Generic Functions in Scott Domains

by

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Abstract

In this thesis we will apply forcing to domain theory. When a Scott domain represents a function space, each function will be a filter in the basis of the domain. By using the partially ordered basis as the forcing relation, each generic filter G yields a model of ZFC in which G is a function, given some other model of ZFC containing this basis. Such generic functions are the main concern of this thesis.

By case studies and general abstractions of these, we will investigate whether G is a total function or not. We will specifically consider the function spaces $C_f \to \mathbb{N}$, $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, $\mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ and $\mathbb{R} \to \mathbb{R}$. In the cases where the domain of G is σ-compact, G is total. For $X \to \mathbb{R}$ where X is a separable complete metric space, the main result is that G is total if and only if X is σ-compact, given some rather weak additional condition on X. When G is not total, we will explicitly construct $x \in X$ for which G is not defined.

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Contents

Chapter 1

Domain Theory

Given some mathematical structure D, domain theory yields a theoretical foundation for the study of computations in D. This is highly relevant in computer science and for the development of programming languages. The obvious facts that only a finite number of computations can be carried out by a computer in finite time and that input and output always are finite, set the premises for these investigations. However, many of the computations we would like to perform are infinite. Getting an approximation of the true answer is then the best we can hope for. An ordering of these approximations is then needed to indicate how much information they contain, relative to each other.

Consider a computer program that produces decimals of π. We would rather like to get the output $\pi \in [3.14, 3.15]$ than $\pi \in [3, 4]$, since this gives us more information about the true answer of the computation. Let, as in this example, D be the set of closed intervals in $\mathbb{R}$. Every $I \in D$ is then representing a piece of information by approximating every $x \in I$, where shorter intervals corresponds to more information. This interpretation gives rise to a *partial information order*. Such partial orders leads to the notion of a *domain*.

One of the main objectives in domain theory is to model recursive computations, or functions, and thus make way for the functional programming languages. This subject will not be treated here. The origination of domain theory is due to the work of Dana S. Scott in the late 1960's.

This brief introduction to domain theory is mainly based on [3] and [6], and partly on [2] and [5]. [5] is an interesting attempt to achieve a foundation for computations with real numbers, by extending the PCF programming language.

1.1 Modelling Information

Definition 1.1.1. *$(D, \prec, \perp_D)$ is the set D partially ordered by the binary relation $\prec$ with a smallest element $\perp_D$, called* bottom*, that is;*

$$\forall p \in D(\perp_D \prec p). \tag{1.1}$$

D is often called a poset *(partially ordered set).*

Notation. *We will use D as an abbriviation for $(D, \prec, \perp_D)$, and $\perp$ for $\perp_D$ when it is clear from the context which partial order is being used.*

In the above example $\perp$ is naturally interpreted as $\mathbb{R} = (-\infty, \infty)$. Then $\perp \prec I$ for every $I \in D$, where the ordering is given by *reverse inclusion*, i.e., $I_1 \prec I_2$ when $I_2 \subseteq I_1$. This illustrates why $\perp$ can be seen to represent 'no information'. That $\perp$ is an approximation of the real interval I only implies the trivial statement $I \subseteq \mathbb{R}$. In general, this is what (1.1) tells us. The element $\perp$ is used to model the output of nonterminating computations.

If the structure does not contain an element $\perp$, it must be added. This would be the case if we only considered intervals of finite length.

All partial orders, as considered above, are not interesting. Before defining which partial orders aspires to be domains, we will look at two simple but important examples. Let

$$\mathbb{N}_\perp = \mathbb{N} \cup \perp,$$

where $\perp \prec n$ and $n \prec n$ for all $n \in \mathbb{N}$, and

$$\mathbb{B}_\perp = \mathbb{B} \cup \perp,$$

where $\perp \prec \mathbf{T}, \mathbf{F}$ and $\mathbf{T} \prec \mathbf{T}$, $\mathbf{F} \prec \mathbf{F}$. Generally we let $A_\perp = A \cup \perp$, which is called the *lifting* of A.

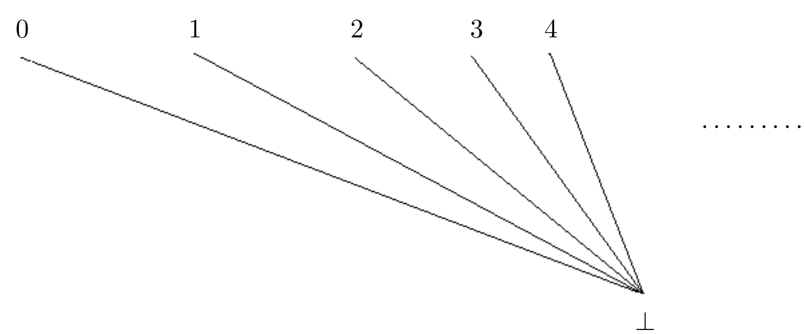
The following definition gives us the opportunity to express that two elements are inconsistent, i.e., that they can not be approximations of the same element.

Definition 1.1.2. *Let D be a poset and $a, b \in D$. $a \perp b$, or a and b are* inconsistent*, if*

$$\neg \exists p \in D (a \prec p \wedge b \prec p).$$

In $\mathbb{N}_\perp$, all m and n such that $m \neq n$ are inconsistent. If a computation gives the answer '7' for input I_1, it should not give the answer '10' for input I_2 when I_1 and I_2 are consistent, because 7 and 10 are inconsistent in $\mathbb{N}_\perp$. In the poset of intervals in $\mathbb{R}$, every I and J such that $I \cap J = \emptyset$ are inconsistent.

$\mathbb{N}_\perp$ and $\mathbb{B}_\perp$ are examples of *flat* domains, which means that all elements are mutually inconsistent, except the bottom element. Such posets can be arranged in a tree structure;



Definition 1.1.3. *Let D be a poset. $A \subseteq D$ is* directed *if for all a and b in A, there exists c in A such that $a \prec c$ and $b \prec c$. If every directed A in D has a least upper bound $\bigsqcup A$, D is* complete*. A complete partial order is called a* cpo*.*

In a cpo, $\emptyset$ is directed and $\bigsqcup\emptyset$ must be smaller than all other elements, so $\bigsqcup\emptyset = \perp$. We included $\perp$ in the partial order from the beginning since we will work exclusively with cpo's. From the discussion above, a directed set A contains only consistent information. If A also has a least upper bound, it is uniquely determined by A, and all elements of A are approximations of $\bigsqcup A$. This suggests that we can identify a directed set A with $\bigsqcup A$ in a cpo. This is important, and will be formalised later.

A directed set is in many aspects like converging sequences in analysis. The elements of the sequence approximates better and better the (possible) limit. Completeness is then a desired property, as we would like such sequences to have a limit. Thus, completeness has a similar meaning in analysis and in domain theory.

1.2 Scott Domains

Definition 1.2.1. *Let D be a cpo and $a, b \in D$. Then a is* way below *b, or $a \ll b$, if for every directed $A \subseteq D$*

$$b \prec \bigsqcup A \Rightarrow \exists c \in A(a \prec c).$$

D is a continuous domain *if the following holds;*

- $\forall b \in D(\{a \mid a \ll b\}$ is directed$)$
- $\forall b \in D(\bigsqcup\{a \mid a \ll b\} = b)$

Note that if $a \ll b$, then $a \prec b$, since $\{b\}$ is trivially directed and $b \prec b = \bigsqcup\{b\}$. Consider again the example where D is the set of closed intervals in $\mathbb{R}$. D is a cpo since $\bigcap\{I \mid I \in A\}$ is nonempty when $A \subseteq D$ is directed, and this will be the least upper bound of A. Then I is way below J if $J \subseteq int(I)$, for assume that I and J have a common endpoint, i.e., $I = [a, c]$ and $J = [a, b]$ where $b < c$. Then the set $A = \{[r, b] \mid r < a\}$ is directed, and $\bigsqcup A = [a, b]$. We have trivially $J \prec J = \bigsqcup A$, but $I \not\prec [r, b]$ for all $[r, b] \in A$. This means that I is not way below J. If however, $J \subset int(I)$, it is clear that for every directed A, there must exist $I' \in A$ such that $J \subset I' \subset I$. Then I is indeed way below J.

In a continuous domain, an element can be characterized by the elements that are way below it. However, sometimes it is sufficient to consider the elements way below belonging to a subset of the domain for such a characterization. Such a subset will be called a basis.

Definition 1.2.2. *Let D be a continuous domain. $B \subseteq D$ is a* basis *for D if*

- $\forall b \in D(\{a \in B \mid a \ll b\}$ is directed$)$
- $\forall b \in D(\bigsqcup\{a \in B \mid a \ll b\} = b)$

D is trivially a basis for itself, but this is of no interest. However, a small basis can give much information about the domain. We will consider those domains that has a basis of compact elements. That an element is compact means that it only contains finite information, in the following sense;

Definition 1.2.3. *Let D be a continuous domain. An element $c \in D$ is* compact *if $c \ll c$.*

Remark 1.2.4. *This means that a compact c can not be a least upper bound of some directed set in which c is not already contained.*

In every cpo $\perp$ is trivially compact, but in the above example there are actually no other compact elements. Assume for contradiction that $[a, b]$ is compact in D. Then we can form the directed set $A = \{[a', b'] \mid a' < a \wedge b < b'\}$. There is no $I \in A$ such that $[a, b] \prec I$, which means $[a, b] \not\ll [a, b]$.

Compactness can be illustrated by letting D be the powerset of $\mathbb{N}$, where $A \prec B$ if $A \subseteq B$. The compact elements of D are the finite subsets of $\mathbb{N}$. If A is infinite, let

$$A' = \{B \mid B \subset_{finite} A\}.$$

Then $A = \bigsqcup A'$, but $A \not\prec B$ for all $B \in A'$. However, if A is finite, A' is directed and $A \prec \bigsqcup A'$, then $A \prec B$ for some $B \in A'$. Otherwise there would be an element $a \in A$ such that $A \setminus \{a\}$ is an upper bound of A', contradicting that $A \prec \bigsqcup A'$.

Compactness leads to the following notions;

Definition 1.2.5. *Let D be a continuous domain. If the compact elements of D, denoted by D_0, forms a basis for D, then D is an* algebraic domain*. If also D_0 is countable, we say that D is* separable *or* ω-algebraic.

There is still one additional property we would like a domain to have, namely that two bounded elements has a least upper bound.

Definition 1.2.6. *Let D be a partial order. If all bounded sets $\{a, b\} \subseteq D$ has a least upper bound, D is* bounded complete*. A* Scott domain *is a separable bounded complete algebraic domain.*

A bounded complete cpo is also often called *consistently complete*. $A \subseteq D$ is *consistent* if it is bounded. This makes sense in the interpretation of D as an information ordering.

Notation. *From this point on we will mean Scott domain when refering to a domain. This should not cause any confusion, since there is no precise definition of a domain.*

Bounded completeness seems like a desirable property for a cpo to possess. And indeed, in the example with closed intervals in $\mathbb{R}$, $I_1 \cap I_2$ is the least upper bound of I_1 and I_2 when they have a nonempty intersection. Note that this cpo is no domain since the compacts, which is only $\perp$, does not form a basis. There is a similar cpo which is not bounded complete, namely the closed discs in $\mathbb{R}^2$ with radius $r \leq \infty$. This is because the intersection of two discs is not a disc. Assume that two discs have an intersection I with a nonempty interieor. Then for every disc d in I we can find another disc in I not contained in d.

Why then use Scott domains, and not only algebraic cpo's? The motivation is in the construction of function spaces.

Definition 1.2.7. *Let X and Y be cpo's. A function $f : X \to Y$ are* continuous *if*

- *f is* monotonic*, i.e.,*
$$f(a) \prec f(b) \text{ whenever } a \prec b.$$

- *f preserves least upper bounds of directed sets;*
$$f(\bigsqcup A) = \bigsqcup f[A]$$
for all directed $A \subseteq X$.

The set of continuous functions from X into Y is denoted by $X \to Y$.

In trying to find an adequate model of computation, restricting our attention to continuous functions is necessary. By giving more information as input, more information about the output should be obtained. Also, a computation should produce consistent output from consistent input. Otherwise a function would not model "real" computations according to our intuition.

The set of continuous functions $X \to Y$ forms a cpo with pointwise ordering. Algebraic cpo's are much more useful than mere cpo's due to the basis of compacts, but $X \to Y$ is in general not an algebraic cpo when X and Y are algebraic cpo's. The problem is that given $f \in (X \to Y)$, the set $\{g \mid g \prec f$ and g is compact$\}$ is generally not directed. However, for Scott domains we have

Theorem 1.2.8. *Let X and Y be Scott domains, then $X \to Y$ is a Scott domain.*

Proof. see [3] or [2]. Essential in the proof is the fact that the following functions are compacts in $A \to B$,

$$f_{\langle p,q\rangle}(x) = \begin{cases} q & \text{if } p \prec x \\ \perp & \text{otherwise} \end{cases}$$

where p and q are compacts in A and B respectively. □

1.3 Ideal Completion

Now we will see how a domain D can be identified with certain subsets of D_0. These subsets are the ideals, and this fact will be of great importance later on.

Definition 1.3.1. *Let D be a partial order with a least element $\perp$. $A \subset D$ is an* ideal *if*

- *$\perp \in A$.*
- *if $a \in A$ and $b \prec a$, then $b \in A$.*
- *if $a, b \in A$ there exists $c \in A$ such that $a, b \prec c$.*

Theorem 1.3.2. *If D is an algebraic cpo, then D and $Id(D_0)$, the ideals in D_0, are isomorphic.*

Proof. This follows easily from D_0 being a basis for D. The canonical embeddings between the two spaces are

$$x \mapsto \{c \in D_0 \mid c \prec x\}$$

and

$$A \mapsto \bigsqcup A.$$

□

From Remark 1.2.4 it follows that the elements of D can be divided into the 'finite' or compacts elements and the rest, the 'infinite' or *total* elements. A compact need not be finite in any literal way, but has properties resembling finiteness. Theorem 1.3.2 tells us that D_0 is the building blocks of D, so that we can work with a domain by only considering D_0 and let D consist of limits of compacts.

Chapter 2

Forcing

Forcing is a technique developed by Paul Cohen in the 1960's to achieve independence and consistency proofs within set theory. The main idea is to consider a countable, transistive model M of ZFC, Zermelo-Fraenkel set theory with the Axiom of Choice. Then, by picking some $G \subset M$ such that $G \notin M$, then M can be extended to the least model $M[G]$ of ZFC to contain G as an element. If G is carefully chosen, $M[G]$ can be shown to have interesting properties. The description of forcing in this chapter is mainly based on Kunen [4].

ZFC is generally accepted as the most natural axiomatic system to let mathematics take place witin. It is formulated in the language of *set theory*, consisting of the binary relations $\in$ and $=$. Relativization and absoluteness is a part of the logical framework for this thesis and will be used both explicitely and implicitely. The formula ϕ *relativized* to the set M, ϕ^M, are ϕ where the existence quantifiers range over M. Then ϕ is *absolute* for M_1,M_2 if

$$\phi^{M_1} \Leftrightarrow \phi^{M_2}.$$

Throughout this chapter we must have in mind that our main objective is to apply forcing to domain theory. So the poset regarded here can be considered as the poset of Chapter 1, namely the compact elements of a domain. This poset is exactly what links domain theory and forcing together.

2.1 Generic Filters and $\mathbb{P}$-Names

Let M be a countable transitive model, abbriviated c.t.m., of ZFC and let $(\mathbb{P}, \leq, \mathbf{1})$ be a partial order with greatest element $\mathbf{1}$, such that $(\mathbb{P}, \leq, \mathbf{1}) \in M$. Note that such a partial order is essentially identical to those considered in Chapter 1, except that the ordering is reverse. So when D has a smallest element $\perp$, $\mathbb{P}$ has greatest element $\mathbf{1}$. We will use the relation symbol $\prec$ for posets with a smallest element, and $\leq$ for posets with a greatest element, following standard notation.

2.1.1 Generic Filters

The notion of an *ideal* from domain theory is equivalent to that of a *filter* in forcing.

Definition 2.1.1. *$A \subseteq \mathbb{P}$ is a* filter *if:*

- $\mathbf{1} \in A$.

- $\forall p \in A \forall q \in \mathbb{P}(p \leq q \Rightarrow q \in A)$.

- $\forall p, q \in A \exists r \in A(r \leq p \wedge r \leq q)$.

Note that $\{\mathbf{1}\}$ is trivially a filter in $\mathbb{P}$. We will be interested in a certain type of filters, that in some sense are rather large.

Definition 2.1.2. *$A \subseteq \mathbb{P}$ is* dense *if*

$$\forall p \in \mathbb{P} \ \exists q \in A(q \leq p).$$

A filter $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M *if for all dense $D \subseteq \mathbb{P}$, $D \in M \Rightarrow G \cap D \neq \emptyset$.*

By this defintion, a generic filter should contain very much information. Just exactly how much, in some special cases, is the main concern of this thesis. Generic filters are not rare, and indeed we have;

Lemma 2.1.3. *For all $p \in \mathbb{P}$ there exists a generic G such that $p \in G$.*

Proof. The idea is that since M is countable, so is the dense subsets of $\mathbb{P}$ in M. Let these subsets be $\{D_n\}_{n \in \mathbb{N}}$. Then, choose inductively $p_i \in D_i$ such that

$$p = p_0 \geq p_1 \geq p_2 \geq \dots$$

This can be done since each D_i is dense. Then let G be the filter generated by $\{p_i\}_{i \in \mathbb{N}}$. From the construction, G is generic. □

The following lemma will be useful in later chapters.

Lemma 2.1.4. *Let G be $\mathbb{P}$-generic, $p \in G$, $A \subset \mathbb{P}$, $A \in M$ and A* dense below *p, i.e.,*

$$\forall q \leq p \exists r \leq q(r \in E).$$

Then $G \cap A \neq \emptyset$.

Proof. see [4]. □

2.1.2 The Construction of M[G]

The ZFC model $M[G]$ will be a hereditary construction in M from the elements of G. This follows from a definition by transfinite recursion;

Definition 2.1.5. *Let τ be a* $\mathbb{P}$-name *if τ is a relation and*

$$\forall \langle \sigma, p \rangle \in \tau(\sigma \text{ is a } \mathbb{P}\text{-name } \wedge p \in \mathbb{P}).$$

This means that $V^{\mathbb{P}}$, the collection of $\mathbb{P}$-names, is to big to be a proper set, but is just a class. We will only use the $\mathbb{P}$-names definable in M, denoted by $M^{\mathbb{P}}$. Then we can use the elements of $M^{\mathbb{P}}$ and G to define $M[G]$.

Definition 2.1.6. *Let τ be a $\mathbb{P}$-name, then*

$$val(\tau, G) = \{val(\sigma, G) \mid \exists p \in G(\langle \sigma, p \rangle \in \tau)\}.$$

τ_G is used as a shorthand for $val(\tau, G)$. Define

$$M[G] = \{\tau_G \mid \tau \in M^{\mathbb{P}}\}.$$

It is clear that τ_G is not definable in M if $G \notin M$, since the definition of τ_G depends on G. This is the case we will be interested in. The following lemma illustrates why.

Lemma 2.1.7. *If $\mathbb{P}$ satisfies*

$$\forall p \in \mathbb{P} \exists q, r \in \mathbb{P}(r \leq p \wedge q \leq p \wedge q \perp r), \tag{2.1}$$

then $G \notin M$ for every generic G.

Proof. see [4]. □

Most of the interesting $\mathbb{P}$'s satisfy (2.1), and if (2.1) does not hold there exists a generic $G \in M$. This would make the use of forcing pointless. If $\mathbb{P} = D_0$ for some domain D as in Chapter 1, (2.1) should indeed hold. This means that for all approximations p there exists two inconsistent approximations extending p. Otherwise p could not approximate more than possibly one infinite element of D(that is, of $D \setminus D_0$), which really does not make p an approximation at all.

We can make two simple but important observations about $M[G]$. First, $M \subset M[G]$. To show this, we must find a suitable $\mathbb{P}$-name for each $x \in M$. Let

$$\hat{x} = \{\langle \hat{y}, \mathbf{1} \rangle \mid y \in x\}.$$

Then $\hat{x}_G = x$. The point is that we know that $\mathbf{1} \in G$ for every generic G, making $\hat{x}$ a copy of x. Secondly, $G \in M[G]$. Define the $\mathbb{P}$-name

$$\Gamma = \{\langle \hat{p}, p \rangle \mid p \in \mathbb{P}\}.$$

Then

$$\Gamma_G = \{\hat{p}_G \mid p \in G\} = \{p \mid p \in G\} = G.$$

To produce forcing results of any real value, it is important that $M[G]$ is a model of ZFC, or some subset thereof;

Theorem 2.1.8. *Let M and $\mathbb{P}$ be as above. If G is $\mathbb{P}$-generic over M, then $M[G]$ satisfies ZFC.*

This is proved in detail in [4]. Proving this comes essentially down to constructing names for the various sets assured to exists by the axioms of ZFC.

Now we can move on to the basic properties of forcing.

2.2 Forcing

Having constructed $M[G]$, one might be interested in the truth or falsity of sentences in $M[G]$. In M this is tricky business, since $\phi^{M[G]}$ depends on G, which M contains no information about. However, there are possible within M to obtain some relative information about $\phi^{M[G]}$. If p is assumed to be contained in G, there is a chance that ϕ must be true in $M[G]$. If this is the case, we say that *p forces ϕ*, or $p \Vdash \phi$. This can be stated precisely as

Definition 2.2.1. *Let M and $\mathbb{P}$ be as above, let $\phi(x_1, \ldots, x_n)$ be a formula and let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. If $p \in \mathbb{P}$, then $p \Vdash_{\mathbb{P},M} (\phi(\tau_1, \ldots, \tau_n))$ if*

$$\forall G[(G \text{ is } \mathbb{P}\text{-generic over } M \wedge p \in G) \Rightarrow \phi^{M[G]}(\tau_{1G}, \ldots, \tau_{nG})].$$

We will omit the subscript $\mathbb{P}, M$ in $\Vdash$ when it is clear from the context which $\mathbb{P}$ and M are under consideration The poset $\mathbb{P}$ is called the *forcing relation*. The following lemma follows from this definition.

Lemma 2.2.2.

1. $(p \Vdash \phi(\tau_1, \ldots, \tau_n) \wedge q \leq p) \rightarrow q \Vdash \phi(\tau_1, \ldots, \tau_n)$.
2. $(p \Vdash \phi(\tau_1, \ldots, \tau_n) \wedge p \Vdash \psi(\tau_1, \ldots, \tau_n)) \Leftrightarrow p \Vdash (\phi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n))$.

Since we have a quantifier over all generic G, $\Vdash$ is not defined in M, but there is possible to define a relation $\Vdash^*$, which relativized to M is equivalent to $\Vdash$. A lengthy and technical argument, depending heavily on the fact that we consider only generic G's, gives the following fundamental theorem.

Theorem 2.2.3. *Let M and $\mathbb{P}$ be as above, let $\phi(x_1, \ldots, x_n)$ be a formula and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Then for all $p \in \mathbb{P}$,*

$$p \Vdash \phi(\tau_1, \ldots, \tau_n) \Leftrightarrow (p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M. \tag{2.2}$$

For every $\mathbb{P}$-generic G,

$$\phi^{M[G]}(\tau_{1G}, \ldots, \tau_{nG}) \Leftrightarrow \exists p \in G(p \Vdash \phi(\tau_1, \ldots, \tau_n)). \tag{2.3}$$

For a proof, see [4]. The equivalence (2.2) tells us that it can be decided within M whether a certain p forces ϕ in $M[G]$ or not. Equivalence (2.3) tells us that for every true ϕ in $M[G]$, there exists some p in G which forces ϕ.

A large number of *relative consistency results* can be derived using forcing. That is, statments like

$$\text{Con(ZFC)} \rightarrow \text{Con(ZFC + A)} \tag{2.4}$$

where A is some property not derivable from ZFC(otherwise (2.4) is trivial). The most famous example of such a result is that CH is independent of ZFC. CH is the *Continuum Hypothesis*, stating that there is no cardinal between that of the natural numbers and of the real numbers, i.e., $\omega_1 = 2^\omega$. For this *independence proof*, it must be shown that both CH and ¬CH are consistent with ZFC, that is

$$\text{Con(ZFC)} \rightarrow \text{Con(ZFC + CH)} \tag{2.5}$$

and

$$\text{Con(ZFC)} \rightarrow \text{Con(ZFC} + \neg\text{CH)}. \tag{2.6}$$

The implication (2.6) is proved using forcing. From the assumption that we have a model M of ZFC, we can construct a new model $M[G]$ of ZFC as above. By choosing $\mathbb{P}$ with care, it can be proved that ¬CH holds in $M[G]$. This shows why our *forcing language* is important, and we are interested in what can be stated about $M[G]$ *in* M. Implication (2.5) is proved by showing that CH holds in L, the universe of constructible sets.

Chapter 3

Domain Theoretical Constructions

3.1 The Algebraic Domain of Ideals

Let $(D, \prec, \perp)$ be a poset, and let F_D be the set of ideals in D ordered by inclusion. We will show that $(F_D, \subseteq)$ is an algebraic domain with a smallest element. First we need a characterization of the compact elements of F_D. We will use F instead of F_D when it is clear which poset is under consideration.

Proposition 3.1.1. *F_0, the compacts in F, is the ideals generated by one element of D, that is*

$$F_0 = \{f_\alpha \mid \alpha \in D\}$$

where f_α is the least ideal in D containing α.

To prove this proposition, the following is useful;

Observation 3.1.2. *When $R \subseteq F$ is directed, $\sqcup_F R = \cup R$.*

Proof of Observation 3.1.2. We must show that $\cup R$ is an ideal. Let $x \in \cup R$, that means there exists $r \in R$ such that $x \in r$. If for some $y \in D$ $y \prec x$ then $y \in r$ since r is an ideal. This implies that we also have $y \in \cup R$. Then let $x, y \in \cup R$, so $x \in r_1$ and $y \in r_2$ for some $r_1, r_2 \in R$. Then, because R is directed, there exists $r \in R$ such that $r_1 \subseteq r$ and $r_2 \subseteq r$. Then $x, y \in r$, and we can find $z \in r$ such that $z \prec x$ and $z \prec y$. Since $z \in \cup R$, $\cup R$ is an ideal. $\cup R$ is obviously the smallest ideal containing R.

□

Proof of Proposition 3.1.1. We have to show that

$$f \in F_0 \Leftrightarrow \exists \alpha \in D(f = f_\alpha). \tag{3.1}$$

For the "$\Rightarrow$"-direction of (3.1), it is enough to show that $f \neq f_\alpha \Rightarrow f \notin F_0$ for all $\alpha \in D$. So let $f \in F$ be given such that $f \neq f_\alpha$ for all $\alpha \in D$. Let $R = \{f_\beta \mid \beta \in f\}$, then R is directed and $f = \cup R = \sqcup R$ follows from Observation 3.1.2. But $f \not\subseteq r$ for all $r \in R$, so from the definition of compactness, f is not compact. For the "$\Leftarrow$"-direction of (3.1), assume $f = f_\alpha$ and let R be a directed set such that $f \subseteq \sqcup R$. Then, since R is directed, $\sqcup R = \cup R$. Hence

$f \subseteq \cup R$

$\Rightarrow \alpha \in \cup R$

$\Rightarrow \exists r \in R(\alpha \in r)$

$\Rightarrow \exists r \in R(f \subseteq r)$

$\Rightarrow f \in F_0.$

This completes the proof.

□

Lemma 3.1.3. *Let $(D, \prec, \perp)$ be a poset, then F_D is an algebraic domain with a smallest element ordered by inclusion.*

Proof. First, $\{\perp_D\}$ is trivially an ideal, and included in all other ideals, so $\perp_F = \{\perp_D\}$. F is algebraic if F_0 is a basis for F , so the following two proporties of F_0 must be verified:

1) $\forall f \in F(\{y \in F_0 \mid y \ll f\}$ is directed)

2) $\forall f \in F\ (f = \bigsqcup\{y \in F_0 \mid y \ll f\})$

For 1), let $f \in F$ and $y_1, y_2 \in F_0$ such that $y_1, y_2 \ll f$. Then there is α and β in D such that $y_1 = f_\alpha$ and $y_2 = f_\beta$. Since $y_1, y_2 \subseteq f$, both α and β are in f. Hence, since f is an ideal, there exists $\gamma \in f$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$. We need to show that $f_\gamma \ll f$, which will imply 1). We have;

$$f_\alpha, f_\beta \subseteq f_\gamma \subseteq f.$$

Then $f_\gamma \ll f$ follows from the fact that generally, $f_1 \subseteq f_2 \Rightarrow f_1 \ll f_2$ when $f_1 \in F_0$. To see this, let R be directed and let $f_2 \subseteq \sqcup R$. Then

$$f_1 = f_\alpha \subseteq f_2 \subseteq \sqcup R = \cup R,$$

which means there is $r \in R$ such that $\alpha \in r$, and hence $f_\alpha \subseteq r$.

For 2) we have

$\sqcup\{y \in F_0 \mid y \ll f\}$

$= \cup\{\ y \in F_0 \mid y \ll f\}$

$= \cup\{\ f_\alpha \mid \alpha \in f\}$

$= f.$

The first equality follows from 1) and Observation 3.1.2. 1) and 2) shows that $(F, \subseteq)$ is an algebraic domain. □

3.2 An Extension of $\mathbb{P}$

Let $(\mathbb{P}, \leq, \mathbf{1})$ be a poset. We will show that $\mathbb{P}$ can be extended to a partial order which is bounded complete without changing it in an essential way for our purposes, that is, so it preserves generic filters. We use $\mathbb{P}$ and the reverse order $\leq$ since this section is not purely domain theoretical, but related to forcing via the generic filters. First, we prove the following;

Proposition 3.2.1. *Every generic filter is maximal.*

Proof. Let G be a generic filter. Assume for contradiction that G can be extended with p, that is, that $G \cup \{p\}$ is a filter. Let

$$E = \{q \in \mathbb{P} \mid q \leq p \vee q \perp p\} \setminus \{p\}$$

Then E is dense below p. This follows from the construction of E as long as E is nonempty. If E was empty, every $q \in \mathbb{P}$ would have been consistent with p, i.e., $p, q \leq r$ for some $r \in \mathbb{P}$. If $r = p$ for every q, $\{p\}$ would be dense and hence be an element of G since G is generic, so we have a contradiction. If on the other hand $r \neq p$, we would have $r \in E$, which contradicts that E is empty. So E is nonempty. Thus there exists some x in $G \cap E$ because G is generic. Since $x \in E$ either $x \leq p$, which contradicts that $p \notin G$, or $x \perp p$, which contradicts that $G \cup \{p\}$ is a filter. $\square$

Definition 3.2.2. *Let $\mathbb{P}^+$ be the set consisting of all finite, bounded subsets $A \subseteq \mathbb{P}$ and let $A \leq B$ if*

$$\forall p \in B \ \exists q \in A \ (q \leq p). \tag{3.2}$$

We will use the relation symbol $\leq$ for both $\mathbb{P}$ and $\mathbb{P}^+$, and try to make sure it is always clear which poset is under consideration.

Lemma 3.2.3. *Given a poset $\mathbb{P}$, then $\mathbb{P}$ can be extended to a bounded complete poset $\mathbb{P}^+$ such that there is a canonical one-to-one relation between the generic filters in $\mathbb{P}$ and $\mathbb{P}^+$.*

Proof. Let $\mathbb{P}^+$ be as in Definition 3.2.2. If G is $\mathbb{P}$-generic, let

$$G^+ = \{A \in \mathbb{P}^+ \mid A \subseteq G\}.$$

We will show that G^+ is $\mathbb{P}^+$-generic. First of all, G^+ is a filter; let $A \in G^+$ and $B \in \mathbb{P}^+$ such that $A \leq B$. Since $A \subseteq G$, (3.2) implies that every $p \in B$ also is an element of G. Hence $B \subseteq G$, and $B \in G^+$. Then let $A, B \in G^+$. Since both A and B are finite and G is a filter, there exists $c \in G$ such that $\forall a \in (A \cup B)(c \leq a)$. Letting $C = \{c\} \in \mathbb{P}^+$, we have $C \leq B$ and $C \leq A$. Hence G^+ is a filter.

Next we have to show that $G^+ \cap \Delta^+ \neq \emptyset$ when $\Delta^+ \subseteq \mathbb{P}^+$ is dense. Let

$$\Delta = \{x \in \mathbb{P} \mid \exists X \in \Delta^+(X \text{ bounded by } x)\}.$$

Then Δ is dense in $\mathbb{P}$; let $p \in \mathbb{P}$, then $\{p\} \in \mathbb{P}^+$. Since Δ^+ is dense, there exists $A \in \Delta^+$ such that $A \leq \{p\}$ and A is bounded by some $x \in \mathbb{P}$ from the definition of $\mathbb{P}^+$. So $x \leq p$ and $x \in \Delta$, since x is a bound of a set in Δ^+. Hence Δ is dense. Then there exists $x \in (\Delta \cap G)$, so x is an upper bound of some set $B \in \Delta^+$. Now

$$\forall y \in B(y \in G)$$

because G is a filter and $x \in G$. So $B \in G^+$, and thus

$$B \in (G^+ \cap \Delta^+).$$

This means that G^+ is $\mathbb{P}^+$-generic.

So a generic filter in $\mathbb{P}$ gives us a generic filter in $\mathbb{P}^+$. Now we want to show that generic sets in $\mathbb{P}^+$ remains generic in $\mathbb{P}$. Let $H \subseteq \mathbb{P}^+$ be $\mathbb{P}^+$-generic. Then

$$H^- = \bigcup\{A \mid A \in H\}$$

will be $\mathbb{P}$-generic. First, H^- is a filter. Let $p \in H^-$ and $q \in \mathbb{P}$ such that $p \leq q$. Then $\{q\} \in \mathbb{P}^+$, and $A \leq \{q\}$ for the $A \in H$ such that $p \in A$. Since H is a filter, $\{q\} \in H$, and from the definition of H^- it follows that $q \in H^-$.

To show the second condition for H^- to be a filter, that two elements of H^- has a common bound, it is surprisingly not enough to use that H is a filter. Given two elements of H^-, a bound can be found, but this may not be in H^-. So let $p, q \in H^-$, we need to find $r \in H^-$ such that $r \leq p$ and $r \leq q$. Let $A, B \in H$ be such that $p \in A$ and $q \in B$. Since H is a filter, we can find $Z \in H$ such that $Z \leq A$ and $Z \leq B$. Let z be a bound for Z, and let

$$\Delta^+ = \{A' \in \mathbb{P}^+ \mid \exists r' \in A'(r' \leq p \wedge r' \leq q)\}.$$

If we can show that $\Delta^+ \subseteq \mathbb{P}^+$ is dense below Z, then $H \cap \Delta^+ \neq \emptyset$ from Lemma 2.1.4, and from the definition of Δ^+ we will have r as required. So let $E \in \mathbb{P}^+$ and assume that $E \leq Z$. E is bounded by some $e \in \mathbb{P}$, and $e \leq z \leq p, q$. This means that $\{e\} \in \Delta^+$, and $\{e\} \leq E$ implies that Δ^+ is dense below Z.

Finally, we must show that H^- is $\mathbb{P}$-generic, so let Δ be dense in $\mathbb{P}$. Define now

$$\Delta^+ = \{\{x\} \mid x \in \Delta.\}$$

Then Δ^+ is dense in $\mathbb{P}^+$; let $E \in \mathbb{P}^+$, where E is bounded by $e \in \mathbb{P}$. Since Δ is dense, there exists $x \in \Delta$ such that $x \leq e$, and we will have $\{x\} \leq \{e\} \leq E$. Then Δ^+ is dense since $\{x\} \in \Delta^+$. Since H is $\mathbb{P}^+$-generic, there exists some $\{x\} \in \Delta^+ \cap H$, which implies that $x \in \Delta \cap H^-$.

Having constructed $\mathbb{P}^+$ and showed that it preserves generic sets, we must make sure that it is bounded complete. Let $A, B \subseteq \mathbb{P}^+$ be bounded by Z, which is bounded in $\mathbb{P}$ by z. Then $C = A \cup B$ is finite and bounded by z, so $C \in \mathbb{P}^+$. Both A and B are obviously bounded by C. To prove minimality of C, let C' be some other bound for A and B. Every $c \in C$ is an element of either A or B, so anyway there exists $c' \in C'$ such that $c' \leq c$. Hence

$$C' \leq \{A, B\} \Rightarrow C' \leq C,$$

and consequently $\mathbb{P}^+$ is bounded complete. □

This result indicates that we should be able to assume that the domain under consideration is bounded complete, and thus a Scott domain, when using forcing.

3.3 Domain Representation

As discussed in Chapter 1, we will be interested in finding domains in which a metric space X can be interpreted as a subset. That is, we will look for a domain D and a surjective mapping

$$f : D \to X.$$

In short, such pairs (D, f) will be called a *domain representation* of X. For a more formal definition, see [7]. The metric spaces $\mathbb{R}$ and $\mathbb{N}^{\mathbb{N}}$ will be in frequent use in this thesis. We will now introduce domain representations for both spaces, before we in section 3.3.3 consider a general method for obtaining domain representations for arbitrary separable metric spaces. We will also briefly take a look at compact elements in domains representing function spaces, as these will be of special interest later on.

3.3.1 Representation of $\mathbb{N}^{\mathbb{N}}$

The space $\mathbb{N}^{\mathbb{N}}$ is the set of functions from $\mathbb{N}$ into $\mathbb{N}$, or equivalently, the set of infinte countable lists of natural numbers;

$$\mathbb{N}^{\mathbb{N}} = \mathbb{N} \to \mathbb{N} = \{\{a_i\}_{i\in\mathbb{N}} \mid a_i \in \mathbb{N}\}.$$

Geometrically, $\mathbb{N}^{\mathbb{N}}$ can be seen as a tree with infinite countable branching, where each infinite branch corresponds to an element of $\mathbb{N}^{\mathbb{N}}$. There is then one node in the tree for each $f(n)$ where $f \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$. We will use the following metric on $\mathbb{N}^{\mathbb{N}}$;

$$d(f_1, f_2) = \begin{cases} 0 & \text{if } f_1 = f_2 \\ 2^{-\mu k.(f_1(k) \neq f_2(k))} & \text{otherwise.} \end{cases}$$

This means that $B(\epsilon, f)$, the open balls of radius ϵ centered at f, has a clear interpretation as the full subtree above $f(n)$, where n is maximal such that $2^{-n} > \epsilon$. The compact subsets of $\mathbb{N}^{\mathbb{N}}$ is given by

Proposition 3.3.1. *$S \subset \mathbb{N}^{\mathbb{N}}$ is compact if and only if $S \subseteq C_f$ for some $f \in \mathbb{N}^{\mathbb{N}}$, where*

$$C_f = \{f' \mid \forall n f'(n) \leq f(n)\}.$$

Now we need a domain representation of $\mathbb{N}^{\mathbb{N}}$. $\mathbb{N}_\perp$ is trivially a domain, where all elements are compact. Then

$$D = (\mathbb{N}_\perp \to \mathbb{N}_\perp) = \mathbb{N}_\perp^{\mathbb{N}_\perp}$$

is a domain according to Theorem 1.2.8, and D will consist of all total and partial functions $\mathbb{N}_\perp \to \mathbb{N}$. The functions f such that $f(n) = \perp$ for some n are strictly speaking total, but we will say that they are partial, since the natural interpretation of $\perp$ is 'undefined'. The induced ordering in this domain is;

$$f_1 \prec f_2 \Leftrightarrow f_1(x) \prec f_2(x) \text{ for all } x \in \mathbb{N}_\perp.$$

Some of the partial functions will be finite branches in the tree structure, i.e., $f(n) = \perp$ when $n > k$, and $f(n) \neq \perp$ when $n \leq k$ for some k. Denote the set of such partial functions by $\mathcal{D}$. Each $f \in \mathcal{D}$ approximates exactly the subtree above the node $f(k)$. All other partial functions $\mathbb{N}_\perp \to \mathbb{N}_\perp$ has 'holes', and have no natural interpretation in the tree structure. The elements of $\mathcal{D}$ will be treated as finite sequences of natural numbers.

Definition 3.3.2. *For $f \in \mathcal{D}$ let length(f) be the unique m such that*

$$f(n) \neq \perp \text{ for all } n < m$$

$$f(n) = \perp \text{ for all } n \geq m.$$

We will use concatenation of sequences the usual way, that is

$$f * \langle i \rangle = \langle f(0), f(1), \ldots, f(n-1), i \rangle.$$

It is easy to verify the following

Proposition 3.3.3. *f is a compact element of D if and only if $f(n) \in \mathbb{N}$ for only finitely many $n \in \mathbb{N}$.*

In a basis for $\mathbb{N}_\perp \to \mathbb{N}_\perp$ we will need all the compacts. However, we are only interested in the total functions $\mathbb{N} \to \mathbb{N}$. To represent these functions, it is sufficient to consider ideals in $\mathcal{D}$, which we will do in section 4.2.1. The elements of $D_0 \setminus \mathcal{D}$ are only needed to approximate the nontotal functions $\mathbb{N} \to \mathbb{N}$.

3.3.2 Representation of $\mathbb{R}$

The real numbers $\mathbb{R}$ will typically be the codomain of the function spaces under consideration. Hence, we will need a representation of $\mathbb{R}$ as a domain.

Lemma 3.3.4. *Let*

$$\mathcal{R} = \{[a, b] \mid a \leq b \wedge a, b \in \mathbb{Q}\} \cup \{(-\infty, \infty)\}$$

and

$$\Re = Id(\mathcal{R}).$$

Then $\perp = (-\infty, \infty)$ and let $\prec$ be the partial order by reverse inclusion, that is

$$I \prec J \Leftrightarrow J \subseteq I \text{ for } I, J \in \mathcal{R}.$$

Then $(Id(\mathcal{R}), \prec, \{\perp\})$ is a domain, and $\mathbb{R}$ can be identified with the set

$$\bar{\Re} = \{A \in Id(\mathcal{R}) \mid \bigsqcup A \in \mathbb{R}\}.$$

Proof. First of all, $(\mathcal{R}, \prec, \perp)$ is obviously a partial order, and $\perp \prec I$ for all $I \in \mathcal{R}$. That $\Re$ is an algebraic domain follows then from Lemma 3.1.3. From this construction, the compacts are the ideals with proper rational intervals as least upper bounds, containing this interval. This set constitutes a countable base for $\Re$, and hence $\Re$ is separable. Since closed rational intervals are closed under intersection, $\Re$ is bounded complete and is thus a Scott domain. ☐

This representation is not completely satisfactory in the sense that an element $q \in \mathbb{Q}$ is not represented by an unique ideal. The element q may, or may not, be an endpoint of the intervals in the ideal. This means that there are tree ideals representing q:

$$A_1 = \{[a, b] \mid a < q \wedge q < b\}$$

$$A_2 = \{[a, b] \mid a \leq q \wedge q < b\}$$

$$A_3 = \{[a, b] \mid a < q \wedge q \leq b\}$$

This ambiguity is an important nuance as $[a, b] \in \mathcal{R}$ can be represented by both of the following ideals;

$$A_1 = \{[a', b'] \mid a' < a \wedge b < b'\}$$

$$A_2 = \{[a', b'] \mid a' \leq a \wedge b \leq b'\}$$

Only A_2 is compact in $Id(\mathcal{R})$ according to Proposition 3.1.1. As explained after Definition 1.2.3, only $\perp$ is compact in the partial order $\mathcal{R}$. However, since we have no interest in representing proper rational intervals, this is no problem. We will work with the elements of $\mathcal{R}$ as compacts approximating real numbers. For a more thoroughly treatment of this subject, see [6].

3.3.3 Representation of Metric Spaces

In the general case in this thesis we will use separable metric spaces (X, d), where *separable* means that there exists a dense countable subset of X denoted by $\{a_i\}_{i\in\mathbb{N}}$. In this section, a domain representation of X will be introduced, which is based on [7].

Definition 3.3.5. *A family F of closed nonempty subsets of X, including X, is a* closed neighbourhood system *if*

1) $A, B \in F$ and $A \cap B \neq \emptyset$ implies $A \cap B \in F$.

2) $x \in U$ for an open set U implies $\exists A \in F(x \in int(A) \wedge A \subseteq U)$.

It is clear that $\mathcal{R}$ in the previous section is a closed neighbourhood system of $\mathbb{R}$. Let F be a closed neighbourhood system of X, then F is a partial order by using reverse inclusion. Then the least element $\perp$ of F is X. F is bounded complete since if $\{A, B\} \subseteq F$ is consistent, then $A \cap B \in F$ is the least upper bound of $\{A, B\}$. Thus, $D = (Id(F), \subseteq)$ is, according to Lemma 3.1.3, an algebraic domain which is also bounded complete. However, D will represent the elements of X and the elements of the closed neighbourhood system, which we will interpret as approximations of elements of X. Let $A \in D$ be a *converging ideal* if A contains sets of arbitrarily small diameter. Denote the set of converging ideals in D by $\bar{D}$. Then every element of $\bar{D}$ will uniquely determine an element of X, so;

Theorem 3.3.6. *Every metric space (X, d) is domain representable.*

The set of all closed subsets of X is clearly a closed neighbourhood system. However, this neglects the fact that we are interested i Scott domains, which also must be separable, and that we consider separable metric spaces. Use the dense subset $\{a_i\}_{i\in\mathbb{N}}$ and define

$$B_{n,q} = \{x \in X \mid d(a_n, x) \leq q\} \text{ for } q \in \mathbb{Q} \text{ and } q > 0.$$

Then we can let $\mathcal{F}$ be the family of finite intersections of such $B_{n,q}$-sets together with X. Now $\mathcal{F}$ satisfies *1)* of Definition 3.3.5 since we consider finite intersections and *2)* is satisfied since $\{a_i\}_{i\in\mathbb{N}}$ is dense. So $\mathcal{F}$ is a closed neighbourhood system and hence $(Id(\mathcal{F}), \subseteq)$ is a bounded complete algebraic domain which also is separable, since $\mathcal{F}$ is countable.

Lemma 3.3.7. *Every separable metric space (X, d) can be represented by a Scott domain.*

We will regard function spaces $A \to B$ where we already have domain representations $\mathcal{A}$ and $\mathcal{B}$ of A and B respectively. According to Theorem 1.2.8, $\mathcal{A} \to \mathcal{B}$ is a domain, which will represent $A \to B$. A basis for $\mathcal{A} \to \mathcal{B}$ is formed by the compacts

$$p = \{\langle I_1, J_1\rangle, \ldots, \langle I_n, J_n\rangle\},$$

where I_i and J_i are $B_{n,q}$-sets of A and B respectively, and

$$I_i \cap I_j \neq \emptyset \Rightarrow J_i \cap J_j \neq \emptyset \tag{3.3}$$

for all $i \leq n$.

Implication (3.3) tells us that p is consistent and thus approximates actual functions. Let $\mathbb{P}$ be the set of these *compact functions* with the following ordering; $p \prec q$ if

$$\forall i \leq n \exists j \leq m(I_i \subseteq I'_j \wedge J'_j \subseteq J_i).$$

when

$$p = \{\langle I_1, J_1\rangle, \ldots, \langle I_n, J_n\rangle\} \text{ and } q = \{\langle I'_1, J'_1\rangle, \ldots, \langle I'_m, J'_m\rangle\}.$$

Given $f : A \to B$, f is approximated by $p \in \mathbb{P}$ if

$$f[I_i] \sqsubseteq J_i \text{ for } i \leq n.$$

Notation. *Even though the elements of* $\mathbb{P}$ *are not functions, we will say that* p is defined for $A' \subseteq A$ *if*

$$\forall x \in A' \exists \langle I_i, J_i\rangle \in p(x \in I_i).$$

Chapter 4

Generic Functions

When the domain D under consideration is a function space, Theorem 1.3.2 (*ideal completion*) tells us that a function is assosiated with a filter of compacts in the domain. Hence, a generic filter G in $\mathbb{P} = D_0$ is interpreted as a *generic function*. If M is a c.t.m. with $(\mathbb{P}, \leq, \mathbf{1}) \in M$, G will be a generic function in $M[G]$, see Chapter 2. These functions will be our main concern, and in this and the next chapter we will investigate under which circumstances they are total.

4.1 A Function Defined From σ_G

Let M be a c.t.m., let $(\mathbb{P}, \leq, \mathbf{1}) \in M$, G be $\mathbb{P}$-generic over M and $D \in M$ be a domain. $F_{\mathbb{P}}$ is the filters in $\mathbb{P}$ ordered by inclusion. $F_{\mathbb{P}}$ will, as before, be denoted by F. Note that $\mathbb{P}$ is ordered by $\leq$ and D by $\prec$. We then have the following;

Lemma 4.1.1. *Given $x \in D^{M[G]}$, there exists a continuous function $f : F \to D$ in M such that $f(G) = x$.*

Proof. Let σ be a $\mathbb{P}$-name for $\{y \in D_0 \mid y \prec x\}$ and let $p \in G$ be such that

$$p \Vdash (\sigma \text{ is an ideal in } D_0).$$

Such a p exists from (2.2) of Theorem 2.2.3. Then we can define in M the function

$$f'(q) = \begin{cases} \sqcup\{c \in D_0 \mid q \Vdash (c \in \sigma)\} & \text{if } p \prec q \\ \perp & \text{otherwise} \end{cases}$$

from $\mathbb{P}$ into D. From (2.3), f' is in M. We must make sure that f' is well defined, which amounts to showing that $K = \{c \in D_0 \mid q \Vdash (c \in \sigma)\}$ is directed when $p \prec q$, and hence has a least upper bound. Note that generally

$$c \prec b \Leftrightarrow c \ll b$$

when c is compact. We will also in the following make use of Lemma 2.2.2. Let $a, b \in K$, i.e.,

$$q \Vdash (a \in \sigma \wedge b \in \sigma \wedge \sigma \text{ is an ideal}).$$

Then
$$q \Vdash (\exists c \in \sigma(a \prec c \wedge b \prec c)). \tag{4.1}$$
However, we must be able to point at a particular such c in K. Since D is a domain, $\{a, b\}$ has a least upper bound c. Then (4.1) gives us
$$q \Vdash (\exists c' \in \sigma(c \prec c')),$$
and since q forces that σ is an ideal,
$$q \Vdash (c \in \sigma).$$
Thus $c \in K$.

Then we can define the function $f : F \to D$ as:
$$f(A) = \bigsqcup\{f(q) \mid q \in A\}.$$
This means that f produces least upper bounds of subideals of $\sigma_G = \{q \in D_0 \mid q \ll x\}$. For f to be well defined, we must show that $\downarrow f(A)$ is an ideal for each $A \in F$, where[1]
$$\downarrow f(A) = \{c \in D_0 \mid c \prec f(A)\}.$$
Assume $b \in\downarrow f(A)$ and $a \prec b \in D_0$. This means that there exists some $q \in A$ such that $q \leq p$ and
$$q \Vdash (b \in \sigma \wedge (\sigma \text{ is an ideal})).$$
This implies
$$q \Vdash (a \in \sigma),$$
and hence $a \in\downarrow f(A)$. Let then $a, b \in\downarrow f(A)$, that is
$$\exists q_1, q_2 \in A(q_1, q_2 \leq p \wedge q_1 \Vdash (a \in \sigma) \wedge q_2 \Vdash (b \in \sigma)).$$
Since A is a filter, there exists $q \in A$ such that $q \leq q_1, q_2$ and then
$$q \Vdash (\exists c \in \sigma(a \prec c \wedge b \prec c)).$$
As for (4.1), this means that the least upper bound of $\{a, b\}$ is in $\downarrow f(A)$. This means that $\downarrow f(A)$ is an ideal, and thus represents an unique element of D.

The informal statement $f(G) = x$ (G is not defined in M) can now be verified. To show that $\sigma_G =\downarrow f(G)$, let $c \in \sigma_G$, that means $(c \in \sigma_G)^{M[G]}$. From (2.2), there exists some $q' \in G$ such that
$$q' \Vdash c \in \sigma.$$
Since G is a filter and p and q' are in G, there exists $q \in G$ that extends both q' and p. Then we have $c \prec f'(q)$ since also q will force $c \in \sigma$. This will again mean that $c \prec f(A)$, and $c \in\downarrow f(A)$.

Now we have $\sigma_G \subseteq\downarrow f(A)$, and since σ_G is maximal in the codomain of f, $\sigma_G =\downarrow f(A)$.

That f is monotonic is rather obvious. Let $A_1 \subseteq A_2$ be filters in $\mathbb{P}$ and let $c \in\downarrow f(A_1)$, which means there exists some q in A_1 such that $q \Vdash (c \in \sigma)$. Since q is also in A_2, $c \in\downarrow f(A_2)$, and hence $f(A_1) \prec f(A_2)$.

[1]The notation $\downarrow x = \{y \in \mathbb{P} \mid y \prec x\}$ is according to [5].

To show that f preserves least upper bounds, let R be a directed set of filters in $\mathbb{P}$. From Observation 3.1.2 we have $\sqcup R = \cup R$, so we must show $\sqcup f(R) = f(\cup R)$. Let first $c \in\downarrow(\sqcup f(R))$. Since f is monotonic, $f(R)$ is directed and therefore, since c is compact, $c \prec f(A)$ for some $A \in R$, and hence $c \prec f(A) \prec f(\cup R)$. This implies $c \in\downarrow f(\cup R)$, which is what we wanted.

Then assume that $c \in\downarrow f(\cup R)$, which means that there exists $A \in R$ and $q \in A$ such that

$$q \Vdash (c \in \sigma).$$

Then

$$c \prec f(A) \prec \sqcup f(R).$$

So $c \in\downarrow(\sqcup f(R))$. This means that f preserves least upper bounds of directed sets and consequently that f is continous. $\square$

4.2 Total Generic Functions

In this section we will consider two kinds of generic functions $f \in X \to Y$. In both cases X is σ-compact, and in the first case X is also compact. As will be shown, X is 'small' enough to assure that f is total. These results points to Theorem 5.3.3, and the strategy in the proofs are similar.

4.2.1 Totalness of Generic $G : C_f \to \mathbb{N}$

Let D be a domain representation of $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ according to section 3.3.1. We use $(D_0, \prec)$ as the forcing relation. The compact elements of this domain are the finite partial functions of the form

$$\sigma = \{\langle \tau_1, n_1 \rangle, \ldots, \langle \tau_k, n_k \rangle\},$$

where $\tau_i : \{0, 1, \ldots, n_i\} \to \mathbb{N}$.

We then have

Lemma 4.2.1. *Let G be D_0-generic and $S = C_f$ for some $f \in \mathbb{N}^{\mathbb{N}}$, then G defines a total function*

$$g : S^{M[G]} \to \mathbb{N}.$$

Proof. Define in $M[G]$

$$g(f') = n \Leftrightarrow \exists \sigma \in G \exists \langle \tau_i, n \rangle \in \sigma(\tau_i \prec f').$$

We say that σ is *total* on C_f if

$$\forall f' \in C_f \exists \langle \tau, n \rangle \in \sigma(\tau \prec f').$$

First we must show in M that $E = \{\sigma \mid \sigma \textit{ is total on } C_f\}$ is dense. Then we will have an element $\sigma \in G \cap E$ since G is generic. This element will then be total on C_f in $M[G]$ as well, since being total on a compact set is absolute, which will be verified below. So σ will in $M[G]$ guarantee that g is total and well defined.

To show that E is dense, let $\sigma_0 \in D_0$ be arbitrary. We need to find $\sigma \in E$ such that $\sigma_0 \prec \sigma$ and σ is total on C_f. Let

$$k = max\{length(\tau) \mid \langle\tau, n\rangle \in \sigma_0\}$$

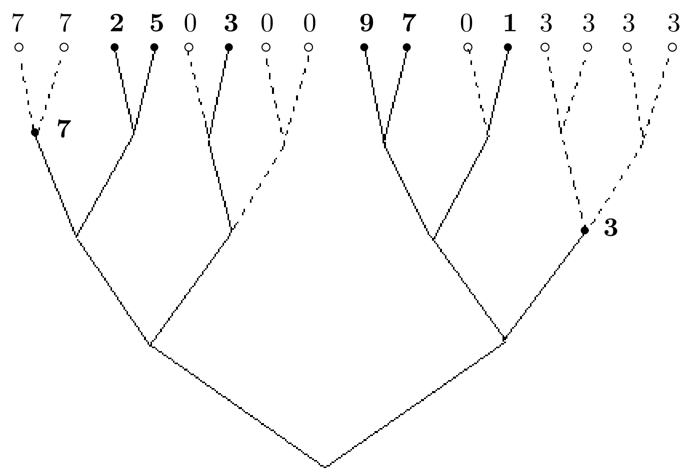
and

$$\sigma = \{\langle\tau, n\rangle \mid \langle\tau, n\rangle \in \sigma_0 \ \vee \ (length(\tau) = k \ \wedge \ n = h(\tau, \sigma_0))\}$$

where

$$h(\tau, \sigma_0) = \begin{cases} m & \text{if } \exists\langle\tau', m\rangle \in \sigma_0(\tau' \prec \tau) \\ 0 & \text{otherwise.} \end{cases}$$

So σ is the tree σ_0 with the 'missing' branches defined consistently with σ_0, giving the value 0 if σ_0 does not give any information about τ. The way σ extends σ_0 can be illustrated as follows;



An example with $k = 4$ and $f(n) = 2$, the strippled branches are added to σ_0.

Then $\sigma_0 \prec \sigma$ because σ contains σ_0. The new elements of σ that extends elements of σ_0 does this consistently, that is;

$$\neg\exists\langle\tau_1, n\rangle, \langle\tau_2, m\rangle \in \sigma(\tau_1 \prec \tau_2 \wedge m \neq n).$$

Thus σ is total on C_f since σ must contain an approximation of all elements in C_f. For σ to be an element of D_0, we must make sure that it is finite. Considering C_f as an infinite tree, the branching in depth n is equal to $f(n)$. Since σ only consists of branches of maximal depth k, there can only be finitely many of them.

Finally we must show that being total on a compact is absolute, i.e.,

$$(\sigma \text{ is total on } C_f)^M \Leftrightarrow (\sigma \text{ is total on } C_f)^{M[G]}. \tag{4.2}$$

The "$\Leftarrow$"-direction of (4.2) is trivial since $M \subset M[G]$. Then assume for contradiction that σ is total on C_f in M and that σ is not total on C_f in $M[G]$. That means there exists some $f' \in C_f$ in $M[G]$ such that there is no approximation of f' in σ. Let k be as above, and let τ be f' restricted to length k. Then τ has neither got an approximation in σ. Since both σ and τ are finite, σ does not approximate τ in M. Then σ will not approximate any $h \in C_f$ in M extending τ, and this contradicts that σ is total on C_f in M. □

4.2.2 Absoluteness of Total Compact Functions

For later arguments, such as the proof of Theorem 4.2.3, we will need a generalization of (4.2). This establishes the main connection between M and $M[G]$ that we will make use of. By showing that a compact function (i.e., an element of $\mathbb{P} = D_0$) is total in a certain sense, on a compact C in M, we will know that this also is the case in $M[G]$. For our purposes we must consider the general case where $D \to \Re$ is a domain for $X \to \mathbb{R}$ where (X, d) is a complete, separable metric space.

Lemma 4.2.2. *Let $p \in (D \to \Re)_0$ be defined with a precision of δ on the compact C in M. That is,*

$$(\forall x \in C \exists \langle I, J\rangle \in p(x \in I \wedge |J| \leq \delta))^M \tag{4.3}$$

Then (4.3) also holds in $M[G]$, i.e., relativized to $M[G]$.

Proof. Let $(x \in C)^{M[G]}$ and let $I = \bigcup_{i \leq m} I_i$ where p is defined with a precision of δ on each I_i. Since C is a compact, there exists $f \in \mathbb{N}^{\mathbb{N}}$ and open balls $B(\epsilon_n, x_{n,i})$ where $i \leq f(n)$ and $\epsilon_n = \frac{1}{n}$, such that

$$C = \{\bigcap_{n \in \mathbb{N}} B(\epsilon_n, x_{n,f'(n)}) \mid f' \leq f\}.$$

This means that C can be seen as a tree structure with finite branching. Let f' be such that

$$x = \bigcap_{n \in \mathbb{N}} B(\epsilon_n, x_{n,f'(n)}).$$

Now, for every n we can find $x_n \in B(\epsilon_{n,f'(n)})$ such that $(x_n \in C)^M$. From the assumption that p is defined with a precision of δ on C, $x_n \in I$. Then $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in I with limit x, and since I is closed, x is also in I. This proves (4.3) for all $(x \in C)^{M[G]}$. □

4.2.3 Totalness of Generic $G : \mathbb{R} \to \mathbb{R}$

Let $\Re$ be a domain representation of $\mathbb{R}$ and let the forcing relation $\mathbb{P}$ be the compact elements of $\Re \to \Re$, i.e., the elements of the form

$$p = \{\langle I_1, J_1\rangle, \ldots, \langle I_n, J_n\rangle\}$$

where I_i and J_i are closed rational intervals and $I_i \cap I_j \neq \emptyset$ implies $J_i \cap J_j \neq \emptyset$.

Theorem 4.2.3. *Let G be $\mathbb{P}$-generic. Then G defines a total $g : \mathbb{R} \to \mathbb{R}$ in $M[G]$.*

Proof. The filter G will in $M[G]$ define the function

$$g(x) = \bigcap \{J \mid \exists \langle I, J\rangle \in \cup G(x \in I)\}.$$

We need to show that $g(x)$ is a nonempty interval of length 0. For every $x \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x \in [-N, N]$. Thus it is sufficient to show that g is total on each $[-N, N]$, since this is a compact in $\mathbb{R}$. First, there is $\langle I, J \rangle \in \cup G$ such that $g(x) \subseteq J$. For let Δ be the set of all $p \in \mathbb{P}$ defined for x, i.e.,

$$\Delta = \{p \mid \exists \langle I, J \rangle \in p(x \in I)\}.$$

Then Δ is dense in M; either an arbitrary p is already defined for x, or p can be extended with some $\langle I, J \rangle$ where $x \in I$. Since G is generic $\Delta \cap G \neq \emptyset$. Knowing this, $g(x)$ must be nonempty.

Now we will show that $|g(x)| = 0$, it is then enough to show that the following sets are dense in M;

$$A_k = \{p \mid \forall x \in [-N, N]\ \exists \langle I, J \rangle \in p(x \in I \wedge |J| \leq 2^{-k})\}$$

for all $k \in \mathbb{N}$. To see this, notice first that $A_k \cap G \neq \emptyset$ for all k since G is generic. Let $p_n \in A_n \cap G$, then there exists $\langle I_n, J_n \rangle \in p_n$ such that $x \in I_n$ and $|J_n| \leq 2^{-n}$ for all $x \in [-N, N]$. From Lemma 4.2.2 each p_n is also defined with precision 2^{-n} in $M[G]$. From the definition of g, it follows that

$$g(x) \subseteq \bigcap_{n \in \mathbb{N}} J_n,$$

and hence

$$|g(x)| \leq |\bigcap_{n \in \mathbb{N}} J_n| = 0.$$

So it comes down to showing that A_k is dense. Let $p \in \mathbb{P}$ be arbitrary, we must then find $p' \in \mathbb{P}$ such that $p \prec p'$ and $p' \in A_k$. We will add certain elements to p to construct p'. These new elements must guarantee that p' is defined with a precision of 2^{-k} on $[-N, N]$. In the end we will have a fine partition of $[-N, N]$ with corresponding short intervals J. During this process it is safe to add $\langle I, J \rangle$ to p' if it does not contradict the construction of p' up until this point, i.e., if not $I \cap I' \neq \emptyset$ and $J \cap J' = \emptyset$ for $\langle I', J' \rangle \in p'$. This is the main idea behind the construction of p'.

If

$$\{[a_i, b_i]\}_{i=0}^{m} = \{I \mid \langle I, J \rangle \in p \wedge I \cap [-N, N] \neq \emptyset\},$$

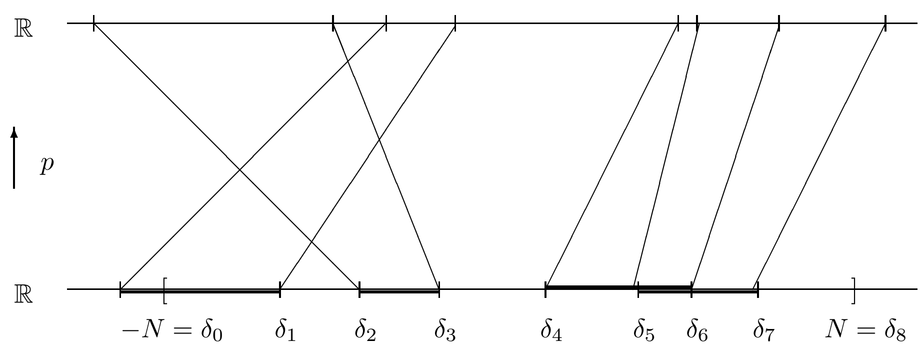
let $\{\delta_i\}_{i=0}^{n}$ be the finite partition of $[-N, N]$ generated by the points $-N$, N, a_i and b_i for $0 \leq i \leq m$. So we use p to create a partition of $[-N, N]$. How p is defined outside of $[-N, N]$ is of no importance.

Each of the intervals $I = [\delta_i, \delta_{i+1}]$ in this partition is of one of the following three types:

1) $(\text{int } I_i) \cap I \neq \emptyset$ for more than one I such that $\langle I, J \rangle \in p$.

2) $(\text{int } I_i) \cap I \neq \emptyset$ for exactly one I such that $\langle I, J \rangle \in p$.

3) $(\text{int } I_i) \cap I = \emptyset$ for all I such that $\langle I, J \rangle \in p$.



$[\delta_5, \delta_6]$ is of type 1), $[\delta_2, \delta_3]$ of type 2) and $[\delta_1, \delta_2]$ of type 3).

We shall first consider all the intervals of type 1) and 2). Here p' must be defined as a refinement of how p is defined on this interval. For the intervals of type 3), we have no such information from p, so here we only need to make sure that the construction of p' is consistent with the definition of p' on the adjacent intervals. It is important to notice that an interval of type 3) always has adjacent intervals of type 1) or 2).

So assume that $I = [\delta_i, \delta_{i+1}]$ is of type 1). Let

$$J = \bigcap \{K \mid \langle I', K\rangle \in p \ \wedge \ I' \cap [\delta_i, \delta_{i+1}] \neq \emptyset\}.$$

This means that p is defined to be J on the interval I. Then make a finite partition $\{\Delta_{i,j}\}_{j=0}^{n_i}$ of J such that

$$|[\Delta_{i,j}, \Delta_{i,j+1}]| < 2^{-k} \text{ for all } j < n_i.$$

This is possible since J is an interval of finite length. Having done this, we must make an arbitrary partition $\{\delta_{i,j}\}_{j=0}^{n_i}$ of $[\delta_i, \delta_{i+1}]$. Then we can add the following elements to p';

$$\langle[\delta_{i,0}, \delta_{i,1}], [\Delta_{i,0}, \Delta_{i,1}]\rangle, \ldots, \langle[\delta_{i,n_i-1}, \delta_{i,n_i}], [\Delta_{i,n_i-1}, \Delta_{i,n_i}]\rangle.$$

We now suppose that $[\delta_i, \delta_{i+1}]$ is of type 2). This case is essentially the same as for intervals of type 1), except that J will not be an intersection. We let J be the interval J for which $\langle I', J\rangle \in p$ and int $[\delta_i, \delta_{i+1}] \cap I' \neq \emptyset$. Then we can make a further extension of p' exactly the same way as we did for intervals of type 1).

Having done this for all intervals of type 1) and 2) we can move on to the intervals of type 3). Assume first that $i \notin \{0, n-1\}$, so that $[\delta_i, \delta_{i+1}]$ is not the first or the last interval of the partition $\{\delta_i\}_{i=0}^{n}$. The adjacent intervals, $[\delta_{i-1}, \delta_i]$ and $[\delta_{i+1}, \delta_{i+2}]$, must be of type 1) or 2), so both

$$J_1 = [\Delta_{i-1,n_{i-1}-1}, \Delta_{i-1,n_{i-1}}]$$

and

$$J_2 = [\Delta_{i+1,n_{i+1}-1}, \Delta_{i+1,n_{i+1}}]$$

are already defined. It is just to check the indices and see that these intervals are the adjacent intervals in the codomain of p'. Let a be the end point of J_1 and let b be the starting point of J_2. Now we must make an extension of p' which in some sense 'fills the gap' between a and b. Let J be the interval between a and b. As before, construct a finite partition $\{\Delta_{i,j}\}_{j=0}^{n_i}$ of J such that each interval in this partition has length less than 2^{-k}. Then make the corresponding arbitrary partition $\{\delta_{i,j}\}_{j=0}^{n_i}$ of $[\delta_i, \delta_{i+1}]$. Now we must consider which elements we want to add to p'. If $a \leq b$, we can do as before and add the following elements;

$$\langle[\delta_{i,0}, \delta_{i,1}], [\Delta_{i,0}, \Delta_{i,1}]\rangle, \ldots, \langle[\delta_{i,n_i-1}, \delta_{i,n_i}], [\Delta_{i,n_i-1}, \Delta_{i,n_i}]\rangle.$$

However, if $b < a$ the intervals in the domain and codomain must be matched the other way around, so add to p' the elements;

$$\langle[\delta_{i,0}, \delta_{i,1}], [\Delta_{i,n_i-1}, \Delta_{i,n_i}]\rangle, \ldots, \langle[\delta_{i,n_i-1}, \delta_{i,n_i}], [\Delta_{i,0}, \Delta_{i,1}]\rangle.$$

This ends the construction of p'.

Now we must verify that p' has the required properties, that is $p' \in \mathbb{P}$, $p \prec p'$ and $p' \in A_k$. This should follow geometrically from the figure. Indeed, p' is finite since it is constructed from finite partitions of a finite partition. If $I_i \cap I_j \neq \emptyset$ for $\langle I_i, I_i\rangle$ and $\langle I_j, I_j\rangle$ new elements in p', I_i and I_j have only a single point in common. Assume, without loss of generality, that $\langle I_i, J_i\rangle$ was added to p' first. From the construction of $\langle I_j, J_j\rangle$ we made sure that J_i and J_j had a common endpoint, so $J_i \cap J_j \neq \emptyset$. If one of $\langle I_i, I_i\rangle$ or $\langle I_j, I_j\rangle$ is an element of p, the conclusion follows from the fact that p' is constructed as a consistent extension of p. This shows that $p' \in \mathbb{P}$. It is also trivially true that $p \prec p'$ since $p \subset p'$.

Let now $x \in [-N, N]$ be arbitrary. Then there is an interval $[\delta_{i,j}, \delta_{i,j+1}]$ which contains x since these intervals is constructed from a partition of $[-N, N]$. From the construction of p', the corresponding interval $[\Delta_{i,j}, \Delta_{i,j+1}]$ has length less than 2^{-k}. This means that $p' \in A_k$, and having shown that A_k is dense, the proof is complete. $\square$

Chapter 5

Nontotal Generic Functions

In this chapter we will first consider two specific function spaces $X \to Y$, both in which generic functions are nontotal in $M[G]$. In both cases we let X be $\mathbb{N}^{\mathbb{N}}$, and we let Y be $\mathbb{N}$ and $\mathbb{R}$ respectively. This means that the result in section 5.2 is a generalization of the result in Section 5.1. In neither case is X σ-compact, and this fact gives us the motivation for the results in Section 5.3, showing that σ-compactness of X is an essential indicator of totalness of generic functions.

5.1 The Function Space $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$

Lemma 4.2.1 states that every generic G defines a total function $g : S \to \mathbb{N}_{\perp}$ in $M[G]$ when $S \subset \mathbb{N}^{\mathbb{N}}$ is compact in M. So from the full tree $\mathbb{N}^{\mathbb{N}}$ with infinite branching, we restricted g to a subtree with only finite branching. Now we will show that g is not total on $\mathbb{N}^{\mathbb{N}}$ by constructing a branch f for which g is not well defined.

Theorem 5.1.1. *Let D be a domain representation for $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, M be a c.t.m. and G be D_0-generic. If*

$$\exists q \in G \exists \langle \tau_i, n_i \rangle, \langle \tau_j, n_j \rangle \in q(n_i \neq n_j) \tag{5.1}$$

then G is not total in $M[G]$.

Proof. Obviously, we must assume that g is not constant, which is assured by assumption (5.1). Otherwise the theorem would be trivially false. The function defined from G in $M[G]$ is;

$$g(f) = n \Leftrightarrow \exists \langle \tau, n \rangle \in \cup G(\tau \prec f).$$

We will construct a sequence $f = \{a_i\}_{i \in \mathbb{N}}$ recursively for which g is not well defined, i.e., does not determine an unique element of $\mathbb{N}$. Let

$$A_0 = \{p \in \mathbb{P} \mid \exists a \in \mathbb{N}\ \exists \langle \tau_i, n_i \rangle, \langle \tau_j, n_j \rangle \in p(\langle a \rangle \prec \tau_i, \tau_j \wedge n_i \neq n_j)\}.$$

So A_0 is the set of compact functions not constant on some $\langle a \rangle$. If we can show that this set is dense below the $q \in G$ assured to exist from (5.1), we are able to define $\langle a_0 \rangle$ for which g is not constant. Then we will try to find an extension $\langle a_0, a_1 \rangle$ of $\langle a_0 \rangle$ where g also is not constant. By continuing this process we end up with a branch f for which g is not well defined.

Let $p \in \mathbb{P}$ be arbitrary such that $q \prec p$. Since p is finite and has no common definition for all of $\mathbb{N}^{\mathbb{N}}$, we can choose the least $a \in \mathbb{N}$ such that p is not defined for $\langle a \rangle$. Then we can make a finite extension $p' \in \mathbb{P}$ of p by adding the two elements $\langle\langle a, 0\rangle, 0\rangle$ and $\langle\langle a, 1\rangle, 1\rangle$ to p. Such an extension will be consistent since p can be given any value where it is not already defined. Now $p \prec p'$ and $p' \in A_0$. This means that A_0 is indeed dense below q, and since G is generic we have $G \cap A_0 \neq \emptyset$ from Lemma 2.1.4. Then let a_0 be the least a such that g is not constant on $\langle a \rangle$, i.e.,

$$a_0 = \mu a.(\exists \langle \tau_i, n_i\rangle, \langle \tau_j, n_j\rangle \in \cup G(\langle a \rangle \prec \tau_i, \tau_j \wedge n_i \neq n_j)).$$

Assume then for induction that $a_0, \ldots, a_n$ has been defined. We make a minor change in the definition of A_0 to define A_{n+1};

$$A_{n+1} = \{p \mid \exists a \in \mathbb{N} \exists \langle \tau_i, n_i\rangle, \langle \tau_j, n_j\rangle \in p(\langle a_0, \ldots, a_n, a\rangle \prec \tau_i, \tau_j \wedge n_i \neq n_j)\}.$$

From the previous step of the induction there exists $q' \in G$ that is an element of A_n. If we can show that A_{n+1} is dense below q' we will have $G \cap A_{n+1} \neq \emptyset$ by Lemma 2.1.4. So let p be arbitrary such that $q' \prec p$. Since p can not have a common definition of the full subtree $\langle a_0, \ldots, a_n\rangle$ and p is finite, we can choose the least a such that p is not defined for $\langle a_0, \ldots, a_n, a\rangle$. Similar to the first induction step, we construct p' by adding the following elements to p;

$$\langle\langle a_0, \ldots, a_n, a, 0\rangle, 0\rangle$$

and

$$\langle\langle a_0, \ldots, a_n, a, 1\rangle, 1\rangle.$$

Then p' is a finite and consistent extension of p, i.e., $p \prec p' \in \mathbb{P}$, and consequently A_{n+1} is dense below q'. Since we know that $G \cap A_i \neq \emptyset$ we can define a_{n+1} to be the least a such that g is not constant on the subtree $\langle a_0, \ldots, a_n, a\rangle$. More precisely, and analogously to the first induction step, we define

$$a_{n+1} = \mu a.(\exists \langle \tau_i, n_i\rangle, \langle \tau_j, n_j\rangle \in \cup G(\langle a_0, \ldots, a_n, a\rangle \prec \tau_i, \tau_j \wedge n_i \neq n_j)).$$

This completes the definition of $f = \{a_i\}_{i \in \mathbb{N}}$.

Now we have an infinite branch f in $\mathbb{N}^{\mathbb{N}}$, and we want to show that g is not defined for f. For each n we can, from the existence of f, find $f_n', f_n'' \in \mathbb{N}^{\mathbb{N}}$ such that

$$d(f_n', f) \leq 2^{-n},\ d(f_n'', f) \leq 2^{-n}$$

and

$$g(f_n') \neq g(f_n'').$$

Then g can not be well defined for f since g is continuous. □

5.2 The Function Space $\mathbb{N}^{\mathbb{N}} \to \mathbb{R}$

In section 5.1 we considered a generic G with codomain $\mathbb{N}$. By making some changes in the argument, this result can be somewhat generalized;

Theorem 5.2.1. *Let D be a domain representing $\mathbb{N}^{\mathbb{N}} \to \mathbb{R}$, M a c.t.m. and G a D_0-generic filter. If G is not a constant function, i.e.,*

$$\exists q \in G \exists r \in \mathbb{R} \exists \langle \tau_i, J_i\rangle, \langle \tau_j, J_j\rangle \in q(d(J_i, J_j) \geq r), \tag{5.2}$$

then G is not total in $M[G]$.

Proof. Now G defines in $M[G]$ the function

$$g(f) = \bigcap\{J \mid \exists\langle\tau, J\rangle \in \cup G(\tau \prec f)\}.$$

We will try to construct an element f of $\mathbb{N}^{\mathbb{N}}$ for which g is not well defined. This f will be such that $g(f) = \emptyset$. As in section 5.1, we will by induction construct a sequence $f = \{a_i\}_{i\in\mathbb{N}}$. To find a_0, consider the set

$$A_0 = \{p \in \mathbb{P} \mid \exists a \in \mathbb{N}\exists\langle\tau_1, J_1\rangle, \langle\tau_2, J_2\rangle \in p(\langle a\rangle \prec \tau_1, \tau_2 \wedge d(J_1, J_2) \geq r)\},$$

where r is fixed and satisfies assumption (5.2). So we look at those compact functions sending subsets of some $\langle a\rangle$ onto sets seperated by at least a distance of r. That g varies with at least r will be preserved along the branch f, and thus make sure that g is not well defined for f.

We will show that A_0 is dense below q, where q is given by (5.2), so let $p \in \mathbb{P}$ be arbitrary such that $q \prec p$. Since p is finite, we can choose the least a such that

$$\forall\langle\tau, J\rangle \in p(\tau \cap \langle a\rangle \neq \emptyset \Rightarrow \tau = \perp).$$

Let

$$J = [j_1, j_2] = \begin{cases} \bigcap\{J' \mid \langle\perp, J'\rangle \in p\} & \text{if } \exists\langle\perp, J'\rangle \in p \\ \text{[0,r]} & \text{otherwise.} \end{cases}$$

This J will be used to make sure that the extension of p is consistent. So let p' be p extended with the elements

$$\langle\langle a, 0\rangle, [j_1 - 1, j_1]\rangle$$

and

$$\langle\langle a, 1\rangle, [j_2, j_2 + 1]\rangle.$$

These elements will not make p' inconsistent, and since $d([j_1-1, j_1], [j_2, j_2+1]) \geq r$, $p' \in A_0$. Hence A_0 is dense. Then we can define

$$a_0 = \mu a.(\exists\langle\tau_i, J_i\rangle, \langle\tau_j, J_j\rangle \in \cup G(\langle a\rangle \prec \tau_i, \tau_j \wedge d(J_i, J_j) \geq r)).$$

Assume then that $a_0, \ldots, a_n$ are defined. Consider the set

$$A_{n+1} = \{p \in \mathbb{P} \mid \exists a \in \mathbb{N}\exists\langle\tau_i, J_i\rangle, \langle\tau_j, J_j\rangle \in p(\tau_i, \tau_j \prec \langle a_0, \ldots, a_n, a\rangle \wedge d(J_i, J_j) \geq r)\}.$$

From the induction hypothesis we know that there exists some $q' \in G \cap A_n$. We want to show that A_{n+1} is dense below q'. So let $p \in \mathbb{P}$ such that $q' \prec p$ be arbitrary and

$$J = [j_1, j_2] = \begin{cases} \bigcap\{J' \mid \langle\tau, J'\rangle \in p\} & \text{if } \exists\langle\tau, J'\rangle \in p(\tau \prec \langle a_0, \ldots, a_n\rangle) \\ \text{[0,r]} & \text{otherwise.} \end{cases}$$

Then we can consistently extend p with

$$\langle\langle a_0, \ldots, a_n, a, 0\rangle, [j_1 - 1, j_1]\rangle$$

and

$$\langle\langle a_0, \ldots, a_n, a, 1\rangle, [j_2, j_2 + 1]\rangle$$

where

$$a = \mu m.(\forall\langle\tau', J'\rangle \in p(\tau' \cap \langle a_0, \ldots, a_n, m\rangle \neq \emptyset \Rightarrow \tau' \prec \langle a_0, \ldots, a_n\rangle)).$$

Thus A_{n+1} is dense below q'. Since then $A_{n+1} \cap G \neq \emptyset$ we can define

$$a_{n+1} = \mu a.(\exists \langle \tau_i, J_i \rangle, \langle \tau_j, J_j \rangle \in \cup G(\langle a_0, \ldots, a_n, a \rangle \prec \tau_i, \tau_j \wedge d(J_i, J_j) \geq r)).$$

This completes the construction of $f = \{a_n\}_{n \in \mathbb{N}}$. Now we can for every n find $f'_n, f''_n \in \mathbb{N}^{\mathbb{N}}$ such that

$$d(f'_n, f) \leq 2^{-n},\ d(f''_n, f) \leq 2^{-n}$$

and

$$|g(f'_n) - g(f''_n)| \geq r.$$

As in the proof of Theorem 5.1.1, this means that g is not well defined for f. □

5.3 The General Case $X \to \mathbb{R}$

The previous results suggests that there is a connection between X being σ-compact and $X \to \mathbb{R}$ (or $\mathbb{N}$) being total. In this section we will investigate this further. First we will show that a generic G will be total for $X \to \mathbb{R}$ when X is a complete separable σ-compact metric space. This is a generalization of Theorem 4.2.3. Then we will try to find out whether the converse holds; that G being generic and total implies that X is σ-compact.

5.3.1 X σ-Compact

Let (X, d) be a σ-compact complete separable metric space, and $\{a_n\}_{n \in \mathbb{N}}$ be a dense set in X. Let D be the domain representing X, according to section 3.3.3, generated by nonempty finite intersections of sets of the form

$$B_{n,r} = \{x \in X \mid d(x, a_n) \leq r\}.$$

This means that $\mathbb{P} = D_0$ is ordered by reverse inclusion and consists of elements

$$p = \bigcap_{i=1}^{k} \{B_{n_i, r_i}\}.$$

In the proof of Theorem 5.3.3 we will make use of a generalization of the Tietze-Urysohn extension theorem.

Theorem 5.3.1. (Tietze-Urysohn) *Let X be a complete, separable metric space. If $A \subseteq X$ and $B \subseteq \mathbb{R}$ are closed and $f : A \to B$ is continuous, then f can be extended to a continuous $g : X \to \mathbb{R}$.*

See [8] for a proof.

Lemma 5.3.2. (Normann) *Let X be a complete separable metric space and let*

$$\langle B_1, I_1 \rangle, \ldots, \langle B_l, I_l \rangle$$

be pairs of closed $B_i \subseteq X$ and closed $I_i \subseteq \mathbb{R}$ such that

$$\bigcap_{i \leq k} B_i \neq \emptyset \Rightarrow \bigcap_{i \leq k} I_i \neq \emptyset.$$

Then there exists a continuous $f : X \to \mathbb{R}$ such that $f[B_i] \subseteq I_i$ for all $i \leq l$.

Proof. For each $x \in X$ let $r(x) = |\{B_i \mid x \in B_i\}|$. Then $r(x)$ is bounded by l, so let $m = max\{r(x) \mid x \in X\}$. We will define f by induction on $m - i$. In the induction start we define f for $A = \{x \in X \mid r(x) = m\}$. Then A is a disjoint union of $\{A_i\}_{i \leq k}$ where $A_i = \bigcap_{j \leq m} B_{i_j}$. The A_i's must be disjoint since they otherwise would have been intersections of more than m B_i's, which would be a contradiction. Fix $x_i \in \bigcap_{j \leq m} I_{i_j}$. Then we can define

$$f(A_i) = x_i.$$

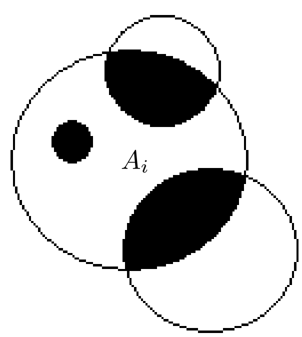
This completes the induction start. Assume then that f has been defined continuously for $\{x \in X \mid r(x) > n\}$, we will then define f for $A = \{x \in X \mid r(x) \geq n\}$. Note that $\{x \in X \mid r(x) > n\} \subseteq A$, so f has already been defined for some parts of this set. As in the induction start, $A = \bigcup_{i \leq k} A_i$. We consider each of the A_i's in turn:

- If $A_i \cap \{x \in X \mid r(x) > n\} = \emptyset$, we have a high degree of freedom to define f, and we can let

  $$f(A_i) = x_i$$

  for some $x_i \in \bigcap_{j \leq n} I_{i_j}$ where $A_i = \bigcap_{j \leq n} B_{i_j}$.

- If $B = A_i \cap \{x \in X \mid r(x) > n\} \neq \emptyset$, B is a finite disjoint union of closed sets, as showed in the figure, so B is closed. From the induction hypothesis, f is already defined for B. Theorem 5.3.1 can then be applied to A_i and B such that f can be continuously extended to all of A_i.



The shaded area shows $A_i \cap \{x \in X \mid r(x) > 0\}$.

However, such an extension g may fail to send A_i into

$$B_i = \bigcap_{j \leq n} I_{i_j} = [b_1, b_2],$$

which is required. But g can be modified for the input values mapped outside of B_i, so let

$$g'(x) = h(x) \circ g(x)$$

where

$$h(x) = \begin{cases} b_1 & \text{if } x < b_1 \\ b_2 & \text{if } x > b_2 \\ x & \text{otherwise} \end{cases}$$

Since continuity is preserved under composition, g' will be continuous and hence f can be extended to g'.

Then f is defined for each A_i and thereby for A. We know that f is continuous on each A_i, but we must make sure that f is also continuous on A. Disjoint A_i's are separated and f is therefore trivially continuous on the union. If A_i and A_j has a nonempty intersection, it is clear from the figure that f is continuous on $A_i \cup A_j$ since f is continuous on A_i, A_j and $A_i \cap A_j$.

The recursion end will define f for $A = \{x \in X \mid r(x) \geq 0\}$. Then f is already defined continuously for $\{x \in X \mid r(x) > 0\}$, which is a closed set. From Theorem 5.3.1, f can be extended to all of X. This completes the proof. □

Theorem 5.3.3. *Let D be a domain representation of $X \to \mathbb{R}$ for a complete separable σ-compact metric space (X, d). Let M be a c.t.m. Then a D_0-generic G will be total in $M[G]$.*

Proof. G defines the function g in $M[G]$ as follows;

$$g(x) = \bigcap\{I \mid \exists\langle I, J\rangle \in \cup G(x \in I)\}. \tag{5.3}$$

First we must show that $g(x)$ is nonempty for all x. For this, we must prove the following.

$$\exists\langle I, J\rangle \in \cup G(x \in I \wedge J \neq \emptyset). \tag{5.4}$$

$$\forall y \in g(x)\forall\langle I, J\rangle \in \cup G(x \in I \Rightarrow y \in J). \tag{5.5}$$

Both follows quite easily from G being a generic filter. For (5.4), let A be the set of those $p \in \mathbb{P}$ defined for x, i.e.,

$$A = \{p \in \mathbb{P} \mid \exists\langle I, J\rangle \in p(x \in I)\}.$$

If A is dense, $G \cap A \neq \emptyset$ and (5.4) will indeed be satisfied. So let $p \in \mathbb{P}$ be arbitrary. Either p is already defined for x, which means there is $\langle I, J\rangle \in p$ such that $x \in I$, or we can make an extension p' of p, which is defined for x. Since p is finite, we can find $n \in \mathbb{N}$ and a sufficiently small $r \in \mathbb{R}$ such that $x \in B_{n,r}$ and $I \cap B_{n,r} = \emptyset$ for each $\langle I, J\rangle \in p$. By adding $\langle B_{n,r}, [0,1]\rangle$ to p, we have constructed $p' \in A$, a finite consistent extension of p defined for x. Statement (5.5) follows trivially from G beeing a filter. If $\langle I_1, J_1\rangle, \langle I_2, J_2\rangle \in \cup G$ and $I_1 \cap I_2 \neq \emptyset$, then $J_1 \cap J_2 \neq \emptyset$.

Now we must show something far less obvious, namely that $g(x)$ is well defined, in the sense that $g(x)$ is an interval of length 0 and thus an unique element of $\mathbb{R}$. Since X is σ-compact, $X = \bigcup_{n\in\mathbb{N}} C_n$, where C_n is compact. It is then sufficient to show that g is well defined for each C_n. Consider the sets

$$A^k_{n,r} = \{q \in \mathbb{P} \mid \forall x \in C_n \exists\langle I, J\rangle \in q(x \in I \wedge |J| \leq 2^{-k})\}.$$

$A^k_{n,r}$ is the set of those $q \in \mathbb{P}$ that are defined for all of C_n with a precision of 2^{-k}. If each $A^k_{n,r}$ is dense, then $A^k_{n,r} \cap G \neq \emptyset$. From Lemma 4.2.2, stating that totalness on compacts is absolute for compact functions, g must then also in $M[G]$ be defined with a precision 2^{-k} for each k. Thus $|g(x)| = 0$ for every $x \in C_n$.

So it is sufficient to show that $A^k_{n,r}$ is dense in M. Let $p \in \mathbb{P}$ be arbitrary, we need to construct an extension p' of p that is an element of $A^k_{n,r}$. Here we can use the continuous f assured to exist by Lemma 5.3.2, where f extends the definition of p. We use f as a foundation for p', without this f, the construction of p' would pose big practical problems.

Now we want to pick $\{a_{n_i}\}_{i=1}^m$ such that $\{B_{n_i,r'} \mid 0 < i < m\}$ is a cover of C_n for a specific $r' \in \mathbb{R}$. This is possible since C_n is compact. This r' must be small enough to allow a very fine definition of p'. Since f is continuous on the compact C_n, f is also uniformly continuous on this set. This is a standard result from analysis, see e.g. [9]. In accordance with the definition of uniform continuity, we can choose a sufficiently small r' such that

$$diam(f(B_{n_i,r'})) < 2^{-(k+1)}$$

for $i \leq m$.

Let p' be p extended with the elements

$$\langle B_{n_i,r'}, [f(a_{n_i}) - 2^{-(k+1)}, f(a_{n_i}) + 2^{-(k+1)}]\rangle$$

for $i \leq m$.

The idea behind this is that if $\langle B_{n_i,r'}, I_i\rangle$ and $\langle B_{n_j,r'}, I_j\rangle$ are new elements in p' and $B_{n_i,r'}$ intersect $B_{n_j,r'}$, the distance $d(a_{n_i}, a_{n_j})$ between a_{n_i} and a_{n_j} is less than $2r'$. From the uniform continuity of f it follows that

$$|f(a_{n_i}) - f(a_{n_j})| < 2 \cdot 2^{-(k+1)} = 2^{-k},$$

and thus $I_i \cap I_j \neq \emptyset$. This means that the new elements in p' is consistent with each other. Assume now that $\langle I, J\rangle \in p$ and that $I \cap B_{n_i,r'} \neq \emptyset$. Then $d(a_{n_i}, I) \leq r'$, and since f is consistent with p, $f(x) \in J$ for all $x \in I$. So $d(f(a_{n_i}), J) < 2^{-(k+1)}$ and thus $I_j \cap J \neq \emptyset$.

We have constructed a finite cover of C_n, and thus ensured that p' is total on C_n. We showed above that $p' \in \mathbb{P}$, and $p \prec p'$ follows trivially from the fact that p' is an extension of p. The construction of p' gives us that p' is defined with a precision of 2^{-k} on C_n. All this implies that $p' \in A^k_{n,r}$ and hence that $A^k_{n,r}$ is dense in $\mathbb{P}$. This completes the proof. □

5.3.2 X not σ-Compact

Now we have proved that if X is σ-compact, every generic G defines a total function. It is natural to ask whether the inverse statement holds. If we assume that every generic G is total, is X then σ-compact?

First we will need some notation.

Definition 5.3.4. *Let* $\mathbf{B}$ *be the set of finite intersections of* $B_{n,q}$*-sets in* X*, i.e.,*

$$\mathbf{B} = \{\bigcap_{i<m} B_{n_i,q_i} \mid m, n_i \in \mathbb{N} \wedge q_i \in \mathbb{Q}\}.$$

Note that $\mathbf{B}$ is countable since we let q range over the rational numbers.

Definition 5.3.5. *Let* $A \in \mathbf{B}$*. Define the predicates* T *and* T^* *as follows;*
$T(A)$ *if there exists a finite set* $\{A_i\}_{i=1}^n$ *where each* $A_i \in \mathbf{B}$ *such that*

- *$A \setminus A_i$ is not σ-compact for all $i \leq n$*

- *$A \setminus \bigcup_{i \leq n} A_i$ is σ-compact*

If also

- *$diam(A_i) \leq \rho \cdot diam(A)$ for some $\rho < 1$,*

then $T^*(A)$.

T^* is stronger than T, so we have;

$$\forall A \in \mathbf{B}(T^*(A) \rightarrow T(A)).$$

We can now formulate one of the main results.

Theorem 5.3.6. *Let (X, d) be a non σ-compact complete separable metric space and let D be a domain representing $X \rightarrow \mathbb{R}$. Let M be a c.t.m. and let G be D_0-generic. If*

$$\forall A \in \mathbf{B}(T(A) \Leftrightarrow T^*(A)) \tag{5.6}$$

and

$$\exists X' \in \mathbf{B}(\neg T^*(X') \wedge \exists p \in G(p \text{ not constant on } X')). \tag{5.7}$$

then G is not total in $M[G]$.

Remark 5.3.7. *The statement 'p not constant on X'' in assumption* (5.7) *means that the information in p contradicts that G defines a total constant function on X'. This implies that there exists some $q \in G$ and $\langle I_i, J_i \rangle, \langle I_j, J_j \rangle \in q$ such that $J_i \cap J_j = \emptyset$ and neither I_i nor I_j is σ-compact. In terms of forcing, p forces that G is not total on X'.*

Proof. The definition of the generic function g from G is as before, and given by (5.3). The main idea is that we can embed $\mathbb{N}^{\mathbb{N}}$ into X' in a certain way. Any embedding will not do, it must have some specific properties. Then a similar argument as in the proof of Theorem 5.2.1 can be carried out. The strategy will be to first find non σ-compact disjoint sets $\{S_n\}_{n \in \mathbb{N}}$ in X'. For each S_n we can find a countable set of non σ-compact disjoint sets contained in S_n. Continuing this process, we obtain a tree structure with countable infinite branching of non σ-compact sets. The main concern in this proof is the way these sets are chosen.

More precisely, we will find non σ-compact sets S_f for all $f : \{0, \ldots, m\} \rightarrow \mathbb{N}$ such that $S_{f_1} \subset S_{f_2}$ whenever $f_2 \prec f_1$. So, by following the branch f to depth $m-1$ in the tree, each $S_{f * \langle n \rangle}$ represents a further branching on depth m. After a recursive definition of S_f, we can define the mapping

$$\bar{f} = \bigcap_{f' \prec f} S_{f'}. \tag{5.8}$$

Since X is complete, $\bar{f}$ will be an element of X if each S_f has diameter less than $\frac{1}{m}$, where $m = length(f)$, since X is complete. This can be assured to hold during the construction. Now we move on to the definition of S_f.

Let $\{B'_n\}_{n \in \mathbb{N}}$ be an enumeration of $\mathbf{B}$. Then let $B_n = B'_n \cap X'$. By going through $\{B_n\}_{n \in \mathbb{N}}$ we shall construct a set S of disjoint subsets of X'. For each $n \in \mathbb{N}$, if

1) B_n is not σ-compact

2) $B_n \subseteq X' \setminus \bigcup S$

3) $\neg T^*(B_n)$

4) $X' \setminus (\bigcup S \cup B_n)$ is not σ-compact

5) $diam(B_n) \leq 1$

we add B_n to S. For S to be as required, we need:

a) $X' \setminus \bigcup S$ contains no non σ-compact B_n.

b) S is an infinite set.

For a), assume that there exists $B_n \subseteq (X' \setminus \bigcup S)$ that is not σ-compact. We will obtain a contradiction by showing that B_n is σ-compact. Since B_n was not added to S during the construction, either 3), 4) or 5) fails. If 4) and 5) fails there must exist $B_m \subset B_n$ for which only 3) fails since $B_m \notin S$. Otherwise there would have been a decreasing sequence $\{B_{m_i}\}_{i \in \mathbb{N}}$ in B_n with

$$\lim_{i \to \infty} diam(B_{m_i}) = 0$$

for which only 4) and 5) (and possibly 3)) fails for each B_{m_i}. This would imply that

$$\bigcup_{i \in \mathbb{N}} (X' \setminus (\bigcup S \cup B_{m_i})) = X' \setminus \bigcup S$$

is σ-compact, since it is a countable union of σ-compact sets, which is a contradiction from the construction of S. So we can without loss of generality assume that only 3) fails for B_n, i.e., $T^*(B_n)$. This means that there exists $\{S'_i\}_{i=1}^m$ such that $S'_i \in \mathbf{B}$, $B_n \setminus S'_i$ is not σ-compact and $B_n \setminus \bigcup_{i \leq m} S'_i$ is σ-compact. We can without loss of generality assume that $S'_i \cap B_n$ is not σ-compact, since S'_i otherwise could be excluded from $\{S'_i\}_{i=1}^m$. Let $S_i = S'_i \cap B_n$, then $S_i \in \mathbf{B}$. We will now construct a branching of sets in B_n. This branching will, contrary to the main construction in this proof, be finite. So we will find sets $K_{f'}$ where f' is finite and $f'(n) \leq f(n)$ for some $f \in \mathbb{N}^{\mathbb{N}}$, i.e., $f' \in C_f$. Define

$$K_{\langle i \rangle} = S_i \text{ for } i \leq m.$$

Assume for induction on the length of f' that $K_{f'}$ has been constructed. Then $T^*(K_{f'})$, since $K_{f'}$ has not been added to S. Then, analogously to the induction start, there is $\{S_i\}_{i=0}^m \subset K_{f'}$ such that $K_{f'} \setminus S_i$ is not σ-compact and $K_{f'} \setminus \bigcup_{i \leq m} S_i$ is σ-compact. Define

$$K_{f' * \langle i \rangle} = S_i \text{ for } i \leq m.$$

Let also

$$F_{f'} = K_{f'} \setminus \bigcup_{i \leq m} K_{f' * \langle i \rangle},$$

which is σ-compact. We now have for some $f \in \mathbb{N}^{\mathbb{N}}$;

$$B_n = \{ \bigcap_{f'' \prec f'} K_{f''} \mid f' \in C_f \} \cup \{ F_{f''} \mid f'' \prec f' \wedge f' \in C_f \}. \tag{5.9}$$

The second set in (5.9) is a countable union of σ-compact sets, and we can show that the first set is compact by using the T^* property;

$$diam(K_{f'}) \leq \rho^{-length(f')}.$$

The proof of this is the same as a standard proof of Lemma 3.3.1. So B_n is σ-compact, contradicting the assumption, and hence $X' \setminus \bigcup S$ contains no non σ-compact B_n.

To show b), assume for contradiction that S is finite. From the construction of S, $X' \setminus \bigcup S$ is not σ-compact. From a), each $B_n \subseteq X \setminus \bigcup S$ is σ-compact, and these sets will constitute a countable cover of $X \setminus \bigcup S$, since this is an open set. This contradicts that $X' \setminus \bigcup S$ is not σ-compact.

Let now $\{S_n\}_{n\in\mathbb{N}}$ be an enumeration of S. Define

$$S_{\langle n\rangle} = S_n \text{ for all } n \in \mathbb{N}.$$

The $S_{\langle n\rangle}$'s constitute the branching on depth 1 in our intended tree structure. This completes the induction start with respect to the depth of the tree, so assume now that $S_{f'}$ has been constructed for a finite f'. We can carry out the exactly same argument as above with $S_{f'}$ in the place of X' since $S_{f'}$ is not σ-compact. The only difference is that we use

$$diam(B_n) \leq 2^{-(length(f')+1)}$$

instead of requirement 5) when adding B_n to S during the construction. So, in $S_{f'}$ we can find $\{S_i\}_{i\in\mathbb{N}}$ of non σ-compact disjoint sets such that $S_{f'} \setminus \bigcup_{i\in\mathbb{N}} S_i$ contains no non σ-compact B_n. This completes the construction of $S_{f'}$ for each

$$f' : \{0, \dots, n\} \to \mathbb{N},\ n \in \mathbb{N}.$$

We have now done the groundwork for the second part of the proof, where we need to find some $f \in \mathbb{N}^{\mathbb{N}}$ in $M[G]$ such that $g(\bar{f}) = \emptyset$.

From the construction of the $S_{f'}$-sets, we have

$$diam(S_{f'}) \leq 2^{-length(f')},$$

which assures that $\bar{f}$ is an uniquely determined element of X. We will first define $f(0)$, and then define $f(n)$ by induction for all $n \in \mathbb{N}$. Consider the set

$$A_0 = \{p \in \mathbb{P} \mid \exists k \in \mathbb{N}\exists\langle I_i, J_i\rangle, \langle I_j, J_j\rangle \in p(I_i, I_j \subset S_{\langle k\rangle} \wedge d(J_i, J_j) \geq r)\},$$

where $r = d(J_1, J_2)$ and J_1, J_2 are such that

$$\exists\langle I_1, J_1\rangle, \langle I_2, J_2\rangle \in q(J_1 \cap J_2 = \emptyset \wedge I_1, I_2 \text{ not } \sigma\text{-compact}), \tag{5.10}$$

according to assumption (5.7).

Now we want to show that A_0 is dense below q, so let $p \in \mathbb{P}$ be arbitrary such that $q \prec p$. Then we can consistently add to p the elements

$$\langle S_{\langle n,0\rangle}, J_1\rangle$$

and

$$\langle S_{\langle n,1\rangle}, J_2\rangle$$

where
$$n = \mu m.(\forall\langle I,J\rangle \in p(S_{\langle m\rangle} \cap I \neq \emptyset \Rightarrow (X' \setminus I)\ \sigma\text{-compact})).$$
This means that A_0 is dense below q, and thus $G \cap A_0 \neq \emptyset$. Then we can define

$$f(0) = \mu m.(\exists\langle I_i, J_i\rangle, \langle I_j, J_j\rangle \in \cup G(I_i, I_j \subset S_{\langle m\rangle} \wedge d(J_i, J_j) \geq r)).$$

Assume then for induction that $f(n)$ has been defined. This means that f now can be treated as a finite sequence. Consider the set

$$A_{n+1} = \{p \in \mathbb{P} \mid \exists k \in \mathbb{N}\exists\langle I_i, J_i\rangle, \langle I_j, J_j\rangle \in p(I_i, I_j \subset S_{f*\langle k\rangle} \wedge d(J_i, J_j) \geq r)\}.$$

We know from the induction hypothesis that there exist $q' \in G \cap A_n$. Then, equivalently as for (5.10), let J_1, J_2 be such that

$$\exists\langle I_1, J_1\rangle, \langle I_2, J_2\rangle \in q'(d(J_1, J_2) \geq r \wedge I_1, I_2 \text{ not } \sigma\text{-compact}).$$

Let p be arbitrary such that $q' \prec p$. Since $\neg T^*(S_f)$ we can, as in the induction start, define

$$k = \mu m.(\forall\langle I,J\rangle \in p(S_{f*\langle m\rangle} \cap I \neq \emptyset \Rightarrow (S_f \setminus I)\ \sigma\text{-compact})).$$

Then we can consistently extend p with the elements

$$\langle S_{f*\langle k,0\rangle}, J_1\rangle$$

and
$$\langle S_{f*\langle k,1\rangle}, J_2\rangle.$$
So A_{n+1} is dense below p and $G \cap A_{n+1} \neq \emptyset$. This means that we can define

$$f(n+1) = \mu m.(\exists\langle I_i, J_i\rangle, \langle I_j, J_j\rangle \in \cup G(I_i, I_j \subset S_{f*\langle m\rangle} \wedge d(J_i, J_j) \geq r)).$$

This completes the construction of f, and it remains to show that $g(\bar{f})$ is not well defined. The existence of f implies that there are sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ in X' such that
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \bar{f}$$
and
$$d(g(a_n), g(b_n)) \geq r \text{ for all } n \in \mathbb{N}.$$
Thus, g is not well defined for $\bar{f}$. This completes the proof of the theorem. □

5.4 An Application to the Urysohn Space

Theorem 5.3.6 can be applied to a wide range of metric spaces. One such space is the Urysohn space, discovered by the russian mathematician Pavel Urysohn in 1925, see [1]. This is the *universal separable metric space.*

Definition 5.4.1. *A metric space (U, d) is called* Urysohn universal *if;*

- *U is a complete, separable metric space.*

- *If $A \subseteq B$ are finite metric spaces and $\phi : A \to U$ is an isometry, then ϕ can be extended to an isometry $\psi : B \to U$.*

Two Urysohn universal metric spaces are always isomorphic. This can be seen by constructing an isomorphism between the dense countable subsets of the spaces by induction. Since these spaces are unique up to isomorphism, they are in some sense the same space, and this is called *the* Urysohn space. This space has got nothing to do with the topological property of a space beeing *a* Urysohn space, which means that it is Hausdorff with a slightly stronger separation axiom.

Lemma 5.4.2. *Let U be the Urysohn space, $\mathcal{U}$ and $\Re$ domains representing U and R respectively. Then an $(\mathcal{U} \to \Re)_0$-generic G is not total.*

Proof. Urysohn proved that every separable metric space can be imbedded isomorphically into U. This means that there is an isomorphism $\psi : \mathbb{N}^{\mathbb{N}} \to U$. The proof of Theorem 5.2.1 then gives us a way of constructing an $x \in U$ for which G is not defined, by considering the subspace $\psi(\mathbb{N}^{\mathbb{N}})$ of U. $\square$

Intuitively, this result is not surprising. It has become clear that 'large' metric spaces give rise to non total generic functions. Thus, as a maximal complete separable metric space, U should indeed have this property. However, to show that

$$T(A) \Leftrightarrow T^*(A)$$

holds in U, so that Theorem 5.3.6 can be applied directly, a more technical argument is required.

Bibliography

[1] P. S. Urysohn. Sur un espace metriq universel, C. R. Acad. Sci. Paris 180, pp. 803-806 (1925).

[2] R. M. Amadio and P. Curien. Domains and Lambda-Calculi. Cambridge University Press, 1998.

[3] D. Normann. λ-kalkyle og domeneteori. University of Oslo, 2005.

[4] K. Kunen. Set Theory: An Introduction to Independence Proofs. Elsevier B.V., 1980.

[5] M.H. Escardo. PCF extended with real numbers: a domain-theoretic approach to higher-order exact real number computation. Ph.d. thesis, University of London, 1997.

[6] V. Stoltenberg-Hansen, I. Lindström and E. R. Griffor. Mathematical Theory of Domains. Cambridge University Press, 1994.

[7] J. Blanck. Domain representability of metric spaces. Annals of Pure and Applied Logic 83, 1997.

[8] J. Dugundji. Topology. Allyn and Bacon, Boston, 1966.

[9] K. R. Davidson and A. P. Donsig. Real Analysis with Real Applications. Prentice Hall, 2002.