Riemann's scale: A puzzle about infinity

Abstract

Ordinarily, the order in which some objects are attached to a scale does not affect the total weight measured by the scale. This principle is shown to fail in certain cases involving infinitely many objects. In these cases, we can produce any desired reading of the scale merely by changing the order in which a fixed collection of objects are attached to the scale. This puzzling phenomenon brings out the metaphysical significance of a theorem about infinite series that is well known by mathematicians but has so far eluded philosophical scrutiny.

Suppose you have an infinite number of iron balls and helium balloons. The balls have mass 1kg, $\frac{1}{3}$kg, $\frac{1}{5}$kg, etc., while the balloons are capable of lifting $\frac{1}{2}$kg, $\frac{1}{4}$kg, $\frac{1}{6}$kg, etc. You also have a scale to which you are able to successively attach the balls and balloons in any chosen order—the whole infinite lot of them. Thus, you are able to weigh the balls and balloons, which make a positive and negative contribution to the reading of the scale, respectively. So far, so unexciting.

Let's take a closer look, however, at how you can go about weighing the balls and balloons. One option is to alternate between attaching one ball and one balloon, each in their standard order (i.e. the order mentioned in the opening paragraph). In this case, the scale will show (in kg):

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \; = \; ln(2) \; \approx \; 0.69$$

Now comes the exciting bit. By adding the balls and balloons to your scale in a different order, while still using all of them, you can, I claim, produce any finite positive or negative reading of the scale whatsoever. For example, instead of making the scale read ca. 0.69kg, you can make it read 1000kg, using precisely the same balls and balloons!

Let me describe what you do to produce the desired reading—and thus to prove my claim. First, you keep adding balls, in their standard order, until the scale has just gone beyond a reading of 1000kg. You are able to do so, using only a finite number of balls, because the series $1 + \frac{1}{3} + \frac{1}{5} + \ldots$ diverges to infinity. Second, you add balloons, in their standard order, until the scale has just gone below a reading of 1000kg. You are able to do so, using only a finite number of balloons, because the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots$ diverges to infinity. Now you repeat the first two steps, except that at every stage you choose objects to attach to the scale from among the infinitely many balls and balloons that remain at that stage. As you proceed, the scale will alternate between a reading just above 1000kg and one just below. But as the individual balls and balloons that remain will be lighter and lighter, the readings will get closer and closer to 1000kg. This ensures that, as you proceed, the scale converges to the desired reading of 1000kg.

Finally, it would be deeply implausible to deny that, when all the balls and balloons have eventually been attached, the reading of the scale is simply the value to which the sequence of intermediate readings converges. This contrasts with Thomson (1954)'s lamp, which in some given one-minute interval is defined to be set to the *on* position when the number of minutes remaining is 1, $\frac{1}{2^2}$, $\frac{1}{2^4}$, etc., and to the *off* position when this number is $\frac{1}{2}$, $\frac{1}{2^3}$, $\frac{1}{2^5}$, etc. In this case, the sequence of intermediate stages fails to converge, which makes it natural to respond, with Benacerraf (1962), that the state of the lamp at the point when the one minute has elapsed is simply undefined. It is important to observe that the analogous response is not available in our case involving weight, where the sequence of intermediate stages *does* converge.

Mathematicians will recognize the heart of my argument as merely a special case of the Riemann Series Theorem (or Rearrangement Theorem), first proved in the 1860s by great German mathematician Bernhard Riemann. The aim of this note is to bring out the metaphysical significance of this theorem, which philosophers have so far failed fully to appreciate.

Indeed, our thought experiment is extremely puzzling. You weigh a fixed collection of balls and balloons—always the same ones. By changing the order in which you attach them to the scale, however, you are able to produce any desired reading of the scale! In short, the weighing process is order sensitive: the order in which you attach the objects to be weighed matters. What are we to make of this?

A finitist will claim that our findings prove the absurdity of our initial assumption that you could have an actual infinity of iron balls and helium balloons. Of course, the finitist is right that there are *in fact* no balls and balloons as described in our imagined scenario. Nor could such an arrangement of objects in practice be produced, since doing so would require infinitely much mass, which we have no reason to suppose available. Stronger yet, our thought experiment appears to conflict with contemporary physics. To attach infinitely many bodies to a scale in a finite time, you would have to move at arbitrarily high velocities, which is prohibited by the special theory of relativity. Still, it would be an astonishing philosophical achievement to discover, entirely from your armchair, without any recourse to observation or experimentation, that our imagined scenario is impossible—indeed, that the scenario is impossible not only in the sense of conflicting with the prevailing laws of physics, but that it is absolutely, or metaphysically, impossible.

An alternative response would be to accept that the sum of a collection of weights can be order sensitive. Doing so is not an appealing option, however, as it would require a dramatic departure from physics as we have it. Physics tells us nothing about any such order sensitivity. It only tells us that to compute the weight of a collection of objects, you add the weights of the individual members of the collection. There is no warning to be mindful of the order in which the members of the collection are considered.

Worse yet, the puzzle generalizes to other physical magnitudes as well. Consider an infinite sequence of possible movements of an object to the right along some given line and another such sequence of possible movements to the left. Suppose the magnitudes of the movements (in meters) match the weights of the described sequences of balls and balloons (in kilograms). We can thus show that the total movement resulting from a fixed infinite collection of component movements is order sensitive: depending on the order in which the component movements are carried out, the object that is moved around can be made to end up *anywhere* on the given line.

To conclude, our considerations leave us with a fairly general dilemma. Either our imagined scenarios are possible, or not. If they are possible, we need a dramatic departure from current physics to deal with the outer reaches of possibility (where these scenarios would be located). If, on the other hand, the imagined scenarios are not possible, then the sphere of possibility turns out to be restricted in a surprising way—likely including a metaphysical prohibition against the existence of actually infinite collections of physical objects or events.[1] Either horn would be surprising and philosophically perilous. But the Riemann Se-

[1]As evidence that this discovery would be surprising, we need only recall a substantial body of work in metaphysics and philosophical logic that readily invokes possibilities of the sort that this discovery would rule out. See Lewis (1986), Sider (2009), and Hawthorne and Uzquiano (2011) for a representative sample.

ries Theorem—once we think through its metaphysical significance—forces us to choose one of the two horns.[2]

References

Benacerraf, P. (1962). Tasks, super-tasks, and the modern eleatics. *Journal of Philosophy*, 59(24):765–784.

Hawthorne, J. and Uzquiano, G. (2011). How many angels can dance on the point of a needle? Transcendental theology meets modal metaphysics. *Mind*, 120(478):53–81.

Lewis, D. (1986). *On the Plurality of Worlds*. Blackwell, Oxford.

Sider, T. (2009). Williamson's many necessary existents. *Analysis*, 69(2):250–258.

Thomson, J. (1954). Tasks and supertasks. *Analysis*, 15(1):1–13.

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