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Noise effects in some stochastic evolution equations: Global existence and dependence on initial data

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Abstract. In this paper, we consider the noise effects on a class of stochastic evolution equations including the stochastic Camassa–Holm equations with or without rotation. We first obtain the existence, uniqueness and a blow-up criterion of pathwise solutions in Sobolev space H^s with $s > 3/2$. Then we prove that strong enough noise can prevent blow-up with probability 1, which justifies the regularization effect of strong nonlinear noise in preventing singularities. Besides, such strengths of noise are estimated in different examples. Finally, for the interplay between regularization effect induced by the noise and the dependence on initial conditions, we introduce and investigate the stability of the exiting time and construct an example to show that the multiplicative noise cannot improve both the stability of the exiting time and the continuity of the dependence on initial data simultaneously.

Résumé. Dans cet article, nous considérons les effets du bruit sur une classe d'équations d'évolution stochastiques y compris les équations stochastiques de Camassa–Holm avec ou sans rotation. Nous obtenons d'abord l'existence, l'unicité et un critère d'explosion de solutions *pathwise* dans l'espace de Sobolev H^s avec $s > 3/2$. Ensuite, nous prouvons qu'un bruit suffisamment fort peut empêcher l'explosion avec probabilité 1, ce qui justifie l'effet régularisant du bruit non linéaire fort dans la prévention des singularités. De plus, de telles forces de bruit sont estimées dans les différents exemples. Enfin, pour l'interaction entre l'effet de régularisation induit par le bruit et la dépendance par rapport aux conditions initiales, nous introduisons et étudions la stabilité du temps de sortie et construisons un exemple pour montrer que le bruit multiplicatif ne peut pas améliorer simultanément la stabilité du temps de sortie et la continuité de la dépendance par rapport aux données initiales.

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1. Introduction and main results

In this paper, we consider a class of stochastic evolution equations under random perturbation. The equation is given by

$$u_t + uu_x + \left(1 - \partial_{xx}^2\right)^{-1}\partial_x\left(a_0 u + a_1 u^2 + a_2 u_x^2 + a_3 u^3 + a_4 u^4\right) = h(t,u)\dot{\mathcal{W}}. \tag{1.1}$$

In (1.1), h is a nonlinear function in (t,u), $a_0, \ldots, a_4$ are constants and $\mathcal{W}$ is a cylindrical Wiener process.

When $h = 0$, $a_1 = 1$, $a_2 = \frac{1}{2}$ and $a_0 = a_3 = a_4 = 0$, (1.1) is the Camassa–Holm (CH) equation [10,33],

$$u_t + uu_x + \left(1 - \partial_{xx}^2\right)^{-1}\partial_x\left(u^2 + \frac{1}{2}u_x^2\right) = 0. \tag{1.2}$$

The CH equation (1.2) models the unidirectional propagation of shallow water waves over a flat bottom and it appeared initially in the context of hereditary symmetries studied by Fuchssteiner and Fokas [33] as a bi-Hamiltonian generalization of KdV equation. Later, Camassa and Holm [10] also derived it by approximating directly in the Hamiltonian for Euler's equations in the shallow water regime. It is well known that (1.2) exhibits both phenomena of (peaked) soliton interaction and wave breaking, and singularities can only occur in the form of breaking waves, cf. [13,17]. We refer to [14–16,58] for the global existence and wave breaking phenomenon of CH equation. Global existences of weak solutions to (1.2) are established in [19,75]. The global conservative and dissipative solutions are obtained in [5,6], and in [45,46].

When $h = 0$, $a_1 = \frac{b}{2}$, $a_2 = \frac{3-b}{2}$ with $b \in \mathbb{R}$ and $a_0 = a_3 = a_4 = 0$, (1.1) becomes the b-family equations, cf. [18,47],

(1.3) $$u_t + uu_x + \left(1 - \partial_{xx}^2\right)^{-1} \partial_x \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2\right) = 0.$$

For general $b \in \mathbb{R}$, there have been extensive studies on (1.3). The well-posedness, blow-up phenomena and global existence (depending on b) of (1.3) can be found in [28,77] and the references therein. Particularly, if $b = 3$, then (1.3) turns out to be the Degasperis–Procesi (DP) equation, cf. [25]. Its complete integrability, bi-Hamilton structure and peakons are studied in [24]. The global existence of strong solutions, weak solutions and the blow-up phenomena for DP equation ($b = 3$ in (1.3)) can be found in [26,27] and the references therein.

When $h = 0$ and a_i $(i = 0, 1, 2, 3, 4)$ are suitably chosen, (1.1) becomes the recently derived rotation-Camassa–Holm equation describing the motion of the fluid with the Coriolis effect from the incompressible shallow water in the equatorial region, cf. [38, equation (4.9)]. In this case, $a_3 \neq 0$ and $a_4 \neq 0$ so that the equation has a cubic and even quartic nonlinearity.

In this paper, we are interested in the stochastic case, i.e., $h \neq 0$. For the deterministic counterpart of (1.1), the so-called weakly dissipative variants have been proposed and studied, cf. [56,73,74] and the reference therein. For example, the weakly disspative Camassa–Holm equation

$$u_t + uu_x + \left(1 - \partial_{xx}^2\right)^{-1} \partial_x \left(u^2 + \frac{1}{2}u_x^2\right) = \lambda u, \quad \lambda < 0$$

is studied in [56,74]. However, the energy exchanging mechanisms in real life can include both energy consuming and energy absorbing and it can also be connected with randomness through external perturbation. Therefore, for some nonlinear function h, we replace λu by $h(t,u)\dot{\mathcal{W}}$ to account for time-dependent, nonlinear and random energy exchanging mechanisms (since $\dot{\mathcal{W}}$ has no fixed sign) rather than simple linear energy dissipation $(\lambda < 0)$.

Hence the first goal of this paper is to consider the Cauchy problem for (1.1) on the whole line $\mathbb{R}$. We reformulate the problem in the following non-local form:

(1.4) $$\begin{cases} \mathrm{d}u + \left[uu_x + F(u)\right]\mathrm{d}t = h(t,u)\,\mathrm{d}\mathcal{W}, & x \in \mathbb{R}, t > 0, \\ u(\omega, 0, x) = u_0(\omega, x), & x \in \mathbb{R}, \end{cases}$$

where $F(u) = \sum_{i=1}^{5} F_i(u)$ with

(1.5) $$\begin{cases} F_1(u) = a_0 \partial_x \left(1 - \partial_{xx}^2\right)^{-1} u, \\ F_2(u) = a_1 \partial_x \left(1 - \partial_{xx}^2\right)^{-1} \left(u^2\right), \\ F_3(u) = a_2 \partial_x \left(1 - \partial_{xx}^2\right)^{-1} \left(u_x^2\right), \\ F_4(u) = a_3 \partial_x \left(1 - \partial_{xx}^2\right)^{-1} \left(u^3\right), \\ F_5(u) = a_4 \partial_x \left(1 - \partial_{xx}^2\right)^{-1} \left(u^4\right). \end{cases}$$

In this paper, under some natural assumptions included in Assumption (A), we obtain the local existence, uniqueness and a blow-up criterion of pathwise solutions to (1.4). The detailed results are stated in Theorem 1.1. For more discussions and comparisons regarding Theorem 1.1, we refer to Remark 1.3. Here we also recall some relevant works on stochastic CH type equations. For the CH equation with additive noise, we refer to [12]. Stochastic CH type equations with nonlinear multiplicative noise are considered in [62–64]. When the noise is of convection type, we refer to [1]. For the stochastic modified CH equation with linear multiplicative noise, we refer to [11].

On the other hand, for SPDEs, the noise effect is one of the probabilistically important questions and the regularization effects have been well observed. For example, it is known that the well-posedness of the linear stochastic transport equation with noise can be established under weaker hypotheses than its deterministic counterpart, cf. [29,31]. For stochastic scalar conservation laws, noise on flux may bring in some regularization effects [35]. In terms of numerics, the regularization effects of noise can be found in [53]. In [37,52,64] the dissipation of energy caused by linear multiplicative noise was analyzed.

Inspired by the above works, the second goal of this paper is to study the effects of nonlinear noise. We mainly consider the following two cases:

- Noise effect on preventing blow-up;
- Noise effect on initial-data dependence.

For the noise effect on blow-up in the current setting, there are essential differences and difficulties because the existing results on the regularization effects by noises for transport type equations are mainly for linear equations or with linear growing (transport) noises. However, our target problem (1.4) is nonlinear and non-local. Indeed, the noise effects become more complicated in nonlinear equations with nonlinear noise. There are both examples in positive direction, i.e., noises can regularize singularities, and negative direction, i.e., noises cannot regularize singularities. For example, for the stochastic 2-D Euler equations, coalescence of vortices disappears (see [32]) but noise cannot prevent the formation of shock in the Burgers equation (see [30]). Moreover, it is observed that regularizing effects depend on the strength of noise, see for example [37,61].

To analyze the validity of the regularization effects by noise in the current setting, we will study how the noise prevents blow-up in (1.4) and how to estimate its strength for this purpose. As is mentioned above, many existing results on the regularization effects by noises are obtained for linear equations or linear growing noises. This particularly motivates us to consider the nonlinear-noise case. Mathematically, searching for nonlinear noise such that blow-up can be prevented is important because it helps us understand the regularizing mechanisms of noise. This in turn brings us one further step to find the really correct and physical noise which provides such regularization. To present our idea, we simply consider the case that $h(t,u)\,\mathrm{d}\mathcal{W} = \sigma(t,u)\,\mathrm{d}W$, where W is a standard 1-D Brownian motion and $\sigma : [0,\infty) \times H^s \to H^s$ is a nonlinear function. Here we use the notation σ rather than h because in this case σ can take values in H^s, whereas h should be a Hilbert–Schmidt operator (see (1.8)) to define stochastic integral with respect to cylindrical Wiener process. Then we focus on

$$\begin{cases} \mathrm{d}u + \left[uu_x + F(u)\right]\mathrm{d}t = \sigma(t,u)\,\mathrm{d}W, & x \in \mathbb{R}, t > 0, \\ u(\omega,0,x) = u_0(\omega,x), & x \in \mathbb{R}, \end{cases} \tag{1.6}$$

where $F(\cdot)$ is given in (1.5). Motivated by [60], we will show in Theorem 1.2 that if $\sigma(t,\cdot)$ grows fast enough (see Assumption (B)), then global existence holds true with probability 1. In different deterministic counterparts of (1.4), blow-up develops even for smooth initial data, we refer to [14–16,58] for the case of CH equation ($h = 0$, $a_1 = 1$, $a_2 = \frac{1}{2}$ and $a_0 = a_3 = a_4 = 0$) and to [28,77] for the b-family equations ($h = 0$, $a_1 = \frac{b}{2}$, $a_2 = \frac{3-b}{2}$ with $b \in \mathbb{R}$ and $a_0 = a_3 = a_4 = 0$). Hence we justify the idea that strong noise has regularization effect in preventing singularities. The strengths of the noise to achieve this in different examples will be given in Section 4.2.

Now we turn to noise effect on the initial-data dependence. We notice that in most of the known results on noise effects, they are studied in terms of the regularity or uniqueness of solutions. Much less is known concerning the noise effect in the direction of dependence on initial data. However, the question whether (and how) noise can affect the dependence on initial data is interesting. Formally, regularization produced by noise may be linked to the regularization effects induced by an additional Laplacian. But if one would add a real Laplacian to the governing equations, then it is possible to improve the continuity of the solution map, i.e., more than continuous, by using parabolic techniques. Indeed, for the deterministic incompressible Euler equations, the solution map $u_0 \mapsto u$ cannot be better than continuity [43], but for the deterministic incompressible Navier–Stokes equations with sufficiently large viscosity, it is at least Lipschitz continuous in sufficiently high Sobolev spaces (see pp. 79–81 in [40]). This motivates us to study whether (and how) noise can affect initial-data dependence. With noise, the interplay between regularization provided by noise and the dependence on initial conditions is first studied in [62,65]. However, in [62,65], the nonlinear terms are of the same order. In this paper, the nonlinear terms are of different orders (see (1.5)), which requires new estimates.

Therefore the final goal of the present work is to investigate the noise effects on the dependence of solution on its initial data. We will consider (1.1) on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. More precisely, we consider

$$\begin{cases} \mathrm{d}u + \left[uu_x + F(u)\right]\mathrm{d}t = h(t,u)\,\mathrm{d}\mathcal{W}, & x \in \mathbb{T}, t > 0, \\ u(\omega,0,x) = u_0(\omega,x), & x \in \mathbb{T}, \end{cases} \tag{1.7}$$

where $F(\cdot)$ is given in (1.5). Following [62,65], we introduce the concept of the stability of the exiting time (this notion refers to the continuous changes of the point in time with respect to the initial condition, where such point is defined as the time when solution leaves a certain range and hence it is called exiting time, see Definition 1.2 below), and then we give a negative statement on the noise effects in terms of initial-data dependence for the problem (1.7). More precisely, in Theorem 1.3, we will show that, when $h(t,u)$ is controlled by the nonlocal term $F(\cdot)$ (see Assumption (C)), the multiplicative noise can *not* improve the stability of the exiting time, *and*, at the same time, improve the continuity of the map $u_0 \mapsto u$ defined by (1.7).

1.1. *Notations, definitions and assumptions*

We now introduce some notations. $L^2(\mathbb{R})$ is the usual space of square–integrable functions on $\mathbb{R}$. For $s \in \mathbb{R}$, $D^s = (1-\partial_{xx}^2)^{s/2}$ is defined by $\widehat{D^s f}(\xi) = (1+\xi^2)^{s/2}\widehat{f}(\xi)$, where $\widehat{f}$ denotes the Fourier transform of a function f. The Sobolev space

$$H^s(\mathbb{R}) \triangleq \left\{ f \in L^2(\mathbb{R}) : \|f\|_{H^s(\mathbb{R})}^2 = \int_{\mathbb{R}} \left(1+\xi^2\right)^s \left|\widehat{f}(\xi)\right|^2 \mathrm{d}\xi < +\infty \right\}$$

is a Hilbert space with inner product

$$(f,g)_{H^s} \triangleq \int_{\mathbb{R}} \left(1+\xi^2\right)^s \widehat{f}(\xi) \cdot \overline{\widehat{g}}(\xi)\,\mathrm{d}\xi = \left(D^s f, D^s g\right)_{L^2}.$$

When the function space refers to $\mathbb{R}$, we will drop $\mathbb{R}$ if there is no ambiguity. We will use $\lesssim$ and $\gtrsim$ to denote estimates that hold up to some universal *deterministic* constant which may change from line to line. For linear operators A and B, we denote $[A,B] = AB - BA$.

In the sequel, $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \mathcal{W})$ is called a stochastic basis where $\mathcal{W}(t) = \mathcal{W}(\omega, t), \omega \in \Omega$ is a cylindrical Wiener process with respect to a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Formally, we consider a separable Hilbert space $\mathfrak{U}$ and let $\{e_k\}$ be a complete orthonormal basis of $\mathfrak{U}$. Let $\{W_k\}_{k\geq 1}$ be a sequence of mutually independent standard 1-D Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Then we define

$$\mathcal{W} \triangleq \sum_{k=1}^{\infty} W_k e_k.$$

Let $\mathcal{X}$ be a separable Hilbert space. As in [21,59], we see that for a predictable process $Z \in \mathcal{L}_2(\mathfrak{U}; \mathcal{X})$ (Hilbert–Schmidt operators from $\mathfrak{U}$ to $\mathcal{X}$),

$$\int_0^t Z\,\mathrm{d}\mathcal{W} \triangleq \sum_{k=1}^{\infty} \int_0^t Z e_k\,\mathrm{d}W_k \tag{1.8}$$

is a well-defined $\mathcal{X}$-valued continuous square integrable martingale. In the sequel, when a stopping time is defined, we set $\inf \varnothing \triangleq \infty$ by convention.

We now give the precise notion of a pathwise solution to (1.4).

Definition 1.1 (Pathwise solutions). Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$ be a fixed stochastic basis. Let $s > 3/2$ and u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable (relative to $\mathcal{S}$).

1. A local pathwise solution to (1.4) is a pair (u, τ), where τ is a stopping time satisfying $\mathbb{P}\{\tau > 0\} = 1$ and $u : \Omega \times [0,\infty] \to H^s$ is an $\mathcal{F}_t$-predictable H^s-valued process satisfying

$$u(\cdot \wedge \tau) \in C\big([0,\infty); H^s\big) \quad \mathbb{P}\text{-a.s.},$$

and for all $t > 0$,

$$u(t\wedge\tau) - u(0) + \int_0^{t\wedge\tau} \big[u u_x + F(u)\big]\,\mathrm{d}t' = \int_0^{t\wedge\tau} h\big(t', u\big)\,\mathrm{d}\mathcal{W} \quad \mathbb{P}\text{-a.s.}$$

2. The local pathwise solutions are said to be pathwise unique, if given any two pairs of local pathwise solutions (u_1, τ_1) and (u_2, τ_2) with $\mathbb{P}\{u_1(0) = u_2(0)\} = 1$, we have

$$\mathbb{P}\big\{u_1(t,x) = u_2(t,x)\ \forall (t,x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{R}\big\} = 1.$$

3. In addition, (u, τ^*) is called a maximal pathwise solution to (1.4) if there is an increasing sequence $\tau_n \to \tau^*$ such that for any $n \in \mathbb{N}$, (u, τ_n) is a pathwise solution to (1.4) and

$$\sup_{t\in[0,\tau_n]} \|u\|_{H^s} \geq n \quad \text{on the set } \big\{\tau^* < \infty\big\}.$$

4. If (u, τ^*) is a maximal pathwise solution and $\tau^* = \infty$ almost surely, then we say that the pathwise solution exists globally.

Next, we introduce the following notion of the stability of exiting time (exiting time is the time when solution leaves a certain range) in Sobolev spaces.

Definition 1.2 (Stability of exiting time). Let $s > 3/2$ and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, \mathcal{W})$ be a fixed stochastic basis. Let u_0 be an H^s-valued $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}\|u_0\|^2_{H^s} < \infty$. Assume that $\{u_{0,n}\}$ is an arbitrary sequence of H^s-valued $\mathcal{F}_0$-measurable random variables satisfying $\mathbb{E}\|u_{0,n}\|^2_{H^s} < \infty$. For each n, let u and u_n be the unique pathwise solutions to (1.4) with initial values u_0 and $u_{0,n}$, respectively. For any $R > 0$ and $n \in \mathbb{N}$, we define the R-exiting times

$$\tau_n^R = \inf\{t \ge 0 : \|u_n(t)\|_{H^s} > R\} \quad \text{and} \quad \tau^R = \inf\{t \ge 0 : \|u(t)\|_{H^s} > R\}.$$

Then we define the following properties:

1. If $u_{0,n} \to u_0$ in H^s $\mathbb{P}$-a.s. implies that

$$\lim_{n\to\infty} \tau_n^R = \tau^R \quad \mathbb{P}\text{-a.s.,} \tag{1.9}$$

   then the R-exiting time of u is said to be stable.
2. If $u_{0,n} \to u_0$ in $H^{s'}$ for all $s' < s$ almost surely implies that (1.9) holds true, the R-exiting time of u is said to be strongly stable.

To study the existence of pathwise solutions to (1.4), we need the following assumptions on the h:

Assumption (A). Let $s > \frac{1}{2}$. We assume that $h : [0,\infty) \times H^s \ni (t,u) \mapsto h(t,u) \in \mathcal{L}_2(\mathfrak{U}; H^s)$ such that if $u : \Omega \times [0,T] \to H^s$ is predictable, then $h(t,u)$ is also predictable. Furthermore, we assume that there are two non-decreasing locally bounded functions $f(\cdot), q(\cdot) : [0,\infty) \to [0,\infty)$ such that the following assumptions hold true:

(A-1) For all $t \ge 0$ and $s > 3/2$,

$$\|h(t,u)\|_{\mathcal{L}_2(\mathfrak{U}; H^s)} \le f(\|u\|_{W^{1,\infty}})(1 + \|u\|_{H^s}).$$

(A-2) For any $t \ge 0$,

$$\sup_{\|u\|_{H^s}, \|v\|_{H^s} \le N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|h(t,u) - h(t,v)\|_{\mathcal{L}_2(\mathfrak{U}, H^s)}}{\|u - v\|_{H^s}} \right\} \le q(N), \quad N \ge 1, s > 3/2.$$

(A-3) For any $t \ge 0$,

$$\sup_{\|u\|_{H^{s+1}}, \|v\|_{H^{s+1}} \le N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|h(t,u) - h(t,v)\|_{\mathcal{L}_2(\mathfrak{U}, H^s)}}{\|u - v\|_{H^s}} \right\} \le q(N), \quad N \ge 1, 3/2 \ge s > 1/2.$$

As for the noise effect vs blow-up, we impose the following condition on σ in (1.6):

Assumption (B). We assume that when $s > \frac{3}{2}$, $\sigma : [0,\infty) \times H^s \ni (t,u) \mapsto \sigma(t,u) \in H^s$ satisfies that if $u : \Omega \times [0,T] \to H^s$ is predictable, then $\sigma(t,u)$ is also predictable. Moreover, we assume the following properties hold true:

(B-1) $\sigma(\cdot, u)$ is locally bounded for all $u \in H^s$ and there is a non-decreasing locally bounded function $l(\cdot) : [0,\infty) \to [0,\infty)$ such that for any $t \ge 0$,

$$\sup_{\|u\|_{H^s}, \|v\|_{H^s} \le N} \left\{ \mathbf{1}_{\{u\neq v\}} \frac{\|\sigma(t,u) - \sigma(t,v)\|_{H^s}}{\|u - v\|_{H^s}} \right\} \le l(N), \quad N \ge 1, s > 3/2.$$

(B-2) Define

$$\mathcal{V} = \left\{ V \in C^2([0,\infty); [0,\infty)) : V(0) = 0, V'(x) > 0, V(x) \le x, V''(x) \le 0 \text{ and } \lim_{x\to\infty} V(x) = \infty \right\},$$

and we assume that there is a function $V \in \mathcal{V}$ and constants $M_1, M_2 > 0$ such that

$$\begin{aligned} &V'\big(\|u\|_{H^s}^2\big)\big\{2Ag\big(\|u\|_{W^{1,\infty}}\big)\|u\|_{H^s}^2 + \big\|\sigma(t,u)\big\|_{H^s}^2\big\} + 2V''\big(\|u\|_{H^s}^2\big)\big|\big(\sigma(t,u),u\big)_{H^s}\big|^2 \\ &\quad \le M_1 - M_2\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t,u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}, \quad (t,u)\in[0,\infty)\times H^s, \end{aligned} \tag{1.10}$$

where A and $g(\cdot)$ are given in Lemma 2.5.

When we consider the initial-data dependence for (1.7) in Section 6, we need the following assumption on $h(t,\cdot)$.

Assumption (C). For $s > \frac{1}{2}$ and $u \in H^s$, $h : [0,\infty)\times H^s \ni (t,u) \mapsto h(t,u) \in \mathcal{L}_2(\mathfrak{U}; H^s)$ satisfies that if $u : \Omega\times[0,T] \to H^s$ is predictable, then $h(t,u)$ is also predictable. Moreover, we assume that there is a constant $C > 0$ such that for all $t \ge 0$ and $u \in H^s$ with $s > \frac{1}{2}$,

$$\big\|h(t,u)\big\|_{\mathcal{L}_2(\mathfrak{U};H^s)} \le C\big\|F(u)\big\|_{H^s}, \qquad \big\|h(t,u)-h(t,v)\big\|_{\mathcal{L}_2(\mathfrak{U};H^s)} \le C\big\|F(u)-F(v)\big\|_{H^s},$$

where $F(\cdot)$ is defined by (1.5).

1.2. *Main results*

Now we are ready to state the main results in this paper.

Theorem 1.1. *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{W})$ be a given stochastic basis. Let $s > 3/2$, $a_i \in \mathbb{R}$ $(i = 1,2,3,4)$ and let $h(t,u)$ satisfy Assumption (A). If u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, then there is a local pathwise solution (u,τ) to (1.4) in the sense of Definition 1.1 with*

$$u(\cdot\wedge\tau) \in L^2\big(\Omega; C\big([0,\infty); H^s\big)\big). \tag{1.11}$$

Moreover, (u,τ) is unique and it can be extended to a unique maximal pathwise solution (u,τ^*) with

$$\mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{H^s}=\infty\}} = \mathbf{1}_{\{\limsup_{t\to\tau^*}\|u(t)\|_{W^{1,\infty}}=\infty\}} \quad \mathbb{P}\text{-}a.s. \tag{1.12}$$

On the noise effect vs blow-up, we consider (1.6) and we have

Theorem 1.2 (Noise vs blow-up). *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, W)$ be a fixed stochastic basis. Let $s > \frac{5}{2}$, $a_i \in \mathbb{R}$ $(i = 1,2,3,4)$ and $u_0 \in H^s$ be an H^s-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. If Assumption (B) holds true, then (1.6) has a unique global solution. Precisely, if τ^* is the maximal existence time of $u \in H^s$, then*

$$\mathbb{P}\big\{\tau^* = \infty\big\} = 1.$$

For the interplay between regularization effects induced by the noise and the dependence on initial conditions, the next result gives a partial (negative) result.

Theorem 1.3 (Weak instability). *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0}, \mathcal{W})$ be a fixed stochastic basis, $s > 3/2$ and $a_i \in \mathbb{R}$ $(i = 1,2,3,4)$. Consider the periodic initial value problem (1.7). If h satisfies Assumption (C), then at least one of the following properties holds true*:

1. *For any $R \gg 1$, the R-exiting time is not strongly stable for the zero solution, i.e., $u \equiv 0$, to (1.7) in the sense of Definition 1.2*;
2. *The solution map $u_0 \mapsto u$ defined by solving (1.7) is not uniformly continuous as a map from $L^2(\Omega, H^s)$ into $L^2(\Omega; C([0,T]; H^s))$ for any $T > 0$. More precisely, there exist two sequences of solutions $u_{1,n}(t)$ and $u_{2,n}(t)$, and two sequences of stopping times $\tau_{1,n}$ and $\tau_{2,n}$, such that*
   - $\mathbb{P}\{\tau_{i,n} > 0\} = 1$ *for each $n > 1$ and $i = 1,2$. Besides,*

$$\lim_{n\to\infty}\tau_{1,n} = \lim_{n\to\infty}\tau_{2,n} = \infty \quad \mathbb{P}\text{-}a.s. \tag{1.13}$$

- *For* $i=1,2$, $u_{i,n}\in C([0,\tau_{i,n}];H^s)$ $\mathbb{P}$*-a.s., and*

$$\mathbb{E}\Big(\sup_{t\in[0,\tau_{1,n}]}\big\|u_{1,n}(t)\big\|^2_{H^s}+\sup_{t\in[0,\tau_{2,n}]}\big\|u_{2,n}(t)\big\|^2_{H^s}\Big)\lesssim 1. \tag{1.14}$$

- *At* $t=0$,

$$\lim_{n\to\infty}\mathbb{E}\big\|u_{1,n}(0)-u_{2,n}(0)\big\|^2_{H^s}=0. \tag{1.15}$$

- *For any* $T>0$,

$$\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{1,n}\wedge\tau_{2,n}]}\big\|u_{1,n}(t)-u_{2,n}(t)\big\|^2_{H^s}\gtrsim\Big(\sup_{t\in[0,T]}|\sin t|\Big)^2. \tag{1.16}$$

1.3. *Remarks on the difficulties, proofs and comparison*

Remark 1.1. To begin with, we notice that our target models involve a difficulty *a priori*, i.e., the classical Itô formulae cannot be directly used to estimate $\mathbb{E}\|u(t)\|^2_{H^s}$. Indeed, to use the Itô formula in a Hilbert space (cf. [21, Theorem 4.32] or [34, Theorem 2.10]), the inner product $(uu_x,u)_{H^s}$ needs to be well-defined. Likewise, to fit into the Itô formula under a Gelfand triplet ([54, Theorem I.3.1] or [59, Theorem 4.2.5]), the dual product ${}_{H^{s-1}}\langle uu_x,u\rangle_{H^{s+1}}$ is required to be well-defined. However, for given $u_0\in H^s$, since the deterministic counterpart of the target models is a transport type PDE, we can *only* expect $u\in H^s$ and therefore $uu_x\in H^{s-1}$. As a result, neither $(uu_x,u)_{H^s}$ nor ${}_{H^{s-1}}\langle uu_x,u\rangle_{H^{s+1}}$ is well-defined in our case. To overcome this difficulty, we will first mollify the equation and then use the Itô formula in a Hilbert space to $\mathbb{E}\|J_\varepsilon u(t)\|^2_{H^s}$, where J_ε is a mollifier (see (2.1) below). Then we take limit to obtain the estimate for $\mathbb{E}\|u(t)\|^2_{H^s}$ (see (3.19), (3.24) and (4.1) for example).

Remark 1.2. Now we give some explanations on the Assumptions (A), (B) and (C).

- Assumption (A-1) allows us to consider general growing noise coefficient. The classical local Lipschtiz condition (A-2) is used to guarantee the existence of approximation solutions in H^s with $s>3/2$ (cf. Proposition 3.1) and the convergence of approximation solutions (cf. Lemma 3.2). As is mentioned in Remark 1.1, the map $u\mapsto uu_x$ is *not* invariant in H^s. Hence the difference between two solutions $u,v\in H^s$ should be measured in H^{s-1} rather than in H^s. Indeed, if H^s-topology is used to measure $u-v$, we will have to deal with $(uu_x-vv_x,u-v)_{H^s}$, which will lead to one term with undefined H^{s+1}-norm. Therefore, we need an assumption on $\|h(t,u)-h(t,v)\|_{\mathcal{L}_2(\mathfrak{U},H^{s-1})}$ for $u,v\in H^s$ with $s>3/2$, which is (A-3). We outline that it is not difficult to construct examples satisfying Assumption (A). Particularly, if $b(t)$ is a continuous and bounded function, $F(u)$ is given in (1.5) and W is a standard 1-D Brownian motion, then $h(t,u)\,\mathrm{d}\mathcal{W}=b(t)F(u)\,\mathrm{d}W$ is an example such that Assumption (A) is verified with $\mathcal{L}_2(\mathfrak{U};H^s)$ replaced by H^s (cf. Lemma 2.4 below).
- Assumption (B) is used in proving Theorem 1.2 and it is highly motivated by [60] (see also [7,63]). Particularly, (B-2) in Assumption (B) is a Lyapunov type condition. Because $V''\le 0$, (B-2) actually requires that the noise is large enough such that the growth of $uu_x+F(u)$ can be cancelled and V can be viewed as a Lyapunov function. However, different from (A-2) and (A-3), we require $s>3/2$ for both (B-1) and (B-2). Mathematically, it seems that one can only require $s>1/2$ in (B-1) (so uniqueness still holds true in H^s with $s>3/2$, in the same reason as uu_x loses one derivative), but if this is the case, it is *not* clear how to construct an example at present. So far we have only given examples (see Section 4.2) with requiring $s>3/2$ in both (B-1) and (B-2). This technical assumption brings a little gap, that is, even though one may use (B-2) to find global existence only in H^s with $s>3/2$, (B-1) means we can *only* prove uniqueness for solutions in H^r with $r>5/2$.
- Finally, Assumption (C) is used in the proof for Theorem 1.3. Indeed, since Theorem 1.3 is proved by constructing counterexample (see Remark 1.5 below), it is natural to first consider the case that the noise is controlled by the non-local term F in (1.4) and hence we need Assumption (C). For more general cases, when the noise is large or the stochastic integral is in the sense of Stratonovich, whether the noise can improve the stability is still unknown.

Remark 1.3. Now we give some remarks regarding the proof for Theorem 1.1.

(I): Difficulties and strategies. We first briefly outline the main difficulties encountered in proving Theorem 1.1 and our strategies.

- (Step 1: Approximation scheme) The starting point of the proof for Theorem 1.1 is to mollify the problem (1.4) such that the mollified version can be viewed as an SDE in Sobolev spaces H^s. Then we obtain the approximation solutions $\{u_\varepsilon\}$. Since the diffusion coefficient satisfies (A-1) and the non-local term $F(\cdot)$ satisfies a nonlinear growth condition (see (2.7) below), the *a priori* estimate for $\mathbb{E}\|u_\varepsilon\|^2_{H^s}$ involves $\mathbb{E}[\Psi(\|u_\varepsilon\|_{W^{1,\infty}})(1+\|u_\varepsilon\|^2_{H^s})]$ for some nonlinear function Ψ. Because the expectation cannot be split generally, we cannot close the *a priori* $L^2(\Omega; H^s)$ estimate for u_ε. To overcome this difficulty, we will add a *cut-off* function into the problem (see (3.1)). Without the *cut-off*, when we consider the $L^2(\Omega; H^s)$ estimate, we have to introduce a sequence of stopping times to localize the process u_ε, that is, if $\tau_\varepsilon = \inf\{\|u_\varepsilon\|_{W^{1,\infty}} > R\}$, then $\mathbb{E}\int_0^{t\wedge\tau_\varepsilon} \Psi(\|u_\varepsilon\|_{W^{1,\infty}})(1+\|u_\varepsilon\|^2_{H^s})\,\mathrm{d}t' \le \Psi(R)\int_0^t \mathbb{E}(1+\|u_\varepsilon\|^2_{H^s})\,\mathrm{d}t'$. However, usually it is *not* clear how to prove $\mathbb{P}\{\inf_\varepsilon \tau_\varepsilon > 0\} = 1$, and we think this is a common difficulty in nonlinear SPDEs. If $\inf_\varepsilon \tau_\varepsilon = 0$ with positive probability, then we can *not* take limit to obtain a pathwise solution in the sense of Definition 1.1.
- (Step 2:Building a pathwise solution) Another main difficulty lies in the lack of compact Sobolev embedding in the whole space $\mathbb{R}$, and this brings us an obstacle to prove the convergence of the approximation solutions $\{u_\varepsilon\}$. Motivated by [57] and based on a careful analysis on the differences between any two approximation layers, we can show that there is a subsequence of the approximation solutions converging in $C([0,T]; H^{s-\frac{3}{2}})$ almost surely. After taking limits to obtain a solution u, one can improve the regularity of u to $C([0,T]; H^s)$ again, and the technical difficulty here is to prove the time continuity of the solution because the classical Itô formula is *not* applicable (cf. Remark 1.1). Under an additional $L^\infty(\Omega)$ condition on u_0, one can *a posteriori* introduce a positive stopping time τ (as in (3.22)) to remove the *cut-off*. To remove the additional condition on u_0 and guarantee that $\tau > 0$ almost surely, the cutting-combining argument will be used (see Section 3.4 for the details). Technically, to take limit in the *cut-off* function, where the $W^{1,\infty}$-norm is involved, we require $s > 3$ as an intermediate requirement such that $H^{s-\frac{3}{2}} \hookrightarrow W^{1,\infty}$. After removing the *cut-off*, the range of s can be extended to $s > 3/2$ by mollifying the initial data, cf. [37,64].

(II): Comparison of approaches. As is mentioned in Remark 1.1, because the map $u \mapsto uu_x$ is *not* invariant in H^s, the concept of (weak) monotonicity fails to be defined for (1.4) and hence the monotone method in a Gelfand-triple is not applicable in our case. This is the first motivation to consider new methods to prove existence. To see the second motivation to do so, and for the convenience of readers, we first briefly review the martingale approach widely used to prove existence of nonlinear SPDEs.

- Roughly speaking, the martingale approach includes obtaining martingale solution first and then establishing pathwise uniqueness to obtain the pathwise solution. As is explained above, when one tries to find a solution to a nonlinear SPDE in some space $\mathcal{X}$, if the growth of the nonlinear terms in $\mathcal{X}$ can be controlled by a product of a linear function of $\|\cdot\|_{\mathcal{X}}$ and a nonlinear function of $\|\cdot\|_{\mathcal{Z}}$ with $\mathcal{X} \hookrightarrow \mathcal{Z}$, one may need to consider a *cut-off* version of the problem first, in which the additional $L^\infty(\Omega; \mathcal{Z})$ condition provided by the *cut-off* enables us to close the $L^2(\Omega; \mathcal{X})$ estimate. It is usually not difficult to approximate the problem and obtain certain uniform estimates for the approximation solutions in $\mathcal{X}$. To obtain a martingale solution to the *cut-off* version of the target SPDE, one can first find a space $\mathcal{Y}$ such that $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$ (here $\hookrightarrow\hookrightarrow$ means the embedding is compact). Then, by $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y}$ and the uniform estimates in $\mathcal{X}$, one can establish the tightness of the measures defined by the approximation solutions in $\mathcal{Y}$. By Prokhorov's Theorem and Skorokhod's Theorem, we can get almost sure convergence in $\mathcal{Y}$. By the martingale representation theorem, we can identify the limit of the stochastic integral. This, together with the embedding $\mathcal{Y} \hookrightarrow \mathcal{Z}$, enables us to take limit to obtain a martingale solution to the *cut-off* problem. Additionally, if pathwise uniqueness holds for the *cut-off* problem, one can use an infinite dimensional Yamada–Watanabe type result (cf. [55]) or use the Gyöngy–Krylov characterization of the convergence in probability (see [39]) to obtain a pathwise solution. Finally one can remove the *cut-off* to obtain a solution to the original problem, cf. [4,36,64] for the techniques.
- If the target SPDE is defined in a *bounded* domain, it is not difficult to find suitable Sobolev spaces $\mathcal{X}, \mathcal{Y}$ such that $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$ and we refer to [2,3,23,37,44,64] for different examples. In *unbounded* domains, compact Sobolev embedding fails to be true and a possible way is to consider compact embedding of local space $\mathcal{X}_{\text{loc}} \hookrightarrow\hookrightarrow \mathcal{Y}_{\text{loc}}$. Repeating the above approach, we can then obtain almost sure convergence in $\mathcal{Y}_{\text{loc}}$. However, as above, to take limit in the *cut-off*, we have to require $\mathcal{Y}_{\text{loc}} \hookrightarrow \mathcal{Z}$. Otherwise, convergence in $\mathcal{Y}_{\text{loc}}$ can *not* imply convergence in $\mathcal{Z}$ and then taking limit will *not* go back to the *cut-off* problem. When the equation has some cancellation properties (for example, divergence free) and the noise grows *linearly*, we refer to [8,9] for different examples such that the construction of $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ can be carried out to satisfy this requirement. However, this can *not* be always guaranteed for general nonlinear SPDEs where strong nonlinear multiplicative noise is involved. For example, in our target problem (1.4), the growth of both the drift and diffusion coefficients involves $W^{1,\infty}$-norm (see (A-1) and Lemma 2.4), and hence we have to introduce the *cut-off* on $\|\cdot\|_{W^{1,\infty}}$ (see (3.2) below). i.e., $\mathcal{Z} = W^{1,\infty}$. We also

establish the uniform estimate of the H^s-norm for approximation solutions in Proposition 3.1. However, even if one can employ Prokhorov's Theorem and Skorokhod's Theorem to find almost sure convergence in local Sobolev space $H^{s'}_{\text{loc}}$ for some $s' < s$, we can *not* pass limit to find a solution to the *cut-off* problem. This is because $\|\cdot\|_{W^{1,\infty}}$ is a *global* quantity involving all $x \in \mathbb{R}$, but the convergence is only a *local* one, i.e., convergence is in $H^{s'}_{\text{loc}}$ (even though s' can be large enough such that $H^{s'} \hookrightarrow W^{1,\infty}$).

- The above unsolved technical issue is the second motivation to propose new approximation schemes to study nonlinear SPDEs with nonlinear noise in unbounded domains. In this work, motivated by [57], we set the target solution space as $C([0,T]; H^s) = \mathcal{X}$, and we mollify the problem to consider its approximation version as an SDE in Sobolev spaces H^s. The advantage of this approximation scheme is that we are able to prove convergence (up to a subsequence) in $C([0,T]; H^{s-\frac{3}{2}}) = \mathcal{Y}$ directly *without* compactness. Indeed, in this case $\mathcal{X}$ is *not* compactly embedded in $\mathcal{Y}$ (compared to the martingale approach, convergence/tightness usually comes from the compact embedding $\mathcal{X} \hookrightarrow\hookrightarrow \mathcal{Y}$). We also notice that the convergence in $C([0,T]; H^{s-\frac{3}{2}})$ *a priori* looks interesting because it is usually expected that the convergence holds true up to some stopping times (see (3.10), (3.11), (3.12)), and as is mentioned above, lower bound for stopping times is difficult to obtain in the stochastic setting. In this work, due to the uniform-in-ε estimate, stopping times can be removed.

Remark 1.4. We notice that blow-up of the solutions actually occurs in the deterministic counterpart of (1.1) (i.e., $h = 0$ in (1.1)) with different a_i $(i = 0, 1, 2, 3, 4)$ and we refer to [14–16,28,38,78] and the references therein. Therefore Theorem 1.2 justifies that strong enough noise can regularize the solutions in terms of preventing singularities. This result is motivated by [7,60,63]. Examples of such noise structure are given in Section 4.2. As a corollary, Theorem 1.2 implies that in the stochastic case, blow-up of pathwise solutions might only be observed if the noise is weak.

Remark 1.5. Theorem 1.3 demonstrates that one cannot improve the stability of the exiting time for the zero solution, and simultaneously improve the continuous dependence of solutions on initial data. It is also worthwhile noticing that Theorem 1.3 is proved under the assumption that noise can *not* grow very fast (cf. Assumption (C)) whereas in Theorem 1.2, fast growing noise can prevent singularities. Now we outline the idea in proving Theorem 1.3.

- Since we are *not* able to get an explicit expression of the solution to (1.7), our idea to obtain (1.16) is to find two sequences of approximation solutions $\{u^{i,n}\}_{n\geq 1}$ $(i \in \{1, 2\})$ such that when n tends to ∞, $u^{i,n}$ tends to the actual solution $u_{i,n}$ for $t > 0$. Then one can prove (1.16) by estimating $u^{i,n}$ rather than $u_{i,n}$. We choose two sequences of approximation solutions $\{u^{i,n}\}$ in the form of the explicit periodic solutions to the incompressible Euler equations, cf. [43,66]. We will see that the actual solutions $u_{i,n}$ starting from $u_{i,n}(0) = u^{i,n}(0)$ satisfy

$$\lim_{n\to\infty} \mathbb{E} \sup_{[0,\tau_{i,n}]} \left\| u_{i,n} - u^{i,n} \right\|_{H^s} = 0, \tag{1.17}$$

where $u_{i,n}$ exists at least on $[0, \tau_{i,n}]$. With (1.17) at hand, one can establish (1.16) by using $u^{i,n}$. However, as before, in this step we again face the problem how to prove $\inf_n \tau_{i,n} > 0$ almost surely. If $\tau_{i,n} \to 0$, we will get nothing. This is one main difference between the deterministic and the stochastic cases. Indeed, such approximation solutions have been used in deterministic CH type equations, see [42,68,69] and the references therein. But in the deterministic cases, one has the lifespan estimate (see (4.7)–(4.8) in [68] and (3.8)–(3.9) in [69] for example), which enables us to find a $T > 0$ independent of n such that all actual solutions $u_{i,n}$ exist on the common interval $[0, T]$.
- The key observation in dealing with the property $\inf_n \tau_{i,n} > 0$ is the connection between this property and the stability property of the exiting time (see Definition 1.2). We find that if for some $R_0 \gg 1$, the R_0-exiting time is strongly stable at the zero solution, then $\tau_{i,n} \to \infty$ (see (6.16)). Technically, to get (1.17), we estimate the error in $H^{2s-\delta}$ and H^{δ} with suitable δ, respectively. Then (1.17) is a consequence of the interpolation. Here we notice that our target problem (1.4) includes nonlinearities from order 1 to order 4, which is different from [62,65], where the nonlinearities are of the same order. Therefore more estimates are involved to balance different orders. Moreover, it is also important to notice that what we have actually obtained is that the solution map $u_0 \mapsto u$ is not uniformly continuous as a map from $L^\infty(\Omega, H^s)$ into $L^1(\Omega; C([0,T], H^s))$. Indeed, because the approximation solutions $u^{i,n}$ are constructed deterministically, (1.15) can be changed into $\lim_{n\to\infty} \|u^{1,n}(0) - u^{2,n}(0)\|_{L^p(\Omega; H^s)} = 0$ for $p \in [1, \infty]$ (see (6.17) for example), and (1.16) can be changed into (6.19). However, to be consist with the existence part, we formulate the result in $L^2(\Omega)$.
- In deterministic cases, the optimal continuity of solution map has been extensively investigated for various nonlinear dispersive and integrable equations. Kato [49] proved that the solution map $u_0 \mapsto u$ of the inviscid Burgers equation is continuous but cannot be Hölder continuous in $H^s(\mathbb{T})$ $(s > 3/2)$, regardless of the Hölder exponent. Since then, various nonlinear evolution PDEs have been studied in terms of this property and here we only mention a few related

results. For the CH equation we refer the readers to [41,42] for the non-uniform dependence on initial data in Sobolev spaces H^s. The results of this type in Besov spaces first appear in [66–69], where the critical case can be also included. Particularly, we extend the recent work [76] on deterministic rotation-CH to the stochastic setting.

We outline the rest of the paper. In next section we formulate and prove some estimates will be employed throughout the paper. In Section 3, we prove Theorem 1.1 by using an approximation method. In Section 4, we study the effect of strong noise and prove Theorem 1.2. Finally, in Section 6, we prove Theorem 1.3.

2. Preliminaries

For any $\varepsilon \in (0,1)$, we let $j(x)$ be a Schwartz function such that $0 \le \widehat{j}(\xi) \le 1$ for all $\xi \in \mathbb{R}$ and $\widehat{j}(\xi) = 1$ for any $|\xi| \le 1$. Then we let $j_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon})$. Using j_ε, we define the Friedrichs mollifier J_ε as

$$[J_\varepsilon f](x) = [j_\varepsilon * f](x), \tag{2.1}$$

where $*$ represents convolution. From the construction of J_ε, we have $\|J_\varepsilon u\|_{L^\infty} \lesssim \|u\|_{L^\infty}$. Let $\mathcal{L}(\mathcal{X};\mathcal{Y})$ be the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. As in [64,65], for any $s > 0$, $\varepsilon > 0$ and $u, v \in H^s$, we have

$$\|I - J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \lesssim \varepsilon^{s-r}, \quad r < s, \tag{2.2}$$

$$\|J_\varepsilon\|_{\mathcal{L}(H^s;H^r)} \lesssim O\big(\varepsilon^{s-r}\big), \quad r > s, \tag{2.3}$$

$$D^s J_\varepsilon = J_\varepsilon D^s, \tag{2.4}$$

$$(J_\varepsilon u, v)_{L^2} = (u, J_\varepsilon v)_{L^2}, \tag{2.5}$$

and

$$\|J_\varepsilon u\|_{H^s} \le \|u\|_{H^s}. \tag{2.6}$$

Lemma 2.1 (Page 3 in [71]). *Let J_ε be defined as in the above. Then there is a constant $C > 0$ such that*

$$\big\|[J_\varepsilon, g]\partial_x f\big\|_{L^2} \le C\|\partial_x g\|_{L^\infty}\|f\|_{L^2}, \quad g \in W^{1,\infty}, f \in L^2.$$

We also recall the following well-known estimates.

Lemma 2.2 ([50,51]). *If $f, g \in H^s \cap W^{1,\infty}$ with $s > 0$, then for $p, p_i \in (1,\infty)$ with $i = 2, 3$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$, there is a $C > 0$ such that*

$$\big\|\big[D^s, f\big]g\big\|_{L^p} \le C\big(\|\nabla f\|_{L^{p_1}}\big\|D^{s-1}g\big\|_{L^{p_2}} + \big\|D^s f\big\|_{L^{p_3}}\|g\|_{L^{p_4}}\big),$$

and

$$\big\|D^s(fg)\big\|_{L^p} \le C\big(\|f\|_{L^{p_1}}\big\|D^s g\big\|_{L^{p_2}} + \big\|D^s f\big\|_{L^{p_3}}\|g\|_{L^{p_4}}\big).$$

Lemma 2.3 (Proposition 4.2, [70]). *If $\rho > 3/2$ and $0 \le \eta + 1 \le \rho$, then for some $c > 0$,*

$$\big\|\big[D^\eta\partial_x, f\big]v\big\|_{L^2} \le c\|f\|_{H^\rho}\|v\|_{H^\eta}\ \forall f \in H^\rho, \quad v \in H^\eta.$$

For the non-local term $F(\cdot)$ defined in (1.5), we have the following lemma.

Lemma 2.4. *Let $s > \frac{1}{2}$. There is a constant $C = C(s) > 0$ such that for any u, v in H^s and $I_s(u,v) = \|u\|_{H^s} + \|v\|_{H^s}$, $F(\cdot)$ satisfies the following estimates*:

- *When $s > 3/2$,*

$$\big\|F(v)\big\|_{H^s} \le C\big(|a_0| + \big(|a_1| + |a_2|\big)\|v\|_{W^{1,\infty}} + |a_3|\|v\|^2_{W^{1,\infty}} + |a_4|\|v\|^3_{W^{1,\infty}}\big)\|v\|_{H^s}, \tag{2.7}$$

$$\big\|F(u) - F(v)\big\|_{H^s} \le C\big[|a_0| + \big(|a_1| + |a_2|\big)I_s(u,v) + |a_3|I_s^2(u,v) + |a_4|I_s^3(u,v)\big]\|u - v\|_{H^s}. \tag{2.8}$$

- *When* $3/2 \geq s > 1/2$,

$$\big\|F(u)-F(v)\big\|_{H^s} \leq C\big[|a_0| + \big(|a_1|+|a_2|\big)I_{s+1}(u,v) + |a_3|I_s^2(u,v) + |a_4|I_s^3(u,v)\big]\|u-v\|_{H^s}. \tag{2.9}$$

Proof. Since $H^s \hookrightarrow W^{1,\infty}$ for $s > 3/2$, (2.7) and (2.8) immediately come from Lemma 2.2 and the fact $\partial_x(1-\partial_{xx}^2)^{-1}$ is bounded from H^s to H^{s+1}. Here we only prove (2.9). We first claim that for $1/2 < \sigma \leq 3/2$,

$$\|fg\|_{H^{\sigma-1}} \leq c_\sigma \|f\|_{H^{\sigma-1}} \|g\|_{H^\sigma}. \tag{2.10}$$

Actually, (2.10) is a special case of the following estimate in Besov space (see [22, (1.4)]). For all $s_1 \leq \frac{1}{p} < s_2 (s_2 \geq \frac{1}{p}$ if $r = 1)$ and $s_1 + s_2 > 0$, it follows that

$$\|fg\|_{B^{s_1}_{p,r}} \leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}} \quad \forall f \in B^{s_1}_{p,r}, g \in B^{s_2}_{p,r}.$$

Since $H^s = B^s_{2,2}$ and $1/2 < \sigma \leq 3/2$, we let $p = r = 2$, $s_1 = \sigma - 1 \leq \frac{1}{2} < \sigma = s_2$ to find (2.10). By (2.10), we arrive at

$$\big\|u_x^2 - v_x^2\big\|_{H^{s-1}} \lesssim \|u_x + v_x\|_{H^s}\|u_x - v_x\|_{H^{s-1}} \lesssim I_{s+1}(u,v)\|u-v\|_{H^s}.$$

When $1/2 < s \leq 3/2$, we use $H^s \hookrightarrow L^\infty$ and the above estimate to derive

$$\begin{aligned}
&\big\|F(u)-F(v)\big\|_{H^s} \\
&\quad\lesssim |a_0|\|u-v\|_{H^{s-1}} + |a_1|\big\|u^2-v^2\big\|_{H^{s-1}} + |a_2|\big\|u_x^2-v_x^2\big\|_{H^{s-1}} + |a_3|\big\|u^3-v^3\big\|_{H^{s-1}} + |a_4|\big\|u^4-v^4\big\|_{H^{s-1}} \\
&\quad\lesssim \big[|a_0| + \big(|a_1|+|a_2|\big)I_{s+1}(u,v) + |a_3|I_s^2(u,v) + |a_4|I_s^3(u,v)\big]\|u-v\|_{H^s},
\end{aligned}$$

which implies (2.9). □

Lemma 2.5. *Let* $s > 3/2$. *Let* $F(\cdot)$ *be given in* (1.5) *and* J_ε *be the Friedrichs mollifier defined in* (2.1). *Let*

$$g(x) = |a_0| + \big(1+|a_1|+|a_2|\big)x + |a_3|x^2 + |a_4|x^3.$$

Then there is a constant $A = A(s) > 0$ *such that for all* $\varepsilon > 0$,

$$\big|\big(J_\varepsilon[uu_x], J_\varepsilon u\big)_{H^s}\big| + \big|\big(J_\varepsilon F(u), J_\varepsilon u\big)_{H^s}\big| \leq Ag\big(\|u\|_{W^{1,\infty}}\big)\|u\|^2_{H^s}.$$

Proof. Due to (2.4) and (2.5), we commute the operator to derive

$$\begin{aligned}
&\big(D^s J_\varepsilon[uu_x], D^s J_\varepsilon u\big)_{L^2} \\
&\quad= \big(\big[D^s, u\big]u_x, D^s J_\varepsilon^2 u\big)_{L^2} + \big([J_\varepsilon, u]D^s u_x, D^s J_\varepsilon u\big)_{L^2} + \big(uD^s J_\varepsilon u_x, D^s J_\varepsilon u\big)_{L^2}.
\end{aligned}$$

Then it follows from Lemmas 2.1 and 2.2, integration by parts, (2.6) and $H^s \hookrightarrow W^{1,\infty}$ that

$$\big|\big(J_\varepsilon[uu_x], J_\varepsilon u\big)_{H^s}\big| \lesssim \|u\|_{W^{1,\infty}}\|u\|^2_{H^s}.$$

Using Lemma 2.4 and (2.6) directly, we have

$$\big|\big(J_\varepsilon F(u), J_\varepsilon u\big)_{H^s}\big| \lesssim \big(|a_0| + \big(|a_1|+|a_2|\big)\|u\|_{W^{1,\infty}} + |a_3|\|u\|^2_{W^{1,\infty}} + |a_4|\|u\|^3_{W^{1,\infty}}\big)\|u\|^2_{H^s}.$$

Combining the above two inequalities gives rise to the desired estimate. □

Finally, on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, the following estimate will be used.

Lemma 2.6 ([42,69]). *Let* $\eta, \rho \in \mathbb{R}$. *If* $n \in \mathbb{Z}^+$ *and* $n \gg 1$, *then*

$$\big\|\sin(nx-\rho)\big\|_{H^\eta(\mathbb{T})} = \big\|\cos(nx-\rho)\big\|_{H^\eta(\mathbb{T})} \approx n^\eta.$$

3. Local-in-time theory

In this section, we prove Theorem 1.1. For clarity, the proof is divided into several subsections.

3.1. *Approximation scheme and uniform estimates*

The first step is to construct a suitable approximation scheme. For any $R > 1$, we let $\chi_R(x) : [0,\infty) \to [0,1]$ be a C^∞ function such that $\chi_R(x) = 1$ for $x \in [0,R]$ and $\chi_R(x) = 0$ for $x > 2R$. Then we consider the following cut-off problem

$$\begin{cases} \mathrm{d}u + \chi_R\big(\|u\|_{W^{1,\infty}}\big)\big[uu_x + F(u)\big]\,\mathrm{d}t = \chi_R\big(\|u\|_{W^{1,\infty}}\big)h(t,u)\,\mathrm{d}\mathcal{W}, \\ u(\omega,0,x) = u_0(\omega,x) \in H^s. \end{cases} \tag{3.1}$$

From Lemma 2.4, we see that the nonlinear term $F(u)$ preserves the H^s-regularity of $u \in H^s$ for any $s > 3/2$. However, to apply the theory of SDEs in Hilbert space to (3.1), we will have to mollify the transport term uu_x since the product uu_x loses one regularity. To this end, we consider the following approximation scheme:

$$\begin{cases} \mathrm{d}u + H_{1,\varepsilon}(u)\,\mathrm{d}t = H_2(t,u)\,\mathrm{d}\mathcal{W}, \\ H_{1,\varepsilon}(u) = \chi_R\big(\|u\|_{W^{1,\infty}}\big)\big[J_\varepsilon(J_\varepsilon u \partial_x J_\varepsilon u) + F(u)\big], \\ H_2(t,u) = \chi_R\big(\|u\|_{W^{1,\infty}}\big)h(t,u), \\ u(0,x) = u_0(x) \in H^s, \end{cases} \tag{3.2}$$

where J_ε is the Friedrichs mollifier defined by (2.1).

Proposition 3.1. *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$ be a fixed stochastic basis. Let $s > 3/2$, $R > 1$ and $\varepsilon \in (0,1)$. Assume h satisfies Assumption (A) and $u_0 \in L^2(\Omega; H^s)$ is an H^s-valued $\mathcal{F}_0$-measurable random variable. Then (3.2) admits a unique solution $u_\varepsilon \in C([0,\infty); H^s)$ $\mathbb{P}$-a.s. Moreover, for any $T > 0$, there is a constant $C > 0$ depending on $a_0, \dots, a_4, R, T$ and u_0 such that*

$$\sup_{\varepsilon>0} \mathbb{E} \sup_{t\in[0,T]} \big\|u_\varepsilon(t)\big\|_{H^s}^2 \leq C \tag{3.3}$$

Proof. *For each fixed ε, it follows from (2.3), (2.8) and (A-2) that $H_{1,\varepsilon}(\cdot)$ and $H_2(t,\cdot)$ are locally Lipschitz continuous in H^s with $s > 3/2$. Moreover, by (A-1), (2.6) and (2.7), there are constants $l_1 = l_1(\varepsilon,R) > 0$ and $l_2 = l_2(R) > 0$ such that for all $t \geq 0$ and $s > 3/2$,*

$$\big\|H_{1,\varepsilon}(u)\big\|_{H^s} \leq l_1\big(1 + \|u\|_{H^s}\big), \ \big\|H_2(t,u)\big\|_{\mathcal{L}_2(\mathfrak{U};H^s)} \leq l_2\big(1 + \|u\|_{H^s}\big), \quad t \in [0,T]. \tag{3.4}$$

Therefore, for $u_0 \in L^2(\Omega; H^s)$ with $s > 3/2$, the existence theory of SDE in Hilbert space (see for example [59, Theorem 4.2.4 with Example 4.1.3] and [48]) implies that (3.2) admits a unique solution $u_\varepsilon \in C([0,\infty), H^s)$ $\mathbb{P}$-a.s. Now we prove (3.3). Using the Itô formula for $\|u_\varepsilon\|_{H^s}^2$, we have

$$\begin{aligned} \mathrm{d}\big\|u_\varepsilon(t)\big\|_{H^s}^2 = {} & 2\chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big(h(t,u_\varepsilon)\,\mathrm{d}\mathcal{W}, u_\varepsilon\big)_{H^s} \\ & - 2\chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big(D^s J_\varepsilon[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon\big)_{L^2}\,\mathrm{d}t \\ & - 2\chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big(D^s F(u_\varepsilon), D^s u_\varepsilon\big)_{L^2}\,\mathrm{d}t \\ & + \chi_R^2\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big\|h(t,u_\varepsilon)\big\|_{\mathcal{L}_2(\mathfrak{U};H^s)}^2\,\mathrm{d}t. \end{aligned}$$

Using the BDG inequality, (A-1) and Lemma 2.4 yields that for some constant $C_1 = C_1(a_0,\dots,a_4,R) > 0$,

$$\begin{aligned} & \mathbb{E}\sup_{t\in[0,T]}\big\|u_\varepsilon(t)\big\|_{H^s}^2 - \mathbb{E}\|u_0\|_{H^s}^2 \\ & \lesssim \mathbb{E}\bigg(\int_0^T \chi_R^2\big(\|u_\varepsilon\|_{W^{1,\infty}}\big) f^2\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big(1 + \|u_\varepsilon\|_{H^s}^2\big)\|u_\varepsilon\|_{H^s}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}} \\ & \quad + 2\mathbb{E}\int_0^T \chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big|\big(D^s J_\varepsilon[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon\big)_{L^2}\big|\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+2\mathbb{E}\int_0^T \chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big|\big(D^s F(u_\varepsilon), D^s u_\varepsilon\big)_{L^2}\big|\,\mathrm{d}t\\
&+\mathbb{E}\int_0^T \chi_R^2\big(\|u_\varepsilon\|_{W^{1,\infty}}\big) f^2\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big(1+\|u_\varepsilon\|_{H^s}^2\big)\,\mathrm{d}t\\
\le\;& \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon\|_{H^s}^2 + C_1\mathbb{E}\int_0^T \big(1+\|u_\varepsilon\|_{H^s}^2\big)\,\mathrm{d}t\\
&+2\mathbb{E}\int_0^T \chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big|\big(D^s J_\varepsilon[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon\big)_{L^2}\big|\,\mathrm{d}t.
\end{aligned}
$$

Let $J_\varepsilon u_\varepsilon = v$. It follows from (2.4), (2.5), Lemma 2.2, integration by parts, $H^s \hookrightarrow W^{1,\infty}$ and (2.6) that

$$
\big|\big(D^s J_\varepsilon[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon\big)_{L^2}\big| \le \big|\big(\big[D^s, v\big]v_x, D^s v\big)_{L^2}\big| + \big|\big(v D^s v_x, D^s v\big)_{L^2}\big| \le C\|u_\varepsilon\|_{W^{1,\infty}}\|u_\varepsilon\|_{H^s}^2,
$$

which implies

$$
2\mathbb{E}\int_0^T \chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big|\big(D^s J_\varepsilon[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon], D^s u_\varepsilon\big)_{L^2}\big|\,\mathrm{d}t \le C(R)\mathbb{E}\int_0^T \|u_\varepsilon\|_{H^s}^2\,\mathrm{d}t.
$$

Therefore we obtain for some constant $C_2 = C_2(a_0, \dots, a_4, R) > 0$ that

$$
\mathbb{E}\sup_{t\in[0,T]}\big\|u_\varepsilon(t)\big\|_{H^s}^2 \le 2\mathbb{E}\|u_0\|_{H^s}^2 + C_2\int_0^T \Big(1+\mathbb{E}\sup_{t'\in[0,t]}\big\|u\big(t'\big)\big\|_{H^s}^2\Big)\,\mathrm{d}t.
$$

Using Grönwall's inequality to the above estimate implies that for some $C = C(a_0, \dots, a_4, R, T, u_0) > 0$,

$$
\mathbb{E}\sup_{t\in[0,T]}\big\|u_\varepsilon(t)\big\|_{H^s}^2 \le C,
$$

which is (3.3). □

3.2. *Convergence of approximation solutions*

The target is to show that when $s > 3$, there is a subsequence of $u_\varepsilon (0 < \varepsilon < 1)$ converging in $C([0, T], H^{s-\frac{3}{2}})$ almost surely. To this end, we consider the difference between two layers u_ε and u_η, where u_ε and u_η are two solutions to (3.2). Let $v_{\varepsilon,\eta} = u_\varepsilon - u_\eta$, then we have

$$
\mathrm{d}v_{\varepsilon,\eta} + \big[H_{1,\varepsilon}(u_\varepsilon) - H_{1,\eta}(u_\eta)\big]\,\mathrm{d}t = \big[H_2(t, u_\varepsilon) - H_2(t, u_\eta)\big]\,\mathrm{d}\mathcal{W}, \quad v_{\varepsilon,\eta} = 0. \tag{3.5}
$$

Direct computation yields that

$$
\begin{aligned}
&H_{1,\varepsilon}(u_\varepsilon) - H_{1,\eta}(u_\eta)\\
&= \chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)\big[J_\varepsilon(J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon)\big] - \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\big[J_\eta(J_\eta u_\eta \partial_x J_\eta u_\eta)\big]\\
&\quad + \chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)F(u_\varepsilon) - \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)F(u_\eta)\\
&= \big[\chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big) - \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\big]J_\varepsilon[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon] + \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)(J_\varepsilon - J_\eta)[J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon]\\
&\quad + \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)J_\eta\big[(J_\varepsilon - J_\eta)u_\varepsilon \partial_x J_\varepsilon u_\varepsilon\big] + \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)J_\eta\big[J_\eta(u_\varepsilon - u_\eta)\partial_x J_\varepsilon u_\varepsilon\big]\\
&\quad + \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)J_\eta\big[J_\eta u_\eta \partial_x (J_\varepsilon - J_\eta)u_\varepsilon\big] + \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)J_\eta\big[J_\eta u_\eta \partial_x J_\eta(u_\varepsilon - u_\eta)\big]\\
&\quad + \big[\chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big) - \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\big]F(u_\varepsilon) + \chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\big[F(u_\varepsilon) - F(u_\eta)\big]\\
&= \sum_{i=1}^{8} R_i.
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
&H_2(t,u_\varepsilon)-H_2(t,u_\eta)\\
&=\chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)h(t,u_\varepsilon)-\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)h(t,u_\eta)\\
&=\big[\chi_R\big(\|u_\varepsilon\|_{W^{1,\infty}}\big)-\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\big]h(t,u_\varepsilon)+\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\big[h(t,u_\varepsilon)-h(t,u_\eta)\big]\\
&=\sum_{i=9}^{10}R_i. \qquad (3.7)
\end{aligned}
$$

Then we use the Itô formula to (3.5) with noticing (3.6) and (3.7) to find that for any $t>0$,

$$\big\|v_{\varepsilon,\eta}(t)\big\|^2_{H^{s-\frac32}}=N_1-\int_0^t N_2\,\mathrm{d}t'+\int_0^t N_3\,\mathrm{d}t', \qquad (3.8)$$

where

$$N_1=2\int_0^t\Bigg(\sum_{i=9}^{10}R_i\,\mathrm{d}\mathcal{W},v_{\varepsilon,\eta}\Bigg)_{H^{s-\frac32}},\qquad N_2=2\sum_{i=1}^{8}(R_i,v_{\varepsilon,\eta})_{H^{s-\frac32}},\qquad N_3=\Bigg\|\sum_{i=9}^{10}R_i\Bigg\|^2_{\mathcal{L}_2(\mathfrak{U};H^{s-\frac32})}. \qquad (3.9)$$

Lemma 3.1. *Let $s>3$. For any $\varepsilon,\eta\in(0,1)$, there is a constant $C>0$ and a locally bounded non-decreasing function $\Phi(\cdot):[0,\infty)\to[0,\infty)$ such that N_2 given by (3.9) satisfies*

$$|N_2|\le C\Phi\big(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\big)\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac32}}+C\Phi\big(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\big)\max\{\varepsilon,\eta\}.$$

Proof. Fix a constant $D=D(a_0,\dots,a_4)\gg1$ and let $\Phi(\cdot)$ be a locally bounded function such that

$$\Phi(x)\ge D\big(1+x^4\big).$$

Using the mean value theorem for $\chi_R(\cdot)$, the embedding $H^{s-\frac32}\hookrightarrow W^{1,\infty}$, Lemma 2.4 and (2.6), and noticing that $0\le\chi_R(\cdot)\le1$, we have

$$
\begin{aligned}
\|R_1\|_{H^{s-\frac32}}&\lesssim\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\|u_\varepsilon\|^2_{H^s},\\
\|R_7\|_{H^{s-\frac32}}&\lesssim\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\big(|a_0|\|u_\varepsilon\|_{H^s}+\big(|a_1|+|a_2|\big)\|u_\varepsilon\|^2_{H^s}+|a_3|\|u_\varepsilon\|^3_{H^s}+|a_4|\|u_\varepsilon\|^4_{H^s}\big)\\
&\lesssim\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\Phi\big(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\|R_8\|_{H^{s-\frac32}}&\lesssim\big(|a_0|+\big(|a_1|+|a_2|\big)I_s(u_\varepsilon,u_\eta)+|a_3|I_s^2(u_\varepsilon,u_\eta)+|a_4|I_s^3(u_\varepsilon,u_\eta)\big)\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\\
&\lesssim\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\Phi\big(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\big).
\end{aligned}
$$

Using (2.2) and (2.6) yields

$$
\begin{aligned}
\|R_i\|_{H^{s-\frac32}}&\lesssim\max\big\{\varepsilon^{1/2},\eta^{1/2}\big\}\|u_\varepsilon\|^2_{H^s},\quad i=2,3,\\
\|R_4\|_{H^{s-\frac32}}&\lesssim\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\|u_\varepsilon\|_{H^s},\\
\|R_5\|_{H^{s-\frac32}}&\lesssim\max\big\{\varepsilon^{1/2},\eta^{1/2}\big\}\|u_\varepsilon\|_{H^s}\|u_\eta\|_{H^s}.
\end{aligned}
$$

For R_6, using (2.4), (2.5) and then integrating by parts, we have

$$
\begin{aligned}
(R_6,v_{\varepsilon,\eta})_{H^{s-\frac32}}&=\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\int_{\mathbb{R}}D^{s-\frac32}[J_\eta u_\eta\partial_xJ_\eta v_{\varepsilon,\eta}]\cdot D^{s-\frac32}J_\eta v_{\varepsilon,\eta}\,\mathrm{d}x\\
&=\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\int_{\mathbb{R}}\big[D^{s-\frac32},J_\eta u_\eta\big]\partial_xJ_\eta v_{\varepsilon,\eta}\cdot D^{s-\frac32}J_\eta v_{\varepsilon,\eta}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\int_{\mathbb{R}} J_\eta u_\eta \partial_x D^{s-\frac32} J_\eta v_{\varepsilon,\eta}\cdot D^{s-\frac32} J_\eta v_{\varepsilon,\eta}\,\mathrm{d}x\\
=&\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\int_{\mathbb{R}}\big[D^{s-\frac32}, J_\eta u_\eta\big]\partial_x J_\eta v_{\varepsilon,\eta}\cdot D^{s-\frac32} J_\eta v_{\varepsilon,\eta}\,\mathrm{d}x\\
&-\frac12\chi_R\big(\|u_\eta\|_{W^{1,\infty}}\big)\int_{\mathbb{R}} J_\eta\partial_x u_\eta\big(D^{s-\frac32} J_\eta v_{\varepsilon,\eta}\big)^2\,\mathrm{d}x.
\end{aligned}
$$

Then Lemma 2.2, (2.6) and the embedding $H^{s-\frac32}\hookrightarrow W^{1,\infty}$ bring us

$$
\begin{aligned}
(R_6, v_{\varepsilon,\eta})_{H^{s-\frac32}} &\lesssim \big(\|u_\eta\|_{H^{s-\frac32}}\|\partial_x J_\eta v_{\varepsilon,\eta}\|_{L^\infty}+\|\partial_x J_\eta u_\eta\|_{L^\infty}\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\big)\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}\\
&\lesssim \|u_\eta\|_{H^s}\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac32}}.
\end{aligned}
$$

Putting all these together, we find a constant $C>0$ such that

$$
|N_2|\le C\Phi\big(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\big)\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac32}}+C\Phi\big(\|u_\varepsilon\|_{H^s}+\|u_\eta\|_{H^s}\big)\max\{\varepsilon,\eta\},
$$

which is the desired estimate. □

Lemma 3.2. *Let $\mathcal{S}=(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\ge0},\mathcal{W})$ be a fixed stochastic basis. Let $s>3$, $R>1$ and $\varepsilon\in(0,1)$. Let $u_\varepsilon\in C([0,\infty);H^s)$ solve (3.2) $\mathbb{P}$-a.s. For any $T>0$ and $K>1$, we define*

$$
\tau^T_{\varepsilon,K}=\inf\big\{t\ge0:\|u_\varepsilon(t)\|_{H^s}\ge K\big\}\wedge T, \tag{3.10}
$$

and

$$
\tau^T_{\varepsilon,\eta,K}=\tau^T_{\varepsilon,K}\wedge\tau^T_{\eta,K}. \tag{3.11}
$$

Then it has that

$$
\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac32}}=0,\quad K>1. \tag{3.12}
$$

Proof. Recalling (3.8) and (3.9), we have

$$
\big\|v_{\varepsilon,\eta}(t)\big\|^2_{H^{s-\frac32}}\le|N_1|+\int_0^t|N_2|\,\mathrm{d}t'+\int_0^t|N_3|\,\mathrm{d}t'. \tag{3.13}
$$

By (3.11), the mean value theorem for $\chi_R(\cdot)$, (A-1) and (A-2), we see that

$$
\|R_9\|_{\mathcal{L}_2(\mathfrak{U};H^{s-\frac32})}\le\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}f(K)(1+K),\quad t\in\big[0,\tau^T_{\varepsilon,\eta,K}\big]\ \mathbb{P}\text{-a.s.},
$$

and

$$
\|R_{10}\|_{\mathcal{L}_2(\mathfrak{U};H^{s-\frac32})}\le\|v_{\varepsilon,\eta}\|_{H^{s-\frac32}}q(K),\quad t\in\big[0,\tau^T_{\varepsilon,\eta,K}\big]\ \mathbb{P}\text{-a.s.},
$$

where R_9 and R_{10} are given in (3.7), $f(\cdot)$ and $q(\cdot)$ are given in Assumption (A). To sum up, there exists a constant $C=C(K)>0$ such that

$$
\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|N_3|\,\mathrm{d}t\le C(K)\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac32}}\,\mathrm{d}t\le C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\big\|v_{\varepsilon,\eta}(t')\big\|^2_{H^{s-\frac32}}\,\mathrm{d}t. \tag{3.14}
$$

Then we employ the BDG inequality to (3.8) to find

$$
\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\big\|v_{\varepsilon,\eta}(t)\big\|^2_{H^{s-\frac32}}
$$

$$\le C(K)\mathbb{E}\bigg(\int_0^{\tau^T_{\varepsilon,\eta,K}}\|v_{\varepsilon,\eta}\|^4_{H^{s-\frac{3}{2}}}\,\mathrm{d}t\bigg)^{\frac{1}{2}}+\sum_{i=2}^{3}\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|N_i|\,\mathrm{d}t$$

$$\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac{3}{2}}}+C(K)\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac{3}{2}}}\,\mathrm{d}t+\sum_{i=2}^{3}\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|N_i|\,\mathrm{d}t$$

$$\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac{3}{2}}}+C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\big\|v_{\varepsilon,\eta}(t')\big\|^2_{H^{s-\frac{3}{2}}}\,\mathrm{d}t+\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|N_2|\,\mathrm{d}t,$$

On account of Lemma 3.1, we arrive at

$$\begin{aligned}\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}|N_2|\,\mathrm{d}t&\le C(K)\mathbb{E}\int_0^{\tau^T_{\varepsilon,\eta,K}}\|v_{\varepsilon,\eta}\|^2_{H^{s-\frac{3}{2}}}\,\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\}\\&\le C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\big\|v_{\varepsilon,\eta}(t')\big\|^2_{H^{s-\frac{3}{2}}}\,\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\}.\end{aligned}$$

Hence we arrive at

$$\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\big\|v_{\varepsilon,\eta}(t)\big\|^2_{H^{s-\frac{3}{2}}}\le C(K)\int_0^T\mathbb{E}\sup_{t'\in[0,\tau^t_{\varepsilon,\eta,K}]}\big\|v_{\varepsilon,\eta}(t')\big\|^2_{H^{s-\frac{3}{2}}}\,\mathrm{d}t+C(K)T\max\{\varepsilon,\eta\},$$

which means that

$$\mathbb{E}\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\big\|v_{\varepsilon,\eta}(t)\big\|^2_{H^{s-\frac{3}{2}}}\le C(K,T)\max\{\varepsilon,\eta\},\tag{3.15}$$

and hence (3.12) holds true. □

Lemma 3.3. *For any fixed $s>3$ and $T>0$, there is a countable subsequence of $\{u_\varepsilon\}$ (still denoted as $\{u_\varepsilon\}$) such that*

$$u_\varepsilon\xrightarrow{\varepsilon\to 0}u\quad\text{in } C\big([0,T];H^{s-\frac{3}{2}}\big)\ \mathbb{P}\text{-a.s.},\tag{3.16}$$

where u is an $\{\mathcal{F}_t\}_{t\ge0}$ progressive measurable H^s-valued process satisfying

$$u\in L^2\big(\Omega;L^\infty\big(0,T;H^s\big)\big).\tag{3.17}$$

Proof. We notice that for each $\varepsilon\in(0,1)$, the approximation problem (3.2) has a solution u_ε almost surely. Now we take $\{\varepsilon\}_{\varepsilon\in(0,1)}$ to be a countable set $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that for all n, u_{ε_n} can be defined on the same set $\widetilde{\Omega}$ with $\mathbb{P}\{\widetilde{\Omega}\}=1$ (otherwise (3.16) may fail). For simplicity, we still use the notation $\{u_\varepsilon\}$. Recall (3.10) and (3.11). For any $\epsilon>0$, we can infer from Proposition 3.1 and Chebyshev's inequality that

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in[0,T]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\Big\}\\&=\mathbb{P}\Big\{\big(\big\{\tau^T_{\varepsilon,\eta,K}<T\big\}\cup\big\{\tau^T_{\varepsilon,\eta,K}=T\big\}\big)\cap\Big\{\sup_{t\in[0,T]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\Big\}\Big\}\\&\le\mathbb{P}\big\{\tau^T_{\varepsilon,K}<T\big\}+\mathbb{P}\big\{\tau^T_{\eta,K}<T\big\}+\mathbb{P}\Big\{\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\Big\}\\&\le\frac{2C(a_0,\dots,a_4,R,T,u_0)}{K^2}+\mathbb{P}\Big\{\sup_{t\in[0,\tau^T_{\varepsilon,\eta,K}]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\Big\}.\end{aligned}$$

It follows from (3.12) that

$$\lim_{\varepsilon\to0}\sup_{\eta\le\varepsilon}\mathbb{P}\Big\{\sup_{t\in[0,T]}\|u_\varepsilon-u_\eta\|_{H^{s-\frac{3}{2}}}>\epsilon\Big\}\le\frac{2C(a_0,\dots,a_4,R,T,u_0)}{K^2},\quad K>1.$$

Letting $K \to \infty$, we see that u_ε converges in probability in $C([0,T]; H^{s-\frac{3}{2}})$. Therefore, there is a subsequence of $\{u_\varepsilon\}$ satisfying (3.16).

It remains to prove (3.17). Since $H^s \hookrightarrow H^{s-3/2}$ is continuous, there exist continuous maps $\phi_m : H^{s-3/2} \to H^s (m \geq 1)$ such that

$$\|\phi_m u\|_{H^s} \leq \|u\|_{H^s} \quad \text{and} \quad \lim_{m\to\infty} \|\phi_m u\|_{H^s} = \|u\|_{H^s}, \quad u \in H^{s-3/2},$$

where $\|u\|_{H^s} \triangleq \infty$ if $u \notin H^s$. Then it follows from Proposition 3.1 and Fatou's lemma that

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|^2_{H^s} &\leq \liminf_{m\to\infty} \mathbb{E} \sup_{t\in[0,T]} \|\phi_m u(t)\|^2_{H^s} \\ &\leq \liminf_{m\to\infty} \liminf_{\varepsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \|\phi_m u_\varepsilon(t)\|^2_{H^s} \\ &\leq \liminf_{m\to\infty} \liminf_{\varepsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|^2_{H^s} < C(a_0, \dots, a_4, R, u_0, T).\end{aligned}$$

Hence we obtain (3.17). Since for each $\varepsilon \in (0,1)$, u_ε is $\{\mathcal{F}_t\}_{t\geq 0}$ progressive measurable, so is u. □

3.3. *Global pathwise solution to the cut-off problem*

Proposition 3.2. *Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{W})$ be a stochastic basis fixed in advance. Let $s > 3$ and $R > 1$. Suppose that Assumption (A) is satisfied. Let $u_0 \in L^2(\Omega; H^s)$ be an H^s-valued $\mathcal{F}_0$ measurable random variable. Then for any $T > 0$, (3.1) has a solution $u \in L^2(\Omega; C([0,T]; H^s))$. That is to say, u solves*

$$\begin{cases} \mathrm{d}u + \chi_R(\|u\|_{W^{1,\infty}})[uu_x + F(u)]\,\mathrm{d}t = \chi_R(\|u\|_{W^{1,\infty}}) h(t,u)\,\mathrm{d}\mathcal{W}, \\ u(\omega, 0, x) = u_0(\omega, x) \in H^s, \end{cases}$$

and there is a constant $C = C(a_0, \dots, a_4, R, T, u_0) > 0$ such that

$$\mathbb{E} \sup_{t\in[0,T]} \|u\|^2_{H^s} \leq C. \tag{3.18}$$

Proof. By Lemma 3.3 and the embedding $H^{s-3/2} \hookrightarrow W^{1,\infty}$, we can send $\varepsilon \to 0$ in (3.2) to conclude that u solves (3.1) and estimate (3.18) holds true.

To finish the proof for this proposition, it remains to prove that $u \in C([0,T]; H^s)$ almost surely. Due to Lemma 3.3, $u \in C([0,T]; H^{s-3/2}) \cap L^\infty(0,T; H^s)$ almost surely. Since H^s is dense in $H^{s-3/2}$, we know that (cf. [72, page 263, Lemma 1.4]) $u \in C_w([0,T]; H^s)$, where $C_w([0,T]; H^s)$ is the space of weakly continuous functions with values in H^s. Therefore we only need to prove the continuity of $[0,T] \ni t \mapsto \|u(t)\|_{H^s}$.

In Remark 1.1, we have noticed that the Itô formula may fail in our problem. In order to use the Itô formula in a Hilbert space, we recall the mollifier J_ε defined in (2.1) and then we arrive at

$$\begin{aligned}\mathrm{d}\|J_\varepsilon u(t)\|^2_{H^s} = {} & 2\chi_R(\|u\|_{W^{1,\infty}})(J_\varepsilon h(t,u)\,\mathrm{d}\mathcal{W}, J_\varepsilon u)_{H^s} \\ & - 2\chi_R(\|u\|_{W^{1,\infty}})(J_\varepsilon[uu_x + F(u)], J_\varepsilon u)_{H^s}\,\mathrm{d}t \\ & + \chi_R^2(\|u\|_{W^{1,\infty}})\|J_\varepsilon h(t,u)\|^2_{\mathcal{L}_2(\mathfrak{U}; H^s)}\,\mathrm{d}t.\end{aligned} \tag{3.19}$$

On account of (3.17), we have

$$\tau_N = \inf\{t \geq 0 : \|u(t)\|_{H^s} > N\} \to \infty \quad \text{as } N \to \infty \ \ \mathbb{P}\text{-a.s.} \tag{3.20}$$

Consequently, it is enough to prove the continuity up to time $\tau_N \wedge T$ for each $N \geq 1$. We notice that J_ε satisfies (2.4), (2.5) and (2.6). Therefore for any $[t_2, t_1] \subset [0,T]$ with $t_1 - t_2 < 1$, we use Lemma 2.5, the BDG inequality and Assumption (A) and (3.20) to find

$$\mathbb{E}\big[\big(\|J_\varepsilon u(t_1 \wedge \tau_N)\|^2_{H^s} - \|J_\varepsilon u(t_2 \wedge \tau_N)\|^2_{H^s}\big)^4\big] \leq C(a_0, \dots, a_4, N, T)|t_1 - t_2|^2.$$

Moreover, since

$$J_\varepsilon u \to u \text{ as } \varepsilon \to 0 \text{ in } C\big([0,T], H^s\big) \quad \mathbb{P}\text{-a.s.,} \tag{3.21}$$

we use Fatou's lemma to find

$$\mathbb{E}\big[\big(\big\|u(t_1\wedge\tau_N)\big\|_{H^s}^2 - \big\|u(t_2\wedge\tau_N)\big\|_{H^s}^2\big)^4\big] \le C(a_0,\ldots,a_4,N,T)|t_1-t_2|^2.$$

Therefore the continuity of $t \mapsto \|u(t\wedge\tau_N)\|_{H^s}$ comes from the above inequality and Kolmogorov's continuity theorem. We complete the proof. □

3.4. *Concluding the proof for Theorem 1.1*

Finally, we are in the position to finish the proof for Theorem 1.1. For clarity, we split the proof into three steps.

Step 1: Existence. For $u_0(\omega,x) \in L^2(\Omega; H^s)$ with $s > 3$, we let

$$\Omega_k = \big\{k-1 \le \|u_0\|_{H^s} < k\big\}, \quad k \in \mathbb{N}, k \ge 1.$$

Since $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we have

$$u_0(\omega,x) = \sum_{k\ge1} \mathbf{1}_{\Omega_k} u_0(\omega,x) = \sum_{k\ge1} u_{0,k}(\omega,x) \quad \mathbb{P}\text{-a.s.}$$

On account of Proposition 3.2, we let $u_{k,R}$ be the pathwise global solution to the cut-off problem (3.1) with initial value $u_{0,k}$ and cut-off function $\chi_R(\cdot)$. Define

$$\tau_{k,R} = \inf\Big\{t > 0 : \sup_{t'\in[0,t]} \big\|u_{k,R}\big(t'\big)\big\|_{H^s}^2 > \|u_{0,k}\|_{H^s}^2 + 2\Big\}. \tag{3.22}$$

Then for any $R > 0$ and $k \ge 1$, we have $\mathbb{P}\{\tau_{k,R} > 0\} = 1$. The difficulty here is that we have to take R to be deterministic. Otherwise Proposition 3.1 will fail. To overcome this difficulty, we let $R = R_k$ be discrete (with $k \ge 1$) and then denote $(u_k, \tau_k) = (u_{k,R_k}, \tau_{k,R_k})$. It is clear that $\mathbb{P}\{\tau_k > 0\ \forall k \ge 1\} = 1$. Let $E > 0$ be the embedding constant such that $\|\cdot\|_{W^{1,\infty}} \le E\|\cdot\|_{H^s}$ for $s > 3$. Particularly, we let $R_k^2 > E^2\|u_{0,k}\|_{H^s}^2 + 2E^2$, and then we have

$$\mathbb{P}\big\{\|u_k\|_{W^{1,\infty}}^2 \le E^2\|u_k\|_{H^s}^2 \le E^2\|u_{0,k}\|_{H^s}^2 + 2E^2 < R_k^2\ \forall t \in [0,\tau_k]\ \forall k \ge 1\big\} = 1,$$

which means

$$\mathbb{P}\big\{\chi_{R_k}\big(\|u_k\|_{W^{1,\infty}}\big) = 1\ \forall t \in [0,\tau_k]\ \forall k \ge 1\big\} = 1.$$

Therefore (u_k, τ_k) is the pathwise solution to (1.4) with initial value $u_{0,k}$. Notice that

$$\mathbf{1}_{\Omega_k} u_k(t\wedge\tau_k) - \mathbf{1}_{\Omega_k} u_{0,k} = -\int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \mathbf{1}_{\Omega_k}\big[u_k\partial_x u_k + F(u_k)\big]\,\mathrm{d}t' + \int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \mathbf{1}_{\Omega_k} h\big(t', u_k\big)\,\mathrm{d}\mathcal{W}.$$

Besides, it has that

$$\mathbf{1}_{\Omega_k} h(t,u_k) = h(t, \mathbf{1}_{\Omega_k} u_k) - \mathbf{1}_{\Omega_k^C} h(t, \mathbf{0})$$

and

$$\mathbf{1}_{\Omega_k}\big[u_k\partial_x u_k + F(u_k)\big] = \big[\mathbf{1}_{\Omega_k} u_k \partial_x \mathbf{1}_{\Omega_k} u_k + F(\mathbf{1}_{\Omega_k} u_k)\big].$$

By Assumption (A), $\|h(t,\mathbf{0})\|_{\mathcal{L}_2(\mathfrak{U}; H^s)} < \infty$. This in turn brings us

$$\begin{aligned}
&\mathbf{1}_{\Omega_k} u_k(t\wedge\tau_k) - \mathbf{1}_{\Omega_k} u_{0,k}\\
&\quad= \mathbf{1}_{\Omega_k} u_k(t\wedge\mathbf{1}_{\Omega_k}\tau_k) - u_{0,k}\\
&\quad= -\int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} \big[(\mathbf{1}_{\Omega_k} u_k)\partial_x(\mathbf{1}_{\Omega_k} u_k) + F(\mathbf{1}_{\Omega_k} u_k)\big]\,\mathrm{d}t' + \int_0^{t\wedge\mathbf{1}_{\Omega_k}\tau_k} h\big(t', \mathbf{1}_{\Omega_k} u_k\big)\,\mathrm{d}\mathcal{W},
\end{aligned}$$

which implies that $(\mathbf{1}_{\Omega_k} u_k, \mathbf{1}_{\Omega_k}\tau_k)$ is a solution to $(1.4)_1$ with initial data $u_{0,k}$.

Since $\Omega_k \cap \Omega_{k'} = \varnothing$ for $k \neq k'$ and $\bigcup_{k\geq 1} \Omega_k$ is a set of full measure, we see that

$$\left(u = \sum_{k\geq 1} \mathbf{1}_{\Omega_k} u_k,\, \tau = \sum_{k\geq 1} \mathbf{1}_{\Omega_k} \tau_k\right)$$

is a pathwise solution to (1.4) corresponding to the initial condition u_0. Besides, by virtue of (3.22), we have

$$\begin{aligned}\sup_{t\in[0,\tau]} \|u\|_{H^s}^2 &= \sum_{k\geq 1} \mathbf{1}_{\Omega_k} \sup_{t\in[0,\tau_k]} \|u_k\|_{H^s}^2 \\ &\leq \sum_{k\geq 1} \mathbf{1}_{\Omega_k} \left(\|u_{0,k}\|_{H^s}^2 + 2\right) \leq 2\|u_0\|_{H^s}^2 + 4.\end{aligned}$$

Taking expectation gives rise to (1.11). Finally, mollifying initial data, analyzing the convergence and employing the argument as in [37,64,65] lead to a local pathwise solution (u,τ) to (1.4) with $u(\cdot\wedge\tau) \in L^2(\Omega; C([0,\infty); H^s))$ for $u_0 \in L^2(\Omega; H^s)$ with $s > 3/2$.

Step 2: Uniqueness. Let (u_1,τ_1) and (u_2,τ_2) be two solutions to (1.4) such that $u_j(0) = u_0$ almost surely and $u_j(\cdot\wedge\tau_j) \in L^2(\Omega; C([0,\infty); H^s))$ with $s > \frac{3}{2}$ for $j = 1,2$. Let $\frac{1}{2} < \delta < \min\{s-1, \frac{3}{2}\}$ and define

$$\tau_K^T = \inf\left\{t \geq 0 : \left\|u_1(t)\right\|_{H^s} + \left\|u_2(t)\right\|_{H^s} \geq K\right\} \wedge T, \quad K \geq 1, T > 0.$$

Using (A-3) instead of (A-2) and using (2.9), then the estimate of $\mathbb{E}\sup_{t\in[0,\tau_K^T]} \|u_1(t) - u_2(t)\|_{H^\delta}^2$ is essential as in the derivation of (3.12) and we have

$$\mathbb{E} \sup_{t\in[0,\tau_K^T]} \left\|u_1(t) - u_2(t)\right\|_{H^\delta}^2 = 0.$$

Since $u_j(\cdot\wedge\tau_j) \in L^2(\Omega; C([0,\infty); H^s))$ for $j = 1,2$ almost surely, we have

$$\mathbb{P}\left\{\liminf_{K,T\to\infty} \tau_K^T \geq \tau_1 \wedge \tau_2\right\} = 1.$$

Hence, by sending $K, T \to \infty$ and using the monotone convergence theorem, we obtain

$$\mathbb{E} \sup_{t\in[0,\tau_1\wedge\tau_2]} \left\|u_1(t) - u_2(t)\right\|_{H^\delta}^2 = 0,$$

which implies the uniqueness of the solution.

Step 3: Blow-up criterion of the maximal solution. With a local pathwise solution (u,τ) in hand, the extending of u to a maximal pathwise solution (u,τ^*) in the sense of Definition 1.1 may be carried out as in [20,36,37,61]. Here we only prove the blow-up criterion (1.12). To this end, we first define

$$\tau_{1,m} = \inf\left\{t \geq 0 : \left\|u(t)\right\|_{H^s} \geq m\right\}, \quad \tau_{2,n} = \inf\left\{t \geq 0 : \left\|u(t)\right\|_{W^{1,\infty}} \geq n\right\},$$

and then let $\tau_1 = \lim_{m\to\infty} \tau_{1,m}$ and $\tau_2 = \lim_{n\to\infty} \tau_{2,n}$. By the continuity of $\|u(t)\|_{H^s}$ and the uniqueness of u, it is easy to check that τ_1 is actually the maximal existence time τ^* of u in the sense of Definition 1.1. Therefore to prove (1.12), we only need to verify that $\tau_1 = \tau_2$ $\mathbb{P}$-a.s. The approach here is motivated by [20,62,65].

Due to the embedding $H^s \hookrightarrow W^{1,\infty}$ for $s > 3/2$, there is a constant $M > 0$ such that,

$$\sup_{t\in[0,\tau_{1,m}]} \left\|u(t)\right\|_{W^{1,\infty}} \leq M \sup_{t\in[0,\tau_{1,m}]} \left\|u(t)\right\|_{H^s} \leq \left([M]+1\right)m,$$

where $[M]$ means the integer part of M. Therefore we have $\tau_{1,m} \leq \tau_{2,([M]+1)m} \leq \tau_2$ $\mathbb{P}$-a.s., which means that $\tau_1 \leq \tau_2$ $\mathbb{P}$-a.s.

Now we prove $\tau_2 \leq \tau_1$ $\mathbb{P}$-a.s. To this end, we first prove the following

Claim:

$$\mathbb{P}\left\{\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \left\|u(t)\right\|_{H^s} < \infty\right\} = 1 \quad \forall n_1, n_2 \in \mathbb{N}. \tag{3.23}$$

As is explained in Remark 1.1, we cannot directly apply the Itô formula for $\|u\|_{H^s}^2$ to control $\mathbb{E}\|u(t)\|_{H^s}^2$. Similar to (3.19), by applying J_ε to (1.4) and using the Itô formula for $\|J_\varepsilon u\|_{H^s}^2$, we have that for any $t > 0$,

$$\begin{aligned}\|J_\varepsilon u(t)\|_{H^s}^2 - \|J_\varepsilon u(0)\|_{H^s}^2 &= 2\int_0^t \big(J_\varepsilon h(t', u)\,\mathrm{d}\mathcal{W}, J_\varepsilon u\big)_{H^s} \\ &\quad - 2\int_0^t \big(D^s J_\varepsilon[uu_x], D^s J_\varepsilon u\big)_{L^2}\,\mathrm{d}t' \\ &\quad - 2\int_0^t \big(D^s J_\varepsilon F(u), D^s J_\varepsilon u\big)_{L^2}\,\mathrm{d}t' \\ &\quad + \int_0^t \|J_\varepsilon h(t', u)\|_{\mathcal{L}_2(\mathfrak{U}; H^s)}^2\,\mathrm{d}t' \\ &= Q_1 + \sum_{i=2}^4 \int_0^t Q_i\,\mathrm{d}t'. \end{aligned} \tag{3.24}$$

Therefore, for any $n_1, n_2 \ge 1$ and $t \in [0, \tau_{2,n_1} \wedge n_2]$, it follows from the BDG inequality that

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \|J_\varepsilon u(t)\|_{H^s}^2 \\ &\quad \le \mathbb{E}\|J_\varepsilon u_0\|_{H^s}^2 + C\mathbb{E}\bigg(\int_0^{\tau_{2,n_1}\wedge n_2} \|J_\varepsilon h(t, u)\|_{\mathcal{L}_2(\mathfrak{U}; H^s)}^2 \|J_\varepsilon u\|_{H^s}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}} + \sum_{i=2}^4 \mathbb{E}\int_0^{\tau_{2,n_1}\wedge n_2} |Q_i|\,\mathrm{d}t.\end{aligned}$$

Then (A-1) and (2.6) lead to

$$\begin{aligned}&C\mathbb{E}\bigg(\int_0^{\tau_{2,n_1}\wedge n_2} \|J_\varepsilon h(t, u)\|_{\mathcal{L}_2(\mathfrak{U}; H^s)}^2 \|J_\varepsilon u\|_{H^s}^2\,\mathrm{d}t\bigg)^{\frac{1}{2}} \\ &\quad \le \frac{1}{2}\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \|J_\varepsilon u\|_{H^s}^2 + Cf^2(n_1)\int_0^{n_2} \big(1 + \mathbb{E}\|u\|_{H^s}^2\big)\,\mathrm{d}t.\end{aligned}$$

For Q_2 and Q_3, we use Lemma 2.5 to find

$$\mathbb{E}\int_0^{\tau_{2,n_1}\wedge n_2} |Q_2| + |Q_3|\,\mathrm{d}t \le Cg(n_1)\int_0^{n_2} \big(1 + \mathbb{E}\|u\|_{H^s}^2\big)\,\mathrm{d}t,$$

where $g(\cdot)$ is given in Lemma 2.5. It follows from (A-1) that

$$\mathbb{E}\int_0^{\tau_{2,n_1}\wedge n_2} |Q_4|\,\mathrm{d}t \le Cf^2(n_1)\int_0^{n_2} \big(1 + \mathbb{E}\|u\|_{H^s}^2\big)\,\mathrm{d}t,$$

Therefore we combine the above estimates with using (2.6), and then send $\varepsilon \to 0$ in the resulting inequality to obtain

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \|u(t)\|_{H^s}^2 \le C\mathbb{E}\|u_0\|_{H^s}^2 + C\int_0^{n_2} \Big(1 + \mathbb{E}\sup_{t'\in[0,t\wedge\tau_{2,n_1}]} \|u(t')\|_{H^s}^2\Big)\,\mathrm{d}t, \tag{3.25}$$

where $C = C(a_0, \dots, a_4, n_1) > 0$. Then Grönwall's inequality shows that for each $n_1, n_2 \in \mathbb{N}$, there is a constant $C = C(a_0, \dots, a_4, n_1, n_2, u_0) > 0$ such that

$$\mathbb{E}\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \|u(t)\|_{H^s}^2 < C,$$

which leads to (3.23).

In view of (3.23), we find for all $n_1, n_2 \in \mathbb{N}$ that,

$$1 = \mathbb{P}\Big\{\sup_{t\in[0,\tau_{2,n_1}\wedge n_2]} \|u(t)\|_{H^s} < \infty\Big\} \le \mathbb{P}\Big\{\bigcup_{m\in\mathbb{N}} \{\tau_{2,n_1} \wedge n_2 \le \tau_{1,m}\}\Big\} \le \mathbb{P}\{\tau_{2,n_1} \wedge n_2 \le \tau_1\}.$$

Consequently, $\mathbb{P}\{\bigcap_{n_1,n_2\in\mathbb{N}}\{\tau_{2,n_1}\wedge n_2\le\tau_1\}\}=1$ and

$$\mathbb{P}\{\tau_2\le\tau_1\}=\mathbb{P}\left\{\bigcap_{n_1\in\mathbb{N}}\{\tau_{2,n_1}\le\tau_1\}\right\}=\mathbb{P}\left\{\bigcap_{n_1,n_2\in\mathbb{N}}\{\tau_{2,n_1}\wedge n_2\le\tau_1\}\right\}=1, \tag{3.26}$$

which means $\tau_1=\tau_2$ $\mathbb{P}$-a.s. We complete the proof for Theorem 1.1.

4. Noise effect on preventing blow-up

Following [60] (see also [7,63]), we are now ready to prove Theorem 1.2. Here we notice that (B-2) in Assumption (B) means that one can find a Lyapunov type function V such that the growth of $uu_x+F(u)$ can be canceled by the noise.

4.1. *Proof for Theorem* 1.2

Proof for Theorem 1.2. As is mentioned in Remark 1.2, by the local Lipschitz continuity of $\sigma(t,\cdot)$ in H^s with $s>3/2$, one can follow the steps as in the proof for Theorem 1.1 to obtain that, if u_0 is an H^s-valued $\mathcal{F}_0$-measurable random variable satisfying $\mathbb{E}\|u_0\|^2_{H^s}<\infty$ with $s>5/2$, then (1.6) has a unique pathwise solution $u\in H^s$ with maximal existence time τ^*. In other words, on $[0,\tau^*)$, u solves the following problem almost surely:

$$\begin{cases}\mathrm{d}u+\big[uu_x+F(u)\big]\,\mathrm{d}t=\sigma(t,u)\,\mathrm{d}W,\\ u(\omega,0,x)=u_0(\omega,x).\end{cases}$$

Now the target is to prove $\mathbb{P}\{\tau^*=\infty\}=1$. To this end, we define

$$\tau_m=\inf\big\{t\ge 0:\big\|u(t)\big\|_{H^s}\ge m\big\}.$$

As is mentioned in Remark 1.1, we have to mollify the equation first such that Itô formula can be employed. Hence we apply the Itô formula to $\|J_\varepsilon u(t)\|^2_{H^s}$ to derive

$$\begin{aligned}\mathrm{d}\|J_\varepsilon u\|^2_{H^s}&=2\big(J_\varepsilon\sigma(t,u),J_\varepsilon u\big)_{H^s}\,\mathrm{d}W-2\big(J_\varepsilon[uu_x],J_\varepsilon u\big)_{H^s}\,\mathrm{d}t\\&\quad-2\big(J_\varepsilon F(u),J_\varepsilon u\big)_{H^s}\,\mathrm{d}t+\big\|J_\varepsilon\sigma(t,u)\big\|^2_{H^s}\,\mathrm{d}t.\end{aligned}\tag{4.1}$$

Let $V\in\mathcal{V}$. Applying the Itô formula again, we find

$$\begin{aligned}\mathrm{d}V\big(\|J_\varepsilon u\|^2_{H^s}\big)&=2V'\big(\|J_\varepsilon u\|^2_{H^s}\big)\big(J_\varepsilon\sigma(t,u),J_\varepsilon u\big)_{H^s}\,\mathrm{d}W\\&\quad+V'\big(\|J_\varepsilon u\|^2_{H^s}\big)\big\{-2\big(J_\varepsilon[uu_x],J_\varepsilon u\big)_{H^s}-2\big(J_\varepsilon F(u),J_\varepsilon u\big)_{H^s}\big\}\,\mathrm{d}t\\&\quad+V'\big(\|J_\varepsilon u\|^2_{H^s}\big)\big\|J_\varepsilon\sigma(t,u)\big\|^2_{H^s}\,\mathrm{d}t+2V''\big(\|J_\varepsilon u\|^2_{H^s}\big)\big|\big(J_\varepsilon\sigma(t,u),J_\varepsilon u\big)_{H^s}\big|^2\,\mathrm{d}t.\end{aligned}$$

Now we take expectation and use (2.6), Lemma 2.5 to find that for any $t>0$,

$$\begin{aligned}&\mathbb{E}V\big(\big\|J_\varepsilon u(t\wedge\tau_m)\big\|^2_{H^s}\big)\\&=\mathbb{E}V\big(\|J_\varepsilon u_0\|^2_{H^s}\big)+\mathbb{E}\int_0^{t\wedge\tau_m}V'\big(\|J_\varepsilon u\|^2_{H^s}\big)\big\{-2\big(J_\varepsilon[uu_x],J_\varepsilon u\big)_{H^s}-2\big(J_\varepsilon F(u),J_\varepsilon u\big)_{H^s}\big\}\,\mathrm{d}t'\\&\quad+\mathbb{E}\int_0^{t\wedge\tau_m}V'\big(\|J_\varepsilon u\|^2_{H^s}\big)\big\|J_\varepsilon\sigma\big(t',u\big)\big\|^2_{H^s}\,\mathrm{d}t'+\mathbb{E}\int_0^{t\wedge\tau_m}2V''\big(\|J_\varepsilon u\|^2_{H^s}\big)\big|\big(J_\varepsilon\sigma\big(t',u\big),J_\varepsilon u\big)_{H^s}\big|^2\,\mathrm{d}t'\\&\le\mathbb{E}V\big(\|u_0\|^2_{H^s}\big)+\mathbb{E}\int_0^{t\wedge\tau_m}V'\big(\|J_\varepsilon u\|^2_{H^s}\big)\big\{2Ag\big(\|u\|_{W^{1,\infty}}\big)\|u\|^2_{H^s}+\big\|\sigma\big(t',u\big)\big\|^2_{H^s}\big\}\,\mathrm{d}t'\\&\quad+\mathbb{E}\int_0^{t\wedge\tau_m}2V''\big(\|J_\varepsilon u\|^2_{H^s}\big)\big|\big(J_\varepsilon\sigma\big(t',u\big),J_\varepsilon u\big)_{H^s}\big|^2\,\mathrm{d}t'.\end{aligned}$$

By virtue of (2.6), (3.21), Fatou's lemma, the dominated convergence theorem and Assumption (B), one has

$$\begin{aligned}
&\mathbb{E}V\big(\|u(t\wedge\tau_m)\|_{H^s}^2\big)\\
&\quad\le \lim_{\varepsilon\to 0}\mathbb{E}V\big(\|J_\varepsilon u(t\wedge\tau_m)\|_{H^s}^2\big)\\
&\quad\le \mathbb{E}V\big(\|u_0\|_{H^s}^2\big)+\lim_{\varepsilon\to 0}\mathbb{E}\int_0^{t\wedge\tau_m}V'\big(\|J_\varepsilon u\|_{H^s}^2\big)\big\{2Ag\big(\|u\|_{W^{1,\infty}}\big)\|u\|_{H^s}^2+\big\|\sigma\big(t',u\big)\big\|_{H^s}^2\big\}\,\mathrm{d}t'\\
&\qquad+\lim_{\varepsilon\to 0}\mathbb{E}\int_0^{t\wedge\tau_m}2V''\big(\|J_\varepsilon u\|_{H^s}^2\big)\big|\big(J_\varepsilon\sigma\big(t',u\big),J_\varepsilon u\big)_{H^s}\big|^2\,\mathrm{d}t'\\
&\quad\le \mathbb{E}V\big(\|u_0\|_{H^s}^2\big)+M_1t-\mathbb{E}\int_0^{t\wedge\tau_m}M_2\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t',u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}\,\mathrm{d}t',
\end{aligned}$$

which means that there is a constant $C=C(u_0,M_1,M_2,t)>0$ such that

$$\mathbb{E}\int_0^{t\wedge\tau_m}\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t',u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}\,\mathrm{d}t'\le C(u_0,M_1,M_2,t). \tag{4.2}$$

For any $t>0$, $m\ge 1$, by (3.21), the dominated convergence theorem and Assumption (B), we can find a positive function $\eta_{t,m}(\varepsilon)$ such that $\eta_{t,m}(\varepsilon)\to 0$ when $\varepsilon\to 0$ and

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau_m}V'\big(\|J_\varepsilon u\|_{H^s}^2\big)\big\{2Ag\big(\|u\|_{W^{1,\infty}}\big)\|u\|_{H^s}^2+\big\|\sigma\big(t',u\big)\big\|_{H^s}^2\big\}+2V''\big(\|J_\varepsilon u\|_{H^s}^2\big)\big|\big(J_\varepsilon\sigma\big(t',u\big),J_\varepsilon u\big)_{H^s}\big|^2\,\mathrm{d}t'\\
&\quad\le \mathbb{E}\int_0^{t\wedge\tau_m}V'\big(\|u\|_{H^s}^2\big)\big\{2Ag\big(\|u\|_{W^{1,\infty}}\big)\|u\|_{H^s}^2+\big\|\sigma\big(t',u\big)\big\|_{H^s}^2\big\}\\
&\qquad+2V''\big(\|u\|_{H^s}^2\big)\big|\big(\sigma\big(t',u\big),u\big)_{H^s}\big|^2\,\mathrm{d}t'+\eta_{t,m}(\varepsilon)\\
&\quad\le M_1t+M_2\mathbb{E}\int_0^{t\wedge\tau_m}\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t',u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}\,\mathrm{d}t'+\eta_{t,m}(\varepsilon).
\end{aligned}\tag{4.3}$$

Therefore, for any $T>0$, it follows from (4.3) and the BDG inequality that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}V\big(\|J_\varepsilon u\|_{H^s}^2\big)\\
&\quad\le \mathbb{E}V\big(\|u_0\|_{H^s}^2\big)+C\mathbb{E}\bigg(\int_0^{T\wedge\tau_m}\big\{V'\big(\|J_\varepsilon u\|_{H^s}^2\big)\big|\big(J_\varepsilon\sigma(t,u),J_\varepsilon u\big)_{H^s}\big|\big\}^2\,\mathrm{d}t\bigg)^{\frac12}\\
&\qquad+M_1T+M_2\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t,u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}\,\mathrm{d}t+\eta_{T,m}(\varepsilon)\\
&\quad\le \mathbb{E}V\big(\|u_0\|_{H^s}^2\big)+\frac12\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}\big(1+V\big(\|J_\varepsilon u\|_{H^s}^2\big)\big)+C\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\{V'(\|J_\varepsilon u\|_{H^s}^2)|(J_\varepsilon\sigma(t,u),J_\varepsilon u)_{H^s}|\}^2}{1+V(\|J_\varepsilon u\|_{H^s}^2)}\,\mathrm{d}t\\
&\qquad+M_1T+M_2\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t,u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}\,\mathrm{d}t+\eta_{T,m}(\varepsilon).
\end{aligned}$$

Thus we use the dominated convergence theorem, Fatou's lemma and (4.2) to obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}V\big(\|u\|_{H^s}^2\big)\\
&\quad\le \lim_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,T\wedge\tau_m]}V\big(\|J_\varepsilon u\|_{H^s}^2\big)\\
&\quad\le 1+2\mathbb{E}V\big(\|u_0\|_{H^s}^2\big)+2C\lim_{\varepsilon\to 0}\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\{V'(\|J_\varepsilon u\|_{H^s}^2)|(J_\varepsilon\sigma(t,u),J_\varepsilon u)_{H^s}|\}^2}{1+V(\|J_\varepsilon u\|_{H^s}^2)}\,\mathrm{d}t
\end{aligned}$$

$$+2M_1T+2M_2\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t,u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}\,\mathrm{d}t$$
$$\le C(u_0,M_1,T)+C(M_2)\mathbb{E}\int_0^{T\wedge\tau_m}\frac{\{V'(\|u\|_{H^s}^2)|(\sigma(t,u),u)_{H^s}|\}^2}{1+V(\|u\|_{H^s}^2)}\,\mathrm{d}t$$
$$\le C(u_0,M_1,M_2,T).$$

As a result, for all $m\ge 1$,

$$\mathbb{P}\{\tau^*<T\}\le\mathbb{P}\{\tau_m<T\}\le\mathbb{P}\Big\{V\big(m^2\big)\le\sup_{t\in[0,T\wedge\tau_m]}V\big(\|u\|_{H^s}^2\big)\Big\}\le\frac{C(u_0,M_1,M_2,T)}{V(m^2)}.$$

Since $\lim_{x\to\infty}V(x)=\infty$, one can send $m\to\infty$ to identify that $\mathbb{P}\{\tau^*<T\}=0$. Since $T>0$ is arbitrary, we have $\mathbb{P}\{\tau^*=\infty\}=1$. □

4.2. *Examples*

Now we give some examples of $\sigma(t,x)$ such that Assumption (B) is verified. As in (1.12), for the solution to (1.4), its H^s-norm blows up if and only if its $W^{1,\infty}$-norm blows up. This and (1.10) in Assumption (B) suggest choosing a noise coefficient involving the $W^{1,\infty}$-norm of u. Therefore we let

$$\sigma(t,u)=\alpha\big(t,\|u\|_{W^{1,\infty}}\big)u, \tag{4.4}$$

and we assume $\alpha(t,x)$ satisfies

Assumption (D). We assume that

- $\alpha(t,x)\in C([0,\infty)\times[0,\infty))$ is locally bounded and $\alpha(t,\cdot)$ is locally Lipschitz continuous;
- $\alpha(t,x)\neq 0$ for all $(t,x)\in[0,\infty)\times[0,\infty)$, and for all $t\ge 0$,

$$\limsup_{x\to+\infty}\frac{2Ag(x)}{\alpha^2(t,x)}<1,$$

  where A and $g(x)$ are given in Lemma 2.5.

We first notice the following algebraic property.

Lemma 4.1. *Let A and $g(\cdot)$ be given in Lemma 2.5. Let $D>0$. If Assumption (D) holds true, then there is an $M_1>0$ such that for any $M_2>0$ and all $0<x\le Dy<\infty$,*

$$\frac{2Ag(x)y^2+\alpha^2(t,x)y^2}{1+y^2}-\frac{2\alpha^2(t,x)y^4}{(1+y^2)^2}\le M_1-M_2\frac{2\alpha^2(t,x)y^4}{(1+y^2)^2(1+\log(1+y^2))}. \tag{4.5}$$

Proof. By Assumption (D), we have

$$\limsup_{x\to+\infty}\frac{2Ag(x)y^2+\alpha^2(t,x)y^2}{1+y^2}-\frac{2\alpha^2(t,x)y^4}{(1+y^2)^2}+M_2\frac{2\alpha^2(t,x)y^4}{(1+y^2)^2(1+\log(1+y^2))}$$
$$\le\limsup_{x\to+\infty}\bigg(\frac{2Ag(x)}{\alpha^2(t,x)}+1-\frac{2(\frac{x}{D})^4}{(1+(\frac{x}{D})^2)^2}+M_2\frac{2}{(1+\log(1+(\frac{x}{D})^2))}\bigg)\alpha^2(t,x)<0,$$

which implies (4.5). □

Lemma 4.2. *If $\alpha(t,x)$ satisfies Assumption (D), then $\sigma(t,u)$ defined by (4.4) satisfies Assumption (B) with $V(x)=\log(1+x)$.*

Proof. Lemma 4.1 implies that (1.10) holds true with $V(x)=\log(1+x)$. The other statements in Assumption (B) obviously hold true. □

We conclude this section with different examples of $\alpha(t,x)$. Let $q:[0,\infty)\to[0,\infty)$ be a continuous function and there are constants $q_*, q^*>0$ such that $q_*\le q^2(t)\le q^*<\infty$ for all t. Then we assume

$$\alpha(t,x)=q(t)(1+x)^{\theta}. \tag{4.6}$$

- When Coriolis effect is involved, we consider the rotation-CH equation, i.e., $a_i\neq 0$ with $i=0,1,2,3,4$, and we assume

$$\theta>3/2,\qquad q^*>q_*>0\quad\text{or}\quad\theta=3/2,\qquad q^*>q_*>2A|a_4|.$$

- When Coriolis effect is *not* involved, we set $a_3=a_4=0$. In this case, we assume

$$\theta>1/2,\qquad q^*>q_*>0\quad\text{or}\quad\theta=1/2,\qquad q^*>q_*>2A\big(1+|a_1|+|a_2|\big).$$

It is easy to see that, for the above two cases, α defined by (4.6) satisfies Assumption (D). Therefore, by Lemma 4.2, Assumption (B) with $V=\log(1+x)$ is verified for $\sigma(t,u)$ defined by (4.4).

5. Remark on Theorems 1.1 and 1.2 on the torus

We point out that both Theorems 1.1 and 1.2 also hold true on the 1-D torus $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, i.e., $x\in\mathbb{T}$. Indeed, in this periodic case, $D^s=(1-\partial_{xx}^2)^{s/2}$ is defined by $\widehat{D^s f}(k)=(1+k^2)^{s/2}\widehat{f}(k)$, where $\widehat{f}(k)$ is the Fourier coefficient of f. The Sobolev space $H^s(\mathbb{T})$ is defined as

$$H^s(\mathbb{T})\triangleq\left\{f\in L^2(\mathbb{T}):\|f\|_{H^s(\mathbb{T})}^2=\sum_{k\in\mathbb{Z}}\big(1+k^2\big)^s\big|\widehat{f}(k)\big|^2<\infty\right\}$$

with inner product $(f,g)_{H^s}\triangleq\sum_{k\in\mathbb{Z}}(1+k^2)^s\widehat{f}(k)\cdot\overline{\widehat{g}}(k)=(D^s f,D^s g)_{L^2}$. Moreover, the mollifier J_ε can be defined as in (2.1). We also define the regularizing operator T_ε on $\mathbb{T}$ as

$$T_\varepsilon f(x)\triangleq\big(1-\varepsilon^2\partial_{xx}^2\big)^{-1}f(x)=\sum_{k\in\mathbb{Z}}\big(1+\varepsilon^2|k|^2\big)^{-1}\widehat{f}(k)\mathrm{e}^{\mathrm{i}xk},\quad \varepsilon\in(0,1). \tag{5.1}$$

Since T_ε is defined by its Fourier multipliers, (2.4)-(2.6) also hold true if J_ε is replaced by T_ε. Furthermore, we have

Lemma 5.1 ([60,65]). *Let $f,g:\mathbb{T}\to\mathbb{R}$ such that $g\in W^{1,\infty}$ and $f\in L^2$. Then for some $C>0$,*

$$\big\|[T_\varepsilon,g]\partial_x f\big\|_{L^2}\le C\|g\|_{W^{1,\infty}}\|f\|_{L^2}.$$

With Lemma 5.1, in the same way as we prove Lemma 2.5, we also have

Lemma 5.2. *Let $s>3/2$. Let $F(\cdot)$ be given in (1.5) and T_ε be given in (5.1). Let*

$$g\big(\|u\|_{W^{1,\infty}}\big)=|a_0|+\big(1+|a_1|+|a_2|\big)\|u\|_{W^{1,\infty}}+|a_3|\|u\|_{W^{1,\infty}}^2+|a_4|\|u\|_{W^{1,\infty}}^3.$$

Then there is a constant $A=A(s)>0$ such that for all $\varepsilon>0$,

$$\big|\big(T_\varepsilon[uu_x],T_\varepsilon u\big)_{H^s}\big|+\big|\big(T_\varepsilon F(u),T_\varepsilon u\big)_{H^s}\big|\le Ag\big(\|u\|_{W^{1,\infty}}\big)\|u\|_{H^s}^2.$$

By using the mollifier J_ε, we construct the same approximation scheme as in (3.2). When the Itô formula is not applicable (see (3.19), (3.24) and (4.1)), one can use T_ε to replace J_ε to proceed the proof for Theorems 1.1 and 1.2 to obtain the same statements, where Lemma 5.2 is needed to replace Lemma 2.5.

6. Noise effect on initial-data dependence

In this section, we will prove Theorem 1.3. Throughout this section, we suppose Assumption (C) holds true. Besides, all the function spaces are over $\mathbb{T}$ in this section and for simplicity, we omit it in the notations of spaces. We will show that if the exiting time of the zero solution is strongly stable, then there are two sequences of pathwise solutions such that (1.13)–(1.16) are satisfied.

6.1. *Approximate and actual solutions*

We define the approximate solutions as

$$u^{l,n} = ln^{-1} + n^{-s}\cos\theta \quad \text{with } \theta = nx - lt, n \in \mathbb{N} \text{ and } l \in \{-1, 1\}.$$

Substituting $u^{l,n}$ into (1.4), we see that the error $\mathcal{E}^{l,n}(t)$ is defined as

$$\mathcal{E}^{l,n}(t) = u^{l,n}(t) - u^{l,n}(0) + \int_0^t \left[u^{l,n}\partial_x u^{l,n} + F\left(u^{l,n}\right)\right] \mathrm{d}t' - \int_0^t h\left(t', u^{l,n}\right) \mathrm{d}\mathcal{W}. \tag{6.1}$$

On the other hand, we consider the following periodic boundary value problem with deterministic initial data $u^{l,n}(0,x)$, i.e.,

$$\begin{cases} \mathrm{d}u + \left[uu_x + F(u)\right]\mathrm{d}t = h(t,u)\,\mathrm{d}\mathcal{W}, & t > 0,\ x \in \mathbb{T}, \\ u(0,x) = u^{l,n}(0,x), & x \in \mathbb{T}. \end{cases} \tag{6.2}$$

Since Assumption (C) and Lemma 2.4 imply Assumption (A), Theorem 1.1 and Section 5 yield that for each $n \in \mathbb{N}$, (6.2) has a uniqueness maximal pathwise solution $(u_{l,n}, \tau^*_{l,n})$.

6.2. *Estimates on the errors*

Before we go further, we estimate $\mathcal{E}^{l,n}(t)$ as follows:

Lemma 6.1. *Let $s > 3/2$. For $n \gg 1$, $\delta \in (1/2, \min\{s-1, 3/2\})$ and any $T > 0$, there is a $C = C(T) > 0$ such that*

$$\mathbb{E} \sup_{t\in[0,T]} \left\|\mathcal{E}^{l,n}(t)\right\|^2_{H^\delta} \le Cn^{-2r_s},$$

where

$$0 < r_s = \begin{cases} 2s - \delta - 1 & \text{if } \frac{3}{2} < s \le 2, \\ s - \delta + 1 & \text{if } s > 2. \end{cases}$$

Proof. Direct computation shows that

$$u^{l,n}(t) - u^{l,n}(0) + \int_0^t u^{l,n}\partial_x u^{l,n}\,\mathrm{d}t' = \int_0^t \left(-n^{-2s+1}\sin\theta\cos\theta\right)\mathrm{d}t'.$$

Then it holds that

$$\mathcal{E}^{l,n}(t) - \int_0^t \left[-n^{-2s+1}\sin\theta\cos\theta + F\left(u^{l,n}\right)\right]\mathrm{d}t' + \int_0^t h\left(t', u^{l,n}\right)\mathrm{d}\mathcal{W} = 0. \tag{6.3}$$

In view of Lemma 2.6, we arrive at

$$\left\|-n^{-2s+1}\sin\theta\cos\theta\right\|_{H^\delta} \lesssim n^{-2s+1+\delta} \lesssim n^{-r_s}. \tag{6.4}$$

Recall that $F(\cdot)$ is given by (1.5). Since $(1-\partial^2_{xx})^{-1}$ is bounded from H^δ to $H^{\delta+2}$, we can use Lemma 2.6 to estimate $\|F_i(u^{l,n})\|_{H^\delta}$ $(i = 1, 2, 3, 4, 5)$ as follows:

$$\left\|F_1\left(u^{l,n}\right)\right\|_{H^\delta} \lesssim \left\|n^{-s+1}\sin\theta\right\|_{H^{\delta-2}} \lesssim n^{-s-1+\delta}, \tag{6.5}$$

$$\left\|F_2\left(u^{l,n}\right)\right\|_{H^\delta} \lesssim \left\|n^{-2s+1}\sin\theta\cos\theta + ln^{-s}\sin\theta\right\|_{H^{\delta-2}} \lesssim n^{-2s-1+\delta} + n^{-s-2+\delta} \lesssim n^{-s-2+\delta}, \tag{6.6}$$

$$\left\|F_3\left(u^{l,n}\right)\right\|_{H^\delta} \lesssim \left\|n^{-2s+3}\sin 2\theta\right\|_{H^{\delta-2}} \lesssim n^{-2s+1+\delta}, \tag{6.7}$$

$$\left\|F_4\left(u^{l,n}\right)\right\|_{H^\delta} \lesssim \sum_{j=0}^{2} n^{-(2-j)}\left\|\left(n^{-s}\cos\theta\right)^j n^{-s+1}\sin\theta\right\|_{H^{\delta-2}}$$

$$\lesssim \sum_{j=0}^{2} n^{-2+j-sj-s+1} \left\| \cos^j \theta \sin\theta \right\|_{H^\delta}$$

$$\lesssim \max_{0\le j\le 2} \left\{ n^{-2+j-sj-s+1+\delta} \right\} \lesssim n^{-1-s+\delta}, \tag{6.8}$$

and

$$\begin{aligned} \left\| F_5\left(u^{l,n}\right) \right\|_{H^\delta} &\lesssim \sum_{j=0}^{3} n^{-(3-j)} \left\| \left(n^{-s}\cos\theta\right)^j n^{-s+1} \sin\theta \right\|_{H^{\delta-2}} \\ &\lesssim \sum_{j=0}^{3} n^{-3+j-sj-s+1} \left\| \cos^j \theta \sin\theta \right\|_{H^\delta} \\ &\lesssim \max_{0\le j\le 2} \left\{ n^{-3+j-sj-s+1+\delta} \right\} \lesssim n^{-2-s+\delta}. \end{aligned} \tag{6.9}$$

Combining (6.5), (6.6), (6.7), (6.8) and (6.9), we have

$$\left\| F\left(u^{l,n}\right) \right\|_{H^\delta} \lesssim \max\left\{ n^{-s-1+\delta}, n^{-2s+1+\delta} \right\} \lesssim n^{-r_s}. \tag{6.10}$$

Then, for any $T > 0$ and $t \in [0, T]$, by virtue of the Itô formula, we arrive at

$$\left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta}^2 \le \left| \int_0^t \left(-2h\left(t', u^{l,n}\right) \mathrm{d}\mathcal{W}, \mathcal{E}^{l,n} \right)_{H^\delta} \right| + \sum_{i=2}^{4} \int_0^t |J_i| \, \mathrm{d}t',$$

where

$$\begin{aligned} J_2 &= 2\left(D^\delta\left(-n^{-2s+1} \sin\theta\cos\theta\right), D^\delta \mathcal{E}^{l,n} \right)_{L^2}, \\ J_3 &= 2\left(D^\delta F\left(u^{l,n}\right), D^\delta \mathcal{E}^{l,n} \right)_{L^2}, \\ J_4 &= \left\| h\left(t', u^{l,n}\right) \right\|_{\mathcal{L}_2(\mathfrak{U}; H^\delta)}^2. \end{aligned}$$

Taking the supremum with respect to $t \in [0, T]$, using the BDG inequality and using Assumption (C) and (6.10) yield

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t \left(-2h\left(t', u^{l,n}\right) \mathrm{d}\mathcal{W}, \mathcal{E}^{l,n} \right)_{H^\delta} \right| &\le 2C\mathbb{E} \left(\int_0^T \left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta}^2 \left\| F\left(u^{l,n}\right) \right\|_{H^\delta}^2 \mathrm{d}t \right)^{\frac{1}{2}} \\ &\le 2C\mathbb{E} \left(\sup_{t\in[0,T]} \left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta}^2 \int_0^T \left\| F\left(u^{l,n}\right) \right\|_{H^\delta}^2 \mathrm{d}t \right)^{\frac{1}{2}} \\ &\le \frac{1}{2} \mathbb{E} \sup_{t\in[0,T]} \left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta}^2 + CTn^{-2r_s}. \end{aligned}$$

By virtue of (6.4) and (6.10), we obtain

$$\begin{aligned} \int_0^T \mathbb{E}|J_2| \, \mathrm{d}t &\le C \int_0^T \mathbb{E}\left(\left\| -n^{-2s+1} \sin\theta\cos\theta \right\|_{H^\delta} \left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta} \right) \mathrm{d}t \\ &\le C \int_0^T \mathbb{E}\left\| -n^{-2s+1} \sin\theta\cos\theta \right\|_{H^\delta}^2 \mathrm{d}t + C \int_0^T \mathbb{E}\left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta}^2 \mathrm{d}t \\ &\le CTn^{-2r_s} + C \int_0^T \mathbb{E}\left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta}^2 \mathrm{d}t, \\ \int_0^T \mathbb{E}|J_3| \, \mathrm{d}t &\le C \int_0^T \mathbb{E}\left(\left\| F\left(u^{l,n}\right) \right\|_{H^\delta} \left\| \mathcal{E}^{l,n}(t) \right\|_{H^\delta} \right) \mathrm{d}t \end{aligned}$$

$$\leq C\int_0^T \mathbb{E}\left\|F\left(u^{l,n}\right)\right\|_{H^\delta}^2 \mathrm{d}t + C\int_0^T \mathbb{E}\left\|\mathcal{E}^{l,n}(t)\right\|_{H^\delta}^2 \mathrm{d}t$$
$$\leq CTn^{-2r_s} + C\int_0^T \mathbb{E}\left\|\mathcal{E}^{l,n}(t)\right\|_{H^\delta}^2 \mathrm{d}t,$$

and by Assumption (C),

$$\int_0^T \mathbb{E}|J_4|\,\mathrm{d}t \leq C\int_0^T \mathbb{E}\left\|F\left(u^{l,n}\right)\right\|_{H^\delta}^2 \mathrm{d}t \leq CTn^{-2r_s}.$$

Combining the above estimates gives

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\mathcal{E}^{l,n}(t)\right\|_{H^\delta}^2 \leq CTn^{-2r_s} + C\int_0^T \mathbb{E}\sup_{t'\in[0,t]}\left\|\mathcal{E}^{l,n}\left(t'\right)\right\|_{H^\delta}^2 \mathrm{d}t.$$

Obviously, for each $n \geq 1$ and $l \in \{-1, 1\}$, $\mathbb{E}\sup_{t'\in[0,t]} \|\mathcal{E}^{l,n}(t')\|_{H^\delta}^2$ is finite. Then it follows from Grönwall's inequality that

$$\mathbb{E}\sup_{t\in[0,T]}\left\|\mathcal{E}^{l,n}(t)\right\|_{H^\delta}^2 \leq Cn^{-2r_s}, \quad C = C(T).$$

This completes the proof for Lemma 6.1. □

For the difference $u^{l,n} - u_{l,n}$, we have the following estimates:

Lemma 6.2. *Let $s > \frac{3}{2}$, $\frac{1}{2} < \delta < \min\{s-1, \frac{3}{2}\}$ and $r_s > 0$ be given as in Lemma 6.1. For any $R > 1$, define*

(6.11) $$\tau_{l,n}^R = \inf\left\{t \geq 0 : \|u_{l,n}\|_{H^s} > R\right\}.$$

Then for any $T > 0$, when $n \gg 1$,

(6.12) $$\mathbb{E}\sup_{t\in[0,T\wedge\tau_{l,n}^R]}\left\|u^{l,n} - u_{l,n}\right\|_{H^\delta}^2 \leq Cn^{-2r_s}, \quad C = C(R,T),$$

and

(6.13) $$\mathbb{E}\sup_{t\in[0,T\wedge\tau_{l,n}^R]}\left\|u^{l,n} - u_{l,n}\right\|_{H^{2s-\delta}}^2 \leq Cn^{2s-2\delta}, \quad C = C(R,T).$$

Proof. In view of Lemma 2.6, we have

(6.14) $$\left\|u^{l,n}(t)\right\|_{H^s} \lesssim 1 \quad \forall t > 0.$$

Let $q = q^{l,n} = u^{l,n} + u_{l,n}$ and $v = v^{l,n} = u^{l,n} - u_{l,n}$. In view of (6.1), (6.2) and (6.3), we have

$$v(t) + \int_0^t \left[\frac{1}{2}\partial_x(qv) - F(u_{l,n})\right]\mathrm{d}t' = -\int_0^t h\left(t', u_{l,n}\right)\mathrm{d}\mathcal{W} - \int_0^t n^{-2s+1}\sin\theta\cos\theta\,\mathrm{d}t'.$$

For any $T > 0$, by Itô formula on $[0, T\wedge\tau_{l,n}^R]$, taking the supremum over $t \in [0, T\wedge\tau_{l,n}^R]$ and the BDG inequality with noticing Assumption (C), we obtain

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau_{l,n}^R]}\left\|v(t)\right\|_{H^\delta}^2 \leq C\mathbb{E}\left(\int_0^{T\wedge\tau_{l,n}^R} |K_1|^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \sum_{i=2}^5 \mathbb{E}\int_0^{T\wedge\tau_{l,n}^R} |K_i|\,\mathrm{d}t,$$

where

$$K_1 = \|v\|_{H^\delta}\left\|F(u_{l,n})\right\|_{H^\delta},$$

$$K_2 = 2\big(D^\delta\big(-n^{-2s+1}\sin\theta\cos\theta\big), D^\delta v\big)_{L^2},$$
$$K_3 = -\big(D^\delta\partial_x[qv], D^\delta v\big)_{L^2},$$
$$K_4 = 2\big(D^\delta F(u_{l,n}), D^\delta v\big)_{L^2},$$
$$K_5 = \big\|h(t,u_{l,n})\big\|^2_{\mathcal{L}_2(\mathfrak{U};H^\delta)}.$$

From Lemma 2.4, we know that for some locally bounded increasing function $\Psi:[0,\infty)\mapsto[0,\infty)$,

$$\begin{aligned}\big\|F(u_{l,n})\big\|^2_{H^\delta} &\lesssim \big(\big\|F\big(u^{l,n}\big)-F(u_{l,n})\big\|_{H^\delta}+\big\|F\big(u^{l,n}\big)\big\|_{H^\delta}\big)^2\\ &\lesssim \Psi\big(\big\|u^{l,n}\big\|_{H^s}+\|u_{l,n}\|_{H^s}\big)\|v\|^2_{H^\delta}+\big\|F\big(u^{l,n}\big)\big\|^2_{H^\delta}.\end{aligned}$$

Therefore, it follows from Lemmas 2.3 and 2.4, Assumption (C), $H^\delta\hookrightarrow L^\infty$, integrating by parts and (6.4) that

$$\begin{aligned}|K_1|^2 &\lesssim \Psi\big(\big\|u^{l,n}\big\|_{H^s}+\|u_{l,n}\|_{H^s}\big)\|v\|^4_{H^\delta}+\big\|F\big(u^{l,n}\big)\big\|^2_{H^\delta}\|v\|^2_{H^\delta},\\ |K_2| &\lesssim n^{-2r_s}+\|v\|^2_{H^\delta},\\ |K_3| &\lesssim \|q\|_{H^s}\|v\|^2_{H^\delta}+\|q_x\|_{L^\infty}\|v\|^2_{H^\delta}\lesssim\|q\|_{H^s}\|v\|^2_{H^\delta},\\ |K_4| &\lesssim \Psi\big(\big\|u^{l,n}\big\|_{H^s}+\|u_{l,n}\|_{H^s}\big)\|v\|^2_{H^\delta}+\big\|F\big(u^{l,n}\big)\big\|^2_{H^\delta}+\|v\|^2_{H^\delta},\end{aligned}$$

and

$$|K_5|\lesssim \Psi\big(\big\|u^{l,n}\big\|_{H^s}+\|u_{l,n}\|_{H^s}\big)\|v\|^2_{H^\delta}+\big\|F\big(u^{l,n}\big)\big\|^2_{H^\delta}.$$

Applying Lemma 2.4, (6.10), (6.11) and (6.14), we have

$$\begin{aligned}&C\mathbb{E}\bigg(\int_0^{T\wedge\tau^R_{l,n}}|K_1|^2\,\mathrm{d}t\bigg)^{\frac12}\\ &\le C\mathbb{E}\bigg(\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^\delta}\int_0^{T\wedge\tau^R_{l,n}}\Psi\big(\big\|u^{l,n}\big\|_{H^s}+\|u_{l,n}\|_{H^s}\big)\|v\|^2_{H^\delta}\,\mathrm{d}t\bigg)^{\frac12}\\ &\quad+C\mathbb{E}\bigg(\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^\delta}\int_0^{T\wedge\tau^R_{l,n}}\big\|F\big(u^{l,n}\big)\big\|^2_{H^\delta}\,\mathrm{d}t\bigg)^{\frac12}\\ &\le\frac12\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^\delta}+C_R\mathbb{E}\int_0^{T\wedge\tau^R_{l,n}}\big\|v(t)\big\|^2_{H^\delta}\,\mathrm{d}t+C\mathbb{E}\int_0^{T\wedge\tau^R_{l,n}}\big\|F\big(u^{l,n}\big)\big\|^2_{H^\delta}\,\mathrm{d}t\\ &\le\frac12\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^\delta}+C_R\mathbb{E}\int_0^T\sup_{t'\in[0,t\wedge\tau^R_{l,n}]}\big\|v\big(t'\big)\big\|^2_{H^\delta}\,\mathrm{d}t+CTn^{-2r_s},\end{aligned}$$

$$\mathbb{E}\int_0^{T\wedge\tau^R_{l,n}}|K_2|+|K_4|+|K_5|\,\mathrm{d}t\le CTn^{-2r_s}+C_R\int_0^T\mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]}\big\|v\big(t'\big)\big\|^2_{H^\delta}\,\mathrm{d}t,$$

and

$$\mathbb{E}\int_0^{T\wedge\tau^R_{l,n}}|K_3|\,\mathrm{d}t\le C_R\int_0^T\mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]}\big\|v\big(t'\big)\big\|^2_{H^\delta}\,\mathrm{d}t.$$

To sum up, we arrive at

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\big\|v(t)\big\|^2_{H^\delta}\le CTn^{-2r_s}+C_R\int_0^T\mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]}\big\|v\big(t'\big)\big\|^2_{H^\delta}\,\mathrm{d}t.$$

Using the Grönwall inequality, we obtain (6.12). For (6.13), we first notice that $u_{l,n}$ is the unique solution to (6.2). Since $2s-\delta>3/2$, we repeat the derivation of (3.25) with using (6.11) to obtain that for each fixed $n\in\mathbb{N}$,

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\left\|u_{l,n}(t)\right\|^2_{H^{2s-\delta}}\leq C\mathbb{E}\left\|u^{l,n}(0)\right\|^2_{H^{2s-\delta}}+C_R\int_0^T\left(\mathbb{E}\sup_{t'\in[0,t\wedge\tau^R_{l,n}]}\left\|u_{l,n}(t')\right\|^2_{H^{2s-\delta}}\right)\mathrm{d}t. \tag{6.15}$$

From (6.15), we can use the Grönwall inequality and Lemma 2.6 to have

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\left\|u_{l,n}(t)\right\|^2_{H^{2s-\delta}}\leq C\mathbb{E}\left\|u^{l,n}(0)\right\|^2_{H^{2s-\delta}}\leq Cn^{2s-2\delta},\quad C=C(R,T).$$

By Lemma 2.6 again, we have that for some $C=C(R,T)$,

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|v\|^2_{H^{2s-\delta}}\leq C\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\|u_{l,n}\|^2_{H^{2s-\delta}}+C\mathbb{E}\sup_{t\in[0,T\wedge\tau^R_{l,n}]}\left\|u^{l,n}\right\|^2_{H^{2s-\delta}}\leq Cn^{2s-2\delta},$$

which is (6.13). We complete the proof for Lemma 6.2. □

6.3. *Concluding the proof for Theorem 1.3*

We first observe the following property:

Lemma 6.3. *Let Assumption (C) hold true. Suppose that there is a $R_0\gg 1$ such that the R_0-exiting time of the zero solution to (1.4) is strongly stable. Then*

$$\lim_{n\to\infty}\tau^{R_0}_{l,n}=\infty\quad \mathbb{P}\text{-}a.s. \tag{6.16}$$

Proof. We notice that for all $s'<s$, $\lim_{n\to\infty}\|u_{l,n}(0)-0\|_{H^{s'}}=\lim_{n\to\infty}\|u^{l,n}(0)\|_{H^{s'}}=0$. Under Assumption (C), it is clear that $u\equiv 0$ is a solution to (1.4) with $u(0)=0$. Since the R_0-exiting time of the zero solution is ∞, we see that (6.16) holds provided the R_0-exiting time of the zero solution to (1.4) is strongly stable. □

Finally we are in the position to finish the proof for Theorem 1.3.

Proof for Theorem 1.3. To prove Theorem 1.3, it suffices to show that if the R_0-exiting time is strongly stable at the zero solution for some $R_0\gg 1$, then the solution map $u_0\mapsto u$ defined by (1.4) can *not* be uniformly continuous. For each $n>1$ and for such fixed $R_0\gg 1$, Lemma 2.6 and (6.11) give

$$\mathbb{P}\left\{\tau^{R_0}_{l,n}>0\right\}=1,$$

and Lemma 6.3 implies (1.13). In addition, it follows from Theorem 1.1 and (6.11) that

$$u_{l,n}\in C\left(\left[0,\tau^{R_0}_{l,n}\right];H^s\right)\quad \mathbb{P}\text{-a.s.},$$

and (1.14) holds. Clearly, (1.15) is given by

$$\left\|u_{-1,n}(0)-u_{1,n}(0)\right\|_{H^s}=\left\|u^{-1,n}(0)-u^{1,n}(0)\right\|_{H^s}\lesssim n^{-1}\to 0,\quad \text{as } n\to\infty. \tag{6.17}$$

It remains to prove (1.16). By interpolation, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T\wedge\tau^{R_0}_{l,n}]}\left\|u^{l,n}-u_{l,n}\right\|_{H^s}\\
&\leq C\left(\mathbb{E}\sup_{t\in[0,T\wedge\tau^{R_0}_{l,n}]}\left\|u^{l,n}-u_{l,n}\right\|_{H^\delta}\right)^{\frac{1}{2}}\left(\mathbb{E}\sup_{t\in[0,T\wedge\tau^{R_0}_{l,n}]}\left\|u^{l,n}-u_{l,n}\right\|_{H^{2s-\delta}}\right)^{\frac{1}{2}}\\
&\leq C\left(\mathbb{E}\sup_{t\in[0,T\wedge\tau^{R_0}_{l,n}]}\left\|u^{l,n}-u_{l,n}\right\|^2_{H^\delta}\right)^{\frac{1}{4}}\left(\mathbb{E}\sup_{t\in[0,T\wedge\tau^{R_0}_{l,n}]}\left\|u^{l,n}-u_{l,n}\right\|^2_{H^{2s-\delta}}\right)^{\frac{1}{4}}.
\end{aligned}$$

For any $T>0$, combining Lemma 6.2 and the above estimate yields

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau_{l,n}^{R_0}]}\|u^{l,n}-u_{l,n}\|_{H^s}\lesssim n^{-\frac{1}{4}\cdot 2r_s}\cdot n^{\frac{1}{4}\cdot(2s-2\delta)}=n^{r_s'}, \tag{6.18}$$

where

$$0>r_s'=-r_s\cdot\frac{1}{2}+(s-\delta)\cdot\frac{1}{2}=\begin{cases}-\frac{s}{2}+\frac{1}{2} & \text{if } \frac{3}{2}<s\le 2,\\ -\frac{1}{2} & \text{if } s>2.\end{cases}$$

Then we find

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{l,n}^{R_0}]}\|u^{l,n}-u_{l,n}\|_{H^s}=0.$$

Consequently, for any $T>0$, we have

$$\begin{aligned}
&\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}\\
&\ge\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]}\|u^{-1,n}(t)-u^{1,n}(t)\|_{H^s}\\
&\quad-\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]}\|u^{-1,n}(t)-u_{-1,n}(t)\|_{H^s}\\
&\quad-\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R}\wedge\tau_{1,n}^{R}]}\|u^{1,n}(t)-u_{1,n}(t)\|_{H^s}\\
&\gtrsim\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]}\|u^{-1,n}(t)-u^{1,n}(t)\|_{H^s}\\
&\gtrsim\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]}\|-2n^{-1}+n^{-s}\cos(nx+t)-n^{-s}\cos(nx-t)\|_{H^s}\\
&\gtrsim\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]}\left(n^{-s}\|\sin(nx)\|_{H^s}|\sin t|-\|2n^{-1}\|_{H^s}\right)\gtrsim\sup_{t\in[0,T]}|\sin t|,
\end{aligned}\tag{6.19}$$

where we have used Fatou's lemma. Therefore,

$$\liminf_{n\to\infty}\mathbb{E}\sup_{t\in[0,T\wedge\tau_{-1,n}^{R_0}\wedge\tau_{1,n}^{R_0}]}\|u_{-1,n}(t)-u_{1,n}(t)\|_{H^s}^2\gtrsim\Big(\sup_{t\in[0,T]}|\sin t|\Big)^2,$$

which implies (1.16). The proof for Theorem 1.3 is completed. □

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