], Lang and Crisan [27], where the Leibniz's rule and integration by parts for classical derivatives are used. However, for a general pseudo-differential operator $\mathcal{Q}_k$ in (1.9), many fundamental properties of classical derivatives no longer hold true, which may present challenges. To illustrate this, we consider the following simple example with $f:\mathbb{R}^d \to \mathbb{R}$. Let $\mathcal{L}_k = (\psi_k \cdot \nabla)$ (the main part of (1.6)) with $\sum_{k=1}^{\infty} \|\mathrm{div}\psi_k\|_{L^\infty} < \infty$. Then, by applying integration by parts, we obtain

$$
\begin{aligned}
\langle \mathcal{L}_k f, f\rangle_{L^2} &= \sum_{i=1}^{d} \langle \psi_k^{(i)} \partial_{x_i} f, f\rangle_{L^2} \\
&= \frac{1}{2}\sum_{i=1}^{d} \int_{\mathbb{R}^d} \psi_k^{(i)} \partial_{x_i}(f^2)\,\mathrm{d}x \le \frac{1}{2}\|\mathrm{div}\psi_k\|_{L^\infty}\|f\|_{L^2}^2.
\end{aligned}
$$

To derive the cancellation property in H^m with integer $m \ge 1$, we also use the Leibniz's rule (cf. $\langle \partial_x^m \mathcal{L}_k f, \partial_x^m f\rangle_{L^2}$). Since the properties of integration by parts and the Leibniz's rule do not hold true for pseudo-differential operators in general, even the validity of (1.13) is not immediately apparent for pseudo-differential operators, let alone the more intricate case (1.14).

(2) Motivated by Alonso-Orán and Bethencourt de León [3], Crisan et al. [11] and the author's works Alonso-Orán et al. [5], Tang and Wang [44], in this paper, we focus on the case where $\{\mathcal{Q}_k\}_{k\ge1}$ is a sequence of operators that are not far from skew-adjoint operators and we use a family of commutators (cf. (3.5)) to achieve the estimates (1.13) and (1.14). Even though we have $\|[\mathcal{Q}, f]g\|_{H^s} \le C\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}$ for some $C > 0$, and some suitable $s, s_1, s_2 \in \mathbb{R}$, when this estimate is applied for Q_k, the constant C may depend on k through Q_k, which may pose some challenges in taking summation (see Remark 3.1 for more details). To address this issue, we need some results on the continuous dependence of commutators on operators (see Lemma A.7). We note that this difficulty does **not** appear for SALT operators because all the dependence on k can be written down in an explicit manner involving η_k in (1.7) and (1.8), where the Leibniz's rule and integration by parts are used. We remark that a special case of (1.13) and (1.14) with $\mathscr{O} = \{J_n(\mathbf{I}-\Delta)^{s/2}\}_{n\ge1}$ for a given sequence of mollifiers $\{J_n\}_{n\ge1}$ and $\mathcal{Q}_k$ being diagonal form was studied in the author's recent work Tang and Wang [44]. This is the first study, to the best of our knowledge, to deal with pseudo-differential noise.

- In Section 4, we prove the local existence, uniqueness, blow-up criterion, and global regularity of solutions to (1.15) (see Theorem 4.1). A key aspect of our approach is the use of conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_0]$ instead of expectation $\mathbb{E}$ in constructing solutions. This avoids the need for any moment condition on initial data. It seems that this technique has been rarely used in the literature of SPDEs. Moreover, our proof for Theorem 4.1 does not require any compactness on Sobolev embeddings, which is necessary in the well-known martingale approach (see Prokhorov's Theorem and Skorokhod's Theorem). As a result, Theorem 4.1 holds true not only on the torus $\mathbb{T}^d$ but also on the whole space $\mathbb{R}^d$. In Section 4.4, we provide further discussions on pseudo-differential noise in terms of global existence. We also propose an intriguing but unsolved problem on stochastic 2-dimensional incompressible Euler equations.
- In Section 5, we investigate the effect of noise on the solution map. Our main result is presented in Theorem 5.1, which demonstrates that (small) multiplicative noise

(in Itô's sense) cannot simultaneously improve the stability of the exit time and the continuity of dependence on initial data. As mentioned before, the lack of lifespan estimate is the main obstacle and now we explain this difficulty in more detail. In the proof for Theorem 5.1, we obtain two sequences of solutions $\{u_{l,n}\}_{n\geq 1}$ $(l=-1,1)$ such that $\{u_{l,n}(0)\}_{n\geq 1}$ $(l=-1,1)$ is bounded in H^s and we need to consider the limit behavior as $n\to\infty$ at time $t>0$. For each n, we know $u_{l,n}$ exists at least on $[0,\tau_{l,n}]$ but we do not know whether or not $\inf_n \tau_{l,n}>0$ $\mathbb{P}$-a.s. In deterministic cases, one can easily obtain the lifespan estimate, which enables us to find a positive lower bound for the existence times of a sequence of $u_{l,n}$, that is, there is a $T>0$ such that $u_{l,n}$ exists on $[0,T]$ for all n (see, for example, (4.7) & (4.8) in Tang et al. [43] and (3.8) & (3.9) in Tang et al. [47]). Further remarks and comparisons can be found in Remark 5.1.

- Appendix A provides necessary estimates/results used in our proofs.

2. Notations, preliminary results and definitions

2.1. Notations and related preliminary results

To begin with, we list some notations used subsequently. Let $\mathbb{N}_0^d:=(\mathbb{N}\cup\{0\})^d$. For two multi-indexes $\alpha=(\alpha_1,\cdots,\alpha_d)$, $\beta=(\beta_1,\cdots,\beta_d)\in\mathbb{N}_0^d$ with $\beta\leq\alpha$ (which means $\beta_i\leq\alpha_i$ with $1\leq i\leq d$), we define

$$|\alpha|_1:=\sum_{k=1}^d\alpha_k,\quad \partial_x^\alpha:=\prod_{k=1}^d\partial_{x_k}^{\alpha_k},\quad \partial_\xi^\alpha:=\prod_{k=1}^d\partial_{\xi_k}^{\alpha_k},\quad \binom{\alpha}{\beta}:=\prod_{i=1}^d\frac{\alpha_i!}{\beta_i!\cdot(\alpha_i-\beta_i)!}.$$

Recall that $\mathbb{K}=\mathbb{R}$ or $\mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}$ and $d,m\in\mathbb{N}$. For $1\leq p<\infty$, we denote by $L^p(\mathbb{K}^d;\mathbb{R}^m)$ the standard Lebesgue space of measurable p-integrable $\mathbb{R}^m$-valued functions with domain $\mathbb{K}^d$, and we let $L^\infty(\mathbb{K}^d;\mathbb{R}^m)$ be the space of essentially bounded functions. Particularly, the inner product in $L^2(\mathbb{K}^d;\mathbb{R}^m)$ is defined by

$$\langle f,g\rangle_{L^2}:=\sum_{i=1}^m\int_{\mathbb{K}^d} f_i\cdot\overline{g}_i\,\mathrm{d}x,$$

where $\overline{g}$ denotes the complex conjugate of g. If there is no ambiguity, in the following we denote by $\langle f,g\rangle_{L^2}$ the inner product for both $f,g\in L^2(\mathbb{K}^d;\mathbb{R}^m)$ and $f,g\in L^2(\mathbb{K}^d;\mathbb{R})$ with the customary abuse of notation.

Let $\mathrm{i}=\sqrt{-1}$ be the imaginary unit. The Fourier transform $\mathscr{F}_{x\to\xi}$ and inverse Fourier transform $\mathscr{F}^{-1}_{\xi\to x}$ on $\mathbb{R}^d$ are defined by

$$(\mathscr{F}_{x\to\xi}f)(\xi):=\int_{\mathbb{R}^d}f(x)\mathrm{e}^{-\mathrm{i}(x\cdot\xi)}\,\mathrm{d}x,\quad (\mathscr{F}^{-1}_{\xi\to x}f)(x):=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}f(\xi)\mathrm{e}^{\mathrm{i}(x\cdot\xi)}\,\mathrm{d}\xi.$$

On torus, i.e., $x \in \mathbb{T}^d$, the Fourier transform $\mathscr{F}_{x\to k}$ and inverse Fourier transform $\mathscr{F}^{-1}_{k\to x}$ are given by

$$(\mathscr{F}_{x\to k} f)(k) := \int_{\mathbb{T}^d} f(x)\mathrm{e}^{-\mathrm{i}(x\cdot k)}\,\mathrm{d}x, \quad (\mathscr{F}^{-1}_{k\to x} f)(x) = \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d} f(k)\mathrm{e}^{\mathrm{i}(x\cdot k)}.$$

For any $s \in \mathbb{R}$, we define the symbol class $\mathbf{S}^s(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m\times m}) \subset C^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m\times m})$ as

$$\mathbf{S}^s(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m\times m}) := \left\{ \mathfrak{p} \,:\, \forall\, \beta,\alpha \in \mathbb{N}_0^d,\ \exists\, C(\beta,\alpha) > 0 \text{ s.t. } \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \frac{\left|\partial_x^\beta \partial_\xi^\alpha \mathfrak{p}(x,\xi)\right|_{m\times m}}{(1+|\xi|)^{s-|\alpha|_1}} < C(\beta,\alpha) \right\}.$$

Here and in the sequel, $|\cdot|_{m\times m}$ and $|\cdot|$ are usual norms in $\mathbb{C}^{m\times m}$ and $\mathbb{R}^d$, respectively. It is well-known that $\mathbf{S}^s(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{m\times m})$ is a Fréchet space equipped with the topology generated by seminorms $\{|\cdot|^{\beta,\alpha;s}_{\mathbb{R}^d\times\mathbb{R}^d}\}_{\beta,\alpha\in\mathbb{N}_0^d}$, where

$$|\mathfrak{p}|^{\beta,\alpha;s}_{\mathbb{R}^d\times\mathbb{R}^d} := \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} |\partial_x^\beta \partial_\xi^\alpha \mathfrak{p}(x,\xi)|_{m\times m}(1+|\xi|)^{-s+|\alpha|_1}.$$

For any $\alpha \in \mathbb{N}_0^d$, we define the partial difference operator $\triangle_k^\alpha$ as

$$(\triangle_k^\alpha g)(k) := \sum_{\gamma\in\mathbb{N}_0^d,\gamma\le\alpha} (-1)^{|\alpha-\gamma|_1}\binom{\alpha}{\gamma} g(k+\gamma), \quad g : \mathbb{Z}^d \to \mathbb{C}, \quad k \in \mathbb{Z}^d.$$

Then the (toroidal) symbol class of order s for $s \in \mathbb{R}$ is defined as (cf. Ruzhansky and Turunen [38]):

$$\mathbf{S}^s(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C}^{m\times m}) := \left\{ \mathfrak{p} \,:\, \begin{array}{c} \mathfrak{p}(\cdot,k) \in C^\infty(\mathbb{T}^d; \mathbb{C}^{m\times m}) \text{ for all } k \in \mathbb{Z}^d; \\ \forall\, \beta,\alpha \in \mathbb{N}_0^d,\ \exists\, C(\beta,\alpha) > 0 \text{ s.t. } \sup_{(x,k)\in\mathbb{T}^d\times\mathbb{Z}^d} \frac{\left|\partial_x^\beta \triangle_k^\alpha \mathfrak{p}(x,k)\right|_{m\times m}}{(1+|k|)^{s-|\alpha|_1}} < C(\beta,\alpha) \end{array} \right\}.$$

Again, $\mathbf{S}^s(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C}^{m\times m})$ is a Fréchet space under the topology given by the following seminorms $\{|\cdot|^{\beta,\alpha;s}_{\mathbb{T}^d\times\mathbb{Z}^d}\}_{\beta,\alpha\in\mathbb{N}_0^d}$:

$$|\mathfrak{p}|^{\beta,\alpha;s}_{\mathbb{T}^d\times\mathbb{Z}^d} := \sup_{(x,k)\in\mathbb{T}^d\times\mathbb{Z}^d} |\partial_x^\beta \triangle_k^\alpha \mathfrak{p}(x,k)|(1+|k|)^{-s+|\alpha|_1}.$$

Then the pseudo-differential operator with symbol $\mathfrak{p}$ is defined by

$$\mathrm{OP}(\mathfrak{p}) := \mathcal{P}, \tag{2.1}$$

$$[\mathcal{P}f](x) := \begin{cases} \left\{\mathscr{F}^{-1}_{\xi\to x}\Big[\mathfrak{p}(z,\xi)\big(\mathscr{F}_{x\to\xi}f\big)(\xi)\Big]\right\}_{z=x}, & \text{if } \mathfrak{p}\in\mathbf{S}^s(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{C}^{m\times m}), \\ \left\{\mathscr{F}^{-1}_{k\to x}\Big[\mathfrak{p}(z,k)\big(\mathscr{F}_{x\to k}f\big)(k)\Big]\right\}_{z=x}, & \text{if } \mathfrak{p}\in\mathbf{S}^s(\mathbb{T}^d\times\mathbb{Z}^d;\mathbb{C}^{m\times m}). \end{cases} \tag{2.2}$$

Throughout this paper, all pseudo-differential operators are assumed to be real-valued, i.e., when f is real, $[\mathrm{OP}(\mathfrak{p})f]$ is also real. Equivalently, it is required that

$$\mathfrak{p}(x,-\xi) = \overline{\mathfrak{p}(x,\xi)} \ \text{ if } \ (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d, \tag{2.3}$$

$$\mathfrak{p}(x,-k) = \overline{\mathfrak{p}(x,k)} \ \text{ if } \ (x,k)\in\mathbb{T}^d\times\mathbb{Z}^d. \tag{2.4}$$

We denote by $\mathbf{S}^s\big(\mathbb{T}^d\times\mathbb{R}^d;\mathbb{C}\big)$ the symbol class such that $\mathfrak{p}\in\mathbf{S}^s\big(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{C}\big)$ means that $\mathfrak{p}(\cdot,\xi)$ is periodic with period 2π for all ξ. Moreover, according to [39, Theorem 5.2] (see also [13, Corollary 2.11]), we see that if

$$|\mathfrak{p}|^{\beta,\alpha;s}_{\mathbb{R}^d\times\mathbb{R}^d} \le c_{\alpha\beta s} \text{ for some } c_{\alpha\beta s}>0, \quad |\alpha|<N_1,\ |\beta|<N_2, \quad N_1,\,N_2\ge 1, \tag{2.5}$$

then $\widetilde{\mathfrak{p}} = \mathfrak{p}\big|_{\mathbb{T}^n\times\mathbb{Z}^n}$ satisfies

$$|\widetilde{\mathfrak{p}}|^{\beta,\alpha;s}_{\mathbb{T}^d\times\mathbb{Z}^d} \le C_{\alpha\beta s} \text{ for some } C_{\alpha\beta s}>0, \quad |\alpha|<N_1,\ |\beta|<N_2, \quad N_1,\,N_2\ge 1. \tag{2.6}$$

Conversely, every symbol $\widetilde{\mathfrak{p}}\in\mathcal{S}^s\big(\mathbb{T}^d\times\mathbb{Z}^d;\mathbb{C}\big)$ satisfying (2.6) is a restriction $\widetilde{\mathfrak{p}} = \mathfrak{p}\big|_{\mathbb{T}^n\times\mathbb{Z}^n}$ of a symbol $\mathfrak{p}\in\mathcal{S}^s\big(\mathbb{T}^d\times\mathbb{R}^d;\mathbb{C}\big)$, where $\mathfrak{p}$ satisfies (2.5). Therefore, we see that

$$\mathrm{OP}\mathcal{S}^s\big(\mathbb{T}^d\times\mathbb{Z}^d;\mathbb{C}\big) = \mathrm{OP}\mathcal{S}^s\big(\mathbb{T}^d\times\mathbb{R}^d;\mathbb{C}\big) \tag{2.7}$$

and any bounded set in $\mathrm{OP}\mathcal{S}^s\big(\mathbb{T}^d\times\mathbb{Z}^d;\mathbb{C}\big)$ coincides with the restriction of a bounded set in $\mathrm{OP}\mathcal{S}^s\big(\mathbb{T}^d\times\mathbb{R}^d;\mathbb{C}\big)$ (see also [13, Theorem 2.10 and Corollary 2.11]). By considering each element in a matrix-valued symbol $\mathfrak{p} = (\mathfrak{p}^{(i,j)})_{1\le i,j\le m}$ with noting that

$$\big(\mathrm{OP}(\mathfrak{p})\big)^{(i,j)} = \mathrm{OP}\big(\mathfrak{p}^{(i,j)}\big),$$

we see that (2.7) also holds true for $\mathrm{OP}\mathbf{S}^s$.

Therefore, we simplify notations if there is no ambiguity in the context and we write

$$\mathbf{S}^s := \big\{\mathbf{S}^s(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{C}^{m\times m}) \,:\, (2.3) \text{ holds}\big\} \ \text{ or } \ \big\{\mathbf{S}^s(\mathbb{T}^d\times\mathbb{Z}^d;\mathbb{C}^{m\times m}) \,:\, (2.4) \text{ holds}\big\},$$

and

$$|\cdot|^{\beta,\alpha;s} := |\cdot|^{\beta,\alpha;s}_{\mathbb{R}^d\times\mathbb{R}^d} \text{ or } |\cdot|^{\beta,\alpha;s}_{\mathbb{T}^d\times\mathbb{Z}^d}.$$

In the following, we will also consider symbols only depending on the frequency variable ξ (if $x\in\mathbb{R}^d$) or k (if $x\in\mathbb{T}^d$). To highlight the differences, we let

$$\mathbf{S}_0 := \left\{ \mathfrak{p} \in \mathbf{S}^s \ : \ \begin{matrix} \mathfrak{p}(x,\xi) = \mathfrak{p}(\xi), \ \text{if } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \\ \mathfrak{p}(x,k) = \mathfrak{p}(k), \ \text{if } (x,k) \in \mathbb{T}^d \times \mathbb{Z}^d \end{matrix} \right\}.$$

To emphasize the scalar symbols (i.e., $m = 1$), we define

$$\mathcal{S}^s := \left\{ \mathbf{S}^s(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}) \ : \ (2.3) \text{ holds} \right\} \ \text{or} \ \left\{ \mathbf{S}^s(\mathbb{T}^d \times \mathbb{Z}^d; \mathbb{C}) \ : \ (2.4) \text{ holds} \right\}.$$

Recalling (2.1) and (2.2), we define

$$\mathrm{OP}\mathbf{S}^s := \left\{ \mathrm{OP}(\mathfrak{p}) \ : \ \mathfrak{p} \in \mathbf{S}^s \right\}, \quad \mathrm{OP}\mathbf{S}^s_0 := \left\{ \mathrm{OP}(\mathfrak{p}) \ : \ \mathfrak{p} \in \mathbf{S}^s_0 \right\}, \quad s \in \mathbb{R}.$$

In the same way, $\mathrm{OP}\mathcal{S}^s$ and $\mathrm{OP}\mathcal{S}^s_0$ can be defined as pseudo-differential operators with symbols in $\mathcal{S}^s$ and $\mathcal{S}^s_0$, respectively.

Recall that $\mathbf{I}$ stands for the identity map. For any $s \in \mathbb{R}$, the operator $\mathcal{D}^s = (\mathbf{I} - \Delta)^{s/2}$ is defined by

$$\mathcal{D}^s := \mathrm{OP}((1 + |\xi|^2)^{s/2}) \ \text{ on } \ \mathbb{R}^d \ \text{ or } \ \mathcal{D}^s := \mathrm{OP}((1 + |k|^2)^{s/2}) \ \text{ on } \ \mathbb{T}^d.$$

For $s \geq 0$, $d, m \geq 1$, the Sobolev spaces H^s on $\mathbb{K}^d$ with values in $\mathbb{R}^m$ are defined as

$$H^s(\mathbb{R}^d; \mathbb{R}^m) := \overline{C_0^\infty(\mathbb{R}^d; \mathbb{R}^m)}^{\|\cdot\|_{H^s}}, \quad H^s(\mathbb{T}^d; \mathbb{R}^m) := \overline{C^\infty(\mathbb{T}^d; \mathbb{R}^m)}^{\|\cdot\|_{H^s}},$$

where

$$\|f\|_{H^s} := \sqrt{\langle f, f \rangle_{H^s}}, \quad \langle f, g \rangle_{H^s} := \sum_{i=1}^m \langle \mathcal{D}^s f_i, \mathcal{D}^s g_i \rangle_{L^2}.$$

Let $W^{1,\infty}(\mathbb{K}^d; \mathbb{R}^m)$ be the set of weakly differential functions $f : \mathbb{K}^d \to \mathbb{R}^m$ with

$$\|f\|_{W^{1,\infty}} := \sum_{j=1}^m \sum_{|\alpha|_1 = 0,1} \|\partial_x^\alpha f\|_{L^\infty} < \infty.$$

We will also be confronted with matrix-valued functions $\Phi(x) = \left(\phi^{(i,j)}(x)\right)_{1 \leq i,j \leq m}$, and for such functions, we define

$$\|\Phi\|_{H^s(\mathbb{R}^d; \mathbb{R}^{m \times m})} := \left(\sum_{i,j=1}^m \|\phi^{(i,j)}\|^2_{H^s(\mathbb{R}^d; \mathbb{R})} \right)^{\frac{1}{2}},$$

and propose the similar definition for $W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^{m \times m})$ and $L^p(\mathbb{R}^d; \mathbb{R}^{m \times m})$. If $d, m \in \mathbb{N}$ are fixed in the context, for $s \geq 0$ and $p \in [1, \infty]$, we will simply write

$$H^s = H^s(\mathbb{K}^d; \mathbb{R}^m), \quad W^{1,\infty} = W^{1,\infty}(\mathbb{K}^d; \mathbb{R}^m), \quad L^p = L^p(\mathbb{K}^d; \mathbb{R}^m),$$

and do the same for the matrix-valued case. In particular, we let $H^\infty := \cap_{s\ge 0} H^s$.

For linear operators $\mathcal{A}$ and $\mathcal{B}$, $[\mathcal{A},\mathcal{B}] := \mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$. $\mathcal{A}^*$ denotes the L^2-adjoint operator of $\mathcal{A}$. Let $\lesssim$ and $\gtrsim$ denote estimates that hold up to some universal *deterministic* constant which may change from line to line. Let X and Y be two Banach spaces. We denote by $\mathscr{L}(X;Y)$ the class of bounded linear operators from X to Y. For two separable Hilbert spaces $\mathbb{U}_1$ and $\mathbb{U}_2$, $\mathscr{L}_2(\mathbb{U}_1;\mathbb{U}_2)$ is class of Hilbert-Schmidt operators from $\mathbb{U}_1$ to $\mathbb{U}_2$.

2.2. Definitions

To begin with, we note that $\mathrm{OP}\mathbf{S}^s$ can be measured using the topology generated by the norm $\|\cdot\|_{\mathscr{L}(H^{r+s};H^r)}$ (cf. **(1)** in Lemma A.6). However, since the map OP is one-to-one (cf. [26, Proposition 1.2, Page 56]), in the paper we will consider the boundedness of $\mathrm{OP}\mathbf{S}^s$ in the following sense:

Definition 2.1. Let $s\in\mathbb{R}$. $\mathscr{O}\subset \mathrm{OP}\mathbf{S}^s$ is said to be bounded if $\{\mathfrak{p} : \mathrm{OP}(\mathfrak{p})\in\mathscr{O}\}\subset \mathbf{S}^s$ is bounded in the sense of boundedness in Fréchet space (cf. Rudin [37]).

To avoid any confusion, for two separable Banach spaces X and Y, $\|\cdot\|_{\mathscr{L}(X;Y)}$ will always be mentioned if boundedness of $\mathscr{L}(X;Y)$ is considered.

Next, we give the precise definition of the solutions. Using (1.11), we rewrite (1.15) as

$$\begin{cases} \mathrm{d}u = \Big[-(u\cdot\nabla)u - F(u) + \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}\mathcal{Q}_k^2 u\Big]\,\mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{k=1}^{\infty}\Big(\mathcal{Q}_k u\,\mathrm{d}\widetilde{W}_k + h_k(t,u)\,\mathrm{d}W_k\Big), \quad x\in\mathbb{K}^d, \\ u\big|_{t=0} = u_0, \quad x\in\mathbb{K}^d. \end{cases} \tag{2.8}$$

Then we will try to find solutions to (2.8) in the following sense:

Definition 2.2. Let $d\ge 2$ and let $\mathbb{K}=\mathbb{R}$ or $\mathbb{T}$. Let u_0 be an $H^s(\mathbb{K}^d;\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable with $s>\frac{d}{2}+1$.

1. A local pathwise solution to (2.8) is a pair (u,τ), where τ is a stopping time satisfying $\mathbb{P}(\tau>0)=1$ and $(u(t))_{t\in[0,\tau)}$ is an $\mathcal{F}_t$-progressively measurable such that

$$\sup_{t'\in[0,t]}\|u(t')\|_{H^s}<\infty,\ t\in[0,\tau)\ \ \mathbb{P}\text{-a.s.},$$

and the following equation holds:

$$u(t)-u_0+\int_0^t\Big[(u\cdot\nabla)\,u+F(u)-\frac{1}{2}\sum_{k=1}^{\infty}\mathcal{Q}_k^2u\Big](t')\,\mathrm{d}t'$$

$$= \int_0^t \sum_{k=1}^{\infty} \mathcal{Q}_k u(t') \, \mathrm{d}\widetilde{W}_k(t') + \int_0^t \sum_{k=1}^{\infty} h_k(t', u) \, \mathrm{d}W_k(t'), \quad t \in [0, \tau) \quad \mathbb{P}\text{-a.s.}$$

2. Additionally, a local solution (u, τ^*) is called maximal, if $\tau^* > 0$ almost surely and

$$\limsup_{t \to \tau^*} \|u(t)\|_{H^s} = \infty \ a.s. \ on \ \{\tau^* < \infty\}.$$

If $\tau^* = \infty$ almost surely, then such a solution is called global.

We also introduce the following notions on the stability of exiting time.

Definition 2.3 *(Stability of exiting time).* Let $d \geq 2$ and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$. Let u_0 be an $H^s(\mathbb{K}^d; \mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variable with $s > \frac{d}{2} + 1$. Assume that $\{u_{0,n}\}$ is an arbitrary sequence of $H^s(\mathbb{K}^d; \mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variables. For each n, let u and u_n be the unique solutions to (1.16) with initial value u_0 and $u_{0,n}$, respectively. For any $R > 0$ and $n \in \mathbb{N}$, define the R-exiting time as

$$\tau_n^R := \inf \{t \geq 0 : \|u_n(t)\|_{H^s} > R\}, \quad \tau^R := \inf \{t \geq 0 : \|u(t)\|_{H^s} > R\},$$

where $\inf \varnothing = \infty$.

1. Let $R > 0$. If $u_{0,n} \to u_0$ in H^s almost surely implies

$$\lim_{n \to \infty} \tau_n^R = \tau^R \quad \mathbb{P}\text{-a.s.}, \tag{2.9}$$

then the R-exiting time is said to be stable at u.
2. Let $R > 0$. If $u_{0,n} \to u_0$ in $H^{s'}$ for all $s' < s$ almost surely also implies (2.9), then the R-exiting time is said to be strongly stable at u.

3. Cancellation of singularities

In this section, we aim to establish cancellation properties in (1.13) and (1.14) for certain cases that generalize the SALT operator.

We remind readers that $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$, and $\mathcal{P}^*$ denotes the L^2-adjoint of the linear operator $\mathcal{P}$. Additionally, all functions and operators are real (cf. (2.3) and (2.4)). These facts will be used frequently without further mention. For $\mathcal{P} \in \mathrm{OP}\mathbf{S}^r$ and a real-valued function h, we sometimes write $h\mathcal{P} := (h\mathbf{I})\mathcal{P}$. We also remind readers of the definition of boundedness in $\mathrm{OP}\mathbf{S}^s$, as per Definition 2.1.

3.1. Case 1: x-dependent operators with order $\alpha \in [0, 1]$

To begin with, we consider (1.13) and we find that it holds true for operators that are close to skew-adjoint operators. The precise statement is the following theorem:

Theorem 3.1. *Let $d, m \geq 1$, $s \geq 0$ and $\alpha \in [0,1]$. Let $\mathscr{O} \subset \mathrm{OP}\mathcal{S}^s$ be a bounded set and we assume that*

$$\{\mathcal{U}_k\}_{k\geq 1} \subset \mathrm{OP}\mathbf{S}^\alpha \quad \textit{and} \quad \left\{\mathcal{U}_k + \mathcal{U}_k^*\right\}_{k\geq 1} \subset \mathrm{OP}\mathbf{S}^0 \quad \textit{are bounded, respectively.} \tag{3.1}$$

Let $s_0 > \left(\frac{d}{2}+1\right) \vee s$ and we assume $\{d_k\}_{k\geq 1} \subset H^{s_0}(\mathbb{K}^d;\mathbb{R})$ such that $\sum_{k=1}^\infty \|d_k\|_{H^{s_0}}^2 < \infty$. Then we have

$$\sup_{\mathcal{P}\in\mathscr{O}} \sum_{k=1}^\infty \left\langle \mathcal{P}(d_k\mathcal{U}_k)f, \mathcal{P}f\right\rangle_{L^2}^2 \lesssim \left(\sum_{k=1}^\infty \|d_k\|_{H^{s_0}}^2\right) \|f\|_{H^s}^4, \quad f \in H^{s+\alpha}.$$

Proof. Let $\mathcal{E}_k := \mathcal{U}_k + \mathcal{U}_k^*$. Then one first infers from **(1)** in Lemma A.6 that

$$\sup_{\mathcal{P}\in\mathscr{O}} \|\mathcal{P}\|_{\mathscr{L}(H^{r+s};H^r)}, \quad \sup_{k\geq 1} \|\mathcal{U}_k\|_{\mathscr{L}(H^{r+\alpha};H^r)}, \quad \sup_{k\geq 1} \|\mathcal{E}_k\|_{\mathscr{L}(H^r;H^r)} < \infty, \quad r \in \mathbb{R}. \tag{3.2}$$

We note that

$$\begin{aligned}
\left\langle \mathcal{P}(d_k\mathcal{U}_k)f, \mathcal{P}f\right\rangle_{L^2} &= I_{1,k} + I_{2,k},\\
I_{1,k} &:= \langle [\mathcal{P}, d_k\mathbf{I}]\mathcal{U}_k f, \mathcal{P}f\rangle_{L^2},\\
I_{2,k} &:= \langle d_k\mathcal{P}\mathcal{U}_k f, \mathcal{P}f\rangle_{L^2}.
\end{aligned}$$

Since $s_0 - s \geq 0$ and $\alpha \leq 1$, we can infer from (A.4) (with $q = 0$ and $\sigma = s_0 \geq r = s$) and (3.2) that

$$\sup_{\mathcal{P}\in\mathscr{O}} |I_{1,k}| \lesssim \|d_k\|_{H^{s_0}} \|\mathcal{U}_k f\|_{H^{s-1}} \|f\|_{H^s} \lesssim \|d_k\|_{H^{s_0}} \|f\|_{H^s}^2.$$

Now we estimate $I_{2,k}$. We observe that

$$\begin{aligned}
I_{2,k} &= \left\langle \left[\mathcal{P}, \mathcal{U}_k\right] f, d_k\mathcal{P}f\right\rangle_{L^2} + \left\langle \mathcal{U}_k\mathcal{P}f, d_k\mathcal{P}f\right\rangle_{L^2}\\
&= \left\langle \left[\mathcal{P}, \mathcal{U}_k\right] f, d_k\mathcal{P}f\right\rangle_{L^2} - \langle \mathcal{P}f, \mathcal{U}_k(d_k\mathcal{P}f)\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{E}_k(d_k\mathcal{P}f)\rangle_{L^2}\\
&= \left\langle \left[\mathcal{P}, \mathcal{U}_k\right] f, d_k\mathcal{P}f\right\rangle_{L^2} - \left\langle \mathcal{P}f, \left[\mathcal{U}_k, d_k\mathbf{I}\right]\mathcal{P}f\right\rangle_{L^2}\\
&\quad - \langle \mathcal{P}f, d_k\mathcal{U}_k\mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{E}_k(d_k\mathcal{P}f)\rangle_{L^2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
I_{2,k} &= \left\langle \left[\mathcal{P}, \mathcal{U}_k\right] f, d_k\mathcal{P}f\right\rangle_{L^2} - \left\langle \left[\mathcal{U}_k, d_k\mathbf{I}\right]\mathcal{P}f, \mathcal{P}f\right\rangle_{L^2}\\
&\quad - \langle \mathcal{U}_k\mathcal{P}f, d_k\mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{E}_k(d_k\mathcal{P}f)\rangle_{L^2}\\
&= 2\left\langle \left[\mathcal{P}, \mathcal{U}_k\right] f, d_k\mathcal{P}f\right\rangle_{L^2} - \left\langle \left[\mathcal{U}_k, d_k\mathbf{I}\right]\mathcal{P}f, \mathcal{P}f\right\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{E}_k(d_k\mathcal{P}f)\rangle_{L^2} - I_{2,k}.
\end{aligned}$$

Hence

$$I_{2,k} = \left\langle \left[\mathcal{P}, \mathcal{U}_k\right] f, d_k \mathcal{P} f \right\rangle_{L^2} - \frac{1}{2} \left\langle \left[\mathcal{U}_k, d_k \mathbf{I}\right] \mathcal{P} f, \mathcal{P} f \right\rangle_{L^2} + \frac{1}{2} \left\langle \mathcal{P} f, \mathcal{E}_k(d_k \mathcal{P} f) \right\rangle_{L^2}.$$

Applying Lemma A.7 to $[\mathcal{P}, \mathcal{U}_k]$, (A.4) to $[\mathcal{U}_k, d_k \mathbf{I}]$ (with $q = 0$ and $\sigma = s_0 > r = \alpha$), (3.2) and $H^{s_0} \hookrightarrow W^{1,\infty}$, we have

$$\sup_{\mathcal{P} \in \mathscr{O}} \left| \left\langle \left[\mathcal{P}, \mathcal{U}_k\right] f, d_k \mathcal{P} f \right\rangle_{L^2} \right| \lesssim \|f\|_{H^{s+\alpha-1}} \|d_k\|_{L^\infty} \|f\|_{H^s} \lesssim \|d_k\|_{H^{s_0}} \|f\|_{H^s}^2,$$

$$\sup_{\mathcal{P} \in \mathscr{O}} \left| \left\langle \left[\mathcal{U}_k, d_k \mathbf{I}\right] \mathcal{P} f, \mathcal{P} f \right\rangle_{L^2} \right| \lesssim \|d_k\|_{H^{s_0}} \|\mathcal{P} f\|_{H^{\alpha-1}} \|f\|_{H^s} \lesssim \|d_k\|_{H^{s_0}} \|f\|_{H^s}^2,$$

and

$$\sup_{\mathcal{P} \in \mathscr{O}} \left| \langle \mathcal{P} f, \mathcal{E}_k(d_k \mathcal{P} f) \rangle_{L^2} \right| \lesssim \|d_k\|_{H^{s_0}} \|f\|_{H^s}^2.$$

In conclusion, we derive

$$\sup_{\mathcal{P} \in \mathscr{O}} |I_{2,k}| \lesssim \|d_k\|_{H^{s_0}} \|f\|_{H^s}^2.$$

Combining the estimates for $I_{i,k}$ with $i = 1, 2$ gives rise to the desired estimate. □

The rest of this section focuses on the more complicated cancellation property in (1.14). For the well-known cases of transport noise and SALT noise, we refer to Alonso-Orán and Bethencourt de León [3], Alonso-Orán et al. [5], Crisan et al. [11], Goodair and Crisan [17] and the references therein.

Hypothesis $(\mathbf{H_1})$. *Let $d, m \geq 1$ and $\alpha \in [0, 1]$. Suppose that*

$$\mathcal{A}_k = a_k \mathcal{L}_k + p_k \mathcal{G}_k, \quad \mathcal{L}_k = \operatorname{diag}(\mathcal{L}_{k,1}, \cdots, \mathcal{L}_{k,m}) \in \mathrm{OP}\mathbf{S}^\alpha, \ \mathcal{G}_k \in \mathrm{OP}\mathbf{S}^0, \quad k \geq 1,$$

where $a_k = a_k(x)$ and $p_k = p_k(x)$ are some functions and the following conditions hold true:

- $(\mathbf{H}_1^a)$ *$\{\mathcal{L}_k\}_{k \geq 1} \subset \mathrm{OP}\mathbf{S}^\alpha$, $\{\mathcal{L}_k + \mathcal{L}_k^*\}_{k \geq 1} \subset \mathrm{OP}\mathbf{S}^0$ and $\{\mathcal{G}_k\}_{k \geq 1} \subset \mathrm{OP}\mathbf{S}^0$ are bounded, respectively.*
- $(\mathbf{H}_1^b)$ *For all $k \geq 1$, the symbols of $\mathcal{L}_k$ and $\mathcal{G}_k$ are commuting matrices.*

We note that $(\mathbf{H}_1^b)$ holds true automatically for two cases: either $\mathcal{G}_k$ only involves operators on the main diagonal, or $\mathcal{L}_{k,i} = \mathcal{L}_{k,j}$, $1 \leq i, j \leq m$ (i.e., $\mathcal{L}_k$ can be treated as a scalar operator). The latter includes cases of SALT operators as in (1.7) and (1.8). For further remarks on assumptions and techniques, see Section 3.3.

Theorem 3.2. *Let $s \geq 0$ and Hypothesis $(\mathbf{H_1})$ hold true. Let $s_0 > (\frac{d}{2} + 1) \vee s$, $\sigma_0 > (\frac{d}{2} + 1) \vee (s + 2\alpha)$ and $\mathscr{O} \subset \mathrm{OP}\mathcal{S}^s$ be bounded.*

(i) *Let* $\{a_k\}_{k\geq 1}, \{p_k\}_{k\geq 1} \subset H^{s_0}(\mathbb{K}^d, \mathbb{R})$ *satisfy* $A_1 := \sum_{k=1}^{\infty}\Big(\|a_k\|^2_{H^{s_0}} + \|p_k\|^2_{H^{s_0}}\Big) < \infty$*, then we have*

$$\sup_{\mathcal{P}\in\mathscr{O}}\sum_{k=1}^{\infty}\big\langle\mathcal{P}\mathcal{A}_k f, \mathcal{P}f\big\rangle^2_{L^2} \lesssim A_1\,\|f\|^4_{H^s}, \quad f\in H^{s+\alpha}. \tag{3.3}$$

(ii) *Let* $\{p_k\}_{k\geq 1} \subset H^{\sigma_0}(\mathbb{K}^d, \mathbb{R})$ *and* $\{a_k\}_{k\geq 1} \subset H^{\infty}(\mathbb{K}^d; \mathbb{R})$ *satisfy* $\sum_{k=1}^{\infty}\Big(\|a_k\|_{H^r} + \|p_k\|^2_{H^{\sigma_0}}\Big) < \infty$ *for all* $r \geq 0$*, then* (3.3) *holds. Besides, for* $s \geq 1-\alpha$ *and* $A_2 := \sum_{k=1}^{\infty}\Big(\|a_k\|_{H^{\sigma_0}} + \|p_k\|^2_{H^{\sigma_0}}\Big)$,

$$\sup_{\mathcal{P}\in\mathscr{O}}\sum_{k=1}^{\infty}\Big|\big\langle\mathcal{P}\mathcal{A}_k^2 f, \mathcal{P}f\big\rangle_{L^2} + \langle\mathcal{P}\mathcal{A}_k f, \mathcal{P}\mathcal{A}_k f\rangle_{L^2}\Big| \lesssim A_2\,\|f\|^2_{H^s}, \quad f\in H^{s+2\alpha}. \tag{3.4}$$

(iii) *If* $\{a_k\}_{k\geq 1}$, $\{p_k\}_{k\geq 1} \in l^2$*, then* (3.3) *and* (3.4) *hold true with* $A_3 := \sum_{k=1}^{\infty}\big(|a_k|^2 + |p_k|^2\big)$ *replacing* A_1 *and* A_2*, respectively.*

Proof. By applying **(2)** in Lemma A.6, we observe that if $(\mathbf{H}_1^a)$ is satisfied, then $\mathcal{L}_k$ and $\mathcal{G}_k$ also satisfy (3.1). Therefore, using Theorem 3.1 with $d_k\mathcal{U}_k = a_k\mathcal{L}_k$ and $d_k\mathcal{U}_k = p_k\mathcal{G}_k$, respectively, we obtain (3.3).

In the following discussion, we will only focus on verifying (3.4) since the other cases can be proved in the same way.

Step (1). *For* $k, n \in \mathbb{N}$*, we let*

$$\mathcal{H}_k := \mathcal{L}_k^* + \mathcal{L}_k,\ \ \mathcal{Z}_k := \big[a_k\mathbf{I}, \mathcal{L}_k\big] + \mathcal{H}_k(a_k\mathbf{I}),$$
$$\mathcal{R}_{1,k} := \big[a_k\mathcal{L}_k, \mathcal{Z}_k\big],\ \ \mathcal{R}_{2,k} := \big[p_k\mathcal{G}_k, a_k\mathcal{L}_k\big],$$

and

$$\mathcal{R}_{3,k,\mathcal{P}} := \big[\mathcal{P}, a_k\mathcal{L}_k\big],\ \ \ \mathcal{R}_{4,k,\mathcal{P}} := \big[\mathcal{R}_{3,k,\mathcal{P}}, a_k\mathcal{L}_k\big].$$

Claim:

$$\big\langle\mathcal{P}\mathcal{A}_k^2 f, \mathcal{P}f\big\rangle_{L^2} + \big\langle\mathcal{P}\mathcal{A}_k f, \mathcal{P}\mathcal{A}_k f\big\rangle_{L^2} = \sum_{l=1}^{11} N_l, \tag{3.5}$$

where

$$
\begin{cases}
N_1 := \langle \mathcal{R}_{4,k,\mathcal{P}} f, \mathcal{P} f\rangle_{L^2}\,, \quad N_2 := \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2}\,, \quad N_3 = 2\,\langle \mathcal{Z}_k \mathcal{P} f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2}\,,\\
N_4 := -\frac{1}{2}\,\langle \mathcal{P} f, \mathcal{R}_{1,k}\mathcal{P} f\rangle_{L^2}\,, \quad N_5 := \frac{1}{2}\left\langle \mathcal{P} f, \mathcal{Z}_k^2 \mathcal{P} f\right\rangle_{L^2}\,,\\
N_6 := 2\,\langle \mathcal{P}(p_k\mathcal{G}_k) f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2}\,, \quad N_7 := 2\,\langle \mathcal{P}(p_k\mathcal{G}_k) f, \mathcal{Z}_k\mathcal{P} f\rangle_{L^2}\,,\\
N_8 := 2\,\langle \mathcal{R}_{3,k,\mathcal{P}}(p_k\mathcal{G}_k) f, \mathcal{P} f\rangle_{L^2}\,, \quad N_9 := \langle \mathcal{P}\mathcal{R}_{2,k} f, \mathcal{P} f\rangle_{L^2}\,,\\
N_{10} := \left\langle \mathcal{P}(p_k\mathcal{G}_k)^2 f, \mathcal{P} f\right\rangle_{L^2}\,, \quad N_{11} := \langle \mathcal{P}(p_k\mathcal{G}_k) f, \mathcal{P}(p_k\mathcal{G}_k) f\rangle_{L^2}\,.
\end{cases}
$$

To simplify notation, we let

$$
\mathcal{T}_k := a_k\mathcal{L}_k, \quad \mathcal{K}_k = p_k\mathcal{G}_k,
$$

and then we have

$$
\begin{aligned}
&\left\langle \mathcal{P}\mathcal{A}_k^2 f, \mathcal{P} f\right\rangle_{L^2} + \left\langle \mathcal{P}\mathcal{A}_k f, \mathcal{P}\mathcal{A}_k f\right\rangle_{L^2}\\
={}& \left\langle \mathcal{P}\mathcal{T}_k^2 f, \mathcal{P} f\right\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k\mathcal{K}_k f, \mathcal{P} f\rangle_{L^2} + \langle \mathcal{P}\mathcal{K}_k\mathcal{T}_k f, \mathcal{P} f\rangle_{L^2} + \left\langle \mathcal{P}\mathcal{K}_k^2 f, \mathcal{P} f\right\rangle_{L^2}\\
&+ \langle \mathcal{P}\mathcal{T}_k f, \mathcal{P}\mathcal{T}_k f\rangle_{L^2} + 2\,\langle \mathcal{P}\mathcal{T}_k f, \mathcal{P}\mathcal{K}_k f\rangle_{L^2} + \langle \mathcal{P}\mathcal{K}_k f, \mathcal{P}\mathcal{K}_k f\rangle_{L^2}\\
={}& \sum_{i=1}^{7} \mathfrak{h}_i.
\end{aligned} \tag{3.6}
$$

By $(\mathbf{H}_1^a)$, one can immediately find that $\mathcal{T}_k^* = -\mathcal{T}_k + \mathcal{Z}_k$, which leads to

$$
\begin{aligned}
\mathfrak{h}_1 &= \left\langle \mathcal{P}\mathcal{T}_k^2 f, \mathcal{P} f\right\rangle_{L^2}\\
&= \left\langle \big(\mathcal{T}_k\mathcal{P} + \mathcal{R}_{3,k,\mathcal{P}}\big)\mathcal{T}_k f, \mathcal{P} f\right\rangle_{L^2}\\
&= \left\langle \mathcal{P}\mathcal{T}_k f, \mathcal{T}_k^*\mathcal{P} f\right\rangle_{L^2} + \left\langle \mathcal{R}_{3,k,\mathcal{P}}\mathcal{T}_k f, \mathcal{P} f\right\rangle_{L^2}\\
&= -\left\langle \mathcal{P}\mathcal{T}_k f, \mathcal{T}_k\mathcal{P} f\right\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k\mathcal{P} f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}}\mathcal{T}_k f, \mathcal{P} f\rangle_{L^2}\\
&= -\langle \mathcal{P}\mathcal{T}_k f, \mathcal{P}\mathcal{T}_k f\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k\mathcal{P} f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}}\mathcal{T}_k f, \mathcal{P} f\rangle_{L^2}\\
&= -\mathfrak{h}_5 + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k\mathcal{P} f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}}\mathcal{T}_k f, \mathcal{P} f\rangle_{L^2}\,.
\end{aligned}
$$

That is to say,

$$
\mathfrak{h}_1 + \mathfrak{h}_5 = \langle \mathcal{P}\mathcal{T}_k f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k\mathcal{P} f\rangle_{L^2} + \left\langle \mathcal{R}_{3,k,\mathcal{P}}\mathcal{T}_k f, \mathcal{P} f\right\rangle_{L^2}.
$$

Note that $\mathcal{P}$ is of order s. Then $\mathcal{P}\mathcal{T}_k$ is of order $s+\alpha \geq s$. Similarly, the order of $\mathcal{R}_{3,k,\mathcal{P}}\mathcal{T}_k$ may be also larger than s. Therefore, by commuting $\mathcal{P}$ and $\mathcal{T}_k$ and using $\mathcal{T}_k^*$ again, we have

$$
\begin{aligned}
&\mathfrak{h}_1 + \mathfrak{h}_5\\
={}& \langle \mathcal{T}_k\mathcal{P} f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2}\\
&+ \langle \mathcal{R}_{3,k,\mathcal{P}}\mathcal{T}_k f, \mathcal{P} f\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k\mathcal{P} f\rangle_{L^2}
\end{aligned}
$$

$$\begin{aligned}
&= -\langle \mathcal{P}f, \mathcal{T}_k \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{Z}_k \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} \\
&\quad + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} \mathcal{T}_k f, \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&= \langle \mathcal{R}_{4,k,\mathcal{P}} f, \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{Z}_k \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} \\
&\quad + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} .
\end{aligned} \tag{3.7}$$

Note that $\mathcal{Z}_k^* = \mathcal{Z}_k$. Then we arrive at

$$\begin{aligned}
&\langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&= \langle \mathcal{T}_k \mathcal{P}f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&= -\langle \mathcal{P}f, \mathcal{T}_k \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{Z}_k^2 \mathcal{P}f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&= -\langle \mathcal{P}f, \mathcal{Z}_k \mathcal{T}_k \mathcal{P}f\rangle_{L^2} - \langle \mathcal{P}f, \mathcal{R}_{1,k} \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{Z}_k^2 \mathcal{P}f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&= -\langle \mathcal{Z}_k \mathcal{P}f, \mathcal{T}_k \mathcal{P}f\rangle_{L^2} - \langle \mathcal{P}f, \mathcal{R}_{1,k} \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}f, \mathcal{Z}_k^2 \mathcal{P}f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} .
\end{aligned}$$

Therefore, using $\mathcal{R}_{3,k,\mathcal{P}} = [\mathcal{P}, \mathcal{T}_k]$ gives rise to

$$\begin{aligned}
&\langle \mathcal{P}\mathcal{T}_k f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&= -\frac{1}{2} \langle \mathcal{P}f, \mathcal{R}_{1,k} \mathcal{P}f\rangle_{L^2} + \frac{1}{2} \langle \mathcal{P}f, \mathcal{Z}_k^2 \mathcal{P}f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} .
\end{aligned} \tag{3.8}$$

Combining (3.7), (3.8) and $\mathcal{Z}_k^* = \mathcal{Z}_k$ gives

$$\begin{aligned}
\mathfrak{h}_1 + \mathfrak{h}_5 &= \langle \mathcal{R}_{4,k,\mathcal{P}} f, \mathcal{P}f\rangle_{L^2} + 2\langle \mathcal{Z}_k \mathcal{P}f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} + \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} \\
&\quad - \frac{1}{2} \langle \mathcal{P}f, \mathcal{R}_{1,k} \mathcal{P}f\rangle_{L^2} + \frac{1}{2} \langle \mathcal{P}f, \mathcal{Z}_k^2 \mathcal{P}f\rangle_{L^2} .
\end{aligned} \tag{3.9}$$

Similarly, we use $\mathcal{T}_k^* = -\mathcal{T}_k + \mathcal{Z}_k$, $\mathcal{R}_{2,k} = [\mathcal{K}_k, \mathcal{T}_k]$ and $\mathcal{R}_{3,k,\mathcal{P}} = [\mathcal{P}, \mathcal{T}_k]$ to derive

$$\begin{aligned}
\mathfrak{h}_2 + \mathfrak{h}_3 &= 2 \langle \mathcal{P}\mathcal{T}_k \mathcal{K}_k f, \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}\mathcal{R}_{2,k} f, \mathcal{P}f\rangle_{L^2} \\
&= 2 \langle \mathcal{T}_k \mathcal{P}\mathcal{K}_k f, \mathcal{P}f\rangle_{L^2} + 2 \langle \mathcal{R}_{3,k,\mathcal{P}} \mathcal{K}_k f, \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}\mathcal{R}_{2,k} f, \mathcal{P}f\rangle_{L^2} \\
&= 2 \langle \mathcal{P}\mathcal{K}_k f, -\mathcal{T}_k \mathcal{P}f\rangle_{L^2} + 2 \langle \mathcal{P}\mathcal{K}_k f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&\quad + 2 \langle \mathcal{R}_{3,k,\mathcal{P}} \mathcal{K}_k f, \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}\mathcal{R}_{2,k} f, \mathcal{P}f\rangle_{L^2} \\
&= 2 \langle \mathcal{P}\mathcal{K}_k f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} - 2 \langle \mathcal{P}\mathcal{K}_k f, \mathcal{P}\mathcal{T}_k f\rangle_{L^2} + 2 \langle \mathcal{P}\mathcal{K}_k f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&\quad + 2 \langle \mathcal{R}_{3,k,\mathcal{P}} \mathcal{K}_k f, \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}\mathcal{R}_{2,k} f, \mathcal{P}f\rangle_{L^2} \\
&= 2 \langle \mathcal{P}\mathcal{K}_k f, \mathcal{R}_{3,k,\mathcal{P}} f\rangle_{L^2} - \mathfrak{h}_6 + 2 \langle \mathcal{P}\mathcal{K}_k f, \mathcal{Z}_k \mathcal{P}f\rangle_{L^2} \\
&\quad + 2 \langle \mathcal{R}_{3,k,\mathcal{P}} \mathcal{K}_k f, \mathcal{P}f\rangle_{L^2} + \langle \mathcal{P}\mathcal{R}_{2,k} f, \mathcal{P}f\rangle_{L^2} .
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathfrak{h}_2+\mathfrak{h}_3+\mathfrak{h}_6\\
&=2\left\langle \mathcal{P}\mathcal{K}_k f,\mathcal{R}_{3,k,\mathcal{P}}f\right\rangle_{L^2}+2\left\langle \mathcal{P}\mathcal{K}_k f,\mathcal{Z}_k\mathcal{P}f\right\rangle_{L^2}\\
&\quad+2\left\langle \mathcal{R}_{3,k,\mathcal{P}}\mathcal{K}_k f,\mathcal{P}f\right\rangle_{L^2}+\left\langle \mathcal{P}\mathcal{R}_{2,k}f,\mathcal{P}f\right\rangle_{L^2}.
\end{aligned}\tag{3.10}$$

On account of (3.6), (3.9) and (3.10), we obtain (3.5).

Step (2). ***Claim:*** *Recall that* $\sigma_0>(\frac{d}{2}+1)\vee(s+2\alpha)$. *There is a constant* $C>0$ *such that for all* $k\geq 1$*,*

$$\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{R}_{3,k,\mathcal{P}}\|_{\mathscr{L}(H^{s+\alpha-1};L^2)},\ \sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{R}_{4,k,\mathcal{P}}\|_{\mathscr{L}(H^{s+2\alpha-2};L^2)},\ \|\mathcal{Z}_k\|_{\mathscr{L}(L^2;L^2)}\leq C\|a_k\|_{H^{\sigma_0}},\tag{3.11}$$

$$\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{P}(p_k\mathcal{G}_k)\|_{\mathscr{L}(H^s;L^2)}\leq C\|p_k\|_{H^{s_0}},\tag{3.12}$$

$$\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{R}_{3,k,\mathcal{P}}(p_k\mathcal{G}_k)\|_{\mathscr{L}(H^s;L^2)}\leq C\|a_k\|_{H^{\sigma_0}}\|p_k\|_{H^{\sigma_0}}.\tag{3.13}$$

Before we prove these estimates, we note that $(\mathbf{H}_1^a)$ implies that for $r\in\mathbb{R}$,

$$\sup_{k\geq1}\|\mathcal{H}_k\|_{\mathscr{L}(H^r;H^r)}<\infty,\quad \sup_{k\geq1}\|\mathcal{G}_k\|_{\mathscr{L}(H^r;H^r)}<\infty,\quad \sup_{k\geq1}\|\mathcal{L}_k\|_{\mathscr{L}(H^{r+\alpha};H^r)}<\infty.\tag{3.14}$$

Verify (3.11). For $\mathcal{R}_{3,k,\mathcal{P}}$, it holds that

$$\|\mathcal{R}_{3,k,\mathcal{P}}g\|_{L^2}=\left\|\left[\mathcal{P},a_k\mathcal{L}_k\right]g\right\|_{L^2}\leq\left\|\left[\mathcal{P},a_k\mathbf{I}\right]\mathcal{L}_k g\right\|_{L^2}+\left\|a_k\left[\mathcal{P},\mathcal{L}_k\right]g\right\|_{L^2}.$$

Since $\mathscr{O}$ is bounded in OP$\mathcal{S}^s$ (in the sense of Definition 2.1), by $\alpha\leq1$, (A.4) (with $\sigma=\sigma_0\geq s=r$ and $q=0$) and (3.14), we obtain that for sufficiently regular function g,

$$\sup_{\mathcal{P}\in\mathscr{O}}\left\|\left[\mathcal{P},a_k\mathbf{I}\right]\mathcal{L}_k g\right\|_{L^2}\leq C\|a_k\|_{H^{\sigma_0}}\|\mathcal{L}_k g\|_{H^{s-1}}\leq C\|a_k\|_{H^{\sigma_0}}\|g\|_{H^{s+\alpha-1}}.$$

Similarly, via $(\mathbf{H}_1^a)$ and Lemma A.7, we see that $\sup_{\mathcal{P}\in\mathscr{O},k\geq1}\left\|\left[\mathcal{P},\mathcal{L}_k\right]\right\|_{\mathscr{L}(H^{s+\alpha-1};L^2)}<\infty$, and therefore,

$$\sup_{\mathcal{P}\in\mathscr{O}}\left\|a_k\left[\mathcal{P},\mathcal{L}_k\right]g\right\|_{L^2}\leq\|a_k\|_{L^\infty}\sup_{\mathcal{P}\in\mathscr{O},k\geq1}\left\|\left[\mathcal{P},\mathcal{L}_k\right]g\right\|_{L^2}\leq C\|a_k\|_{H^{\sigma_0}}\|g\|_{H^{s+\alpha-1}}.$$

Collecting the above estimates, we obtain

$$\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{R}_{3,k,\mathcal{P}}\|_{\mathscr{L}(H^{s+\alpha-1};L^2)}\leq C\|a_k\|_{H^{\sigma_0}}.$$

Following the proof for (A.2), we have that $\{\mathcal{R}_{3,k,\mathcal{P}}\}_{k\geq1,\mathcal{P}\in\mathscr{O}}$ is bounded in OP$\mathbf{S}^{s+\alpha-1}$. Moreover, we observe that $\mathcal{R}_{3,k,\mathcal{P}}$ is of diagonal form. Since $s+\alpha\geq1$, by repeating the above procedure and replacing $\mathcal{P}$ with $\mathcal{R}_{3,k,\mathcal{P}}$, we obtain

$$\sup_{\mathcal{P}\in\mathscr{O}} \|\mathcal{R}_{4,k,\mathcal{P}}\|_{\mathscr{L}(H^{s+2\alpha-2};L^2)} \le C\,\|a_k\|_{H^{\sigma_0}}\,.$$

One can also obtain the estimate for $\mathcal{Z}_k$ in much the same way. Indeed, applying (A.4) to $\left[\mathcal{L}_k, a_k\mathbf{I}\right]$ (with $q=0$ and $r=\alpha<\sigma=\sigma_0$), we arrive at

$$\left\|\left[\mathcal{L}_k, a_k\mathbf{I}\right]g\right\|_{L^2} \le C\|a_k\|_{H^{\sigma_0}}\|g\|_{H^{\alpha-1}} \le C\|a_k\|_{H^{\sigma_0}}\|g\|_{L^2}.$$

From this, $H^{\sigma_0}\hookrightarrow L^\infty$, and (3.14), we obtain

$$\|\mathcal{Z}_k\|_{\mathscr{L}(L^2;L^2)} \le C\,\|a_k\|_{H^{\sigma_0}}\,.$$

Hence (3.11) holds.

Verify (3.12) **and** (3.13)**.** Remember that $\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{P}\|_{\mathscr{L}(H^{r+s};H^r)}<\infty$ for any $r\in\mathbb{R}$ (see (3.2)). If $s\le\frac{d}{2}$, we use Lemma A.3 (with $s_1=s$ and $s_2=\sigma_0$) and (3.14) to derive

$$\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{P}(p_k\mathcal{G}_k)g\|_{L^2} \le C\|p_k\mathcal{G}_kg\|_{H^s} \le C\|p_k\|_{H^{\sigma_0}}\|\mathcal{G}_kg\|_{H^s} \le C\|p_k\|_{H^{\sigma_0}}\|g\|_{H^s}.$$

If $s>\frac{d}{2}$, one has $H^s\hookrightarrow L^\infty$ and this means

$$\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{P}(p_k\mathcal{G}_k)g\|_{L^2} \le C\|p_k\mathcal{G}_kg\|_{H^s} \le C\|p_k\|_{H^s}\|\mathcal{G}_kg\|_{H^s} \le C\|p_k\|_{H^{\sigma_0}}\|g\|_{H^s}.$$

Hence (3.12) holds. By the same argument leading to the estimate for $\mathcal{P}(p_k\mathcal{G}_k)$ with noting (3.11), we also have that for all $s\ge 1-\alpha\ge 0$,

$$\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{R}_{3,k,\mathcal{P}}(p_k\mathcal{G}_k)g\|_{L^2} \le C\|a_k\|_{H^{\sigma_0}}\|p_k\mathcal{G}_kg\|_{H^s} \le C\|a_k\|_{H^{\sigma_0}}\|p_k\|_{H^{\sigma_0}}\|g\|_{H^s}.$$

Hence (3.13) is proved.

Step (3). *In this step we finish the proof for* (3.4).

In the following, we will frequently use the fact that $\sup_{\mathcal{P}\in\mathscr{O}}\|\mathcal{P}\|_{\mathscr{L}(H^{r+s};H^r)}<\infty$ for any $r\in\mathbb{R}$ (see (3.2)) without further mention. Moreover, we recall that $\sigma_0>(\frac{d}{2}+1)\vee(s+2\alpha)$.

To begin with, we simply use (3.11) to derive

$$\sup_{\mathcal{P}\in\mathscr{O}}|N_1| \le C\,\|a_k\|_{H^{\sigma_0}}\,\|f\|_{H^s}^2\,,\quad \sup_{\mathcal{P}\in\mathscr{O}}\left\{|N_2|+|N_3|+|N_5|\right\} \le C\,\|a_k\|_{H^{\sigma_0}}^2\,\|f\|_{H^s}^2\,.$$

For N_4, we observe that

$$\left[a_k\mathcal{L}_k, \mathcal{Z}_k\right] = \left[a_k\mathcal{L}_k, a_k\mathcal{L}_k-\mathcal{L}_k(a_k\mathbf{I})+\mathcal{H}_k(a_k\mathbf{I})\right] = \left[(a_k\mathbf{I})\mathcal{L}_k, \mathcal{L}_k^*(a_k\mathbf{I})\right]. \quad (3.15)$$

Therefore, **(2)** in Lemma A.6 and (A.3) in Lemma A.7 give rise to

$$\sup_{\mathcal{P}\in\mathscr{O}} |N_4| \lesssim \|a_k\|^2_{H^{\sigma_0}} \|f\|^2_{H^s}.$$

For N_6, N_7, N_8 and N_{11}, we use (3.11), (3.12) and (3.13) to obtain

$$\sup_{\mathcal{P}\in\mathscr{O}} \left\{|N_6| + |N_7| + |N_8| + |N_{11}|\right\} \le C \left(\|a_k\|^2_{H^{\sigma_0}} + \|p_k\|^2_{H^{\sigma_0}}\right) \|f\|^2_{H^s}.$$

Now we consider N_9. Recalling that $s \ge 0$, $\alpha \in [0,1]$ and $\mathcal{R}_{2,k} = \left[p_k\mathcal{G}_k, a_k\mathcal{L}_k\right]$, we use (A.3) in Lemma A.7 to find that

$$\sup_{\mathcal{P}\in\mathscr{O}} |N_9| = \sup_{\mathcal{P}\in\mathscr{O}} \left|\left\langle \mathcal{P}\mathcal{R}_{2,k}f, \mathcal{P}f\right\rangle_{L^2}\right| \le \|a_k\|_{H^{\sigma_0}} \|p_k\|_{H^{\sigma_0}} \|f\|^2_{H^s}.$$

Finally, for N_{10}, the proof for the estimate on $\mathcal{P}(p_k\mathcal{G}_k)$ in (3.12) actually gives

$$\sup_{\mathcal{P}\in\mathscr{O}} \left\|\mathcal{P}(p_k\mathcal{G}_k)^2\right\|_{\mathscr{L}(H^s;L^2)} \le C \|p_k\|^2_{H^{\sigma_0}},$$

and this implies

$$\sup_{\mathcal{P}\in\mathscr{O}} |N_{10}| = \sup_{\mathcal{P}\in\mathscr{O}} \left|\left\langle \mathcal{P}(p_k\mathcal{G}_k)^2 f, \mathcal{P}f\right\rangle_{L^2}\right| \lesssim \|p_k\|^2_{H^{\sigma_0}} \|f\|^2_{H^s}.$$

Collecting all these estimates for (3.5), we see that

$$\sup_{\mathcal{P}\in\mathscr{O}} \left|\left\langle \mathcal{P}\mathcal{A}_k^2 f, \mathcal{P}f\right\rangle_{L^2} + \left\langle \mathcal{P}\mathcal{A}_k f, \mathcal{P}\mathcal{A}_k f\right\rangle_{L^2}\right| \lesssim \left(\|a_k\|_{H^{\sigma_0}} + \|a_k\|^2_{H^{\sigma_0}} + \|p_k\|^2_{H^{\sigma_0}}\right) \|f\|^2_{H^s}.$$

Hence we obtain (3.4). □

3.2. Case 2: x-independent operators with order $\beta \ge 0$

The proof for Theorem 3.2 also implies that the cancellation properties hold true for another class of operators when operators are x-independent.

Hypothesis ($\mathbf{H_2}$). *Let $d, m \ge 1$ and $\beta \ge 0$. We assume the following:*

($\mathbf{H}_2^a$) $\{\mathcal{B}_k\}_{k\ge1} \subset \mathrm{OP}\mathbf{S}_0^\beta$ *and* $\left\{\mathcal{B}_k + \mathcal{B}_k^*\right\}_{k\ge1} \subset \mathrm{OP}\mathbf{S}_0^0$ *are bounded, respectively.*
($\mathbf{H}_2^b$) $\mathcal{B}_k = b_k\mathcal{J}_k + q_k\mathcal{V}_k$*, where* $b_k, q_k \in \mathbb{R}$*,* $\mathcal{J}_k = \mathrm{diag}(\mathcal{J}_{k,1}, \cdots, \mathcal{J}_{k,m})$*,*

$$\{\mathcal{J}_k\}_{k\ge1} \subset \mathrm{OP}\mathbf{S}_0^\beta \quad \textit{and} \quad \{\mathcal{V}_k\}_{k\ge1} \subset \mathrm{OP}\mathbf{S}_0^{0\wedge(1-\beta)} \quad \textit{are bounded, respectively.}$$

Besides, $\{\mathcal{J}_k + \mathcal{J}_k^*\}_{k\ge1} \subset \mathrm{OP}\mathbf{S}_0^0$ *is bounded and for all* $k \ge 1$*, the symbols of* $\mathcal{J}_k$ *and* $\mathcal{V}_k$ *are commuting matrices.*

Now we state the following cancellation properties for x-independent operators:

Theorem 3.3. *Let $s \geq 0$ and $\mathscr{O} \subset \mathrm{OP}\mathcal{S}_0^s$ be a bounded set. Let $\{b_k\}_{k\geq 1}$, $\{q_k\}_{k\geq 1}$, $\{c_k\}_{k\geq 1}$ $\in l^2$ with $B_1 := \sum_{k=1}^{\infty} c_k^2$ and $B_2 := \sum_{k=1}^{\infty} b_k^2 + q_k^2$. Then we have the following properties:*

- *If $(\mathbf{H}_2^a)$ holds, then*

$$\sup_{\mathcal{P}\in\mathscr{O}} \sum_{k=1}^{\infty} \left\langle \mathcal{P} c_k \mathcal{B}_k f, \mathcal{P} f\right\rangle_{L^2}^2 \lesssim B_1 \|f\|_{H^s}^4, \quad f \in H^{s+\beta}. \tag{3.16}$$

- *If $(\mathbf{H}_2^b)$ holds, then (3.16) holds with B_2 replacing B_1 and*

$$\sup_{\mathcal{P}\in\mathscr{O}} \sum_{k=1}^{\infty} \left| \left\langle \mathcal{P}\mathcal{B}_k^2 f, \mathcal{P} f\right\rangle_{L^2} + \left\langle \mathcal{P}\mathcal{B}_k f, \mathcal{P}\mathcal{B}_k f\right\rangle_{L^2} \right| \lesssim B_2 \|f\|_{H^s}^2, \quad f \in H^{s+2\beta}. \tag{3.17}$$

Proof. We first recall that $\sup_{\mathcal{P}\in\mathscr{O}} \|\mathcal{P}\|_{\mathscr{L}(H^s;L^2)} < \infty$. Under condition $(\mathbf{H}_2^a)$, we can follow the proof for Theorem 3.1 and use the fact that $[\mathcal{P}, \mathcal{B}_k] = [\mathcal{P}, c_k] = [\mathcal{B}_k, c_k] = 0$ to find

$$\left\langle \mathcal{P}\mathcal{B}_k f, \mathcal{P} f\right\rangle_{L^2} = \frac{1}{2} \left\langle \mathcal{P} f, \mathcal{M}_k \mathcal{P} f\right\rangle_{L^2}, \quad \mathcal{M}_k := \mathcal{B}_k + \mathcal{B}_k^*.$$

Since $\{\mathcal{M}_k\}_{k\geq 1} \subset \mathrm{OP}\mathcal{S}_0^0$ are bounded, we obtain (3.16). When $(\mathbf{H}_2^b)$ is satisfied, it is clear that $\mathcal{J}_k$ and $\mathcal{V}_k$ also enjoys $(\mathbf{H}_2^a)$, and hence (3.16) also holds with B_2 replacing B_1. Now we prove (3.17). Following the same procedure as in **Step (1)** of the proof for (3.4), and utilizing the fact that $[\mathcal{P}, \mathcal{J}_k] = [\mathcal{P}, \mathcal{V}_k] = [\mathcal{P}, b_k] = [\mathcal{J}_k, b_k] = [\mathcal{J}_k, \mathcal{J}_k^*] = 0$, we identify that

$$\left\langle \mathcal{P}\mathcal{B}_k^2 f, \mathcal{P} f\right\rangle_{L^2} + \left\langle \mathcal{P}\mathcal{B}_k f, \mathcal{P}\mathcal{B}_k f\right\rangle_{L^2} = \sum_{i=1}^{5} M_i, \tag{3.18}$$

where

$$\begin{cases} \widetilde{\mathcal{J}}_k := \mathcal{J}_k^* + \mathcal{J}_k, \quad M_1 := \frac{b_k^2}{2} \left\langle \mathcal{P} f, (\widetilde{\mathcal{J}}_k)^2 \mathcal{P} f\right\rangle_{L^2}, \\ M_2 := 2 b_k q_k \left\langle \mathcal{P}\mathcal{V}_k f, \widetilde{\mathcal{J}}_k \mathcal{P} f\right\rangle_{L^2}, \quad M_3 := b_k q_k \left\langle \mathcal{P}\left[\mathcal{V}_k, \mathcal{J}_k\right] f, \mathcal{P} f\right\rangle_{L^2}, \\ M_4 := q_k^2 \left\langle \mathcal{P}\mathcal{V}_k^2 f, \mathcal{P} f\right\rangle_{L^2}, \quad M_5 := q_k^2 \left\langle \mathcal{P}\mathcal{V}_k f, \mathcal{P}\mathcal{V}_k f\right\rangle_{L^2}. \end{cases}$$

Then $(\mathbf{H}_2^b)$ and Lemma A.7 yield that for all $r \in \mathbb{R}$,

$$\sup_{k\geq 1} \left\{ \|\widetilde{\mathcal{J}}_k\|_{\mathscr{L}(H^r;H^r)} + \|\mathcal{V}_k\|_{\mathscr{L}(H^r;H^r)} + \|\mathcal{J}_k\|_{\mathscr{L}(H^{r+\beta};H^r)} + \left\| \left[\mathcal{V}_k, \mathcal{J}_k\right] \right\|_{\mathscr{L}(H^r;H^r)} \right\} < \infty.$$

From the above estimate, one can easily obtain (3.17). □

3.3. Remarks, examples and other extensions

We will begin by providing some comments on Theorems 3.1, 3.2, and 3.3. Subsequently, we will present examples and discuss other extensions that generalize the SALT operator.

Remark 3.1. We assume that the dominant part of the operators (the part with positive order) is close to skew-adjoint operators. This is expressed by the conditions that $\mathcal{A}_k+\mathcal{A}_k^*$, $\mathcal{L}_k+\mathcal{L}_k^*$, $\mathcal{B}_k+\mathcal{B}_k^*$ and $\mathcal{J}_k+\mathcal{J}_k^*$ are zero-order operators (see (3.1), $(\mathbf{H}_1^a)$, $(\mathbf{H}_2^a)$ and $(\mathbf{H}_2^b)$), respectively. This assumption can be interpreted as a form of "integration by parts".

In Theorems 3.1 and 3.2, the coefficients d_k, a_k, and p_k are assumed to be functions of x. At first glance, this may not seem necessary since $\mathcal{L}_k$ and $\mathcal{G}_k$ already depend on x (their symbols depend on x). However, we include the cases where $d_k = d_k(x)$, $a_k = a_k(x)$, and $p_k = p_k(x)$. This generalization is *non-trivial* and requires delicate modifications (see explanation and question (3.19) below). Besides, the proof for this extended case can help us quickly extend SALT operators and establish cancellation properties for them, cf. Hypothesis $(\mathbf{H_3})$ and Theorem 3.4. We now make a few remarks concerning other hypotheses and some comparisons.

(1) *On the choice of* $\mathscr{O}\subset \mathrm{OP}\mathcal{S}^s$*.* When constructing an approximation scheme for (2.8) (as shown in (4.9) below), the mollifier J_n cannot commute with $\mathcal{A}_k$. In certain cases, to obtain a uniform estimate in H^s, we must address $\mathscr{O}=\{\mathcal{D}^sJ_n\}_{n\geq 1}$ (as demonstrated in Lemma 4.3 below). For this reason, we state the uniform (in n) estimate for $\mathcal{P}\in\mathscr{O}$ rather than just one $\mathcal{P}$.

(2) *On* $\{a_k\}_{k\geq 1}$ *and diagonal form of* $\mathcal{L}_k$*.* We begin by posing the following question:
Question: Given $s\geq 0$, $\mathcal{P}\in \mathrm{OP}\mathcal{S}^s$, and $\mathcal{P}_1,\mathcal{P}_2\in \mathrm{OP}\mathcal{S}^1$, is there a sufficiently large $\zeta_0=\zeta_0(s)$ (ζ_0 depends on s) such that for $h_1,h_2\in H^{\zeta_0}$,

$$\big\|\big[[\mathcal{P},h_1\mathcal{P}_1],h_2\mathcal{P}_2\big]\big\|_{\mathscr{L}(H^s;L^2)}\lesssim \|h_1\|_{H^{\zeta_0}}\|h_2\|_{H^{\zeta_0}}? \tag{3.19}$$

We conjecture that this is true, but it is not the focus of this paper. In this work, we treat $\big[[\mathcal{P},h_1\mathcal{P}_1],h_2\mathcal{P}_2\big]$ as follows: let $\widetilde{\mathcal{P}}=[\mathcal{P},h_1\mathcal{P}_1]$ and assume $h_1\in H^\infty$. Then, by **(3)** and **(4)** in Lemma A.6, we have $\widetilde{\mathcal{P}}\in \mathrm{OP}\mathcal{S}^s$. Let $\rho_0>(\frac{d}{2}+1)\vee s$. By (A.4) (with $q=0$, $r=s<\sigma=\rho_0$), $H^{\rho_0}\hookrightarrow L^\infty$, and **(4)** in Lemma A.6 again, we find a constant $C>0$ that depends on h_1 (i.e., $C=C(h_1)>0$):

$$\begin{aligned}\big\|\big[[\mathcal{P},h_1\mathcal{P}_1],h_2\mathcal{P}_2\big]f\big\|_{L^2}\leq&\big\|\big[\widetilde{\mathcal{P}},h_2\mathbf{I}\big]\mathcal{P}_2f\big\|_{L^2}+\big\|h_2\big[\widetilde{\mathcal{P}},\mathcal{P}_2\big]f\big\|_{L^2}\\ \leq& C(h_1)\Big(\|h_2\|_{H^{\rho_0}}\|\mathcal{P}_2f\|_{H^{s-1}}+\|h_2\|_{L^\infty}\big\|\big[\widetilde{\mathcal{P}},\mathcal{P}_2\big]f\big\|_{L^2}\Big)\\ \leq& C(h_1)\|h_2\|_{H^{\rho_0}}\|f\|_{H^s},\quad f\in H^s.\end{aligned}$$

The above analysis illustrates the scenario when estimating $\mathcal{R}_{4,k,\mathcal{P}}$. To apply **(4)** from Lemma A.6 to $\mathcal{R}_{4,k,\mathcal{P}}$, $\mathcal{R}_{3,k,\mathcal{P}}$ must be a pseudo-differential operator, and its symbol must commute with $\mathcal{L}_k$. This is why we require that $a_k \in H^\infty$ and that $\mathcal{L}_k$ takes a diagonal form. The condition $\sum_{k=1}^\infty \|a_k\|_{H^r} < \infty$ for all $r \geq 0$ is necessary because it ensures that all constants in the estimate can be chosen uniformly for k (note that in the above example $C = C(h_1)$ and see (A.4)) and that one can take the summation $\sum_{k=1}^\infty$. For $\mathcal{R}_{4,k,\mathcal{P}}$, if **Question** (3.19) has a **positive** answer, then one can relax the conditions $a_k \in H^\infty$ and $\sum_{k=1}^\infty \|a_k\|_{H^r} < \infty$ for all $r \geq 0$ to $a_k \in H^{\varsigma_0}$ and $\sum_{k=1}^\infty \|a_k\|^2_{H^{\varsigma_0}} < \infty$, respectively. For the cancellation properties of SALT type operators (as in (1.7) and (1.8)) in H^m with integer order $m \geq 0$, this problem seems to disappear since this double commutator can be explicitly split by Leibniz's rule (cf. [17, Appendix II]). In [17, Equation (20)], the $W^{m+2\times 1,\infty}$ norm is required (the SALT operator is of order 1) for the cancellation property of type (1.14). In this work, we need $\sigma_0 > (\frac{d}{2}+1) \vee (s + 2 \times \alpha)$, where α is the order of $\mathcal{L}_k$ and this condition arises from (3.15) and (A.3).

(3) *On* $(\mathbf{H}_1^b)$. As far as current knowledge is concerned, existing literature results pertain to situations where $\mathcal{L}_{k,i} = \mathcal{L}_{k,j}$ for $1 \leq i, j \leq m$. This implies that $(\mathbf{H}_1^b)$ is satisfied because such $\mathcal{L}_k$ is equivalent to a scalar operator. For instance, in the case of SALT operator (see (1.7) and (1.8)), the derivative operator on the main diagonal is the same one $(\eta_k \cdot \nabla)$. However, when $\mathcal{L}_{k,i_0} \neq \mathcal{L}_{k,j_0}$ for some $1 \leq i_0,\, j_0 \leq m$, Lemma A.7 can only be applied to $\left[p_k \mathcal{G}_k, \mathcal{L}_k\right]$ by invoking $(\mathbf{H}_1^b)$. When $\mathcal{G}_k$ is also an operator with only diagonal elements, $(\mathbf{H}_1^b)$ remains valid.

(4) *On* $(\mathbf{H}_2^b)$. Similar to the above explanation, we require the commutative property of the symbols of J_k and $\mathcal{V}_k$. Since $\mathcal{J}_k \in \mathrm{OP}\mathcal{S}_0^\beta$ and there is no upper bound on β, to estimate $[\mathcal{J}_k, \mathcal{V}_k]$ in M_3 in (3.18), we must assume $\mathcal{V}_k \in \mathrm{OP}\mathbf{S}_0^{0\wedge(1-\beta)}$ (cf. Lemma A.6). This is stronger than the assumption $\mathcal{G}_k \in \mathrm{OP}\mathcal{S}^0$ in $(\mathbf{H}_1^a)$. However, it appears that the cancellation of singularities in such a case has not been reported in the literature.

Example 3.1. Now we construct some examples involving non-local operators such that Theorems 3.1, 3.2 and 3.3 are satisfied. As explained before, it suffices to construct examples satisfying Hypotheses $(\mathbf{H_1})$ and $(\mathbf{H}_2^b)$.

Let $m \geq 1$, $\{\psi_k^{(i)}(x)\}_{k\geq 1}$ and $\{\phi_k^{i,j}(x)\}_{k\geq 1}$ $(1 \leq i, j \leq m)$ be two families of functions such that for $1 \leq i, j \leq m$ and $k \geq 1$,

$$\begin{cases} \psi_k^{(i)}(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d), \quad \phi_k^{i,j}(x) \in C_c^\infty(\mathbb{R}^d;\mathbb{R}), \quad \text{if } \ x \in \mathbb{R}^d, \\ \psi_k^{(i)}(x) \in C^\infty(\mathbb{T}^d;\mathbb{R}^d), \quad \phi_k^{i,j}(x) \in C^\infty(\mathbb{T}^d;\mathbb{R}), \quad \text{if } \ x \in \mathbb{T}^d. \end{cases}$$

We assume that

$$\sum_{k=1}^\infty \sum_{i=1}^m \left\|\psi_k^{(i)}\right\|_{H^r(\mathbb{K}^d;\mathbb{R})} + \sum_{k=1}^\infty \sum_{i,j=1}^m \left\|\phi_k^{(i,j)}\right\|_{H^r(\mathbb{K}^d;\mathbb{R})} < \infty \quad \forall r \geq 0.$$

Let $s_i \geq 0$, σ_i, $s^{(i,j)} \in \mathbb{R}$ $(1 \leq i, j \leq m)$ and define

$$\begin{cases} \mathscr{H}_k := \operatorname{diag}\left(\big(\psi_k^{(1)}(x)\cdot\nabla\big)(-\Delta)^{s_1}(\mathbf{I}-\Delta)^{\sigma_1},\cdots,\big(\psi_k^{(m)}(x)\cdot\nabla\big)(-\Delta)^{s_m}(\mathbf{I}-\Delta)^{\sigma_m}\right), \\ \mathscr{T}_k := \big(\mathscr{T}_k^{(i,j)}\big)_{1\le i,j\le m}, \quad \mathscr{T}_k^{(i,j)} = \phi_k^{(i,j)}(x)(\mathbf{I}-\Delta)^{s^{(i,j)}}, \quad 1\le i,j\le m. \end{cases}$$

By Lemma A.6, we have the following examples:

- Let $s_i+\sigma_i+1\in[0,1]$ and $s^{(i,j)}\le 0$ for all $1\le i,j\le m$. If either $\psi_k^{(i)}=\psi_k^{(j)}$ with $1\le i,j\le m$ or $\phi_k^{(l,n)}=0$ with $1\le l\ne n\le m$, then $\mathcal{L}_k:=\mathscr{H}_k$ and $\mathcal{G}_k:=\mathscr{T}_k$ satisfy Hypothesis $(\mathbf{H_1})$.
- When $s_i+\sigma_i+1\ge 0$ and $s^{(i,j)}=0\vee(-s_1-s_2)$ for all $1\le i,j\le m$, $\psi_k^{(i)}(x)$ and $\phi_k^{(i,j)}(x)$ are *invariant* in x (i.e., they do *not* depend on x), and either $\psi_k^{(i)}=\psi_k^{(j)}$ with $1\le i,j\le m$ or $\phi_k^{(l,n)}=0$ with $1\le l\ne n\le m$, then $\mathcal{J}_k:=\mathscr{H}_k$ and $\mathcal{V}_k:=\mathscr{T}_k$ satisfy $(\mathbf{H}_2^b)$.

To conclude this section, we present another generalization of SALT operators. This generalization may differ slightly from the one in Hypothesis $(\mathbf{H_1})$; however, the corresponding cancellation properties can be established by using the same approach as we used to prove Theorem 3.2. Throughout this discussion, we denote the norms of vector-valued functions or matrix-valued functions simply by $\|\cdot\|_{H^s}$, provided that $d,m\in\mathbb{N}$ are apparent from the context.

As previously noted, in the case of SALT operator (see (1.7) and (1.8)), the derivative operator is a *scalar* operator $(\eta_k\cdot\nabla)$. Theorem 3.2 extends the results on the scalar operator $(\eta_k\cdot\nabla)$ in the SALT operator (see (1.7) and (1.8)) to the operator $a_k\mathcal{L}_k$, where $\mathcal{L}_k=\operatorname{diag}(\mathcal{L}_{k,1},\cdots,\mathcal{L}_{k,m})$ is of order $\alpha\in[0,1]$.

Next, we consider the case where $(\eta_k\cdot\nabla)=\sum_{j=1}^d\eta_{k,j}\partial_{x_j}$ is generalized to $\sum_{j=1}^\infty a_{k,j}\mathcal{L}_{k,j}$. We propose the following Hypothesis:

Hypothesis $(\mathbf{H_3})$ *(Another generalization of the SALT operator). Let $d,m\ge 1$ and $\gamma\in[0,1]$. For $k,j\ge 1$, we let $a_{k,j}=a_{k,j}(x)$ and $\Phi_k=\Phi_k(x)$ be scalar function and matrix-valued functions, respectively. Suppose that*

$$\widetilde{\mathcal{A}}_k=\sum_{j=1}^\infty a_{k,j}\mathcal{L}_{k,j}+\Phi_k\mathbf{I},\quad k\ge 1,$$

where $\{\mathcal{L}_{k,i}\}_{k,i\ge1}\subset \mathrm{OP}\mathcal{S}^\gamma$ and $\{\mathcal{L}_{k,i}+\mathcal{L}_{k,i}^*\}_{k,i\ge1}\subset\mathrm{OP}\mathcal{S}^0$ are bounded, respectively.

Obviously, operators in (1.7) and (1.8) are just special cases of $\widetilde{\mathcal{A}}_k$ ($\mathcal{L}_{k,j}=\partial_{x_j}$ for $1\le j\le d$ and $\mathcal{L}_{k,j}=0$ for $j\ge d+1$). Similar to Example 3.1, one can construct many examples satisfying Hypothesis $(\mathbf{H_3})$.

Theorem 3.4. *Let $s\ge 0$, $\mathscr{O}\subset\mathrm{OP}\mathcal{S}^s$ be bounded and Hypothesis $(\mathbf{H_3})$ hold true. Let $s_0>(\frac{d}{2}+1)\vee s$ and $\widetilde{\sigma}_0>(\frac{d}{2}+1)\vee(s+2\gamma)$.*

(i) *If* $\{a_{k,j}\}_{k,j\geq 1} \subset H^{s_0}(\mathbb{K}^d;\mathbb{R})$ *and* $\{\Phi_k\}_{k\geq 1} \subset H^{s_0}(\mathbb{K}^d;\mathbb{R}^{m\times m})$ *satisfy*

$$\widetilde{A}_1 := \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\|a_{k,j}\|_{H^{s_0}}^2 + \|\Phi_k\|_{H^{s_0}}^2\right) < \infty,$$

then we have that for $f \in H^{s+\gamma}$,

$$\sup_{\mathcal{P}\in\mathscr{O}}\sum_{k=1}^{\infty}\left\langle \mathcal{P}\widetilde{\mathcal{A}}_k f, \mathcal{P}f\right\rangle_{L^2}^2 \lesssim \widetilde{A}_1\,\|f\|_{H^s}^4\,. \tag{3.20}$$

(ii) *If* $\{a_{k,j}\}_{k,j\geq 1} \subset H^{\infty}(\mathbb{K}^d;\mathbb{R})$ *and* $\{\Phi_k\}_{k\geq 1} \subset H^{\sigma_0}(\mathbb{K}^d;\mathbb{R}^{m\times m})$ *satisfy that for all* $r \geq 0$,

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\|a_{k,j}\|_{H^r} + \|\Phi_k\|_{H^{\tilde{\sigma}_0}}^2\right) < \infty,$$

then (3.20) *holds. Moreover, for* $\widetilde{A}_2 := \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\|a_{k,j}\|_{H^{\tilde{\sigma}_0}} + \|\Phi_k\|_{H^{\tilde{\sigma}_0}}^2\right)$, $s \geq 1-\gamma$ *and* $f \in H^{s+2\gamma}$,

$$\sup_{\mathcal{P}\in\mathscr{O}}\sum_{k=1}^{\infty}\left|\left\langle \mathcal{P}\widetilde{\mathcal{A}}_k^2 f, \mathcal{P}f\right\rangle_{L^2} + \left\langle \mathcal{P}\widetilde{\mathcal{A}}_k f, \mathcal{P}\widetilde{\mathcal{A}}_k f\right\rangle_{L^2}\right| \lesssim \widetilde{A}_2\,\|f\|_{H^s}^2\,. \tag{3.21}$$

Proof. We only provide a sketch of the proof for (3.21), since (3.20) can be proved in the same way as we prove Theorem 3.1. For $k, i \geq 1$, we let

$$\mathcal{H}_{k,i} := \mathcal{L}_{k,i}^* + \mathcal{L}_{k,i}, \quad \mathcal{Z}_{k,i} := \left[a_{k,i}\mathbf{I}, \mathcal{L}_{k,i}\right] + \mathcal{H}_{k,i}(a_{k,i}\mathbf{I}).$$

With the customary abuse of notation, we also define

$$\mathcal{Z}_k := \sum_{i=1}^{\infty}\mathcal{Z}_{k,i}, \quad \mathcal{R}_{1,k} := \sum_{i,j=1}^{\infty}\left[a_{k,j}\mathcal{L}_{k,j}, \mathcal{Z}_{k,i}\right], \quad \mathcal{R}_{2,k} := \sum_{i=1}^{\infty}\left[\Phi_k\mathbf{I}, a_{k,i}\mathcal{L}_{k,i}\right],$$

and

$$\mathcal{R}_{3,k,\mathcal{P}} := \sum_{i=1}^{\infty}\left[\mathcal{P}, a_{k,i}\mathcal{L}_{k,i}\right], \quad \mathcal{R}_{4,k,\mathcal{P}} := \sum_{i,j=1}^{\infty}\left[\left[\mathcal{P}, a_{k,i}\mathcal{L}_{k,i}\right], a_{k,j}\mathcal{L}_{k,j}\right].$$

Then the analysis in **Step (1)** in the proof for (3.4) yields

$$\left\langle \mathcal{P}\big(\widetilde{\mathcal{A}}_k\big)^2 f, \mathcal{P}f\right\rangle_{L^2} + \left\langle \mathcal{P}\widetilde{\mathcal{A}}_k f, \mathcal{P}\widetilde{\mathcal{A}}_k f\right\rangle_{L^2} = \sum_{l=1}^{11} N_l,$$

where

$$\begin{cases} N_1 := \langle \mathcal{R}_{4,k,\mathcal{P}} f, \mathcal{P} f \rangle_{L^2}, \quad N_2 := \langle \mathcal{R}_{3,k,\mathcal{P}} f, \mathcal{R}_{3,k,\mathcal{P}} f \rangle_{L^2}, \quad N_3 = 2 \langle \mathcal{Z}_k \mathcal{P} f, \mathcal{R}_{3,k,\mathcal{P}} f \rangle_{L^2}, \\ N_4 := -\frac{1}{2} \langle \mathcal{P} f, \mathcal{R}_{1,k} \mathcal{P} f \rangle_{L^2}, \quad N_5 := \frac{1}{2} \left\langle \mathcal{P} f, \mathcal{Z}_k^2 \mathcal{P} f \right\rangle_{L^2}, \\ N_6 := 2 \langle \mathcal{P}(\Phi_k \mathbf{I}) f, \mathcal{R}_{3,k,\mathcal{P}} f \rangle_{L^2}, \quad N_7 := 2 \langle \mathcal{P}(\Phi_k \mathbf{I}) f, \mathcal{Z}_k \mathcal{P} f \rangle_{L^2}, \\ N_8 := 2 \langle \mathcal{R}_{3,k,\mathcal{P}} (\Phi_k \mathbf{I}) f, \mathcal{P} f \rangle_{L^2}, \quad N_9 := \langle \mathcal{P} \mathcal{R}_{2,k} f, \mathcal{P} f \rangle_{L^2}, \\ N_{10} := \left\langle \mathcal{P}(\Phi_k^2 \mathbf{I}) f, \mathcal{P} f \right\rangle_{L^2}, \quad N_{11} := \langle \mathcal{P}(\Phi_k \mathbf{I}) f, \mathcal{P}(\Phi_k \mathbf{I}) f \rangle_{L^2}. \end{cases}$$

Note that $\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \|a_{k,j}\|_{H^r} + \|\Phi_k\|_{H^{\tilde{\sigma}_0}}^2 \right) < \infty$. Then the same argument in **Steps (2)** and **(3)** in the proof for Theorem 3.2 leads to (3.21). □

Remark 3.2. By considering each element on the main diagonal, one can extend Theorem 3.2 to cover

$$\mathcal{A}_k = \widetilde{a}_k \mathcal{L}_k + p_k \mathcal{G}_k, \quad \widetilde{a}_k = \operatorname{diag}\left(a_k^{(1)}, \cdots, a_k^{(m)} \right), \tag{3.22}$$

where $\mathcal{L}_k$ and $\mathcal{G}_k$ are given in Hypothesis $(\mathbf{H_1})$. Furthermore, it is also possible to combine the above case and Theorem 3.4 to consider

$$\widetilde{\mathcal{A}}_k = \operatorname{diag}\left(\sum_{j=1}^{\infty} a_{k,j}^{(1)} \mathcal{L}_{k,j}^1, \cdots, \sum_{j=1}^{\infty} a_{k,j}^{(m)} \mathcal{L}_{k,j}^m \right) + \Phi_k \mathbf{I}, \quad k \geq 1.$$

This seems to be a special case of (3.22). However, even if we know that the summation converges, i.e., $\widetilde{A}_k$ equals $\operatorname{diag}(\mathcal{L}_{k,1}, \cdots, \mathcal{L}_{k,m}) + \Phi_k \mathbf{I}$ and we denote $\operatorname{diag}(\mathcal{L}_{k,1}, \cdots, \mathcal{L}_{k,m}) = \operatorname{OP}(\mathfrak{p}_k)$, we do not know if $\operatorname{OP}(\mathfrak{p}_k) = \widetilde{a}_k \operatorname{OP}(\mathfrak{q}_k)$ for some $\widetilde{a}_k$ (as in (3.22)) and $\mathfrak{q}_k$.

4. Local and global solutions

In this section, we focus on (1.15) and we will use Theorems 3.1, 3.2 and 3.3 with $m = d$. As before we simply write

$$H^s = H^s(\mathbb{K}^d; \mathbb{R}^d), \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{T}.$$

We recall the following estimates for $F(\cdot)$:

Lemma 4.1 *(Yan and Yin [52], Zhao et al. [53]). Let $s > d/2$ with $d \geq 2$ and let $F(\cdot)$ be the non-local term defined in (1.3). For all $s > d/2 + 1$ and $v, v_1, v_2 \in H^s$, we have*

$$\|F(v)\|_{H^s} \lesssim \|v\|_{W^{1,\infty}} \|v\|_{H^s},$$
$$\|F(v_1) - F(v_2)\|_{H^s} \lesssim (\|v_1\|_{H^s} + \|v_2\|_{H^s}) \|v_1 - v_2\|_{H^s}$$

Besides, for all $s \in (d/2, d/2+1]$ *and* $v_1, v_2 \in H^{s+1}$*, we have*

$$\|F(v_1) - F(v_2)\|_{H^s} \lesssim (\|v_1\|_{H^{s+1}} + \|v_2\|_{H^{s+1}}) \|v_1 - v_2\|_{H^s}.$$

Let J_n be the Friedrichs mollifier defined in Appendix A (cf. (A.1)).

Lemma 4.2. *For all* $\sigma > \frac{d}{2} + 1$*, there is a constant* $\Lambda = \Lambda(\sigma, d) > 0$ *such that*

$$\left|\left\langle (u \cdot \nabla)u + F(u), u \right\rangle_{H^\sigma}\right| \leq \Lambda \|u\|_{H^\sigma}^2 \|u\|_{W^{1,\infty}}, \quad u \in H^{\sigma+1}, \tag{4.1}$$

$$\left|\left\langle J_n[(u \cdot \nabla)u] + J_n F(u), J_n u \right\rangle_{H^\sigma}\right| \leq \Lambda \|u\|_{H^\sigma}^2 \|u\|_{W^{1,\infty}}, \quad u \in H^\sigma. \tag{4.2}$$

Proof. We only prove (4.2) since (4.1) can be proved in the same way. Using Lemmas A.1, A.2 and A.4, integration by parts and $H^s \hookrightarrow W^{1,\infty}$, we obtain that for some $\Lambda = \Lambda(\sigma, d) > 0$,

$$\begin{aligned}
&\left\langle \mathcal{D}^\sigma J_n \left[(u \cdot \nabla)u\right], \mathcal{D}^\sigma J_n u \right\rangle_{L^2} \\
=& \left\langle \left[\mathcal{D}^\sigma, (u \cdot \nabla)\right] u, \mathcal{D}^\sigma J_n^2 u \right\rangle_{L^2} + \left\langle [J_n, (u \cdot \nabla)] D^\sigma u, \mathcal{D}^\sigma J_n u \right\rangle_{L^2} \\
&+ \left\langle (u \cdot \nabla) \mathcal{D}^\sigma J_n u, \mathcal{D}^\sigma J_n u \right\rangle_{L^2} \\
\leq& \Lambda \left(\|u\|_{H^\sigma} \|\nabla u\|_{L^\infty} \|J_n u\|_{H^\sigma} + \|u\|_{H^\sigma} \|\nabla u\|_{L^\infty} \|J_n u\|_{H^\sigma} + \|J_n u\|_{H^\sigma}^2 \|\nabla u\|_{L^\infty} \right) \\
\leq& \Lambda \|u\|_{H^\sigma}^2 \|u\|_{W^{1,\infty}}.
\end{aligned}$$

Similarly, Lemma 4.1 implies

$$\langle \mathcal{D}^\sigma J_n F(u), \mathcal{D}^\sigma J_n u \rangle_{L^2} \leq \Lambda \|u\|_{H^\sigma}^2 \|u\|_{W^{1,\infty}}.$$

Combining the above estimates gives (4.2). □

To obtain a solution, we need the following hypothesis:

Hypothesis ($\mathbf{H_4}$). *Let* $d \geq 1$ *and define*

$$\mathcal{Q}_k = \mathcal{A}_k + \mathcal{B}_k, \quad k \geq 1.$$

Furthermore, $\mathcal{A}_k$ *and* $\mathcal{B}_k$ *satisfy the following conditions:*

- $\mathcal{A}_k = a_k \mathcal{L}_k + p_k \mathcal{G}_k$ *satisfies Hypothesis* ($\mathbf{H_1}$) *with* $m = d$*. Either* $\{a_k\}_{k\geq 1}, \{p_k\}_{k\geq 1} \in l^2$*, or* $\{p_k\}_{k\geq 1} \subset H^{\sigma_0}(\mathbb{K}^d, \mathbb{R})$ *and* $\{a_k\}_{k\geq 1} \in H^\infty(\mathbb{K}^d; \mathbb{R})$ *satisfy* $\sum_{k=1}^\infty \Big(\|a_k\|_{H^r} + \|p_k\|_{H^{\sigma_0}}^2 \Big) < \infty$ *for all* $r \geq 0$*.*
- $\mathcal{B}_k = b_k \mathcal{J}_k + q_k \mathcal{V}_k$ *satisfies* ($\mathbf{H}_2^b$) *with* $m = d$ *and* $\{b_k\}_{k\geq 1}, \{q_k\}_{k\geq 1} \in l^2$*.*
- *For all* $k \geq 1$*, either* $\mathcal{A}_k = 0$ *or* $\mathcal{B}_k = 0$*.*

Hypothesis (H$_5$). *Let $s > \frac{d}{2}+1$. For all $k \geq 1$, $h_k : [0,\infty) \times H^s \ni (t,u) \mapsto h_k(t,u) \in H^s$ is continuous. Moreover, there is a function $K : [0,\infty) \times [0,\infty) \to (0,\infty)$ increasing in both variables such that for all $t \geq 0$ and $u, v \in H^s$,*

$$\sum_{k=1}^{\infty} \|h_k(t,u)\|_{H^s}^2 \leq K(t, \|u\|_{W^{1,\infty}})(1+\|u\|_{H^s}^2),$$

$$\sum_{k=1}^{\infty} \|h_k(t,u)-h_k(t,v)\|_{H^s}^2 \leq K(t, \|u\|_{H^s}+\|v\|_{H^s})\|u-v\|_{H^s}^2.$$

Recall $\alpha \in [0,1]$ and $\beta \geq 0$ given in Hypotheses (H$_1$) and (H$_2$), respectively. Let

$$\gamma_0 := \begin{cases} \max\left\{\alpha \mathbf{1}_{\{\sum_{k=1}^{\infty}\|a_k\|_{H^s}>0\}},\ \beta \mathbf{1}_{\{\sum_{k=1}^{\infty}|b_k|^2>0\}}\right\}, & \text{if } \{a_k\}_{k\geq 1} \subset H^\infty(\mathbb{K}^d;\mathbb{R}), \\ \max\left\{\alpha \mathbf{1}_{\{\sum_{k=1}^{\infty}|a_k|^2>0\}},\ \beta \mathbf{1}_{\{\sum_{k=1}^{\infty}|b_k|^2>0\}}\right\}, & \text{if } \{a_k\}_{k\geq 1} \in l^2. \end{cases} \tag{4.3}$$

The main results for (1.15) (or (2.8)) are stated as follows:

Theorem 4.1. *Let Hypotheses (H$_4$) and (H$_5$) be verified. Let $s > \frac{d}{2}+1+\max\{2\gamma_0, 1\}$ with $d \geq 2$. For any H^s-valued $\mathcal{F}_0$-measurable random variable u_0,*

- **(I)** (1.15) *admits a unique maximal solution (u,τ^*) in the sense of Definition 2.2. Besides, (u,τ^*) defines a map $H^s \ni u_0 \mapsto u(t) \in C([0,\tau^*);H^s)$ $\mathbb{P}$-a.s., where τ^* does not depend on s, and*

$$\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\}} = \mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{W^{1,\infty}}=\infty\}} \quad \mathbb{P}\text{-a.s.} \tag{4.4}$$

- **(II)** *Let $T>0$, $\sigma > \frac{d}{2}+1$ and define*

$$\Xi(T,v,\sigma) := \sup_{t\in[0,T]} \sum_{k=1}^{\infty}\left(\|h_k(t,v)\|_{H^\sigma}^2 - \frac{2\langle h_k(t,v),v\rangle_{H^\sigma}^2}{\mathrm{e}+\|v\|_{H^\sigma}^2}\right). \tag{4.5}$$

  Then u exists globally, i.e., $\mathbb{P}(\tau^*=\infty)=1$, provided that

$$\limsup_{\|f\|_{H^\eta}\to\infty} \frac{\Xi(T,f,\eta)}{2\Lambda\|f\|_{W^{1,\infty}}\|f\|_{H^\eta}^2} < -1, \quad \eta \in \left(\frac{d}{2}+1,\ s-\max\{2\gamma_0,1\}\right), \tag{4.6}$$

  where Λ is given in Lemma 4.2.

The proof for Theorem 4.1 can be carried out in a way similar to Tang and Wang [44]. However, since the pseudo-differential noise in this paper is extended, we also provide the details here. The proof is divided into three subsections.

4.1. Approximation scheme and estimates

For convenience, we recall that (1.15) is equivalent to (2.8). Since either $\mathcal{A}_k = 0$ or $\mathcal{B}_k = 0$ for all $k \ge 1$, we can rewrite

$$\sum_{k=1}^{\infty} \mathcal{Q}_k u \,\mathrm{d}\widetilde{W}_k(t) = \sum_{k=1}^{\infty} \mathcal{A}_k u \,\mathrm{d}\overline{W}_k(t) + \sum_{k=1}^{\infty} \mathcal{B}_k u \,\mathrm{d}\widehat{W}_k(t)$$

with a family of independent standard 1-dimensional Brownian motions $\{\overline{W}_k(t), \widehat{W}_k(t)\}_{k\ge 1}$, which are also independent of $\{W_k(t)\}_{k\ge 1}$. Then we use (1.11) to rewrite (2.8) as

$$\begin{cases} \mathrm{d}u = \Big\{ -(u\cdot\nabla)u - F(u) + \dfrac{1}{2}\displaystyle\sum_{k=1}^{\infty}\left[\mathcal{A}_k^2 u + \mathcal{B}_k^2 u\right] \Big\}\,\mathrm{d}t \\ \qquad\qquad + \displaystyle\sum_{k=1}^{\infty} \Big\{ \mathcal{A}_k u \,\mathrm{d}\overline{W}_k + \mathcal{B}_k u \,\mathrm{d}\widehat{W}_k + h_k(t,u)\,\mathrm{d}W_k(t) \Big\}, \quad t>0, \quad x \in \mathbb{K}^d, \\ u\big|_{t=0} = u_0, \quad x \in \mathbb{K}^d. \end{cases} \tag{4.7}$$

Let $\mathbb{U}$ be a separable Hilbert space with a complete orthonormal basis $\{e_k\}_{k\ge1}$. Let

$$\begin{cases} G(u) := -(u\cdot\nabla)u + \frac{1}{2}\sum_{k=1}^{\infty}\left[\mathcal{A}_k^2 u + \mathcal{B}_k^2 u\right], \\ H(t,u)e_{3k-2} := \mathcal{A}_k u, \quad k \ge 1, \\ H(t,u)e_{3k-1} := \mathcal{B}_k u, \quad k \ge 1, \\ H(t,u)e_{3k} := h_k(t,u), \quad k \ge 1, \\ \mathcal{W}(t) := \sum_{k=1}^{\infty}\left(\overline{W}_k(t)e_{3k-2} + \widehat{W}_k(t)e_{3k-1} + W_k(t)e_{3k}\right). \end{cases} \tag{4.8}$$

With the above notations, (4.7) reduces to

$$\mathrm{d}u = [G(u) - F(u)]\,\mathrm{d}t + H(t,u)\,\mathrm{d}\mathcal{W}(t), \quad u\big|_{t=0} = u_0, \quad t > 0.$$

Let $d \ge 2$ and recall γ_0 in (4.3). Let $s > \frac{d}{2} + 1 + \max\{2\gamma_0, 1\}$. According to Hypotheses **($\mathbf{H_4}$)** and **($\mathbf{H_5}$)**, if $u \in H^s$, then $G(u) \in H^{(s-1)\wedge(s-2\gamma_0)}$ and $H(t,u) \in \mathscr{L}_2(\mathbb{U}; H^{s-\gamma_0})$, while by Lemma 4.1, $F(u) \in H^s$. To apply the theory for SDEs in Hilbert space, we need to mollify $G(u)$ and $H(t,u)$. To this end, we will use the mollifier J_n defined in (A.1) and construct the following regularization:

$$\begin{cases} G_n(u) := -J_n[(J_n u\cdot\nabla J_n u)] + \frac{1}{2}\sum_{k=1}^\infty J_n^3\mathcal{A}_k^2 J_n u + \frac{1}{2}\sum_{k=1}^\infty J_n^3\mathcal{B}_k^2 J_n u, \\ H_n(t,u)e_{3k-2} := J_n\mathcal{A}_k J_n u, \quad k\ge 1, \\ H_n(t,u)e_{3k-1} := J_n\mathcal{B}_k J_n u, \quad k\ge 1, \\ H_n(t,u)e_{3k} := h_k(t,u), \ k\ge 1. \end{cases} \tag{4.9}$$

We also need a cut-off function to split the expectation. Hence for any $R>1$, we take a cut-off function $\chi_R\in C^\infty([0,\infty);[0,1])$ such that

$$\chi_R(y)=1 \text{ for } |y|\le R, \text{ and } \chi_R(y)=0 \text{ for } y>2R, \tag{4.10}$$

and then we consider

$$\begin{aligned} \mathrm{d}u = &\chi_R^2\big(\|u(t)-u_0\|_{W^{1,\infty}}\big)\,[G_n(u)-F(u)]\,\mathrm{d}t \\ &+\chi_R\big(\|u(t)-u_0\|_{W^{1,\infty}}\big)H_n(t,u)\,\mathrm{d}\mathcal{W}(t), \quad u\big|_{t=0}=u_0. \end{aligned} \tag{4.11}$$

Keep in mind that γ_0 is in (4.3) and we have the following:

Lemma 4.3. *Let $d\ge 2$ and $s>\frac{d}{2}+1+\max\{2\gamma_0,1\}$. Let Hypotheses* **(H$_4$)** *and* **(H$_5$)** *be verified. For any $R>1$, $n\ge 1$ and $\mathcal{F}_0$-measurable H^s-valued random variable u_0, (4.11) has a unique global solution $u_n=u_n^{(R)}(t,x)\in C([0,\infty);H^s)$. Besides, there exists a function $V:[0,\infty)\times[0,\infty)\to(0,\infty)$ increasing in both variables such that for any $R>1, T>0$,*

$$\sup_{n\ge 1}\mathbb{E}\Big[\sup_{t\in[0,T]}\|u_n\|_{H^s}^2\big|\mathcal{F}_0\Big]\le V(T,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_0\|_{H^s}^2). \tag{4.12}$$

Proof. By Lemma A.1, it is easy to see that $G_n:H^s\to H^s$ and $H_n:[0,\infty)\times H^s\to\mathscr{L}_2(\mathbb{U};H^s)$ are locally Lipschitz. This, together with Lemma 4.1, means that for any deterministic initial data, (4.11) admits a unique solution, and the solution is continuous in H^s (See for instance Prévôt and Röckner [34], Wang [51]). Combining this with the fact that $\mathcal{F}_0$ is independent of the equation, we see that for any $\mathcal{F}_0$-measurable H^s-valued random variable u_0, (4.11) also admits a unique solution $u_n=u_n(t)$, which is continuous in H^s.

Now we verify (4.12). To begin with, we can infer from (4.9), Hypothesis **(H$_5$)**, Lemma A.1, Theorems 3.2 and 3.3 (with $\mathscr{O}=\{\mathcal{D}^s\}$, $f=J_nu_n$) that

$$\begin{aligned} &\sum_{k=1}^\infty\langle H_n(t,u_n)e_k,u_n\rangle_{H^s}^2 \\ =&\sum_{k=1}^\infty\Big(\langle J_n\mathcal{A}_kJ_nu_n,u_n\rangle_{H^s}^2+\langle J_n\mathcal{B}_kJ_nu_n,u_n\rangle_{H^s}^2+\langle h_k(t,u_n),u_n\rangle_{H^s}^2\Big) \\ \lesssim&\,\big(1+K^2(t,\|u_n\|_{W^{1,\infty}})\big)\,(1+\|u_n\|_{H^s}^4). \end{aligned}$$

Besides, it follows from (4.9) that

$$\begin{aligned}
&2\langle G_n(u_n)-F(u_n),u_n\rangle_{H^s}+\|H_n(t,u_n)\|^2_{\mathscr{L}_2(\mathbb{U};H^s)}\\
=&-2\langle J_n[(J_nu_n\cdot\nabla)J_nu_n],u_n\rangle_{H^s}-2\langle F(u_n),u_n\rangle_{H^s}\\
&+\sum_{k=1}^{\infty}\left\langle J_n^3\mathcal{A}_k^2J_nu_n,u_n\right\rangle_{H^s}+\sum_{k=1}^{\infty}\left\langle J_n^3\mathcal{B}_k^2J_nu_n,u_n\right\rangle_{H^s}\\
&+\sum_{k=1}^{\infty}\|J_n\mathcal{A}_kJ_nu_n\|^2_{H^s}+\sum_{k=1}^{\infty}\|J_n\mathcal{B}_kJ_nu_n\|^2_{H^s}+\sum_{k=1}^{\infty}\|h_k(t,u_n)\|^2_{H^s}\\
:=&\sum_{i=1}^{7}I_i.
\end{aligned}$$

On account of Hypothesis $(\mathbf{H_5})$, Lemmas 4.1, A.1 and A.4, it holds that

$$|I_1|+|I_2|\lesssim\|u_n\|_{W^{1,\infty}}\|u_n\|^2_{H^s},\qquad |I_7|\le K(t,\|u_n\|_{W^{1,\infty}})(1+\|u_n\|^2_{H^s}).$$

It follows from Theorems 3.2 and 3.3 (with $\mathscr{O}=\{\mathcal{D}^sJ_n\}_{n\ge1}$ and $f=J_nu_n$) that

$$|I_3+I_5|+|I_4+I_6|\lesssim\|J_nu_n\|^2_{H^s}\le\|u_n\|^2_{H^s}.$$

By the above estimates and Itô's formula, we find a function $\widetilde{V}:[0,\infty)\times[0,\infty)\to(0,\infty)$ such that

$$\begin{aligned}
&\mathrm{d}\|u_n(t)\|^2_{H^s}-\mathrm{d}M_n(t)\\
=&\chi_R^2\big(\|u_n(t)-u_0\|_{W^{1,\infty}}\big)\left\{\sum_{i=1}^{7}I_i\right\}\mathrm{d}t\le\widetilde{V}(t,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_n(t)\|^2_{H^s})\,\mathrm{d}t,
\end{aligned}$$

where

$$\mathrm{d}M_n(t):=\;2\chi_R\big(\|u_n(t)-u_0\|_{W^{1,\infty}}\big)\big\langle u_n(t),\,H_n(t,u_n)\,\mathrm{d}\mathcal{W}(t)\big\rangle_{H^s}$$

satisfies

$$\mathrm{d}\big\langle M_n(t)\big\rangle\le\widetilde{V}(t,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_n(t)\|^4_{H^s}\big)\,\mathrm{d}t.$$

Define

$$\tau_n:=\lim_{N\to\infty}\tau_{n,N},\quad \tau_{n,N}:=\inf\big\{t\ge0:\|u_n(t)\|_{H^s}\ge N\big\},\quad n,\,N\ge1.$$

For any $T>0$, we use BDG's inequality to find constants $c_1,c_2>0$ such that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t'\in[0,t\wedge\tau_{n,N}]}\|u_n(t)\|_{H^s}^2\Big|\mathcal{F}_0\Big]-\|u_0\|_{H^s}^2\\
&\leq c_1\mathbb{E}\Bigg[\Bigg(\int_0^{t\wedge\tau_{n,N}}\widetilde{V}(t',2R+\|u_0\|_{W^{1,\infty}})\Big(1+\|u_n(t')\|_{H^s}^4\Big)\,\mathrm{d}t'\Bigg)^{\frac{1}{2}}\Bigg|\mathcal{F}_0\Bigg]\\
&\quad+c_1\mathbb{E}\Bigg[\int_0^{t\wedge\tau_{n,N}}\widetilde{V}(t',2R+\|u_0\|_{W^{1,\infty}})\Big(1+\|u_n(t')\|_{H^s}^2\Big)\,\mathrm{d}t'\Bigg|\mathcal{F}_0\Bigg]\\
&\leq\frac{1}{2}\mathbb{E}\Big[\sup_{t'\in[0,t\wedge\tau_{n,N}]}\|u_n(t')\|_{H^s}^2\Big|\mathcal{F}_0\Big]+c_2\\
&\quad+c_2\int_0^t\widetilde{V}(t',2R+\|u_0\|_{W^{1,\infty}})\mathbb{E}\Big[\sup_{r\in[0,t'\wedge\tau_{n,N}]}\|u_n(r)\|_{H^s}^2\Big|\mathcal{F}_0\Big]\,\mathrm{d}t',\quad t\in[0,T],\quad N\geq 1.
\end{aligned}
$$

By Grönwall's inequality, there exists a function $V:[0,\infty)\times[0,\infty)\to(0,\infty)$ increasing in both variables such that for all $n,N\geq 1$,

$$
\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau_{n,N}]}\|u_n(t)\|_{H^s}^2\Big|\mathcal{F}_0\Big]\leq V(T,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_0\|_{H^s}^2). \tag{4.13}
$$

This implies that for all $n,N\geq 1$,

$$
\mathbb{P}\left(\tau_{n,N}<T\big|\mathcal{F}_0\right)\leq\frac{V(T,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_0\|_{H^s}^2)}{N^2},
$$

so that $\tau_n=\lim_{N\to\infty}\tau_{n,N}$ satisfies

$$
\mathbb{P}\big(\tau_n<T\big|\mathcal{F}_0\big)\leq\lim_{N\to\infty}\mathbb{P}(\tau_{n,N}<T|\mathcal{F}_0)=0.
$$

Hence, $\mathbb{P}\big(\tau_n\geq T\big)=\mathbb{E}\big[\mathbb{P}(\tau_n\geq T|\mathcal{F}_0)\big]=1$ for all $T>0$, which means $\mathbb{P}(\tau_n=\infty)=1$. Letting $N\to\infty$ in (4.13) yields (4.12). □

4.2. Solve the cut-off problem

In this section, we will take limit in (4.11) to find a solution to the following cut-off problem:

$$
\begin{aligned}
\mathrm{d}u=&\,\chi_R^2\big(\|u(t)-u_0\|_{W^{1,\infty}}\big)\,[G(u)-F(u)]\,\mathrm{d}t\\
&+\chi_R^2\big(\|u(t)-u_0\|_{W^{1,\infty}}\big)H(t,u)\,\mathrm{d}\mathcal{W}(t),\quad u\big|_{t=0}=u_0,\quad t>0,
\end{aligned} \tag{4.14}
$$

where χ_R, F and (G,H) are given in (4.10), (1.3) and (4.8), respectively.

Lemma 4.4. *Let u_n be the approximate solution as in Lemma 4.3. For any $n, l \geq 1$, $\delta_0 \in \left(\frac{d}{2}+1, s-\max\{2\gamma_0, 1\}\right)$ and T, $N>0$, let*

$$\tau_N^{n,l,T} := T \wedge \inf\left\{t \geq 0 : \|u_n(t)\|_{H^s} \vee \|u_l(t)\|_{H^s} \geq N\right\}.$$

Then $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty} \sup_{l\geq n} \mathbb{E}\left[\sup_{t\in[0,\tau_N^{n,l,T}]} \|u_n(t)-u_l(t)\|^2_{H^{\delta_0}} \middle| \mathcal{F}_0\right] = 0, \quad T,\ N>0. \tag{4.15}$$

Proof. Let $v_{n,l} = u_n - u_l$ for $n, l \geq 1$. We have that

$$\mathrm{d}v_{n,l}(t) = \sum_{i=1}^{4} A_i^{n,l}(t)\,\mathrm{d}t + \sum_{i=1}^{2} B_i^{n,l}(t)\,\mathrm{d}\mathcal{W}(t), \quad v_{n,l}(0)=0, \tag{4.16}$$

where

$$\begin{aligned}
A_1^{n,l}(t) &:= -\left[\chi_R^2\big(\|u_n(t)-u_0\|_{W^{1,\infty}}\big) - \chi_R^2\big(\|u_l(t)-u_0\|_{W^{1,\infty}}\big)\right] F(t,u_n(t)),\\
A_2^{n,l}(t) &:= -\chi_R^2\big(\|u_l(t)-u_0\|_{W^{1,\infty}}\big)\left[F(t,u_n(t)) - F(t,u_l(t))\right],\\
A_3^{n,l}(t) &:= \left[\chi_R^2\left(\|u_n(t)-u_0\|_{W^{1,\infty}}\right) - \chi_R^2\big(\|u_l(t)-u_0\|_{W^{1,\infty}}\big)\right] G_n(t,u_n(t)),\\
A_4^{n,l}(t) &:= \chi_R^2\big(\|u_l(t)-u_0\|_{W^{1,\infty}}\big)\left[G_n(t,u_n(t)) - G_l(t,u_l(t))\right],
\end{aligned}$$

and

$$\begin{aligned}
B_1^{n,l}(t) &:= \left[\chi_R(\|u_n(t)-u_0\|_{W^{1,\infty}}) - \chi_R\big(\|u_l(t)-u_0\|_{W^{1,\infty}}\big)\right] H_n(t,u_n(t)),\\
B_2^{n,l}(t) &:= \chi_R\big(\|u_l(t)-u_0\|_{W^{1,\infty}}\big)[H_n(t,u_n(t)) - H_l(t,u_l(t))].
\end{aligned}$$

By the Itô formula, we obtain

$$\begin{aligned}
\mathrm{d}\,\|v_{n,l}(t)\|^2_{H^{\delta_0}} =& 2\sum_{i=1}^{2}\left\langle v_{n,l}(t),\ B_i^{n,l}(t)\,\mathrm{d}\mathcal{W}(t)\right\rangle_{H^{\delta_0}}\\
&+\left\{\sum_{i=1}^{2}\left\|B_i^{n,l}(t)\right\|^2_{\mathcal{L}_2(\mathbb{U};H^{\delta_0})} + 2\sum_{i=1}^{4}\left\langle A_i^{n,l}(t), v_{n,l}(t)\right\rangle_{H^{\delta_0}}\right\}\mathrm{d}t.
\end{aligned}$$

Claim: There is a function $Q : [0,\infty)\times[0,\infty) \to (0,\infty)$ increasing in both variables and a function $\lambda : \mathbb{N}\times\mathbb{N} \to [0,\infty)$ with $\lim\limits_{n,l\to\infty} \lambda_{n,l} = 0$ such that for all $n, l \geq 1$ and $t \in [0, \tau_N^{n,l,T}]$,

$$\sum_{i=1}^{2}\sum_{k=1}^{\infty}\left\langle v_{n,l}(t), B_i^{n,l}(t)e_k\right\rangle^2_{H^{\delta_0}} \leq Q(t,N)\,\|v_{n,l}(t)\|^2_{H^{\delta_0}}\left\{\lambda_{n,l} + \|v_{n,l}(t)\|^2_{H^{\delta_0}}\right\}, \tag{4.17}$$

$$\sum_{i=1}^{2}\left\|B_i^{n,l}(t)\right\|^2_{\mathcal{L}_2(\mathbb{U};H^{\delta_0})}+2\sum_{i=1}^{4}\left\langle A_i^{n,l}(t),\,v_{n,l}(t)\right\rangle_{H^{\delta_0}}$$
$$\leq Q(t,N)\left\{\lambda_{n,l}+\|v_{n,l}(t)\|^2_{H^{\delta_0}}\right\}. \tag{4.18}$$

If (4.17) and (4.18) hold true, then we use BDG's inequality to (4.16) to find constants $a_1, a_2>0$ depending on N and T such that for all $n,l\geq 1$ and $t\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t'\in[0,t\wedge\tau_N^{n,l,T}]}\|v_{n,l}(t')\|^2_{H^{\delta_0}}\middle|\mathcal{F}_0\right]\\
&\leq a_1\mathbb{E}\left[\int_0^{t\wedge\tau_N^{n,l,T}} Q(t',N)\left\{\lambda_{n,l}+\|v_{n,l}(t')\|^2_{H^{\delta_0}}\right\}\mathrm{d}t'\middle|\mathcal{F}_0\right]\\
&\quad+a_1\mathbb{E}\left[\left(\int_0^{t\wedge\tau_N^{n,l,T}} Q(t',N)\,\|v_{n,l}(t')\|^2_{H^{\delta_0}}\left\{\lambda_{n,l}+\|v_{n,l}(t')\|^2_{H^{\delta_0}}\right\}\mathrm{d}t'\right)^{\frac12}\middle|\mathcal{F}_0\right]\\
&\leq\frac12\mathbb{E}\left[\sup_{t'\in[0,t\wedge\tau_N^{n,l,T}]}\|v_{n,l}(t')\|^2_{H^{\delta_0}}\middle|\mathcal{F}_0\right]+a_2\lambda_{n,l}\\
&\quad+a_2\int_0^t Q(t',N)\mathbb{E}\left[\sup_{r\in[0,t'\wedge\tau_N^{n,l,T}]}\|v_{n,l}(r)\|^2_{H^{\delta_0}}\middle|\mathcal{F}_0\right]\mathrm{d}t'.
\end{aligned}\tag{4.19}$$

By Grönwall's inequality and noting $\lambda_{n,l}\to 0$ as $n,l\to\infty$, we prove (4.15). Therefore, it suffices to prove (4.17) and (4.18).

We only prove (4.18) since (4.17) can be verified similarly. We note that $\chi_R(\cdot)$ is bounded and Lipschitz, $F(\cdot)$ is locally Lipschitz (cf. Lemma 4.1) and $H^{\delta_0}\hookrightarrow W^{1,\infty}$. Then we use Hypothesis **($\mathbf{H_4}$)**, (4.9) and Lemma A.1 to obtain that for all $n,l\geq 1$ and $t\in[0,\tau_N^{n,l,T}]$,

$$\left\|B_1^{n,l}(t)\right\|^2_{\mathcal{L}_2(\mathbb{U};H^{\delta_0})}+2\sum_{i=1}^{3}\left\langle A_i^{n,l}(t),\,v_{n,l}(t)\right\rangle_{H^{\delta_0}}\leq Q(t,N)\,\|v_{n,l}(t)\|^2_{H^{\delta_0}}$$

for some increasing function $Q:[0,\infty)\times[0,\infty)\to(0,\infty)$ increasing in both variables. Once again, since $\chi_R(\cdot)\leq 1$, we only need to prove that for all $n,l\geq 1$ and $t\in[0,\tau_N^{n,l,T}]$,

$$2\left\langle G_n(u_n)-G_l(u_l),v_{n,l}\right\rangle_{H^{\delta_0}}+\left\|H_n(t,u_n)-H_l(t,u_l)\right\|^2_{\mathscr{L}_2(\mathbb{U};H^{\delta_0})}$$
$$\leq Q(t,N)\left\{\lambda_{n,l}+\|v_{n,l}(t)\|^2_{H^{\delta_0}}\right\}. \tag{4.20}$$

To this end, we find

$$2\left\langle G_n(u_n)-G_l(u_l),v_{n,l}\right\rangle_{H^{\delta_0}}+\left\|H_n(t,u_n)-H_l(t,u_l)\right\|^2_{\mathscr{L}_2(\mathbb{U};H^{\delta_0})}=\Psi_1+\sum_{k=1}^{\infty}\sum_{i=2}^{6}\Psi_{i,k},$$

where

$$\begin{aligned}
\Psi_1=\Psi_1^{n,l}&:=2\left\langle J_n[(J_nu_n\cdot\nabla)J_nu_n]-J_l[(J_lu_l\cdot\nabla)J_lu_l],\,u_n-u_l\right\rangle_{H^{\delta_0}},\\
\Psi_{2,k}=\Psi_{2,k}^{n,l}&:=\left\langle J_n^3\mathcal{A}_k^2J_nu_n-J_l^3\mathcal{A}_k^2J_lu_l,\,u_n-u_l\right\rangle_{H^{\delta_0}},\\
\Psi_{3,k}=\Psi_{3,k}^{n,l}&:=\left\langle J_n^3\mathcal{B}_k^2J_nu_n-J_l^3\mathcal{B}_k^2J_lu_l,\,u_n-u_l\right\rangle_{H^{\delta_0}},\\
\Psi_{4,k}=\Psi_{4,k}^{n,l}&:=\|H_n(t,u_n)e_{3k-2}-H_l(t,u_l)e_{3k-2}\|^2_{H^{\delta_0}},\\
\Psi_{5,k}=\Psi_{5,k}^{n,l}&:=\|H_n(t,u_n)e_{3k-1}-H_l(t,u_l)e_{3k-1}\|^2_{H^{\delta_0}},\\
\Psi_{6,k}=\Psi_{6,k}^{n,l}&:=\|h_k(t,u_n)-h_k(t,u_l)\|^2_{H^{\delta_0}}.
\end{aligned}$$

For Ψ_1, one can show that

$$|\Psi_1|\lesssim\left((\|u_n\|_{H^s}+\|u_l\|_{H^s})^4+1\right)\left(\|v_{n,l}\|^2_{H^{\delta_0}}+(l\wedge n)^{-2(s-1-\delta_0-\epsilon)}\right),\ \ \epsilon\in(0,s-1-\delta_0).$$

The proof for this estimate is similar to [46, Lemma 3.1] (see also Miao et al. [30]), and here we omit the details to save space. Now we estimate the other terms. To control $\sum_{k=1}^{\infty}\{\Psi_{3,k}+\Psi_{5,k}\}$, we find

$$\Psi_{3,k}=\sum_{j=1}^{3}\Psi_{3,k,j},\quad \Psi_{5,k}=\sum_{i,j=1}^{3}\left\langle\Psi_{5,k,i},\Psi_{5,k,j}\right\rangle_{H^{\delta_0}},$$

where

$$\begin{cases}
\Psi_{3,k,1}:=\left\langle(J_n^3-J_l^3)\mathcal{B}_k^2J_nu_n,v_{n,l}\right\rangle_{H^{\delta_0}}, & \Psi_{5,k,1}:=(J_n-J_l)\mathcal{B}_kJ_nu_n,\\
\Psi_{3,k,2}:=\left\langle J_l^3\mathcal{B}_k^2(J_n-J_l)u_n,v_{n,l}\right\rangle_{H^{\delta_0}}, & \Psi_{5,k,2}:=J_l\mathcal{B}_k(J_n-J_l)u_n,\\
\Psi_{3,k,3}:=\left\langle J_l^3\mathcal{B}_k^2J_lv_{n,l},v_{n,l}\right\rangle_{H^{\delta_0}}, & \Psi_{5,k,3}:=J_l\mathcal{B}_kJ_lv_{n,l}.
\end{cases}$$

By Hypothesis $(\mathbf{H_4})$ and Lemma A.1, we have for any $\epsilon\in(0,s-2\gamma_0-\delta_0)$,

$$\begin{aligned}
&\sum_{k=1}^{\infty}|\Psi_{3,k,1}|+\sum_{k=1}^{\infty}|\Psi_{3,k,2}|+\sum_{k=1}^{\infty}\sum_{i\in\{1,2\}}\left|\left\langle\Psi_{5,k,i},\Psi_{5,k,3}\right\rangle_{H^{\delta_0}}\right|\\
&\qquad\qquad\lesssim(l\wedge n)^{-(s-2\gamma_0-\delta_0-\epsilon)}\left\|u_n\right\|_{H^s}\left\|v_{n,l}\right\|_{H^{\delta_0}},\\
&\sum_{k=1}^{\infty}\sum_{i,j\in\{1,2\}}\left|\left\langle\Psi_{5,k,i},\Psi_{5,k,j}\right\rangle_{H^{\delta_0}}\right|\lesssim(l\wedge n)^{-2(s-2\gamma_0-\delta_0-\epsilon)}\left\|u_n\right\|^2_{H^s}.
\end{aligned}$$

Then we apply Theorem 3.3 (with $s=\delta_0$, $\mathscr{O}=\{\mathcal{D}^{\delta_0}J_l\}$ and $f=J_lv_{n,l}$) to find

$$\sum_{k=1}^{\infty}\left\{\Psi_{3,k,3}+\langle\Psi_{5,k,3},\Psi_{5,k,3}\rangle_{H^{\delta_0}}\right\}\lesssim\|v_{n,l}\|^2_{H^{\delta_0}}.$$

Hence we find an increasing function $Q:[0,\infty)\times[0,\infty)\to(0,\infty)$ increasing in both variables such that for all n, $l\geq 1$ and $t\in[0,\tau_N^{n,l,T}]$,

$$\sum_{k=1}^{\infty}\{\Psi_{3,k}+\Psi_{5,k}\}\lesssim Q(t,N)\left\{(l\wedge n)^{-(s-2\gamma_0-\delta_0-\epsilon)}+\|v_{n,l}\|^2_{H^{\delta_0}}\right\}.$$

Similarly, the same estimate holds for $\sum_{k=1}^{\infty}\{\Psi_{2,k}+\Psi_{4,k}\}$. Obviously, the desired upper bound of $\Psi_{6,k}$ follows from Hypothesis $(\mathbf{H_5})$. In conclusion, (4.20) holds true. □

Lemma 4.5. *Let u_n be the approximate solution as in Lemma 4.3 and V be given in Lemma 4.3. There exists an $\mathcal{F}_t$-progressive measurable H^s-valued process $u(t)=(u^{(R)}(t))_{t\geq 0}$ such that, up to a subsequence,* $\mathbb{P}$-a.s.,

$$u_n\xrightarrow{n\to\infty}u\text{ in }C([0,\infty);H^{\delta_0}),\tag{4.21}$$

and u satisfies

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\|u(t)\|^2_{H^s}\Big|\mathcal{F}_0\Big]\leq V(T,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_0\|^2_{H^s}).\tag{4.22}$$

Proof. For any $T>0$, $N\geq 1$ and $\epsilon>0$, by using (4.12) in Lemma 4.3 and Chebyshev's inequality, we have

$$\begin{aligned}&\mathbb{P}(\tau_N^{n,l,T}<T|\mathcal{F}_0)\\ \leq&\,\mathbb{P}\left(\sup_{t\in[0,T]}\|u_n(t)\|_{H^s}\geq N\Big|\mathcal{F}_0\right)+\mathbb{P}\left(\sup_{t\in[0,T]}\|u_l(t)\|_{H^s}\geq N\Big|\mathcal{F}_0\right)\\ \leq&\,\frac{2V(T,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_0\|^2_{H^s})}{N^2}.\end{aligned}$$

Since $\tau_N^{n,l,T}\leq T$ $\mathbb{P}$-a.s., for any $T>0$, $N\geq 1$ and n, $l\geq 1$, we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[0,T]}\|u_n(t)-u_l(t)\|_{H^{\delta_0}}>\epsilon\Big|\mathcal{F}_0\right)\\ \leq&\,\mathbb{P}\left(\tau_N^{n,l,T}<T\Big|\mathcal{F}_0\right)+\mathbb{P}\left(\sup_{t\in[0,\tau_N^{n,l,T}]}\|u_n(t)-u_l(t)\|_{H^{\delta_0}}>\epsilon\Big|\mathcal{F}_0\right)\\ \leq&\,\frac{2V(T,2R+\|u_0\|_{W^{1,\infty}})(1+\|u_0\|^2_{H^s})}{N^2}+\mathbb{P}\left(\sup_{t\in[0,\tau_N^{n,l,T}]}\|u_n(t)-u_l(t)\|_{H^{\delta_0}}>\epsilon\Big|\mathcal{F}_0\right).\end{aligned}$$

On account of Lemma 4.4, we first let $n,l\to\infty$ and then $N\to\infty$ to find

$$\lim_{n,l\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \|u_n(t)-u_l(t)\|_{H^{\delta_0}} > \epsilon \middle| \mathcal{F}_0 \right) = 0, \quad \epsilon,\, T>0.$$

According to the reverse Fatou lemma, we obtain

$$\begin{aligned}
&\limsup_{n,l\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \|u_n(t)-u_l(t)\|_{H^{\delta_0}} > \epsilon \right) \\
=&\limsup_{n,l\to\infty} \mathbb{E}\left[\mathbb{P}\left(\sup_{t\in[0,T]} \|u_n(t)-u_l(t)\|_{H^{\delta_0}} > \epsilon \middle| \mathcal{F}_0 \right)\right] \\
\le& \mathbb{E}\left[\limsup_{n,l\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \|u_n(t)-u_l(t)\|_{H^{\delta_0}} > \epsilon \middle| \mathcal{F}_0 \right)\right] = 0, \quad \epsilon,\, T>0.
\end{aligned}$$

Therefore, up to a subsequence, (4.21) holds for certain progressively measurable process u. Furthermore, (4.22) follows from Fatou's lemma, (4.21) and (4.12). □

Lemma 4.6. *Let $d \ge 2$ and $s > \frac{d}{2} + 1 + \max\{2\gamma_0, 1\}$. Let Hypotheses* **($\mathbf{H_4}$)** *and* **($\mathbf{H_5}$)** *hold. For any $R > 1$, $n \ge 1$ and $\mathcal{F}_0$-measurable H^s-valued random variable,* (4.14) *has a unique global solution $u = u^{(R)}$ such that for any $T > 0$,*

$$\mathbb{P}\big(u \in C([0,T];H^s)\big) = 1. \tag{4.23}$$

Proof. For any $R \ge 1$, by Lemma 4.5, as in Tang and Wang [44], we can take limit to see that the limit process u obtained in Lemma 4.5 is a solution to (4.14). Uniqueness of solution can be obtained in the same way as we estimate (4.19).

Now we prove (4.23). By (4.21), we know that $u \in C([0,T];H^{\delta_0})$, which, together with the fact that $H^s \hookrightarrow H^{\delta_0}$ is dense, means that u is weakly continuous in H^s. In order to prove (4.23), we only need to prove that $[0,T] \ni t \mapsto \|u(t)\|_{H^s}$ is continuous almost surely. Let

$$\tau_N := N \wedge \inf\big\{t \ge 0 : \|u(t)\|_{H^s} \ge N\big\}, \quad N \ge 1.$$

Note that (4.22) implies $\lim_{N\to\infty}\tau_N = \infty$ $\mathbb{P}$-a.s. It suffices to prove

$$\|u(\cdot)\|_{H^s} \in C\big([0,\tau_N \wedge T];\mathbb{R}\big), \qquad N \ge 1. \tag{4.24}$$

However, the two terms $\langle H(t,u)e_k, u\rangle_{H^s}$ and $\big\langle [G(u)-F(u)], u\big\rangle_{H^s} + \|H(t,u)\|^2_{\mathscr{L}_2(\mathbb{U};H^s)}$ are not well-defined since we only know $u \in H^s$ (by (4.22)). Hence one cannot apply Itô's formula to $\|u\|^2_{H^s}$. Then we apply Lemma 4.2, (4.8), Hypothesis ($\mathbf{H_5}$), Lemma A.1, Theorems 3.1, 3.2 and 3.3 (with $\mathscr{O} = \{\mathcal{D}^s J_n\}_{n\ge1}$, $f = u$) to find

$$\sum_{k=1}^{\infty} \langle J_n H(t,u)e_k, J_n u\rangle^2_{H^s}$$

$$= \sum_{k=1}^{\infty} \left(\langle J_n \mathcal{A}_k u, J_n u \rangle_{H^s}^2 + \langle J_n \mathcal{B}_k u, J_n u \rangle_{H^s}^2 + \langle J_n h_k(t,u), J_n u \rangle_{H^s}^2 \right)$$
$$\lesssim \left(1 + K^2(t, \|u\|_{W^{1,\infty}})\right) (1 + \|u\|_{H^s}^4),$$

and

$$\left| 2\left\langle J_n \left[G(u) - F(u)\right], J_n u \right\rangle_{H^s} + \|J_n H(t,u)\|^2_{\mathscr{L}_2(\mathbb{U};H^s)} \right|$$
$$\leq 2\Lambda \|u\|_{H^s}^2 \|u\|_{W^{1,\infty}} + \left| \sum_{k=1}^{\infty} \left\langle J_n \mathcal{A}_k^2 u, J_n u \right\rangle_{H^s} + \sum_{k=1}^{\infty} \|J_n \mathcal{A}_k u\|_{H^s}^2 \right|$$
$$+ \left| \sum_{k=1}^{\infty} \left\langle J_n \mathcal{B}_k^2 u, J_n u \right\rangle_{H^s} + \sum_{k=1}^{\infty} \|J_n \mathcal{B}_k u\|_{H^s}^2 \right| + \sum_{k=1}^{\infty} \|h_k(t,u)\|_{H^s}^2$$
$$\lesssim \left(1 + \|u\|_{W^{1,\infty}} + K^2(t, \|u\|_{W^{1,\infty}})\right) (1 + \|u\|_{H^s}^4).$$

Therefore, by applying Itô's formula to $\|J_n u\|_{H^s}^2$, for any n, $N \geq 1$, we find a martingale $M_t^{(n)}$ such that for some constant $Q_N > 0$,

$$-Q_N \,\mathrm{d}t \leq \mathrm{d}\|J_n u(t)\|_{H^s}^2 + \mathrm{d}M^{(n)}(t) \leq Q_N \,\mathrm{d}t, \quad t \in [0, \tau_N], \quad n \geq 1,$$

and

$$\mathrm{d}\left\langle M^{(n)} \right\rangle(t) \leq Q_N \,\mathrm{d}t, \quad t \in [0, \tau_N], \quad n \geq 1.$$

Therefore, there is a constant $C_N > 0$ such that for all $t, t' \geq 0$, $|t - t'| < 1$,

$$\mathbb{E}\left[\left| \|J_n u(t \wedge \tau_N)\|_{H^s}^2 - \|J_n u(t' \wedge \tau_N)\|_{H^s}^2 \right|^4 \right] \leq C_N |t - t'|^2, \quad n \geq 1.$$

By Lemma A.1 and Fatou's lemma with $n \to \infty$, we derive

$$\mathbb{E}\left[\left| \|u(t \wedge \tau_N)\|_{H^s}^2 - \|u(t' \wedge \tau_N)\|_{H^s}^2 \right|^4 \right] \leq C_N |t - t'|^2.$$

From this and Kolmogorov's continuity theorem, we obtain (4.24). □

4.3. Finish the proof for Theorem 4.1

Now we are in the position to prove Theorem 4.1.

Proof for Theorem 4.1. We will verify (**I**) and (**II**) as follows:

(**I**). Let $u = u^{(R)}$ be the solution to (4.14) as in Lemma 4.6. Now we remove the cut-off. To this end, we let

$$\tau^{(R)} := \inf\left\{ t \geq 0 : \|u^{(R)}(t) - u_0\|_{W^{1,\infty}} \geq R \right\}.$$

By the continuity of $u^{(R)}(t)$ in H^{δ_0} and $H^{\delta_0} \hookrightarrow W^{1,\infty}$, we have $\mathbb{P}(\tau^{(R)} > 0) = 1$ for any $R > 0$. Since $\chi_R^2(\|u^{(R)}(t) - u_0\|_{W^{1,\infty}}) = 1$ for $t \le \tau$, $(u^{(R)}, \tau^{(R)})$ is a local solution to (2.8) (or equivalently, (4.7)). The uniqueness of $u^{(R)}$ (to the cut-off problem (4.14), cf. Lemma 4.6) implies

$$u^{(R)}(t) = u^{(R+1)}(t), \quad t \le \tau^{(R)}, \ R \ge 1 \ \ \mathbb{P}\text{-a.s.}$$

Define

$$\tau^* := \lim_{R\to\infty} \tau^{(R)}, \quad \tau^{(0)} := 0, \quad u(t) := \sum_{R=1}^{\infty} u^{(R)}(t) \mathbf{1}_{[\tau^{(R-1)}, \tau^{(R)})}(t), \quad t \in [0, \tau^*).$$

Then one can conclude that (u, τ^*) is a local solution to (2.8). Again, by the uniqueness of $u^{(R)}$ and (4.23), $\mathbb{P}\big(u \in C([0,\tau^*); H^s)\big) = 1$. Moreover, the construction of τ^* and (4.22) immediately tell us

$$\limsup_{t\to\tau^*} \|u(t)\|_{W^{1,\infty}} = \limsup_{t\to\tau^*} \|u(t)\|_{H^s} = \infty \text{ on } \{\tau^* < \infty\} \ \ \mathbb{P}\text{-a.s.},$$

which gives (4.4).

(**II**). Recall that $\eta \in \left(\frac{d}{2} + 1,\, s - \max\{2\gamma_0, 1\}\right)$ and define

$$\widetilde{\tau}^* := \lim_{N\to\infty} \widetilde{\tau}_N, \quad \widetilde{\tau}_N := N \wedge \inf\big\{t \ge 0 : \|u(t)\|_{H^\eta} \ge N\big\}, \quad N \ge 1.$$

From (4.4) and $H^\eta \hookrightarrow W^{1,\infty}$, we have $\widetilde{\tau}^* = \tau^*$ $\mathbb{P}$-a.s. Then it suffices to prove $\mathbb{P}(\widetilde{\tau}^* < \infty) = 0$.

Recall (4.8). Then we apply Itô's formula to $\log(\mathrm{e} + \|u(t)\|_{H^\eta}^2)$ with noting (4.1) in Lemma 4.2, Theorems 3.2 and 3.3 to derive

$$\begin{aligned}
&\mathrm{d}\log(\mathrm{e} + \|u(t)\|_{H^\eta}^2) \\
&= \frac{1}{\mathrm{e} + \|u(t)\|_{H^\eta}^2} \left\{ 2\big\langle G(u(t)) + F(u(t)), u(t)\big\rangle_{H^\eta} + \|H(t,u(t))\|_{\mathcal{L}_2(\mathbb{U};H^\eta)}^2 \right\} \mathrm{d}t \\
&\quad - \frac{2}{\left(\mathrm{e} + \|u(t)\|_{H^\eta}^2\right)^2} \sum_{k=1}^{\infty} \big\langle H(t,u(t))e_k, u(t)\big\rangle_{H^\eta}^2 \,\mathrm{d}t + \mathrm{d}M_t \\
&\le \frac{1}{\mathrm{e} + \|u(t)\|_{H^\eta}^2} \left\{ 2\Lambda \|u(t)\|_{H^\eta}^2 \|u(t)\|_{W^{1,\infty}} + C\, \|u(t)\|_{H^\eta} + \sum_{k=1}^{\infty} \|h_k(t,u(t))\|_{H^\eta}^2 \right\} \mathrm{d}t \\
&\quad - \frac{2}{\left(\mathrm{e} + \|u(t)\|_{H^\eta}^2\right)^2} \sum_{k=1}^{\infty} \big\langle h_k(t,u(t)), u(t)\big\rangle_{H^\eta}^2 \,\mathrm{d}t + \mathrm{d}M_t, \quad t \in [0, \widetilde{\tau}^*),
\end{aligned}$$

where M_t is a martingale up to $\widetilde{\tau}_N$ with $N \ge 1$. According to (4.6) and (4.5), one can find a bounded function $Q : [0;\infty) \to (0,\infty)$ such that

$$\frac{1}{\mathrm{e}+\|u(t)\|_{H^\eta}^2}\Big\{2\Lambda\|u\|_{H^\eta}^2\|u\|_{W^{1,\infty}}+C\|u(t)\|_{H^\eta}^2+\sum_{k=1}^{\infty}\|h_k(t,u(t))\|_{H^\eta}^2$$
$$-\frac{2}{(\mathrm{e}+\|u(t)\|_{H^\eta}^2)}\sum_{k=1}^{\infty}\big\langle h_k(t,u(t)),u(t)\big\rangle_{H^\eta}^2\Big\}\le Q(t),\quad t\in[0,\tau^*).$$

Consequently, we infer from the above estimate that

$$\mathrm{d}\log(\mathrm{e}+\|u(t)\|_{H^\eta}^2)\le Q(t)\,\mathrm{d}t+\,\mathrm{d}M_t,\quad t\in[0,\tau^*),$$

which means that for some function $V:[0,\infty)\times[0,\infty)\to(0,\infty)$ increasing in both variables,

$$\mathbb{E}\Big[\log(\mathrm{e}+\|u(t\wedge\widetilde{\tau}_N)\|_{H^\eta}^2)\big|\mathcal{F}_0\Big]\le V(t,\|u_0\|_{H^s}),\quad t\ge 0,\quad N\ge 1.$$

Consequently, by the continuity of u in H^s (hence also in H^η), we derive that

$$\begin{aligned}\mathbb{P}\left(\widetilde{\tau}^*<t\big|\mathcal{F}_0\right)&\le\mathbb{P}\left(\widetilde{\tau}_N<t\big|\mathcal{F}_0\right)\\&\le\frac{\mathbb{E}\Big[\log(\mathrm{e}+\|u(t\wedge\widetilde{\tau}_N)\|_{H^\eta}^2)\big|\mathcal{F}_0\Big]}{\log(\mathrm{e}+N^2)}\le\frac{V(t,\|u_0\|_{H^s})}{\log(\mathrm{e}+N^2)},\quad N\ge 1,\ t>0.\end{aligned}$$

Letting $N\to\infty$ and then $t\to\infty$, we see that $\mathbb{P}(\widetilde{\tau}^*<\infty|\mathcal{F}_0)=0$ and hence $\mathbb{P}(\widetilde{\tau}^*<\infty)=0$. □

Remark 4.1. Since we have obtained cancellation properties for another generalized form of SALT operators, as described in Theorem 3.4, using the same procedure as we used to prove Theorem 4.1, we can also obtain results parallel to (**I**) and (**II**) in Theorem 4.1 for the following case:

$$\mathcal{Q}_k=\mathcal{A}_k+\mathcal{B}_k+\widetilde{\mathcal{A}}_k.\ \ \widetilde{\mathcal{A}}_k=\sum_{j=1}^{\infty}a_{k,j}\mathcal{L}_{k,j}+\Phi_k\mathbf{I}.$$

One can require that:

- $\mathcal{A}_k$ and $\mathcal{B}_k$ satisfy Hypothesis $(\mathbf{H_4})$, and $\mathcal{L}_{k,j}$ satisfies Hypothesis $(\mathbf{H_3})$ with $m=d$.
- $\{a_{k,j}\}_{k,j\ge1}\subset H^\infty(\mathbb{K}^d;\mathbb{R})$, and $\{\Phi_k\}_{k\ge1}\subset H^{\widetilde{\sigma}_0}(\mathbb{K}^d;\mathbb{R}^{m\times m})$ with $\widetilde{\sigma}_0>(\frac{d}{2}+1)\vee(s+2\gamma)$ satisfy that $\sum_{k=1}^\infty\left(\sum_{j=1}^\infty\|a_{k,j}\|_{H^r}+\|\Phi_k\|_{H^{\widetilde{\sigma}_0}}^2\right)<\infty$ for all $r\ge0$.
- For all $k\ge1$, only one of the operators $\mathcal{A}_k$, $\mathcal{B}_k$, or $\widetilde{\mathcal{A}}_k$ is non-zero.

4.4. Further discussion on pseudo-differential noise

To conclude Section 4, we discuss the role of the pseudo-differential noise structure in global existence in SPDEs in the following remark. To simplify our analysis, we will focus on the noise structure $\mathcal{Q}u \circ \mathrm{d}W(t)$ with only one pseudo-differential operator $\mathcal{Q}$ and one standard Brownian motion $W(t)$.

(1) From (**II**) in Theorem 4.1, we can infer that the "largeness" of the noise $\sum_{k=1}^{\infty} h_k \,\mathrm{d}W_k(t)$ can lead to global existence by canceling out the growth of the other terms in (1.4). This means that sufficiently large multiplicative noise, in the sense of Itô, can prevent blow-up. We note that blow-up may actually occur as wave-breaking when the multiplicative noise is linear (cf. Rohde and Tang [36]). Additional results in this direction can be found in Brzeźniak et al. [8], Ren et al. [35], Rohde and Tang [36], Tang and Wang [44]. For the case of transport noise, regularization effects have also been identified. For example, it has been demonstrated that transport noise can delay blow-up in certain SPDEs (cf. Flandoli et al. [14], Flandoli and Luo [16]).

(2) However, understanding the effects of general pseudo-differential noise $\mathcal{Q}u \circ \mathrm{d}W(t)$ in nonlinear SPDEs can be challenging. To provide a comparison, we consider the following 2-dimensional stochastic incompressible Euler equations in vorticity form:

$$\mathrm{d}w+(u\cdot\nabla)w\,\mathrm{d}t+\mathcal{Q}w\circ\mathrm{d}W(t)=0,\quad u=\mathscr{B}w,\quad w\big|_{t=0}=w(0)\in H^s,\ \ s\gg 1. \tag{4.25}$$

Here, $\mathscr{B}$ is the Biot-Savart operator, u represents the velocity of an incompressible fluid, and w is the corresponding vorticity. We note the following situations in (4.25):

(**a**) The global existence of solutions to (4.25) without noise, i.e., $\mathcal{Q} \equiv 0$, is well-known, cf. Bahouri et al. [6], Taylor [50]. The transport nature of the system plays a crucial role in the proof, which yields $\|w\|_{L^\infty} = \|w_0\|_{L^\infty}$, as shown in [50, Proposition 2.5, Page 547].

(**b**) Consider the case of transport noise when $\mathcal{Q} = (\eta\cdot\nabla)$ for a well-behaved function η. Global existence of solutions has been obtained in Lang and Crisan [27], where the spatial locality of $\mathcal{Q}$ is essential to derive $\|w\|_{L^\infty} = \|w_0\|_{L^\infty}$ (almost surely), cf. [27, Equations (21) and (22)].

(**c**) Motivated by the above cases, we consider $\mathcal{Q} \in \mathrm{OP}\mathcal{S}^1$ involving non-locality in x, for example, $\mathcal{Q} = (\eta\cdot\nabla)(-\Delta)^{\gamma}(\mathbf{I}-\Delta)^{-\gamma}$ with $\gamma > 0$, and propose the following question:

Question: Does the solution to (4.25) with $\mathcal{Q} \in \mathrm{OP}\mathcal{S}^1$ exist globally?

Unfortunately, a proof or counterexample has not yet been found. However, this question is very interesting when considering the following:

- The deterministic case has a global solution and the additional noise term is **only** a linear term that can be controlled, as guaranteed by Theorems 3.1 and 3.2. It is *highly reasonable* to expect that (4.25) will still have a global solution. This is because a linear term should **not** accelerate the growth of the H^s-norm of the solution.

- However, the introduction of noise results in additional non-local interactions. It remains unclear whether $\|w\|_{L^\infty} = \|w_0\|_{L^\infty}$, as the usual transport structure is disrupted when compared to the transport noise case (see also the previous example (1.12)).

We believe that this question can serve as a starting point for a deeper understanding of the mechanisms behind more complex non-local random perturbations.

5. Noise effect on the dependence on initial data

In this section, we consider the problem (1.16) on $\mathbb{T}^d$. For simplicity, we fix a separable Hilbert space $\mathbb{U}$ with the complete orthonormal basis $\{e_k\}_{k\geq 1}$. Then we reformulate (1.16) on $\mathbb{T}^d$ as

$$\begin{cases} \mathrm{d}u + [(u\cdot\nabla)\,u + F(u)]\;\mathrm{d}t = B(t,u)\,\mathrm{d}\mathcal{W}(t), \quad t>0, \quad u\big|_{t=0} = u_0, \\ \mathcal{W}(t) := \displaystyle\sum_{k=1}^{\infty} W_k(t)e_k, \\ B(t,u)e_k := h_k(t,u). \end{cases} \tag{5.1}$$

We assume that $h_k(t,\cdot)$ is controlled by F in the following sense:

Hypothesis ($\mathbf{H_6}$). *For all k, $h_k : [0,\infty)\times H^s \ni (t,u) \mapsto h_k(t,u) \in H^s$ is continuous for $s > \frac{d}{2}$, and*

$$\sum_{k=1}^{\infty} \|h_k(t,u)\|_{H^s}^2 \leq \|F(u)\|_{H^s}^2, \quad \sum_{k=1}^{\infty} \|h_k(t,u) - h_k(t,v)\|_{H^s}^2 \leq \|F(u) - F(v)\|_{H^s}^2,$$

where F is defined in (1.3).

Obviously, in terms of $B(t,u)$ such that $B(t,u)e_k = h_k(t,u)$, Hypothesis ($\mathbf{H_6}$) is equivalent to the following hypothesis:

Hypothesis ($\mathbf{H_B}$). *Let $s > \frac{d}{2}$. We suppose that $B : (t,u) \mapsto B(t,u) \in \mathscr{L}_2(\mathbb{U}; H^s)$ is continuous and*

$$\|B(t,u)\|_{\mathscr{L}_2(\mathbb{U};H^s)} \leq \|F(u)\|_{H^s}, \quad \|B(t,u) - B(t,v)\|_{\mathscr{L}_2(\mathbb{U};H^s)} \leq \|F(u) - F(v)\|_{H^s}.$$

For (5.1), we have

Proposition 5.1. *Let $s > \frac{d}{2} + 1$. Let Hypothesis* $(\mathbf{H_B})$ *(or equivalently Hypothesis* $(\mathbf{H_6})$*) hold. If u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, then there is a unique maximal solution (u, τ^*) to (5.1) in the sense of Definition 2.2, and (u, τ^*) satisfies (4.4).*

Proof. By Lemma 4.1, Hypothesis $(\mathbf{H_6})$ implies Hypothesis $(\mathbf{H_5})$. So existence, uniqueness and the blow-up criterion (4.4) in H^s with $s > \frac{d}{2} + 2$ follow from Theorem 4.1. To extend the result from $s > \frac{d}{2} + 2$ to $s > \frac{d}{2} + 1$, we can use the same method as in Miao et al. [30], Tang [40], which involves mollifying initial data and then passing to the limit. We omit the details here to avoid redundancy. □

For the noise effect on the solution map $u_0 \mapsto (u, \tau)$, we consider (1.16) and the results can be stated as follows:

Theorem 5.1. *Let $s > d/2 + 1$ with $d \geq 2$. Let Hypothesis* $(\mathbf{H_B})$ *be satisfied. Then there is at least one of the following properties holding true for the problem (5.1):*

(1) *For any $R \gg 1$, the R-exiting time is* **not** *strongly stable at the zero solution in the sense of Definition 2.3.*

(2) *For any $T > 0$, the solution map $u_0 \mapsto u$ defined by (5.1) is* **not** *uniformly continuous, as a map from $L^p(\Omega, H^s)$ $(p \in [1, \infty])$ into $L^1(\Omega; C([0, T]; H^s))$. More precisely, there exist two sequences of solutions $u_{1,n}(t)$ and $u_{2,n}(t)$, and two sequences of stopping times $\tau_{1,n}$ and $\tau_{2,n}$, such that*

(**a**) *$\mathbb{P}\{\tau_{i,n} > 0\} = 1$ for each $n > 1$ and $i = 1, 2$. Besides,*

$$\lim_{n\to\infty} \tau_{1,n} = \lim_{n\to\infty} \tau_{2,n} = \infty \quad \mathbb{P}\text{-a.s.}$$

(**b**) *For $i = 1, 2$, $u_{i,n} \in C([0, \tau_{i,n}]; H^s)$ $\mathbb{P}$-a.s., and*

$$\left\| \sup_{t\in[0,\tau_{1,n}]} \|u_{1,n}(t)\|_{H^s} \right\|_{L^p(\Omega)} + \left\| \sup_{t\in[0,\tau_{2,n}]} \|u_{2,n}(t)\|_{H^s} \right\|_{L^p(\Omega)} \lesssim 1, \quad p \in [1, \infty].$$

(**c**) *At $t = 0$, $\lim_{n\to\infty} \|u_{1,n}(0) - u_{2,n}(0)\|_{L^p(\Omega; H^s)} = 0$, $p \in [1, \infty]$.*

(**d**) *For any $T > 0$, we have*

$$\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{1,n}\wedge\tau_{2,n}]} \|u_{1,n}(t) - u_{2,n}(t)\|_{H^s} \gtrsim \sup_{t\in[0,T]} |\sin t|.$$

Remark 5.1. We give the following remarks concerning Theorem 5.1.

(1) To prove Theorem 5.1, we suppose that for some $R_0 \gg 1$, the R_0-exiting time of the zero solution is strongly stable. We then construct an example to show that the

solution map $u_0 \mapsto u$ defined by (1.16) is not uniformly continuous. This example involves the construction of two sequences of solutions, which converge at time zero but stay far apart at any later time for each $s > d/2+1$. Specifically, we first construct two sequences of approximate solutions $u^{l,n}$ ($l \in -1, 1$) such that the actual solutions $u_{l,n}$ ($l \in -1, 1$) starting from $u_{l,n}(0) = u^{l,n}(0)$ satisfy

$$\lim_{n\to\infty} \mathbb{E} \sup_{[0,\tau_{l,n}]} \|u_{l,n} - u^{l,n}\|^2_{H^s} = 0, \tag{5.2}$$

where $u_{l,n}$ exists at least on $[0, \tau_{l,n}]$. However, due to the lack of a lifespan estimate in the stochastic setting (see explanation in Section 1.3), we first relate the property $\inf_n \tau_{l,n} > 0$ to the stability property of the exiting time of the zero solution. If (5.2) holds, then we can estimate the approximate solutions instead of the actual solutions and obtain (**d**) by showing that the error in $H^{2s-\sigma}$ behaves like $n^{s-\sigma}$, whereas the error in H^σ is $O(1/n^{r_s})$, where $d/2 < \sigma < s-1$ and $-r_s + s - \sigma < 0$. These two estimates and interpolation imply (5.2). We prove Theorem 5.1 for $d \geq 2$. However, the proof also works for $d = 1$, that is, the stochastic CH equation case (see Remark 5.2).

(2) Theorem 5.1 implies that we cannot expect (small) multiplicative noise (in the Itô's sense) to simultaneously improve the stability of the exiting time of the zero solution and the continuity of the dependence on initial data. We refer to Alonso-Orán et al. [4], Miao et al. [30], Tang and Yang [46] for similar results in this direction. The question of whether (and how) noise in the Stratonovich sense can improve the dependence on initial data is a topic for future work.

(3) The non-uniform dependence of solutions on initial data for various deterministic fluid PDEs has been studied extensively in the literature. For example, this phenomenon has been examined for the incompressible Euler equations in the Sobolev spaces H^s in Himonas and Misiołek [20] and for the CH equation in Himonas et al. [19]. The first results of this kind for Besov spaces were obtained in Tang and Liu [41], Tang et al. [47]. In particular, non-uniform dependence of solutions on initial data in the critical Besov space first appears in Tang and Liu [42], Tang et al. [43].

Now we proceed to prove Theorem 5.1. We assume that for some $R_0 \gg 1$, the R_0-exiting time is strongly stable at the zero solution. Then we will show that the solution map $u_0 \mapsto u$ defined by (1.16) is not uniformly continuous. We will firstly assume that the dimension $d \geq 2$ is even.

5.1. Approximate solution and error

Let $l \in \{-1, 1\}$. Define

$$u^{l,n} := (ln^{-1} + n^{-s}\cos\theta_1,\, ln^{-1} + n^{-s}\cos\theta_2,\, \cdots,\, ln^{-1} + n^{-s}\cos\theta_d), \tag{5.3}$$

where $\theta_i = nx_{d+1-i} - lt$ with $1 \le i \le d$ and $n \ge 1$. Substituting $u^{l,n}$ into (1.16), we see that the (vector) error $\mathbf{E}^{l,n}(t)$ can be defined as

$$\begin{aligned}&\mathbf{E}^{l,n}(t)\\&:= u^{l,n}(t) - u^{l,n}(0) + \int_0^t \left[(u^{l,n}\cdot\nabla)u^{l,n} + F(u^{l,n})\right] \mathrm{d}t' - \int_0^t B(t',u^{l,n})\,\mathrm{d}\mathcal{W}(t).\end{aligned} \tag{5.4}$$

Now we analyze the error.

Lemma 5.1. *Let $d \ge 2$ be even and $s > 1 + \frac{d}{2} \ge 2$. For $\sigma \in \left(\frac{d}{2}, \min\left\{s-1, \frac{d}{2}+1\right\}\right)$, we have that for any $T > 0$ and $n \gg 1$,*

$$\mathbb{E}\sup_{t\in[0,T]}\|\mathbf{E}^{l,n}(t)\|^2_{H^\sigma} \le Cn^{-2r_s}, \quad C = C(T), \tag{5.5}$$

where

$$r_s = \begin{cases} 2s-\sigma-1 & \text{if } 1+\frac{d}{2} < s \le 3,\\ s-\sigma+2 & \text{if } s > 3.\end{cases}$$

Proof. Direct computation shows that

$$(u^{l,n}\cdot\nabla)u^{l,n} = \left(-ln^{-s}\sin\theta_i - n^{-2s+1}\sin\theta_i\cos\theta_{d+1-i}\right)_{1\le i\le d},$$

which means that

$$u^{l,n}(t) - u^{l,n}(0) + \int_0^t (u^{l,n}\cdot\nabla)u^{l,n}\,\mathrm{d}t' = \int_0^t \left(-n^{-2s+1}\sin\theta_i\cos\theta_{d+1-i}\right)_{1\le i\le d}\mathrm{d}t'.$$

Then we have the following equation

$$\begin{aligned}&\mathbf{E}^{l,n}(t)\\&= \int_0^t \left[-\left(n^{-2s+1}\sin\theta_i\cos\theta_{d+1-i}\right)_{1\le i\le d} + F(u^{l,n})\right]\mathrm{d}t' - \int_0^t B(t,u^{l,n})\,\mathrm{d}\mathcal{W}(t').\end{aligned} \tag{5.6}$$

We note that by Lemma A.5,

$$\left\|\left(-n^{-2s+1}\sin\theta_i\cos\theta_{d+1-i}\right)_{1\le i\le d}\right\|_{H^\sigma} \lesssim n^{-2s+1+\sigma} \lesssim n^{-r_s}. \tag{5.7}$$

For $F(\cdot) = (\mathbf{I}-\Delta)^{-1}\mathrm{div}F_1(u) + (\mathbf{I}-\Delta)^{-1}F_2(u)$ given by (1.3), some calculations reveal that $F_1(u^{l,n})$ is a diagonal matrix such that

$$F_1(u^{l,n}) = n^{-2s+2} \times \mathrm{diag}(\kappa_1, \cdots, \kappa_d),$$

where for $1 \le i \le d$,

$$\kappa_i := \sin\theta_i(\sin\theta_i + \sin\theta_{d+1-i}) - \sin^2\theta_{d+1-i} + \frac{1}{2}\left(\sin^2\theta_1 + \cdots + \sin^2\theta_d\right).$$

Therefore, we have

$$\mathrm{div}F_1(u^{l,n}) = n^{-2s+3}\big(\sin\theta_i\cos\theta_{d+1-i} - \sin\theta_{d+1-i}\cos\theta_{d+1-i}\big)_{1\le i\le d}.$$

Similarly, since $\mathrm{div}u^{l,n} = 0$, we have

$$F_2(u^{l,n}) = \big(-ln^{-s}\sin\theta_{d+1-i} - n^{-2s+1}\sin\theta_{d+1-i}\cos\theta_{d+1-i}\big)_{1\le i\le d}.$$

Therefore

$$F(u^{l,n}) = \big((\mathbf{I}-\Delta)^{-1}\Gamma_i\big)_{1\le i\le d},$$

where

$$\Gamma_i = \left(n^{-2s+3}\sin\theta_i\cos\theta_{d+1-i} - \frac{n^{-2s+1}+n^{-2s+3}}{2}\sin 2\theta_{d+1-i} - ln^{-s}\sin\theta_{d+1-i}\right).$$

Since $(\mathbf{I}-\Delta)^{-1}$ is bounded from H^σ to $H^{\sigma+2}$, we can use Lemma A.5 to derive that

$$\begin{aligned}
\big\|F(u^{l,n})\big\|_{H^\sigma} \le\, & C\sum_{i=1}^d \left(\big\|n^{-2s+3}\sin\theta_i\cos\theta_{d+1-i}\big\|_{H^{\sigma-2}} + \big\|n^{-2s+3}\sin 2\theta_{d+1-i}\big\|_{H^{\sigma-2}}\right)\\
& + C\sum_{i=1}^d \left(\big\|n^{-2s+1}\sin 2\theta_{d+1-i}\big\|_{H^{\sigma-2}} + \|n^{-s}\sin\theta_{d+1-i}\|_{H^{\sigma-2}}\right)\\
\lesssim\, & n^{-2s+3+\sigma-2} + n^{-2s+1+\sigma-2} + n^{-s+\sigma-2} \lesssim n^{-r_s}.
\end{aligned} \tag{5.8}$$

Applying the Itô formula to (5.6), we find that for any $T > 0$ and $t \in [0, T]$,

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|\mathbf{E}^{l,n}(t)\|^2_{H^\sigma}\\
&\le \mathbb{E}\sup_{t\in[0,T]}\left|-2\int_0^t \big\langle B(t',u^{l,n})\,\mathrm{d}\mathcal{W}(t'), \mathbf{E}^{l,n}(t')\big\rangle_{H^\sigma}\right| + \sum_{i=2}^4\int_0^T \mathbb{E}|P_i|\,\mathrm{d}t,
\end{aligned} \tag{5.9}$$

where

$$P_2 = -2\left\langle \mathcal{D}^\sigma \left(n^{-2s+1}\sin\theta_i\cos\theta_{d+1-i}\right)_{1\le i\le d}, \mathcal{D}^\sigma \mathbf{E}^{l,n}\right\rangle_{L^2},$$
$$P_3 = 2\big\langle \mathcal{D}^\sigma F(u^{l,n}), \mathcal{D}^\sigma \mathbf{E}^{l,n}\big\rangle_{L^2},\ \ P_4 = \|B(t,u^{l,n})\|^2_{\mathscr{L}_2(\mathbb{U};H^\sigma)}.$$

Using Hypothesis $(\mathbf{H_B})$ and the BDG inequality, we find that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t \left\langle -2B(t',u^{l,n})\,\mathrm{d}\mathcal{W}(t'), \mathbf{E}^{l,n}\right\rangle_{H^\sigma}\right| \\
&\le 2\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{E}^{l,n}(t)\|^2_{H^\sigma}\int_0^T \|F(u^{l,n})\|^2_{H^\sigma}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|\mathbf{E}^{l,n}(t)\|^2_{H^\sigma} + CTn^{-2r_s}.
\end{aligned}$$

We use (5.7), (5.8) and Hypothesis $(\mathbf{H_B})$ to find that,

$$\int_0^T \mathbb{E}\left(|P_2|+|P_3|+|P_4|\right)\mathrm{d}t \le CTn^{-2r_s} + C\int_0^T \mathbb{E}\|\mathbf{E}^{l,n}(t)\|^2_{H^\sigma}\,\mathrm{d}t.$$

Collecting the above estimates into (5.9), we arrive at

$$\mathbb{E}\sup_{t\in[0,T]}\|\mathbf{E}^{l,n}(t)\|^2_{H^\sigma} \le CTn^{-2r_s} + C\int_0^T \mathbb{E}\sup_{t'\in[0,t]}\|\mathbf{E}^{l,n}(t')\|^2_{H^\sigma}\,\mathrm{d}t.$$

Then it follows from the Grönwall inequality that

$$\mathbb{E}\sup_{t\in[0,T]}\|\mathbf{E}^{l,n}(t)\|^2_{H^\sigma} \le Cn^{-2r_s},\quad C=C(T),$$

which is the desired result. □

5.2. Actual solution and associated estimates

Now we consider the problem (1.16) with deterministic initial data $u^{l,n}(0,x)$, i.e.,

$$\begin{cases}\mathrm{d}u + \left[(u\cdot\nabla)u + F(u)\right]\mathrm{d}t = B(t,u)\,\mathrm{d}\mathcal{W}(t), & x\in\mathbb{T}^d,\ t>0,\\ u\big|_{t=0} = u^{l,n}(0,x) = \left(ln^{-1} + n^{-s}\cos nx_{d+1-i}\right)_{1\le i\le d}, & x\in\mathbb{T}^d.\end{cases} \tag{5.10}$$

Then Proposition 5.1 means that for each n, (5.10) has a unique maximal solution $(u_{l,n},\tau^*_{l,n})$.

Lemma 5.2. *Let $d \geq 2$ be even, $s > 1 + \frac{d}{2}$, $\sigma \in \left(\frac{d}{2}, \min\left\{s-1, \frac{d}{2}+1\right\}\right)$ and $r_s > 0$ be given in Lemma 5.1. For $R \gg 1$, we define*

$$\tau_{l,n}^R := \inf\left\{t \geq 0 : \|u_{l,n}\|_{H^s} > R\right\}, \quad l \in \{-1, 1\}. \tag{5.11}$$

Then for any $T > 0$ and $n \gg 1$, we have that for $l \in \{-1, 1\}$,

$$\mathbb{E} \sup_{t \in [0, T \wedge \tau_{l,n}^R]} \|u^{l,n} - u_{l,n}\|_{H^\sigma}^2 \leq C n^{-2r_s}, \quad C = C(R, T), \tag{5.12}$$

and

$$\mathbb{E} \sup_{t \in [0, T \wedge \tau_{l,n}^R]} \|u^{l,n} - u_{l,n}\|_{H^{2s-\sigma}}^2 \leq C n^{2s-2\sigma}, \quad C = C(R, T). \tag{5.13}$$

Proof. We first note that by Lemma A.5, for $l \in \{1, -1\}$,

$$\|u^{l,n}(t)\|_{H^s} \lesssim 1, \quad t \geq 0, \ n \geq 1, \tag{5.14}$$

which means $\mathbb{P}\{\tau_{l,n}^R > 0\} = 1$ for any $n \geq 1$ and $l \in \{-1, 1\}$. Let $v = v^{l,n} = u^{l,n} - u_{l,n}$. In view of (5.4), (5.6) and (5.10), we see that v satisfies

$$\begin{aligned} & v(t) + \int_0^t \left[(u^{l,n} \cdot \nabla) v + (v \cdot \nabla) u_{l,n} + (-F(u_{l,n}))\right] \mathrm{d}t' \\ & = \int_0^t \left[-B(t', u_{l,n})\right] \mathrm{d}\mathcal{W}(t') - \int_0^t \left[\left(n^{-2s+1} \sin\theta_i \cos\theta_{d+1-i}\right)_{1 \leq i \leq d}\right] \mathrm{d}t'. \end{aligned}$$

For any $T > 0$, we use the Itô formula on $[0, T \wedge \tau_{l,n}^R]$, take a supremum over $t \in [0, T \wedge \tau_{l,n}^R]$ and use the BDG inequality to find

$$\begin{aligned} & \mathbb{E} \sup_{t \in [0, T \wedge \tau_{l,n}^R]} \|v\|_{H^\sigma}^2 \\ & \leq 2\mathbb{E} \sup_{t \in [0, T \wedge \tau_{l,n}^R]} \left| \int_0^t \left\langle -B(t', u_{l,n}) \, \mathrm{d}\mathcal{W}(t'), v \right\rangle_{H^\sigma} \right| + \sum_{i=2}^6 \mathbb{E} \int_0^{T \wedge \tau_{l,n}^R} |S_i| \, \mathrm{d}t, \end{aligned}$$

where

$$\begin{aligned} & S_2 := 2\left\langle \mathcal{D}^\sigma \left(-n^{-2s+1} \sin\theta_i \cos\theta_{d+1-i}\right)_{1 \leq i \leq d}, \mathcal{D}^\sigma v \right\rangle_{L^2}, \\ & S_3 := -2\left\langle \mathcal{D}^\sigma [(v \cdot \nabla) u_{l,n}], \mathcal{D}^\sigma v \right\rangle_{L^2}, \quad S_4 := -2\left\langle \mathcal{D}^\sigma [(u^{l,n} \cdot \nabla) v], \mathcal{D}^\sigma v \right\rangle_{L^2}, \\ & S_5 := 2\left\langle \mathcal{D}^\sigma F(u_{l,n}), \mathcal{D}^\sigma v \right\rangle_{L^2}, \quad S_6 := \|B(t, u_{l,n})\|_{\mathscr{L}_2(\mathbb{U}; H^\sigma)}^2. \end{aligned}$$

We can first infer from Lemma 4.1 that

$$\begin{aligned}\|F(u_{l,n})\|^2_{H^\sigma} &\lesssim \left(\|F(u^{l,n})-F(u_{l,n})\|_{H^\sigma}+\|F(u^{l,n})\|_{H^\sigma}\right)^2\\&\lesssim (\|u^{l,n}\|_{H^s}+\|u_{l,n}\|_{H^s})^2\|v\|^2_{H^\sigma}+\|F(u^{l,n})\|^2_{H^\sigma}.\end{aligned}$$

From the above estimate, Hypothesis $(\mathbf{H_B})$, the BDG inequality, (5.8), (5.11) and (5.14), we have

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\left|\int_0^t\left\langle -2B(t',u_{l,n})\,\mathrm{d}\mathcal{W}(t'),v\right\rangle_{H^s}\right|\\&\le C\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^\sigma}\int_0^{T\wedge\tau^R_{l,n}}(\|u^{l,n}\|_{H^s}+\|u_{l,n}\|_{H^s})^2\|v\|^2_{H^\sigma}\,\mathrm{d}t\right)^{\frac12}\\&\quad+C\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^\sigma}\int_0^{T\wedge\tau^R_{l,n}}\|F(u^{l,n})\|^2_{H^\sigma}\,\mathrm{d}t\right)^{\frac12}\\&\le\frac12\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^\sigma}+C_R\mathbb{E}\int_0^T\sup_{t'\in[0,t\wedge\tau^R_{l,n}]}\|v(t')\|^2_{H^\sigma}\,\mathrm{d}t+CTn^{-2r_s}.\end{aligned}$$

Applying Lemma 4.1, $H^\sigma\hookrightarrow L^\infty$, integration by parts and (5.7), we have

$$\begin{aligned}|S_2|&\lesssim\left\|(n^{-2s+1}\sin\theta_i\cos\theta_{d+1-i})_{1\le i\le d}\right\|^2_{H^\sigma}+\|v\|^2_{H^\sigma}\lesssim n^{-2r_s}+\|v\|^2_{H^\sigma},\\|S_3|&\lesssim\|(v\cdot\nabla)u_{l,n}\|_{H^\sigma}\|v\|_{H^\sigma}\lesssim\|v\|^2_{H^\sigma}\|u_{l,n}\|_{H^s},\\|S_5|&\lesssim(\|u^{l,n}\|_{H^s}+\|u_{l,n}\|_{H^s})\|v\|^2_{H^\sigma}+\|F(u^{l,n})\|^2_{H^\sigma}+\|v\|^2_{H^\sigma},\end{aligned}$$

and

$$|S_6|\lesssim(\|u^{l,n}\|_{H^s}+\|u_{l,n}\|_{H^s})^2\|v\|^2_{H^\sigma}+\|F(u^{l,n})\|^2_{H^\sigma}.$$

With Lemma A.4 at hand, we consider the following two cases:

$$\begin{aligned}|S_4|&\lesssim\|u^{l,n}\|_{W^{\sigma,\frac{2d}{d-2}}}\|\nabla v\|_{L^d}\|v\|_{H^\sigma}+\|\nabla u^{l,n}\|_{L^\infty}\|v\|^2_{H^\sigma}\\&\lesssim\|u^{l,n}\|_{H^s}\|v\|^2_{H^\sigma}\quad\text{for even } d\ge4,\end{aligned}$$

and

$$|S_4|\lesssim\|u^{l,n}\|_{W^{\sigma,q}}\|\nabla v\|_{L^p}\|v\|_{H^\sigma}+\|\nabla u^{l,n}\|_{L^\infty}\|v\|^2_{H^\sigma}\quad\text{for } d=2,$$

where in the case $d = 2$, p will be chosen such that $\sigma - \frac{d}{2} = \sigma - 1 > 1 - \frac{2}{p} > 0$ and q is determined by $\frac{1}{2} = \frac{1}{q} + \frac{1}{p}$. We use $H^s \hookrightarrow H^{\sigma+1} \hookrightarrow W^{\sigma,\frac{2d}{d-2}}$, $H^\sigma \hookrightarrow W^{1,d}$ for the case $d \geq 4$ and use $H^s \hookrightarrow W^{\sigma+\frac{2}{q},q} \hookrightarrow W^{\sigma,q}$ and $H^\sigma \hookrightarrow W^{1,p}$ for the case $d = 2$ to obtain

$$|S_4| \lesssim \|u^{l,n}\|_{H^s}\|v\|^2_{H^\sigma}.$$

Therefore, we can infer from Lemma 4.1, (5.8), (5.11) and (5.14) that

$$\mathbb{E}\int_0^{T\wedge\tau^R_{l,n}} (|S_2| + |S_5| + |S_6|)\, \mathrm{d}t \leq CTn^{-2r_s} + C_R\int_0^T \mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]} \|v(t')\|^2_{H^\sigma}\, \mathrm{d}t,$$

and

$$\mathbb{E}\int_0^{T\wedge\tau^R_{l,n}} (|S_3| + |S_4|)\, \mathrm{d}t \leq C_R\int_0^T \mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]} \|v(t')\|^2_{H^\sigma}\, \mathrm{d}t.$$

Over all, we arrive at

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]} \|v(t)\|^2_{H^\sigma} \leq CTn^{-2r_s} + C_R\int_0^T \mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]} \|v(t')\|^2_{H^\sigma}\, \mathrm{d}t.$$

Via the Grönwall inequality, we have

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]} \|v(t)\|^2_{H^\sigma} \leq Cn^{-2r_s}, \quad C = C(R,T),$$

which is (5.12). For (5.13), we first note that $u_{l,n}$ is the unique solution to (5.10) and $2s-\sigma > d/2+1$. For each fixed $n \geq 1$, similar to the analysis in the proof for Lemma 4.3, we find constant $C = C(R,T) > 0$ such that

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]} \|u_{l,n}(t)\|^2_{H^{2s-\sigma}} \leq 2\mathbb{E}\|u^{l,n}(0)\|^2_{H^{2s-\sigma}} + C\int_0^T \left(\mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]} \|u(t')\|^2_{H^{2s-\sigma}}\right) \mathrm{d}t.$$

From the above estimate, we can use the Grönwall inequality and Lemma A.5 to infer

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]} \|u_{l,n}(t)\|^2_{H^{2s-\sigma}} \leq C\mathbb{E}\|u^{l,n}(0)\|^2_{H^{2s-\sigma}} \leq Cn^{2s-2\sigma}, \quad C = C(R,T).$$

Then it follows from Lemma A.5 that for some $C = C(R,T)$ and $l \in \{-1,1\}$,

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^{2s-\sigma}} \le C\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|u_{l,n}\|^2_{H^{2s-\sigma}} + C\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|u^{l,n}\|^2_{H^{2s-\sigma}} \le Cn^{2s-2\sigma},$$

which is (5.13). □

5.3. Proof for Theorem 5.1

Lemma 5.3. *Let $d \ge 2$ be even and $B(t,u)$ satisfy Hypothesis* **(H$_6$)**. *If for some $R_0 \gg 1$, the R_0-exiting time is strongly stable at the zero solution to (1.16), then for $l \in \{1,-1\}$, we have*

$$\lim_{n\to\infty}\tau^{R_0}_{l,n} = \infty \quad \mathbb{P}\text{-a.s.,} \tag{5.15}$$

where $\tau^{R_0}_{l,n}$ is given in (5.11).

Proof. Since $F(0) = 0$, it is clear that zero is the unique solution to (1.16) with zero initial data under Hypothesis **(H$_6$)**. Due to (5.10), it follows that

$$\lim_{n\to\infty}\|u_{l,n}(0) - 0\|_{H^{s'}} = \lim_{n\to\infty}\|u^{l,n}(0)\|_{H^{s'}} = 0 \quad \forall\ s' < s.$$

Note that the R_0-exiting time at the zero solution is ∞. As a result, we see that if the R_0-exiting time is strongly stable at the zero solution to (1.16), then (5.15) holds true. □

With the above result at our disposal, now we can prove Theorem 5.1.

Proof for Theorem 5.1. Let us first consider the case $d \ge 2$ is even. We will show that, if the R_0-exiting time is strongly stable at the zero solution for some $R_0 \gg 1$, then $(u_{-1,n}, \tau^{R_0}_{-1,n})$ and $(u_{1,n}, \tau^{R_0}_{1,n})$ satisfy (**a**)–(**d**) in Theorem 5.1.

Verify (**a**)**.** For each $n > 1$, for $l \in \{1,-1\}$ and for the fixed $R_0 \gg 1$, Lemma A.5 and (5.11) give us $\mathbb{P}\{\tau^{R_0}_{l,n} > 0\} = 1$ and Lemma 5.3 implies the desired estimate in (**a**).

Verify (**b**)**.** Theorem 4.1 and (5.11) show that $u_{l,n} \in C([0,\tau^{R_0}_{l,n}]; H^s)$ $\mathbb{P}$-a.s. and

$$\sup_{t\in[0,\tau^{R_0}_{l,n}]}\|u_{l,n}\|_{H^s} \le R_0 \quad \mathbb{P}\text{-a.s.,}$$

which gives (**b**).

Verify (**c**)**.** Since $u^{-1,n}(0)$ and $u^{1,n}(0)$ are deterministic and

$$\|u_{-1,n}(0) - u_{1,n}(0)\|_{H^s} = \|u^{-1,n}(0) - u^{1,n}(0)\|_{H^s} \lesssim n^{-1},$$

we obtain (**c**).

Verify (**d**)**.** For any $T > 0$, using the interpolation inequality and Lemma 5.2, we see that for $l \in \{-1, 1\}$ and $v = v^{l,n} = u^{l,n} - u_{l,n}$,

$$\left(\mathbb{E} \sup_{t\in[0,T\wedge\tau_{l,n}^{R_0}]} \|v\|_{H^s}\right)^2 \leq \left(\mathbb{E} \sup_{t\in[0,T\wedge\tau_{l,n}^{R_0}]} \|v\|_{H^\sigma}^2\right)^{\frac{1}{2}} \left(\mathbb{E} \sup_{t\in[0,T\wedge\tau_{l,n}^{R_0}]} \|v\|_{H^{2s-\sigma}}^2\right)^{\frac{1}{2}}$$
$$\lesssim n^{-r_s+(s-\sigma)}.$$

It follows from

$$0 > -r_s + s - \sigma = \begin{cases} 1 - s & \text{if } 1 + \frac{d}{2} < s \leq 3, \\ -2 & \text{if } s > 3, \end{cases}$$

that for $l \in \{-1, 1\}$,

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{l,n}^{R_0}]} \|u^{l,n} - u_{l,n}\|_{H^s} = 0. \tag{5.16}$$

For any given $T > 0$, on account of (5.16), Lemmas A.5 and 5.3, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s} \\
\gtrsim\ &\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]} \|u^{-1,n}(t) - u^{1,n}(t)\|_{H^s} \\
\gtrsim\ &\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]} \left\|n^{-s}\cos\left(nx_{d+1-i} + t\right) - n^{-s}\cos\left(nx_{d+1-i} - t\right)\right\|_{H^s} \\
\gtrsim\ &\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]} \left(n^{-s}\|\sin nx_{d+1-i}\|_{H^s}|\sin t| - \|2n^{-1}\|_{H^s}\right).
\end{aligned}$$

Using Fatou's lemma, we arrive at

$$\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]} \|u_{-1,n}(t) - u_{1,n}(t)\|_{H^s} \gtrsim \sup_{t\in[0,T]} |\sin t|,$$

which implies (**d**).

Now we consider the case that $d \geq 3$ is odd. Instead of (5.3), we define

$$u^{l,n} = (ln^{-1} + n^{-s}\cos\theta_1,\ ln^{-1} + n^{-s}\cos\theta_2,\ \cdots,\ ln^{-1} + n^{-s}\cos\theta_{d-1},\ 0),$$

where $\theta_i = nx_{d-i} - lt$ with $1 \leq i \leq d-1$, $n \geq 1$, $l \in \{-1, 1\}$. In this case, $d-1$ is even and we can repeat the proof for Lemma 5.1 to find that the error $\mathbf{E}^{l,n}(t)$ also enjoys (5.5). Moreover, for the pathwise solutions $u_{l,n}$ to (5.10) with

$$u_{l,n}(0) = u^{l,n}(0) = \big(ln^{-1} + n^{-s}\cos nx_{d-i},\ 0\big)_{1\le i\le d-1},$$

we can basically repeat the previous procedure to show that Lemmas 5.2 and 5.3 also hold true. Therefore, one can establish (**a**)–(**d**) for $u_{l,n}$ similarly.

In conclusion, we see that if for some $R_0 \gg 1$, the R_0-exiting time is strongly stable at the zero solution, then the solution map defined by (1.16) is not uniformly continuous when $B(t,\cdot)$ satisfies Hypothesis $(\mathbf{H_B})$. □

Remark 5.2. From the above proof for Theorem 5.1, it is clear that if $d = 1$, one can use

$$u^{l,n} = ln^{-1} + n^{-s}\cos(nx - lt)$$

as a sequence of approximation solutions and repeat the other part of the proof correspondingly to obtain the similar statements in $d = 1$. Therefore, Theorem 5.1 also holds true for $d = 1$, namely the stochastic CH equation case.

Data availability

No data was used for the research described in the article.

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Appendix A. Auxiliary results

In this appendix, we recall and establish some auxiliary results from analysis employed in the proofs above. We begin with introducing mollifiers. For $n \ge 1$, we define the Friedrichs mollifier

$$J_n := \mathrm{OP}\big(j(\cdot/n)\big), \quad n \ge 1, \tag{A.1}$$

where $j \in \mathscr{S}(\mathbb{R}^d;\mathbb{R})$ (the Schwarz space of rapidly decreasing C^∞ functions on $\mathbb{R}^d$) satisfies $0 \le j(y) \le 1$ for all $y \in \mathbb{R}^d$ and $j(y) = 1$ for any $|y| \le 1$.

From the construction of J_n, it is easy to find the following lemma:

Lemma A.1. *The following properties for J_n hold true:*

$$\|\mathbf{I} - J_n\|_{\mathscr{L}(H^s;H^r)} \lesssim \frac{1}{n^{s-r}}, \quad r < s,$$
$$\|J_n\|_{\mathscr{L}(H^s;H^r)} \sim O(n^{r-s}), \quad r > s,$$

and for all $n \geq 1$, $s \geq 0$,

$$[\mathcal{D}^s, J_n] = 0, \quad \langle J_n f, g\rangle_{L^2} = \langle f, J_n g\rangle_{L^2}, \quad \|J_n\|_{\mathscr{L}(L^\infty;L^\infty)} \lesssim 1, \quad \|J_n\|_{\mathscr{L}(H^s;H^s)} \leq 1.$$

Lemma A.2 *(Page 3 in Taylor [50]). Let $d \geq 1$ and $f, g : \mathbb{K}^d \to \mathbb{R}^d$ such that $g \in W^{1,\infty}$ and $f \in L^2$. Then for some $C = C(d) > 0$,*

$$\|[J_n, (g \cdot \nabla)]f\|_{L^2} \leq C\|g\|_{W^{1,\infty}}\|f\|_{L^2}, \quad n \geq 1.$$

Then we recall some estimates in Sobolev spaces H^s.

Lemma A.3 *(Bahouri et al. [6]). Let $s_1, s_2 \in \mathbb{R}$ with $s_1 + s_2 > 0$ and $s_1 \leq \frac{d}{2} < s_2$,*

$$\|fg\|_{H^{s_1}} \lesssim \|f\|_{H^{s_1}}\|g\|_{H^{s_2}}, \quad f \in H^{s_1}, g \in H^{s_2}.$$

Lemma A.4 *(Kenig et al. [25]). If $f, g \in H^s \bigcap W^{1,\infty}$ with $s > 0$, then for $p, p_i \in (1, \infty)$ with $i = 2, 3$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, we have*

$$\| [\mathcal{D}^s, f\mathbf{I}]\, g\|_{L^p} \leq C_s(\|\nabla f\|_{L^{p_1}}\|\mathcal{D}^{s-1}g\|_{L^{p_2}} + \|\mathcal{D}^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}),$$

and

$$\|\mathcal{D}^s(fg)\|_{L^p} \leq C_s(\|f\|_{L^{p_1}}\|\mathcal{D}^s g\|_{L^{p_2}} + \|\mathcal{D}^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}).$$

Lemma A.5 *(Tang and Liu [42], Zhao et al. [53]). Let $\sigma, r \in \mathbb{R}$. If $n \gg 1$, then*

$$\|\sin(nx - r)\|_{H^\sigma(\mathbb{T};\mathbb{R})}, \ \|\cos(nx - r)\|_{H^\sigma(\mathbb{T};\mathbb{R})} \approx n^\sigma,$$
$$\|\cos(nx - r)\sin(ny - r)\|_{H^\sigma(\mathbb{T}^2;\mathbb{R})} \approx n^\sigma.$$

The following lemmas with single $\mathcal{P}$ and a pair $(\mathcal{P}_1, \mathcal{P}_2)$ on $\mathbb{R}^d$ are well known in the literature. By (2.7) in previous sections, they also hold true on $\mathbb{T}^d$. Recall that $\mathbf{S}^s$ is Fréchet space with seminorms $\{|\cdot|^{\beta,\alpha;s}\}_{\beta,\alpha\in\mathbb{N}_0^d}$.

Lemma A.6. *Let $r, r_1, r_2 \in \mathbb{R}$, $\mathfrak{p} \in \mathbf{S}^r$, $\mathfrak{p}_1 \in \mathbf{S}^{r_1}, \mathfrak{p}_2 \in \mathbf{S}^{r_2}$, and let OP be given in (2.1) and (2.2). Then we have the following results:*

(1) **(Continuity of** OP**)** *For any $q, s \in \mathbb{R}$, $\mathrm{OP} : \mathbf{S}^s \to \mathscr{L}(H^{q+s}; H^q)$ is bounded and there are $\widetilde{\beta}, \widetilde{\alpha} \in \mathbb{N}_0^d$ and a constant $C = C(s, q) > 0$ such that*

$$\|\mathrm{OP}(\mathfrak{q})\|_{\mathscr{L}(H^{q+s};H^q)} \leq C(s, q)|\mathfrak{q}|^{\widetilde{\beta},\widetilde{\alpha};s}.$$

(2) **(Adjoint)** $\left(\mathrm{OP}(\mathfrak{p})\right)^* \in \mathrm{OP}\mathbf{S}^r$*, and the map*

$$\mathbf{S}^r \ni \mathfrak{p} \mapsto \widetilde{\mathfrak{p}} \in \mathbf{S}^r \ \textit{is continuous},$$

where $\big(\mathrm{OP}(\mathfrak{p})\big)^* = \mathrm{OP}\big(\widetilde{\mathfrak{p}}\big)$.

(3) **(Composition)** $\mathrm{OP}(\mathfrak{p}_1)\mathrm{OP}(\mathfrak{p}_2) \in \mathrm{OP}\mathbf{S}^{r_1+r_2}$, *and the map*

$$\mathbf{S}^{r_1} \times \mathbf{S}^{r_2} \ni (\mathfrak{p}_1, \mathfrak{p}_2) \mapsto \mathfrak{p}_1 \# \mathfrak{p}_2 \in \mathbf{S}^{r_1+r_2} \text{ is continuous,}$$

where $\mathrm{OP}(\mathfrak{p}_1)\mathrm{OP}(\mathfrak{p}_2) = \mathrm{OP}(\mathfrak{p}_1 \# \mathfrak{p}_2)$.

(4) **(Commutator)** *If* $\mathfrak{p}_1 \in \mathbf{S}^{r_1}$ *and* $\mathfrak{p}_2 \in \mathbf{S}^{r_2}$ *are commuting matrices, then*

$$\big[\mathrm{OP}(\mathfrak{p}_1), \mathrm{OP}(\mathfrak{p}_2)\big] \in \mathrm{OP}\mathbf{S}^{r_1+r_2-1}.$$

Proof. These properties are well-known in the literature. However, for the convenience of readers, we provide some references where the details of each property can be found. The first property can be found in [26, Theorem 2.7, Page 124] and [1, Theorem 3.41]. The second property is detailed in [28, Theorems 1.1.8 & 1.1.21] and [7, Theorem C.1]. The third property is given in [33, Theorem 1.2.16] and [1, Page 72]. Lastly, we refer to [48, Page 32] and [7, Theorem C.3] for the final property. □

Lemma A.7. *Let* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{T}$, $d \geq 1$, s, r_1, $r_2 \in \mathbb{R}$. *Suppose that* $\mathscr{M} \times \mathscr{O} \subset \mathbf{S}^{r_1} \times \mathbf{S}^{r_2}$ *is a bounded subset such that for any* $(\mathfrak{p}_1, \mathfrak{p}_2) \in \mathscr{M} \times \mathscr{O}$, $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *are commuting matrices. Then we have:*

$$\sup_{(\mathfrak{p}_1,\mathfrak{p}_2)\in\mathscr{M}\times\mathscr{O}} \big\| \big[\mathrm{OP}(\mathfrak{p}_1), \mathrm{OP}(\mathfrak{p}_2)\big] \big\|_{\mathscr{L}(H^{s+r_1+r_2-1}; H^s)} < \infty. \tag{A.2}$$

If additionally $s, r_1, r_2 \geq 0$, *then for all* $f, g \in H^{r_0}(\mathbb{K}^d; \mathbb{R})$ *with* $r_0 > \max\big\{\frac{d}{2}+1, s+r_1+r_2\big\}$, *the following estimate holds:*

$$\sup_{(\mathfrak{p}_1,\mathfrak{p}_2)\in\mathscr{M}\times\mathscr{O}} \|\mathcal{Q}\|_{\mathscr{L}(H^{s+r_1+r_2-1}; H^s)} \lesssim \|f\|_{H^{r_0}} \|g\|_{H^{r_0}}, \quad \mathcal{Q} \in S, \tag{A.3}$$

where for $\mathcal{P}_i = \mathrm{OP}(\mathfrak{p}_i)$ *with* $i = 1, 2$,

$$S := \Big\{ \big[(f\mathbf{I})\mathcal{P}_1, (g\mathbf{I})\mathcal{P}_2\big], \big[\mathcal{P}_1(f\mathbf{I}), (g\mathbf{I})\mathcal{P}_2\big], \big[(f\mathbf{I})\mathcal{P}_1, \mathcal{P}_2(g\mathbf{I})\big], \big[\mathcal{P}_1(f\mathbf{I}), \mathcal{P}_2(g\mathbf{I})\big] \Big\}.$$

Proof. When $\mathfrak{p}$ and $\mathfrak{q}$ are commuting matrices, some direct computations (cf. [28, Corollary 1.1.22] or [7, Theorem C.3]) yield $\mathfrak{p}_1 \# \mathfrak{p}_2 - \mathfrak{p}_2 \# \mathfrak{p}_1 \in \mathbf{S}^{r_1+r_2-1}$. From this and Lemma A.6, we see that

$$(\mathfrak{p}_1, \mathfrak{p}_2) \mapsto \big[\mathrm{OP}(\mathfrak{p}_1), \mathrm{OP}(\mathfrak{p}_2)\big] = \mathrm{OP}\big(\mathfrak{p}_1 \# \mathfrak{p}_2 - \mathfrak{p}_2 \# \mathfrak{p}_1\big)$$

is continuous from $\mathbf{S}^{r_1} \times \mathbf{S}^{r_2}$ to $\mathscr{L}(H^{s+r_1+r_2-1}; H^s)$, which implies (A.2).

Now we will prove (A.3). To begin with, we have the following:

Claim: If $\mathscr{M} \subset \mathbf{S}^r$ is bounded with $r \geq 0$, then there is a constant $C > 0$ such that

$$\sup_{\mathfrak{p}\in\mathscr{M}} \left\| \left[\mathrm{OP}(\mathfrak{p}), g\mathbf{I}\right] \right\|_{\mathscr{L}(H^{q+r-1};H^q)} \leq C\|g\|_{H^\sigma},\ \sigma > \frac{d}{2}+1,\ q \in [0, \sigma - r],\quad g \in H^\sigma. \quad \text{(A.4)}$$

Indeed, for a single operator $\mathcal{P} = \mathrm{OP}(\mathfrak{p}) \in \mathrm{OP}\mathcal{S}^r$, by [49, Proposition 4.2], we can find a constant $C > 0$ such that

$$\|[\mathcal{P}, g\mathbf{I}]\, u\|_{H^q} \leq C\|g\|_{H^\sigma}\|u\|_{H^{q+r_1-1}}.$$

From its proof (see Taylor [49]) and **(1)** in Lemma A.6, we observe that the constant C depends on the seminorms of $\mathfrak{p}$. This implies that the bilinear map

$$\mathfrak{T} : \mathcal{S}^r \times H^\sigma \ni (\mathfrak{p}, f) \mapsto [\mathrm{OP}(\mathfrak{p}), f\mathbf{I}] \in \mathscr{L}(H^{q+r-1}; H^q)$$

is continuous separately in $\mathfrak{p}$ and f. By [37, Theorem 2.17], $\mathfrak{T}$ is continuous, which implies (A.4) with $\mathfrak{p} \in \mathcal{S}^r$. Recalling the fact

$$\left(\mathrm{OP}(\mathfrak{p})\right)^{(i,j)} = \mathrm{OP}\left(\mathfrak{p}^{(i,j)}\right),$$

and then summing over i and j, we obtain (A.4) with $\mathfrak{p} \in \mathbf{S}^r$.

To prove (A.3), we observe that

$$[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B.$$

This and (A.4) lead to (A.3). For brevity, we will only verify the case of $\left[(f\mathbf{I})\mathcal{P}_1, \mathcal{P}_2(g\mathbf{I})\right]$. In this case, we have

$$\left[(f\mathbf{I})\mathcal{P}_1, \mathcal{P}_2(g\mathbf{I})\right] = (f\mathbf{I})[\mathcal{P}_1, \mathcal{P}_2](g\mathbf{I}) + (f\mathbf{I})\mathcal{P}_2[\mathcal{P}_1, g\mathbf{I}] + [f\mathbf{I}, \mathcal{P}_2](g\mathbf{I})\mathcal{P}_1.$$

Let $\eta_1 > s \vee \frac{d}{2}$. Using either the algebra property of H^s (when $s > \frac{d}{2}$) or Lemma A.3 (when $s \leq \frac{d}{2}$), we obtain

$$\|(f\mathbf{I})[\mathcal{P}_1, \mathcal{P}_2](g\mathbf{I})h\|_{H^s} \lesssim \|f\|_{H^{\eta_1}} \|[\mathcal{P}_1, \mathcal{P}_2](gh)\|_{H^s}.$$

Once again, let $\eta_2 > (s + r_1 + r_2 - 1) \vee \frac{d}{2} \vee 1$. Then (A.2) gives rise to

$$\begin{aligned}\sup_{(\mathfrak{p}_1,\mathfrak{p}_2)\in\mathscr{M}\times\mathscr{O}} \|(f\mathbf{I})[\mathcal{P}_1, \mathcal{P}_2](g\mathbf{I})h\|_{H^s} &\lesssim \|f\|_{H^{\eta_1}} \|gh\|_{H^{s+r_1+r_2-1}} \\ &\lesssim \|f\|_{H^{\eta_1}} \|g\|_{H^{\eta_2}} \|h\|_{H^{s+r_1+r_2-1}}.\end{aligned}$$

Let $\eta_3 > (s + r_1 + r_2) \vee (\frac{d}{2} + 1)$. Similarly, for $(f\mathbf{I})\mathcal{P}_2[\mathcal{P}_1, g\mathbf{I}]$, we use **(1)** in Lemma A.6 and apply (A.4) to $[\mathcal{P}_1, g\mathbf{I}]h$ (with $u = h$, $0 \leq q = s + r_2 \leq \sigma - r$, $\sigma = \eta_3$ and $r = r_1$) to find

$$\sup_{(\mathfrak{p}_1,\mathfrak{p}_2)\in\mathscr{M}\times\mathscr{O}} \|(f\mathbf{I})\mathcal{P}_2[\mathcal{P}_1,g\mathbf{I}]h\|_{H^s} \lesssim \|f\|_{H^{\eta_1}} \sup_{\mathfrak{p}_1\in\mathscr{M}} \|[\mathcal{P}_1,g\mathbf{I}]h\|_{H^{s+r_2}}$$
$$\lesssim \|f\|_{H^{\eta_1}} \|g\|_{H^{\eta_3}} \|h\|_{H^{s+r_1+r_2-1}}.$$

For $[f\mathbf{I},\mathcal{P}_2](g\mathbf{I})\mathcal{P}_1$, we note that $\eta_3 > s+r_2-1$ and $\eta_3+s+r_2-1>0$. Then, (A.4) (with $u=g\mathcal{P}_1h$, $0\le q=s\le\sigma-r$, $\sigma=\eta_3$ and $r=r_2$) leads to

$$\sup_{(\mathfrak{p}_1,\mathfrak{p}_2)\in\mathscr{M}\times\mathscr{O}} \left\|[f\mathbf{I},\mathcal{P}_2](g\mathbf{I})\mathcal{P}_1h\right\|_{H^s} \lesssim \|f\|_{H^{\eta_3}} \sup_{\mathfrak{p}_1\in\mathscr{M}} \|g\mathcal{P}_1h\|_{H^{s+r_2-1}}$$
$$\lesssim \|f\|_{H^{\eta_3}} \|g\|_{H^{\eta_3}} \|h\|_{H^{s+r_1+r_2-1}}.$$

To sum up, we obtain (A.3) for $\left[(f\mathbf{I})\mathcal{P}_1,\mathcal{P}_2(g\mathbf{I})\right]$. The other cases can be proved in the same way. □

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