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Well-posedness for a stochastic Camassa–Holm type equation with higher order nonlinearities

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Abstract
This paper aims at studying a generalized Camassa–Holm equation under random perturbation. We establish a local well-posedness result in the sense of Hadamard, i.e., existence, uniqueness and continuous dependence on initial data, as well as blow-up criteria for pathwise solutions in the Sobolev spaces H^s with $s > 3/2$ for $x \in \mathbb{R}$. The analysis on continuous dependence on initial data for nonlinear stochastic partial differential equations has gained less attention in the literature so far. In this work, we first show that the solution map is continuous. Then we introduce a notion of stability of exiting time. We provide an example showing that one cannot improve the stability of the exiting time and simultaneously improve the continuity of the dependence on initial data. Finally, we analyze the regularization effect of nonlinear noise in preventing blow-up. Precisely, we demonstrate that global existence holds true almost surely provided that the noise is strong enough.

Keywords Stochastic generalized Camassa–Holm equation · Noise effect · Exiting time · Dependence on initial data · Global existence

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1 Introduction and main results

We consider the following stochastic generalized Camassa–Holm (CH) equation on $\mathbb{R}$:

$$u_t - u_{xxt} + (k+2)u^k u_x \ -(1-\partial_x^2)h(t,u)\dot{\mathcal{W}} \\ = (k+1)u^{k-1}u_x u_{xx} + u^k u_{xxx}, \quad k \in \mathbb{N}_{>0}. \tag{1.1}$$

In (1.1), $\mathcal{W}$ is a cylindrical Wiener process.

For $h = 0$ and $k = 1$, Eq. (1.1) reduces to the deterministic CH equation given by

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

Equation (1.2) was introduced by Fokas and Fuchssteiner [21] to study completely integrable generalizations of the Korteweg–de Vries equation with bi-Hamiltonian structure. In [10], Camassa and Holm proved that (1.2) can be connected to the unidirectional propagation of shallow water waves over a flat bottom. Since then, (1.2) has been studied intensively, and we only mention a few related results here. The CH equation exhibits both phenomena of soliton interaction (peaked soliton solutions) and wave breaking (the solution remains bounded while its slope becomes unbounded in finite time [16]).

When $h = 0$ and $k = 2$, Eq. (1.1) becomes the so-called Novikov equation

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{1.3}$$

which was derived in [44]. Equation (1.3) also possesses a bi-Hamiltonian structure with an infinite sequence of conserved quantities, and it admits peaked solutions [24], as well as multipeakon solutions with explicit formulas [34]. For the study of other deterministic instances of (1.1), we refer to [28, 60].

When additional noise is included, as in [46], the noise term can be used to account for the randomness arising from the energy exchange mechanisms. Indeed, in [40, 59], the weakly dissipative term $(1-\partial_x^2)(\lambda u)$ with $\lambda > 0$ was added to the governing equations. In [46], such weakly dissipative term is assumed to be time-dependent, nonlinear in u and random. Therefore, $(1-\partial_x^2)h(t,u)\dot{\mathcal{W}}$ is proposed to describe random energy exchange mechanisms.

In this work, we consider the Cauchy problem for (1.1) on the whole space $\mathbb{R}$. Applying the operator $(1-\partial_x^2)^{-1}$ to (1.1), we reformulate the equation as

$$\begin{cases} \mathrm{d}u + \left[u^k \partial_x u + F(u)\right]\mathrm{d}t = h(t,u)\,\mathrm{d}\mathcal{W}, \quad x \in \mathbb{R},\ t > 0,\ k \in \mathbb{N}_{>0}, \\ u(\omega, 0, x) = u_0(\omega, x), \quad x \in \mathbb{R}, \end{cases} \tag{1.4}$$

with

$$F(u) := F_1(u) + F_2(u) + F_3(u)$$
$$\text{and} \quad \begin{cases} F_1(u) := (1-\partial_x^2)^{-1}\partial_x\left(u^{k+1}\right), \\ F_2(u) := \dfrac{2k-1}{2}(1-\partial_x^2)^{-1}\partial_x\left(u^{k-1}u_x^2\right), \\ F_3(u) := \dfrac{k-1}{2}(1-\partial_x^2)^{-1}\left(u^{k-2}u_x^3\right). \end{cases} \tag{1.5}$$

Here we remark that $F_3(u)$ in (1.5) will disappear for the CH case (i.e., when $k = 1$). The operator $(1-\partial_x^2)^{-1}$ in $F(\cdot)$ is understood as

$$\left[(1-\partial_x^2)^{-1} f\right](x) = \left[\frac{1}{2}\mathrm{e}^{-|\cdot|} \star f\right](x),$$

where $\star$ stands for the convolution.

In this paper, regarding (1.4), we focus on the following issues:

- Local well-posedness, in the sense of Hadamard (existence, uniqueness and continuous dependence on initial data), and blow-up criterion of (1.4).
- Understanding the dependence on initial data, and in particular how continuous the solution map $u_0 \mapsto u$ is.
- Analyzing the effect of noise vs blow-up of the deterministic counterpart of (1.4).

For the first and second issue, we refer to Theorems 1.1 and 1.2, respectively. Extended remarks, explanations of difficulties, and a review of literature are given in Remarks 1.1, 1.2, 1.3 and 1.4.

The third question in our targets is on the impact of noise, which is one of the central questions in the study of stochastic partial differential equations (SPDEs). Regularization effects of noise have been observed for many different models. For example, it is known that the well-posedness of linear stochastic transport equations with noise can be established under weaker hypotheses than its deterministic counterpart, cf. [20]. Particularly, for the impact of linear noise in different models, we refer to [2, 14, 15, 26, 38, 47, 54].

Notably, the existing results on regularization by noise are largely restricted to linear equations or linear noise. Hence we have particular interest in the nonlinear noise case. Finding such noise is important as it helps us to understand the stabilizing mechanisms of noise. This is the first step to characterize relevant noise which provides regularization effects for the CH-type equations. In order to emphasize our ideas in a simple way, we only consider the noise as a 1-D Brownian motion in the current setting. That is, we consider the case that $h(t,u)\,\mathrm{d}\mathcal{W} = q(t,u)\,\mathrm{d}W$, where W is a standard 1-D Brownian motion and $q : [0,\infty) \times H^s \to H^s$ is a nonlinear function. Here we use the notation q rather than h because h needs to be a Hilbert–Schmidt operator [see (1.8)] to define the stochastic integral with respect to a cylindrical Wiener

process $\mathcal{W}$. Then we will focus on

$$\begin{cases} \mathrm{d}u + \left[u^k u_x + F(u)\right]\mathrm{d}t = q(t,u)\,\mathrm{d}W, \quad x \in \mathbb{R},\ t > 0,\ k \in \mathbb{N}_{>0}, \\ u(\omega, 0, x) = u_0(\omega, x), \quad x \in \mathbb{R}. \end{cases} \tag{1.6}$$

In Theorem 1.3, we provide a sufficient condition on q such that global existence can be guaranteed. We refer to Remark 1.5 for further remarks on Theorem 1.3.

Before we introduce the notations, definitions and assumptions, we recall some recent results on stochastic CH-type equations. For the stochastic CH type equation with multiplicative noise, we refer to [46–48], where global existence and wave breaking were studied in the periodic case, i.e., $x \in \mathbb{T}$. In particular, when the noise is of transport type, we refer to [1, 4, 22, 32, 33]. We also refer to [12, 13, 45] for more results in stochastic CH type equations.

1.1 Notations

We begin by introducing some notations. Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a right-continuous complete filtration probability space. Formally, we consider a separable Hilbert space $\mathfrak{U}$ and let $\{e_n\}$ be a complete orthonormal basis of $\mathfrak{U}$. Let $\{W_n\}_{n\geq 1}$ be a sequence of mutually independent standard 1-D Brownian motions on $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Then we define the cylindrical Wiener process $\mathcal{W}$ as

$$\mathcal{W} := \sum_{n=1}^{\infty} W_n e_n. \tag{1.7}$$

Let $\mathcal{X}$ be a separable Hilbert space. $\mathcal{L}_2(\mathfrak{U}; \mathcal{X})$ stands for the Hilbert-Schmidt operators from $\mathfrak{U}$ to $\mathcal{X}$. If $Z \in L^2(\Omega; L^2_{\mathrm{loc}}([0,\infty); \mathcal{L}_2(\mathfrak{U}; \mathcal{X})))$ is progressively measurable, then the integral

$$\int_0^t Z\,\mathrm{d}\mathcal{W} := \sum_{n=1}^{\infty} \int_0^t Z e_n\,\mathrm{d}W_n \tag{1.8}$$

is a well-defined $\mathcal{X}$-valued continuous square-integrable martingale [see [5, 23] for example]. Throughout the paper, when a stopping time is defined, we set $\inf \emptyset := \infty$ by convention.

For $s \in \mathbb{R}$, the differential operator $D^s := (1 - \partial_x^2)^{s/2}$ is defined by $\widehat{D^s f}(\xi) = (1+\xi^2)^{s/2}\widehat{f}(\xi)$, where $\widehat{f}$ denotes the Fourier transform of f. The Sobolev space $H^s(\mathbb{R})$ is defined as

$$H^s(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : \|f\|^2_{H^s(\mathbb{R})} := \int_{\mathbb{R}} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2\,\mathrm{d}\xi < +\infty \right\},$$

and the inner product on $H^s(\mathbb{R})$ is $(f,g)_{H^s} := (D^s f, D^s g)_{L^2}$. In the sequel, for simplicity, we will drop $\mathbb{R}$ if there is no ambiguity. We will use $\lesssim$ to denote estimates

that hold up to some universal *deterministic* constant which may change from line to line but whose meaning is clear from the context. For linear operators A and B, $[A, B] := AB - BA$ is the commutator of A and B.

1.2 Definitions and assumptions

We first make the precise notion of a solution to (1.4).

Definition 1.1 Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ be a fixed in advance. Let $s > 3/2$, $k \in \mathbb{N}_{>0}$ and u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable.

1. A local solution to (1.4) is a pair (u, τ), where τ is a stopping time satisfying $\mathbb{P}\{\tau > 0\} = 1$ and $u : \Omega \times [0, \infty) \to H^s$ is an $\mathcal{F}_t$-predictable H^s-valued process satisfying

$$u(\cdot \wedge \tau) \in C([0, \infty); H^s) \ \ \mathbb{P}\text{-a.s.},$$

and for all $t > 0$,

$$u(t \wedge \tau) - u(0) + \int_0^{t\wedge\tau} \left[u^k \partial_x u + F(u)\right] \mathrm{d}t' = \int_0^{t\wedge\tau} h(t', u)\, \mathrm{d}\mathcal{W} \ \ \mathbb{P}\text{-a.s.}$$

2. The local solutions are said to be unique, if given any two pairs of local solutions (u_1, τ_1) and (u_2, τ_2) with $\mathbb{P}\{u_1(0) = u_2(0)\} = 1$, we have

$$\mathbb{P}\{u_1(t, x) = u_2(t, x), \ (t, x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{R}\} = 1.$$

3. Additionally, (u, τ^*) is called a maximal solution to (1.4) if $\tau^* > 0$ almost surely and if there is an increasing sequence $\tau_n \to \tau^*$ such that for any $n \in \mathbb{N}$, (u, τ_n) is a solution to (1.4) and on the set $\{\tau^* < \infty\}$, we have

$$\sup_{t\in[0,\tau_n]} \|u\|_{H^s} \geq n.$$

4. If (u, τ^*) is a maximal solution and $\tau^* = \infty$ almost surely, then we say that the solution exists globally.

Motivated by [46, 49], we introduce the concept on stability of exiting time in Sobolev spaces. Exiting time, as its name would suggest, is defined as the time when solution leaves a certain range.

Definition 1.2 (Stability of exiting time) Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ be fixed, $s > 3/2$ and $k \in \mathbb{N}_{>0}$. Let u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. Assume that $\{u_{0,n}\}$ is a sequence of H^s-valued $\mathcal{F}_0$-measurable random variables satisfying $\mathbb{E}\|u_{0,n}\|_{H^s}^2 < \infty$. For each n, let u and u_n be the unique solutions to (1.4), as in Definition 1.1, with initial values u_0 and $u_{0,n}$, respectively. For any $R > 0$, define the R-exiting times

$$\tau_n^R := \inf\{t \geq 0 : \|u_n\|_{H^s} > R\}, \quad \tau^R := \inf\{t \geq 0 : \|u\|_{H^s} > R\}.$$

Now we define the following properties on stability:

1. If $u_{0,n} \to u_0$ in H^s $\mathbb{P}$-a.s. implies that

$$\lim_{n\to\infty} \tau_n^R = \tau^R \ \ \mathbb{P}\text{-a.s.}, \tag{1.9}$$

then the R-exiting time of u is said to be stable.
2. If $u_{0,n} \to u_0$ in $H^{s'}$ for all $s' < s$ almost surely implies that (1.9) holds true, the R-exiting time of u is said to be strongly stable.

Our main results rely on the following assumptions concerning the noise coefficient $h(t, u)$ in (1.1).

Hypothesis $\mathbf{H_1}$ For $s > 1/2$, we assume that $h : [0, \infty) \times H^s \ni (t, u) \mapsto h(t, u) \in \mathcal{L}_2(\mathfrak{U}; H^s)$ is measurable and satisfies the following conditions:

$H_1(1)$ There is a non-decreasing function $f(\cdot) : [0, +\infty) \to [0, +\infty)$ such that for any $u \in H^s$ with $s > 3/2$, we have the following growth condition

$$\sup_{t\geq 0} \|h(t, u)\|_{\mathcal{L}_2(\mathfrak{U};H^s)} \leq f(\|u\|_{W^{1,\infty}})(1 + \|u\|_{H^s}).$$

$H_1(2)$ There is a non-decreasing function $g_1(\cdot) : [0, \infty) \to [0, \infty)$ such that for all $N \geq 1$,

$$\sup_{t\geq 0,\, \|u\|_{H^s},\, \|v\|_{H^s}\leq N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|h(t, u) - h(t, v)\|_{\mathcal{L}_2(\mathfrak{U},H^s)}}{\|u - v\|_{H^s}} \right\} \leq g_1(N), \quad s > 3/2.$$

$H_1(3)$ There is a non-decreasing function $g_2(\cdot) : [0, \infty) \to [0, \infty)$ such that for all $N \geq 1$ and $3/2 \geq s > 1/2$,

$$\sup_{t\geq 0,\, \|u\|_{H^{s+1}},\, \|v\|_{H^{s+1}}\leq N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|h(t, u) - h(t, v)\|_{\mathcal{L}_2(\mathfrak{U},H^s)}}{\|u - v\|_{H^s}} \right\} \leq g_2(N).$$

Here we outline $H_1(2)$ is the classical local Lipschitz condition. $H_1(3)$ is needed to prove uniqueness in Lemma 3.1. Indeed, if one finds two solutions $u, v \in H^s$ to (1.4), one can only estimate $u - v$ in $H^{s'}$ for $s' \leq s - 1$ because the term $u^k u_x$ loses one derivative. We refer to Remark 1.1 for more details.

Hypothesis $\mathbf{H_2}$ When we consider (1.4) in Sect. 4, we assume that there is a real number $\rho_0 \in (1/2, 1)$ such that for $s \geq \rho_0$, $h : [0, \infty) \times H^s \ni (t, u) \mapsto h(t, u) \in \mathcal{L}_2(\mathfrak{U}; H^s)$ is measurable. Besides, we suppose the following:

$H_2(1)$ There exists a non-decreasing function $l(\cdot) : [0, +\infty) \to [0, +\infty)$ such that for any $u \in H^s$ with $s > 3/2$,

$$\sup_{t\geq 0} \|h(t, u)\|_{\mathcal{L}_2(\mathfrak{U};H^s)} \leq l(\|u\|_{W^{1,\infty}})\|u\|_{H^s},$$

and $H_1(2)$ holds.

$H_2(2)$ There is a non-decreasing function $g_3(\cdot) : [0, +\infty) \to [0, +\infty)$ such that for all $N \geq 1$,

$$\sup_{t\geq 0,\ \|u\|_{H^s}\leq N} \|h(t,u)\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})} \leq g_3(N)\mathrm{e}^{-\frac{1}{\|u\|_{H^{\rho_0}}}},\quad s > 3/2, \tag{1.10}$$

and

$$\sup_{t\geq 0,\ \|u\|_{H^{\rho_0}},\ \|v\|_{H^{\rho_0}}\leq N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|h(t,u)-h(t,v)\|_{\mathcal{L}_2(\mathfrak{U},H^{\rho_0})}}{\|u-v\|_{H^{\rho_0}}} \right\} \leq g_3(N).$$

We remark here that (1.10) means that there is a $\rho_0 \in (1/2, 1)$ such that, if u_n is bounded in H^s and u_n tends to zero in the topology of H^{ρ_0} as n tends to ∞, then $\|h(t,u_n)\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}$ tends to zero exponentially as n tends to ∞. Examples of such noise structure can be found in Sect. 4.4.

As for the regularization effect of noise, we impose the following condition on q in (1.6):

Hypothesis $\mathbf{H_3}$ We assume that when $s > 3/2, q : [0,\infty) \times H^s \ni (t,u) \mapsto q(t,u) \in H^s$ is measurable. Define the set $\mathcal{V}$ as a subset of $C^2([0,\infty);[0,\infty))$ such that

$$\mathcal{V} := \left\{ V(0) = 0,\ V'(x) > 0,\ V''(x) \leq 0 \text{ and } \lim_{x\to\infty} V(x) = \infty \right\}.$$

Then we assume the following:

$H_3(1)$ There is a non-decreasing function $g_4(\cdot) : [0, +\infty) \to [0, +\infty)$ such that for any $u \in H^s$ with $s > 3/2$, we have the following growth condition

$$\sup_{t\geq 0} \|q(t,u)\|_{H^s} \leq g_4(\|u\|_{W^{1,\infty}})(1 + \|u\|_{H^s}).$$

$H_3(2)$ $q(\cdot,u)$ is bounded for all $u \in H^s$ and there is a non-decreasing function $g_4(\cdot) : [0,\infty) \to [0,\infty)$, such that

$$\sup_{t\geq 0,\ \|u\|_{H^s},\ \|v\|_{H^s}\leq N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|q(t,u)-q(t,v)\|_{H^s}}{\|u-v\|_{H^s}} \right\} \leq g_4(N),\quad N \geq 1,\ s > 3/2.$$

$H_3(3)$ There is a $V \in \mathcal{V}$ and constants $N_1, N_2 > 0$ such that for all $(t,u) \in [0,\infty) \times H^s$ with $s > 3/2$,

$$\mathcal{H}_s(t,u) \leq N_1 - N_2 \frac{\left\{ V'(\|u\|_{H^s}^2) \left| (q(t,u),u)_{H^s} \right| \right\}^2}{1 + V(\|u\|_{H^s}^2)},$$

where

$$\begin{aligned}&\mathcal{H}_s(t,u)\\&:=V'(\|u\|_{H^s}^2)\left\{2\lambda_s\|u\|_{W^{1,\infty}}^k\|u\|_{H^s}^2+\|q(t,u)\|_{H^s}^2\right\}+2V''(\|u\|_{H^s}^2)\left|(q(t,u),u)_{H^s}\right|^2\end{aligned}$$

and $\lambda_s > 0$ is the constant given in Lemma A.6 below.

Examples of the noise structure satisfying Hypothesis H_3 can be found in Sect. 5.2.

1.3 Main results and remarks

Now we summarize our major contributions providing proofs later in the remainder of the paper.

Theorem 1.1 *Let $s > 3/2$, $k \geq 1$ and let $h(t,u)$ satisfy Hypothesis H_1. Assume that u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. Then*

(i) (*Existence and uniqueness*) *There is a unique local solution (u,τ) to (1.4) in the sense of Definition 1.1 with*

$$\mathbb{E}\sup_{t\in[0,\tau]}\|u(t)\|_{H^s}^2 < \infty. \tag{1.11}$$

(ii) (*Blow-up criterion*) *The local solution (u,τ) can be extended to a unique maximal solution (u,τ^*) with*

$$\boldsymbol{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\}} = \boldsymbol{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{W^{1,\infty}}=\infty\}}\ \mathbb{P}\text{-a.s.} \tag{1.12}$$

(iii) (*Stability for almost surely bounded initial data*) *Assume additionally that $u_0 \in L^\infty(\Omega;H^s)$. Let $v_0 \in L^\infty(\Omega;H^s)$ be another H^s-valued $\mathcal{F}_0$-measurable random variable. For any $T > 0$ and any $\epsilon > 0$, there is a $\delta = \delta(\epsilon,u_0,T) > 0$ such that if*

$$\|u_0 - v_0\|_{L^\infty(\Omega;H^s)} < \delta, \tag{1.13}$$

then there is a stopping time $\tau \in (0,T]$ $\mathbb{P}$-a.s. and

$$\mathbb{E}\sup_{t\in[0,\tau]}\|u(t)-v(t)\|_{H^s}^2 < \epsilon, \tag{1.14}$$

where u and v are the solutions to (1.4) with initial data u_0 and v_0, respectively.

Remark 1.1 Existence and uniqueness have been studied for abundant SPDEs. In many works, the authors did not address the continuous dependence on initial data. In this work, our Theorem 1.1 provides a local well-posedness result in the sense of Hadamard including the continuous dependence on initial data. Moreover, a blow-up criterion is also obtained. We refer to [11, 19, 42] for the study about the dependence on the

initial data for cases that solutions to the target problems exist globally. However, it is necessary to point out that almost nothing is known on the analysis for dependence on initial data for SPDEs whose solutions may blow up in finite time.

The key difficulty for such a case is as follows: on one hand, if solutions to a nonlinear stochastic partial differential equation (SPDE) blow up in finite time, it is usually very difficult to obtain the lifespan estimates. On the other hand, we have to find a positive time τ to obtain an inequality like (1.14). In addition, the target problem (1.4) is more difficult because the classical Itô formulae are *not* applicable. Indeed, for $u_0 \in H^s$, we can *only* know $u \in H^s$ because this is a transport type equation, then $u^k u_x \in H^{s-1}$. However, the inner product $\left(u^k u_x, u\right)_{H^s}$ appears if one uses the Itô formula in a Hilbert space (cf. [23, Theorem 2.10]) and the dual product ${}_{H^{s-1}}\langle u^k u_x, u\rangle_{H^{s+1}}$ appears in the Itô formula under a Gelfand triplet (cf. [39, Theorem I.3.1]). Since we only have $u \in H^s$ and $u^k u_x \in H^{s-1}$, neither of them are well-defined. Likewise, when we consider the H^s-norm for the difference between two solutions $u, v \in H^s$ to (1.4), we will have to handle $(u^k u_x - v^k v_x, u - v)_{H^s}$, which gives rise to control either $\|u\|_{H^{s+1}}$ or $\|v\|_{H^{s+1}}$.

Remark 1.2 Now we list some technical remarks on the statements of Theorem 1.1.

(1). Our proof for (i) in Theorem 1.1 is motivated by the recent results in [55]. For the convenience of the reader, here we also give a brief comparison between our approach and the framework employed in many previous works.

– We first briefly review the martingale approach used to prove existence of nonlinear SPDEs. Roughly speaking, in searching for a solution to a nonlinear SPDE in some space $\mathcal{X}$, the martingale approach, as its name would suggest, includes obtaining martingale solution first and then establishing (pathwise) uniqueness to obtain the (pathwise) solution. To begin with, one needs to approximate the equation and establish uniform estimate. For nonlinear problems, one may have to add a *cut-off* function to cut the nonlinear parts growing in some space $\mathcal{Z}$ with $\mathcal{X} \hookrightarrow \mathcal{Z}$ (such choice of $\mathcal{Z}$ depends on concrete problems). As far as we know, the technique of *cut-off* first appears in [17] for the stochastic Schrödinger equation. This *cut-off* enables us to split the expectation of nonlinear terms, and then the $L^2(\Omega; \mathcal{X})$ estimate can be closed. For example, for (1.4), the estimate for $\mathbb{E}\|u\|_{H^s}^2$ will give rise to $\mathbb{E}\left(\|u\|_{W^{1,\infty}}\|u\|_{H^s}^2\right)$, and hence we need to add a function to cut $\|\cdot\|_{W^{1,\infty}}$. With this additional *cut-off*, we need to consider the *cut-off* version of the problem first and remove it then. The first main step in the martingale approach is finding a martingale solution. Usually, this can be done by first obtaining tightness of the measures defined by the approximative solutions in some space $\mathcal{Y}$, and then using Prokhorov's Theorem and Skorokhod's Theorem to obtain the convergence in $\mathcal{Y}$. Since $\mathcal{X}$ is usually infinite dimensional (usually, $\mathcal{X}$ is a Sobolev space), to obtain tightness, it is required that $\mathcal{X}$ is compactly embedded into $\mathcal{Y}$, i.e, $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y}$. This brings another requirement to specify $\mathcal{Z}$, that is, $\mathcal{Y} \hookrightarrow \mathcal{Z}$. Otherwise, taking limits will *not* bring us back to the *cut-off* problem due to the additional *cut-off* term $\|\cdot\|_{\mathcal{Z}}$ (in some cases, the choice of $\mathcal{Z}$ may only give rise to a semi-norm and here we use this notation $\|\cdot\|_{\mathcal{Z}}$ only for simplicity). Usually, in bounded

domains, it is not difficult to pick $\mathcal{Y}$ and $\mathcal{Z}$ such that $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$ (Sobolev spaces enjoy compact embeddings in bounded domains), see for example [4, 9, 18, 26, 48]. In unbounded domains, the difficulty lies in the choice of $\mathcal{Y}$ and $\mathcal{Z}$ such that $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$. We refer to [7, 8] for fluid models with certain cancellation properties (for example, divergence free) and linear growing noise. However, it is difficult to achieve this for SPDEs with general nonlinear terms and nonlinear noise. For instance, the *cut-off* in our case will have to involve $\|\cdot\|_{\mathcal{Z}} = \|\cdot\|_{W^{1,\infty}}$ [see $H_1(1)$ and (2.3)]. Even though we can get the convergence in $H^{s'}_{\text{loc}}$ with some $\frac{3}{2} < s' < s$, it is still not clear whether the convergence holds true in $W^{1,\infty}$, and this is because *local* convergence can *not* control a *global* object $\|\cdot\|_{W^{1,\infty}}$. Therefore, technically speaking, nonlinear SPDEs are more non-local than its deterministic counterpart.

– Due to the above unsolved technical issue, the martingale approach is difficult to apply in our problem and we will try to prove convergence directly, which is motivated by [41, 55] [see also [49, 53, 54] for recent developments]. Generally speaking, we will analyze the difference between two approximative solutions and *directly* find a space $\mathcal{Y}$ such that $\mathcal{X} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$ and convergence (up to a subsequence) holds true in $\mathcal{Y}$. The difficult part is finding convergence in $\mathcal{Y}$ *without* compactness $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y}$ (compared to the martingale approach, tightness comes from the compact embedding $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y}$). In this paper, the target path space is $C([0,T];H^s) = \mathcal{X}$, and we are able to prove convergence (up to a subsequence) in $C([0,T];H^{s-\frac{3}{2}}) = \mathcal{Y}$ directly. After taking limits to obtain a solution, one can improve the regularity to H^s again, and the technical difficulty in this step is to prove the time continuity of the solution because the classical Itô formula is *not* applicable (see in Remark 1.1). To overcome this difficulty, we apply a mollifier J_ε to equation and estimate $\mathbb{E}\|J_\varepsilon u\|^2_{H^s}$ first [see (2.11)]. We also remark that the techniques in removing the *cut-off* have been used in [5, 25, 54]. Here we formulate such a technical result in Lemma A.7 in an abstract way.

(2). Now we give a remark on (iii) in Theorem 1.1. For the question on dependence on initial data, there are some delicate differences between the stochastic and the deterministic case. In the deterministic counterpart of (1.4), due to the lifespan estimate [see (4.10) for instance], for given $u_0 \in H^s$, it can be shown that if $\|u_0 - v_0\|_{H^s}$ is small enough, then there is a $T > 0$ depending on u_0 such that $\sup_{t\in[0,T]} \|u(t) - v(t)\|^2_{H^s}$ is also small. In stochastic setting, since existence and uniqueness are obtained in the framework of $L^2(\Omega; H^s)$, it is therefore very natural to expect that, for given $u_0 \in L^2(\Omega; H^s)$, if $\mathbb{E}\|u_0 - v_0\|^2_{H^s}$ is small enough, then for some almost surely positive τ depending u_0, $\mathbb{E}\sup_{t\in[0,\tau]} \|u(t) - v(t)\|^2_{H^s}$ is also small. However, so far we have only proved it with assuming the smallness of $\|u_0 - v_0\|_{L^\infty(\Omega;H^s)}$. Since $L^\infty(\Omega; H^s)$ can be viewed as being *less* random than $L^2(\Omega; H^s)$, one may roughly conclude that what the solution map needs to be continuous/stable (the initial data and its perturbation are $L^\infty(\Omega; H^s)$) is more "picky" in determinism than what the existence of such a solution map requires

(existence and uniqueness guarantee that a solution map can be defined). For the technical difficulties involved, we have the following explanations:

– As is mentioned in Remark 1.1, when we estimate the H^s-norm for the difference between two solutions u and v, H^{s+1}-norm will appear. Hence, we have to use smooth approximations to make the analysis valid. More precisely, we approximate u and v by smooth process u_ε and v_ε and consider

$$\|u - v\|_{H^s} \le \|u - u_\varepsilon\|_{H^s} + \|u_\varepsilon - v_\varepsilon\|_{H^s} + \|v_\varepsilon - v\|_{H^s}. \tag{1.15}$$

Then all terms can be estimated because $\|u_\varepsilon\|_{H^{s+1}}$ and $\|v_\varepsilon\|_{H^{s+1}}$ make sense. Here we refer to Remark 3.2 for more details on the construction of such an approximation.

– In dealing with the above three terms in the stochastic case, two sequences of stopping times (exiting times) are needed to control $\|u_\varepsilon\|_{H^s}$ and $\|v_\varepsilon\|_{H^s}$ [see (3.20) below]. Since we aim at obtaining $\tau > 0$ almost surely in (1.14) (otherwise the difference between two solutions on the set $\{\tau = 0\}$ can *not* be measured), we will have to guarantee that those stopping times used in bounding $\|u_\varepsilon\|_{H^s}$ and $\|v_\varepsilon\|_{H^s}$ have positive lower bounds almost surely. Up to now, we have only achieved this for initial values belonging to $L^\infty(\Omega; H^s)$. We also remark that this is different from the proof for existence. In the proof for existence, u_ε exists on a common interval $[0, T]$ for all ε and enjoys a uniform-in-ε estimate (2.4), hence we can get rid of stopping times in convergence (from (2.8) to (2.9)). Here we do *not* have such common existence interval due to the lack of a lifespan estimate, which is a significant difference between the stochastic and the deterministic cases. Indeed, we can easily find the lifespan estimate for the deterministic counterpart of (1.4) [see (4.10) below].

– Moreover, even if the above issue can be handled, in dealing with the three terms in (1.15), we are confronted with $\frac{1}{\varepsilon^2}\mathbb{E}\|u_0 - v_0\|^2_{H^{s'}}\|u_0\|^2_{H^s}$ for some suitably chosen s' (cf. (3.27)). After ε is fixed, the smallness of $\mathbb{E}\|u_0 - v_0\|^2_{H^s}$ is *not* enough to control $\frac{1}{\varepsilon^2}\mathbb{E}\|u_0 - v_0\|^2_{H^{s'}}\|u_0\|^2_{H^s}$, either. We use the $L^\infty(\Omega; H^s)$ condition to take $\|u_0\|^2_{H^s}$ out of $\frac{1}{\varepsilon^2}\mathbb{E}\|u_0 - v_0\|^2_{H^{s'}}\|u_0\|^2_{H^s}$. In deterministic case, no expectation is involved, $\frac{1}{\varepsilon^2}\|u_0 - v_0\|^2_{H^{s'}}\|u_0\|^2_{H^s}$ can be controlled by $\|u_0 - v_0\|^2_{H^s}$.

Roughly speaking, (iii) in Theorem 1.1 means that for any fixed $u_0 \in L^\infty(\Omega; H^s)$ and any $T > 0$, if $\|u_0 - v_0\|_{L^\infty(\Omega; H^s)} \to 0$, then

$$\exists\, \tau \in (0, T]\ \mathbb{P}\text{-a.s. such that } \mathbb{E}\|u(\cdot \wedge \tau) - v(\cdot \wedge \tau)\|^2_{C([0,T]; H^s)} \to 0,$$

where u, v are solutions corresponding to u_0, v_0, respectively. Below we will study this issue quantitatively. The next result addresses at least a partially negative answer.

Theorem 1.2 (Weak instability) *Let $s > 5/2$ and $k \ge 1$. If h satisfies Hypothesis H_2, then at least one of the following properties holds true:*

(i) For any $R \gg 1$, the R-exiting time is not strongly stable for the zero solution to (1.4) in the sense of Definition 1.2;

(ii) There is a $T > 0$ such that the solution map $u_0 \mapsto u$ defined by (1.4) is not uniformly continuous as a map from $L^\infty(\Omega; H^s)$ into $L^1(\Omega; C([0,T]; H^s))$. More precisely, there exist two sequences of solutions $u^{1,n}$ and $u^{2,n}$, and two sequences of stopping time $\tau_{1,n}$ and $\tau_{2,n}$, such that

– *For $i = 1, 2$, $\mathbb{P}\{\tau_{i,n} > 0\} = 1$ for each $n > 1$. Besides,*

$$\lim_{n\to\infty} \tau_{1,n} = \lim_{n\to\infty} \tau_{2,n} = \infty \ \mathbb{P}\text{-a.s.} \tag{1.16}$$

– *For $i = 1, 2$, $u^{i,n} \in C([0, \tau_{i,n}]; H^s)$ $\mathbb{P}$-a.s., and*

$$\mathbb{E}\left(\sup_{t\in[0,\tau_{1,n}]} \|u^{1,n}(t)\|_{H^s} + \sup_{t\in[0,\tau_{2,n}]} \|u^{2,n}(t)\|_{H^s} \right) \lesssim 1. \tag{1.17}$$

– *At initial time $t = 0$, for any $p \in [1, \infty]$,*

$$\lim_{n\to\infty} \|u^{1,n}(0) - u^{2,n}(0)\|_{L^p(\Omega;H^s)} = 0. \tag{1.18}$$

– *When $t > 0$,*

$$\liminf_{n\to\infty} \mathbb{E} \sup_{t\in[0,T\wedge\tau_{1,n}\wedge\tau_{2,n}]} \|u^{1,n}(t) - u^{2,n}(t)\|_{H^s} \gtrsim \begin{cases} \sup\limits_{t\in[0,T]} |\sin(t)|, & \text{if } k \text{ is odd}, \\ \sup\limits_{t\in[0,T]} \left|\sin\left(\frac{t}{2}\right)\right|, & \text{if } k \text{ is even}. \end{cases} \tag{1.19}$$

Remark 1.3 We first briefly outline the main difficulties encountered in the proof for Theorem 1.2 and the main strategies we used.

(1). Because we can *not* get an explicit expression of the solution to (1.4), to obtain (1.19), we will construct two sequences of approximative solutions $\{u_{m,n}\}$ ($m \in \{1, 2\}$) such that the actual solutions $\{u^{m,n}\}$ with $u^{m,n}(0) = u_{m,n}(0)$ satisfy

$$\lim_{n\to\infty} \mathbb{E} \sup_{[0,\tau_{m,n}]} \|u^{m,n} - u_{m,n}\|_{H^s} = 0, \tag{1.20}$$

where $u^{m,n}$ exists at least on $[0, \tau_{m,n}]$. Then, one can establish (1.19) by estimating $\{u_{m,n}\}$ rather than $\{u^{m,n}\}$. We also remark that the construction of approximative solution $u_{m,n}$ for $x \in \mathbb{R}$ is more difficult than the construction of approximative solution for $x \in \mathbb{T}$ [see [46]] since the approximative solution involves both high and low frequency parts (high frequency part is already enough for the case $x \in \mathbb{T}$, cf. [46, 55]). The key point is that we need to guarantee $\inf_n \tau_{m,n} > 0$ almost surely in dealing with (1.20). Hence we are confronted with a common difficulty

in SPDEs again, that is, the lack of lifespan estimate. In deterministic cases, one can easily obtain the lifespan estimate, which enables us to find a common interval $[0, T]$ such that all actual solutions exist on $[0, T]$ (see for example Lemma 4.1). In the stochastic case, so far we have *not* been able to prove this.

(2). To settle the above difficulty, we observe that the bound $\inf_n \tau_{m,n} > 0$ can be connected to the stability property of the exiting time (see Definition 1.2). The condition that the R_0-exiting time is strongly stable at the zero solution will be used to provide a common existence time $T > 0$ such that for all n, $u^{m,n}$ exists up to T (see Lemma 4.4 below). Therefore, to prove Theorem 1.2, we will show that, if the R_0-exiting time is strongly stable at the zero solution for some $R_0 \gg 1$, then the solution map $u_0 \mapsto u$ defined by (1.4) can *not* be uniformly continuous. To get (1.20), we estimate the error in $H^{2s-\rho_0}$ and H^{ρ_0}, respectively, where ρ_0 is given in H_2. Then (1.20) is a consequence of the interpolation. We remark that (1.18) holds because the approximative solutions are constructed deterministically.

Remark 1.4 With regard to similar results in the literature and further hypotheses, we give some more remarks on Theorem 1.2.

(1). In deterministic cases, the issue of the (optimal) initial-data dependence of solutions has been extensively investigated for various nonlinear dispersive and integrable equations. We refer to [35] for the inviscid Burgers equation and to [37] for the Benjamin–Ono equation. For the CH equation we refer the readers to [29, 30] concerning the non-uniform dependence on initial data in Sobolev spaces H^s with $s > 3/2$. For the first results of this type in Besov spaces, we refer to [50, 56]. Particularly, non-uniform dependence on initial data in critical Besov space first appears in [51, 52]. In this work, Theorem 1.2 and (iii) in Theorem 1.1 demonstrate that the continuity of the solution map $u_0 \mapsto u$ is almost an optimal result in the sense that, when the growth of the noise coefficient satisfies certain conditions (cf. Hypothesis H_2), the map $u_0 \mapsto u$ is continuous, but one can *not* improve the stability of the exiting time and simultaneously the continuity of the map $u_0 \mapsto u$. Up to our knowledge, results of this type for SPDEs first appeared in [46, 49]. We also refer to [3, 43, 55] for recent developments.

(2). It is worthwhile mentioning that, as noted in **(1)** of Remark 1.3, the strong stability of exiting times is used as a technical "assumption" to handle the lower bound of a sequence of stopping times. So far we have not been able to verify the non-emptyness of this strong stability assumption for the current model. However, if the transport noise $u_x \circ \mathrm{d}W$ is considered (W is a standard 1-D Brownian motion and $\circ \mathrm{d}W$ means the Stratonovich stochastic differential), we might conjecture that either the notion of strong stability of exiting times can be captured, or the solution map $u_0 \mapsto u$ can become more regular than being continuous. Indeed, if $h(t, u)\,\mathrm{d}\mathcal{W}$ is replaced by $u_x \circ \mathrm{d}W$ in (1.4), one can rewrite the equation into Itô's form with an additional viscous term $-\frac{1}{2}u_{xx}$ on the left hand side of the equation. Therefore, it is reasonable to expect that in this case, either the strong stability of exiting times or the continuity of the solution map $u_0 \mapsto u$ can be improved. We refer to [31] and [27] for deterministic examples on the continuity of the solution map.

Theorem 1.3 (Noise prevents blow-up) *Let $s > 5/2$, $k \geq 1$ and $u_0 \in H^s$ be an $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. If Hypothesis H_3 holds true, then the corresponding maximal solution (u, τ^*) to (1.6) satisfies*

$$\mathbb{P}\left\{\tau^* = \infty\right\} = 1.$$

Remark 1.5 We notice that many of the existing results on regularization effects by noise are essentially restricted to linear equations or linear growing noise. In Theorem 1.3, both the drift and diffusion term are nonlinear. We also remark that the blow-up can actually occur in the deterministic counterpart of (1.6). For example, when $k = 1$, blow-up (as wave breaking) of solutions to the CH equation can be found in [16]. Therefore, Theorem 1.3 demonstrates that large enough noise can prevent singularities. Indeed, $H_3(3)$ means that the growth of $u^k u_x + F(u)$ can be controlled provided that the noise grows fast enough in terms of a Lyapunov type function V. In contrast to $H_1(2)$ and $H_1(3)$, we require $s > 3/2$ in both $H_3(2)$ and $H_3(3)$. As is stated in Hypothesis H_1, $H_3(2)$ implies that uniqueness holds true for solutions in H^s with $s > 5/2$. It seems that one can require $s > 1/2$ in $H_3(2)$ to guarantee uniqueness in H^ρ with $\rho > 3/2$, but at present we can only construct examples for the case that $s > 3/2$ is required in both $H_3(2)$ and $H_3(3)$.

We outline the remainder of the paper. In Sect. 2, we study the cut-off version of (1.4) and then we remove the cut-off and prove Theorem 1.1 in Sect. 3. We prove Theorem 1.2 in Sect. 4. Concerning the interplay of noise vs blow-up, we prove Theorem 1.3 in Sect. 5.

2 Cut-off version: regular solutions

We first consider a cut-off version of (1.4). To this end, for any $R > 1$, we let $\chi_R(x) : [0, \infty) \to [0, 1]$ be a C^∞-function such that $\chi_R(x) = 1$ for $x \in [0, R]$ and $\chi_R(x) = 0$ for $x > 2R$. Then we consider the following cut-off problem

$$\begin{cases} \mathrm{d}u + \chi_R(\|u\|_{W^{1,\infty}})\left[u^k \partial_x u + F(u)\right]\mathrm{d}t = \chi_R(\|u\|_{W^{1,\infty}})h(t,u)\,\mathrm{d}\mathcal{W}, \\ u(\omega, 0, x) = u_0(\omega, x) \in H^s. \end{cases} \tag{2.1}$$

In this section, we aim at proving the following result:

Proposition 2.1 *Let $s > 3$, $k \geq 1$, $R > 1$ and Hypothesis H_1 be satisfied. Assume that $u_0 \in L^2(\Omega; H^s)$ is an H^s-valued $\mathcal{F}_0$-measurable random variable. Then, for any $T > 0$, (2.1) has a solution $u \in L^2(\Omega; C([0, T]; H^s))$. More precisely, there is a constant $C(R, T, u_0) > 0$ such that*

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^s}^2 \leq C(R, T, u_0). \tag{2.2}$$

The proof for Proposition 2.1 is given in the following subsections.

2.1 The approximation scheme

The first step is to construct a suitable approximation scheme. From Lemma A.5, we see that the nonlinear term $F(u)$ preserves the H^s-regularity of $u \in H^s$ for any $s > 3/2$. However, to apply the theory of SDEs in Hilbert space to (2.1), we will have to mollify the transport term $u^k \partial_x u$ since the product $u^k \partial_x u$ loses one regularity. To this end, we consider the following approximation scheme:

$$\begin{cases} \mathrm{d}u + H_{1,\varepsilon}(u)\,\mathrm{d}t = H_2(t,u)\,\mathrm{d}\mathcal{W}, \\ \qquad H_{1,\varepsilon}(u) = \chi_R(\|u\|_{W^{1,\infty}})\left[J_\varepsilon\left((J_\varepsilon u)^k \partial_x J_\varepsilon u\right) + F(u)\right], \\ \qquad H_2(t,u) = \chi_R(\|u\|_{W^{1,\infty}})h(t,u), \\ \qquad u(0,x) = u_0(x) \in H^s, \end{cases} \tag{2.3}$$

where J_ε is the Friedrichs mollifier defined in Appendix A. After mollifying the transport term $u^k \partial_x u$, it follows from $H_1(2)$ and Lemmas A.1 and A.5 that for any $\varepsilon \in (0,1)$, $H_{1,\varepsilon}(\cdot)$ and $H_2(t,\cdot)$ are locally Lipschitz continuous in H^s with $s > \frac{3}{2}$. Besides, we notice that the cut-off function $\chi_R(\|\cdot\|_{W^{1,\infty}})$ guarantees the linear growth condition (cf. Lemma A.5 and $H_1(1)$). Thus, for fixed $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ and for $u_0 \in L^2(\Omega; H^s)$ with $s > 3/2$, the existence theory of SDE in Hilbert space (see for example [23]) means that (2.3) admits a unique solution $u_\varepsilon \in C([0,\infty); H^s)$ $\mathbb{P}$-a.s..

2.2 Uniform estimates

Now we establish some uniform-in-ε estimates for u_ε.

Lemma 2.1 *Let $k \geq 1$, $s > 3/2$, $R > 1$ and $\varepsilon \in (0,1)$. Assume that h satisfies Hypothesis H_1 and $u_0 \in L^2(\Omega; H^s)$ is an H^s-valued $\mathcal{F}_0$-measurable random variable. Let $u_\varepsilon \in C([0,\infty); H^s)$ be the unique solution to (2.3). Then for any $T > 0$, there is a constant $C = C(R,T,u_0) > 0$ such that*

$$\sup_{\varepsilon>0} \mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s}^2 \leq C. \tag{2.4}$$

Proof Using the Itô formula for $\|u_\varepsilon\|_{H^s}^2$, we have that for any $t > 0$,

$$\begin{aligned} \mathrm{d}\|u_\varepsilon(t)\|_{H^s}^2 =\ & 2\chi_R\left(\|u_\varepsilon\|_{W^{1,\infty}}\right)\left(h(t,u_\varepsilon)\,\mathrm{d}\mathcal{W}, u_\varepsilon\right)_{H^s} \\ & - 2\chi_R\left(\|u_\varepsilon\|_{W^{1,\infty}}\right)\left(D^s J_\varepsilon\left[(J_\varepsilon u_\varepsilon)^k \partial_x J_\varepsilon u_\varepsilon\right], D^s u_\varepsilon\right)_{L^2}\mathrm{d}t \\ & - 2\chi_R\left(\|u_\varepsilon\|_{W^{1,\infty}}\right)\left(D^s F(u_\varepsilon), D^s u_\varepsilon\right)_{L^2}\mathrm{d}t \\ & + \chi_R^2(\|u_\varepsilon\|_{W^{1,\infty}})\|h(t,u_\varepsilon)\|_{\mathcal{L}_2(\mathfrak{U};H^s)}^2\,\mathrm{d}t. \end{aligned}$$

On account of Lemmas A.1 and A.3, we derive

$$\left|\left(D^s J_\varepsilon\left[(J_\varepsilon u_\varepsilon)^k \partial_x J_\varepsilon u_\varepsilon\right], D^s u_\varepsilon\right)_{L^2}\right| \leq C\|u_\varepsilon\|_{W^{1,\infty}}^k \|u_\varepsilon\|_{H^s}^2,$$

Therefore, one can infer from the BDG inequality, $H_1(1)$, Lemma A.5 and the above estimate that

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2-\mathbb{E}\|u_0\|_{H^s}^2\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon\|_{H^s}^2+C_R\mathbb{E}\int_0^T\left(1+\|u_\varepsilon\|_{H^s}^2\right)\mathrm{d}t,$$

which implies

$$\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}^2\le 2\mathbb{E}\|u_0\|_{H^s}^2+C_R\int_0^T\left(1+\mathbb{E}\sup_{t'\in[0,t]}\|u_\varepsilon(t')\|_{H^s}^2\right)\mathrm{d}t. \quad (2.5)$$

Using Grönwall's inequality in (2.5) implies (2.4). □

2.3 Convergence of approximative solutions

Now we are going to show that the family $\{u_\varepsilon\}$ contains a convergent subsequence. For different layers u_ε and u_η, we see that $v_{\varepsilon,\eta}:=u_\varepsilon-u_\eta$ satisfies the following problem:

$$\mathrm{d}v_{\varepsilon,\eta}+\Big(\sum_{i=1}^{8}q_i\Big)\mathrm{d}t=\Big(\sum_{i=9}^{10}q_i\Big)\mathrm{d}\mathcal{W},\quad v_{\varepsilon,\eta}(0,x)=0, \quad (2.6)$$

where

$$\begin{aligned}
q_1&:=\left[\chi_R(\|u_\varepsilon\|_{W^{1,\infty}})-\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)\right]J_\varepsilon[(J_\varepsilon u_\varepsilon)^k\partial_x J_\varepsilon u_\varepsilon],\\
q_2&:=\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)(J_\varepsilon-J_\eta)[(J_\varepsilon u_\varepsilon)^k\partial_x J_\varepsilon u_\varepsilon],\\
q_3&:=\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)J_\eta[((J_\varepsilon u_\varepsilon)^k-(J_\eta u_\varepsilon)^k)\partial_x J_\varepsilon u_\varepsilon],\\
q_4&:=\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)J_\eta[((J_\eta u_\varepsilon)^k-(J_\eta u_\eta)^k)\partial_x J_\varepsilon u_\varepsilon],\\
q_5&:=\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)J_\eta[(J_\eta u_\eta)^k\partial_x(J_\varepsilon-J_\eta)u_\varepsilon],\\
q_6&:=\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)J_\eta[(J_\eta u_\eta)^k\partial_x J_\eta(u_\varepsilon-u_\eta)],\\
q_7&:=\left[\chi_R(\|u_\varepsilon\|_{W^{1,\infty}})-\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)\right]F(u_\varepsilon),\\
q_8&:=\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)[F(u_\varepsilon)-F(u_\eta)],\\
q_9&:=\left[\chi_R(\|u_\varepsilon\|_{W^{1,\infty}})-\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)\right]h(t,u_\varepsilon),\\
q_{10}&:=\chi_R\left(\|u_\eta\|_{W^{1,\infty}}\right)[h(t,u_\varepsilon)-h(t,u_\eta)].
\end{aligned}$$

Lemma 2.2 *Let $s>3$ and $k\ge 1$ and let $\mathcal{G}(x):=x^{2k+2}+1$. For any $\varepsilon,\eta\in(0,1)$, we find a constant $C>0$ such that*

$$\sum_{i=1}^{8}\left|(q_i,v_{\varepsilon,\eta})_{H^{s-\frac{3}{2}}}\right|\le C\mathcal{G}\left(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\right)\left(\|v_{\varepsilon,\eta}\|_{H^{s-\frac{3}{2}}}^2+\max\{\varepsilon,\eta\}\right).$$

Proof Using Lemmas A.1, A.3 and A.5, the mean value theorem for $\chi_R(\cdot)$, and the embedding $H^{s-\frac{3}{2}} \hookrightarrow W^{1,\infty}$, we have that for some $C > 0$,

$$\left\| D^{s-\frac{3}{2}} q_1 \right\|_{L^2}, \ \left\| D^{s-\frac{3}{2}} q_7 \right\|_{L^2} \leq C \|v_{\varepsilon,\eta}\|_{H^{s-\frac{3}{2}}} \|u_\varepsilon\|_{H^s}^{k+1},$$

and

$$\left\| D^{s-\frac{3}{2}} q_8 \right\|_{L^2} \leq C \left(\|u_\varepsilon\|_{H^s} + \|u_\eta\|_{H^s} \right)^k \|v_{\varepsilon,\eta}\|_{H^{s-\frac{3}{2}}}.$$

Using Lemma A.1, we see that

$$\begin{aligned}
\left\| D^{s-\frac{3}{2}} q_i \right\|_{L^2} &\leq C \max\{\varepsilon^{1/2}, \eta^{1/2}\} \|u_\varepsilon\|_{H^s}^{k+1}, \ i = 2, 3, \\
\left\| D^{s-\frac{3}{2}} q_4 \right\|_{L^2} &\leq C \left(\|u_\varepsilon\|_{H^s} + \|u_\eta\|_{H^s} \right)^{k-1} \|v_{\varepsilon,\eta}\|_{H^{s-\frac{3}{2}}} \|u_\varepsilon\|_{H^s}, \\
\left\| D^{s-\frac{3}{2}} q_5 \right\|_{L^2} &\leq C \max\{\varepsilon^{1/2}, \eta^{1/2}\} \|u_\varepsilon\|_{H^s} \|u_\eta\|_{H^s}^k.
\end{aligned}$$

For q_6, using Lemma A.1 and then integrating by part, we have

$$\begin{aligned}
&\left(D^{s-\frac{3}{2}} q_6, D^{s-\frac{3}{2}} v_{\varepsilon,\eta} \right)_{L^2} \\
&= \chi_R \left(\|u_\eta\|_{W^{1,\infty}} \right) \int_{\mathbb{R}} [D^{s-\frac{3}{2}}, (J_\eta u_\eta)^k] \partial_x J_\eta v_{\varepsilon,\eta} \cdot D^{s-\frac{3}{2}} J_\eta v_{\varepsilon,\eta} \, \mathrm{d}x \\
&\quad - \frac{1}{2} \chi_R \left(\|u_\eta\|_{W^{1,\infty}} \right) \int_{\mathbb{R}} \partial_x (J_\eta u_\eta)^k (D^{s-\frac{3}{2}} J_\eta v_{\varepsilon,\eta})^2 \, \mathrm{d}x.
\end{aligned}$$

Via the embedding $H^{s-\frac{3}{2}} \hookrightarrow W^{1,\infty}$ and Lemmas A.1 and A.3, we obtain

$$\left| \left(D^{s-\frac{3}{2}} q_6, D^{s-\frac{3}{2}} v_{\varepsilon,\eta} \right)_{L^2} \right| \lesssim \|u_\eta\|_{H^s}^k \|v_{\varepsilon,\eta}\|_{H^{s-\frac{3}{2}}}^2.$$

Therefore, we can put this all together to find

$$\begin{aligned}
\sum_{i=1}^{8} \left| (q_i, v_{\varepsilon,\eta})_{H^{s-\frac{3}{2}}} \right| \leq\ & C \left((\|u_\varepsilon\|_{H^s} + \|u_\eta\|_{H^s})^{k+1} + 1 \right) \|v_{\varepsilon,\eta}\|_{H^{s-\frac{3}{2}}}^2 \\
& + C (\|u_\varepsilon\|_{H^s} + \|u_\eta\|_{H^s})^{2k+2} \max\{\varepsilon, \eta\},
\end{aligned}$$

which gives rise to the desired estimate. □

Lemma 2.3 *Let $s > 3$, $R > 1$ and $\varepsilon \in (0, 1)$. For any $T > 0$ and $K > 1$, we define*

$$\tau_{\varepsilon,K}^T := \inf \{ t \geq 0 : \|u_\varepsilon(t)\|_{H^s} \geq K \} \wedge T, \quad \tau_{\varepsilon,\eta,K}^T := \tau_{\varepsilon,K}^T \wedge \tau_{\eta,K}^T. \tag{2.7}$$

Then we have

$$\lim_{\varepsilon\to 0}\sup_{\eta\le\varepsilon}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}=0. \tag{2.8}$$

Proof By employing the BDG inequality to (2.6), for some constant $C>0$, we arrive at

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-\frac{3}{2}}}\\
&\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac{3}{2}}}+C\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\sum_{i=1}^{8}\left|(q_i,v_{\varepsilon,\eta})_{H^{s-\frac{3}{2}}}\right|\mathrm{d}t\\
&\quad+C\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\sum_{i=9}^{10}\|q_i\|^2_{\mathcal{L}_2\left(\mathfrak{U};H^{s-\frac{3}{2}}\right)}\mathrm{d}t.
\end{aligned}$$

For q_9 and q_{10}, we use (2.7), the mean value theorem for $\chi_R(\cdot)$, $H_1(1)$ and $H_1(2)$ to find a constant $C=C(K)>0$ such that

$$\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\sum_{i=9}^{10}\|q_i\|^2_{\mathcal{L}_2\left(\mathfrak{U};H^{s-\frac{3}{2}}\right)}\mathrm{d}t\le C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t')\|^2_{H^{s-\frac{3}{2}}}\mathrm{d}t.$$

On account of Lemma 2.2 and the above estimate, we find

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t)\|^2_{H^{s-\frac{3}{2}}}\\
&\le C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}(t')\|^2_{H^{s-\frac{3}{2}}}\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\}.
\end{aligned}$$

Therefore, (2.8) holds true. □

Lemma 2.4 *For any fixed $s>3$ and $T>0$, there is an $\{\mathcal{F}_t\}_{t\ge0}$ progressive measurable $H^{s-3/2}$-valued process u and a countable subsequence of $\{u_\varepsilon\}$ (still denoted as $\{u_\varepsilon\}$) such that*

$$u_\varepsilon\xrightarrow{\varepsilon\to0}u\text{ in }C\left([0,T];H^{s-\frac{3}{2}}\right)\ \mathbb{P}\text{-a.s.} \tag{2.9}$$

Proof We first let ε be discrete, i.e., $\varepsilon=\varepsilon_n(n\ge1)$ such that $\varepsilon_n\to0$ as $n\to\infty$. In this way, for all n, u_{ε_n} can be defined on the same set $\widetilde{\Omega}$ with $\mathbb{P}\{\widetilde{\Omega}\}=1$. For brevity, u_{ε_n} is still denoted as u_ε. For any $\epsilon>0$, by using (2.7), Lemma 2.1 and Chebyshev's inequality, we see that

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\right\}$$

$$\leq \mathbb{P}\left\{\tau_{\varepsilon,K}^T < T\right\} + \mathbb{P}\left\{\tau_{\eta,K}^T < T\right\} + \mathbb{P}\left\{\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]} \|u_\varepsilon - u_\eta\|_{H^{s-\frac{3}{2}}} > \epsilon\right\}$$

$$\leq \frac{2C(R,T,u_0)}{K^2} + \mathbb{P}\left\{\sup_{t\in[0,\tau_{\varepsilon,\eta,K}^T]} \|u_\varepsilon - u_\eta\|_{H^{s-\frac{3}{2}}} > \epsilon\right\}.$$

Now (2.8) clearly forces

$$\lim_{\varepsilon\to 0}\sup_{\eta\leq\varepsilon}\mathbb{P}\left\{\sup_{t\in[0,T]} \|u_\varepsilon - u_\eta\|_{H^{s-\frac{3}{2}}} > \epsilon\right\} \leq \frac{2C(R,T,u_0)}{K^2}.$$

Letting $K\to\infty$, we see that u_ε converges in probability in $C\left([0,T];H^{s-\frac{3}{2}}\right)$. Therefore, up to a further subsequence, (2.9) holds. □

2.4 Proof for Proposition 2.1

By (2.9), since for each $\varepsilon\in(0,1)$, u_ε is $\{\mathcal{F}_t\}_{t\geq 0}$ progressive measurable, so is u. Notice that $H^{s-3/2}\hookrightarrow W^{1,\infty}$. Then one can send $\varepsilon\to 0$ in (2.3) to prove that u solves (2.1). Furthermore, it follows from Lemma 2.1 and Fatou's lemma that

$$\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_{H^s}^2 < C(R,u_0,T). \tag{2.10}$$

With (2.10), to prove (2.2), we only need to prove $u\in C([0,T];H^s)$, $\mathbb{P}$-a.s. Due to Lemma 2.4 and (1.11), $u\in C([0,T];H^{s-3/2})\cap L^\infty(0,T;H^s)$ almost surely. Since H^s is dense in $H^{s-3/2}$, we see that $u\in C_w([0,T];H^s)$ ($C_w([0,T];H^s)$ is the set of weakly continuous functions with values in H^s). Therefore, we only need to prove the continuity of $[0,T]\ni t\mapsto\|u(t)\|_{H^s}$. As is mentioned in Remark 1.1, we first consider the following mollified version with J_ε being defined in (A.1):

$$\begin{aligned}\mathrm{d}\|J_\varepsilon u(t)\|_{H^s}^2 =\ & 2\chi_R(\|u\|_{W^{1,\infty}})\,(J_\varepsilon h(t,u)\,\mathrm{d}\mathcal{W}, J_\varepsilon u)_{H^s}\\ & - 2\chi_R(\|u\|_{W^{1,\infty}})\left(J_\varepsilon\left[u^k u_x + F(u)\right], J_\varepsilon u\right)_{H^s}\mathrm{d}t\\ & + \chi_R^2(\|u\|_{W^{1,\infty}})\|J_\varepsilon h(t',u)\|_{\mathcal{L}_2(\mathfrak{U};H^s)}^2\,\mathrm{d}t.\end{aligned} \tag{2.11}$$

By (2.10),

$$\tau_N := \inf\{t\geq 0 : \|u(t)\|_{H^s} > N\}\to\infty \text{ as } N\to\infty\ \ \mathbb{P}\text{-a.s.} \tag{2.12}$$

Then we only need to prove the continuity up to time $\tau_N\wedge T$ for each $N\geq 1$. Let $[t_2,t_1]\subset[0,T]$ with $t_1-t_2<1$. We use Lemma A.6, the BDG inequality, Hypothesis H_1 and (2.12) to find

$$\mathbb{E}\left[\left(\|J_\varepsilon u(t_1\wedge\tau_N)\|_{H^s}^2-\|J_\varepsilon u(t_2\wedge\tau_N)\|_{H^s}^2\right)^4\right]\le C(N,T)|t_1-t_2|^2.$$

We notice that for any $T>0$, $J_\varepsilon u$ tends to u in $C([0,T];H^s)$ as $\varepsilon\to 0$. This, together with Fatou's lemma, implies

$$\mathbb{E}\left[\left(\|u(t_1\wedge\tau_N)\|_{H^s}^2-\|u(t_2\wedge\tau_N)\|_{H^s}^2\right)^4\right]\le C(N,T)|t_1-t_2|^2.$$

This and Kolmogorov's continuity theorem ensure the continuity of $t\mapsto\|u(t\wedge\tau_N)\|_{H^s}$.

3 Proof for Theorem 1.1

Now we can prove Theorem 1.1. For the sake of clarity, we provide the proof in several subsections.

3.1 Proof for (i) in Theorem 1.1: Existence and uniqueness

3.1.1 Uniqueness

Before we prove the existence of a solution in H^s with $s>3/2$, we first prove uniqueness since some estimates here will be used later.

Lemma 3.1 *Let $s>3/2$, $k\ge 1$, and Hypothesis H_1 hold. Suppose that u_0 and v_0 are two H^s-valued $\mathcal{F}_0$-measurable random variables satisfying $u_0,v_0\in L^2(\Omega;H^s)$. Let (u,τ_1) and (v,τ_2) be two local solutions to (1.4) in the sense of Definition 1.1 such that $u(0)=u_0$, $v(0)=v_0$ almost surely. For any $N>0$ and $T>0$, we denote*

$$\tau_u:=\inf\{t\ge 0:\|u(t)\|_{H^s}>N\},\quad \tau_v:=\inf\{t\ge 0:\|v(t)\|_{H^s}>N\},$$

and $\tau_{u,v}^T:=\tau_u\wedge\tau_v\wedge T$. Then for $s'\in\left(\frac{1}{2},\min\left\{s-1,\frac{3}{2}\right\}\right)$, we have that

$$\mathbb{E}\sup_{t\in[0,\tau_{u,v}^T]}\|u(t)-v(t)\|_{H^{s'}}^2\le C(N,T)\mathbb{E}\|u_0-v_0\|_{H^{s'}}^2. \tag{3.1}$$

Proof Let $w(t)=u(t)-v(t)$ for $t\in[0,\tau_1\wedge\tau_2]$. We have

$$\mathrm{d}w+\frac{1}{k+1}\partial_x\left[u^{k+1}-v^{k+1}\right]\mathrm{d}t+[F(u)-F(v)]\,\mathrm{d}t=[h(t,u)-h(t,v)]\,\mathrm{d}\mathcal{W}.$$

Then we use the Itô formula for $\|w\|_{H^{s'}}^2$ with $s'\in\left(\frac{1}{2},\min\left\{s-1,\frac{3}{2}\right\}\right)$ to find that

$$\mathrm{d}\|w\|_{H^{s'}}^2=2\left([h(t,u)-h(t,v)]\,\mathrm{d}\mathcal{W},w\right)_{H^{s'}}-\frac{2}{k+1}\left(\partial_x(P_kw),w\right)_{H^{s'}}\mathrm{d}t$$

$$
\begin{aligned}
&- 2\left([F(u) - F(v)], w\right)_{H^{s'}} \mathrm{d}t + \|h(t,u) - h(t,v)\|^2_{\mathcal{L}_2(\mathfrak{U}; H^{s'})} \mathrm{d}t \\
&:= R_1 + \sum_{i=2}^{4} R_i \, \mathrm{d}t,
\end{aligned}
$$

where $P_k = u^k + u^{k-1}v + \cdots + uv^{k-1} + v^k$. Taking the supremum over $t \in [0, \tau^T_{u,v}]$ and using the BDG inequality, $H_1(3)$ and the Cauchy–Schwarz inequality yield

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\tau^T_{u,v}]} \|w(t)\|^2_{H^{s'}} - \mathbb{E}\|w(0)\|^2_{H^{s'}} \\
&\leq \frac{1}{2}\mathbb{E} \sup_{t\in[0,\tau^T_{u,v}]} \|w\|^2_{H^{s'}} + Cg_2(N) \int_0^T \mathbb{E} \sup_{t'\in[0,\tau^t_{u,v}]} \|w(t')\|^2_{H^{s'}} \mathrm{d}t \\
&\quad + \sum_{i=2}^{4} \mathbb{E} \int_0^{\tau^T_{u,v}} |R_i| \, \mathrm{d}t.
\end{aligned}
$$

Using Lemma A.4, integration by parts and $H^s \hookrightarrow W^{1,\infty}$, we have

$$
\begin{aligned}
|R_2| &\lesssim \left|\left([D^{s'}\partial_x, P_k]w, D^{s'}w\right)_{L^2}\right| + \left|\left(P_k D^{s'}\partial_x w, D^{s'}w\right)_{L^2}\right| \\
&\lesssim \|w\|^2_{H^{s'}} \left(\|u\|_{H^s} + \|v\|_{H^s}\right)^k.
\end{aligned}
$$

Therefore, for some constant $C(N) > 0$, we have that

$$
\mathbb{E} \int_0^{\tau^T_{u,v}} |R_2| \, \mathrm{d}t \leq C(N) \int_0^T \mathbb{E} \sup_{t'\in[0,\tau^t_{u,v}]} \|w(t')\|^2_{H^{s'}} \mathrm{d}t.
$$

Similarly, Lemma A.5 and $H_1(3)$ yield

$$
\sum_{i=3}^{4} \mathbb{E} \int_0^{\tau^T_{u,v}} |R_i| \, \mathrm{d}t \leq C(N) \int_0^T \mathbb{E} \sup_{t'\in[0,\tau^t_{u,v}]} \|w(t')\|^2_{H^{s'}} \mathrm{d}t.
$$

Therefore, we combine the above estimates to find

$$
\mathbb{E} \sup_{t\in[0,\tau^T_{u,v}]} \|w(t)\|^2_{H^{s'}} \leq 2\mathbb{E}\|w(0)\|^2_{H^{s'}} + C(N) \int_0^T \mathbb{E} \sup_{t'\in[0,\tau^t_{u,v}]} \|w(t')\|^2_{H^{s'}} \mathrm{d}t.
$$

Using the Grönwall inequality in the above estimate leads to (3.1). □

Similarly, one can obtain the following uniqueness result for the original problem (1.4), and we omit the details for simplicity.

Lemma 3.2 *Let $s > 3/2$, and let Hypothesis H_1 be true. Let u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable such that $u_0 \in L^2(\Omega; H^s)$. If (u_1, τ_1) and (u_2, τ_2) are*

two local solutions to (1.4) *satisfying* $u_i(\cdot \wedge \tau_i) \in L^2(\Omega; C([0,\infty); H^s))$ *for* $i = 1, 2$ *and* $\mathbb{P}\{u_1(0) = u_2(0) = u_0(x)\} = 1$, *then*

$$\mathbb{P}\{u_1(t,x) = u_2(t,x), \quad (t,x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{R}\} = 1.$$

3.1.2 The case $s > 3$

To begin with, we first state the following existence and uniqueness results in H^s with $s > 3$ for the Cauchy problem (1.4):

Proposition 3.1 *Let* $s > 3$, $k \geq 1$, *and* $h(t,u)$ *satisfy Hypothesis* H_1. *If* u_0 *is an* H^s*-valued* $\mathcal{F}_0$*-measurable random variable satisfying* $\mathbb{E}\|u_0\|^2_{H^s} < \infty$, *then there is a unique local solution* (u, τ) *to* (1.4) *in the sense of Definition* 1.1 *with*

$$u(\cdot \wedge \tau) \in L^2\left(\Omega; C\left([0,\infty); H^s\right)\right). \tag{3.2}$$

Proof Since uniqueness has been obtained in Lemma 3.2, via Proposition 2.1, we only need to remove the cut-off function. For $u_0(\omega, x) \in L^2(\Omega; H^s)$, we let

$$\Omega_m := \{m - 1 \leq \|u_0\|_{H^s} < m\}, \ m \geq 1.$$

Let $u_{0,m} := u_0 \mathbf{1}_{\{m-1 \leq \|u_0\|_{H^s} < m\}}$. For any $R > 0$, on account of Proposition 2.1, we let $u_{m,R}$ be the global solution to the cut-off problem (2.1) with initial value $u_{0,m}$ and cut-off function $\chi_R(\cdot)$. Define

$$\tau_{m,R} := \inf\left\{t > 0 : \sup_{t' \in [0,t]} \|u_{m,R}(t')\|^2_{H^s} > \|u_{0,m}\|^2_{H^s} + 2\right\}.$$

Then for any $R > 0$ and $m \in \mathbb{N}$, it follows from the time continuity of the solution that $\mathbb{P}\{\tau_{m,R} > 0\} = 1$. Particularly, for any $m \in \mathbb{N}$, we assign $R = R_m$ such that $R_m^2 > c^2 m^2 + 2c^2$, where $c > 0$ is the embedding constant such that $\|\cdot\|_{W^{1,\infty}} \leq c\|\cdot\|_{H^s}$ for $s > 3$. For simplicity, we denote $(u_m, \tau_m) := (u_{m,R_m}, \tau_{m,R_m})$. Then we have

$$\mathbb{P}\left\{\|u_m\|^2_{W^{1,\infty}} \leq c^2\|u_m\|^2_{H^s} \leq c^2\|u_{0,m}\|^2_{H^s} + 2c^2 < R_m^2, \ t \in [0, \tau_m], \ m \geq 1\right\} = 1,$$

which means $\mathbb{P}\left\{\chi_{R_m}(\|u_m\|_{W^{1,\infty}}) = 1, \ t \in [0, \tau_m], \ m \geq 1\right\} = 1$. Therefore, (u_m, τ_m) is the solution to (1.4) with initial value $u_{0,m}$. Since $\mathbb{E}\|u_0\|^2_{H^s} < \infty$, the condition (A.5) is satisfied with $I = \mathbb{N}^+$. Applying Lemma A.7 means that

$$\left(u = \sum_{m \geq 1} \mathbf{1}_{\{m-1 \leq \|u_0\|_{H^s} < m\}} u_m, \quad \tau = \sum_{m \geq 1} \mathbf{1}_{\{m-1 \leq \|u_0\|_{H^s} < m\}} \tau_m\right)$$

is a solution to (1.4) corresponding to the initial condition u_0. Besides,

$$\sup_{t\in[0,\tau]}\|u\|_{H^s}^2=\sum_{m\geq 1}\mathbf{1}_{\{m-1\leq\|u_0\|_{H^s}<m\}}\sup_{t\in[0,\tau_m]}\|u_m\|_{H^s}^2\leq 2\|u_0\|_{H^s}^2+4\ \ \mathbb{P}\text{-a.s.}$$

Taking expectation gives rise to (3.2). □

3.1.3 The case $s > 3/2$

When $s > 3/2$, we first consider the following problem

$$\begin{cases}\mathrm{d}u+\left[u^k\partial_x u+F(u)\right]\mathrm{d}t=h(t,u)\,\mathrm{d}\mathcal{W},\ \ k\geq 1,\ \ x\in\mathbb{R},\ t>0,\\ u(\omega,0,x)=J_\varepsilon u_0(\omega,x)\in H^\infty,\ \ x\in\mathbb{R},\ \varepsilon\in(0,1),\end{cases}\tag{3.3}$$

where J_ε is the mollifier defined in (A.1). Proposition 3.1 implies that for each $\varepsilon\in(0,1)$, (3.3) has a local pathwise solution $(u_\varepsilon,\tau_\varepsilon)$ such that $u_\varepsilon\in L^2(\Omega;C([0,\tau_\varepsilon];H^s))$.

Lemma 3.3 *Assume u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable such that $\|u_0\|_{H^s}\leq M$ for some $M>0$. For any $T>0$ and $s>3/2$, we define*

$$\tau_\varepsilon^T:=\inf\{t\geq 0:\|u_\varepsilon\|_{H^s}\geq\|J_\varepsilon u_0\|_{H^s}+2\}\wedge T,\ \ \tau_{\varepsilon,\eta}^T:=\tau_\varepsilon^T\wedge\tau_\eta^T,\ \ \varepsilon,\eta\in(0,1).\tag{3.4}$$

Let $K\geq 2M+5$ be fixed and let $s'\in\left(\frac{1}{2},\min\left\{s-1,\frac{3}{2}\right\}\right)$. Then, there is a constant $C(K,T)>0$ such that $w_{\varepsilon,\eta}=u_\varepsilon-u_\eta$ satisfies

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}(t)\|_{H^s}^2\\&\leq C(K,T)\mathbb{E}\left\{\|w_{\varepsilon,\eta}(0)\|_{H^s}^2+\|w_{\varepsilon,\eta}(0)\|_{H^{s'}}^2\|u_\varepsilon(0)\|_{H^{s+1}}^2\right\}\\&\quad+C(K,T)\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2.\end{aligned}\tag{3.5}$$

Proof To start with, we notice that Lemma A.1 implies

$$\|J_\varepsilon u_0\|_{H^s}\leq M,\ \ \varepsilon\in(0,1)\ \ \mathbb{P}\text{-a.s.}\tag{3.6}$$

Since (3.4) and (3.6) are used frequently in the following, they will be used without further notice. Let

$$P_l=P_l(u_\varepsilon,u_\eta):=\begin{cases}u_\varepsilon^l+u_\varepsilon^{l-1}u_\eta+\cdots+u_\varepsilon u_\eta^{l-1}+u_\eta^l,\ \ \text{if}\ \ l\geq 1,\\ 1,\ \ \text{if}\ \ l=0.\end{cases}\tag{3.7}$$

Applying the Itô formula to $\|w_{\varepsilon,\eta}\|_{H^s}^2$ gives rise to

$$\begin{aligned}\mathrm{d}\|w_{\varepsilon,\eta}\|_{H^s}^2 =\ & 2\left(\left[h(t,u_\varepsilon)-h(t,u_\eta)\right]\mathrm{d}\mathcal{W}, w_{\varepsilon,\eta}\right)_{H^s} - 2\left(w_{\varepsilon,\eta}P_{k-1}\partial_x u_\varepsilon, w_{\varepsilon,\eta}\right)_{H^s}\mathrm{d}t\\ & - 2\left(u_\eta^k\partial_x w_{\varepsilon,\eta}, w_{\varepsilon,\eta}\right)_{H^s}\mathrm{d}t - 2\left(\left[F(u_\varepsilon)-F(u_\eta)\right], w_{\varepsilon,\eta}\right)_{H^s}\mathrm{d}t\\ & + \left\|h(t,u_\varepsilon)-h(t,u_\eta)\right\|_{\mathcal{L}_2(\mathfrak{U};H^s)}^2\mathrm{d}t := Q_{1,s} + \sum_{i=2}^{5} Q_{i,s}\,\mathrm{d}t. \end{aligned} \tag{3.8}$$

Since $H^{s'} \hookrightarrow L^\infty$ and $H^s \hookrightarrow W^{1,\infty}$, we can use Lemmas A.3 and A.5 to find

$$\begin{aligned}\left|Q_{2,s}\right| &\lesssim \left(\|w_{\varepsilon,\eta}P_{k-1}\|_{H^s}\|\partial_x u_\varepsilon\|_{L^\infty} + \|w_{\varepsilon,\eta}P_{k-1}\|_{L^\infty}\|\partial_x u_\varepsilon\|_{H^s}\right)\|w_{\varepsilon,\eta}\|_{H^s}\\ &\lesssim \|w_{\varepsilon,\eta}\|_{H^s}^2\left(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\right)^k + \|w_{\varepsilon,\eta}\|_{H^{s'}}^2\|u_\varepsilon\|_{H^{s+1}}^2\\ &\quad + \left(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\right)^{2k-2}\|w_{\varepsilon,\eta}\|_{H^s}^2\\ \left|Q_{3,s}\right| &\lesssim \left|\left([D^s,u_\eta^k]\partial_x w_{\varepsilon,\eta}, D^s w_{\varepsilon,\eta}\right)_{L^2}\right| + \left|\left(u_\eta^k\partial_x D^s w_{\varepsilon,\eta}, D^s w_{\varepsilon,\eta}\right)_{L^2}\right|\\ &\lesssim \|w_{\varepsilon,\eta}\|_{H^s}^2\|u_\eta\|_{H^s}^k, \end{aligned}$$

and

$$\left|Q_{4,s}\right| \lesssim \|w_{\varepsilon,\eta}\|_{H^s}^2\left(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\right)^k.$$

The above estimates and $H_1(2)$ imply that there is a constant $C(K)>0$ such that

$$\begin{aligned}&\sum_{i=2}^{5}\mathbb{E}\int_0^{\tau_{\varepsilon,\eta}^T}|Q_{i,s}|\,\mathrm{d}t\\ &\lesssim \mathbb{E}\int_0^{\tau_{\varepsilon,\eta}^T}\left[\left(\left(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\right)^{2k}+1\right)\|w_{\varepsilon,\eta}\|_{H^s}^2 + \|w_{\varepsilon,\eta}\|_{H^{s'}}^2\|u_\varepsilon\|_{H^{s+1}}^2\right]\mathrm{d}t\\ &\quad + \mathbb{E}\int_0^{\tau_{\varepsilon,\eta}^T} g_1^2(K)\|w_{\varepsilon,\eta}\|_{H^s}^2\,\mathrm{d}t\\ &\le C(K)\int_0^T \mathbb{E}\sup_{t'\in[0,\tau_{\varepsilon,\eta}^t]}\|w_{\varepsilon,\eta}(t')\|_{H^s}^2\,\mathrm{d}t + C(K)T\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2\|u_\varepsilon(t)\|_{H^{s+1}}^2. \end{aligned}$$

For $Q_{1,s}$, applying the BDG inequality and $H_1(2)$, we derive

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\int_0^t\left(\left[h(t,u_\varepsilon)-h(t,u_\eta)\right]\mathrm{d}\mathcal{W}, w_{\varepsilon,\eta}\right)_{H^s}\right)\\ &\quad\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}\|_{H^s}^2 + Cg_1^2(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau_{\varepsilon,\eta}^t]}\|w_{\varepsilon,\eta}(t')\|_{H^s}^2\,\mathrm{d}t. \end{aligned}$$

Summarizing the above estimates and then using Grönwall's inequality, we find some constant $C = C(K, T) > 0$ such that

$$\begin{aligned}&\mathbb{E} \sup_{t\in[0,\tau_{\varepsilon,\eta}^T]} \|w_{\varepsilon,\eta}(t)\|_{H^s}^2 \\ &\quad\le C\left(\mathbb{E}\|w_{\varepsilon,\eta}(0)\|_{H^s}^2 + \mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]} \|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2 \|u_\varepsilon(t)\|_{H^{s+1}}^2\right). \end{aligned} \tag{3.9}$$

Now we estimate $\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]} \|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2 \|u_\varepsilon(t)\|_{H^{s+1}}^2$. To this end, we first recall (1.7) and then apply the Itô formula to deduce that for any $\rho > 0$,

$$\begin{aligned}\mathrm{d}\|u_\varepsilon\|_{H^\rho}^2 =& 2\sum_{l=1}^\infty (h(t,u_\varepsilon)e_l, u_\varepsilon)_{H^\rho}\,\mathrm{d}W_l - 2\left(D^\rho\left[(u_\varepsilon)^k\partial_x u_\varepsilon\right], D^\rho u_\varepsilon\right)_{L^2}\mathrm{d}t \\ &- 2\left(D^\rho F(u_\varepsilon), D^\rho u_\varepsilon\right)_{L^2}\mathrm{d}t + \|h(t,u_\varepsilon)\|_{\mathcal{L}_2(\mathfrak{U};H^\rho)}^2\mathrm{d}t \\ :=& \sum_{l=1}^\infty Z_{1,\rho,l}\,\mathrm{d}W_l + \sum_{i=2}^4 Z_{i,\rho}\,\mathrm{d}t. \end{aligned} \tag{3.10}$$

In the same way, we also rewrite $Q_{1,s}$ in (3.8) as

$$Q_{1,s} = 2\sum_{j=1}^\infty \left(\left[h(t,u_\varepsilon) - h(t,u_\eta)\right]e_j, w_{\varepsilon,\eta}\right)_{H^s}\mathrm{d}W_j := \sum_{j=1}^\infty Q_{1,s,j}\,\mathrm{d}W_j. \tag{3.11}$$

With the summation form (3.11) at hand, applying the Itô product rule to (3.8) and (3.10), we derive

$$\begin{aligned}\mathrm{d}\|w_{\varepsilon,\eta}\|_{H^{s'}}^2\|u_\varepsilon\|_{H^{s+1}}^2 =& \sum_{j=1}^\infty\left(\|w_{\varepsilon,\eta}\|_{H^{s'}}^2 Z_{1,s+1,j} + \|u_\varepsilon\|_{H^{s+1}}^2 Q_{1,s',j}\right)\mathrm{d}W_j \\ &+ \sum_{i=2}^4 \|w_{\varepsilon,\eta}\|_{H^{s'}}^2 Z_{i,s+1}\,\mathrm{d}t + \sum_{i=2}^5 \|u_\varepsilon\|_{H^{s+1}}^2 Q_{i,s'}\,\mathrm{d}t \\ &+ \sum_{j=1}^\infty Q_{1,s',j}Z_{1,s+1,j}\,\mathrm{d}t. \end{aligned}$$

We first notice that

$$Q_{2,s'} + Q_{3,s'} = \frac{2}{k+1}\left(\partial_x(P_k w_{\varepsilon,\eta}), w_{\varepsilon,\eta}\right)_{H^{s'}},$$

where P_k is defined by (3.7). As a result, Lemma A.4, integration by parts and $H^s \hookrightarrow W^{1,\infty}$ give rise to

$$|Q_{2,s'} + Q_{3,s'}| \lesssim \|w_{\varepsilon,\eta}\|^2_{H^{s'}} \left(\|u_\varepsilon\|_{H^s} + \|u_\eta\|_{H^s}\right)^k.$$

Using Lemma A.3, Hypothesis H_1, Lemma A.5 as well as the embedding of $H^s \hookrightarrow W^{1,\infty}$ for $s > 3/2$, we obtain that for some $C(K) > 0$,

$$\sum_{i=2}^{4} \|w_{\varepsilon,\eta}\|^2_{H^{s'}} |Z_{i,s+1}| \lesssim \|w_{\varepsilon,\eta}\|^2_{H^{s'}} \left[\|u_\varepsilon\|^k_{H^s} \|u_\varepsilon\|^2_{H^{s+1}} + f^2(\|u_\varepsilon\|_{H^s})(1 + \|u_\varepsilon\|^2_{H^{s+1}})\right],$$

$$\sum_{i=4}^{5} \mathbb{E} \int_0^{\tau^T_{\varepsilon,\eta}} \|u_\varepsilon\|^2_{H^{s+1}} |Q_{i,s'}| \, \mathrm{d}t \le C(K) \int_0^T \mathbb{E} \sup_{t' \in [0,\tau^t_{\varepsilon,\eta}]} \|u_\varepsilon(t')\|^2_{H^{s+1}} \|w_{\varepsilon,\eta}(t')\|^2_{H^{s'}} \, \mathrm{d}t.$$

Then one can infer from the above three inequalities, the BDG inequality and Hypothesis H_1 that for some constant $C(K) > 0$,

$$\begin{aligned}
&\mathbb{E} \sup_{t \in [0,\tau^T_{\varepsilon,\eta}]} \|w_{\varepsilon,\eta}\|^2_{H^{s'}} \|u_\varepsilon\|^2_{H^{s+1}} - \mathbb{E}\|w_{\varepsilon,\eta}(0)\|^2_{H^{s'}} \|u_\varepsilon(0)\|^2_{H^{s+1}} \\
&\lesssim \mathbb{E} \left(\int_0^{\tau^T_{\varepsilon,\eta}} \|w_{\varepsilon,\eta}\|^4_{H^{s'}} \|h(t,u_\varepsilon)\|^2_{\mathcal{L}_2(\mathfrak{U};H^{s+1})} \|u_\varepsilon\|^2_{H^{s+1}} \, \mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad + \mathbb{E} \left(\int_0^{\tau^T_{\varepsilon,\eta}} \|u_\varepsilon\|^4_{H^{s+1}} \|h(t,u_\varepsilon) - h(t,u_\eta)\|^2_{\mathcal{L}_2(\mathfrak{U};H^{s'})} \|w_{\varepsilon,\eta}\|^2_{H^{s'}} \, \mathrm{d}t \right)^{\frac{1}{2}} \\
&\quad + \sum_{i=2}^{4} \mathbb{E} \int_0^{\tau^T_{\varepsilon,\eta}} \|w_{\varepsilon,\eta}\|^2_{H^{s'}} |Z_{i,s+1}| \, \mathrm{d}t + \mathbb{E} \int_0^{\tau^T_{\varepsilon,\eta}} \|u_\varepsilon\|^2_{H^{s+1}} |Q_{2,s'} + Q_{3,s'}| \, \mathrm{d}t \\
&\quad + \sum_{i=4}^{5} \mathbb{E} \int_0^{\tau^T_{\varepsilon,\eta}} \|u_\varepsilon\|^2_{H^{s+1}} |Q_{i,s'}| \, \mathrm{d}t + \mathbb{E} \int_0^{\tau^T_{\varepsilon,\eta}} \sum_{j=1}^{\infty} |Q_{1,s',j} Z_{1,s+1,j}| \, \mathrm{d}t \\
&\le \frac{1}{2} \mathbb{E} \sup_{t \in [0,\tau^T_{\varepsilon,\eta}]} \|w_{\varepsilon,\eta}\|^2_{H^{s'}} \|u_\varepsilon\|^2_{H^{s+1}} \\
&\quad + C(K) \int_0^T \mathbb{E} \sup_{t' \in [0,\tau^t_{\varepsilon,\eta}]} \|w_{\varepsilon,\eta}(t')\|^2_{H^{s'}} \|u_\varepsilon(t')\|^2_{H^{s+1}} \, \mathrm{d}t \\
&\quad + C(K) T \mathbb{E} \sup_{t \in [0,\tau^T_{\varepsilon,\eta}]} \|w_{\varepsilon,\eta}(t)\|^2_{H^{s'}} + \mathbb{E} \int_0^{\tau^T_{\varepsilon,\eta}} \sum_{j=1}^{\infty} |Q_{1,s',j} Z_{1,s+1,j}| \, \mathrm{d}t. \qquad (3.12)
\end{aligned}$$

For the last term, we proceed as follows:

$$\mathbb{E} \int_0^{\tau^T_{\varepsilon,\eta}} \sum_{j=1}^{\infty} \left| Q_{1,s',j} Z_{1,s+1,j} \right| \, \mathrm{d}t$$

$$
\begin{aligned}
&\lesssim \mathbb{E}\int_0^{\tau_{\varepsilon,\eta}^T} \|h(t,u_\varepsilon)-h(t,u_\eta)\|_{\mathcal{L}_2(\mathfrak{U};H^{s'})}\|w_{\varepsilon,\eta}\|_{H^{s'}}\|h(t,u_\varepsilon)\|_{\mathcal{L}_2(\mathfrak{U};H^{s+1})}\|u_\varepsilon\|_{H^{s+1}}\,\mathrm{d}t.\\
&\le C(K)T\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2\\
&\quad + C(K)\int_0^T \mathbb{E}\sup_{t'\in[0,\tau_{\varepsilon,\eta}^t]}\|w_{\varepsilon,\eta}(t')\|_{H^{s'}}^2\|u_\varepsilon(t')\|_{H^{s+1}}^2\,\mathrm{d}t
\end{aligned}
$$

Consequently, (3.12) reduces to

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}\|_{H^{s'}}^2\|u_\varepsilon\|_{H^{s+1}}^2 - 2\mathbb{E}\|w_{\varepsilon,\eta}(0)\|_{H^{s'}}^2\|u_\varepsilon(0)\|_{H^{s+1}}^2\\
&\quad\le C(K)T\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2\\
&\qquad + C(K)\int_0^T \mathbb{E}\sup_{t'\in[0,\tau_{\varepsilon,\eta}^t]}\|w_{\varepsilon,\eta}(t')\|_{H^{s'}}^2\|u_\varepsilon(t')\|_{H^{s+1}}^2\,\mathrm{d}t,
\end{aligned}
$$

which means that for some $C(K,T)>0$,

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}\|_{H^{s'}}^2\|u_\varepsilon\|_{H^{s+1}}^2\\
&\quad\le C\left(\mathbb{E}\|w_{\varepsilon,\eta}(0)\|_{H^{s'}}^2\|u_\varepsilon(0)\|_{H^{s+1}}^2 + \mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2\right).
\end{aligned}
\tag{3.13}
$$

Combining (3.9) and (3.13), we obtain (3.5). □

To proceed further, we state the following lemma in [25] as a form which is convenient for our purposes.

Lemma 3.4 (Lemma 5.1, [25]) *Let all the conditions in Lemma 3.3 hold true. Assume*

$$
\lim_{\varepsilon\to 0}\sup_{\eta\le\varepsilon}\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|u_\varepsilon-u_\eta\|_{H^s}=0 \tag{3.14}
$$

and

$$
\lim_{a\to 0}\sup_{\varepsilon\in(0,1)}\mathbb{P}\left\{\sup_{t\in[0,\tau_\varepsilon^T\wedge a]}\|u_\varepsilon\|_{H^s}\ge\|J_\varepsilon u_0\|_{H^s}+1\right\}=0 \tag{3.15}
$$

hold true. Then we have:

(a) *There exists a sequence of stopping times ξ_{ε_n}, for some countable sequence $\{\varepsilon_n\}$ with $\varepsilon_n\to 0$ as $n\to\infty$, and a stopping time τ such that*

$$
\xi_{\varepsilon_n}\le\tau_{\varepsilon_n}^T,\quad \lim_{n\to\infty}\xi_{\varepsilon_n}=\tau\in(0,T]\ \ \mathbb{P}\text{-a.s.}
$$

(b) *There is a process* $u \in C([0,\tau];H^s)$ *such that*

$$\lim_{n\to\infty}\sup_{t\in[0,\tau]}\|u_{\varepsilon_n}-u\|_{H^s}=0,\quad \sup_{t\in[0,\tau]}\|u\|_{H^s}\le\|u_0\|_{H^s}+2\ \ \mathbb{P}\text{-a.s.}$$

(c) *There is a sequence of sets* $\Omega_n\uparrow\Omega$ *such that for any* $p\in[1,\infty)$,

$$\mathbf{1}_{\Omega_n}\sup_{t\in[0,\tau]}\|u_{\varepsilon_n}\|_{H^s}\le\|u_0\|_{H^s}+2\ \ \mathbb{P}\text{-a.s., and }\sup_n\mathbb{E}\left(\mathbf{1}_{\Omega_n}\sup_{t\in[0,\tau]}\|u_{\varepsilon_n}\|_{H^s}^p\right)<\infty.$$

Remark 3.1 In the original form of [25, Lemma 5.1], the authors only emphasize the existence of stopping time $\tau\in(0,T]$ such that (b) and (c) in Lemma 3.4 hold true. However, here we point out that they obtained such τ by constructing stopping times ξ_{ε_n}. We refer to (5.2), (5.12), (5.15), (5.20) and (5.24) in [25] for the details. The properties (a) and (c) in Lemma 3.4 will be used in the proof for (iii) in Theorem 1.1.

Proposition 3.2 *Let Hypothesis* H_1 *hold. Assume that* $s>3/2$, $k\ge1$ *and let* u_0 *is an* H^s*-valued* $\mathcal{F}_0$*-measurable random variable such that* $\|u_0\|_{H^s}\le M$ *for some* $M>0$. *Then* (1.4) *has a unique pathwise solution* (u,τ) *in the sense of Definition 1.1 such that*

$$\sup_{t\in[0,\tau]}\|u\|_{H^s}\le\|u_0\|_{H^s}+2\ \ \mathbb{P}\text{-a.s.}$$

Proof We first prove that $\{u_\varepsilon\}$ satisfies the estimates (3.14) and (3.15).

(i) (3.14) is satisfied. Lemma A.1 tells us that

$$\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}\mathbb{E}\|w_{\varepsilon,\eta}(0)\|_{H^s}^2=0. \tag{3.16}$$

Since $\|J_\varepsilon u_0\|_{H^s}\le M$, as in Lemma 3.1, we have

$$\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}\mathbb{E}\sup_{t\in[0,\tau_{\varepsilon,\eta}^T]}\|w_{\varepsilon,\eta}(t)\|_{H^{s'}}^2\le C(M,T)\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}\mathbb{E}\|w_{\varepsilon,\eta}(0)\|_{H^s}^2=0. \tag{3.17}$$

Moreover, it follows from Lemma A.1 that

$$\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}\|w_{\varepsilon,\eta}(0)\|_{H^{s'}}^2\|u_\varepsilon(0)\|_{H^{s+1}}^2\lesssim\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}o\left(\varepsilon^{2s-2s'}\right)O\left(\frac{1}{\varepsilon^2}\right)=0. \tag{3.18}$$

Summarizing (3.16), (3.17), (3.18) and Lemma 3.3, (3.14) holds true.

(ii) (3.15) is satisfied. Recall (3.10) and let $a>0$. We have

$$\sup_{t\in[0,\tau_\varepsilon^T\wedge a]}\|u_\varepsilon(t)\|_{H^s}^2 \le \|J_\varepsilon u_0\|_{H^s}^2 + \sup_{t\in[0,\tau_\varepsilon^T\wedge a]}\left|\int_0^t\sum_{j=1}^{\infty} Z_{1,s,j}\,\mathrm{d}W_j\right| + \sum_{i=2}^{4}\int_0^{\tau_\varepsilon^T\wedge a}|Z_{i,s}|\,\mathrm{d}t,$$

which clearly forces that

$$\mathbb{P}\left\{\sup_{t\in[0,\tau_\varepsilon^T\wedge a]}\|u_\varepsilon(t)\|_{H^s}^2 > \|J_\varepsilon u_0\|_{H^s}^2 + 1\right\} \le \mathbb{P}\left\{\sup_{t\in[0,\tau_\varepsilon^T\wedge a]}\left|\int_0^t\sum_{j=1}^{\infty} Z_{1,s,j}\,\mathrm{d}W_j\right| > \frac{1}{2}\right\} + \mathbb{P}\left\{\sum_{i=2}^{4}\int_0^{\tau_\varepsilon^T\wedge a}|Z_{i,s}|\,\mathrm{d}t > \frac{1}{2}\right\}.$$

Due to the Chebyshev inequality, Lemmas A.3 and A.5, Hypothesis H_1, the embedding of $H^s \hookrightarrow W^{1,\infty}$ for $s > 3/2$, (3.4) and (3.6), we have

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=2}^{4}\int_0^{\tau_\varepsilon^T\wedge a}|Z_{i,s}|\,\mathrm{d}t > \frac{1}{2}\right\} &\le C\sum_{i=2}^{4}\mathbb{E}\int_0^{\tau_\varepsilon^T\wedge a}|Z_{i,s}|\,\mathrm{d}t\\ &\le C\mathbb{E}\int_0^{\tau_\varepsilon^T\wedge a}\left[\|u_\varepsilon\|_{H^s}^{k+2} + f^2(\|u_\varepsilon\|_{H^s})(1+\|u_\varepsilon\|_{H^s}^2)\right]\mathrm{d}t\\ &\le C\mathbb{E}\int_0^{\tau_\varepsilon^T\wedge a} C(M,T)\,\mathrm{d}t \le C(M,T)a.\end{aligned}$$

Then we can infer from the Doob's maximal inequality and the Itô isometry that

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in[0,\tau_\varepsilon^T\wedge a]}\left|\int_0^t\sum_{j=1}^{\infty} Z_{1,s,j}\,\mathrm{d}W_j\right| > \frac{1}{2}\right\}\\ &\le C\mathbb{E}\left(\int_0^{\tau_\varepsilon^T\wedge a}\sum_{j=1}^{\infty} Z_{1,s,j}\,\mathrm{d}W_j\right)^2\\ &\le C\mathbb{E}\int_0^{\tau_\varepsilon^T\wedge a}\left[f^2(\|u_\varepsilon\|_{W^{1,\infty}})(1+\|u_\varepsilon\|_{H^s})^2\|u_\varepsilon\|_{H^s}^2\right]\mathrm{d}t\\ &\le C\mathbb{E}\int_0^{\tau_\varepsilon^T\wedge a} C(M,T)\,\mathrm{d}t \le C(M,T)a.\end{aligned}$$

Hence we have

$$\mathbb{P}\left\{\sup_{t\in[0,\tau_\varepsilon^T\wedge a]}\|u_\varepsilon(t)\|_{H^s}^2 > \|J_\varepsilon u_0\|_{H^s}^2 + 1\right\} \le C(M,T)a,$$

which gives (3.15).

(iii) Applying Lemma 3.4. By Lemma 3.4, we can take limit in some subsequence of $\{u_{\varepsilon_n}\}$ to build a solution u to (1.4) such that $u \in C([0,\tau];H^s)$ and $\sup_{t\in[0,\tau]}\|u\|_{H^s} \le \|u_0\|_{H^s}+2$. Uniqueness is a direct corollary of Lemma 3.2. □

Now we can finish the proof for (i) in Theorem 1.1.

Proof for (i) in Theorem 1.1. As in Proposition 3.1, we let

$$u_0(\omega,x) := \sum_{m\ge 1} u_{0,m}(\omega,x) := \sum_{m\ge 1} u_0(\omega,x)\mathbf{1}_{\{m-1\le\|u_0\|_{H^s}<m\}} \ \mathbb{P}\text{-a.s.}$$

For each $m \ge 1$, we can infer from Proposition 3.2 that (1.4) has a unique solution (u_m,τ_m) with $u_m(0)=u_{0,m}$ almost surely. Furthermore, $\sup_{t\in[0,\tau_m]}\|u_m\|_{H^s} \le \|u_{0,m}\|_{H^s}+2\,\mathbb{P}$-a.s. Using Lemma A.7 in a similar way as in Proposition 3.2, we find that

$$\left(u=\sum_{m\ge 1}\mathbf{1}_{\{m-1\le\|u_0\|_{H^s}<m\}}u_m,\quad \tau=\sum_{m\ge 1}\mathbf{1}_{\{m-1\le\|u_0\|_{H^s}<m\}}\tau_m\right)$$

is a solution to (1.4) satisfying (1.11) and $u(0)=u_0$ almost surely. Uniqueness is given by Lemma 3.2. □

3.2 Proof for (ii) in Theorem 1.1: Blow-up criterion

With a local solution (u,τ) at hand, one may pass from (u,τ) to the maximal solution (u,τ^*) as in [5, 26]. In the periodic setting, i.e., $x\in\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, the blow-up criterion (1.12) for a maximal solution has been proved in [46] by using energy estimate and some stopping-time techniques. When $x\in\mathbb{R}$, (1.12) can be also obtained in the same way, and we omit the details for brevity.

3.3 Proof for (iii) in Theorem 1.1: Stability

Let $u_0, v_0 \in L^\infty(\Omega;H^s)$ be two H^s-valued $\mathcal{F}_0$-measurable random variables. Let u and v be the corresponding solutions with initial conditions u_0 and v_0. To prove (iii) in Theorem 1.1, for any $\epsilon>0$ and $T>0$, we need to find a $\delta=\delta(\epsilon,u_0,T)>0$ and a $\tau\in(0,T]$ $\mathbb{P}$-a.s. such that (1.14) holds true as long as (1.13) is satisfied. Without loss of generality, by (1.13), we can first assume

$$\|v_0\|_{L^\infty(\Omega;H^s)} \le \|u_0\|_{L^\infty(\Omega;H^s)}+1. \tag{3.19}$$

From now on $\epsilon>0$ and $T>0$ are given.

However, as is mentioned in Remark 1.1, the term $u^k u_x$ loses one regularity and the estimate for $\mathbb{E}\sup_{t\in[0,\tau]}\|u(t)-v(t)\|_{H^s}^2$ will involve $\|u\|_{H^{s+1}}$ or $\|v\|_{H^{s+1}}$, which might be infinite since we only know $u,v\in H^s$. To overcome this difficulty, we will

consider (3.3). Let $\varepsilon \in (0,1)$. By (i) in Theorem 1.1, there is a unique solution u_ε (resp. v_ε) to the problem (3.3) with initial data $J_\varepsilon u_0$ (resp. $J_\varepsilon v_0$). Then the H^{s+1}-norm is well-defined for the smooth solution u_ε and v_ε. Similar to (3.4), for any $T > 0$, we define

$$\tau_\varepsilon^{f,T} := \inf\{t \geq 0 : \|f_\varepsilon\|_{H^s} \geq \|J_\varepsilon f_0\|_{H^s} + 2\} \wedge T, \quad f \in \{u, v\}. \tag{3.20}$$

Recalling the analysis in Lemma 3.3 and Proposition 3.2 (for the case $f = v$, we notice (3.19)), we can use Lemma 3.4 to find that there exists a unified subsequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that for $f \in \{u, v\}$, there is a sequence of stopping times $\xi_{\varepsilon_n}^f$ and a stopping time τ^f satisfying

$$\xi_{\varepsilon_n}^f \leq \tau_{\varepsilon_n}^{f,T},\ n \geq 1 \ \text{ and } \ \lim_{n\to\infty} \xi_{\varepsilon_n}^f = \tau^f \in (0, T] \ \mathbb{P}\text{-a.s.}, \tag{3.21}$$

and

$$\lim_{n\to\infty} \sup_{t\in[0,\tau^f]} \|f - f_{\varepsilon_n}\|_{H^s} = 0, \quad \sup_{t\in[0,\tau^f]} \|f\|_{H^s} \leq \|f_0\|_{H^s} + 2 \ \mathbb{P}\text{-a.s.} \tag{3.22}$$

Moreover, for $f \in \{u, v\}$, there exists $\Omega_n^f \uparrow \Omega$ such that

$$\mathbf{1}_{\Omega_n^f} \sup_{t\in[0,\tau^f]} \|f_{\varepsilon_n}\|_{H^s} \leq \|f_0\|_{H^s} + 2 \ \mathbb{P}\text{-a.s.} \tag{3.23}$$

Next, we let $\Omega_n := \Omega_n^u \cap \Omega_n^v$. Then $\Omega_n \uparrow \Omega$. This, (3.22), (3.23) and Lebesgue's dominated convergence theorem yield

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\in[0,\tau^f]} \|f - \mathbf{1}_{\Omega_n} f_{\varepsilon_n}\|_{H^s}^2 = 0, \quad f \in \{u, v\}.$$

Therefore, we have, when n is large enough, that

$$\mathbb{E} \sup_{t\in[0,\tau^f]} \|f - \mathbf{1}_{\Omega_n} f_{\varepsilon_n}\|_{H^s}^2 < \frac{\epsilon}{9}, \quad f \in \{u, v\}. \tag{3.24}$$

Now we consider $\mathbb{E}\sup_{t\in[0,\tau^u\wedge\tau^v]} \|\mathbf{1}_{\Omega_n} u_{\varepsilon_n} - \mathbf{1}_{\Omega_n} v_{\varepsilon_n}\|_{H^s}^2$. It follows from (3.21) that for all $n \geq 1$,

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\tau^u\wedge\tau^v]} \|\mathbf{1}_{\Omega_n} u_{\varepsilon_n} - \mathbf{1}_{\Omega_n} v_{\varepsilon_n}\|_{H^s}^2 \\
&\leq \mathbb{E} \sup_{t\in[0,\xi_{\varepsilon_n}^u\wedge\xi_{\varepsilon_n}^v\wedge\tau^u\wedge\tau^v]} \|\mathbf{1}_{\Omega_n} u_{\varepsilon_n} - \mathbf{1}_{\Omega_n} v_{\varepsilon_n}\|_{H^s}^2 \\
&\quad + \mathbb{E} \sup_{t\in[\xi_{\varepsilon_n}^u\wedge\xi_{\varepsilon_n}^v\wedge\tau^u\wedge\tau^v,\tau^u\wedge\tau^v]} \|\mathbf{1}_{\Omega_n} u_{\varepsilon_n} - \mathbf{1}_{\Omega_n} v_{\varepsilon_n}\|_{H^s}^2 \\
&\leq \mathbb{E} \sup_{t\in[0,\tau_{\varepsilon_n}^{u,T}\wedge\tau_{\varepsilon_n}^{v,T}]} \|u_{\varepsilon_n}(t) - v_{\varepsilon_n}(t)\|_{H^s}^2
\end{aligned}$$

$$+\mathbb{E}\sup_{t\in[\xi^u_{\varepsilon_n}\wedge\xi^v_{\varepsilon_n}\wedge\tau^u\wedge\tau^v,\tau^u\wedge\tau^v]}\|\mathbf{1}_{\Omega_n}u_{\varepsilon_n}-\mathbf{1}_{\Omega_n}v_{\varepsilon_n}\|^2_{H^s}. \tag{3.25}$$

By (3.23),

$$\sup_{t\in[\xi^u_{\varepsilon_n}\wedge\xi^v_{\varepsilon_n}\wedge\tau^u\wedge\tau^v,\tau^u\wedge\tau^v]}\|\mathbf{1}_{\Omega_n}u_{\varepsilon_n}-\mathbf{1}_{\Omega_n}v_{\varepsilon_n}\|^2_{H^s}\le 32\left(\|u_0\|^2_{H^s}+\|v_0\|^2_{H^s}+1\right).$$

Consequently, by Lebesgue's dominated convergence theorem and (3.21), we have for $n\gg 1$ that,

$$\mathbb{E}\sup_{t\in[\xi^u_{\varepsilon_n}\wedge\xi^v_{\varepsilon_n}\wedge\tau^u\wedge\tau^v,\tau^u\wedge\tau^v]}\|\mathbf{1}_{\Omega_n}u_{\varepsilon_n}-\mathbf{1}_{\Omega_n}v_{\varepsilon_n}\|^2_{H^s}<\frac{\epsilon}{18}. \tag{3.26}$$

Now we estimate $\mathbb{E}\sup_{t\in[0,\tau^{u,T}_{\varepsilon_n}\wedge\tau^{v,T}_{\varepsilon_n}]}\|u_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\|^2_{H^s}$. Similar to (3.5), by using (3.19), one can show that for $s'\in\left(\frac{1}{2},\min\left\{s-1,\frac{3}{2}\right\}\right)$,

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau^{u,T}_{\varepsilon_n}\wedge\tau^{v,T}_{\varepsilon_n}]}\|u_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\|^2_{H^s}\\
&\le C\mathbb{E}\left\{\|J_{\varepsilon_n}u_0-J_{\varepsilon_n}v_0\|^2_{H^s}+\|J_{\varepsilon_n}u_0-J_{\varepsilon_n}v_0\|^2_{H^{s'}}\|J_{\varepsilon_n}u_0\|^2_{H^{s+1}}\right\}\\
&\quad+C\mathbb{E}\sup_{t\in[0,\tau^{u,T}_{\varepsilon_n}\wedge\tau^{v,T}_{\varepsilon_n}]}\|u_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\|^2_{H^{s'}}\\
&\le C\mathbb{E}\left\{\|u_0-v_0\|^2_{H^s}+\frac{1}{\varepsilon_n^2}\|u_0-v_0\|^2_{H^{s'}}\|u_0\|^2_{H^s}\right\}\\
&\quad+C\mathbb{E}\sup_{t\in[0,\tau^{u,T}_{\varepsilon_n}\wedge\tau^{v,T}_{\varepsilon_n}]}\|u_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\|^2_{H^{s'}},
\end{aligned}\tag{3.27}$$

where $C=C\left(\|u_0\|_{L^\infty(\Omega;H^s)},T\right)$ and Lemma A.1 is used in the last step. Since $u_0\in L^\infty(\Omega;H^s)$, by Lemmas 3.1 and A.1 again, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau^{u,T}_{\varepsilon_n}\wedge\tau^{v,T}_{\varepsilon_n}]}\|u_{\varepsilon_n}(t)-v_{\varepsilon_n}(t)\|^2_{H^s}\\
&\le C\mathbb{E}\left\{\|u_0-v_0\|^2_{H^s}+\frac{1}{\varepsilon_n^2}\|u_0-v_0\|^2_{H^{s'}}\|u_0\|^2_{H^s}\right\}+C\mathbb{E}\|J_{\varepsilon_n}u_0-J_{\varepsilon_n}v_0\|^2_{H^{s'}}\\
&\le C\mathbb{E}\|u_0-v_0\|^2_{H^s}+C\frac{1}{\varepsilon_n^2}\mathbb{E}\|u_0-v_0\|^2_{H^{s'}}+C\mathbb{E}\|u_0-v_0\|^2_{H^{s'}},
\end{aligned}\tag{3.28}$$

where $C = C\left(\|u_0\|_{L^\infty(\Omega;H^s)}, T\right)$ as before. Fix $n = n_0 \gg 1$ such that (3.24) and (3.26) are satisfied, i.e.,

$$\begin{cases} \mathbb{E} \sup\limits_{t\in[0,\tau^f]} \|f - \mathbf{1}_{\Omega_{n_0}} f_{\varepsilon_{n_0}}\|_{H^s}^2 < \dfrac{\epsilon}{9}, \quad f \in \{u, v\}, \\ \mathbb{E} \sup\limits_{t\in[\xi^u_{\varepsilon_{n_0}} \wedge \xi^v_{\varepsilon_{n_0}} \wedge \tau^u \wedge \tau^v, \tau^u \wedge \tau^v]} \|\mathbf{1}_{\Omega_{n_0}} u_{\varepsilon_{n_0}} - \mathbf{1}_{\Omega_{n_0}} v_{\varepsilon_{n_0}}\|_{H^s}^2 < \dfrac{\epsilon}{18}. \end{cases} \tag{3.29}$$

Then, for (3.28) with $n = n_0$, we can find a $\delta = \delta(\epsilon, u_0, T) \in (0, 1)$ such that (3.19) is satisfied and

$$\mathbb{E} \sup_{t\in[0,\tau^{u,T}_{\varepsilon_{n_0}} \wedge \tau^{v,T}_{\varepsilon_{n_0}}]} \|u_{\varepsilon_{n_0}}(t) - v_{\varepsilon_{n_0}}(t)\|_{H^s}^2 < \frac{\epsilon}{18}, \quad \text{if } \|u_0 - v_0\|_{L^\infty(\Omega;H^s)} < \delta. \tag{3.30}$$

As a result, for (3.25) with fixed $n = n_0$, we use $(3.29)_2$ and (3.30) to derive that

$$\mathbb{E} \sup_{t\in[0,\tau^u\wedge\tau^v]} \|\mathbf{1}_{\Omega_{n_0}} u_{\varepsilon_{n_0}} - \mathbf{1}_{\Omega_{n_0}} v_{\varepsilon_{n_0}}\|_{H^s}^2 \quad \le \frac{\epsilon}{18} + \frac{\epsilon}{18} = \frac{\epsilon}{9}, \quad \text{if } \|u_0 - v_0\|_{L^\infty(\Omega;H^s)} < \delta.$$

This inequality and $(3.29)_1$ yield that

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,\tau^u\wedge\tau^v]} \|u - v\|_{H^s}^2 \\ &\le 3 \sum_{f\in\{u,v\}} \mathbb{E} \sup_{t\in[0,\tau^u\wedge\tau^v]} \|f - \mathbf{1}_{\Omega_{n_0}} f_{\varepsilon_{n_0}}\|_{H^s}^2 \\ &\quad + 3\mathbb{E} \sup_{t\in[0,\tau^u\wedge\tau^v]} \|\mathbf{1}_{\Omega_{n_0}} u_{\varepsilon_{n_0}} - \mathbf{1}_{\Omega_{n_0}} v_{\varepsilon_{n_0}}\|_{H^s}^2 \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \quad \text{if } \|u_0 - v_0\|_{L^\infty(\Omega;H^s)} < \delta. \end{aligned}$$

Hence we obtain (1.14) with $\tau = \tau^u \wedge \tau^v$. Due to (3.21), $\tau \in (0, T]$ almost surely.

Remark 3.2 Here we remark that the restriction $\mathbf{1}_{\Omega_n}$ is needed to estimate

$$\mathbb{E} \sup_{t\in[0,\tau^f]} \|f - \mathbf{1}_{\Omega_n} f_{\varepsilon_n}\|_{H^s}^2$$

for $f \in \{u, v\}$. This is because we only have $\lim_{n\to\infty} \sup_{t\in[0,\tau^f]} \|f - f_{\varepsilon_n}\|_{H^s} = 0$ $\mathbb{P}$-a.s. (cf. (b) in Lemma 3.4), and we need to interchange limit and expectation. By (c) in Lemma 3.4,

$$\sup_{t\in[0,\tau^f]} \|f - \mathbf{1}_{\Omega_n} f_{\varepsilon_n}\|_{H^s}^2 \le 2 \sup_{t\in[0,\tau^f]} \|f\|_{H^s}^2 + 2\mathbf{1}_{\Omega_n} \sup_{t\in[0,\tau^f]} \|f_{\varepsilon_n}\|_{H^s}^2 \le 4\|f_0\|_{H^s}^2 + 16.$$

Hence Lebesgue's dominated convergence theorem can be used. In the deterministic case, one can directly consider $\|f - f_{\varepsilon_n}\|_{H^s}^2$.

4 Weak instability

Now we prove Theorem 1.2. As is mentioned in Remark 1.3, since we can not get an explicit expression of the solution to (1.4), we start with constructing some approximative solutions from which (1.19) can be established.

4.1 Approximative solutions and actual solutions

Following the approach in [28, 46], now we construct the approximative solutions. We fix two functions $\phi, \tilde{\phi} \in C_c^\infty$ such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 2, \end{cases} \quad \text{and} \quad \tilde{\phi}(x) = 1 \text{ if } x \in \operatorname{supp} \phi. \tag{4.1}$$

Let $k \geq 1$ and

$$m \in \{-1, 1\} \text{ if } k \text{ is odd and } m \in \{0, 1\} \text{ if } k \text{ is even.} \tag{4.2}$$

Then we consider the following sequence of approximative solutions

$$u_{m,n} = u_l + u_h, \tag{4.3}$$

where $u_h = u_{h,m,n}$ is the high-frequency part defined by

$$u_h = u_{h,m,n}(t, x) = n^{-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \cos(nx - mt), \quad n \in \mathbb{N}, \tag{4.4}$$

and $u_l = u_{l,m,n}$ is the low-frequency part constructed such that u_l is the solution to the following problem:

$$\begin{cases} \partial_t u_l + u_l^k \partial_x u_l + F(u_l) = 0, \quad x \in \mathbb{R},\ t > 0,\ k \geq 1, \\ u_l(0, x) = m n^{-\frac{1}{k}} \tilde{\phi}\left(\frac{x}{n^\delta}\right), \quad x \in \mathbb{R}. \end{cases} \tag{4.5}$$

The parameter $\delta > 0$ in (4.4) and (4.5) will be determined later for different $k \geq 1$. Particularly, when $m = 0$, we have $u_l = 0$. In this case the approximative solution $u_{0,n}$ has *no* low-frequency part and

$$u_{0,n}(t, x) = n^{-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \cos(nx).$$

Next, we consider the problem (1.4) with initial data $u_{m,n}(0, x)$, i.e.,

$$\begin{cases} \mathrm{d}u + [u^k \partial_x u + F(u)]\,\mathrm{d}t = h(t, u)\,\mathrm{d}\mathcal{W}, \quad t > 0,\ x \in \mathbb{R}, \\ u(0, x) = m n^{-\frac{1}{k}} \tilde{\phi}\left(\frac{x}{n^\delta}\right) + n^{-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \cos(nx), \quad x \in \mathbb{R}, \end{cases} \tag{4.6}$$

where $F(\cdot)$ is defined by (1.5). Since h satisfies $H_2(1)$, similar to the proof for Theorem 1.1, we see that for each fixed $n \in \mathbb{N}$, (4.6) has a unique solution $(u^{m,n}, \tau^{m,n})$ such that $u^{m,n} \in C([0, \tau^{m,n}]; H^s)$ $\mathbb{P}$-a.s. with $s > 5/2$.

4.2 Estimates on the errors

Substituting (4.3) into (1.4), we define the error $\mathcal{E}(\omega, t, x)$ as

$$\begin{aligned}\mathcal{E}(\omega, t, x) :=\ & u_{m,n}(t, x) - u_{m,n}(0, x) \\ & + \int_0^t \left[u_{m,n}^k \partial_x u_{m,n} + F(u_{m,n}) \right] \mathrm{d}t' - \int_0^t h(t', u_{m,n}) \, \mathrm{d}\mathcal{W} \ \ \mathbb{P}\text{-a.s.}\end{aligned}$$

For simplicity, we let

$$\mathcal{Z}_q = \mathcal{Z}_q(u_h, u_l) = \begin{cases} \displaystyle\sum_{j=1}^{q} C_q^j u_l^{q-j} u_h^j, \ \text{if } q \geq 1, \\ 0, \ \text{if } q = 0, \end{cases} \tag{4.7}$$

where C_q^j is the binomial coefficient. By using (4.3), (4.5) and (4.7), $\mathcal{E}(\omega, t, x)$ can be reformulated as

$$\begin{aligned}&\mathcal{E}(\omega, t, x) \\ &= u_l(t, x) - u_l(0, x) + \int_0^t u_l^k \partial_x u_l \, \mathrm{d}t' + \int_0^t F(u_l) \, \mathrm{d}t' \\ &\quad + u_h(t, x) - u_h(0, x) + \int_0^t \left[u_l^k \partial_x u_h + \mathcal{Z}_k(\partial_x u_l + \partial_x u_h) \right] \mathrm{d}t' \\ &\quad + \int_0^t \left[F(u_l + u_h) - F(u_l) \right] \mathrm{d}t' - \int_0^t h(t', u_{m,n}) \, \mathrm{d}\mathcal{W} \\ &= u_h(t, x) - u_h(0, x) + \int_0^t \left[u_l^k \partial_x u_h + \mathcal{Z}_k(\partial_x u_l + \partial_x u_h) \right] \mathrm{d}t' \\ &\quad + \int_0^t \left[F(u_l + u_h) - F(u_l) \right] \mathrm{d}t' - \int_0^t h(t', u_{m,n}) \, \mathrm{d}\mathcal{W} \ \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.8}$$

4.2.1 Estimates on the low-frequency part

The following lemma gives a decay estimate for the low-frequency part of $u_{m,n}$, that is, u_l.

Lemma 4.1 *Let $k \geq 1$, $|m| = 1$ or $m = 0$, $s > 3/2$, $\delta \in (0, 2/k)$ and $n \gg 1$. Then there is a $T_l > 0$ such that for all $n \gg 1$, the initial value problem (4.5) has a unique smooth solution $u_l = u_{l,m,n} \in C([0, T_l]; H^s)$ such that T_l does not depend on n.*

Besides, for all $r > 0$, there is a constant $C = C_{r,\tilde{\phi},T_l} > 0$ such that u_l satisfies

$$\|u_l(t)\|_{H^r} \le C|m|n^{\frac{\delta}{2}-\frac{1}{k}}, \quad t \in [0, T_l]. \tag{4.9}$$

Proof When $m = 0$, as is mentioned above, $u_l \equiv 0$ for all $t \ge 0$. It remains to prove the case $|m| = 1$. For any fixed $n \ge 1$, since $u_l(0, x) \in H^\infty$, by applying Theorem 1.1 with $h = 0$ and deterministic initial data, we see that for any $s > 3/2$, (4.5) has a unique (deterministic) solution $u_l = u_{l,m,n} \in C\left([0, T_{m,n}]; H^s\right)$. Different from the stochastic case, here we will show that there is a lower bound of the existence time, i.e., there is a $T_l > 0$ such that for all $n \gg 1$, $u_l = u_{l,m,n}$ exists on $[0, T_l]$ and satisfies (4.9).

Step 1: Estimate $\|u_l(0, x)\|_{H^r}$. When $n \gg 1$, we have

$$\begin{aligned}\|u_l(0,x)\|_{H^r}^2 &= m^2 n^{2\delta-\frac{2}{k}} \int_{\mathbb{R}} (1+|\xi|^2)^r \left|\widehat{\tilde{\phi}}(n^\delta \xi)\right|^2 \mathrm{d}\xi \\ &= m^2 n^{\delta-\frac{2}{k}} \int_{\mathbb{R}} \left(1+\left|\frac{z}{n^\delta}\right|^2\right)^r \left|\widehat{\tilde{\phi}}(z)\right|^2 \mathrm{d}z \le C m^2 n^{\delta-\frac{2}{k}}\end{aligned}$$

for some constant $C = C_{r,\tilde{\phi}} > 0$. As a result, we have

$$\|u_l(0, x)\|_{H^r} \le C|m|n^{\frac{\delta}{2}-\frac{1}{k}}.$$

Step 2: Prove (4.9) **for** $r > 3/2$. In this case, we apply Lemmas A.3 and A.5, $H^r \hookrightarrow W^{1,\infty}$ to find

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r}^2 \\ &\le \left|\left(D^r u_l, D^r\left(u_l^k \partial_x u_l\right)\right)_{L^2}\right| + \left|\left(D^r u_l, D^r F(u_l)\right)_{L^2}\right| \\ &\le \left|\left([D^r, u_l^k]\partial_x u_l, D^r u_l\right)_{L^2}\right| + \left|\left(u_l^k D^r \partial_x u_l, D^r u_l\right)_{L^2}\right| + \|u_l\|_{H^r}\|F(u_l)\|_{H^r} \\ &\lesssim \|u_l^k\|_{H^r}\|\partial_x u_l\|_{L^\infty}\|u_l\|_{H^r} + \|\partial_x u_l\|_{L^\infty}\|u_l\|_{L^\infty}^{k-1}\|u_l\|_{H^r}^2 + \|u_l\|_{W^{1,\infty}}^k\|u_l\|_{H^r}^2 \\ &\le C\|u_l\|_{H^r}^{k+2}, \quad C = C_r > 0.\end{aligned}$$

Solving the above inequality gives

$$\|u_l\|_{H^r} \le \frac{\|u_l(0)\|_{H^r}}{\left(1 - Ckt\|u_l(0)\|_{H^r}^k\right)^{\frac{1}{k}}}, \quad 0 \le t < \frac{1}{Ck\|u_l(0)\|_{H^r}^k}.$$

Therefore, we arrive at

$$\|u_l\|_{H^r} \le 2\|u_l(0)\|_{H^r}, \quad t \in [0, T_{m,n}], \quad T_{m,n} = \frac{1}{2Ck\|u_l(0)\|_{H^r}^k}. \tag{4.10}$$

By **Step 1**, we have $T_{m,n} \gtrsim \frac{1}{2Ckn^{k(\frac{\delta}{2}-\frac{1}{k})}} \to \infty$, as $n \to \infty$. Consequently, we can find a common time interval $[0, T_l]$ such that

$$\|u_l\|_{H^r} \leq 2\|u_l(0)\|_{H^r} \leq C|m|n^{\frac{\delta}{2}-\frac{1}{k}}, \quad t \in [0, T_l],$$

which is (4.9).

Step 3: Prove (4.9) **for** $0 < r \leq 3/2$. Similarly, by applying Lemmas A.3 and A.5, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r}^2 \\
&\quad \leq \left|\left(D^r u_l, D^r\left(u_l^k \partial_x u_l\right)\right)_{L^2}\right| + \left|\left(D^r u_l, D^r F(u_l)\right)_{L^2}\right| \\
&\quad \leq \left|\left([D^r, u_l^k]\partial_x u_l, D^r u_l\right)_{L^2}\right| + \left|\left(u_l^k D^r \partial_x u_l, D^r u_l\right)_{L^2}\right| + \|u_l\|_{H^r}\|F(u_l)\|_{H^r} \\
&\quad \lesssim \|u_l^k\|_{H^r}\|\partial_x u_l\|_{L^\infty}\|u_l\|_{H^r} + \|\partial_x u_l\|_{L^\infty}\|u_l\|_{L^\infty}^{k-1}\|u_l\|_{H^r}^2 \\
&\qquad + \|u_l\|_{H^r}\|u_l\|_{W^{1,\infty}}^k\left(\|u_l\|_{H^r} + \|\partial_x u_l\|_{H^r}\right).
\end{aligned}$$

It follows from the embedding $H^{r+\frac{3}{2}} \hookrightarrow H^{r+1}$ and $H^{r+\frac{3}{2}} \hookrightarrow W^{1,\infty}$ that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r}^2 &\lesssim \|u_l^k\|_{H^r}\|\partial_x u_l\|_{L^\infty}\|u_l\|_{H^r} + \|\partial_x u_l\|_{L^\infty}\|u_l\|_{L^\infty}^{k-1}\|u_l\|_{H^r}^2 \\
&\quad + \|u_l\|_{H^r}\|u_l\|_{W^{1,\infty}}^k\left(\|u_l\|_{H^r} + \|\partial_x u_l\|_{H^r}\right) \\
&\lesssim \|u_l\|_{W^{1,\infty}}^k\|u_l\|_{H^r}^2 + \|u_l\|_{W^{1,\infty}}^k\|u_l\|_{H^r}\|u_l\|_{H^{r+1}} \\
&\lesssim \|u_l\|_{H^{r+\frac{3}{2}}}^k\|u_l\|_{H^r}^2 + \|u_l\|_{H^r}\|u_l\|_{H^{r+\frac{3}{2}}}^{k+1}.
\end{aligned}$$

Using the conclusion of **Step 2** for $r + \frac{3}{2} > \frac{3}{2}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_l\|_{H^r} \lesssim \|u_l\|_{H^r}\|u_l(0)\|_{H^{r+\frac{3}{2}}}^k + \|u_l(0)\|_{H^{r+\frac{3}{2}}}^{k+1}, \quad t \in [0, T_l],$$

and hence

$$\|u_l(t)\|_{H^r} \lesssim \|u_l(0)\|_{H^r} + \|u_l(0)\|_{H^{r+\frac{3}{2}}}^{k+1} T_l + \int_0^t \|u_l\|_{H^r}\|u_l(0)\|_{H^{r+\frac{3}{2}}}^k \mathrm{d}t', \quad t \in [0, T_l].$$

Applying Grönwall's inequality to the above inequality, we have

$$\|u_l\|_{H^r} \lesssim \left(\|u_l(0)\|_{H^r} + \|u_l(0)\|_{H^{r+\frac{3}{2}}}^{k+1} T_l\right) \exp\left\{\|u_l(0)\|_{H^{r+\frac{3}{2}}}^k T_l\right\}, \quad t \in [0, T_l].$$

Since $\delta \in (0, 2/k)$, we can infer from **Step 1** that $\exp\left\{\|u_l(0)\|_{H^{r+\frac{3}{2}}}^k T_l\right\} < C(T_l)$ for some constant $C(T_l) > 0$ and $\|u_l(0)\|_{H^{r+\frac{3}{2}}}^{k+1} \leq \|u_l(0)\|_{H^{r+\frac{3}{2}}} \leq C|m|n^{\frac{\delta}{2}-\frac{1}{k}}$. Hence we

see that there is a constant $C = C_{r,\tilde{\phi},T_l} > 0$ such that

$$\|u_l\|_{H^r} \le C|m|n^{\frac{\delta}{2}-\frac{1}{k}}, \quad t \in [0, T_l],$$

which is (4.9). □

Recall the approximative solution defined by (4.3). The above result means that the H^s-norm of the low-frequency part u_l is decaying. For the high-frequency part u_h, as in Lemma A.8, its H^s-norm is bounded.

4.2.2 Estimate on $\mathcal{E}$

Recall the error $\mathcal{E}$ given in (4.8). By using (4.1) and (4.2), we have $m = m^k$ and $\phi = \tilde{\phi}^k\phi$ for all $k \ge 1$. Then by (4.4) and $u_l(0, x)$ in (4.5), we see that as long as $m \ne 0$,

$$\begin{aligned}
&u_h(t, x) - u_h(0, x)\\
&= n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt) - n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx)\\
&= m^{-1}m^k\tilde{\phi}^k\left(\frac{x}{n^\delta}\right)n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt)\\
&\quad - m^{-1}m^k\tilde{\phi}^k\left(\frac{x}{n^\delta}\right)n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx)\\
&= m^{-1}u_l^k(0, x)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx - mt)\\
&\quad - m^{-1}u_l^k(0, x)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\cos(nx)\\
&= \int_0^t u_l^k(0, x)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx - mt')\,\mathrm{d}t'.
\end{aligned}$$

If $m = 0$, then $u_l = 0$ and we also have

$$u_h(t, x) - u_h(0, x) = \int_0^t 0\ \mathrm{d}t' = \int_0^t u_l^k(0, x)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx - mt')\,\mathrm{d}t'.$$

To sum up, we find that for all $k \ge 1$, m satisfying (4.2), u_h given by (4.4) and $u_l(0, x)$ in (4.5),

$$u_h(t, x) - u_h(0, x) = \int_0^t u_l^k(0, x)n^{1-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx - mt')\,\mathrm{d}t'. \tag{4.11}$$

On the other hand, for all $k \ge 1$,

$$\int_0^t u_l^k \partial_x u_h \, \mathrm{d}t' = -\int_0^t u_l^k(t') n^{1-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \sin(nx - mt') \, \mathrm{d}t' + \int_0^t u_l^k(t') n^{-\frac{3\delta}{2}-s} \partial_x \phi\left(\frac{x}{n^\delta}\right) \cos(nx - mt') \, \mathrm{d}t'. \tag{4.12}$$

Combining (4.11), (4.12) and (1.5) into (4.8) yields

$$\mathcal{E}(\omega, t, x) = \sum_{i=1}^4 \int_0^t E_i \, \mathrm{d}t' - \int_0^t h(t', u_{m,n}) \, \mathrm{d}\mathcal{W}, \quad k \geq 1 \ \ \mathbb{P}\text{-a.s.}, \tag{4.13}$$

where

$$\begin{aligned}
E_1 :=& [u_l^k(0) - u_l^k(t)] n^{1-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \sin(nx - mt) \\
&+ u_l^k(t) n^{-\frac{3\delta}{2}-s} \partial_x \phi\left(\frac{x}{n^\delta}\right) \cos(nx - mt) + \mathcal{Z}_k(\partial_x u_l + \partial_x u_h), \\
E_2 :=& F_1(u_l + u_h) - F_1(u_l) = D^{-2} \partial_x \mathcal{Z}_{k+1}, \\
E_3 :=& F_2(u_l + u_h) - F_2(u_l) \\
=& \frac{2k-1}{2} D^{-2} \partial_x \left\{ u_l^{k-1} \left[2(\partial_x u_l)(\partial_x u_h) + (\partial_x u_h)^2 \right] + \mathcal{Z}_{k-1}(\partial_x u_l + \partial_x u_h)^2 \right\}, \\
E_4 :=& F_3(u_l + u_h) - F_3(u_l) \\
=& \frac{k-1}{2} D^{-2} \Big\{ u_l^{k-2} \big[3(\partial_x u_l)^2 (\partial_x u_h) + 3(\partial_x u_l)(\partial_x u_h)^2 + (\partial_x u_h)^3 \big] \\
&+ \mathcal{Z}_{k-2}(\partial_x u_l + \partial_x u_h)^3 \Big\}.
\end{aligned}$$

We remark here that E_4 disappears when $k = 1$. Recalling $\rho_0 \in (1/2, 1)$ in Hypothesis H_2, now we shall estimate the H^{ρ_0}-norm of the error $\mathcal{E}$. Actually, we will show that the H^{ρ_0}-norm of $\mathcal{E}$ is decaying.

Lemma 4.2 *Let $T_l > 0$ be given in Lemma 4.1, and $\rho_0 \in (1/2, 1)$ be given in H_2. Let $n \gg 1$, $s > 5/2$. Let*

$$\begin{cases} \dfrac{2}{3} < \delta < 1, \ \textit{when } k = 1, \\ \dfrac{2}{k} - \dfrac{2}{2k-1} < \delta < \dfrac{1}{k}, \ \textit{when } k \geq 2, \end{cases} \tag{4.14}$$

and

$$0 > r_s = -s - 1 + \rho_0 + k\delta, \quad k \geq 1. \tag{4.15}$$

Then the error $\mathcal{E}$ given by (4.13) satisfies that for some $C = C(T_l) > 0$,

$$\mathbb{E} \sup_{t \in [0, T_l]} \|\mathcal{E}(t)\|_{H^{\rho_0}}^2 \leq C n^{2r_s}.$$

Proof The proof is technical and it is given in Appendix B. □

4.2.3 Estimate on $u_{m,n} - u^{m,n}$

Recall the approximative solutions $u_{m,n}$ given by (4.3). Then we have the following estimates on the difference between the actual solutions and the approximative solutions.

Lemma 4.3 *Let $k \geq 1$, $s > 5/2$ and ρ_0 be given in H_2. Let (4.14) hold true and $r_s < 0$ be given in (4.15). For any $R > 1$, we define*

$$\tau_R^{m,n} := \inf\{t > 0 : \|u^{m,n}\|_{H^s} > R\}. \tag{4.16}$$

Then for $n \gg 1$,

$$\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|(u_{m,n} - u^{m,n})(t)\|^2_{H^{\rho_0}} \leq Cn^{2r_s}, \tag{4.17}$$

$$\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|(u_{m,n} - u^{m,n})(t)\|^2_{H^{2s-\rho_0}} \leq Cn^{2s-2\rho_0}, \tag{4.18}$$

where $T_l > 0$ is given in Lemma 4.1 and $C = C(R, T_l) > 0$.

Proof Let $v = v_{m,n} = u_{m,n} - u^{m,n}$. Then v satisfies $v(0) = 0$ and

$$v(t) + \int_0^t \left(\frac{1}{k+1}\partial_x(Pv) + F(u_{m,n}) - F(u^{m,n})\right) \mathrm{d}t' = -\int_0^t h(t', u^{m,n})\,\mathrm{d}\mathcal{W} + \sum_{i=1}^4 \int_0^t E_i\,\mathrm{d}t',$$

where

$$P = P_{m,n} = u_{m,n}^k + u_{m,n}^{k-1}u^{m,n} + \cdots + u_{m,n}(u^{m,n})^{k-1} + (u^{m,n})^k, \quad k \geq 1.$$

On $[0, T_l]$, by the Itô formula, we have that

$$\begin{aligned}\|v(t)\|^2_{H^{\rho_0}} = &-2\int_0^t \left(h(t', u^{m,n})\,\mathrm{d}\mathcal{W}, v\right)_{H^{\rho_0}} + 2\sum_{i=1}^4 \int_0^t (E_i, v)_{H^{\rho_0}}\,\mathrm{d}t' \\ &- \frac{2}{k+1}\int_0^t (\partial_x(Pv), v)_{H^{\rho_0}}\,\mathrm{d}t' - 2\int_0^t ([F(u_{m,n}) - F(u^{m,n})], v)_{H^{\rho_0}}\,\mathrm{d}t' \\ &+ \int_0^t \|h(t', u^{m,n})\|^2_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}\,\mathrm{d}t'.\end{aligned}$$

Taking supremum with respect to $t \in [0, T_l \wedge \tau_R^{m,n}]$, and then using the BDG inequality yield

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|v(t)\|_{H^{\rho_0}}^2 \\
&\le C\mathbb{E}\left(\int_0^{T_l\wedge\tau_R^{m,n}} \|v\|_{H^{\rho_0}}^2 \|h(t,u^{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2 \,\mathrm{d}t\right)^{1/2} \\
&\quad + 2\sum_{i=1}^4 \mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} |(E_i, v)_{H^{\rho_0}}| \,\mathrm{d}t \\
&\quad + \frac{2}{k+1}\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} |(\partial_x(Pv), v)_{H^{\rho_0}}| \,\mathrm{d}t \\
&\quad + 2\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \left|\left(F(u_{m,n}) - F(u^{m,n}), v\right)_{H^{\rho_0}}\right| \mathrm{d}t \\
&\quad + \mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \|h(t,u^{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2 \,\mathrm{d}t.
\end{aligned}$$

It follows from Lemmas A.8 and 4.1 that $\|u_{m,n}\|_{H^s} \lesssim 1$ on $[0, \tau_R^{m,n} \wedge T_l]$. Hence we can infer from Hypothesis H_2 that

$$\begin{aligned}
\|h(t,u^{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2 &\lesssim \|h(t,u_{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2 + \|h(t,u_{m,n}) - h(t,u^{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2 \\
&\lesssim \left(\mathrm{e}^{\frac{-1}{\|u_{m,n}\|_{H^{\rho_0}}}}\right)^2 + g_3^2(CR)\|v\|_{H^{\rho_0}}^2, \quad t \in [0, \tau_R^{m,n} \wedge T_l] \ \mathbb{P}\text{-a.s.},
\end{aligned}$$

where $g_3(\cdot)$ is given in $H_2(2)$. As a result, for any fixed $s > 5/2$, by applying Lemmas A.8 and 4.1 again, we can pick $N = N(s,k) \gg 1$ to derive

$$\begin{aligned}
\|h(t,u^{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2 &\lesssim \|u_{m,n}\|_{H^{\rho_0}}^{2N} + g_3^2(CR)\|v\|_{H^{\rho_0}}^2 \\
&\lesssim \left(n^{N(-s+\rho_0)} + n^{N(\frac{\delta}{2}-\frac{1}{k})}\right)^2 + g_3^2(CR)\|v\|_{H^{\rho_0}}^2 \\
&\lesssim n^{2r_s} + g_3^2(CR)\|v\|_{H^{\rho_0}}^2, \quad t \in [0, \tau_R^{m,n} \wedge T_l] \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Consequently, we can infer from the above inequalities that

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|v(t)\|_{H^{\rho_0}}^2 \\
&\le \frac{1}{2}\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|v(t)\|_{H^{\rho_0}}^2 + 2\sum_{i=1}^4 \mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} |(E_i, v)_{H^{\rho_0}}| \,\mathrm{d}t \\
&\quad + \frac{2}{k+1}\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} |(\partial_x(Pv), v)_{H^{\rho_0}}| \,\mathrm{d}t
\end{aligned}$$

$$+ 2\mathbb{E}\int_0^{T_l\wedge\tau_R^{m,n}} \left|\left(F(u_{m,n}) - F(u^{m,n}),\, v\right)_{H^{\rho_0}}\right| \mathrm{d}t$$
$$+ C(T_l)n^{2r_s} + C_R\int_0^{T_l} \mathbb{E}\sup_{t'\in[0,t\wedge\tau_R^{m,n}]} \|v(t')\|_{H^{\rho_0}}^2 \,\mathrm{d}t.$$

Via Lemma 4.2, we have

$$2\sum_{i=1}^4 |(E_i, v)_{H^{\rho_0}}| \le 2\sum_{i=1}^4 \|E_i\|_{H^{\rho_0}}\|v\|_{H^{\rho_0}}$$
$$\lesssim \sum_{i=1}^4 \|E_i\|_{H^{\rho_0}}^2 + \|v\|_{H^{\rho_0}}^2 \lesssim C(T_l)n^{2r_s} + \|v\|_{H^{\rho_0}}^2.$$

Using Lemma A.4 and integration by parts, we obtain that

$$\begin{aligned}&\left|(D^{\rho_0}\partial_x(Pv), D^{\rho_0}v)_{L^2}\right|\\ &= \left|([D^{\rho_0}\partial_x, P]v, D^{\rho_0}v)_{L^2} + (PD^{\rho_0}\partial_x v, D^{\rho_0}v)_{L^2}\right|\\ &\lesssim \|P\|_{H^s}\|v\|_{H^{\rho_0}}^2 + \|P_x\|_{L^\infty}\|v\|_{H^{\rho_0}}^2 \lesssim (\|u^{m,n}\|_{H^s} + \|u_{m,n}\|_{H^s})^k\|v\|_{H^{\rho_0}}^2.\end{aligned}$$

Then, we use Lemma A.5 to find that

$$\begin{aligned}\left|\left(F(u_{m,n}) - F(u^{m,n}), v\right)_{H^{\rho_0}}\right| &\lesssim \|F(u_{m,n}) - F(u^{m,n})\|_{H^{\rho_0}}\|v\|_{H^{\rho_0}}\\ &\lesssim \|F(u_{m,n}) - F(u^{m,n})\|_{H^{\rho_0}}^2 + \|v\|_{H^{\rho_0}}^2\\ &\lesssim (\|u^{m,n}\|_{H^s} + \|u_{m,n}\|_{H^s})^{2k}\|v\|_{H^{\rho_0}}^2 + \|v\|_{H^{\rho_0}}^2,\end{aligned}$$

To sum up, by (4.16), Lemmas 4.1 and A.8, we arrive at

$$\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|v(t)\|_{H^{\rho_0}}^2 \le C(T_l)n^{2r_s} + C_R\int_0^{T_l} \mathbb{E}\sup_{t'\in[0,t\wedge\tau_R^{m,n}]} \|v(t')\|_{H^{\rho_0}}^2 \,\mathrm{d}t.$$

Using the Grönwall inequality, we obtain (4.17).

Now we prove (4.18). Since $2s - \rho_0 > s > \frac{5}{2}$ and $u^{m,n}$ is the unique solution to (4.6), similar to (2.5), we can use (4.16) and $H_2(1)$ to find for each fixed $n \in \mathbb{N}$ that

$$\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|u^{m,n}(t)\|_{H^{2s-\rho_0}}^2$$
$$\le C\mathbb{E}\|u_{m,n}(0)\|_{H^{2s-\rho_0}}^2 + C_R\int_0^{T_l} \mathbb{E}\sup_{t'\in[0,t\wedge\tau_R^{m,n}]} \|u^{m,n}(t')\|_{H^{2s-\rho_0}}^2 \,\mathrm{d}t.$$

Using the Grönwall inequality and Lemmas 4.1 and A.8, we find a constant $C = C(R, T_l)$ such that for all $n \geq 1$,

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|u^{m,n}(t)\|^2_{H^{2s-\rho_0}} &\leq C\mathbb{E}\|u_{m,n}(0)\|^2_{H^{2s-\rho_0}} \\ &\leq C(n^{\frac{\delta}{2}-\frac{1}{k}} + n^{s-\rho_0})^2 \leq Cn^{2s-2\rho_0}.\end{aligned}$$

Hence, by Lemmas 4.1 and A.8 again, we arrive at

$$\begin{aligned}&\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|(u^{m,n} - u_{m,n})(t)\|^2_{H^{2s-\rho_0}} \\ &\leq 2\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|u^{m,n}(t)\|^2_{H^{2s-\rho_0}} + 2\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_R^{m,n}]} \|u_{m,n}(t)\|^2_{H^{2s-\rho_0}} \\ &\leq Cn^{2s-2\rho_0}, \quad n \geq 1.\end{aligned}$$

The proof is therefore completed. □

4.3 Finish the proof for Theorem 1.2

To begin with, we observe the following property:

Lemma 4.4 *Let $H_2(1)$ hold true. Suppose that for some $R_0 \gg 1$, the R_0-exiting time of the zero solution to (1.4) is strongly stable. Then we have*

$$\lim_{n\to\infty} \tau_{R_0}^{m,n} = \infty \ \mathbb{P}\text{-a.s.} \tag{4.19}$$

Proof By $H_2(1)$, the unique solution with zero initial data to (1.4) is zero. On the other hand, we notice that for all $s' < s$, $\lim_{n\to\infty} \|u_{m,n}(0)\|_{H^{s'}} = \lim_{n\to\infty} \|u_{m,n}(0) - 0\|_{H^{s'}} = 0$. Since the R_0-exiting time of the zero solution is ∞, we see that (4.19) holds provided that the R_0-exiting time of the zero solution to (1.4) is strongly stable. □

Proof for Theorem 1.2 Our strategy is to show that if the R_0-exiting time is strongly stable at the zero solution for some $R_0 \gg 1$, then $\{u^{-1,n}\}$ and $\{u^{1,n}\}$ (if k is odd) or $\{u^{0,n}\}$ and $\{u^{1,n}\}$ (if k is even) are two sequences of solutions such that (1.16), (1.17), (1.18) and (1.19) are satisfied.

For each $n > 1$ and for fixed $R_0 \gg 1$, Lemmas 4.1, A.8 and (4.16) give $\mathbb{P}\{\tau_{R_0}^{m,n} > 0\} = 1$, and Lemma 4.4 implies (1.16). Then, it follows from (4.16) that $u^{m,n} \in C([0, \tau_{R_0}^{m,n}]; H^s)$ $\mathbb{P}$-a.s. and (1.17) holds true. Next, we check (1.18). By interpolation, we have

$$\begin{aligned}&\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^s} \\ &\leq C\left(\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^{\rho_0}}\right)^{\frac{1}{2}} \left(\mathbb{E} \sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]} \|u_{m,n} - u^{m,n}\|_{H^{2s-\rho_0}}\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq C\left(\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]}\|u_{m,n}-u^{m,n}\|_{H^{\rho_0}}^2\right)^{\frac{1}{4}}\left(\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]}\|u_{m,n}-u^{m,n}\|_{H^{2s-\rho_0}}^2\right)^{\frac{1}{4}}.$$

For $T_l>0$, combining Lemma 4.3 and the above estimate yields

$$\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]}\|u_{m,n}-u^{m,n}\|_{H^s}\leq C(R_0,T_l)n^{\frac{1}{4}\cdot 2r_s}\cdot n^{\frac{1}{4}\cdot(2s-2\rho_0)}=C(R_0,T_l)n^{r_s'},$$

where r_s is defined by (4.14) and $r_s'=r_s\cdot\frac{1}{2}+(s-\rho_0)\cdot\frac{1}{2}=\frac{k\delta-1}{2}<0$. Consequently, we can deduce that

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{m,n}]}\|u_{m,n}-u^{m,n}\|_{H^s}=0. \tag{4.20}$$

When k is odd,

$$\begin{aligned}\|u^{-1,n}(0)-u^{1,n}(0)\|_{H^s}&=\|u_{-1,n}(0)-u_{1,n}(0)\|_{H^s}\\&=2\left\|n^{-\frac{1}{k}}\tilde{\phi}\left(\frac{x}{n^\delta}\right)\right\|_{H^s}\to 0,\ \text{as } n\to\infty.\end{aligned}$$

When k is even

$$\begin{aligned}\|u^{0,n}(0)-u^{1,n}(0)\|_{H^s}&=\|u_{0,n}(0)-u_{1,n}(0)\|_{H^s}\\&=\left\|n^{-\frac{1}{k}}\tilde{\phi}\left(\frac{x}{n^\delta}\right)\right\|_{H^s}\to 0,\ \text{as } n\to\infty.\end{aligned}$$

The above two estimates imply that (1.18) holds true.

Now we prove (1.19). Let $T_l>0$ be given in Lemma 4.1. When k is odd, we use (4.20) to derive

$$\begin{aligned}&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u^{-1,n}(t)-u^{1,n}(t)\|_{H^s}\\&\gtrsim\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}\\&\quad-\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u^{-1,n}(t)\|_{H^s}\\&\quad-\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{-1,n}^{R}\wedge\tau_{1,n}^{R}]}\|u_{1,n}(t)-u^{1,n}(t)\|_{H^s}\\&\gtrsim\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}.\end{aligned}$$

It follows from the construction of $u_{m,n}$, Fatou's lemma, Lemmas 4.1, A.8 and 4.4 that

$$\begin{aligned}
&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}\\
&=\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}\Big\|-2n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx)\sin(t)\\
&\qquad\qquad+[u_{l,-1,n}(t)-u_{l,1,n}(t)]\Big\|_{H^s}\\
&\gtrsim\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T_l\wedge\tau_{R_0}^{-1,n}\wedge\tau_{R_0}^{1,n}]}n^{-\frac{\delta}{2}-s}\left\|\phi\left(\frac{x}{n^\delta}\right)\sin(nx)\right\|_{H^s}|\sin t|-\limsup_{n\to\infty}n^{\frac{\delta}{2}-\frac{1}{k}}\\
&\gtrsim\sup_{t\in[0,T_l]}|\sin t|,
\end{aligned}\tag{4.21}$$

which is (1.19) in the case that k is odd. When k is even, one has

$$\begin{aligned}
\|u_{0,n}(t)-u_{1,n}(t)\|_{H^s}&=\left\|-2n^{-\frac{\delta}{2}-s}\phi\left(\frac{x}{n^\delta}\right)\sin(nx-t/2)\sin(t/2)-u_{l,1,n}(t)\right\|_{H^s}\\
&\gtrsim n^{-\frac{\delta}{2}-s}\left\|\phi\left(\frac{x}{n^\delta}\right)\sin(nx-t/2)\right\|_{H^s}|\sin(t/2)|-n^{\frac{\delta}{2}-\frac{1}{k}}.
\end{aligned}$$

Similar to (4.21), we can also obtain (1.19) in the case that k is even. The proof is completed. □

4.4 Example

Now we give an example of noise structure satisfying Hypothesis H_2. For simplicity, we consider the case that $h(t,u)\,\mathrm{d}\mathcal{W}=b(t,u)\,\mathrm{d}W$, where W is a standard 1-D Brownian motion. Let $m\geq 1$ and $f(\cdot)$ be a continuous and bounded function, then

$$b(t,u)=f(t)\mathrm{e}^{-\frac{1}{\|u\|_{H^{\rho_0}}}}u^m,$$

satisfies Hypothesis H_2.

5 Noise prevents blow up

5.1 Proof for Theorem 1.3

Our approach is motivated by [6, 45]. Let $s>5/2$ and u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2<\infty$. With $H_3(1)$ and $H_3(2)$ at hand, one can follow the steps in the proof for Theorem 1.1 to obtain a unique solution u to (1.6) such that $u\in C([0,\tau^*);H^s)$ $\mathbb{P}$-a.s. and

$$\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\}}=\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{W^{1,\infty}}=\infty\}}\ \mathbb{P}\text{-a.s.}\tag{5.1}$$

Here we remark that $H_3(2)$ is the condition of locally Lipschitz continuous in H^σ with $\sigma > 3/2$, hence uniqueness can only be considered for solution in H^s with $s > 5/2$. This is because, if two solutions to (1.6) belong to H^s, the difference between them can be only estimated in $H^{s'}$ for $s' \le s-1$ (Recalling (3.9), H^{s+1}-norm appears).

Define

$$\tau_m := \inf\left\{t \ge 0 : \|u(t)\|_{H^{s-1}} \ge m\right\}, \quad m \ge 1 \ \text{ and } \ \widetilde{\tau^*} := \lim_{m\to\infty} \tau_m.$$

Due to (5.1), we have $\tau_m < \widetilde{\tau^*} = \tau^*$ $\mathbb{P}$-a.s. and hence we only need to show

$$\widetilde{\tau^*} = \infty \ \ \mathbb{P}\text{-a.s.}. \tag{5.2}$$

For $V \in \mathcal{V}$, applying the Itô formula to $\|u(t)\|^2_{H^{s-1}}$ and then to $V(\|u\|^2_{H^{s-1}})$, we find

$$\begin{aligned}
\mathrm{d}V(\|u\|^2_{H^{s-1}}) =& 2V'(\|u\|^2_{H^{s-1}})\,(q(t,u),u)_{H^{s-1}}\,\mathrm{d}W \\
&+ V'(\|u\|^2_{H^{s-1}})\left\{-2\left(u^k u_x, u\right)_{H^{s-1}} - 2\,(F(u),u)_{H^{s-1}}\right\}\mathrm{d}t \\
&+ V'(\|u\|^2_{H^{s-1}})\|q(t,u)\|^2_{H^{s-1}}\,\mathrm{d}t \\
&+ 2V''(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|^2\mathrm{d}t.
\end{aligned}$$

Next, we recall $\tau_m < \widetilde{\tau^*} = \tau^*$ and $s-1 > 3/2$, take expectation and then use Hypothesis H_3 and Lemma A.6 to find that

$$\begin{aligned}
&\mathbb{E}V(\|u(t\wedge\tau_m)\|^2_{H^{s-1}}) \\
&= \mathbb{E}V(\|u_0\|^2_{H^{s-1}}) \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_m} V'(\|u\|^2_{H^{s-1}})\left\{-2\left(u^k u_x,u\right)_{H^{s-1}} - 2\,(F(u),u)_{H^{s-1}}\right\}\mathrm{d}t' \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_m} V'(\|u\|^2_{H^{s-1}})\|q(t',u)\|^2_{H^{s-1}}\,\mathrm{d}t' \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_m} 2V''(\|u\|^2_{H^{s-1}})\left|\left(q(t',u),u\right)_{H^{s-1}}\right|^2\mathrm{d}t' \\
&\le \mathbb{E}V(\|u_0\|^2_{H^{s-1}}) + \mathbb{E}\int_0^{t\wedge\tau_m} \mathcal{H}_{s-1}(t',u)\,\mathrm{d}t' \\
&\le \mathbb{E}V(\|u_0\|^2_{H^{s-1}}) + N_1 t - \mathbb{E}\int_0^{t\wedge\tau_m} N_2\frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|\left(q(t',u),u\right)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t',
\end{aligned}$$

where $\mathcal{H}_\sigma(t,u)$ ($u \in H^\sigma$ and $\sigma > 3/2$) is defined in Hypothesis $H_3(3)$. Then we can infer from the above estimate that there is a constant $C(u_0, N_1, N_2, t) > 0$ such that

$$\mathbb{E}\int_0^{t\wedge\tau_m}\frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|\left(q(t',u),u\right)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t' \le C(u_0,N_1,N_2,t). \tag{5.3}$$

Next, for any $T > 0$, it follows from the BDG inequality that

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T\wedge\tau_m]} V(\|u\|^2_{H^{s-1}}) - \mathbb{E}V(\|u_0\|^2_{H^{s-1}}) \\
&\quad\le C\mathbb{E}\left(\int_0^{T\wedge\tau_m} \left\{V'(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|\right\}^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad + N_1 T + N_2\mathbb{E}\int_0^{T\wedge\tau_m} \frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\quad\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\left(1+V(\|u\|^2_{H^{s-1}})\right) \\
&\qquad + C\mathbb{E}\int_0^{T\wedge\tau_m} \frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\qquad + N_1 T + N_2\mathbb{E}\int_0^{T\wedge\tau_m} \frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t.
\end{aligned}$$

Thus we use (5.3) to obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]} V(\|u\|^2_{H^{s-1}}) \\
&\quad\le 1 + 2\mathbb{E}V(\|u_0\|^2_{H^{s-1}}) + C\mathbb{E}\int_0^{T\wedge\tau_m} \frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\qquad + 2N_1 T + 2N_2\mathbb{E}\int_0^{T\wedge\tau_m} \frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\quad\le C(u_0,N_1,T) + C(N_2)\mathbb{E}\int_0^{T\wedge\tau_m} \frac{\left\{V'(\|u\|^2_{H^{s-1}})\left|(q(t,u),u)_{H^{s-1}}\right|\right\}^2}{1+V(\|u\|^2_{H^{s-1}})}\,\mathrm{d}t \\
&\quad\le C(u_0,N_1,N_2,T).
\end{aligned}$$

As a result, for all $m \ge 1$,

$$\begin{aligned}
\mathbb{P}\{\widetilde{\tau^*} < T\} &\le \mathbb{P}\{\tau_m < T\} \\
&\le \mathbb{P}\left\{\sup_{t\in[0,T\wedge\tau_m]} V(\|u\|^2_{H^{s-1}}) \ge V(m^2)\right\} \le \frac{C(u_0,N_1,N_2,T)}{V(m^2)}.
\end{aligned}$$

Since $\mathbb{P}\{\widetilde{\tau^*} < T\}$ does not depend on m, sending $m \to \infty$ gives rise to $\mathbb{P}\{\tau^* < T\} = 0$. Since $T > 0$ is arbitrary, we obtain (5.2), which completes the proof for Theorem 1.3.

5.2 Example

As in (1.12), for the solution to (1.4), its H^s-norm blows up if and only if its $W^{1,\infty}$-norm blows up. On the other hand, $H_3(3)$ means that the growth of $2\lambda_s\|u\|^k_{W^{1,\infty}}\|u\|^2_{H^s}$ can be canceled by $2V''(\|u\|^2_{H^s})|(q(t,u),u)_{H^s}|$. Motivated by these two observations, we consider the following examples where the $W^{1,\infty}$-norm of u will be involved, that is,

$$q(t,u)=\beta(t,\|u\|_{W^{1,\infty}})u, \tag{5.4}$$

where $\beta(t,x)$ satisfies the following conditions:

Hypothesis $\mathbf{H_4}$ We assume that

- The function $\beta(t,x)\in C([0,\infty)\times[0,\infty))$ such that for any $x\geq 0$, $\beta(\cdot,x)$ is bounded as a function of t, and for all $t\geq 0$, $\beta(t,\cdot)$ is locally Lipschitz continuous as a function of x;
- The function $\beta(t,x)\neq 0$ for all $(t,x)\in[0,\infty)\times[0,\infty)$, and $\limsup_{x\to+\infty}\frac{2\lambda_s x^k}{\beta^2(t,x)}<1$ for all $t\geq 0$, where $\lambda_s>0$ is given in Lemma A.6.

Now we give a concrete example $\beta(t,x)$ satisfying Hypothesis H_4. Let $b:[0,\infty)\to[0,\infty)$ be a continuous function and there are constants $b_*, b^*>0$ such that $b_*\leq b^2(t)\leq b^*<\infty$ for all t. For all $k\geq 1$, if

$$\text{either } \theta>k/2,\ b^*>b_*>0 \text{ or } \theta=k/2,\ b^*>b_*>2\lambda_s,$$

then $\beta(t,x)=b(t)(1+x)^\theta$ satisfies Hypothesis H_4. Moreover, by the following two lemmas, we will see that $q(t,u)=b(t)(1+\|u\|_{W^{1,\infty}})^\theta u$ satisfies Hypothesis H_3.

Lemma 5.1 *Let λ_s be given in Lemma A.6. Let $K>0$. If Hypothesis H_4 holds true, then there is an $M_1>0$ such that for any $M_2>0$ and all $0<x\leq Ky<\infty$,*

$$\frac{2\lambda_s x^k y^2+\beta^2(t,x)y^2}{1+y^2}-\frac{2\beta^2(t,x)y^4}{(1+y^2)^2}\leq M_1-M_2\frac{2\beta^2(t,x)y^4}{(1+y^2)^2(1+\log(1+y^2))}. \tag{5.5}$$

Proof By Hypothesis H_4, we have

$$\begin{aligned}&\limsup_{x\to+\infty}\frac{2\lambda_s x^k y^2+\beta^2(t,x)y^2}{1+y^2}-\frac{2\beta^2(t,x)y^4}{(1+y^2)^2}+M_2\frac{2\beta^2(t,x)y^4}{(1+y^2)^2(1+\log(1+y^2))}\\ &\leq\limsup_{x\to+\infty}\left(\frac{2\lambda_s x^k}{\beta^2(t,x)}+1-\frac{2\left(\frac{x}{K}\right)^4}{\left(1+\left(\frac{x}{K}\right)^2\right)^2}+M_2\frac{2}{\left(1+\log\left(1+\left(\frac{x}{K}\right)^2\right)\right)}\right)\beta^2(t,x)<0,\end{aligned}$$

which implies (5.5). □

Lemma 5.2 *If $\beta(t,x)$ satisfies Hypothesis H_4, then $q(t,u)$ defined by (5.4) satisfies Hypothesis H_3.*

Proof It follows from Lemma 5.1 that $H_3(3)$ holds true with the choice $V(x) = \log(1+x) \in \mathcal{V}$. Since $H^s \hookrightarrow W^{1,\infty}$ with $s > 3/2$, it is obvious that the other requirements in Hypothesis H_3 are verified. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest and data sharing is not applicable to this article since no datasets were generated or analyzed during the current study.

A Appendix: Auxiliary results

In this appendix we formulate and prove some estimates employed in the above proofs. We first recall the Friedrichs mollifier J_ε defined as

$$[J_\varepsilon f](x) = [j_\varepsilon \star f](x), \quad \varepsilon \in (0,1), \tag{A.1}$$

where $\star$ stands for the convolution, $j_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon})$ and $j(x)$ is a Schwartz function satisfying $\widehat{j}(\xi) : \mathbb{R} \to [0,1]$ and $\widehat{j}(\xi) = 1$ for $\xi \in [-1,1]$. From the above construction, we have

Lemma A.1 [41, 48] *For all $\varepsilon \in (0,1)$, $s, r \in \mathbb{R}$ and $u \in H^s$, J_ε constructed in (A.1) satisfies*

$$\|I - J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \lesssim \varepsilon^{s-r}, \quad \|u - J_\varepsilon u\|_{H^r} \sim o(\varepsilon^{s-r}), \quad r \le s,$$

$$\|J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \sim O(\varepsilon^{s-r}), \quad r > s,$$

and

$$[D^s, J_\varepsilon] = 0, \quad (J_\varepsilon f, g)_{L^2} = (f, J_\varepsilon g)_{L^2}, \quad \|J_\varepsilon\|_{\mathcal{L}(L^\infty;L^\infty)} \lesssim 1, \quad \|J_\varepsilon\|_{\mathcal{L}(H^s;H^s)} \le 1,$$

where $\mathcal{L}(\mathcal{X};\mathcal{Y})$ is the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

Lemma A.2 [58] *Let f, g be two functions such that $g \in W^{1,\infty}$ and $f \in L^2$. Then for some $C > 0$,*

$$\|[J_\varepsilon, g] f_x\|_{L^2} \leq C \|g_x\|_{L^\infty} \|f\|_{L^2}.$$

Lemma A.3 [36] *If $f \in H^s \bigcap W^{1,\infty}$, $g \in H^{s-1} \bigcap L^\infty$ for $s > 0$, then there exists a constant $C_s > 0$ such that*

$$\left\| \left[D^s, f \right] g \right\|_{L^2} \leq C_s (\|D^s f\|_{L^2} \|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty} \|D^{s-1} g\|_{L^2}).$$

Besides, if $s > 0$, then we have for all $f, g \in H^s \bigcap L^\infty$,

$$\|fg\|_{H^s} \leq C_s (\|f\|_{H^s} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{H^s}).$$

Lemma A.4 (Proposition 4.2, [57]) *Let $\rho > 3/2$ and $0 \leq \eta + 1 \leq \rho$. We have for some $c > 0$,*

$$\|[D^\eta \partial_x, f] v\|_{L^2} \leq c \|f\|_{H^\rho} \|v\|_{H^\eta} \quad \forall\, f \in H^\rho, v \in H^\eta.$$

Lemma A.5 *For $F(\cdot)$ defined in (1.5), we have for all $k \geq 1$ the following estimates:*

$$\begin{aligned}
\|F(v)\|_{H^s} &\lesssim \|v\|_{W^{1,\infty}}^k \|v\|_{H^s}, \quad s > 3/2, \\
\|F(v)\|_{H^s} &\lesssim \|v\|_{W^{1,\infty}}^k \left(\|v\|_{H^s} + \|v_x\|_{H^s} \right), \quad 0 < s \leq 3/2, \\
\|F(u) - F(v)\|_{H^s} &\lesssim \left(\|u\|_{H^s} + \|v\|_{H^s} \right)^k \|u - v\|_{H^s}, \quad s > 3/2, \\
\|F(u) - F(v)\|_{H^s} &\lesssim \left(\|u\|_{H^{s+1}} + \|v\|_{H^{s+1}} \right)^k \|u - v\|_{H^s}, \quad 1/2 < s \leq 3/2.
\end{aligned}$$

Proof We only estimate $\|F(v)\|_{H^s}$ for $0 < s \leq 3/2$ since the other cases can be found in [46, 52, 56]. When $s > 0$, by using (1.5) and Lemma A.3, we derive

$$\|F_1(v)\|_{H^s} \lesssim \|v^{k+1}\|_{H^s} \lesssim \|v\|_{L^\infty}^k \|v\|_{H^s}, \quad k \geq 1. \tag{A.2}$$

When $k \geq 2$, we have

$$\begin{aligned}
\|F_2(v)\|_{H^s} &\lesssim \left\| v^{k-1} v_x^2 \right\|_{H^s} \\
&\lesssim \|v\|_{H^s} \|v\|_{L^\infty}^{k-2} \|v_x\|_{L^\infty}^2 + \|v\|_{L^\infty}^{k-1} \|v_x\|_{H^s} \|v_x\|_{L^\infty} \\
&\lesssim \|v\|_{W^{1,\infty}}^k \left(\|v\|_{H^s} + \|v_x\|_{H^s} \right).
\end{aligned}$$

When $k = 1$, $F_2(v) = \frac{1}{2}(1 - \partial_x^2)^{-1} \partial_x \left(v_x^2 \right)$ and hence

$$\|F_2(v)\|_{H^s} \lesssim \left\| v_x^2 \right\|_{H^s} \lesssim \|v_x\|_{L^\infty} \|v_x\|_{H^s} \lesssim \|v\|_{W^{1,\infty}} \left(\|v\|_{H^s} + \|v_x\|_{H^s} \right).$$

Combining the above two cases for F_2, we arrive at

$$\|F_2(v)\|_{H^s} \lesssim \|v\|_{W^{1,\infty}}^k \left(\|v\|_{H^s} + \|v_x\|_{H^s}\right), \quad k \geq 1. \tag{A.3}$$

Now we consider F_3. When $k \geq 3$, we have

$$\begin{aligned}\|F_3(v)\|_{H^s} &\lesssim \left\|v^{k-2}v_x^3\right\|_{H^s} \\ &\lesssim \|v\|_{H^s}\|v\|_{L^\infty}^{k-3}\|v_x\|_{L^\infty}^3 + \|v\|_{L^\infty}^{k-2}\|v_x\|_{H^s}\|v_x\|_{L^\infty}^2 \\ &\lesssim \|v\|_{W^{1,\infty}}^k \left(\|v\|_{H^s} + \|v_x\|_{H^s}\right).\end{aligned}$$

When $k = 2$, we have $F_3(v) = \frac{1}{2}(1-\partial_x^2)^{-1}\left(v_x^3\right)$ and then

$$\|F_3(v)\|_{H^s} \lesssim \left\|v_x^3\right\|_{H^s} \lesssim \|v_x\|_{H^s}\|v_x\|_{L^\infty}^2 \lesssim \|v\|_{W^{1,\infty}}^2 \left(\|v\|_{H^s} + \|v_x\|_{H^s}\right).$$

Combining the above two cases for F_3 with noticing that $F_3 = 0$ for $k = 1$, we find

$$\|F_3(v)\|_{H^s} \lesssim \|v\|_{W^{1,\infty}}^k \left(\|v\|_{H^s} + \|v_x\|_{H^s}\right), \quad k \geq 1. \tag{A.4}$$

Then the desired estimate is a consequence of (A.2), (A.3) and (A.4). □

Lemma A.6 *Let $s > 3/2$, $k \geq 1$, $F(\cdot)$ be given in (1.5) and J_ε be the mollifier defined in (A.1). There exists a constant $\lambda_s > 0$ such that for all $\varepsilon > 0$,*

$$\begin{aligned}&\left|\left(D^s J_\varepsilon\left[u^k u_x\right], D^s J_\varepsilon u\right)_{L^2}\right| + \left|\left(D^s J_\varepsilon F(u), D^s J_\varepsilon u\right)_{L^2}\right| \\ &\quad \leq \lambda_s \|u\|_{W^{1,\infty}}^k \|u\|_{H^s}^2, \quad u \in H^s, \quad s > 3/2.\end{aligned}$$

If $u \in H^{s+1}$, then $u^k u_x \in H^s$, and the above estimate also holds true without J_ε.

Proof We only prove the case that $u \in H^s$. It follows from Lemmas A.1, A.2 and A.3, integration by parts and $H^s \hookrightarrow W^{1,\infty}$ that

$$\begin{aligned}&\left|\left(D^s J_\varepsilon\left[u^k u_x\right], D^s J_\varepsilon u\right)_{L^2}\right| \\ &\quad \leq \left|\left(\left[D^s, u^k\right]u_x, D^s J_\varepsilon^2 u\right)_{L^2}\right| + \left|\left([J_\varepsilon, u^k]D^s u_x, D^s J_\varepsilon u\right)_{L^2}\right| \\ &\qquad + \left|\left(u^k D^s J_\varepsilon u_x, D^s J_\varepsilon u\right)_{L^2}\right| \\ &\quad \leq C(s)\|u\|_{W^{1,\infty}}^k \|u\|_{H^s}^2.\end{aligned}$$

From Lemma A.5, we also have

$$\left|\left(D^s J_\varepsilon F(u), D^s J_\varepsilon u\right)_{L^2}\right| \leq C(s)\|u\|_{W^{1,\infty}}^k \|u\|_{H^s}^2.$$

Combining the above two inequalities gives rise to the desired estimate of the lemma. □

The following technique has been used in [4, 5, 25]. Here we formulate such a technique result in an abstract way.

Lemma A.7 *Suppose u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable, and suppose $H_1(1)$ holds true. Let I be a countable index set and let $\{\Omega_i\}_{i\in I}$ satisfy*

$$\Omega_i \subset \Omega,\ \mathbb{P}\{\cup_{i\in I}\Omega_i\} = 1 \text{ and } \Omega_i \cap \Omega_j = \emptyset \text{ for all } i, j \in I,\ i \neq j. \tag{A.5}$$

If (u_i, τ_i) with $i \in I$ is a solution to (1.4) with initial value $\boldsymbol{I}_{\Omega_i} u_0$, then

$$\left(u = \sum_{i\in I} \boldsymbol{I}_{\Omega_i} u_i,\quad \tau = \sum_{i\in I} \boldsymbol{I}_{\Omega_i}\tau_i\right) \tag{A.6}$$

is a solution to (1.4) with initial data u_0.

Proof Since (u_i, τ_i) is a solution to (1.4) with initial value $u_0\mathbf{1}_{\Omega_i}$, we find

$$u_i(t\wedge\tau_i) - \mathbf{1}_{\Omega_i}u_0 = -\int_0^{t\wedge\tau_i}\left[u_i^k\partial_x u_i + F(u_i)\right]\mathrm{d}t' + \int_0^{t\wedge\tau_i} h(t,u_i)\,\mathrm{d}\mathcal{W}\ \ \mathbb{P}\text{-a.s.}$$

Therefore, we restrict the above equation to Ω_i and we obtain

$$\begin{aligned}\mathbf{1}_{\Omega_i}u_i(t\wedge\tau_i) - \mathbf{1}_{\Omega_i}u_0 = &-\int_0^{t\wedge\mathbf{1}_{\Omega_i}\tau_i}\mathbf{1}_{\Omega_i}\left[u_i^k\partial_x u_i + F(u_i)\right]\mathrm{d}t'\\ &+\int_0^{t\wedge\mathbf{1}_{\Omega_i}\tau_i}\mathbf{1}_{\Omega_i}h(t,u_i)\,\mathrm{d}\mathcal{W}\ \ \mathbb{P}\text{-a.s.}\end{aligned}$$

It is clear that almost surely,

$$\begin{aligned}\mathbf{1}_{\Omega_i}h(t,u_i) &= h(t,\mathbf{1}_{\Omega_i}u_i) - \mathbf{1}_{\Omega_i^c}h(t,\mathbf{0}),\\ \mathbf{1}_{\Omega_i}\left[u_i^k\partial_x u_i + F(u_i)\right] &= \left[(\mathbf{1}_{\Omega_i}u_i)^k\partial_x\left(\mathbf{1}_{\Omega_i}u_i\right) + F(\mathbf{1}_{\Omega_i}u_i)\right].\end{aligned}$$

By $H_1(1)$, we have $\|h(t,\mathbf{0})\|_{\mathcal{L}_2(\mathfrak{U};H^s)} < \infty$. Then, from the above three equations, we have that almost surely

$$\begin{aligned}\mathbf{1}_{\Omega_i}u_i(t\wedge\tau_i) - \mathbf{1}_{\Omega_i}u_0 &= \mathbf{1}_{\Omega_i}u_i(t\wedge\mathbf{1}_{\Omega_i}\tau_i) - \mathbf{1}_{\Omega_i}u_0\\ &= -\int_0^{t\wedge\mathbf{1}_{\Omega_i}\tau_i}\left[(\mathbf{1}_{\Omega_i}u_i)^k\partial_x(\mathbf{1}_{\Omega_i}u_i) + F(\mathbf{1}_{\Omega_i}u_i)\right]\mathrm{d}t'\\ &\quad+\int_0^{t\wedge\mathbf{1}_{\Omega_i}\tau_i} h(t,\mathbf{1}_{\Omega_i}u_i)\,\mathrm{d}\mathcal{W},\end{aligned}$$

which means $(\mathbf{1}_{\Omega_i}u_i, \mathbf{1}_{\Omega_i}\tau_i)$ also solves (1.4) with initial data $\mathbf{1}_{\Omega_i}u_0$. By summing up both sides of the above equation with noticing (A.5), we derive that (A.6) is a

solution to (1.4) with initial data u_0 almost surely. Indeed, for the initial data, we have $u_0 = \sum_{i\in I} \mathbf{1}_{\Omega_i} u_0$ $\mathbb{P}$-a.s. For the nonlinear term $u^k \partial_x u$, by (A.5), we have that $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\sum_{i\in I} \int_0^{t\wedge \mathbf{1}_{\Omega_i}\tau_i} (\mathbf{1}_{\Omega_i} u_i)^k \partial_x (\mathbf{1}_{\Omega_i} u_i)\, \mathrm{d}t' \\
&= \sum_{i\in I} \int_0^{t\wedge \sum_{j\in I}\mathbf{1}_{\Omega_j}\tau_j} (\mathbf{1}_{\Omega_i} u_i)^k \partial_x \left(\sum_{l\in I} \mathbf{1}_{\Omega_l} u_l\right) \mathrm{d}t' \\
&= \int_0^{t\wedge\tau} \sum_{i\in I} (\mathbf{1}_{\Omega_i} u_i)^k \partial_x \left(\sum_{l\in I} \mathbf{1}_{\Omega_l} u_l\right) \mathrm{d}t' \\
&= \int_0^{t\wedge\tau} \left(\sum_{i\in I} \mathbf{1}_{\Omega_i} u_i\right)^k \partial_x \left(\sum_{l\in I} \mathbf{1}_{\Omega_l} u_l\right) \mathrm{d}t' = \int_0^{t\wedge\tau} u^k \partial_x u\, \mathrm{d}t'.
\end{aligned}$$

The other terms can also be justified in the same way, here we omit the details. □

Finally, we recall the following estimate on the product of a Schwartz function and a trigonometric function.

Lemma A.8 [29, 37] *Let $\mathscr{S}(\mathbb{R})$ be the set of Schwartz functions. Let $\delta > 0$ and $\alpha \in \mathbb{R}$. Then for any $r \geq 0$ and $\psi \in \mathscr{S}(\mathbb{R})$, we have that*

$$\lim_{n\to\infty} n^{-\frac{\delta}{2}-r} \left\| \psi\left(\frac{x}{n^\delta}\right) \cos(nx-\alpha) \right\|_{H^r} = \frac{1}{\sqrt{2}} \|\psi\|_{L^2}. \tag{A.7}$$

Relation (A.7) *is also true if* cos *is replaced by* sin.

B Appendix: Proof for Lemma 4.2

As $u_{m,n} = u_l + u_h$ is explictly given, we will firstly estimate E_i $(i = 1, 2, 3, 4)$. Let $T_l > 0$ be given in Lemma 4.1 such that u_l exits on $[0, T_l]$ for all $n \gg 1$ and (4.9) is satisfied.

(i) Estimating $\|E_1\|_{H^{\rho_0}}$. We apply the embedding $H^{\rho_0} \hookrightarrow L^\infty$, Lemmas 4.1 and A.8 to obtain

$$\begin{aligned}
&\|E_1\|_{H^{\rho_0}} \\
&\leq \left\| \left[u_l^k(0) - u_l^k(t)\right] n^{1-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \sin(nx-mt) \right. \\
&\quad \left. + u_l^k(t) n^{-\frac{3\delta}{2}-s} \partial_x \phi\left(\frac{x}{n^\delta}\right) \cos(nx-mt) \right\|_{H^{\rho_0}} \\
&\quad + \|\mathcal{Z}_k \partial_x u_l\|_{H^{\rho_0}} + \|\mathcal{Z}_k \partial_x u_h\|_{H^{\rho_0}} \\
&\lesssim n^{1-\frac{\delta}{2}-s} \left\| u_l^k(0) - u_l^k(t) \right\|_{H^{\rho_0}} \left\| \phi\left(\frac{x}{n^\delta}\right) \sin(nx-mt) \right\|_{H^{\rho_0}} \\
&\quad + n^{-\frac{3\delta}{2}-s} \|u_l(t)\|_{H^{\rho_0}}^k \left\| \partial_x \phi\left(\frac{x}{n^\delta}\right) \cos(nx-mt') \right\|_{H^{\rho_0}} + \|\mathcal{Z}_k \partial_x u_l\|_{H^{\rho_0}}
\end{aligned}$$

$$
\begin{aligned}
&+ \|\mathcal{Z}_k \partial_x u_h\|_{H^{\rho_0}} \\
&\lesssim n^{1-s+\rho_0} \left\| u_l^k(0) - u_l^k(t) \right\|_{H^{\rho_0}} + n^{-s-1+\rho_0+(\frac{k}{2}-1)\delta} + \|\mathcal{Z}_k \partial_x u_l\|_{H^{\rho_0}} + \|\mathcal{Z}_k \partial_x u_h\|_{H^{\rho_0}} \\
&\lesssim n^{1-s+\rho_0} \left\| u_l^k(0) - u_l^k(t) \right\|_{H^{\rho_0}} + n^{r_s} + \|\mathcal{Z}_k \partial_x u_l\|_{H^{\rho_0}} + \|\mathcal{Z}_k \partial_x u_h\|_{H^{\rho_0}}, \quad t \in [0, T_l].
\end{aligned} \tag{B.1}
$$

Next, we estimate $\|u_l^k(0) - u_l^k(t)\|_{H^{\rho_0}}$. Using the fundamental theorem of calculus and the algebra property, we have that for all $k \geq 1$ and $t \in [0, T_l]$,

$$
\begin{aligned}
\left\| u_l^k(0) - u_l^k(t) \right\|_{H^{\rho_0}} &= \left\| k \int_0^t u_l^{k-1}(t') \partial_t u_l(t') \, \mathrm{d}t' \right\|_{H^{\rho_0}} \\
&\lesssim \int_0^t \|u_l(t')\|_{H^{\rho_0}}^{k-1} \|\partial_t u_l(t')\|_{H^{\rho_0}} \, \mathrm{d}t'.
\end{aligned}
$$

Using (4.5) with $t \in [0, T_l]$, (1.5), Lemmas A.5 and 4.1 and the embedding $H^{\rho_0+1} \hookrightarrow W^{1,\infty}$, we get

$$
\begin{aligned}
\left\| u_l^k(0) - u_l^k(t) \right\|_{H^{\rho_0}} &\lesssim \int_0^t \|u_l\|_{H^{\rho_0}}^{k-1} \left(\|u_l^k \partial_x u_l\|_{H^{\rho_0}} + \|F(u_l)\|_{H^{\rho_0+1}} \right) \mathrm{d}t' \\
&\lesssim \int_0^t \|u_l\|_{H^{\rho_0+1}}^{k-1} \|u_l\|_{H^{\rho_0+1}}^{k+1} \, \mathrm{d}t' \lesssim n^{(\frac{\delta}{2}-\frac{1}{k})2k} T_l, \quad k \geq 1, \quad t \in [0, T_l],
\end{aligned}
$$

which implies

$$
n^{1-s+\rho_0} \left\| u_l^k(0) - u_l^k(t) \right\|_{H^{\rho_0}} \lesssim n^{1-s+\rho_0+k\delta-2} T_l = n^{r_s} T_l, \quad k \geq 1, \quad t \in [0, T_l]. \tag{B.2}
$$

Again, applying the algebra property and using Lemmas 4.1, A.8 and (4.7), we have that for all $k \geq 1$ and $t \in [0, T_l]$,

$$
\begin{aligned}
\|\mathcal{Z}_k \partial_x u_l\|_{H^{\rho_0}} &\lesssim \sum_{j=1}^{k} \|u_l\|_{H^{\rho_0}}^{k-j} \|u_h\|_{H^{\rho_0}}^{j} \|u_l\|_{H^{\rho_0+1}} \\
&\lesssim \left(\sum_{j=1}^{k} n^{(\frac{\delta}{2}-\frac{1}{k})(k-j)} n^{(-s+\rho_0)j} \right) n^{\frac{\delta}{2}-\frac{1}{k}} \\
&= \sum_{j=1}^{k} n^{j(-s+\rho_0-\frac{\delta}{2}+\frac{1}{k})-1-\frac{1}{k}+(k+1)\frac{\delta}{2}} \lesssim n^{-s-1+\rho_0+k\frac{\delta}{2}} \lesssim n^{r_s}.
\end{aligned} \tag{B.3}
$$

Here we used the facts that $-s + \rho_0 - \frac{\delta}{2} + \frac{1}{k} < -s + 1 - \frac{\delta}{2} + 1 = -s + 2 - \frac{\delta}{2} < -\frac{1}{2} - \frac{\delta}{2} < 0$ for all $k \geq 1$, which means that the term corresponding to $j = 1$ dominates.

For the last term $\mathcal{Z}_k \partial_x u_h$, by using (4.4), (4.7), Lemma A.3, Lemmas 4.1 and A.8, we obtain that

$$
\begin{aligned}
&\|\mathcal{Z}_k \partial_x u_h\|_{H^{\rho_0}} \\
&\lesssim \sum_{j=1}^{k} \|u_l\|_{H^{\rho_0}}^{k-j} \|u_h^j \partial_x u_h\|_{H^{\rho_0}} \\
&\lesssim \sum_{j=1}^{k} \|u_l\|_{H^{\rho_0}}^{k-j} \left[\|u_h^j\|_{H^{\rho_0}} \|\partial_x u_h\|_{L^\infty} + \|u_h^j\|_{L^\infty} \|u_h\|_{H^{\rho_0+1}} \right] \\
&\lesssim \sum_{j=1}^{k} n^{(\frac{\delta}{2}-\frac{1}{k})(k-j)} \Bigg\{ n^{(-s+\rho_0)j} \Bigg[\left\| -n^{1-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \sin(nx-mt) \right\|_{L^\infty} \\
&\qquad\qquad + \left\| n^{-\frac{3\delta}{2}-s} \partial_x \phi\left(\frac{x}{n^\delta}\right) \cos(nx-mt) \right\|_{L^\infty} \Bigg] \\
&\qquad + \left\| n^{-\frac{\delta}{2}-s} \phi\left(\frac{x}{n^\delta}\right) \cos(nx-mt) \right\|_{L^\infty}^{j} n^{-s+\rho_0+1} \Bigg\} \\
&\lesssim \sum_{j=1}^{k} n^{(\frac{\delta}{2}-\frac{1}{k})(k-j)} \left\{ n^{(-s+\rho_0)j} \left[n^{-s-\frac{\delta}{2}+1} + n^{-\frac{3\delta}{2}-s} \right] + n^{(-\frac{\delta}{2}-s)j} n^{-s+\rho_0+1} \right\} \\
&\lesssim \sum_{j=1}^{k} n^{j(-s+\rho_0-\frac{\delta}{2}+\frac{1}{k})-s+(k-1)\frac{\delta}{2}} + \sum_{j=1}^{k} n^{j(-s-\delta+\frac{1}{k})-s+\rho_0+k\frac{\delta}{2}}, \quad k \geq 1, \quad t \in [0, T_l].
\end{aligned}
$$

When $k \geq 1$, $-s+\rho_0-\frac{\delta}{2}+\frac{1}{k} < 0$ and $-s-\delta+\frac{1}{k} < 0$, therefore, both sums are bounded by $n^{-2s+\rho_0+\frac{1}{k}+(k-2)\frac{\delta}{2}}$. Furthermore, when $k \geq 1$, $-2s + \rho_0 + \frac{1}{k} + (k-2)\frac{\delta}{2} \leq r_s$, which means

$$
\|\mathcal{Z}_k \partial_x u_h\|_{H^{\rho_0}} \lesssim n^{-2s+\rho_0+\frac{1}{k}+(k-2)\frac{\delta}{2}} \lesssim n^{r_s}, \quad k \geq 1, \quad t \in [0, T_l]. \tag{B.4}
$$

Finally, inserting (B.2), (B.3) and (B.4) into (B.1), we arrive at

$$
\|E_1\|_{H^{\rho_0}} \lesssim n^{r_s}, \quad k \geq 1, \quad t \in [0, T_l]. \tag{B.5}
$$

(ii) Estimating $\|E_2\|_{H^{\rho_0}}$. For E_2, we first recall (4.7). Applying the embedding $H^{\rho_0} \hookrightarrow L^\infty$ and Lemma 4.1, and then taking the dominated term $j = 1$, we find that for all $k \geq 1$ and $t \in [0, T_l]$,

$$
\begin{aligned}
\|E_2\|_{H^{\rho_0}} &\lesssim \left\| \sum_{j=1}^{k+1} C_{k+1}^j u_l^{k+1-j} u_h^j \right\|_{H^{\rho_0}} \\
&\lesssim \sum_{j=1}^{k+1} n^{(k+1-j)(\frac{\delta}{2}-\frac{1}{k})} n^{(-s+\rho_0)j} \lesssim n^{-s-1+\rho_0+k\frac{\delta}{2}} \lesssim n^{r_s}.
\end{aligned} \tag{B.6}
$$

(iii) Estimating $\|E_3\|_{H^{\rho_0}}$. As in the estimate for (B.4), we have obtained that

$$\|\partial_x u_h\|_{L^\infty} \lesssim n^{-s-\frac{\delta}{2}+1} + n^{-\frac{3\delta}{2}-s} \lesssim n^{-s-\frac{\delta}{2}+1}, \quad t \in [0, T_l].$$

When $k = 1$, $Z_{k-1} = 0$ and then we find

$$\begin{aligned}
\|E_3\|_{H^{\rho_0}} &\lesssim \|2(\partial_x u_l)(\partial_x u_h) + (\partial_x u_h)^2\|_{H^{\rho_0-1}} \\
&\lesssim \|2(\partial_x u_l)(\partial_x u_h) + (\partial_x u_h)^2\|_{L^2} \\
&\lesssim \|u_l\|_{H^2}\|\partial_x u_h\|_{L^\infty} + \|\partial_x u_h\|_{L^2}\|\partial_x u_h\|_{L^\infty} \\
&\lesssim \|u_l\|_{H^2}\|\partial_x u_h\|_{L^\infty} + \|u_h\|_{H^1}\|\partial_x u_h\|_{L^\infty} \\
&\lesssim n^{\frac{\delta}{2}-1} n^{1-\frac{\delta}{2}-s} + n^{-s+1} n^{1-\frac{\delta}{2}-s} \lesssim n^{-s} + n^{-2s+2-\frac{\delta}{2}}, \quad t \in [0, T_l].
\end{aligned}$$

Since $\delta > 0$, $-2s+2-\frac{\delta}{2} - r_s = -s+3-\rho_0-\frac{3}{2}\delta < -\frac{5}{2}+3-\frac{1}{2}-\frac{3}{2}\delta = -\frac{3}{2}\delta < 0$, hence

$$\|E_3\|_{H^{\rho_0}} \lesssim n^{-s} + n^{-2s+2-\frac{\delta}{2}} \lesssim n^{r_s}, \quad k = 1, \quad t \in [0, T_l]. \tag{B.7}$$

When $k \geq 2$, we can use the above estimate, Lemma 4.1, the facts $\|f\|_{H^{\rho_0-1}} \leq \|f\|_{L^2}$ and $\|fg\|_{L^2} \leq \|f\|_{L^2}\|g\|_{L^\infty}$ and take the dominate term $j = 1$ to obtain

$$\begin{aligned}
&\|E_3\|_{H^{\rho_0}} \\
&\lesssim \left\|u_l^{k-1}[2(\partial_x u_l)(\partial_x u_h) + (\partial_x u_h)^2]\right\|_{H^{\rho_0-1}} + \left\|\mathcal{Z}_{k-1}(\partial_x u_l + \partial_x u_h)^2\right\|_{H^{\rho_0-1}} \\
&\lesssim \left\|u_l^{k-1}[2(\partial_x u_l)(\partial_x u_h) + (\partial_x u_h)^2]\right\|_{L^2} + \left\|\left(\sum_{j=1}^{k-1} C_{k-1}^j u_l^{k-1-j} u_h^j\right)(\partial_x u_l + \partial_x u_h)^2\right\|_{L^2} \\
&\lesssim \|u_l\|_{H^2}^k \|\partial_x u_h\|_{L^\infty} + \|u_l\|_{H^2}^{k-1}\|\partial_x u_h\|_{L^\infty}^2 + \sum_{j=1}^{k-1}\|u_l\|_{H^2}^{k+1-j}\|u_h\|_{L^\infty}^j \\
&\quad + \sum_{j=1}^{k-1}\|u_l\|_{H^2}^{k-j}\|u_h^j\partial_x u_h\|_{L^\infty} + \sum_{j=1}^{k-1}\|u_l\|_{H^2}^{k-1-j}\|u_h^j(\partial_x u_h)^2\|_{L^\infty} \\
&\lesssim n^{k(\frac{\delta}{2}-\frac{1}{k})} n^{1-\frac{\delta}{2}-s} + n^{(k-1)(\frac{\delta}{2}-\frac{1}{k})} n^{2-\delta-2s} + \sum_{j=1}^{k-1} n^{(k+1-j)(\frac{\delta}{2}-\frac{1}{k})} n^{j(-\frac{\delta}{2}-s)} \\
&\quad + \sum_{j=1}^{k-1} n^{(k-j)(\frac{\delta}{2}-\frac{1}{k})} n^{j(-\frac{\delta}{2}-s)+1-\frac{\delta}{2}-s} + \sum_{j=1}^{k-1} n^{(k-1-j)(\frac{\delta}{2}-\frac{1}{k})} n^{j(-\frac{\delta}{2}-s)+2-\delta-2s} \\
&= n^{-s+(k-1)\frac{\delta}{2}} + n^{-2s+1+\frac{1}{k}+(k-3)\frac{\delta}{2}} + \sum_{j=1}^{k-1} n^{j(-s-\delta+\frac{1}{k})-1-\frac{1}{k}+(k+1)\frac{\delta}{2}}
\end{aligned}$$

$$+\sum_{j=1}^{k-1} n^{j(-s-\delta+\frac{1}{k})-s+(k-1)\frac{\delta}{2}} + \sum_{j=1}^{k-1} n^{j(-s-\delta+\frac{1}{k})-2s+1+\frac{1}{k}+(k-3)\frac{\delta}{2}}$$
$$\lesssim n^{r_s}, \quad k \geq 2, \quad t \in [0, T_l]. \tag{B.8}$$

Combining (B.8) and (B.7), we have the following conclusion for E_3:

$$\|E_3\|_{H^{\rho_0}} \lesssim n^{r_s}, \quad k \geq 1, \quad t \in [0, T_l]. \tag{B.9}$$

(iv) Estimating $\|E_4\|_{H^{\rho_0}}$. For $k = 1$, $E_4 = 0$ since F_3 disappears. When $k = 2$, $\mathcal{Z}_{k-2} = 0$ and then

$$\begin{aligned}
\|E_4\|_{H^{\rho_0}} &\lesssim \left\| 3(\partial_x u_l)^2(\partial_x u_h) + 3(\partial_x u_l)(\partial_x u_h)^2 + (\partial_x u_h)^3 \right\|_{H^{\rho_0-2}} \\
&\lesssim \|u_l\|_{H^2}^2 \|\partial_x u_h\|_{L^\infty} + \|u_l\|_{H^2} \|\partial_x u_h\|_{L^\infty}^2 + \|u_h\|_{H^1} \|\partial_x u_h\|_{L^\infty}^2 \\
&\lesssim n^{2(\frac{\delta}{2}-\frac{1}{2})} n^{1-\frac{\delta}{2}-s} + n^{\frac{\delta}{2}-\frac{1}{2}} n^{2(1-\frac{\delta}{2}-s)} + n^{1-s} n^{2(1-\frac{\delta}{2}-s)} \\
&= n^{-s+\frac{\delta}{2}} + n^{-2s-\frac{\delta}{2}+\frac{3}{2}} + n^{3-\delta-3s} \lesssim n^{r_s}, \quad k = 2, \quad t \in [0, T_l].
\end{aligned}$$

Finally, for $k \geq 3$ and $t \in [0, T_l]$,

$$\begin{aligned}
\|E_4\|_{H^{\rho_0}} \lesssim & \left\| u_l^{k-2} [3(\partial_x u_l)^2(\partial_x u_h) + 3(\partial_x u_l)(\partial_x u_h)^2 + (\partial_x u_h)^3] \right\|_{H^{\rho_0-2}} \\
& + \left\| \mathcal{Z}_{k-2} (\partial_x u_l + \partial_x u_h)^3 \right\|_{H^{\rho_0-2}} \\
\lesssim & \left\| u_l^{k-2} [3(\partial_x u_l)^2(\partial_x u_h) + 3(\partial_x u_l)(\partial_x u_h)^2 + (\partial_x u_h)^3] \right\|_{L^2} \\
& + \left\| \left(\sum_{j=1}^{k-2} C_{k-2}^j u_l^{k-2-j} u_h^j \right) (\partial_x u_l + \partial_x u_h)^3 \right\|_{L^2} \\
\lesssim & \|u_l\|_{H^2}^k \|\partial_x u_h\|_{L^\infty} + \|u_l\|_{H^2}^{k-1} \|\partial_x u_h\|_{L^\infty}^2 \\
& + \|u_l\|_{H^2}^{k-2} \|\partial_x u_h\|_{L^\infty}^3 + \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k+1-j} \|u_h\|_{L^\infty}^j \\
& + \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k-j} \|u_h^j \partial_x u_h\|_{L^\infty} + \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k-1-j} \|u_h^j (\partial_x u_h)^2\|_{L^\infty} \\
& + \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k-2-j} \|u_h^j (\partial_x u_h)^3\|_{L^\infty}.
\end{aligned}$$

Repeating the analysis in (B.8), one has that for $k \geq 3$ and $t \in [0, T_l]$,

$$n^{r_s} \gtrsim \begin{cases} \|u_l\|_{H^2}^k \|\partial_x u_h\|_{L^\infty}, \ \|u_l\|_{H^2}^{k-1} \|\partial_x u_h\|_{L^\infty}^2, \\ \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k+1-j} \|u_h\|_{L^\infty}^j, \ \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k-j} \|u_h^j \partial_x u_h\|_{L^\infty}, \\ \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k-1-j} \|u_h^j (\partial_x u_h)^2\|_{L^\infty}, \end{cases}$$

and therefore it suffices to estimate the different terms, which are

$$\begin{aligned} \|u_l\|_{H^2}^{k-2} \|\partial_x u_h\|_{L^\infty}^3 &\lesssim n^{(k-2)(\frac{\delta}{2}-\frac{1}{k})} n^{3(1-\frac{\delta}{2}-s)} \\ &= n^{-3s+2+\frac{2}{k}+(k-5)\frac{\delta}{2}} \lesssim n^{r_s}, \quad k \geq 3, \quad t \in [0, T_l], \\ \sum_{j=1}^{k-2} \|u_l\|_{H^2}^{k-2-j} \|u_h^j (\partial_x u_h)^3\|_{L^\infty} &\lesssim \sum_{j=1}^{k-2} n^{(k-2-j)(\frac{\delta}{2}-\frac{1}{k})} n^{j(-\frac{\delta}{2}-s)+3(1-\frac{\delta}{2}-s)} \\ &\lesssim \sum_{j=1}^{k-2} n^{j(-s-\delta+\frac{1}{k})+2+\frac{2}{k}-3s+(k-5)\frac{\delta}{2}} \\ &\lesssim n^{-4s+\frac{3}{k}+2+(k-7)\frac{\delta}{2}} \lesssim n^{r_s}, \quad k \geq 3, \quad t \in [0, T_l]. \end{aligned}$$

Combining the above estimations, we get

$$\|E_4\|_{H^{\rho_0}} \lesssim n^{r_s}, \quad k \geq 1, \quad t \in [0, T_l]. \tag{B.10}$$

(v) Estimating $\|\mathcal{E}\|_{H^{\rho_0}}$. Let $T_l > 0$ be given in Lemma 4.1 such that T_l does not depend on n. Let $t \in [0, T_l]$, by virtue of the Itô formula and (4.13), we derive that

$$\begin{aligned} \|\mathcal{E}(t,x)\|_{H^{\rho_0}}^2 \leq &\left| -2 \int_0^t (h(t', u_{m,n}) \, \mathrm{d}\mathcal{W}, \mathcal{E})_{H^{\rho_0}} \right| + 2 \sum_{i=1}^4 \int_0^t |(E_i, \mathcal{E})_{H^{\rho_0}}| \, \mathrm{d}t' \\ &+ \int_0^t \|h(t', u_{m,n})\|_{\mathcal{L}_2(\mathfrak{U}; H^{\rho_0})}^2 \, \mathrm{d}t'. \end{aligned}$$

Taking supremum with respect to $t \in [0, T_l]$ and then using the BDG inequality give rise to

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0,T_l]} \|\mathcal{E}(t)\|_{H^{\rho_0}}^2 \\ &\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0,T_l]} \|\mathcal{E}(t)\|_{H^{\rho_0}}^2 + C \mathbb{E} \int_0^{T_l} \|h(t, u_{m,n})\|_{\mathcal{L}_2(\mathfrak{U}; H^{\rho_0})}^2 \, \mathrm{d}t \end{aligned}$$

$$+C\int_0^{T_l}\Big[\sum_{i=1}^{4}\mathbb{E}\|E_i\|_{H^{\rho_0}}^2+\mathbb{E}\|\mathcal{E}(t)\|_{H^{\rho_0}}^2\Big]\,\mathrm{d}t.$$

For any fixed $s>\frac{5}{2}$, since $\|u_{m,n}\|_{H^s}\lesssim 1$, on account of (1.10), Lemmas A.8 and 4.1, we can pick $N=N(s,k)\gg 1$ such that

$$\|h(t,u_{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2\lesssim\left(\mathrm{e}^{\frac{-1}{\|u_{m,n}\|_{H^{\rho_0}}}}\right)^2\lesssim\left(n^{N(-s+\rho_0)}+n^{N(\frac{\delta}{2}-\frac{1}{k})}\right)^2\lesssim n^{2r_s}.$$

This, (B.5), (B.6), (B.9) and (B.10) yield

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\rho_0}}^2&\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\rho_0}}^2+C\mathbb{E}\int_0^{T_l}\|h(t,u_{m,n})\|_{\mathcal{L}_2(\mathfrak{U};H^{\rho_0})}^2\,\mathrm{d}t\\&\quad+C\int_0^{T_l}\Big[\sum_{i=1}^{4}\mathbb{E}\|E_i\|_{H^{\rho_0}}^2+\mathbb{E}\|\mathcal{E}(t)\|_{H^{\rho_0}}^2\Big]\,\mathrm{d}t\\&\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\rho_0}}^2+C(T_l)n^{2r_s}\\&\quad+C\int_0^{T_l}\mathbb{E}\sup_{t'\in[0,t]}\|\mathcal{E}(t')\|_{H^{\rho_0}}^2\,\mathrm{d}t\end{aligned}$$

Obviously, for each $n\ge 1$, $\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\rho_0}}^2$ is finite. Then by the Grönwall inequality, we have

$$\mathbb{E}\sup_{t\in[0,T_l]}\|\mathcal{E}(t)\|_{H^{\rho_0}}^2\le Cn^{2r_s},\ C=C(T_l).$$

References

1. Albeverio, S., Brzeźniak, Z., Daletskii, A.: Stochastic Camassa–Holm equation with convection type noise. J. Differ. Equ. **276**, 404–432 (2021)
2. Alonso-Orán, D., Bethencourt de León, A., Takao, S.: The Burgers' equation with stochastic transport: shock formation, local and global existence of smooth solutions. NoDEA Nonlinear Differ. Equ. Appl. **26**(6), Paper No. 57, 33 (2019)
3. Alonso-Orán, D., Miao, Y., Tang, H.: Global existence, blow-up and stability for a stochastic transport equation with non-local velocity. J. Differ. Equ. **335**, 244–293 (2022)
4. Alonso-Orán, D., Rohde, C., Tang, H.: A local-in-time theory for singular SDEs with applications to fluid models with transport noise. J. Nonlinear Sci. **31**(6), Paper No. 98,55 (2021)
5. Breit, D., Feireisl, E., Hofmanová, M.: Stochastically forced compressible fluid flows. De Gruyter Series in Applied and Numerical Mathematics, vol. 3. De Gruyter, Berlin (2018)
6. Brzeźniak, Z., Maslowski, B., Seidler, J.: Stochastic nonlinear beam equations. Probab. Theory Relat. Fields **132**(1), 119–149 (2005)
7. Brzeźniak, Z., Motyl, E.: Existence of a martingale solution of the stochastic Navier–Stokes equations in unbounded 2D and 3D domains. J. Differ. Equ. **254**(4), 1627–1685 (2013)
8. Brzeźniak, Z., Motyl, E.: Fractionally dissipative stochastic quasi-geostrophic type equations on $\mathbb{R}^d$. SIAM J. Math. Anal. **51**(3), 2306–2358 (2019)

9. Brzeźniak, Z., Ondreját, M.: Strong solutions to stochastic wave equations with values in Riemannian manifolds. J. Funct. Anal. **253**(2), 449–481 (2007)
10. Camassa, R., Holm, D.D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**(11), 1661–1664 (1993)
11. Chen, G.-Q.G., Pang, P.H.C.: Nonlinear anisotropic degenerate parabolic–hyperbolic equations with stochastic forcing. J. Funct. Anal. **281**(12), Paper No. 109222,48 (2021)
12. Chen, Y., Duan, J., Gao, H.: Global well-posedness of the stochastic Camassa–Holm equation. Commun. Math. Sci. **19**(3), 607–627 (2021)
13. Chen, Y., Duan, J., Gao, H.: Wave-breaking and moderate deviations of the stochastic Camassa–Holm equation with pure jump noise. Phys. D **424**, Paper No. 132944, 12 (2021)
14. Chen, Y., Gao, H.: Well-posedness and large deviations of the stochastic modified Camassa–Holm equation. Potential Anal. **45**(2), 331–354 (2016)
15. Chen, Y., Miao, Y., Shi, S.: Global existence and wave breaking for a stochastic two-component Camassa–Holm system. J. Math. Phys. **64**(1), Paper No. 011505,28 (2023)
16. Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. **181**(2), 229–243 (1998)
17. de Bouard, A., Debussche, A.: A stochastic nonlinear Schrödinger equation with multiplicative noise. Commun. Math. Phys. **205**(1), 161–181 (1999)
18. Debussche, A., Glatt-Holtz, N.E., Temam, R.: Local martingale and pathwise solutions for an abstract fluids model. Physica D **240**(14–15), 1123–1144 (2011)
19. Fedrizzi, E., Flandoli, F.: Pathwise uniqueness and continuous dependence of SDEs with non-regular drift. Stochastics **83**(3), 241–257 (2011)
20. Flandoli, F., Gubinelli, M., Priola, E.: Well-posedness of the transport equation by stochastic perturbation. Invent. Math. **180**(1), 1–53 (2010)
21. Fuchssteiner, B., Fokas, A.S.: Symplectic structures, their Bäcklund transformations and hereditary symmetries. Physica D **4**(1), 47–66 (1981/82)
22. Galimberti, L., Holden, H., Karlsen, K.H., Pang, P.H.C.: Global existence of dissipative solutions to the Camassa–Holm equation with transport noise (2022)
23. Gawarecki, L., Mandrekar, V.: Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations Probability and Its Applications (New York). Springer, Heidelberg (2011)
24. Geng, X., Xue, B.: An extension of integrable Peakon equations with cubic nonlinearity. Nonlinearity **22**(8), 1847–1856 (2009)
25. Glatt-Holtz, N., Ziane, M.: Strong pathwise solutions of the stochastic Navier–Stokes system. Adv. Differ. Equ. **14**(5–6), 567–600 (2009)
26. Glatt-Holtz, N.E., Vicol, V.C.: Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise. Ann. Probab. **42**(1), 80–145 (2014)
27. Henry, D.: Geometric theory of semilinear parabolic equations. 840:iv+348 (1981)
28. Himonas, A.A., Holliman, C.: The Cauchy problem for a generalized Camassa–Holm equation. Adv. Differ. Equ. **19**(1–2), 161–200 (2014)
29. Himonas, A.A., Kenig, C.: Non-uniform dependence on initial data for the CH equation on the line. Differ. Integral Equ. **22**(3–4), 201–224 (2009)
30. Himonas, A.A., Kenig, C., Misiołek, G.: Non-uniform dependence for the periodic CH equation. Commun. Partial Differ. Equ. **35**(6), 1145–1162 (2010)
31. Himonas, A.A., Misiołek, G.: Non-uniform dependence on initial data of solutions to the Euler equations of hydrodynamics. Commun. Math. Phys. **296**(1), 285–301 (2010)
32. Holden, H., Karlsen, K.H., Pang, P.H.C.: The Hunter–Saxton equation with noise. J. Differ. Equ. **270**, 725–786 (2021)
33. Holden, H., Karlsen, K.H., Pang, P.H.C.: Global well-posedness of the viscous Camassa–Holm equation with gradient noise. Discrete Contin. Dyn. Syst. **43**(2), 568–618 (2023)
34. Hone, A.N.W., Lundmark, H., Szmigielski, J.: Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa–Holm type equation. Dyn. Partial Differ. Equ. **6**(3), 253–289 (2009)
35. Kato, T.: The Cauchy problem for quasi-linear symmetric hyperbolic systems. Arch. Ration. Mech. Anal. **58**(3), 181–205 (1975)
36. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. Commun. Pure Appl. Math. **41**(7), 891–907 (1988)

37. Koch, H., Tzvetkov, N.: Nonlinear wave interactions for the Benjamin–Ono equation. Int. Math. Res. Not. **30**, 1833–1847 (2005)
38. Kröker, I., Rohde, C.: Finite volume schemes for hyperbolic balance laws with multiplicative noise. Appl. Numer. Math. **62**(4), 441–456 (2012)
39. Krylov, N.V., Rozovskiĭ, B.L.: Stochastic evolution equations. In: Current problems in mathematics, Vol. 14 (Russian), pp. 71–147, 256. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow (1979)
40. Lenells, J., Wunsch, M.: On the weakly dissipative Camassa–Holm, Degasperis–Procesi, and Novikov equations. J. Differ. Equ. **255**(3), 441–448 (2013)
41. Li, J., Liu, H., Tang, H.: Stochastic MHD equations with fractional kinematic dissipation and partial magnetic diffusion in $\mathbb{R}^2$. Stoch. Process. Appl. **135**, 139–182 (2021)
42. Marinelli, C., Prévôt, C., Röckner, M.: Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise. J. Funct. Anal. **258**(2), 616–649 (2010)
43. Miao, Y., Wang, Z., Zhao, Y.: Noise effect in a stochastic generalized Camassa–Holm equation. Commun. Pure Appl. Anal. **21**(10), 3529–3558 (2022)
44. Novikov, V.: Generalizations of the Camassa–Holm equation. J. Phys. A **42**(34), 342002, 14 (2009)
45. Ren, P., Tang, H., Wang, F.-Y.: Distribution-path dependent nonlinear SPDEs with application to stochastic transport type equations. arXiv:2007.09188 (2020)
46. Rohde, C., Tang, H.: On a stochastic Camassa–Holm type equation with higher order nonlinearities. J. Dyn. Differ. Equ. **33**(4), 1823–1852 (2021)
47. Rohde, C., Tang, H.: On the stochastic Dullin–Gottwald–Holm equation: global existence and wave-breaking phenomena. NoDEA Nonlinear Differ. Equ. Appl. **28**(1), Paper No. 5,34 (2021)
48. Tang, H.: On the pathwise solutions to the Camassa–Holm equation with multiplicative noise. SIAM J. Math. Anal. **50**(1), 1322–1366 (2018)
49. Tang, H.: On stochastic Euler–Poincaré equations driven by pseudo-differential/multiplicative noise (2022). arXiv:2002.08719v4
50. Tang, H., Liu, Z.: Continuous properties of the solution map for the Euler equations. J. Math. Phys. **55**(3), 031504, 10 (2014)
51. Tang, H., Liu, Z.: Well-posedness of the modified Camassa–Holm equation in Besov spaces. Z. Angew. Math. Phys. **66**(4), 1559–1580 (2015)
52. Tang, H., Shi, S., Liu, Z.: The dependences on initial data for the b-family equation in critical Besov space. Monatsh. Math. **177**(3), 471–492 (2015)
53. Tang, H., Wang, F.-Y.: A general framework for solving singular spdes with applications to fluid models driven by pseudo-differential noise (2022). arXiv:2208.08312
54. Tang, H., Wang, Z.-A.: Strong solutions to a nonlinear stochastic aggregation–diffusion equation. Commun. Contemp. Math. (2023). https://doi.org/10.1142/S0219199722500730
55. Tang, H., Yang, A.: Noise effects in some stochastic evolution equations: global existence and dependence on initial data. Ann. Inst. Henri Poincaré Probab. Stat. **59**(1), 378–410 (2023)
56. Tang, H., Zhao, Y., Liu, Z.: A note on the solution map for the periodic Camassa–Holm equation. Appl. Anal. **93**(8), 1745–1760 (2014)
57. Taylor, M.: Commutator estimates. Proc. Am. Math. Soc. **131**(5), 1501–1507 (2003)
58. Taylor, M.: Partial Differential Equations III. Applied Mathematical Sciences, vol. 117. Springer, New York (2011)
59. Wu, S., Yin, Z.: Blow-up and decay of the solution of the weakly dissipative Degasperis–Procesi equation. SIAM J. Math. Anal. **40**(2), 475–490 (2008)
60. Zhou, S., Mu, C.: The properties of solutions for a generalized *b*-family equation with Peakons. J. Nonlinear Sci. **23**(5), 863–889 (2013)