Cox Rings of Projective Varieties

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Contents

Abstract

The aim of this paper is to survey some known properties of Cox rings of projective surfaces and also present some new results. These results are explicit descriptions of Cox rings of

- Del Pezzo surfaces and other rational surfaces occurring as blow-ups of $\mathbf{P}^2$ in special configurations of points. In particular, we show that the Cox ring of the blow up of $\mathbf{P}^2$ in n points lying on a line is a complete intersection ring with $2n+1$ generators.
- Rational threefolds arising as blow-ups of $\mathbf{P}^3$ in points. In case of five points, we show that the Cox ring is isomorphic to the coordinate ring of the Grassmannian $G(2,6)$. We show using a correspondence of Mukai to invariant theory, that this phenomenon generalizes to higher dimensions, i.e $n+2$ points in $\mathbf{P}^n$ give Cox ring isomorphic to the coordinate ring of $G(2,n+3)$.
- K3 surfaces of Picard number 2. We give a new proof of a result of Artebani, Hausen, and Laface on the finite generation of the Cox rings in this case. We also give strong bounds for the number of generators and investigate explicit models of Cox rings of some classical K3 surfaces including quartic surfaces in $\mathbf{P}^3$ and double covers of $\mathbf{P}^2$.

Chapter 1
Generalities on Cox rings

1.1 Introduction

The coordinate ring of a variety is a fundamental object in algebraic geometry. Indeed, classical projective geometry can be loosely described as the theory of translating geometric properties of the variety into algebraic properties of the coordinate ring, that is, in terms of commutative algebra. That being said, given a projective variety X, there is no *canonical* candidate for its coordinate ring, since it depends on the embedding of the variety into projective space, i.e it depends on a choice of a very ample line bundle and a generating set of sections.

In the early 1990s David Cox [Cox95] constructed the *homogeneous coordinate ring* of a toric variety to remedy this. The idea was to construct a multigraded polynomial ring that encodes much of the combinatorics of the defining fan. Loosely speaking, this ring is to a toric variety as the ring of polynomials $k[x_0, \dots, x_n]$ is to $\mathbf{P}^n$. When the toric variety is projective, the ring also gives information about the various projective embeddings. In addition, the ring realized many new similarities between toric varieties and projective space $\mathbf{P}^n$. For example, for a smooth projective toric variety X, the homogeneous coordinate ring is a polynomial ring R such that

- Every closed subvariety of X correspond to a graded ideal of R,
- X can be recovered as a geometric invariant theory quotient of $\operatorname{Spec}(R)$ by a torus action[1], and
- Coherent sheaves on X correspond to R-modules.

Aiming to generalize Cox' construction to a broader class of varieties, Hu and Keel [HK00] introduced the *Cox ring*, or *total coordinate ring* of a variety. The ring is essentially defined by

$$\operatorname{Cox}(X) = \bigoplus_{\mathscr{L} \in \operatorname{Pic}(X)} H^0(X, \mathscr{L}),$$

with some mild restrictions on X. See Section 1.2 for a precise definition. The Cox ring is thus the huge graded algebra consisting of all global sections from all line bundles on X. This ring need not be finitely generated in general. It is a nice exercise to check that the above definition coincides with Cox' homogeneous coordinate ring when X is toric, and $\operatorname{Pic}(X)$ is free (see Section 1.3).

[1] As is the case for $\mathbf{P}^n = (\operatorname{Spec} k[x_0, \dots, x_n] - 0) / \mathbb{G}_m$.

The first natural question is: *For which varieties X is* $\mathrm{Cox}(X)$ *a finitely generated k-algebra?*. This question has been a main focus in recent algebraic geometry, mainly because finite generation of the ring has important implications on the birational geometry of X. For example, the Cox ring being finitely generated means that the effective cone and nef cone are both polyhedral and there are only finitely many *small modifications*, i.e contracting birational maps $f : X \dashrightarrow X'$ with X' projective and $\mathbf{Q}-$factorial and f an isomorphism in codimension one. Also, the condition ensures that the Mori program can be carried out for any divisor on X [HK00, Prop. 1.11]. For these reasons, Hu and Keel call varieties with finitely generated Cox ring *Mori dream spaces*. It was conjectured in [HK00] that any log-Fano variety has a finitely generated Cox ring. This was recently proved by Birkar, Cascini, Hacon and McKernan in their groundbreaking paper [BCHM06].

There is also a surprising link between finite generation of the Cox ring and Hilbert's fourteenth problem. In its classical form, this problem asks if the ring of invariants S^G is finitely generated, where S is a polynomial ring and G is an algebraic group acting linearly on S. The answer was shown to be positive by Hilbert himself in the case when G is finite. In general however, the result does not hold. The first counterexample was found in 1958 by Nagata: Consider a linear subspace $G \subset \mathbb{G}_a^9$ of codimension 3. The group G induces a so-called *Nagata action* on $S = \mathbf{C}[x_1, \dots, x_9, y_1, \dots, y_9]$ by $x_i \to x_i$, $y_i \to y_i + t_i x_i$ for $1 \leq i \leq 9$. The ingenious idea of Nagata was to relate S^G to (what we today know as) a Cox ring of the blow-up of $\mathbf{P}^2$ in 9 general points. It is not hard to see that this variety has infinitely many (-1)-curves, all of which are extremal in the effective cone. The effective cone is then not finitely generated, contradicting the finiteness of the Cox ring. This example was generalized by Mukai in [Muk01], who considered more general blow-ups of projective space $\mathbf{P}^n$. His precise result is

Theorem (Mukai). *Let X be the blow-up of the projective space $\mathbf{P}^{r-1}$ in n points in general position. Then* $\mathrm{Cox}(X)$ *is not finitely generated if* $\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} \geq 1$.

In particular, we need $n \geq 9$ general points in $\mathbf{P}^2$, and $n \geq 8$ points in $\mathbf{P}^3$ for infinite generation. See [Muk01] for more details.

The second natural question is: *Given that* $\mathrm{Cox}(X)$ *is finitely generated, can we find its explicit generators and relations?* This means that we choose generating sections $x_1, \dots, x_n$ from the vector spaces $H^0(X, D_1), \dots, H^0(X, D_n)$ (here some of the D_i may coincide) and regard $\mathrm{Cox}(X)$ as a quotient

$$\mathrm{Cox}(X) = k[x_1, \dots, x_n]/I_X.$$

Here we consider the natural $\mathrm{Pic}(X)-$grading on $k[x_1, \dots, x_n]$ and I_X given by letting $\deg(x_i) = D_i$, so that $\mathrm{Cox}(X)$ is in fact a *multigraded* ring. The ideal I_X is always a prime ideal, since $\mathrm{Cox}(X)$ has no zero divisors. In fact, under our assumptions, $\mathrm{Cox}(X)$ is an UFD by the results in [Arz08].

In spite of the fact that the definition of the Cox ring is very explicit, finding its presentation is in general a very hard problem. It requires a lot of information about linear systems and divisors of the variety X. An important example is the calculation of the Cox ring of Del Pezzo surfaces, which is the content of the Batyrev-Popov conjecture. This problem, originally formulated in [BP04], has gained a formidable amount of attention in recent literature in algebraic geometry [Der06, STV06, LV07, TVV08, SX08], and shows that describing the behaviour of the Cox ring under blow-ups is a highly non-trivial problem. The ideals of relations quickly become very complicated, and computer calculations are infeasible. For example, when X is a degree one Del Pezzo surface, the Cox ring is minimally generated by 242 sections, and the ideal I_X above is generated by 17399 quadrics [TVV08].

One may ask which varieties correspond to ”simple” Cox rings. Toric varieties are the simplest in this respect since their Cox ring is a polynomial ring. In fact, using GIT, Hu and Keel show that also the converse is true: A variety whose Cox ring is a polynomial ring is also toric. The next step is to study varieties whose Cox rings have a unique defining relation. Some examples of such spaces are given in [BH07] and [Der06]. Other than this, few actual computations of Cox rings has been carried out.

There exists one method which in principle works for any surface, namely Laface and Velasco's complex. This method was introduced in [LV07] to study the Cox rings of Del Pezzo surfaces. Recently, Artebani, Laface and Hausen [AHL09] also investigated Cox rings of certain K3 surfaces using this method. The basic idea is to reduce the problem of finding minimal relations in the ideal to the vanishing of certain homology groups. In the case of Del Pezzo surfaces, the latter problem in turn reduces to an interesting combinatorial game on the graph of exceptional curves. The method relies on a predefined set of generators for the ring and also heavily on vanishing theorems like the Kawamata-Vieweg vanishing theorem. Although this approach is appealing, the methods for computation of the homology groups are usually very ad hoc and so the method is hard to apply in general.

One of the aims of this thesis is to provide more computations of Cox rings and apply them to study the varieties in question. We will avoid the methods of [LV07], searching for new techniques of computation. That being said, it seems futile to hope for a general strategy for computing a Cox ring: Each of the varieties we study has its own special geometric properties which must be employed to get information about the generators and the ideal of relations of the Cox ring. In particular, one needs to choose explicit generators wisely in order to say something at all about the relations. Our main focus will be on surfaces, since there is already a great deal of classical theory to utilize for these purposes. Also, studying numerical traits such as nefness and ampleness is easier on surfaces, since divisors are curves: This allows us to easily apply numerical criteria like the Nakai-Moishezon criterion.

The computation of $\mathrm{Cox}(X)$ can be divided into two subproblems as follows:

- Show that $\mathrm{Cox}(X)$ is finitely generated and find explicit generating sections.
- Find the relations between these and prove that they generate the whole ideal.

We think that both of these questions are hard problems in general. Finite generation of the ring is perhaps a more important question in itself, and has been a main focus in the study of Cox rings in recent literature. On the other hand, very few authors actually find explicit generators. A few notable exceptions are Castravet's articles [Cas07, CT06] and [BP04]. The main tools we use to approach this question are Koszul cohomology, Zariski's theorem and induction.

The second question is perhaps somewhat more delicate, since there are no familiar techniques to apply or natural line of attack. Finding some relations is easy, by looking at the multigraded Hilbert function of $\mathrm{Cox}(X)$ - the main difficulty here is proving that these generate the entire ideal. Although we will try many different approaches to it in this thesis, we have not found any preferred method.

The thesis is organized as follows. In Chapter 2, we investigate Cox rings of certain rational surfaces with effective anticanonical divisor. We show that Cox rings of blow-ups of $\mathbf{P}^2$ in ≤ 8 distinct points are finitely generated. In particular, when the points are in general position, this is a converse of Mukai's result above. Our results are somewhat constructive in the sense that they tell us where to look for generators of the ring. For example, we will see that we need a generator for each curve with negative self-intersection. In the rest of the chapter, we study Del Pezzo surfaces. We give a proof of the Batyrev-Popov conjecture for Del Pezzo surfaces of degree ≥ 3 and give geometric interpretations for the defining relations of the ideal I_X. We also study the Cox ring of degree 5 Del Pezzo surfaces in greater detail. In particular, we study the syzygies and the resolution of $\mathrm{Cox}(X)$ and find a quadratic Gröbner basis for the ideal.

In Chapter 3, we investigate rational surfaces X which arise as blow-ups of $\mathbf{P}^2$ in special configurations of points. We find that the Cox rings actually become simpler as the points move into special positions. We study in detail the extreme case when all the points lie on a line. In contrast to earlier results, we show that in this case, the Cox ring is always finitely generated for any number of blown-up points. We find explicit generators and calculate the defining ideal using Gröbner bases and combinatorics. The main result is that $\mathrm{Cox}(X)$ is a complete intersection ring with defining ideal generated by quadrics.

In Chapter 4, we try to extend the techniques in Chapter 2 to threefolds occurring as blow-ups of $\mathbf{P}^3$ in points. We find that as in the case of the quintic Del Pezzo, $\mathbf{P}^3$ blown up in 5 general points has a Cox ring isomorphic to the coordinate ring of a Grassmannian variety. This turns out to be true in higher dimensions as well, as we show using invariant theory.

In Chapter 5, we study Cox rings of K3 surfaces with Picard number 2. We find that it is hard to say something in general about the defining ideal in this case, although the Cox ring is generally finitely generated if its effective cone is. However, we are able to compute the Cox ring in some cases, for example if if we assume that the Picard group is generated by two projective lines or two elliptic curves.

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Notations and basic results

We fix some notations. We denote by X a normal $n-$dimensional projective variety over an algebraically closed field k of characteristic zero. We assume further that X has finitely generated free Picard group $\mathrm{Pic}(X)$. Since X is normal, we will freely pass between the three notions *divisor class*, *line bundle* and *invertible sheaf*. In general, the notation will follow Hartshorne [Har77].

Let $A(X) = \bigoplus_{i=0}^{n} A^i(X)$ be the Chow ring, graded by codimension. We set

$$N^1(X) = A^1(X)/\equiv, \qquad N_1(X) = A^{n-1}(X)/\equiv,$$

where $\equiv$ denotes numerical equivalence. The pair $(N^1(X), N_1(X))$ is the *Neron-Severi* bilattice of X and comes with a pairing

$$N^1(X) \times N_1(X) \to \mathbf{Z}, \qquad (C, D) \mapsto C.D$$

defined by the intersection of cycles. We extend this pairing to the real vector spaces $N^1(X) \otimes \mathbf{R}$ and $N_1(X) \otimes \mathbf{R}$. These are finite-dimensional vector spaces and we call their dimension, ρ_X the *Picard number* of X. In all cases in this thesis numerical equivalence will equal linear equivalence, so that we will have $\mathrm{Pic}(X) \cong N^1(X, \mathbf{Z})$. We will use capital letters for divisors and divisor class interchangeably - hopefully this sloppy notation will be clear from the context. We will also use the standard short-hand notations $H^0(X, D) = H^0(X, \mathscr{O}_X(D))$ and $h^0(X, D) = h^0(X, \mathscr{O}_X(D))$.

A divisor class D is said to be *nef* (or numerically eventually free) if $D.C \geq 0$ for each curve $C \subset X$, and is *big* if $D^n > 0$. Let $NE^1(X) \subset N^1(X)$ denote the monoid of effective divisors and $NM^1(X)$ the monoid of nef divisors. We let $NE^1(X, \mathbf{R})$ denote the (pseudo)effective cone, i.e the smallest real closed cone containing all the effective divisors of X. Similarly, we define $NM^1(X, \mathbf{R})$ as the *nef cone* of X. Note that for surfaces these cones are dual in the sense that

$$D \in NM^1(X, \mathbf{R}) \Longleftrightarrow D.C \geq 0, \text{ for all } C \in NE^1(X, \mathbf{R}).$$

These cones will usually be finitely generated in this thesis since this is a necessary condition for finite generation of the Cox rings. At this point it is appropriate to mention the following general results:

Theorem (Kleiman) *The interior of the nef cone, $NM^1(X, \mathbf{R})^\circ$ is the ample cone of X, i.e the cone generated by ample divisor classes.*

Theorem (Hodge Index Theorem) *If E is a divisor on X such that $E^2 > 0$, then for every divisor D on X such that $E.D = 0$ we have $D^2 \leq 0$. Furthermore, $D^2 = 0$ if and only if $D \equiv 0$.*

We will often use the following equivalent result: If D_1, D_2 are numerically independent divisors such that $(aD_1 + bD_2)^2 > 0$ for some $a, b \in \mathbf{R}$, then

$$\begin{vmatrix} D_1^2 & D_1D_2 \\ D_1D_2 & D_2^2 \end{vmatrix} < 0.$$

The next theorem along with Riemann-Roch will be our main tool for computing ranks of cohomology groups. By Kleiman's theorem, it can be seen as a generalization of the Kodaira vanishing theorem.

Theorem (Kawamata-Vieweg Vanishing) *Let D be a nef and big divisor on a smooth projective variety. Then $H^p(X, K + D) = 0$ for all $p > 0$.*

A divisor D is said to be semiample if the linear system $|nD|$ is base-point free for large n. Note that if D is base-point free then $D.C \geq 0$ for any curve C, since we can choose a representative of D not passing through a given point $p \in C$. So semiample divisors are nef. It turns out that it will be important to prove the converse to this in some cases. This is mainly because of the next result, namely Zariski's theorem [Laz05, Ex. 2.1.30], which concerns finite generation of the section ring for semiample line bundles.

Theorem (Zariski) *Suppose that $\mathscr{L}$ is a semiample line bundle on a normal projective variety X. Then the section ring of $\mathscr{L}$*

$$\bigoplus_{k\geq 0} H^0(X, \mathscr{L}^{\otimes k})$$

is finitely generated.

We also recall a theorem due to Mori [Mor79], at the heart of the famous 'bend and break' technique:

Theorem (Mori) *Let $C \subset X$ be a rational curve such that $-K.C \geq n + 2$. Then C can be deformed into a cycle which is the sum of ≥ 2 rational curves.*

This theorem will be important in studying effective cones in Chapter 2.

1.2 Cox rings of Projective Varieties

In this and the remaining sections we survey some well-known properties of Cox rings. The formal definition goes as follows:

Definition 1.1. Let X be a projective variety whose Picard group $\mathrm{Pic}(X)$ is free of rank r and coincides with $N^1(X)$. We define the Cox ring of X to be

$$\mathrm{Cox}(X) = \bigoplus_{(m_1,\ldots,m_r)\in\mathbf{Z}^r} H^0\left(X, \mathscr{L}_1^{\otimes m_1} \otimes \cdots \otimes \mathscr{L}_r^{\otimes m_r}\right)$$

where we have chosen a collection $\mathscr{L}_1, \ldots, \mathscr{L}_r$ of line bundles on X whose classes form a $\mathbf{Z}$-basis of $\mathrm{Pic}(X)$. The ring product is given via the canonical multiplication map

$$H^0(X, \mathscr{L}) \otimes H^0(X, \mathscr{L}') \to H^0(X, \mathscr{L} \otimes \mathscr{L}').$$

It is possible to define the Cox ring also when $\mathrm{Pic}(X)$ differs from $N^1(X)$. See [Arz08] and [BH07] for definitions using $\mathrm{Cl}(X)$. Note by the way that the assumptions of the definition are fulfilled if $H^1(X, \mathscr{O}_X) = 0$ (say, when X is rational or a K3 surface) since by [Bea96, I.10], $\mathrm{Pic}^0(X)$ is isomorphic to the quotient $H^1(X, \mathscr{O}_X)/H^1(X, \mathbf{Z})$.

In this thesis all varieties will be normal, so every line bundle will be of the form $\mathscr{O}_X(D)$ for some divisor D. In this setting the Cox ring can be equivalently defined by choosing a finite set of Cartier divisors $D_1, \ldots, D_r$ generating $\mathrm{CaCl}(X)$, and defining

$$\mathrm{Cox}(X) = \bigoplus_{(m_1,\ldots,m_r)\in\mathbf{Z}^r} H^0\left(X, \mathscr{O}_X(m_1 D_1 + \ldots + m_r D_r)\right).$$

In this setting, the ring product coincides with multiplication of sections as functions in $k(X)$.

At first the definition of $\mathrm{Cox}(X)$ may seem a bit unsettling, since it depends both on the basis of $\mathrm{Pic}(X)$ and the choice of particular representatives of each isomorphism class. In fact, there is no canonical way of choosing a $\mathbf{Z}$-basis for $\mathrm{Pic}(X)$. Moreover, even after such a choice has been made we still need to choose specific divisors, rather than divisor classes. This is because even if D and D' are linearly equivalent, there exist no natural isomorphism between the vector spaces $H^0(X, \mathscr{O}_X(D))$ and $H^0(X, \mathscr{O}_X(D'))$. This lack of naturality destroys some functorial properties of $\mathrm{Cox}(X)$. However, as one might expect, all of these choices yield isomorphic rings and there is nothing to worry about.

We note that the Cox ring contains a lot of geometric information about our variety X. For example, suppose D is a very ample divisor, giving an embedding $i : X \hookrightarrow \mathbf{P}^n$ and $\mathscr{O}_X(D) = i^*\mathscr{O}_{\mathbf{P}^n}(1)$. We have a subring of $\mathrm{Cox}(X)$ given by

$$R(X, D) = \bigoplus_{m\in\mathbf{Z}} H^0\left(X, \mathscr{O}_X(mD)\right).$$

We recognize this from [Har77, II.5, Ex. 5.13-14 and II.7] where it is shown that X is completely determined by $X \cong \mathrm{Proj}(R(X, D))$. In this respect, the Cox ring has all the 'coordinate rings' as subrings, which explains the name 'total coordinate ring'. In particular, when $\mathrm{Pic}(X) \cong \mathbf{Z} \cdot H$ is generated by a very ample divisor H, and the homogeneous coordinate ring $S(X)$ under the projective embedding in $|H|$ is an UFD, $\mathrm{Cox}(X) \cong S(X)$. This happens for example for all Grassmannians $G(m, n)$ [LV09].

Proposition 1.2. *If* $\mathrm{Cox}(X)$ *is finitely generated, then its dimension is given by* $\mathrm{rk}\,\mathrm{Pic}(X) + \dim X$.

Proof. See [BP04, Remark 1.4].

1.3 Toric Varieties and Cox' Construction

We recall basic facts on toric varieties. The main references are [Ful93] and [Cox95].

A toric variety is a normal variety containing an open dense algebraic torus $T \cong \mathbb{G}_m^n$, whose action extends to an action $T \times X \to X$. Such varieties are determined by data from convex geometry. To be precise, let $N = \mathrm{Hom}(k, T) \cong \mathbf{Z}^n$ be the character lattice of T and $N_{\mathbf{R}} = N \otimes \mathbf{R} \cong \mathbf{R}^n$ the induced vector space. The dual lattice (resp. space) is denoted by M (resp. $M_{\mathbf{R}}$), and there is a natural pairing $\langle\ ,\rangle : M \times N \to \mathbf{Z}$. A cone σ

is a subset $N_{\mathbf{R}}$ generated by non-negative linear combinations of a finite set of integer vectors $\{v_1, \ldots, v_r\}$. We assume that cones are strictly convex, i.e they contain no line through the origin. Each cone σ has a dual cone $\sigma^\vee = \{m \in M | \langle m, v\rangle \geq 0, \forall v \in \sigma\}$ in M. A fan Δ consists of a finite collection of cones such that each face of a cone is also in Δ, and any pair of cones σ, σ' intersect in a common face.

To each cone σ in N we can associate an affine variety U_σ. More precisely, define $S_\sigma = \sigma^\vee \cap M$, which is a monoid, and consider the group algebra $k[\sigma^\vee \cap M]$, which is a finitely generated k-algebra. It consists of linear combinations of monomials χ^m, $m \in S_\sigma$, and multiplication is induced by the addition in S_σ. We define $U_\sigma = \operatorname{Spec} k[S_\sigma]$. Given a fan Δ we obtain a variety X by gluing together the affine varieties $U_\sigma, \sigma \in \Delta$. All toric varieties arise in this way.

Example 1.3. The fan corresponding to $X = \mathbf{P}^2$ is shown below. The corresponding cones and affine varieties are given in the following table

σ	S_σ	U_σ
$\langle e_1, e_2\rangle$	$\langle e_1^*, e_2^*\rangle$	$\operatorname{Spec} k[X, Y] \cong k^2$
$\langle e_2, -e_1 - e_2\rangle$	$\langle -e_1^*, -e_1^* + e_2^*\rangle$	$\operatorname{Spec} k[X^{-1}, YX^{-1}] \cong k^2$
$\langle e_1, -e_1 - e_2\rangle$	$\langle -e_2^*, e_1^* - e_2^*\rangle$	$\operatorname{Spec} k[Y^{-1}, XY^{-1}] \cong k^2$

Note how the affine toric varieties coincide with the affines in the standard covering of $\mathbf{P}^2$.

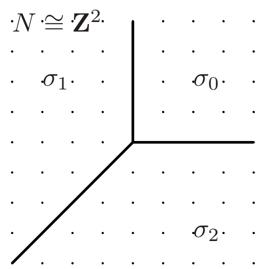


Fig. 1.1 The fan of $\mathbf{P}^2$.

Given a fan Δ in $N_{\mathbf{R}}$, let $\Delta(1)$ be the set of 1-dimensional cones in Δ. By the 'orbit-cone correspondence' [Ful93], these correspond to torus-invariant divisors D_ρ on X. Let $\mathbf{Z}^{\Delta(1)}$ be the free group on the D_ρ, and for each cone $\rho \in \Delta(1)$, let $v_p \in N$ be its unique generator. We then have an exact sequence

$$M \to \mathbf{Z}^{\Delta(1)} \to \operatorname{Pic}(X) \to 0$$

where the first map is $m \mapsto \sum_{\rho\in\Delta(1)} \langle m, v_p\rangle D_p$. In particular, $\operatorname{Pic}(X)$ is generated by D_ρ for $\rho \in \Delta(1)$.

The original construction of Cox was to consider the polynomial ring

$$R = k[x_\rho : \rho \in \Delta(1)],$$

with multigrading given by $\deg(x_\rho) = D_\rho$. So for example, in the case $X = \mathbf{P}^2$ we have three such cones, and we recover the standard coordinate ring $S = k[x_0, x_1, x_2]$. Note also that the fan Δ is recovered by the multigrading. We show that this ring coincides with the previous definition of $\operatorname{Cox}(X)$:

Proposition 1.4. *The degree D part of R coincides with $H^0(X, D)$.*

Proof. Let $D = \sum a_\rho D_\rho$. By [Ful93, §3.4], $H^0(X, D)$ is spanned by monomials x^m such that $\langle m, v_\rho \rangle \geq -a_\rho$ for all $\rho \in \Delta(1)$. We then have a bijective map between monomials $x^m \in H^0(X, D)$ and monomials $v_1^{\langle m, v_1 \rangle + a_1} \cdots v_n^{\langle m, v_n \rangle + a_n}$ in R_D. It is clearly injective, since the v_ρ span $N_{\mathbf{R}}$, and also surjective: Let $x_{\rho_1}^{b_1} \cdots x_{\rho_n}^{b_n}$ a monomial in R_D. Then $\sum (b_\rho - a_\rho) D_\rho = D - D = 0$, so by the exact sequence above, there exists an $m \in M$ such that $b_\rho - a_\rho = \langle m, v_\rho \rangle$ for all ρ, and m satisfies the above inequality since $\langle m, v_\rho \rangle = b_\rho - a_\rho \geq -a_\rho$. □

1.4 Cox rings in Geometric Invariant Theory

We recall some basics of GIT. For ease of exposition we take X to be a projective variety, although the GIT applies to general irreducible schemes with some modifications. Let G be an algebraic group acting on X. A G-invariant map $p : X \to Y$ is called a *categorical quotient* by G, if for every G-invariant map $f : X \to Z$ there exist a unique $\bar{f} : Y \to Z$ such that $\bar{f} \circ p = f$.

A $G-$equivariant map $p : X \to Y$ is a *good quotient* if p satisfies:

- For all open sets $U \subseteq Y$, $p : \mathscr{O}_Y(U) \to \mathscr{O}_X(p^{-1}(U))$ is an isomorphism onto the subring $\mathscr{O}_X(p^{-1}(U))^G$ of $G-$invariant functions.
- If $W \subseteq X$ is closed and $G-$invariant, then $p(W)$ is closed.
- If $U, V \subseteq X$ are both closed, disjoint and $G-$invariant, then $p^{-1}(U) \cap p^{-1}(V) = \emptyset$.

The main objective of GIT is to study possible good quotients of the orbit space of X by G. For this, one considers the data $(\mathscr{L}, \pi, \overline{\sigma})$, where $\mathscr{L}$ is a line bundle on X with projection $\pi : \mathscr{L} \to X$, and $\overline{\sigma}$ is a $G-$linearization of $\mathscr{L}$, i.e., an extension of the action $\sigma : G \times X \to X$ to an action $\bar{\sigma} : \mathscr{L} \to \mathscr{L}$ such that the zero-section of π is G invariant, and the following diagram commutes:

$$\begin{array}{ccc} G \times \mathscr{L} & \xrightarrow{\bar{\sigma}} & \mathscr{L} \\ \big\downarrow{\scriptstyle id \times \pi} & & \big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

If such a linearization is given, we get a linearization on all tensor powers $\mathscr{L}^{\otimes N}$. We define the set of *semi-stable points* with respect to $\mathscr{L}$ as

$$X^{ss}(\mathscr{L}) = \bigcup_{N \geq 0} \bigcup_{s \in H^0(X, \mathscr{L}^{\otimes N})} \operatorname{supp}(s)$$

(More generally, when X is not necessarily projective, we also require the sets X_s to be affine). We define the set of unstable points as the complement $X \setminus X^{ss}(\mathscr{L})$.

By fundamental theorems of Mumford in [MFK94] the *GIT quotient* $X^{ss}//G$ always exists as a quasiprojective variety, and in case X is projective and $\mathscr{L}$ is ample, $X^{ss}//_G \cong \operatorname{Proj}(R_{\mathscr{L}})$, where $R_{\mathscr{L}}$ is the ring

$$\bigoplus_{n \geq 0} H^0(X, \mathscr{L}^{\otimes n})^G.$$

where $H^0(X, \mathscr{L}^{\otimes n})^G$ are the G-invariant sections of $\mathscr{L}^{\otimes n}$. The GIT quotient is different from the topological quotient since it is not an orbit space in general. Topologically, it is the quotient of X^{ss} by the new equivalence relation

$$x \sim y \Leftrightarrow \overline{G \cdot x} \cap \overline{G \cdot y} \cap X^{ss} \neq \emptyset$$

Consider the case where $X = \mathrm{Spec}(R)$, where $R = \mathrm{Cox}(X)$, and $G = \mathrm{Hom}(\mathrm{Pic}(X), k^*) \cong \mathbb{G}_m^\rho$ is an algebraic torus. For any $D = a_1 D_1 + \ldots + a_\rho D_\rho \in \mathrm{Pic}(X)$, G acts naturally on each $H^0(X, D)$ by

$$G \times H^0(X, D) \to H^0(X, D)$$
$$(x, s) \mapsto x^D s$$

where $x = (x_1, \ldots, x_\rho)$ and $x^D = x_1^{a_1} \cdots x_\rho^{a_\rho}$. Thus G acts naturally on the affine variety V. We will consider GIT quotients of V by G.

Consider the trivial bundle $\mathscr{L} = V \times \mathrm{Spec}(k[t])$ on V. A linearization of $\mathscr{T}$ can be given by choosing a divisor class D and its character $\chi_D \in \mathrm{Hom}(G, k^*) \cong Pic(X)$. Explicitly, the linearization corresponds to a homomorphism

$$R[t] \to R[t] \otimes_k k[G] \tag{1.1}$$
$$t \mapsto t \otimes x^D \tag{1.2}$$

We consider the ring $\bigoplus_{n \geq 0} H^0(V, \mathscr{T}^{\otimes n})$, and its subring, $\mathscr{R}_{\mathscr{T}}$ of G-invariant sections. Note that such sections correspond to R-algebra homomorphisms $s : R[t] \to R$ which make the following diagram commutative:

$$\begin{array}{ccc} R[t] & \longrightarrow & R[t] \otimes k[G] \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s \otimes id} \\ R & \longrightarrow & R \otimes k[G]. \end{array}$$

Such homomorphisms are determined by where they send t, and by looking at the $\mathrm{Pic}(X)$ grading, we see that they are in 1-1 correspondence with the sections in $H^0(X, D)$, thus

$$H^0(V, \mathscr{T})^G = H^0(X, D).$$

Taking higher tensor powers, we have

$$\left(\bigoplus_{n \geq 0} H^0(V, \mathscr{T}_D^{\otimes n}) \right)^G \cong \bigoplus_{n \geq 0} H^0(X, nD)$$

Thus the GIT quotient of V by G is nothing but the scheme $\mathrm{Proj}(\bigoplus_{n \geq 0} H^0(X, nD))$. When D is ample, this equals X.

The semi-stable points $X^{ss} \subseteq V$ are now of interest. We define the *irrelevant ideal* I_X as the ideal of the points which are not semi-stable, i.e the points where $s(x) = 0$, for all s in $\bigoplus_{n \geq 0} H^0(V, \mathscr{T}^{\otimes n})$. Concretely, let $s_1, \ldots, s_N$ be generators for $H^0(X, D)$. Then the ideal of the unstable points is given by

$$I_X = \sqrt{(s_1, \ldots, s_N)}.$$

When D is an ample divisor, we see that X is a good geometric quotient of $X^{ss} = \mathrm{Spec}(R) - V(I_X)$ by G.

When X is a toric variety, the ideal I_X coincides with the ideal $(x^{\bar{\sigma}} : \sigma \in \Delta)$ where $x^{\bar{\sigma}}$ is the product of all variables x_ρ such that $\rho \not\subset \sigma$. The latter ideal is the *toric irrelevant ideal* of Cox(X) and by the above, we recover a theorem of Cox [Cox95, Thm. 2.1].

It is natural to study how this quotient changes as we vary the divisor D. Here we do not require D to be ample. This problem is central in the so-called *variational geometric invariant theory* and is investigated in [HK00].

1.5 Examples when Cox(X) is not finitely generated.

In this section we present some classical examples of surfaces whose Cox ring is not finitely generated.

It is a standard fact that if D is an effective divisor on a surface, Γ a curve and $D.\Gamma < 0$, then Γ is a fixed component of the linear system $|D|$ and $\Gamma^2 < 0$. This is because we may write $D = a\Gamma + D'$ where $a \geq 0, \Gamma \not\subset D'$ and hence $D'.\Gamma \geq 0$. Then $D.\Gamma = a\Gamma^2 + D'.\Gamma$ can only be negative if $\Gamma^2 < 0$ and $a > 0$.

Lemma 1.5. *Let X be a surface containing an infinite number of curves of negative self-intersection. Then* Cox(X) *is not finitely generated.*

Proof. It suffices show that $NE^1(X, \mathbf{Z}) = \{D \in N^1(X) : \mathrm{Cox}(X)_D \neq 0\}$ is not finitely generated, since Cox(X) is graded by this monoid. Suppose that the classes of the divisors $C_1, \dots, C_N$ generate $NE^1(X, \mathbf{Z})$. Let E be a curve on X with negative self-intersection. Then $E \sim \sum_i m_i C_i$ for $m_i \geq 0$, since E is effective. Note that

$$E^2 = \sum_i m_i (C_i.E).$$

The right-hand side can only be negative if some $C_i.E < 0$, so E is a component of C_i. Since each of the C_i can only have finitely many fixed components, this contradicts the assumption that X had infinitely such E. □

Definition 1.6. A curve E on X is called an exceptional curve (of the first kind) if it is smooth and rational and $E^2 = -1$. Or alternatively, by the genus formula, it is an integral curve E satisfying $E^2 = -1$ and $-K.E = 1$.

Lemma 1.7. *Irreducible curves E, with $E^2 < 0$ are extremal in the effective cone, i.e., If $E = A + B$, for $A, B \in NE^1(X, \mathbf{Z})$, then either $A = 0$ or $B = 0$.*

Proof. This is a well-known result in Mori theory. See [Deb01, p. 145]. □

We present some classical examples due to Nagata [Nag60] of varieties with infinitely many exceptional curves. Let $p_1, \dots, p_9$ be points in $\mathbf{P}^2$ which are the intersection of two cubic curves. Let $\pi : X \to \mathbf{P}^2$ be the blow-up of the plane in these points, and let $E_1, \dots, E_9$ be the exceptional divisors. It is well-known that X has infinitely many exceptional curves, so by Lemma 1.5, Cox(X) cannot be finitely generated. The usual proof of this is based on computing the Mordell-Weil group (the group of sections) of the morphism $X \to \mathbf{P}^1$ given by the anticanonical system $|-K_X| = |3L - E_1 - \ldots - E_9|$, since sections of this morphism correspond to exceptional curves. This group is known to be isomorphic to $\mathbf{Z}^8$ [Deb01], so in particular there are infinitely many of them.

Suppose now that the points are in general position. Also here we get infinitely many exceptional curves. We give a proof of this based on the Cremona transformation, following an exercise in Hartshorne [Har77, V.4.15]. Suppose there are only finitely many exceptional curves. In particular there exists a divisor D with divisor class $aL - b_1E_1 - \ldots - b_9E_9$, with $b_1 \leq b_2 \leq \ldots \leq b_9$ and maximal $a > 0$. Consider the divisor class

$$\tilde{D} = (2a-b_1-b_2-b_3)L-(a-b_2-b_3)E_1-(a-b_1-b_3)E_2-(a-b_1-b_2)E_3-b_4E_4-\ldots-b_9E_9.$$

This divisor class corresponds to the image of D after performing a Cremona transformation based at p_1, p_2, p_3, and in particular, $\tilde{D}$ is the class of an exceptional curve. We claim that $2a - b_1 - b_2 - b_3 > a$, so that $\tilde{D}$ has higher coefficient of L than D, contradicting the maximality of a. Suppose to the contrary that $a - b_1 - b_2 - b_3 \leq 0$. Then

$$\begin{aligned} -K.\tilde{D} &= 3a - b_1 - \ldots - b_9 \\ &\leq (a - b_1 - b_2 - b_3) + (a - b_1 - b_2 - b_3) + (a - b_1 - b_2 - b_3) \\ &\leq 0. \end{aligned}$$

This contradicts the genus formula since $-K.\tilde{D} = 1$. Hence $2a - b_1 - b_2 - b_3 > a$ and we are done.

Remark. It is possible to make a formula parameterizing infinitely many exceptional curves on X by looking at the system of Diophantine equations $D^2 = -1$ $-K.D = 1$. One possibility is

$$3k(k+1)L - k(k+2)E_1 - k^2E_2 - k(k+1)(E_3 + \ldots + E_9) \qquad k = 0, 1, 2, \ldots$$

Remark. It is also well-known that a K3 surface of Picard number 20 (e.g the Fermat quartic $X = Z(x_0^4 + x_1^4 + x_2^4 + x_3^4) \subset \mathbf{P}^3$) has infinitely many curves of self-intersection -2 (see [Kov94, §7]) and hence have infinitely generated Cox ring.

Chapter 2
Cox rings of Rational Surfaces with effective anticanonical divisor

In this chapter we investigate Cox rings of certain rational surfaces with effective anticanonical divisor $-K$. Standard examples are Del Pezzo surfaces and Hirzebruch surfaces. Cox rings of such surfaces were first approached in this generality by Testa, Várilly and Velasco in the recent paper [TVV09]. Their main result is the finite generation of Cox rings of rational surfaces for which $-K$ is big. The aim of this chapter is to give related results on finite generation of Cox rings of anticanonical surfaces. In particular, we study blow-ups of $\mathbf{P}^2$ in distinct points and also study to Del Pezzo surfaces in greater detail.

2.1 Complete linear systems and vanishing on an anticanonical surface

Throughout this chapter, an *anticanonical rational surface* will refer to a non-singular rational surface with $-K$ effective. These have been thoroughly studied by Harbourne in [Har97, Har98]. Here we recall some basic facts and vanishing theorems on such surfaces. The results of this section are mostly standard and follow in some way from results in Chapter V in [Har77].

Proposition 2.1. *Let $\pi : Y \to X$ be a birational map of non-singular projective surfaces. Let $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} Y$ be the pullback. Then the higher direct images $R^i\pi_*\mathscr{O}_X$ vanish, and for any $\mathscr{L} \in \operatorname{Pic}(X)$ and $i \geq 0$,*

$$H^i(Y, \pi^*\mathscr{L}) \cong H^i(X, \mathscr{L})$$

Proof. It is well-known that any birational map can be realized as a composition of finitely may blow-ups in points and contractions [Har77, V.5.5]. The first result $R^i\pi_*\mathscr{O}_X = 0$ now follows from [Har77, III.8]. From the projection formula, we get

$$R^i\pi_*(\mathscr{O}_Y \otimes_{\mathscr{O}_Y} \pi^*\mathscr{L}) \cong R^i\pi_*\mathscr{O}_Y \otimes_{\mathscr{O}_X} \mathscr{L},$$

so $R^i\pi_*\pi^*\mathscr{L} = 0$. The vanishing of $R^i\pi_*\pi^*\mathscr{L}$ and the isomorphism $\pi_*\pi^*\mathscr{L} \cong \mathscr{L}$ imply that $H^i(Y, \pi^*\mathscr{L}) \cong H^i(X, \mathscr{L})$ by [Har77, Ex. III. 8.1]. □

Note that the proposition implies that π^* also preserves effectiveness of divisors in the sense that D is effective if and only $\pi^*(D)$ is.

Lemma 2.2. *Let X be a non-singular rational projective surface and let D be an effective divisor class on X. Then $h^2(X, D) = 0$.*

Proof. This is a standard argument. We claim that $K - D$ cannot be effective. This is because multiplication by a section in $H^0(X, D)$ gives an injection

$$H^0(X, K - D) \to H^0(X, K).$$

But $H^0(X, K) = H^2(X, \mathscr{O}_X) = H^2(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}) = 0$ by rationality and the previous proposition. So $H^0(X, K-D)$ must be zero and hence also $H^2(X, D) = H^0(X, K-D) = 0$ by Serre duality. □

Lemma 2.3. *Let D be a nef divisor class on a rational anticanonical surface. Then $D^2 \geq 0$ and D is effective.*

Proof. $D^2 \geq 0$ follows from Kleiman's criterion. Then effectiveness follows from the previous lemma and Riemann-Roch and from the fact that $-K$ is effective:

$$h^0(X, D) \geq \frac{1}{2}\left(D^2 - K.D\right) + 1 \geq 1.$$

since $-K$ is effective. □

Hence all nef divisors are effective on an anticanonical surface. Note that if $-K$ is ample, i.e., X is a Del Pezzo surface, then the above inequality shows that $h^0(X, D) \geq 2$ for all nef classes D. We need a result about the base-point freeness of nef divisors on X:

Lemma 2.4. *If N is a nef divisor such that $-K.N \geq 2$, then the linear system $|N|$ is basepoint free.*

Proof. This is Theorem III, part a) in [Har97]. □

Note that if $-K.D > 0$, then the lemma implies that nD is base-point free for $n \geq 2$. In particular, D is semiample.

The next result is also well-known.

Proposition 2.5. *Let X be a non-singular projective surface whose anticanonical divisor $-K$ is nef. Suppose that D is a nef divisor on X. Then $H^i(X, \mathscr{O}_X(D)) = 0$ for each $i > 0$.*

Proof. This is a consequence of the Kawamata-Vieweg vanishing theorem, since $D = (D - K) + K$ and $D - K$ is nef and big: This follows by the inequalities

$$\begin{aligned} (D - K).C &= D.C + (-K).C \geq 0, \\ (D - K)^2 &= D^2 - 2D.K + K^2 > 0, \qquad \forall C \in NE^1(X, \mathbf{R}) \end{aligned}$$

where all the last terms are non-negative since D is nef and $K^2 > 0$. □

Surfaces with $-K$ nef and big are called *generalized Del Pezzo surfaces*. They can be characterized as blow-ups of $\mathbf{P}^2$ in ≤ 8 points in *almost general position*, i.e point sets containing infinitely near points, no more than three collinear points or six points on a conic. Using the previous proposition and Riemann-Roch, we determine the dimension of $\mathrm{Cox}(X)_D$ for D nef:

Corollary 2.6. *For nef divisor classes D we have*

$$\dim_k \operatorname{Cox}(X)_D = \chi(\mathscr{O}_X(D)) = \frac{1}{2}D.(D-K)+1$$

This result will help us to find generators of $\operatorname{Cox}(X)$.

Lemma 2.7. *Let E be an irreducible effective divisor with negative self-intersection. Then $H^0(X,E)$ is one-dimensional. In particular, any generating set of $\operatorname{Cox}(X)$ must contain some section of degree E.*

Proof. Since E is effective $H^0(X,E) \geq 1$. If $H^0(X,E) \geq 2$, let s,t be two linearly independent sections. The number of intersection points of $(s)_0$ and $(t)_0$ is non-negative since they have no component in common, and this contradicts $E^2 < 0$. The last part of the lemma is clear since E is irreducible. □

2.2 Anticanonical rational surfaces with finitely generated Cox ring

In this section we prove that a relatively large class of rational surfaces have finitely generated Cox ring. We focus on blow-ups of $\mathbf{P}^2$ in a finite number of points although some of the results hold in greater generality. For example the next theorem holds for general projective surfaces with finitely generated effective cone. The proposition shows that when studying elements of $\operatorname{Cox}(X)$, we may 'chop off' the negative curves which are fixed components of D and we may assume D to be nef. This is good for our purposes, since nef divisors have nice vanishing properties.

Proposition 2.8. *Let X be a smooth projective surface with finitely generated effective cone, and let $\mathscr{N} = \{\Gamma_1,\ldots,\Gamma_N\}$ be the set of integral curves with negative self-intersection. Let x_i be a generator for $H^0(X,\Gamma_i)$. Let D be an effective divisor class with decomposition $D = F + M$ where F is the fixed part and M is nef. Write $F = a_1\Gamma_1 + \ldots + a_n\Gamma_n$. Then*

$$\operatorname{Cox}(X)_D = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}\operatorname{Cox}(X)_M. \tag{2.1}$$

If $\operatorname{Cox}(X)$ is finitely generated, say by sections $s_1,\ldots,s_N$, then also

$$k[s_1,\ldots,s_N]_D = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}k[s_1,\ldots,s_N]_M$$

In particular, $\operatorname{Cox}(X)$ is finitely generated if and only if the subalgebra

$$\bigoplus_{N\in NM^1(X,\mathbf{Z})} H^0(X,N)$$

is.

Proof. Let D be an effective divisor and fix an ample divisor H on X so that we may induct on the number $H.D \geq 0$. Let s be a section in $H^0(X,D)$. For $H.D = 0$, (2.1) trivially holds since D is the zero divisor, which is nef. Suppose that $H.D > 0$. If D is not nef, there exists some negative curve, say $\Gamma \in \mathscr{N}$ such that that $D.\Gamma < 0$, and Γ is necessarily in the fixed part of D. Multiplication by x_Γ induces an exact sequence

$$0 \to H^0(X, D - \Gamma) \to H^0(X, D) \to H^0(\Gamma, D|_\Gamma) = 0$$

and so $s = x_\Gamma \cdot t$ for some t in $H^0(D - \Gamma)$. Replacing D by $D - \Gamma$, we iterate the process until we reach the divisor M, a nef divisor. This proves the first part.

If $\mathrm{Cox}(X)$ is finitely generated, then any monomial $m = s_1^{a_1} s_2^{a_2} \cdots s_N^{a_N}$ in $k[s_1, \ldots, s_N]_D$ corresponds to writing D as a sum of effective divisors corresponding to the x_i. If $D.\Gamma < 0$, then as above we must have that Γ is a fixed component of D and Γ occurs in the sum. This means that x_Γ divides m. Replacing D by $D - \Gamma$, the result now follows by induction on the degree of m.

The last part of the theorem is clear since X has only finitely many curves of negative self-intersection. □

The above theorem will be very powerful in our study of Cox rings. For example, it tells us that generators for $\mathrm{Cox}(X)$ are either sections corresponding to negative curves or nef classes. This observation and Corollary 2.6 will help us to find explicit generators for $\mathrm{Cox}(X)$. Also,

Corollary 2.9. *If* $\mathrm{Cox}(X)$ *is finitely generated, the ideal* I_X *is generated in degrees corresponding to nef divisor classes.*

Proof. Write $D = N + F$ as before. Then since the ideal is homogeneous with respect to the $\mathrm{Pic}\, X$-grading, any relation $f \in I_D$ can be written as a product of a monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and a relation in $f' \in I_N$. □

It was shown in [TVV09] by Testa, Várilly and Velasco that all rational surfaces with $-K$ effective and big, has finitely generated Cox rings. The authors show that in this case, there are finitely many curves of negative self-intersection and that this fact is sufficient to ensure finite generation of $\mathrm{Cox}(X)$. Some similar results appear in [GM05] for blow-ups of $\mathbf{P}^2$.

We present a new proof of a weakened version of this result, namely, we restrict to anticanonical surfaces X with finitely generated effective cone, in particular, surfaces which are blow-ups of $\mathbf{P}^2$ in $r \leq 8$ points in arbitrary position. This includes all smooth and generalized Del Pezzo surfaces, and is the main result of this section.

Lemma 2.10. *Let* X *be the blow up of* $\mathbf{P}^2$ *in* $r \leq 7$ *distinct points, then* $-K$ *can be written as a sum of classes of rational curves with negative self-intersection.*

Proof. In terms of the standard basis for $L, E_1 \ldots, E_r$ (see Section 2.3), $-K$ is given by $3L - E_1 - \ldots - E_r$ by [Har77, V.3].

If $r \leq 6$, $-K$ can be written as a sum of classes $L - E_i - E_j$ and E_i. If $r = 7$, $-K$ can be written as $(2L - E_1 - \ldots - E_5) + (L - E_6 - E_7)$. These classes correspond to (possibly reducible) rational effective divisors, so by further decomposing the summands, we get the result. □

If $r = 8$, the result does not hold unless the points are in a special configuration (see Chapter 3).

We now show that the effective cone of blow-ups of $\mathbf{P}^2$ in ≤ 8 points is finitely generated. This result is folklore (it is in fact provable using Mori's cone theorem [Deb01]), but we present a (rather vulgar) proof in lack of reference for the form we need.

Proposition 2.11. *Let X be the blow-up of $\mathbf{P}^2$ in $r \leq 8$ distinct points. Then the effective monoid $NE^1(X, \mathbf{Z})$ is finitely generated.*

If $r \leq 7$, it is generated by classes of negative rational curves if $r \leq 7$. If $r = 8$, then one needs in addition the class $-K$.

Proof. We argue by induction, by fixing an ample divisor H on X and defining the degree of an effective divisor D as the number $H.D \geq 0$. Note that when $H.D = 0$, then D is the zero-divisor.

Let

$$\mathcal{R} = \{C \in NE^1(X, \mathbf{Z}) \mid C \text{ rational and } -K.C \leq 3\} \cup \{-K\}.$$

Note that since $-K$ has at most finitely many fixed components, there can be only finitely many values of $-K.C$ for $C \in \mathcal{R}$. We show that $\mathcal{R}$ is finite and that it generates $NE^1(X, \mathbf{Z})$.

Let $aL - \sum_{i=1}^r b_i E_r$ be the class of C in $\mathrm{Pic}(X)$. Assuming C is not one of the exceptional divisors $E_1, \ldots, E_r$, we must have $b_i \geq 0$. Write $\rho = -K.C$, so that $3a - \rho = \sum b_i$. By the genus formula we have $D^2 = a^2 - \sum b_i^2 = \rho - 2$.

Recall the Quadratic Mean-Arithmetic Mean Inequality,

$$\frac{\sum_{i=1}^n x_i^2}{n} \geq \left(\frac{\sum_{i=1}^n x_i}{n}\right)^2$$

which holds for non-negative real numbers x_i. Using this we get a bound on the number a:

$$(3a - \rho)^2 = (b_1 + \ldots + b_r)^2 \leq r \cdot (b_1^2 + \ldots + b_r^2) \leq 8(a^2 - \rho + 2)$$

since $r \leq 8$. This shows that $a^2 - 6\,a\rho + \rho^2 + 8\,\rho - 16 \leq 0$, and hence there are only finitely many such a for each ρ. Now, for each fixed a and ρ, the conditions $p_a(C) = 0$ and $-K.C = \rho$ translate into a system of diophantine equations with only finitely many solutions as in [Har77, V.4]. Since there are only finitely many possibilities for a and ρ, $\mathcal{R}$ is finite.

We now show that $\mathcal{R}$ generates $NE^1(X, \mathbf{Z})$. Suppose C is an effective divisor which we may take to be irreducible. If C is rational, and $-K.C \geq 4$ then by Mori's theorem, the curve degenerates into a sum of rational curves of lower degree and we are done by induction. If $-K.C \leq 3$, then $C \in \mathcal{R}$.

If C is not rational, we claim that $C + K = C - (-K)$ is effective. Since $-K \in \mathcal{R}$, we are done by induction since $C + K$ has lower degree than C. To prove the claim, consider the exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

The long exact sequence in cohomology and duality gives $h^0(X, C-(-K)) = h^2(X, -C) = h^1(C, \mathcal{O}_C) = p_a(C) \geq 1$. This shows that the effective monoid is generated (non- minimally) by elements of $\mathcal{R}$.

It is possible to give a smaller generating set for $NE^1(X, \mathbf{Z})$. We use an idea by Harbourne [Har98]. Let D be an effective divisor. By subtracting if necessary the negative curves E such that $D.E < 0$, we reduce to the case D nef. When the points $p_1, \ldots, p_r$ are in general position, i.e the resulting blow-up is Del Pezzo, the proposition is well-known, and the effective monoid is generated by the exceptional curves on X (see Section 2.3).

When the points move into into special position, the effective cone can only get 'larger' (this is because of the upper semicontinuity theorem [Har77, III. §8]), and its dual, the nef cone gets 'smaller'. It follows that, the divisor class $D = aL - \sum b_i E_i$ on X remains nef when when the points are in general position. Since every nef divisor is effective, we may then write D as a sum of classes of exceptional curves. On X, the classes of the exceptional curves may be further reducible. In any case, there exists a rational negative curve E with negative self-intersection such that $D - E$ is effective. By induction we get the result. □

We are now in position to prove the main theorem of this chapter.

Theorem 2.12. *Let X be an anticanonical rational surface with finitely generated effective cone $NE^1(X, \mathbf{R})$. Then* $\mathrm{Cox}(X)$ *is finitely generated.*

In particular, all blow-ups of $\mathbf{P}^2$ in $r < 9$ distinct points in arbitrary position have finitely generated Cox ring.

Proof. Since $NE^1(X, \mathbf{R})$ is finitely generated, so is $NE^1(X, \mathbf{Z})$ by Gordan's lemma [Ful93], and the set of integral negative curves is finite. Hence by Proposition 2.8 above, it is enough now to show that the subring

$$S = \bigoplus_{D \in NM^1(X, \mathbf{Z})} H^0(X, D)$$

is finitely generated. We show first that all nef divisors are semiample, and then apply Zariski's theorem. Let $N \neq 0$ be a nef divisor on X. Recall that $N^2 \geq 0$ and $-K.N \geq 0$ since $-K$ is effective. If $N^2 = 0$, then N is base-point free since two curves in $|N|$ intersect in $N^2 = 0$ points. Suppose $N^2 > 0$. If $N = -nK$, we have $K^2 > 0$, then N is semiample by Lemma 2.4. If $N \neq -nK$, then for some ϵ $(N + \epsilon(-K))^2 = N^2 - 2\epsilon N.K + \epsilon^2 K^2 > 0$. Now the Hodge Index Theorem implies that $(-K.N)^2 > N^2 \geq 0$. Since in any case $-K.N \geq 0$ (N is nef) we have $(-K.N) > 0$. So nef divisors are semiample by Lemma 2.4.

Since $NE^1(X, \mathbf{Z})$ is finitely generated, so is its dual, the nef monoid $NM^1(X, \mathbf{Z})$, say by classes $D_1, \ldots, D_r$. Now, we apply the following trick from the proof of Lemma 2.8 in [HK00]. Consider the projectivized bundle

$$\mathbf{P} = \mathbf{P}(\mathscr{O}_X(D_1) \oplus \cdots \oplus \mathscr{O}_X(D_r)).$$

We have $S \cong \bigoplus_{n \in \mathbf{Z}} H^0(\mathbf{P}, \mathscr{O}_{\mathbf{P}}(n))$. Since $\mathscr{O}_{\mathbf{P}}(1)$ is semiample, the latter algebra is finitely generated by Zariski's theorem, and so is $\mathrm{Cox}(X)$.

The last statement follows by the previous proposition. □

Remark. The finite generation of $\mathrm{Cox}(X)$ is usually not a formal consequence of the finite generation of the effective cone, although we do not know of any counter-examples in the case of surfaces. But an analogue of the theorem above would need additional assumptions on X if $\dim X \geq 3$. For example, for surfaces, $NE^1(X, \mathbf{Z})$ is finitely generated if and only if $NM^1(X, \mathbf{Z})$ is, which is used in the proof. This no longer holds in higher dimensions: For example, blowing up $\mathbf{P}^3$ in 9 distinct points lying on the intersection of two cubic curves lying in a hyperplane gives an example of a variety for which $NE^1(X, \mathbf{Z})$ is finitely generated, but $NM^1(X, \mathbf{Z})$ is not, since X has infinitely many curves of self-intersection -1. See [HT04] for more details.

Of course, the requirement $r \leq 8$ is not possible to avoid, since 9 points gives Nagata's counterexample. However, blowing up special configurations of points in $\mathbf{P}^2$ may still give finitely generated Cox rings, provided that the effective cone is finitely generated. Castravet and Tevelev [CT06] show that $\mathbf{P}^2$ blown-up in any number of points lying on a smooth conic has finitely generated Cox ring, and give explicit generators. In Chapter 3, we show this for $\mathbf{P}^2$ blown up in any number of points on a line.

We continue this chapter with a detailed study of the case of Del Pezzo surfaces, and investigate blow-ups of special configurations in Chapter 3.

2.3 Cox Rings of Del Pezzo surfaces

A smooth projective surface X with an ample anticanonical class $-K$ is called a *Del Pezzo surface*. The standard examples are $\mathbf{P}^2$ and cubic surfaces in $\mathbf{P}^3$. Del Pezzo surfaces and their Cox rings have been studied extensively [BP04, STV06, LV07, SX08, TVV08]. We recall some of their basic properties.

It is well-known that all Del Pezzo surfaces except $\mathbf{P}^1 \times \mathbf{P}^1$ arise as blow-ups X_r of the projective plane in $r \leq 8$ in *general position*, i.e no three points collinear, no six on a conic and no eight points on a cubic curve. Since $\mathbf{P}^1 \times \mathbf{P}^1$ is a toric variety it is immediate that $\mathrm{Cox}(X)$ is a polynomial ring, and we will only consider the surfaces X_r arising as blow-ups in the following. The degree of X_r is defined as the number $K^2 = 9 - r$, so for example any smooth cubic Del Pezzo surface is isomorphic to a blow-up of $\mathbf{P}^2$ in six general points. For $r \leq 6$, $-K$ is very ample and gives an embedding of X_r as a surface of degree $9 - r$ in $\mathbf{P}^{9-r}$.

The Picard group of X_r has rank $r + 1$ and is generated by the classes of the exceptional divisors $E_1, \ldots, E_r$ and L which is the pullback of a general line in $\mathbf{P}^2$ via the blow-up morphism $\pi : X \to \mathbf{P}^2$. The intersection form on $N^1(X) = \operatorname{Pic} X$ is given by

$$E_i \cdot E_j = -\delta_{ij}, \qquad E_i \cdot L = 1, \qquad L^2 = 1.$$

As before, the anticanonical class equals $-K = 3L - E_1 - \ldots - E_r$ in this basis.

Recall that an exceptional curve is defined as a smooth rational curve of self-intersection -1. Note that the exceptional divisors E_i are exceptional curves. A fundamental theorem in the Enriques classification of surfaces is Castelnuovo's theorem which states a form for the converse statement: For any exceptional curve $E \subset X$, there exists a surface Y and a morphism $\pi : X \to Y$ which is a blow-up of Y in a point with E as the exceptional divisor. The proof is even constructive: It gives an algorithm for constructing Y explicitly [Har77, V.5]. It is shown in [Man86, Cor. 24.5.2] that in case of Del Pezzo surfaces, the contraction $\pi : X \to Y$ gives a new Del Pezzo surface, so in this respect, Del Pezzo surfaces form an own class of rational surfaces.

Note that the only negative curves on X_r are the exceptional curves: This follows by the genus formula: $C^2 = 2g - 2 - K.C \geq -1$ since $g \geq 0$ and $-K$ is ample. We denote the set of exceptional curves on X_r by $\mathscr{E}_r$.

Because of Castelnuovo's remarkable theorem, and since $\mathrm{Cox}(X)$ must contain generators corresponding to negative curves, it seems natural to understand the exceptional curves on X. On Del Pezzo surfaces these have been classified (see [Man86]), and there is a rich theory devoted to them. The following theorem gives a geometric description and the divisor class of each curve:

Theorem 2.13. *(See [Man86]) Let $2 \leq r \leq 8$. The exceptional curves on X_r are strict transforms of:*

- *Points, the exceptional divisors, E_i.*
- *Lines through pairs of points, $L_{ij} = L - E_i - E_j$.*
- *Conics through 5 points, $2L - \sum_5 E_i$.*
- *Cubics through 7 points, vanishing doubly at E_j, $3L - E_j - \sum_7 E_i$.*
- *Quartics through 8 points, vanishing doubly at E_j, E_k, E_l, $4L - E_j - E_k - E_l - \sum_8 E_i$.*
- *Quintics through 8 points, vanishing doubly at 6 points, $5L - 2\sum_8 E_i + E_i + E_j$.*
- *Sextics through 8 points, vanishing doubly at 7 points, and one triply $6L - 2\sum_8 E_i - E_j$.*

There are interesting symmetries in the graph G_r of the exceptional curves. This graph is constructed by taking the exceptional curves as nodes and adding an edge between intersecting curves. For $r = 4$, the graph G_4 is isomorphic to the Petersen graph shown in Figure 2.1. These symmetries are encoded in the Weyl group W_r of X_r, which is the subgroup of $\mathrm{Aut}(\mathrm{Pic}(X_r))$ which acts on divisor classes by permutations and preserves K and the intersection form. This action restricts to the set $\mathscr{E}_r$, by permuting the lines, thus providing the graph automorphisms of G_r. This explains the nice symmetry in the graphs G_r

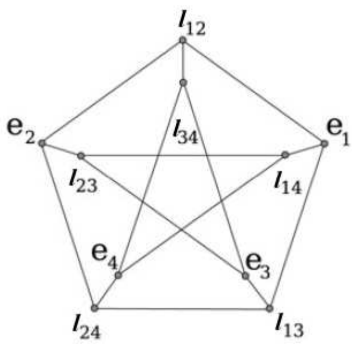


Fig. 2.1 Graphs of exceptional curves on X_4.

Explicitly, one finds that the Weyl group W_r is generated by permutations of the exceptional curves E_i for $r \geq 3$, and the *Cremona element* σ given on the generators as

$$\sigma(L) = 2L - E_1 - E_2 - E_2 \qquad \sigma(E_1) = L - E_2 - E_3 \qquad \sigma(E_2) = L - E_1 - E_3$$
$$\sigma(E_3) = L - E_1 - E_2 \qquad \sigma(E_i) = E_i \quad \forall i \notin \{1, 2, 3\}.$$

Compare this with Section 1.5.

2.3.1 Generators for $\mathrm{Cox}(X_r)$

Let X be a Del Pezzo surface. By Theorem 2.12, $\mathrm{Cox}(X)$ is finitely generated, and using Lemma 2.7, we see that the number of generators must be at least the number of exceptional curves on X. In fact, Batyrev and Popov prove in [BP04] that their sections are almost sufficient to generate the ring. More precisely, they prove

Theorem 2.14. *[BP04] For $3 \leq r \leq 8$, the ring $\mathrm{Cox}(X_r)$ is generated by elements of degree 1. If $r \leq 7$, the generators of $\mathrm{Cox}(X_r)$ are global sections of line bundles defining the exceptional curves. If $r = 8$, then we must add to the above set of generators two linearly independent sections of degree $-K$.*

Note in particular that this implies that the effective monoid $NE^1(X, \mathbf{Z})$ is generated by the exceptional curves and $-K$, which fits nicely with Proposition 2.11. The above theorem allows us to view $\mathrm{Cox}(X)$ as a quotient of $k[\mathscr{E}_r] = k[x_E : E \in \mathscr{E}_r]$ by some homogeneous prime ideal:

$$\mathrm{Cox}(X) = k[\mathscr{E}_r]/I_r.$$

Note that monomials in $k[\mathscr{E}_r]_D$ correspond to ways of writing D as a sum of exceptional curves. We will also sometimes speak of a coarser grading on $\mathrm{Cox}(X)$ given by $\mathrm{Cox}(X)_n = \bigoplus_{-K.D=n} \mathrm{Cox}(X)_D$. Note that all variables have degree 1 with respect to this grading.

We are interested in explicit generators for I_r in this presentation. Moreover, we wish to prove

The Conjecture of Batyrev and Popov: [BP04] *Cox rings of Del Pezzo surfaces are quadratic algebras, i.e for $4 \leq r \leq 8$, the ideal I_r above is generated by quadratic polynomials.*

We will restrict ourselves to the case where $r \leq 6$, to avoid the extra complications with the two extra sections from $-K$. The remaining cases $r = 7, 8$ were studied in detail by Testa, Velasco and Várilly in [TVV08].

Notation. For simplicity we will label the variables according to their description in Theorem 2.13:

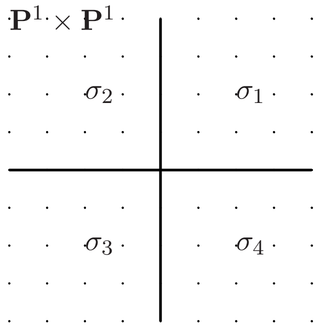
variable	divisor class
e_i	E_i
l_{ij}	$L - E_i - E_j$
g	$2L - E_1 - \ldots - E_5$ if $r = 5$.
g_i	$2L - E_1 - \ldots - E_6 + E_i$ if $r = 6$.

2.3.2 Toric Del Pezzo Surfaces.

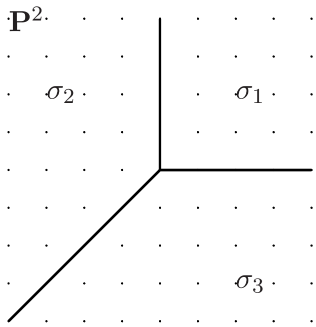
The surfaces X_r for $r \leq 3$ are, along with $\mathbf{P}^2$ and $X = \mathbf{P}^1 \times \mathbf{P}^1$, the toric Del Pezzo surfaces. This can be seen as follows: Note that the action of $PGL(3)$ act transitively on triples in $\mathbf{P}^2$, so that it is possible to move any ≤ 3 points in general position to the "distinguished points", or "torus invariant" points $(1:0:0), (0:1:0), (0:0:1)$.

By the universal property of blowing-up [Har77, II.8], there is a unique isomorphism between the blow-ups extending this action.

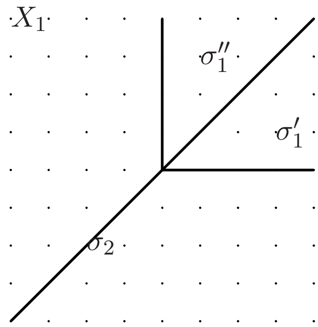
Consider the fan of $\mathbf{P}^2$ as shown in the figure. As explained in [Ful93], blowing up $\mathbf{P}^2$ in the distinguished point $p_1 = [1, 0, 0]$ gives a toric variety (since the torus action extends to the complement of the distinguished points), and the corresponding fan is the "star subdivision" of the cone σ_1 into $\{\sigma_1', \sigma_1''\}$. This fan is obtained by inserting a ray $\langle e_1 + e_2 \rangle$, as shown in the fan for X_1. Continuing this way we find the following fans for the toric Del Pezzo surfaces. Note the corresponding Cox rings for each fan.



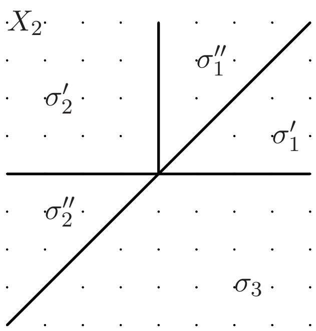
$\mathrm{Cox}(X) = k[x, y, s, t]$



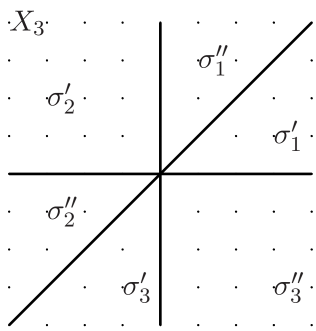
$\mathrm{Cox}(X) = k[x, y, z]$



$\mathrm{Cox}(X_1) = k[e_1, x, y, z]$



$\mathrm{Cox}(X_2) = k[e_1, e_2, x, y, z]$



$\mathrm{Cox}(X_3) = k[e_1, e_2, e_3, x, y, z]$

Note that the rays $e_1 + e_2, -e_1, -e_2$ correspond to the exceptional divisors on X_3 and that the remaining rays corresponds to the strict transforms of the lines through pairs of blown-up points. These are precisely the exceptional curves on X_3, in agreement with Batyrev and Popov's result.

2.4 Conic Bundles on X_r

We now turn to Del Pezzo surfaces arising as blow-ups of $\mathbf{P}^2$ in ≥ 4 general points. These are not toric anymore, so we expect relations in their Cox rings. Before we are able to say anything about the relations in I_r, we must examine which $\mathrm{Pic}(X)$-degrees they arise in.

Definition 2.15. A divisor class D is *conic* if $D^2 = 0$ and $-K.D = 2$.

The name conic comes from the geometric fact that their sections give conic bundles $X_r \to \mathbf{P}^1$. This will be shown below.

First, we classify all conic divisor classes in terms of the standard basis for Pic X:

Lemma 2.16. *For $2 \leq r \leq 7$, the conic divisor classes are given in the following table:*

r	*Total*	#	*Divisor (up to permutation of E_is)*
4	*5*	*4*	$L - E_1$
		1	$2L - E_1 - E_2 - E_3 - E_4$
5	*10*	*5*	$L - E_1$
		5	$2L - E_1 - E_2 - E_3 - E_4$
6	*27*	*6*	$L - E_1$
		15	$2L - E_1 - E_2 - E_3 - E_4$
		6	$3L - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6$
7	*126*	*7*	$L - E_1$
		35	$2L - E_1 - E_2 - E_3 - E_4$
		42	$3L - 2E_1 - E_2 - E_3 - E_4 - E_5 - E_6$
		35	$4L - 2E_1 - 2E_2 - 2E_3 - E_4 - E_5 - E_6 - E_7$
		7	$5L - 2E_1 - 2E_2 - 2E_3 - 2E_4 - 2E_5 - 2E_6 - E_7$

Proof. Set $P = aL - b_1E_1 - \ldots - b_rE_r$. From $P^2 = 0, P.K = -2$ we get the equations

$$a^2 = b_1^2 + \ldots + b_r^2, \;\; 3a - 2 = b_1 + \ldots + b_r \tag{2.2}$$

By the Quadratic Mean-Arithmetic Mean Inequality, we get a bound on a:

$$(3a-2)^2 = (b_1 + \ldots + b_r)^2 \leq r \cdot (b_1^2 + \ldots + b_r^2) \leq ra^2$$

whence $1 \leq a \leq \frac{2}{3-\sqrt{r}}$. Plugging in each of these values of a in equation (2.2), gives us the b_i and we recover the list of conics. □

Note that since Cox(X) is generated by sections corresponding to exceptional curves, the Weyl group acts in a natural way on $k[\mathscr{E}_r]$ and Cox(X). In fact, the Weyl group acts transitively on conics as well:

Lemma 2.17. *The Weyl group W_r acts transitively on conics.*

Proof. The Weyl group permutes conics since $\sigma(-K) = -K$, $\sigma(Q)^2 = Q^2 = 0$ and $-K.\sigma(Q) = -K.Q = 2$. Recall that the Cremona element σ acts on a divisor class by

$$\sigma(aL - \sum_{i=1}^{r} b_iE_i) = (2a - b_1 - b_2 - b_3)L + \ldots.$$

Note that $2a - b_1 - b_2 - b_3 < a$ for all conics in the table except $L - E_1$, hence we have an algorithm for running through the table of all conics: Start with the conic $Q = aL - \sum_{i=1}^{r} b_iE_i$ with greatest a in the list, apply σ to it to reduce a, permute the

E_i so that $b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_r$ (this permutation is an element of W_r), and reapply σ. The process will stop when it is not possible to reduce a anymore, i.e when $a = 1$ and $Q = L - E_1$. This shows that it is possible to run through all conics on the list using the action of the Weyl group. The rest of the conics can be reached by permuting the E_i. □

Lemma 2.18. *A divisor class D is conic if and only if $D = E + E'$ for exceptional curves $E, E' \in \mathscr{E}_r$ intersecting in a point. Any conic can be represented this way in exactly $r - 1$ ways.*

Proof. Since the effective monoid $NE^1(X_r, \mathbf{Z})$ is generated by the elements of $\mathscr{E}_r$ and $-K.E = 1$ for any $E \in E_r$, we may write D as the sum of two exceptional curves E and E'. Note that $2E.E' = (E + E')^2 - E^2 - E'^2 = 0 + 2 = 2$, hence $E.E' = 1$ and so E, E' are classes of distinct exceptional curves that intersect in a point.

For the last part, we may (after possibly acting by the Weyl group) assume that $D = L - E_1$. Here it is obvious that $L - E_1$ can be written only in the form $(L - E_1 - E_j) + E_j$ for $j = 2, \ldots, r$. □

Lemma 2.19. *Let Q be a conic divisor class. Then Q is nef, and the linear system $|Q|$ has no base points and determines a morphism $X \to \mathbf{P}^1$, which is a conic bundle.*

Proof. It is clear that $D = E + E'$ is nef since if there exists an exceptional curve F such that $(E + E').F \leq 0$, we must have $E = F$ or $E' = F$, which gives $(E + E').F = -1 + 1 = 0$. Then by Riemann-Roch and Kawamata-Vieweg, we have that $h^0(X, Q) = 2$, so we get a rational map $\phi : X_r \to \mathbf{P}^1$. Take two generic sections s_1, s_2 in $H^0(X, Q)$ and note that they do not intersect since $Q^2 = 0$. In particular, this shows that $|Q|$ has no base points, and hence ϕ is a morphism. The generic fiber of this morphism is a smooth conic, so the map is indeed a conic bundle. □

2.5 Quadratic relations in Cox(X).

Our interest in linear systems of conics D on X_r lies in the fact that they will provide us with generators for the ideal I_r. More specifically, each conic has exactly $r-1$ reducible sections by Lemma 2.18, namely the sections $\xi \cdot \xi'$ for $\xi \in H^0(X, E), \xi' \in H^0(X, E')$ for each decomposition $D = E + E'$. Geometrically, these correspond to the singular fibers of the conic bundle $X \to \mathbf{P}^1$. Now, since $h^0(X, \mathscr{O}_X(D)) = 2$, this means that we have $r - 3$ linear relations between the sections, and each give a relation in I_r. This happens for every conic on X, so the number of relations in I_r is at least $(r - 3)$ times the number of conics. We denote the ideal of these relations in Cox(X) by J_r.

Lemma 2.20. *The ideal J_r generates the degree 2 part of I_r, i.e the degrees D for which $-K.D = 2$.*

Proof. We are only interested in the case where D is effective and $-K.D = 2$. We may write as before $D = E + E'$ where E, E' are divisor classes of lines in Pic X. Note that $E, E' \in \{\pm 1, 0\}$. If $E.E' = 1$, then D is a conic by Lemma 2.18. If $E.E' \leq 0$, we claim E, E' are uniquely determined by $E + E' = D$. This follows since $E.D = E.(E + E') =$

$-1 + E.E' \leq -1$, hence D has both E and E' in its fixed components. Therefore we have only one monomial of degree D, namely $\xi \cdot \xi'$ where $\xi \in H^0(X, E), \xi' \in H^0(X, E')$ and of course no relations in I_D. □

2.5.1 Degree 5 Del Pezzo surfaces

The degree 5 Del Pezzo surface X_4 is the first non-toric Del Pezzo surface. It can be realized as a fourfold hyperplane section of the Grassmannian $G(2,5)$ by the Plücker embedding. In this section, we study the relations in J_4 in detail.

Consider first the conic $D = L - E_1$ on X_4. There are 3 monomials of degree D and they are of the form $l_{1i}e_i$ for $i = 2, \ldots, 4$. Since $h^0(X, D) = 2$, there must be one linear relation between them. To see what it is, consider the blow-up morphism $\pi : X_4 \to \mathbf{P}^2$, and the points $p_1, \ldots, p_4 \in \mathbf{P}^2$. We may assume that these points are as follows

$$p_1 = (1:0:0),\ p_2 = (0:1:0),\ p_3 = (0:0:1),\ p_4 = (1:1:1).$$

since $PGL(3)$ acts transitively on quadruples of points in general position.

The projection of the section l_{1j} by π is a section h_{1j} in $H^0(\mathbf{P}^2, \mathscr{O}(1))$, whose zero set is the line through p_1 and p_j. This is shown in the figure

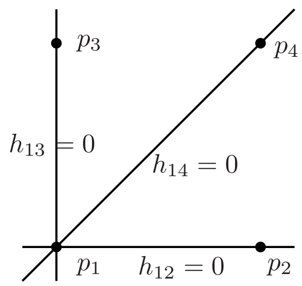


Fig. 2.2 $l_{14}l_{34} + l_{13}l_{24} - l_{14}l_{23} = 0$.

As global sections of $\mathscr{O}_{\mathbf{P}^2}(1)$, any three sections h_{1j} going through p_1 are linearly dependent. This makes the computation of the relations just a task of linear algebra, using simple determinants to find representatives for the sections and finding their dependence relation. For example, we may choose the sections $h_{12} = Z$, $h_{13} = Y$, $h_{14} = Y - Z$,xs such that we have $h_{12} - h_{13} + h_{14} = 0$. The total inverse image of h_{1j} is the divisor $l_{1j}e_1e_j$, so pulling back the relation with π^* and cancelling e_1, this gives us the following relation in J_4:

$$l_{12}e_2 - l_{13}e_3 + l_{14}e_4 = 0.$$

The same thing happens for all the conics $L - E_i$, so we get 4 quadratic relations. There is also an additional conic, of degree $2L - E_1 - E_2 - E_3 - E_4$. Here we have the monomials $l_{12}l_{34}, l_{13}l_{24}, l_{14}l_{23}$ and a linear dependence relation between them. In $\mathbf{P}^2$, the projections satisfy a relation $h_{14}h_{23} - h_{13}h_{24} + h_{12}h_{34} = (Y - Z)X - (X - Z)Y + Z(X - Y) = 0$.

Pulling back these sections and cancelling $e_1e_2e_3e_4$, we see that the relation is

$$l_{14}l_{23} - l_{13}l_{24} + l_{12}l_{34} = 0.$$

There is a similar geometric picture in this case. In all, we find the following relations:

Degree	Relation
$L - E_1$	$e_2l_{12} - e_3l_{13} + e_4l_{14}$
$L - E_2$	$e_1l_{12} - e_3l_{23} + e_4l_{24}$
$L - E_3$	$e_1l_{13} - e_2l_{23} + e_4l_{34}$
$L - E_4$	$e_1l_{14} - e_2l_{24} + e_3l_{34}$
$2L - E_1 - E_2 - E_3 - E_4$	$l_{14}l_{23} - l_{13}l_{24} + l_{12}l_{34}$

We note that these are exactly the Plücker relations of the Grassmannian $G(2,5)$, and so the relations in J_4 are generated by the Pfaffians of the matrix

$$\begin{pmatrix} 0 & e_1 & e_2 & e_3 & e_4 \\ -e_1 & 0 & l_{34} & l_{24} & l_{23} \\ -e_2 & -l_{34} & 0 & l_{14} & l_{13} \\ -e_3 & -l_{24} & -l_{14} & 0 & l_{12} \\ -e_4 & -l_{23} & -l_{13} & -l_{12} & 0 \end{pmatrix} \tag{2.3}$$

Remark. Once we show the equality $I_4 = J_4$, we have shown that X_4 can be obtained as a GIT quotient of the Grassmannian $G(2,5)$. This is a classical fact, see [Sko93].

2.6 A Proof of the Batyrev-Popov Conjecture for $r \leq 6$

In this section we give a proof that the ideal I_r is generated by relations coming from the conic divisor classes. This was shown in [BP04] up to radical. The proof is inspired by Laface and Velasco's article [LV07], and is somewhat computational by the fact that we check that the section e_r is not a zero-divisor modulo the ideal J_r by computer.

Theorem 2.21. *Let X_r be a Del Pezzo surface of degree $9 - r$, and $4 \leq r \leq 6$. The ideal I_r is generated by quadrics coming from the conic divisor classes.*

Proof. By induction on r, the number of blown-up points. We show that $J_N = I_N$ for all nef classes $N = aL - b_1E_1 - \ldots - b_rE_r$. Since N is nef we have $b_r = N.E_r \geq 0$. Let $D = N + b_rE_r$, and note that $D.E_r = 0$. Let $\pi : X_r \to X_{r-1}$ be the contraction of E_r, where X_{r-1} is a Del Pezzo surface of degree $8 - r$. Note that since $D.E_r = 0$, $D = \pi^*B$ for some divisor B on X_{r-1}. So we get, by fixing appropriate generators for $\mathrm{Cox}(X_r)$ as a k-algebra, and abuse of notation, a map $\pi^* : k[\mathscr{E}_{r-1}]_B \to k[\mathscr{E}_r]_D$, mapping x_E to x_{π^*E} for each i. Set $J_3 = 0$. For each r there is a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & k[\mathscr{E}_{r-1}]_B & \xrightarrow{\pi^*} & k[\mathscr{E}_r]_D & \longrightarrow & \cdots \\ & & \downarrow & \searrow^{p\circ\pi^*} & \downarrow p & & \\ \cdots & \longrightarrow & \left(k[\mathscr{E}_{r-1}]/J_{r-1}\right)_B & \xrightarrow{\pi^*} & \left(k[\mathscr{E}_r]/J_r\right)_D & \longrightarrow & \cdots \end{array}$$

Claim: The bottom map is surjective for each $4 \leq r \leq 6$.

Since $\pi^*(J_{r-1}) \subset J_r$ this will follow if we show that $p \circ \pi^*$ is surjective.

To see why this implies the theorem, note that we will have a composition of surjections

$$\Big(k[\mathscr{E}_{r-1}]/J_{r-1}\Big)_B \xrightarrow{\pi^*} \Big(k[\mathscr{E}_r]/J_r\Big)_D \longrightarrow \Big(k[\mathscr{E}_r]/I_r\Big)_D \tag{2.4}$$

Or equivalently, since by induction, $H^0(X_{r-1}, B) = (k[x_1, \dots, x_{n_{r-1}}]/J_{r-1})_B$,

$$H^0(X_{r-1}, B) \longrightarrow \Big(k[\mathscr{E}_r]/J_r\Big)_D \longrightarrow H^0(X_r, D)$$

By lemma 2.1, this composition is an isomorphism, so this implies that the rightmost map in (2.4) is an isomorphism and that $J_D = I_D$.

We conclude that $e_r^{b_r} I_N \subseteq I_D = J_D$. It remains to check that e_r is not a zero divisor modulo the ideal J_r for $r = 4, 5, 6$, this is done by a quick calculation in Macaulay 2. So we have $I_N = J_N$.

Proof of claim. Let $e_r^n m$ be a monomial in $k[\mathscr{E}_r]_D$, where $e_r \not| m$, and let $M = \deg m$ be the divisor class corresponding to m. We show that we may (modulo the ideal J_r) reduce the monomial to sections not intersecting E_r.

Assume $n > 0$. Then since $D.E_r = 0$, and $D = \deg e_r^n m = nE_r + M$ we must have $M.E_r = n$. In particular, there must be an exceptional curve $F \subset M$ such that x_F divides m and $F.E_r > 0$. Note that for $r \leq 6$, the condition $E.E' > 0$ implies $E.E' = 1$ (see Theorem 2.13). This means that $F + E_r$ is a conic, so there is a relation

$$x_F e_r = \sum c_{EE'} x_E x_{E'}.$$

Note that if $Q = E + E'$ and $Q.F = 0$, then either one of E, E' is equal to F or $E.F = E'.F = 0$. This means that the lines $E, E' \in \mathscr{E}_r$ in the sum above do not intersect E_r. Consequently, we have shown that modulo the ideal J_r, we may write $e_r^n m$ as a sum of monomials with smaller exponent in e_r. Iterating this, we reduce n for each step, until we arrive at the base case $n = 0$. Now, if $n = 0$, $M.E_r = D.E_r = 0$ and so M does not intersect E_r. This means that m is a product of sections coming from X_{r-1}. This finishes the proof. □

Remark. This theorem can be proved in another way using a result of Popov from [Pop05], where it is shown that the the ring $k[\mathscr{E}_r]/J_r$ is Cohen-Macaulay for $r \leq 7$. We can use this to show that J_r is a prime ideal. A dimension argument can then be used to conclude that $J_r = I_r$. Specifically, the Cohen-Macaulay property allows us to apply Serre's criterion [Eis95, Thm. 25.20] which states that in this case J_r is prime iff the ideal generated by the $c-$minors of the Jacobian $J = \left(\frac{\partial f_i}{\partial x_j}\right)$ has codimension ≥ 2. Here c is the codimension of J_r in $k[\mathscr{E}_r]$. This hypothesis is tested for the ideals J_4, J_5, J_6 in Macaulay2, and after a few days of computation the process stops and yields an affirmative answer. Of course, this computerised *deus ex machina* may seem a little unsatisfactory, but it is worth noting that this is a general approach to test if we have found all the relations in the Cox ring. Also, the problem of proving the primality of an ideal is in general a very difficult problem and Serre's criterion is one of the few methods we know.

2.7 Syzygies of Cox(X_r)

Just as conic divisor classes gave the defining relations for the ideal of Cox(X_r), divisor classes satisfying $D^2 = 1$, $-K.D = 3$ will play the role for finding the first syzygies. We will call such divisor classes *cubic*.

Note that $C^2 = 1, -K.C = 3 \Leftrightarrow (C-E)^2 = 0, -K.(C-E) = 2$ for every line E such that $C.E = 0$, i.e., $C - E$ is a conic. Hence by running through all Q and E such that $Q.E = 1$, we get the cubics from the previous table of conics. Note that for $r = 4$, D is conic if and only if $-K - D$ is cubic. The cubic divisor classes X_r are given in the following list for $r = 4, 5$:

r	#	Divisors(up to permutation of E_i's)
4	5	L
		$2L - E_1 - E_2 - E_3$
5	16	L
		$2L - E_1 - E_2 - E_3$
		$3L - 2E_1 - E_2 - E_3 - E_4 - E_5$

Consider the Del Pezzo surface X_4. We claim that there is exactly one syzygy for every cubic divisor class C. First of all, $C^2 = 1, -K.C = 3$ implies by Riemann-Roch, that $h^0(X, C) = 3$. Now, $k[\mathscr{E}_r]_C$ has dimension 6 ($k[\mathscr{E}_r]_L = \text{span}\{l_{ij}e_ie_j\}$, now use the Weyl group), and there are four conics Q such that $C - Q$ is effective (a line), hence there must be one linear relation between the four relations coming from the Q's for each C. There are five cubic bundles on X_4, and we find the following syzygies:

D	Syzygy
L	$e_2g_1 - e_1g_2 - e_3g_3 + e_4g_4$
$2L - E_1 - E_2 - E_3$	$l_{13}g_1 - l_{23}g_2 - l_{12}g_3 + e_4g_5$
$2L - E_1 - E_2 - E_4$	$l_{14}g_1 - l_{24}g_2 - l_{12}g_4 + e_3g_5$
$2L - E_1 - E_3 - E_4$	$l_{34}g_2 - l_{14}g_3 + l_{13}g_4 - e_2g_5$
$2L - E_2 - E_3 - E_4$	$l_{34}g_1 - l_{24}g_3 + l_{23}g_4 - e_1g_5$

The next proposition shows that the above syzygies generate the whole syzygy module.

Proposition 2.22. *The set $\{g_1, \ldots, g_5\}$ forms a Gröbner basis for I with respect to the monomial ordering given by*

$$e_1 \succ e_2 \succ e_3 \succ e_4 \succ l_{12} \succ l_{13} \succ l_{14} \succ l_{23} \succ l_{24} \succ l_{34}$$

and $J = in(e_1l_{12}, e_1l_{13}, e_1l_{14}, e_2l_{12}, l_{14}l_{23})$ is an initial ideal for I.

Also, the above relations generate the entire syzygy module of I.

Proof. We apply Buchberger's criterion. We must show that the S-polynomials of the pairs g_i, g_j reduce to zero modulo the ideal I for $i \neq j$. This can be done by looking at the syzygies. For example, the S-polynomial of the pair $\{g_1, g_2\}$ reduces to zero, since the first syzygy in table 2.7 can be written

$$e_2(e_1l_{12}-e_3l_{23}+e_4l_{24})-e_1(e_2l_{12}-e_3l_{13}+e_4l_{14}) = e_3(e_1l_{13}-e_2l_{23}+e_4l_{34})-e_4(e_1l_{14}-e_2l_{24}+e_3l_{34})$$

showing that the remainder is zero. The same thing happens for the pairs

$$\{g_1, g_3\}, \{g_1, g_4\}, \{g_2, g_3\}, \{g_4, g_5\}.$$

Now the leading terms of the remaining pairs are relatively prime, hence their S-polynomials also reduce to zero [AL94]. It follows that the $\{g_1, \ldots, g_5\}$ form a Gröbner basis for I.

The last part follows from Schreyer's theorem [Eis95], which states that the coefficients of the S-polynomials generate the syzygy module of the ideal. □

Since we have a Gröbner basis, this makes the study of the ideal I easier. For example, we can verify, using Stanley's criterion [Eis95], that Cox(X) is Gorenstein. We are also able to write down the multigraded minimal resolution of Cox(X):

$$0 \longrightarrow R(K) \xrightarrow{G^t} \bigoplus_Q R(K+Q) \xrightarrow{M} \bigoplus_Q R(-Q) \xrightarrow{G} R \longrightarrow \mathrm{Cox}(X) \longrightarrow 0.$$

where $R = k[\mathscr{E}_r]$ and Q run through all conics and M is the matrix form in 2.3

2.7.1 The Cox Rings of X_5 and X_6

Note that in the case $r = 5$, there are a total of $2 \cdot 10 = 20$ relations in I_5 (2 for each conic), and $3 \cdot 27 = 81$ relations in I_6. We may, without loss of generality, assume that the first five blown-up points of $\mathbf{P}^2$ are in the positions

$$p_1 = (1:0:0),\ p_2 = (0:1:0),\ p_3 = (0:0:1),\ p_4 = (1:1:1),\ p_5 = (1:a:b).$$

Using the method of Section 2.5, we compute the following minimal relations in I_5.

$$\begin{array}{rc}
l_{14}l_{23} + l_{12}l_{34} - l_{13}l_{24} & e_5l_{15} + ae_3l_{13} - be_2l_{12} \\
l_{23}e_3 + l_{24}e_4 - l_{12}e_1 & al_{23}e_3 + l_{25}e_5 - l_{12}e_1 \\
l_{12}l_{35} - l_{13}l_{25} + l_{15}l_{23} & e_3l_{34} + e_1l_{14} - e_2l_{24} \\
l_{12}l_{45} + l_{14}l_{25} - l_{15}l_{24} & ge_3 + bl_{14}l_{25} - l_{15}l_{24} \\
l_{13}l_{45} + l_{14}l_{35} - l_{15}l_{34} & ge_2 + al_{14}l_{35} - l_{15}l_{34} \\
l_{23}l_{45} + l_{24}l_{35} - l_{25}l_{34} & ge_1 + al_{24}l_{35} - bl_{25}l_{34} \\
e_4l_{34} + e_2l_{23} - e_1l_{13} & e_5l_{35} + be_2l_{23} - e_1l_{13} \\
be_2l_{25} - ae_3l_{35} - e_1l_{15} & e_4l_{14} + e_3l_{13} - e_2l_{12} \\
(a-1)l_{12}l_{35} + ge_4 - (b-1)l_{13}l_{25} & ge_5 + b(a-1)l_{12}l_{34} - a(b-1)l_{13}l_{24} \\
(a-1)e_3l_{34} + e_5l_{45} - (b-1)e_2l_{24} & (b-1)e_2l_{25} - (a-1)e_3l_{35} - e_4l_{45}
\end{array}$$

Proposition 2.23. *The generators for the ideals I_5 and I_6 form a quadratic Gröbner basis with respect to the ordering given by $e_i \succ l_{ij} \succ g$. The syzygy module of I_r is generated in degrees ≤ 2. For $r = 5$, all non-Koszul relations come from conic divisor classes.*

Proof. A computation in Macaulay2 shows the first part. Now, since I_r have quadratic Gröbner basis, this bounds the degrees of syzygies by 2. By inspecting the initial ideal of I_r we see that there are no essential quadratic syzygies except the Koszul syzygies $g_ig_i' - g_i'g_i$ where g_i, g_i' are the two relations coming from a conic. □

Note that for $r = 6$, there are more syzygies than the ones occurring in degree C for C cubic (although all are generated in degrees ≤ 2). For example in degree $-K$, there must be at least 25 syzygies: The number of monomials of degree $-K$ is 60 ($l_{1i}l_{jk}l_{lm}$ give $5 \cdot \binom{4}{2} = 30$ and $g_i l_{ik} e_k$ give $6 \cdot 5 = 30$), $h^0(X, -K) = 4$, so $\dim I_{-K} = 56$. Also, since $K^2 = 3$, for every conic D, there is a unique exceptional line E $(= -K - D)$ such that $D + E = -K$. This means that we get a total 81 relations of degree $-K$ by multiplying a relation of degree D by some x_E. Hence there are $81 - 56 = 25$ linear syzygies between them.

Chapter 3
The Cox Ring of $\mathbf{P}^2$ Blown Up in Special Configurations

In this chapter we will investigate Cox rings of $\mathbf{P}^2$ blown up in special configurations of points. The first four sections give a detailed study of the case when the all the points lie on a line. This case was studied first by Elizondo, Kurano, and Watanabe in [EKW04] who show that the ring is noetherian. We generalize their result substantially by showing that the Cox ring is finitely generated for any number of points, and give an explicit presentation of the ring. The main result is that the ring is in fact a complete intersection ring. In the last section we give a classification of the Cox rings of $\mathbf{P}^2$ blown up in any ≤ 5 distinct points.

3.1 n points on a line

Let X be the blow-up of $\mathbf{P}^2$ in n distinct points $p_1, \ldots, p_n$ lying on a line Y in $\mathbf{P}^2$. The Picard group $\operatorname{Pic} X$ has rank $n+1$ and is generated by the divisor classes of the exceptional curves $E_1, \ldots, E_n$ and L which is the pullback of a general line H in $\mathbf{P}^2$ not passing through any of the $p_1, \ldots, p_n$. The main difference between X and the Del Pezzo surfaces is that we have more negative effective divisors, in particular, a curve with self-intersection $-(n-1)$: This corresponds to the pullback of the line Y, and has the divisor class $L - E_1 - \ldots - E_n$.

Lemma 3.1. *The monoid of effective divisor classes of X is finitely generated as follows:*

$$NE^1(X, \mathbf{Z}) = \mathbf{Z}_{\geq 0}\{L - E_1 - \ldots - E_n, E_1, E_2, \ldots, E_n\}.$$

Proof. It is clear that the generators above are all effective, hence we have the "$\supseteq$" inclusion. Conversely, note that these divisor classes actually form a $\mathbf{Z}$-basis for $\operatorname{Pic} X$. So let D be an *irreducible* effective divisor, and let

$$m(L - \sum_{i=1}^{n} E_i) + \sum_{i=1}^{n} a_i E_i$$

represent the corresponding divisor class. We show that all the coefficients are non-negative. If D is not one of the generators above we have $D.E_i = m - a_i \geq 0$ and $D.(L - E_1 - \ldots - E_n) = -(n-1)m + \sum_{i=1}^{n} a_i = m - \sum_{i=1}^{n}(m - a_i) \geq 0$. These inequalities imply that $m \geq 0$ and $m \geq m - a_i \geq 0,\ \forall i = 1, \ldots, n$. Hence $m, a_i \geq 0$, and we are done. ∎

Note that the lemma and Theorem 2.12 imply that $\mathrm{Cox}(X)$ is finitely generated.

Lemma 3.2. *The nef monoid $NM^1(X,\mathbf{Z})$ is generated by the divisor classes $L, L-E_1, L-E_2, \dots, L-E_n$.*

Proof. The above divisor classes are base-point free, hence nef, so their cone is included in $NM^1(X,\mathbf{Z})$. Conversely, note that the nef condition and the generating set of $NE^1(X)$ translates into the following set of inequalities on a nef divisor class $D = aL - \sum b_i$:

$$a \geq b_1 + b_2 + \ldots + b_n, \qquad b_i \geq 0, \ \forall i = 1, 2, \dots, n$$

Now it is easy to see that we can decompose each D as a sum of the $L - E_i$'s by using b_i of $L - E_i$ and finally add L $a - b_1 - b_2 - \ldots - b_n \geq 0$ times. □

Note that $L - E_i = (L - E_1 - \ldots - E_n) + E_1 + \ldots + \widehat{E_i} + \ldots + E_n$, hence every nef divisor D on X is effective.

3.2 Cohomology vanishing for Nef Divisors on X

Note first that $H^2(X,D) = H^0(X,K-D) = 0$ by Serre duality, since K cannot be effective on X. We now turn to $H^1(X,D)$, by recalling a result of Harbourne [Har98].

Lemma 3.3. *Let X be a smooth projective surface and let $\mathscr{N}$ be the class of a non-trivial effective divisor N on X. If $\mathscr{N}+K$ is not effective, and D meets every component of N non-negatively, then $h^1(N, D|_N) = 0$.*

Proof. See [Har98, Lemma 2.4]. □

Lemma 3.4. *If D is a nef divisor class on X, then $h^1(X,D) = 0$.*

Proof. The proof is done by induction on the number m of $(L-E_i)$-classes in the decomposition of D in the nef cone. For $m = 0$ we have $D = kL$ and the result is trivial since $h^1(X,kL) = h^1(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(k)) = 0$. So suppose $m > 0$ and w.l.o.g that $L - E_1$ occurs in the decomposition. Let C be an irreducible curve of $|L - E_1|$. Note that $L.(K + (L - E_1)) < 0$, hence $K + (L - E_1)$ is not effective (since L is nef). By the above lemma we get that $h^1(C, D|_C) = 0$. Now we take the exact sequence

$$0 \to \mathscr{O}_X(D - C) \to \mathscr{O}_X(D) \to \mathscr{O}_C(D|_C) \to 0$$

and the long exact sequence gives us $h^1(X,D) = 0$, since by the induction assumption we have $h^1(X, D - C) = 0$. □

3.3 Generators for $\mathrm{Cox}(X)$.

We need some preparatory lemmas before we can find the generators for $\mathrm{Cox}(X)$ as a $k-$algebra. The following lemma is the content of Exercise 17.18 in [Eis95]. We include the proof here by lack of reference

Lemma 3.5 (Castelnuovo's base-point free pencil trick). *Let X be an algebraic variety over a field k, let $\mathscr{F}$ be any sheaf of $\mathscr{O}_X$-modules on X, let $\mathscr{L}$ be an invertible sheaf on X and V a two-dimensional base-point free subspace of $H^0(X,\mathscr{L})$. If $H^1(X,\mathscr{F}\otimes\mathscr{L}^{-1})=0$, then the multiplication map*

$$V\otimes H^0(X,\mathscr{F})\to H^0(X,\mathscr{L}\otimes\mathscr{F}) \tag{3.1}$$

is surjective.

Proof. Since $\mathscr{L}$ is generated by global sections, there exists generators $s_1,s_2\in V$ that generate $\mathscr{L}$ locally everywhere. Taking the Koszul complex of the sequence s_1,s_2, we get the following exact sequence

$$0\to\mathscr{L}^{-1}\to V\otimes\mathscr{O}_X\to\mathscr{L}\to 0.$$

which is exact since V is base-point free. Now, $\mathscr{L}$ is locally free, so we may tensor the sequence with $\mathscr{F}$, giving

$$0\to\mathscr{F}\otimes\mathscr{L}^{-1}\to\mathscr{F}\otimes V\to\mathscr{F}\otimes\mathscr{L}\to 0$$

and taking the long exact sequence of cohomology we have

$$0\to H^0(X,\mathscr{L}^{-1}\otimes\mathscr{F})\to V\otimes H^0(X,\mathscr{F})\to H^0(X,\mathscr{L}\otimes\mathscr{F})\to H^1(X,\mathscr{F}\otimes\mathscr{L}^{-1})\to\cdots$$

The vanishing of $H^1(X,\mathscr{F}\otimes\mathscr{L}^{-1})$ proves the surjection. □

Note that if $\mathscr{L}=\mathscr{O}_X(L-E_1),\mathscr{F}=\mathscr{O}_X(D-(L-E_i))$ and $H^1(X,D-2(L-E_i))=0$, under the assumptions above we have an exact sequence

$$0\to H^0(X,D-2(L-E_1))\to H^0(X,L-E_1)\otimes H^0(X,D-L+E_1)\to H^0(X,D)\to 0$$

We need a technical lemma,

Lemma 3.6. $h^1(X,nL-2E_1-E_2-\ldots-E_n)=0$.

Proof. Let $C=L-E_1-E_2-\ldots-E_n$ and $D=nL-2E_1-E_2-\ldots-E_n=C+(n-1)L-E_1$. C is an irreducible rational curve, and so $h^1(C,\mathscr{O}_C(-1))=0$ by Riemann-Roch. Taking the long exact sequence of $0\to\mathscr{O}_X(D-C)\to\mathscr{O}_X(D)\to\mathscr{O}_C(D|_C)\to 0$ gives

$$\cdots\to H^1(X,(n-1)L-E_1)\to H^1(X,D)\to H^1(C,D|_C)\to\cdots.$$

Now $\deg D|_C=D.C=-1$, and $(n-1)L-E_1$ is nef for $n\geq 1$, so the first and third cohomology groups vanish and so $H^1(X,D)=0$, by exactness. □

We now proceed to find generators for Cox(X). We first choose some generators $e_1,\ldots,e_n$ for the 1-dimensional vector spaces $H^0(X,E_i)$, for $i=1,\ldots,n$. Also, let l be a generator for $H^0(X,L-E_1-\ldots-E_n)$. We now consider generators for classes generating the nef cone. These are of course of the form $L-E_i$, and since $H^0(X,L-E_i)$ is 2-dimensional, we need in addition to the section $le_1\cdots e_{i-1}e_{i+1}\cdots e_n$, a new section s_i to form a basis. Suppose we choose these sections such that their projections to $\mathbf{P}^2$ are as the three lines shown in Figure 3.1. That is, we choose a point $q\in\mathbf{P}^2$, and for each i take a section corresponding to the strict transform of the line going through q and p_i. We now claim that these sections generate Cox(X).

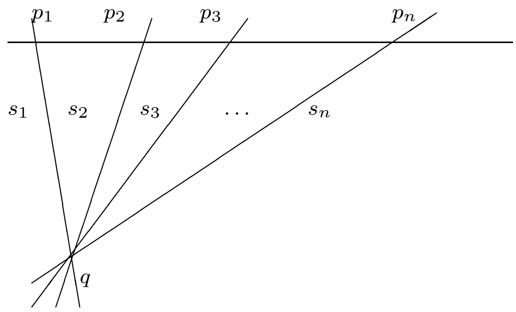


Fig. 3.1 The choice of the sections $s_1, s_2, \ldots, s_n$.

Proposition 3.7. *Let X be the blow-up of $\mathbf{P}^2$ in n distinct points on a line. Then there is a multigraded surjection*

$$p : k[l, e_1, e_2, \ldots, e_n, s_1, s_2, \ldots, s_n] \to \mathrm{Cox}(X).$$

Proof. By Proposition 2.7, we may take D to be nef. Write D (uniquely) as a sum of the nef cone generators,

$$D = a_1(L - E_1) + a_2(L - E_2) + \ldots + a_n(L - E_n) + aL$$

where $a, a_i \geq 0$. Note that all the nef cone generators of the form $L - E_i$ are indeed base-point free pencils, so we may apply lemma 3.5. We proceed by induction on n and $a + a_1 + \ldots + a_n$. Let C be an irreducible conic in $|L - E_i|$.

Case 1: $a_i \geq 2$ for some $1 \leq i \leq n$. Suppose this is $a_1 \geq 2$. First of all, $H^1(X, D - 2C) = 0$, by nefness, and so we get by Castelnuovo's base-point free pencil trick a surjection

$$H^0(X, D - C) \otimes H^0(X, C) \to H^0(X, D).$$

By the induction hypothesis, $H^0(X, D - C)$ is generated by the above sections and so the the claim follows by induction on $a + a_1 + \ldots + a_n$.

Case 2: $a_i = 1$ for some $1 \leq i \leq n$ and $a \geq 1$. Suppose $D.E_1 = 1$ we need the exact sequence $0 \to \mathscr{O}_X(D - 2C - E_1) \to \mathscr{O}_X(D - 2C) \to \mathscr{O}_{E_1}(-1) \to 0$. This gives

$$H^1(X, \mathscr{O}_X(D - 2C - E_1)) \to H^1(X, \mathscr{O}_X(D - 2C)) \to H^1(E_1, \mathscr{O}_{E_1}(-1)) \to 0$$

The first H^1 is zero since $D - 2C - E_1 = (a-1)L + a_2(L - E_2) + \ldots + a_n(L - E_n)$ is nef, while the last one is zero since $h^1(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(-1)) = 0$ by Riemann-Roch. Exactness gives $H^1(X, \mathscr{O}_X(D - 2C)) = 0$, and we get a surjection as above by the base-point free pencil trick.

Case 3: $a_i = 0$ for some $1 \leq i \leq n$. Here we must have say, $D.E_n = 0$, and $\mathscr{O}_X(D) \cong \pi^*\mathscr{O}_{X'}(D')$ where $\pi : X \to X'$ is the morphism contradicting E_n and X' is a rational surface isomorphic to the blow-up of $\mathbf{P}^2$ in the points $p_1, \ldots, p_{n-1}$. For $n = 2$, the result is clear since $\mathrm{Cox}(X)$ is the polynomial ring $k[e_1, e_2, l_{12}, s_1, s_2]$. Now, by induction on n a generating set of $\mathrm{Cox}(X')$, is $l', s'_1, \ldots, s'_{n-1}, e'_1, \ldots, e'_{n-1}$, and we have $\pi^*(e'_i) = e_i, \pi^*(s'_i) = s_i$ and $\pi^*(l') = le_n$. Hence we can choose a basis of $H^0(X, \mathscr{O}_X(D))$ of monomials in the variables $\{l, e_i, s_i\}_{i=1,\ldots,n}$.

Case 4: $D = nL - E_1 - \ldots - E_n$. This is the case $a_1 = \cdots = a_n = 1$, $a = 0$ above. Note that we have an exact sequence

$$0 \to \mathscr{O}_X(nL - 2E_1 - E_2 - \ldots - E_n) \to \mathscr{O}_X(nL - 2E_1 - \ldots - E_n) \to \mathscr{O}_{E_1}(1) \to 0.$$

Taking the long exact sequence of cohomology and using the fact that $H^1(X, nL - 2E_1 - \ldots - E_n) = 0$ by the lemma, we get

$$0 \to H^0(X, nL - 2E_1 - \ldots - E_n) \to H^0(X, nL - E_1 - \ldots - E_n) \to H^0(E_1, \mathscr{O}_{E_1}(1)) \to 0.$$

in which $H^0(E_1, \mathscr{O}_{E_1}(1))$ is 2-dimensional. By induction $H^0(X, nL - 2E_1 - \ldots - E_n) \cong l \cdot H^0(X, (n-1)L - E_1)$ is generated by the above sections, so we need only show that $H^0(X, nL - E_1 - \ldots - E_n)$ has two sections that restrict to a basis of $H^0(E_1, \mathscr{O}_{E_1}(1))$.

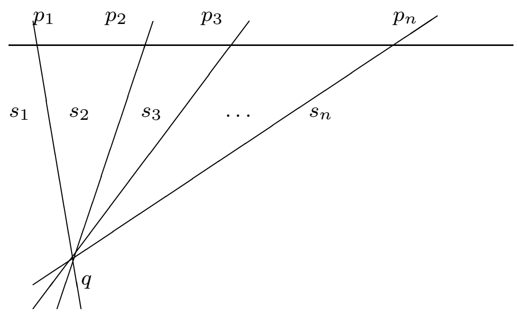
Consider the sections $s = s_1(s_2 s_3 \cdots s_n)$ and $t = le_2 \cdots e_n(s_2 s_3 \cdots s_n)$. We claim that the restrictions of s and t to E_1 are linearly independent. Note that s meets the line E_1 in the intersection between s_1 and E_1, while t meets E_1 in the intersection with l. Recall that s_1 was defined as the pullback of a line through $p_1, q \in \mathbf{P}^2$, where q was not lying on the line C. These lines have different tangent directions through p_1 which implies that s_1 meets e_1 in a different point than l in the blow-up. Hence the two sections s, t vanish at different points on E_1 and hence restrict to linearly independent sections of $H^0(E_1, \mathscr{O}_{E_1}(1))$. $\square$

3.4 Relations

Now, consider the divisor class L. We have $h^0(X, L) = 3$, while there are $n+1$ monomials of degree L in $k[l, e_1, e_2, \ldots, e_n, s_1, s_2, \ldots, s_n]$:

$$s_1 e_1, \quad s_2 e_2, \quad \cdots \quad s_n e_n, \quad l e_1 e_2 e_3 \cdots e_n$$

Hence there are $n - 2$ linear dependence relations between them. Consider again the projection of these sections in Figure 3.1. Of course any three of these lines are linearly



independent, since they belong to the subsystem of $|L|$ of lines through q. It follows that we have minimal relations of the following form:

$$\begin{aligned} g_1 &= \underline{s_1 e_1} + a_1 s_{n-1} e_{n-1} + b_1 s_n e_n = 0 \\ g_2 &= \underline{s_2 e_2} + a_2 s_{n-1} e_{n-1} + b_2 s_n e_n = 0 \\ &\vdots \qquad\qquad \vdots \\ g_{n-2} &= \underline{s_{n-2} e_{n-2}} + a_{n-2} s_{n-1} e_{n-1} + b_{n-2} s_n e_n = 0 \end{aligned} \tag{3.2}$$

where all of the coefficients a_i, b_i are non-zero. We denote the ideal generated by these relations by J. The leftmost terms above are underlined since, as the next lemma shows, they form an initial ideal for J.

Lemma 3.8. *The set $\{g_1, \ldots, g_{n-2}\}$ is a Gröbner basis for J with respect to the graded lexicographical order, and* in $J = (s_1 e_1, \ldots, s_{n-2} e_{n-2})$ *is an initial ideal of J.*

Proof. It is well-known (e.g see [AL94]) that a collection of polynomials with relatively prime leading terms is a Gröbner basis for the ideal they generate. □

Theorem 3.9. *Let X be the blow-up of n distinct points on a line. Then* $\mathrm{Cox}(X)$ *is a complete intersection, with $n-2$ quadratic relations.*

Proof. We show that $I_D = J_D$ for all nef classes $D = mL - a_1E_1 - a_2E_2 - \ldots - a_nE_n$, where $m \geq a_1 + \ldots + a_n$ and $a_i \geq 0$. Note that since $J \subseteq I$, we have in any case a multigraded surjection

$$k[l, e_1, e_2, \ldots, e_n, s_1, s_2, \ldots, s_n]/J \to \mathrm{Cox}(X).$$

To show that this is an isomorphism in degree D we calculate the (multigraded) Hilbert function of both sides. From Riemann-Roch we have

$$\begin{aligned} \dim_k \mathrm{Cox}(X)_D &= h^0(X, mL - a_1E_1 - a_2E_2 - \ldots - a_nE_n) \\ &= \binom{m+2}{2} - \binom{a_1+1}{2} - \ldots - \binom{a_n+1}{2}. \end{aligned}$$

To calculate $\dim_k R/J_D$, we use the Gröbner basis for J. Since the Hilbert function is preserved when going to initial ideals, we have

$$\dim_k R/(f)_D = \dim_k R/(s_1e_1, \ldots, s_{n-2}e_{n-2})_D$$

Now, any monomial m in $R/(s_1e_1, \ldots, s_{n-2}e_{n-2})_D$ corresponds to a way of writing D as a non-negative sum of divisor classes from

$$L - E_1 - \ldots - E_n, \qquad L - E_1, \qquad \ldots \qquad L - E_n, \qquad E_1, \qquad \ldots, \qquad E_n$$

such that not both $L - E_i$ and E_i occur in the sum for $i = 1, \ldots, n-2$. Thus the problem of finding $\dim_k(R/J)_D$ reduces to the following counting problem: finding the number of non-negative solutions of

$$\begin{aligned} s_1 + s_2 + \ldots + s_n + l &= m \\ s_1 + l - e_1 &= a_1 \\ s_2 + l - e_2 &= a_2 \\ \vdots \quad &\vdots \\ s_n + l - e_n &= a_n \end{aligned} \tag{3.3}$$

such that

$$s_i \cdot e_i = 0 \text{ for } i = 1, \ldots, n-2.$$

Here, by horrible abuse of notation, the numbers s_i, e_i, l represents respectively the non-negative coefficients of $L - E_i, E_i, L - E_1 - \ldots - E_n$ in the sum. Now, fix $l \geq 0$ and subtract l from each of the equations in (3.3) to get

$$\begin{aligned} s_1 + s_2 + \ldots + s_n &= m - l \\ s_1 - e_1 &= a_1 - l \\ s_2 - e_2 &= a_2 - l \\ \vdots \quad &\vdots \\ s_n - e_n &= a_n - l \end{aligned} \tag{3.4}$$

We count the number of non-negative solutions $S(l)$ to this system. First we claim that

$$m - l \geq \sum_{k=1}^{n} \max(a_i - l, 0)$$

for $l \leq m$. This is by induction on l: For $l = 0$, this is the nef condition on D. Now, increasing l by one decreases the left hand side by one, and if there is some $a_i - l > 0$, then $\max(a_i - l, 0)$ is also decreased by 1, if not, the right hand side is zero, so in any case the inequality is preserved.

Now, note that s_i for $1 \leq i \leq n-2$ is completely determined by $s_i.e_i = 0$, in fact $s_i = \max(a_i - l, 0)$. Hence by the first equation in (3.4) we are looking for non negative solutions to

$$s_n + s_{n-1} = m - l - \sum_{k=1}^{n-2} \max(a_k - l, 0)$$

such that $s_n \geq \max(a_n - l, 0)$ and $s_{n-1} \geq \max(a_{n-1} - l, 0)$, of which there are in total

$$m - l - \sum_{k=1}^{n-2} \max(a_{n-2} - l, 0) + 1 - \sum_{k=n-1,n} \max(a_k - l, 0).$$

Hence the total number of solutions to (3.3) is

$$\begin{aligned} \sum_{l=0}^{m} S(l) &= \sum_{l=0}^{m} \left(m + 1 - l - \sum_{k=1}^{n} \max(a_k - l, 0) \right) \\ &= \binom{m+2}{2} - \sum_{i=0}^{a_1} (a_1 - i) - \sum_{i=0}^{a_1} (a_1 - i) - \ldots - \sum_{i=0}^{a_n} (a_n - i) \\ &= \binom{m+2}{2} - \binom{a_1+1}{2} - \binom{a_2+1}{2} - \ldots - \binom{a_n+1}{2}. \end{aligned}$$

This finishes the proof that $I = J$. Now, from [BP04, Remark 1.4] we have $\dim \operatorname{Cox}(X) = n+3$, furthermore by Proposition 3.7 we have that $\operatorname{codim} \operatorname{Cox}(X) = (2n+1) - (n+3) = n-2$, which is exactly the number of relations in I. □

Corollary 3.10. *The ring* $\mathrm{Cox}(X)$ *is a Koszul algebra and is Gorenstein.*

Proof. It is well known that any $G-$quadratic algebra is Koszul and that any complete intersection is Gorenstein. □

Remark. It was shown in [Pop05] that the Cox rings of Del Pezzo surfaces are also Gorenstein.

Remark. The above theorem can also be proved in another way, using the following lemma, proved by Stillman in [ST05]:

Lemma 3.11. *Let* $J \subset k[x_1, x_2, \ldots, x_n]$ *be an ideal containing a polynomial* $f = gx_1 + h$*, with* g*,* h *not involving* x_1 *and* g *a non-zero divisor modulo* J*. Then,* J *is prime if and only if the elimination ideal* $J \cap k[x_2, \ldots, x_n]$ *is prime.*

Note that $(g_1, \ldots, g_{n-2}) \cap k[s_2, \ldots, s_n, e_2 \ldots, e_n, l] = (g_2, \ldots, g_{n-2})$ since $\{g_1, \ldots, g_{n-2}\}$ is a Gröbner basis. Then the above lemma can be applied inductively, to prove that J is prime (take $x_1 = e_1, g = s_1$, $h = a_1 s_{n-1} e_{n-1} + b_1 s_n e_n$). For $n = 3$, the result is obvious since $s_1 e_2 - s_2 e_2 + s_3 e_3$ is irreducible. Then, since $I \subseteq J$ are two prime ideals with the same Krull dimension, it follows that they are in fact equal.

3.5 Three points on a line: Explicit computations

In this section we consider the case of three points on a line in more detail, using Macaulay 2 to exhibit the relations. This is a 'limiting case' in the sense that it is the first case where the anticanonical divisor $-K = 3L - E_1 - E_2 - E_3$ ceases to be ample, since it contracts the line L. This makes the surface X a generalized Del Pezzo surface. Also, this gives a simple example of a Cox ring with a single defining relation.

We exhibit the generators and relations by looking at the anticanonical map of X. We have $h^0(X, -K) = 7$ and so a generating set of sections of $H^0(X, -K)$ give a rational map $\phi : X \to \mathbf{P}^6$. We get explicit equations by taking the linear system of cubics through the points $(1:0:0), (0:1:0), (1:1:0)$. This gives us the linear series

$$x_0 = Z^3, x_1 = Z^2 X, x_2 = Z^2 Y, x_3 = XYZ, x_4 = ZX^2, x_5 = ZY^2, x_6 = XY(Y - X)$$

Using Macaulay 2, we compute the following elimination ideal, which defines the blow-up as a surface Y in $\mathbf{P}^6$.

$$\begin{array}{lll} x_2x_6 - x_3x_5 + x_4x_5 & x_3^2 - x_4x_5 & x_2^2 - x_0x_5 \\ x_3x_4 - x_4x_5 + x_1x_6 & x_2x_3 - x_1x_5 & x_1x_3 - x_1x_5 + x_0x_6 \\ x_2x_4 - x_1x_5 + x_0x_6 & x_1x_2 - x_0x_3 & x_1^2 - x_0x_4 \end{array}$$

The surface Y has its singularity in the point $(0:0:0:0:0:0:1)$, which is the image of L under ϕ. The intersection of Y with the hyperplane $Z(x_0, x_1, x_2)$ splits into three projective lines, and so we find that the lines E_1, E_2, E_3 are given by

$$E_1 = Z(x_0, x_1, x_2, x_3, x_4) \quad E_2 = Z(x_0, x_1, x_2, x_3, x_5) \quad E_3 = Z(x_0, x_1, x_2, x_3 - x_5, x_3 - x_4)$$

Furthermore, by intersecting Y with $Z(x_6)$, we get three conics

$$Q_1 = Z(x_3, x_1, x_4, x_6, -x_2^2 + x_0x_5), \quad Q_2 = Z(x_3, x_2, x_5, x_6, -x_1^2 + x_0x_4)$$

$$Q_3 = Z(-x_1 + x_2, -x_3 + x_5, -x_3 + x_4, x_6, x_1^2 - x_0x_3)$$

These correspond to conics of degree $L - E_i$. Using the equations above we determine the anticanonical embedding

$$\begin{aligned} &\phi^*(x_0) = l^3e_1^2e_2^2e_3^2 \quad \phi^*(x_1) = l^2e_1^2e_2e_3s_1 \quad \phi^*(x_2) = l^2e_1e_2^2e_3s_2 \\ &\phi^*(x_3) = le_1e_2s_1s_2 \qquad \phi^*(x_4) = le_1^2s_1^2 \qquad \phi^*(x_5) = le_2^2s_2^2 \\ &\phi^*(x_6) = s_1s_2s_3 \end{aligned}$$

where $Q_i = (\phi_* s_i)_0$. Hence we find the following relation:

$$\begin{aligned} 0 = \phi^*(x_2x_4 - x_1x_5 + x_0x_6) &= l^3e_1^3e_2^2e_3s_1^2s_2 - l^3e_1^2e_2^3e_3s_1s_2^2 + l^3e_1^2e_2^2e_3s_1s_2s_3 \\ &= l^3e_1^2e_2^2e_3s_1s_2(e_1s_1 - e_2s_2 + e_3s_3), \end{aligned}$$

clearly showing the relation $e_1s_1 - e_2s_2 + e_3s_3 = 0$ in $H^0(X, L)$.

3.6 Singularities

We now study the singularities of $\operatorname{Proj}\operatorname{Cox}(X)$ and $\operatorname{Spec}\operatorname{Cox}(X)$. These are highly singular, and the singular locus increases with n: Note for example that $\operatorname{Cox}(X)$ is singular along the codimension 5 subscheme defined by

$$Z(e_1 = e_2 = \ldots = e_n = s_1 = s_{n-1} = s_n) \cap Z(I)$$

Proposition 3.12. *The singular locus of* $\operatorname{Proj}\operatorname{Cox}(X_n)$ *has codimension 5.*

Proof. The proof is by induction on n. For $n = 3$, $\operatorname{Proj}\operatorname{Cox}(X)$ is the hypersurface $Z(s_1e_1 - s_2e_2 + s_3e_3) \subset \mathbf{P}^6$. There is an isolated singularity in the point $p = (0:0:0:0:0:0:1)$. Now, the proof of [BP04, Prop. 4.4] extends to this case and shows that

$$U_{x_E} \cap \operatorname{Proj}\operatorname{Cox}(X_n) \cong \operatorname{Spec}(X_{n-1})$$

where E is a exceptional curve. Inductively this shows that the dimension of the singular locus increases by one for each blow-up. □

3.7 Classification of Cox rings of $\mathbf{P}^2$ blown up in few points

In [Der06], Derenthal studied Cox rings of generalized Del Pezzo surfaces, whose Cox rings have a unique defining relation. In this section, we provide related results by giving a complete description of Cox rings of $\mathbf{P}^2$ blown in ≤ 5 points.

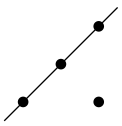
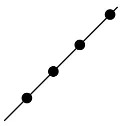
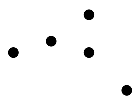
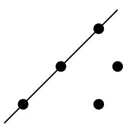
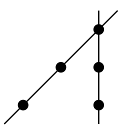
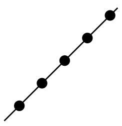
We are interested in studying how the Cox ring changes when the points blown up vary. Already for the case of three points in the plane, we see that there is an interesting phenomenon occurring: Here the 'general fiber' in the family of Cox rings is a polynomial ring, while the 'special fiber' is a quadric hypersurface. The reason for this is mainly because of the (−2)-curve, which becomes an extra generator. We think that it would be an interesting problem to find some general framework for studying such families.

Special configurations other than points on a line provide no new difficulties in the computation of their Cox rings. On the contrary, as the points move into 'more general' position, it turns out that we need fewer 'extra sections' like the sections s_i above. This is because in the new setting, the effective cone needs more generators and the new divisor classes provide enough generators for the Cox ring. For example, when $r = 3$ and the points are general, the sections $l_{12}e_2, l_{13}e_2$ constitute a basis for $\mathrm{Cox}(X)_{L-E_1}$. This reflects the fact that $L - E_1 - E_2$ and $L - E_1 - E_3$ are minimal divisor classes in the former case, but not in the latter.

In all the cases below, the effective cone will be generated by negative curves, and there will be (base-point free) pencils in a generating set for the nef cone. This will allows us to find generators for $\mathrm{Cox}(X)$ using the previous technique. We then find relations among the generators in low-degree nef divisor classes by using explicit equations for their projections to $\mathbf{P}^2$ and using elimination theory to find their linear dependencies. Since there are only ≤ 5 relations in each case, we can proceed using a dimension argument and a primality test to check that we have found all the minimal relations.

The results are presented in the table on the opposite page.

Note the equations for the sixth case where one of the points lie on the intersection of the lines through the two other pairs of points. Making the substitution $u = e_5 l_{125}$ and $v = e_5 l_{135}$, we recover the homogeneous coordinate ring of the Grassmannian $G(2,5)$. Thus the equations seem to define some $\mathbf{P}^1$-bundle over the Grassmannian.

Configuration	Generators	Relations
 Points: (1:0:0),(0:1:0),(0:0:1), (1:1:1)	$e_1, e_2, e_3, e_4,$ $l_{12}, l_{13}, l_{14}, l_{24}, l_{34}$	$e_1 l_{12} - e_3 l_{23} + e_4 l_{24}$ $e_2 l_{12} - e_3 l_{13} + e_4 l_{14}$ $e_1 l_{13} - e_2 l_{23} + e_4 l_{34}$ $e_1 l_{14} - e_2 l_{24} + e_3 l_{34}$ $l_{14} l_{23} - l_{13} l_{24} + l_{12} l_{34}$
 Points: (1:0:0),(0:1:0),(0:0:1), (1:1:0)	$e_1, e_2, e_3, e_4,$ $l_{14}, l_{24}, l_{34}, l_{123}$	$e_1 l_{14} + e_2 l_{24} + e_3 l_{34}$
 Points: (1:0:0),(0:1:0),(1:1:1), (1:r:1)	$e_1, e_2, e_3, e_4,$ $s_1, s_2, s_3, s_4, l_{1234}$	$e_1 s_1 + s_3 e_3 + s_4 e_4$ $e_2 s_2 + a s_3 e_3 + b s_4 e_4$
 Points: (1:0:0),(0:1:0),(0:0:1), (1:1:1),(1:a:b)	$e_1, e_2, e_3, e_4, e_5, l_{12}$ $l_{13}, l_{14}, l_{23}, l_{24}, l_{34}, g$	20 quadrics. See Section 2.7.1.
 3 on a line, Points: Points: (1:0:0),(0:1:0),(0:0:1), (1:1:1),(1:0:c)	$e_1, e_2, e_3, e_4, e_5, l_{12}$ $l_{135}, l_{14}, l_{23}, l_{24}, l_{34}, l_{45}, l_{25}$	$l_{14} e_1 - l_{24} e_2 + l_{34} e_3$ $l_{12} e_1 - l_{23} e_2 + l_{24} e_4$ $l_{13} e_1 - l_{23} e_2 + l_{34} e_4$ $(c-1) l_{34} e_3 + l_{24} e_2 - l_{45} e_5$ $(c-1) l_{23} e_3 + l_{24} e_4 - l_{25} e_5$ $l_{23} l_{24} - l_{24} l_{34} - l_{23} l_{45} + l_{34} l_{25}$
 Points: (1:0:0),(0:1:0),(0:0:1), (1:1:0),(1:0:1)	$e_1, e_2, e_3, e_4, e_5,$ $l_{135}, l_{245}, l_{14}, l_{23}, l_{34}$	$e_1 l_{12} - e_3 l_{23} + e_4 e_5 l_{245}$ $e_2 l_{12} - e_3 l_{135} e_5 + e_4 l_{14}$ $e_1 e_5 l_{13} - e_2 l_{23} + e_4 l_{34}$ $e_1 l_{14} - e_2 e_5 l_{245} + e_3 l_{34}$ $l_{14} l_{23} - e_5^2 l_{135} l_{245} + l_{12} l_{34}$
 Points: (1:0:0),(0:1:0),(0:0:1), (1:a:0),(1:b:0)	$e_1, e_2, e_3, e_4, e_5,$ $s_1, s_2, s_3, s_4, s_5, l_{12345}$	$e_1 s_1 + s_4 e_4 + s_5 e_5$ $e_2 s_2 + a s_4 e_4 + b s_5 e_5$ $e_3 s_3 + c s_4 e_4 + d s_5 e_5$

Chapter 4
Cox Rings of Blow-ups of $\mathbf{P}^3$

Let X_r denote a blow-up of $\mathbf{P}^3$ in r points $p_1, \ldots, p_r$ in general position. It follows from Proposition 6.7 in [Ful93], that we have an isomorphism

$$A_k(X) \cong A_k(\mathbf{P}^3) \oplus A_k E_1 \oplus \cdots \oplus A_k E_r, \qquad k = 1, 2$$

We choose a basis for $A_2(X) = \mathrm{Pic}(X)$ by taking the pullback H of a plane in $\mathbf{P}^3$ and the exceptional planes $E_1, \ldots E_5$. Similarly, we choose a basis for the group of $1-$cycles $A_1(X)$ by taking $l = H^2$ to be the pullback of a line in $\mathbf{P}^3$ and $l_i = E_i^2$ lines in E_i for $i = 1, \ldots, 5$. The intersection pairing is given by

$$H.l = 1, \qquad H.l_i = 0, \qquad E_i.l = 0, \qquad E_i.l_j = -\delta_{ij}. \tag{4.1}$$

Lemma 4.1. *The Chern classes of X_r are given by*

$$\begin{aligned} c_1(X) &= -K_X = 4H^2 - 2E_1 - \ldots - 2E_r \\ c_2(X) &= \pi^* c_2 = 6H^2 \end{aligned}$$

Proof. This follows at once from the blow-up for Chern classes in [Ful93, §15.4] or [GH78, §4.6], using the values $c_1(\mathbf{P}^3) = 4H$ and $c_2(\mathbf{P}^3) = 6H^2$. □

Proposition 4.2. *Let X be the blow-up of $\mathbf{P}^3$ in r points in general position, and let $D = aH - b_1E_1 - \ldots - b_rE_r$ be a divisor class on X. Then the following formula for $\chi(\mathscr{O}_X(D))$ holds:*

$$\chi(\mathscr{O}_X(D)) = \binom{a+3}{3} - \binom{b_1+2}{3} - \ldots - \binom{b_r+2}{3} \tag{4.2}$$

Proof. By the Hirzebruch-Riemann-Roch theorem [Ful84], we have the following formula for $\chi(\mathscr{L})$ on a threefold

$$\begin{aligned} \chi(\mathscr{O}_X(D)) &= \int_X D^3 + \frac{1}{4} c_1 \cdot D^2 + \frac{1}{2}(c_1^2 + c_2) \cdot D + \frac{1}{24} c_1 c_2 \\ &= \int_X D^3 + \frac{1}{4} c_1 \cdot D^2 + \frac{1}{2}(c_1^2 + c_2) \cdot D + 1 \end{aligned}$$

where we have used that $\frac{1}{24} c_1 c_2 = \chi(\mathscr{O}_X)$ (which is obtained by setting $D = 0$ above) and $\chi(\mathscr{O}_X) = 1 - p_a = 1$. The above formula for $\chi(\mathscr{O}_X(D))$ is then obtained by a rather tedious substitution using Lemma 4.1 for the Chern classes and the relations (4.1). □

4.1 $\mathbf{P}^3$ blown up in ≤ 4 distinct points

Blow-ups of $\mathbf{P}^3$ in $r \leq 4$ points are toric and their fans and Cox rings are computed as in Section 2.3.2.

r	$\mathrm{Cox}(X_r)$
0	$k[z, y, z, w]$
1	$k[z, y, z, w, e_1]$
2	$k[z, y, z, w, e_1, e_2]$
3	$k[z, y, z, w, e_1, e_2, e_3]$
4	$k[z, y, z, w, e_1, e_2, e_3, e_4]$

4.2 $\mathbf{P}^3$ blown up in five distinct points

Let $X = X_5$ be the blow-up of $\mathbf{P}^3$ in points $p_1, \ldots, p_5$ in general position. By the transitive action of $PGL(4)$ on general quintuples, we may take the five points to be

$$p_1 = (1, 0 : 0 : 0),\ \ p_2 = (0 : 1 : 0 : 0),\ \ p_3 = (0 : 0 : 1 : 0)$$
$$p_4 = (0 : 0 : 0 : 1),\ \ p_5 = (1 : 1 : 1 : 1).$$

Let $x_1, \ldots, x_5$ be the generators for the cohomology groups $H^0(X, E_1), \ldots, H^0(X, E_5)$ respectively, and let h_{ijk} denote a generator for $H^0(h - E_i - E_j - E_j)$. Geometrically the zero-sections of $x_1, \ldots, x_5$ correspond to the exceptional planes and h_{ijk} corresponds to pullbacks of planes through p_i, p_j, p_k in $\mathbf{P}^3$. We will for this reason henceforth refer to x_i, h_{ijk} as the *planar sections.*

In this notation, $\mathrm{Cox}(X_4)$ is generated by the sections $x_1, \ldots, x_4, h_{ijk}$, $\{i, j, k\} \subset \{1, 2, 3, 4\}$ distinct.

Lemma 4.3. *If $D = aH - \sum b_i E_i$ is an effective divisor class and $b_1, \ldots, b_5 \geq 0$, then $a \geq b_i$ for all $i = 1, \ldots, 5$.*

Proof. We show that the 1-cycle $l - l_i$ has positive intersection number with any effective divisor, i.e., it is a 'nef' curve. This is because a curve with class $l - l_i$ is the pullback of a line in $\mathbf{P}^3$ through p_i, which means that for any point p in X, there is a curve with class $l - e_i$ passing through p. Therefore $D.(l - l_i) = a - b_i \geq 0$. □

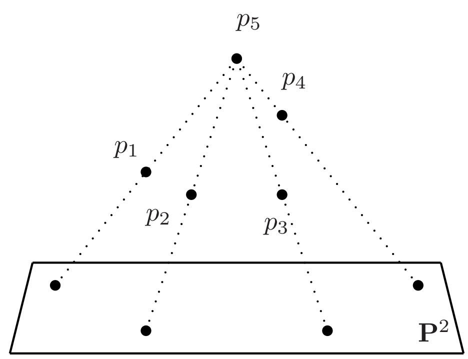
Lemma 4.4. *The class of curve $3l - l_1 - l_2 - l_3 - l_4 - l_5$ is nef.*

Proof. Note that for any given point p of $\mathbf{P}^3$ there is a conic with class $(2l - l_1 - l_2 - l_3 - l_4)$ going through p, so it is base-point free. The class $3l - l_1 - l_2 - l_3 - l_4 - l_5$ is now the sum of $(2l - l_1 - l_2 - l_3 - l_4)$ and $(l - l_5)$, both of which are base-point free, hence nef.

We are now in position to calculate the Cox ring of X_5. The approach is somewhat different than the technique used in the proof of Theorem 3.7, and owes debt to Castravet and Tevelev's work in [CT06], where we got the idea for the approach used in the proof.

Theorem 4.5. *Let X be the blow-up of $\mathbf{P}^3$ in the five general points $p_1, \ldots, p_5$. Then* $\mathrm{Cox}(X)$ *is generated by the sections x_i, h_{ijk} from the respective divisor classes $E_1, \ldots, E_5$ and $H - E_i - E_j - E_k$, $1 \leq i, j, k \leq 5$ distinct.*

Proof. Let $p : \mathbf{P}^3 \dashrightarrow \mathbf{P}^2$ be the projection from $p_5 = (0:0:0:1)$ and let $q_1 = (1:0:0)$, $q_2 = (0:1:0), q_3 = (0:0:1), q_4 = (1:1:1)$ denote the images of $p_1, \ldots, p_4$ under p.



Let Y be the Del Pezzo surface obtained by blowing up the points $q_1, \ldots, q_4$. Let $E'_1, \ldots, E'_4$ denote the respective exceptional divisors and let L denote the pullback of a line in $\mathbf{P}^2$. As before, we fix generating sections $e_1, \ldots, e_4, l_{12}, \ldots, l_{34}$ generating $\mathrm{Cox}(Y)$. We construct a map $\phi : H^0(X, D) \to \mathrm{Cox}(Y)$.

Let $aH - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4 - b_5E_5$ be the divisor class of D. We may assume the following ordering on the b_i:

$$b_1 \geq b_2 \geq \cdots \geq b_5 > 0.$$

This follows by permutation of the $p_1, \ldots, p_5$ and since in case $b_5 \leq 0$, we may consider the divisor $D' = D - b_5E_5$: Note that $D' = \pi^*(B)$ for some effective divisor B on X_4. Since generators for X_4 pull back to generators for $\mathrm{Cox}(X)$ via the blow-up $\pi : X \to X_4$, this means that $H^0(X, D')$ is generated by the planar sections and hence also $H^0(X, D)$ via multiplication by $e_5^{b_5}$. We may by Lemma 4.3 also assume that $a \geq b_i$.

To define the map ϕ in degree D, first identify $E_5 \cong \mathbf{P}^2$ with the image of the projection p and regard the restriction map r as a map

$$r : H^0(X, D) \to H^0(E_5, D|_{E_5}) = H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(b_5)) = H^0(Y, b_5L).$$

Note that if for some $i = 1, \ldots, 4$ one has $(l - l_i - l_5).D = a - b_i - b_5 < 0$, then $l - e_i - e_5$ is a fixed component in the linear system $|D|$ and consequently, the projection of any section in $H^0(X, D)$ to $\mathbf{P}^2$ is a curve through the point q_i with multiplicity $b_i + b_5 - a$. This means that the image of r lies in the linear subsystem $|b_5L - (b_i + b_5 - a)E'_i| \subset |b_5L|$, hence $r(s)$ is divisible by $e_i^{b_i + b_5 - a}$ for any $s \in H^0(X, D)$. It therefore makes sense (as in [CT06]) to formally define

$$\phi(s) = r(s) \cdot e_1^{a-b_1-b_5} e_2^{a-b_2-b_5} e_3^{a-b_3-b_5} e_4^{a-b_4-b_5}.$$

Note that the kernel of the map ϕ is precisely $\ker r \cong H^0(X, D - E)$.

The map ϕ induces a linear map $\alpha : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ given by

$$\alpha : aH - \sum_{i=1}^{5} a_i E_i \mapsto b_5 L - \sum_{i=1}^{4} (b_i + b_5 - a) E_i'$$

making the following diagram commute (cf. [CT06]):

$$\begin{array}{ccc} H^0(X, D) & \xrightarrow{\phi} & H^0(Y, \alpha(D)) \\ \downarrow r & & \downarrow b_* \\ H^0(E_5, D|_{E_5}) & = & H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(b_5)) \end{array}$$

Here b_* is the map given by push-forward via the blow-up $b : Y \to \mathbf{P}^2$.

Note that $\alpha(E_i) = E_i'$ for $i = 1, 2, 3, 4$, so $\phi(x_i)$ is some scalar multiple of e_i. By replacing section x_i by a scalar multiple, we may assume that $\phi(x_i) = e_i$. Similarly, $\alpha(H - E_i - E_j - E_5) = L - E_i' - E_j'$ $i \neq j$, so we may take $\phi(h_{ij5}) = l_{ij}$.

Since $\ker \phi \cong H^0(X, D - E)$, we have an exact sequence:

$$0 \to H^0(X, D - E) \to H^0(X, D) \to H^0(Y, \alpha(D)).$$

We claim that the right hand map is surjective and that any section of $H^0(Y, \alpha(D))$ can be lifted to a linear combination of products of sections x_i, h_{ijk}.

Let us first explain why this implies the theorem. Let s be any section of $H^0(X, D)$. Then $r(s)$ is a section in $H^0(Y, \alpha(D))$ which by hypothesis lifts to a section $s' \in H^0(X, D)$ which a polynomial in the x_i, h_{ijk}. Since $r(s) = r(s')$, this means that $s - s' \in \ker r = H^0(X, D - E)$, i.e $s - s' = e_5 t$ for some $t \in H^0(X, D - E)$. This means that we reduce to showing that $H^0(X, D - E)$ is generated by the planar sections. Continuing this process, we must reach a divisor such that $D - E$ is not effective anymore, and at this point we have $H^0(X, D) \cong H^0(Y, \alpha(D))$ and we are done (since generators for $H^0(Y, \alpha(D))$ lift isomorphically to generators of $H^0(X, D)$).

Since $\mathrm{Cox}(Y)$ is generated by sections corresponding to exceptional curves, it suffices to show that any section $s = \prod_i e_i^{n_i} \prod_{i \neq j} l_{ij}^{n_{ij}} \in H^0(Y, \alpha(D))$ can be lifted to a section in $H^0(X, D)$, as above.

The section s has $\mathrm{Pic}(Y)$-degree

$$\alpha(D) = \sum_i n_i E_i' + \sum_{i \neq j} n_{ij} (L - E_i' - E_j').$$

We will lift the sections l_{ij} of degree $L - E_i - E_j$ to the sections h_{ij5}. This gives us a divisor class on X,

$$F = \sum_{i \neq j} n_{ij} (H - E_i - E_j - E_5)$$

Consider the divisor class $D' = D - F$. We will show that D' is an effective divisor on X and that there exist a degree D' monomial in the planar sections that that map to the remaining part $\prod_i e_i^{n_i}$. Note that since $\sum_{i \neq j} n_{ij} = b_5$, D' is a divisor on X not containing E_5. Write

$$D' = cH - d_1 E_1 - \ldots - d_4 E_4,$$

where $c = a - b_5$.

Note that both x_i and $\hat{h_i} := h_{jkl}$, $\{j, k, l\} = \{1, 2, 3, 4\} \setminus \{i\}$, map to the variable e_i via ϕ. The plan is to lift the monomial $m = \prod_i e_i^{n_i}$ to a monomial in $H^0(X, D')$, by taking a product of c $\hat{h_i}$'s and the remaining of the form x_i. By construction, the product we get will map to m via ϕ.

For example, if $D = 2H - E_1 - E_2 - E_3 - E_4 - E_5$, then $\alpha(D) = L$ and we must lift say, the section $l_{12}e_1e_2 \in H^0(Y, L)$. We lift l_{12} to h_{125}, so we have to find a monomial of degree

$$D' = (2H - E_1 - E_2 - E_3 - E_4 - E_5) - (H - E_1 - E_2 - E_5) = H - E_3 - E_4$$

mapping to the monomial $e_1e_2 \in H^0(Y, L)$. We decompose $H - E_3 - E_4$ as $(H - E_2 - E_3 - E_4) + E_2$ which gives the section $h_{234}x_2 \in H^0(X, D')$. Finally, the section $s = h_{125}h_{234}x_2$ has degree D and is mapped by ϕ to the section $l_{12}e_1e_2$ in $H^0(X, L)$.

We show that it is always possible to decompose D' as above. For this, we first show that $\sum n_i \geq c = a - b_5$. This essentially means that we have enough E_i's to decompose D'. The explicit algorithm to decompose D' as above is given below. Note that

$$\alpha(D) = \sum_i n_i E_i' + \sum_{i \neq j} n_{ij}(L - E_i' - E_j') = b_5 L + \sum_{i=1}^{4} (a - b_i - b_5) E_i'. \tag{4.3}$$

Hence we get by counting E_i's,

$$\begin{aligned} \sum_{i=1}^{4} n_i - c &= \sum_{i=1}^{4} (a - b_i - b_5) + 2 \sum_{i \neq j} n_{ij} - (a - b_5) \\ &= 4a - 4b_5 - \sum_{i=1}^{4} b_i + 2 \cdot b_5 - a + b_5 \\ &= 3a - \sum_{i=1}^{5} b_i = D.(3l - l_1 - \ldots - l_5) \\ &\geq 0 \end{aligned}$$

Where the last inequality follows by the nefness of the curve $3l - l_1 - \ldots - l_5$, by Lemma 4.4. Hence $\sum n_i \geq c$.

Also, by intersecting both sides of the equation (4.3) with the divisor class $L - E_i$, we see that also

$$0 \leq n_i \leq a - b_i \leq a - b_5 = c.$$

Because of these inequalities, it is possible to decompose $\sum_{i=1}^{4} n_i E_i'$ as a sum $\sum_{i=1}^{4} n_i' E_i' + \sum_{i=1}^{4} n_i'' E_i'$, where $0 \leq n_i' \leq n_i$ such that $\sum_{i=1}^{4} n_i' = c$, and consider the section

$$u = \prod_{i=1}^{4} \hat{h}_i^{n_i} \cdot \prod_{i=1}^{4} x_i^{n_i''} \in H^0(X, D'').$$

where D'' is some divisor on X. By the construction, u is a section that will map to $\prod_i e_i^{n_i}$ via ϕ.

It remains to check that the 'lifted' divisor class D'' actually equals D' on X. Consider their difference $M = D'' - D'$. Note that the map α is surjective, so by comparing ranks of the Picard groups, we see that the kernel of α is generated by one element, namely $H - E_1 - \ldots - E_5$. Since both D'' and D' map to the same divisor class on Y, we must have that $\alpha(M) = 0$ and so M is of the form $m(H - E_1 - \ldots - E_5)$ for some $m \in \mathbf{Z}$. But in fact $m = l.M = c - c = 0$, so $M = 0$ and $D' = D''$. This shows that D' is effective, and that there exists a section u mapping to $\prod_i e_i^{n_i}$ via ϕ.

This completes the proof. □

Corollary 4.6. *The effective cone* $NE^1(X, \mathbf{Z})$ *is generated by* $E_1, \ldots, E_5$ *and* $H - E_i - E_j - E_k$, $1 \le i, j, k \le 5$ *distinct.*

4.2.1 Relations.

Consider the divisor class $D = H - E_1 - E_2$. In $\mathbf{P}^3$, this corresponds to the linear system of planes through the points p_1 and p_2. Using this observation, or using Riemann-Roch, we find that $\dim H^0(X, D) = 2$. Of course, there are 3 monomials of degree D, namely $h_{123}x_3, h_{124}x_4, h_{125}x_5$ corresponding to ways of writing D as a sum of effective divisor classes $(H - E_1 - E_2 - E_i) + E_i$. This shows that there is exactly one quadratic relation in I_D, in fact,

$$h_{123}x_3 - h_{124}x_4 + h_{125}x_5 = 0.$$

The same thing happens for all classes $H - E_i - E_j$, so we have 10 relations.

There is a similar argument for the divisor classes $2H - 2E_1 - E_2 - E_3 - E_4 - E_5$. This gives five relations. In all, we find a total of 15 quadrics in the quadratic part of I. Moreover, we recognize these as the Plücker quadrics of the Grassmannian $G(2, 6)$. We thus have a surjective map between the coordinate ring of $G(2, 6)$ and $\mathrm{Cox}(X)$. It is clear that this is the entire ideal since we have a surjection $k[G(2, 6)] \to \mathrm{Cox}(X)$ and their dimensions agree:

$$\dim R/J = \dim G(2, 6) + 1 = 9 = \dim \mathrm{Pic}(X) + \dim X = \dim \mathrm{Cox}(X).$$

This shows the following theorem:

Theorem 4.7. *The Cox ring of* X *is isomorphic to the coordinate ring of the Grassmannian variety* $G(2, 6)$.

Here it is understood that the isomorphism is taken with respect to the coarser grading given by letting all the x_i, h_{ijk} have degree 1.

The theorem shows that the blow-up of $\mathbf{P}^3$ in 5 general points realizes a GIT quotient of $G(2, 6)$ by the maximal torus in $SL(2)$. This is at least very intuitive: By the Gelfand-MacPherson correspondence [Kap93], GIT quotients of $G(2, 6)$ correspond to quotients $(\mathbf{P}^3)^6/PSL(4)$. Think of the moduli space of 6 points $p_1, \ldots, p_6$ in $\mathbf{P}^3$. Fix 5 of the points to get rid of the action of $PSL(4)$ - the remaining point p_6 moves freely and so the moduli space is some compactification of $\mathbf{P}^3 \setminus \{p_1, \ldots, p_5\}$.

4.3 Cox(X) as an invariant ring

In this section we calculate the Cox ring of the blow-up $\mathbf{P}^n$ in $n + 2$ points, using Mukai's correspondence. This is an interesting case, since blow-ups of $\mathbf{P}^m$ in $\le n + 1$

general points are toric so this represents some boundary case. We will see that the Cox ring is isomorphic to the Grassmannian $G(2, n+3)$, thus generalizing the previous theorem. Let us first recall Mukai's correspondence. Let $G \subset \mathbb{G}_a^m$ be the nullspace of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rm} \end{pmatrix}$$

Elements $(t_1, \dots, t_m) \in G$ act on $R = k[x_1, \dots, x_m, y_1, \dots, y_m]$ by the following *Nagata action*:

$$x_i \mapsto x_i, \qquad y_i \mapsto y_i + t_i x_i \tag{4.4}$$

We are interested in the polynomials invariant under this action:

Theorem 4.8 (Mukai). *The ring R^G coincides with the Cox ring of the blow-up of $\mathbf{P}^{r-1}$ in the points $p_1, \dots, p_m$ whose coordinates are the column vectors of A.*

We want to study the Cox ring of $\mathbf{P}^n$ blown up in $n+2$ general points from this point of view. This means that we consider the case $m = n+2$, $r = n+1$. In this case, since the points are in general position, G is one-dimensional and we may after a linear change of variables of R, take

$$G = \operatorname{Span}\{(1, 1, \dots, 1)\} \subset \mathbb{G}_a^{n+2}.$$

Theorem 4.9. *The Cox ring of $\mathbf{P}^n$ blown up in $n+2$ distinct general points is isomorphic to the coordinate ring of the Grassmannian $G(2, n+3)$.*

Proof. By Mukai's theorem we want to find all polynomials invariant under G, i.e all $f \in R$ such that for all $t \in k$,

$$f(x_1, \dots, x_{n+2}, y_1 + tx_1, \dots, y_{n+2} + tx_{n+2}) = f(x_1, \dots, x_{n+2}, y_1, \dots, y_{n+2}). \tag{4.5}$$

Of course the x_i are invariant under (4.4), as are the determinants

$$p_{ij} = x_i y_j - x_j y_i$$

We claim that the invariant ring is generated by these, i.e.

$$R^G = k[x_1, \dots, x_n, p_{12}, p_{13}, \dots, p_{(n+1)(n+2)}].$$

Using the Taylor formula, we see that a polynomial $f \in R^G$ is invariant if and only it lies in the intersection

$$k[x_1, \dots, x_n, y_1 + tx_1, \dots, y_n + tx_n] \cap k[x_1, \dots, x_n, y_1, \dots, y_n].$$

We view this intersection as a subalgebra of $R[t]$ with the monomial ordering: $t \succ x_1 \succ \cdots \succ x_n \succ y_1 \succ \cdots \succ y_n$.

Lemma 4.10. *Any leading monomial in $k[x_i, x_i t + y_i]$ is a product of the monomials*

$$x_i,\ x_i y_j,\ x_i t,\ 1 \leq i < j \leq n+2.$$

Proof. Proving this lemma is essentially is the same as showing that $\{x_i, tx_i + y_i, p_{ij}\}$ forms a sagbi basis for $k[x_i, x_it + y_i]$ with respect to the ordering above (cf. [RS90]). This follows from the sagbi basis algorithm [RS90], and the following 'straightening relations'

$$\begin{aligned} x_ip_{jk} - x_jp_{ik} &= -x_kp_{ij} & p_{ik}p_{jl} - p_{il}p_{jk} &= p_{ij}p_{kl} \\ q_ip_{jk} - q_jp_{ik} &= -q_kp_{ij} & x_iq_j - x_jq_i &= p_{ij} \end{aligned} \tag{4.6}$$

where $q_i = tx_i + y_i$. □

Note that two equations (4.6) are exactly the Plücker relations on the polynomials x_i, p_{ij}.

Let f be an arbitrary element in the intersection above. Note that its leading term cannot be divisible by t, and so the leading term is a product of x_i's and x_iy_j's for $i < j$. Suppose the leading term is

$$cx_1^{a_1} \cdots x_{n+2}^{a_{n+2}} \cdot (x_1y_2)^{b_{12}} \cdots (x_{n+1}y_{n+2})^{b_{(n+1)(n+2)}}$$

and consider the following polynomial:

$$g = f - cx_1^{a_1} \cdots x_{n+2}^{a_{n+2}} \cdot (p_{12})^{b_{12}} \cdots (p_{(n+1)(n+2)})^{b_{(n+1)(n+2)}}.$$

The polynomial g is clearly invariant under (4.5) and has leading term strictly smaller than that of f. Repeating the process with g we eventually reach a polynomial which is invariant under G, which has constant leading term, that is, a constant polynomial. This shows that we may write any invariant f as a polynomial in the x_i, p_{ij} and so $f \in k[x_1, \dots, x_n, p_{12}, p_{13}, \dots, p_{(n+1)(n+2)}]$. Hence

$$R^G = k[x_1, \dots, x_n, p_{12}, p_{13}, \dots, p_{(n+1)(n+2)}].$$

A dimension argument now completes the proof[1],

$$\begin{aligned} \dim \mathrm{Cox}(X) = \mathrm{rank}\ \mathrm{Pic}(X) + \dim X &= (n+3) + n \\ &= 2(n+2) + 1 = \dim k[G(2, n+3)]. \end{aligned}$$

□

[1] Alternatively, one could argue here directly using a sagbi basis argument, since by [RS90], the defining ideal is generated by the straightening relations (4.6).

Chapter 5
K3 Surfaces with $\rho = 2$

Let X be a K3 surface, i.e., a projective surface with $K_X = \mathscr{O}_X$ and $H^1(X, \mathscr{O}_X) = 0$. It is well-known that the generic K3 surface has $\operatorname{Pic} X = \mathbf{Z}$, and so the Cox ring is simply the coordinate ring $\bigoplus_{n\geq 0} H^0(X, nH)$. Generators and relations of this ring are investigated by Saint-Donat in [SD74]. We assume henceforth that $\rho = \operatorname{rank} \operatorname{Pic} X = 2$.

Cox rings of K3 surfaces were first studied by Abertani, Hausen and Laface in the recent paper [AHL09]. In this paper it is proved that K3 surfaces with finitely generated effective cone have finitely generated Cox ring. The authors make some attempt in finding some explicit generators, although they do not prove the sufficiency of these. The authors also discuss the problem of finding relations for a special class K3 surfaces using Laface and Velasco's complex.

In this chapter, we also address some of the problems investigated in [AHL09]. We we will prove finite generation of the Cox rings of K3 surfaces with $\rho = 2$. We will study in detail K3 surfaces arising as double covers of $\mathbf{P}^2$ and some quartic surfaces. Some of the results on finite generation coincide with results from [AHL09], although there does not seem to be much overlap between their methods and the following.

5.1 Complete linear systems and vanishing on K3 Surfaces

We recall some standard results on linear systems on K3 surfaces. Most of the results here are due to Saint-Donat [SD74].

Lemma 5.1. *[SD74, Corollary 3.2] Let D be an effective divisor on a K3 surface. Then $|D|$ has no base-points outside its fixed components.*

Lemma 5.2. *[SD74, Corollary 2.6] Let D be a nef divisor on a K3 surface. If $D^2 > 0$, then $|D|$ is base-point free, $h^1(X, D) = 0$ and the generic member of $|D|$ is smooth and irreducible. Furthermore, if $D^2 = 0$, then $|D|$ is composed with a pencil, i.e $D = kE$, where E is an elliptic pencil.*

The vanishing of $h^2(X, D)$ for D effective is immediate by duality: $h^2(X, D) = h^0(X, -D) = 0$.

Lemma 5.3. *[SD74, Proposition 5.2] Let D be a nef divisor such that $D^2 \geq 4$. Then D is hyperelliptic only if there exists an elliptic curve E with $D.E = 1$, or $D = 2B$ for some genus 2 curve B.*

We also have the following result about the ideal of the embedding given by D.

Proposition 5.4. *[SD74, Theorem 7.2] Let H be an effective divisor class, such that $H^2 \geq 8$ such that the general member of $|H|$ is smooth and non-hyperelliptic. Then the algebra $A = \bigoplus_{n\geq 0} H^0(X, nH)$ is generated in degree 1, and the kernel of the map $SymH^0(X, H) \to A$ is generated by elements of degree 2, except if there is a curve E such that $E^2 = 0$ and $E.L = 3$ in which case the ideal is generated in degrees 2 and 3.*

The following result due to Kovacs is a special case of Theorem 2 of [Kov94]. It gives information about the effective divisor classes in $\mathrm{Pic}(X)$.

Proposition 5.5. *[Kov94] Let X be a K3 surface with $\rho_X = 2$. The effective cone $NE^1(X, \mathbf{R})$ is generated by the classes of curves with self intersection -2 or 0.*

Note that if $NE^1(X, \mathbf{R}) = \mathbf{R}_{\geq 0}\Gamma_1 \oplus \mathbf{R}_{\geq 0}\Gamma_2$, then Γ_1, Γ_2 are linearly independent, and hence form a basis for $\mathrm{Pic}(X)$. In particular this implies that we need only consider the cases: $\mathrm{Pic}(X) = \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2$ where $\Gamma_i^2 \in \{-2, 0\}$.

5.2 K3 Surfaces with two smooth rational curves

In this section we consider the case where the Picard group of X is generated by two smooth rational curves, say, Γ_1 and Γ_2. By the adjunction formula we have $\Gamma_1^2 = \Gamma_2^2 = -2$. Let $d = \Gamma_1.\Gamma_2$ be their intersection number. Note that the Hodge Index Theorem implies that $\begin{vmatrix} \Gamma_1^2 & \Gamma_1\Gamma_2 \\ \Gamma_1\Gamma_2 & \Gamma_2^2 \end{vmatrix} = 4 - d^2 < 0$, so $d \geq 3$. Furthermore $d = 3$ is attainable (see section 5.3).

Lemma 5.6. *The effective monoid $NE^1(X, \mathbf{Z})$ is generated by Γ_1 and Γ_2.*

If $d = 2n$ the nef monoid is generated by the classes of $j\Gamma_1 + \Gamma_2$ and $\Gamma_1 + j\Gamma_2$ for $j = 1, \ldots, n$ and if $d = 2n + 1$, also the classes of $\Gamma_1 + d\Gamma_2$ and $d\Gamma_1 + \Gamma_2$.

Proof. Of course the cone $\tau = \mathbf{Z}_{\geq 0}\{\Gamma_1, \Gamma_2\} \subseteq NE^1(X, \mathbf{Z})$. Let τ^* be the dual cone of τ. Note that

$$(a\Gamma_1 + b\Gamma_2) \cdot \Gamma_1 \geq 0 \iff -2a + db \geq 0$$
$$(a\Gamma_1 + b\Gamma_2) \cdot \Gamma_2 \geq 0 \iff da - 2b \geq 0.$$

These inequalities imply that τ^* is generated over $\mathbf{R}$ by the classes $d\Gamma_1 + 2\Gamma_1$ and $2\Gamma_1 + d\Gamma_2$. Over $\mathbf{Z}$ this means that the dual monoid is generated by the classes listed in the lemma. Note in particular that all of these classes are effective, being positive integer combinations of Γ_1, Γ_2. Now, let $D \in \mathrm{Pic}(X)$ be the class of an effective curve. We can write

$$D = n\Gamma_1 + m\Gamma_2 + M$$

where M is an effective divisor with $M.\Gamma_i \geq 0$, i.e $M \in \tau^*$. Since all elements of τ^* are positive integer combinations of Γ_1, Γ_2. This shows that $D \in \tau$, as required. $\square$

Theorem 5.7. *The Cox ring of X is finitely generated, and a generating set of* $\mathrm{Cox}(X)$ *contains sections of degrees $\Gamma_1 + a\Gamma_2, a\Gamma_1 + \Gamma_2$ for $a = 0, \ldots, \lfloor \frac{d}{2} \rfloor$ and also $2\Gamma_1 + d\Gamma_2$ and $d\Gamma_1 + 2\Gamma_2$) if d is odd. In particular, such a set must contain at least $\frac{d(d-2)}{2} + 3$ elements if d is even, and $\frac{(d-1)^2}{2} + 4$ if d is odd. These bounds are sharp.*

Proof. We first show that any nef divisor is base-point free. By Lemma 5.2, the divisor D is base-point free if it is big, i.e $D^2 > 0$. But note that all of the generators of the nef cone are big, so the same must apply for any positive linear combination of them. This means that any nef divisor is base-point free, hence semi-ample, so the finite generation follows by arguing as in the proof of Theorem 2.12.

We now look for generators. Note that we need two generators s, t in degrees Γ_1, Γ_2 respectively. Consider the classes $D = a\Gamma_1 + \Gamma_2$. These are nef by the previous lemma. Since also all nef divisors are big, it follows from the Kawamata-Vieweg vanishing theorem and Riemann-Roch that

$$h^0(X, a\Gamma_1 + \Gamma_2) = a(d-a) + 1.$$

For $a = 1$, note that we need at least $d-1$ new generators in addition to $s \cdot t$ of degree $\Gamma_1 + \Gamma_2$, since $h^0(X, \Gamma_1 + \Gamma_2) = d \geq 3$. In fact, the multiplication map

$$H^0(X, (a-1)\Gamma_1 + \Gamma_2) \otimes H^0(X, \Gamma_1) \to H^0(X, a\Gamma_1 + \Gamma_2)$$

is never surjective, since $h^0(X, \Gamma_1) = 1$ and since $h^0(a\Gamma_1 + \Gamma_2) - h^0((a-1)\Gamma_1 + \Gamma_2) = d - 2a + 1 > 0$. This means that we need $d - 2a + 1$ new generators in the degrees listed above. Summing the differences gives the bound on the number of generators:

If $d = 2n$ is even:

$$\underbrace{1+1}_{\Gamma_1,\Gamma_2} + \underbrace{d-1}_{\Gamma_1+\Gamma_2} + 2\sum_{a=2}^{n} \underbrace{(d-2a+1)}_{a\Gamma_1+\Gamma_2} = \frac{d(d-1)}{2} + 3$$

If $d = 2n+1$ is odd, we need at least one generator in each of the degrees $d\Gamma_1 + 2\Gamma_2$ and $2\Gamma_1 + d\Gamma_2$, and so we need at least

$$\underbrace{1+1}_{\Gamma_1,\Gamma_2} + \underbrace{1+1}_{d\Gamma_1+2\Gamma_2, 2\Gamma_1+d\Gamma_2} + \underbrace{d-1}_{\Gamma_1+\Gamma_2} + 2\sum_{a=2}^{n} \underbrace{(d-2a+1)}_{a\Gamma_1+\Gamma_2} = \frac{(d-1)^2}{2} + 4$$

generators. The example for Section 5.4 shows that the bound above is sharp. □

Note that the theorem does not reveal anything about the sufficiency of these sections in generating the Cox ring, it merely states that $\mathrm{Cox}(X)$ needs minimal generators of the $\mathrm{Pic}(X)$-degrees generating $NE^1(X, \mathbf{Z})$ and $NM^1(X, \mathbf{Z})$. Note that this number of variables increases with d, so the rings become increasingly more complicated. Also, this indicates that the number of minimal generators grows (at least) quadratically with d. For example, for $d = 5$, we need at least 12 generators, and hence at least $12 - \dim \mathrm{Cox}\, X = 12 - 4 = 8$ relations. Another problem is that we haven't chosen explicit sections for the generators - all we know is their multidegrees. That means that we don't know anything about the relations in the ideal, except their multidegrees. Hence the computation of the ideal I_d for large d is not a very manageable problem.

In the next section we investigate the Cox ring of X, when $d = 3$, to give some idea of the complexity of the problem.

5.3 K3 surfaces arising as double covers of $\mathbf{P}^2$

We give an explicit description of a K3 surface with intersection matrix $\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$ and study its Cox ring.

Let $\pi : X \to \mathbf{P}^2$ be a double cover of $\mathbf{P}^2$ ramified over a sextic curve $C = \mathcal{V}(f)$. An example of such surface X can be realized by taking a subscheme of $\mathbf{P}(3,1,1,1)$, with defining equation

$$w^2 = f(x_0, x_1, x_2)$$

where $f(x_0, x_1, x_2)$ is the equation of the sextic curve. The surface X comes equipped with an ample divisor, $H = \pi^*(L)$ with self-intersection 2.

The generic K3 surface has $\rho_X = 1$, so we are dealing with a very special case. This is also reflected in the nature of the sextic curve: we assume that there is a line $L \subset \mathbf{P}^2$ which is tritangent to the sextic C. The restriction of f to L is the square of a section g in $\mathscr{O}_L(3)$, and we may write $f = g^2 + lP$ where $P \in \mathscr{O}_{\mathbf{P}^2}(5)$ and $L = V(l)$. The pullback of L is given by $\pi^*(L) = \Gamma_1 + \Gamma_2$, where Γ_1, Γ_2 corresponds to the curves given by $w = \pm g$. Γ_1 and Γ_2 are lines, since they are isomorphic to L. For a simple example one could take

$$f = x_2^5 x_0 - x_1^2(x_1^2 - x_2^2)^2.$$

Here $L = V(x_0)$, $\Gamma_1 = V(w - x_1(x_1^2 - x_2^2))$, $\Gamma_2 = V(w + x_1(x_1^2 - x_2^2))$.

As before, by the adjunction formula we have $\Gamma_i^2 = -2$. Also, since $l^2 = 1$ in $\mathbf{P}^2$, we have $(\Gamma_1 + \Gamma_2)^2 = (\deg \pi) l^2 = 2$, giving $d = \Gamma_1.\Gamma_2 = 3$. Let $\sigma : X \to X$ be the involution that switches the sheets of X over $\mathbf{P}^2$, i.e sends w to $-w$ above. This induces an automorphism $\sigma^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ such that $\sigma(\Gamma_1) = \Gamma_2$. This means that $\Gamma = \{\Gamma_1, \Gamma_2\}$ is a nice $\mathbf{Z}$-basis for $\mathrm{Pic}\, X$ and we will use this in the following.

By Lemma 5.6, we have $NE^1(X, \mathbf{R}) = \mathbf{R}_{\geq 0}\Gamma_1 + \mathbf{R}_{\geq 0}\Gamma_2$ and the nef monoid is generated by the divisor classes

$$H := \Gamma_1 + \Gamma_2, \qquad N_1 := 2\Gamma_1 + 3\Gamma_2, \qquad N_2 = 3\Gamma_1 + 2\Gamma_2.$$

The two cones are plotted in Figure 5.1.

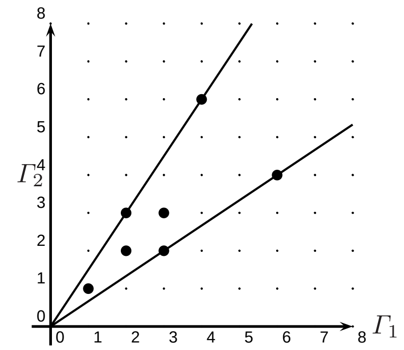


Fig. 5.1 $NM^1(X)$ as a subcone of $NE^1(X)$ with the 'special' divisor classes plotted.

Note that all the above classes are nef and big, which means that we have nice vanishing on X. By Kawamata-Vieweg and the Riemann-Roch formula, we have for $a\Gamma_1 + b\Gamma_2$ nef:

$$h^0(X, a\Gamma_1 + b\Gamma_2) = 3ab - a^2 - b^2 + 2.$$

Lemma 5.8. *Let s_R be a section defining the ramification divisor on X. Then*

$$H^0(X, \mathscr{O}_X(kH)) \cong \pi^* H^0(\mathbf{P}^2, \mathscr{O}(k)) \oplus \pi^* H^0(\mathbf{P}^2, \mathscr{O}(k-3)) s_R$$

Proof. This follows from [BPV84, I.17.2] and the projection formula. □

In particular, we may choose a basis of $H^0(X, H)$ and $H^0(X, 2H)$ consisting of pullbacks of sections from $\mathscr{O}_{\mathbf{P}^2}(1)$.

5.3.1 Generators

We now look for generators for $\mathrm{Cox}(X)$. Of course we need at least two sections corresponding to the (-2)-curves: We let x, y denote generators for $H^0(X, \Gamma_1)$ and $H^0(X, \Gamma_2)$, respectively.

In degree 2, we have the divisor class H, in which $H^0(X, H)$ is 3-dimensional by Riemann-Roch, hence we need two new generators for a basis. Call these z_1, z_2. Note that these are the pullback of sections from $H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(1))$ by Lemma 5.8, and in particular the sections in $H^0(X, H)$ are invariant under the involution σ.

The divisor class $D = 2H$ needs no new sections, since also these sections are pullbacks: $H^0(X, D) \cong H^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(2)) = \mathrm{Sym}^2(H^0(X, H))$. Consider now the divisor class $D = 2\Gamma_1 + 3\Gamma_3$. This has a 7-dimensional cohomology group, while we can only create 6 monomials in degree D with the generators created so far: these are all from the basis for $H^0(X, 2H)$ multiplied with y. Hence we need one new generator for a basis. Call this monomial v.

The same thing happens for the divisor class $D = 3\Gamma_1 + 2\Gamma_2$, and we need another section, say w. In fact we may choose $w = \sigma(v)$, since this cannot be linearly dependent on the previous monomials (since by applying σ, the same would apply to v).

In all we have shown that we need generators x, y, z_1, z_2, v, w in degrees $\Gamma_1, \Gamma_2, \Gamma_1 + \Gamma_2, 2\Gamma_1 + 3\Gamma_2, 3\Gamma_1 + 2\Gamma_2$ respectively, in accordance with Lemma 5.6. We now claim that these sections are sufficient to generate the Cox ring.

Proposition 5.9. $\mathrm{Cox}(X)$ *is generated by the sections x, y, z_1, z_2, v, w.*

Proof. Koszul cohomology and induction. We first look at some more "special" divisor classes D, where it is not so obvious that we do not need additional generators. These will also play the role of base cases for the induction.

$D = 3H$. Since $h^0(X, D) = 11$ and $\dim_k \mathrm{Sym}^3(H^0(X, H)) = 10$, we need one more section to produce a basis for $H^0(X, D)$. Consider the section $v \cdot x$. We claim that this cannot be linearly dependent on the previous monomials. This follows since these are in fact $\sigma-$invariant, while $\sigma(vx) = wy \neq vx$, since $\mathrm{Cox}(X)$ is an UFD. Hence these 11 monomials form a basis for $H^0(X, D)$.

$D = 4\Gamma_1 + 6\Gamma_2$ or $D = 6\Gamma_1 + 4\Gamma_2$. Note that $h^0(X, 4\Gamma_1 + 6\Gamma_2) = 22$ and that $4\Gamma_1 + 6\Gamma_2 = (4\Gamma_1 + 5\Gamma_2) + \Gamma_2$. Consider the divisor $D' = 4\Gamma_1 + 5\Gamma_2$. D' is nef and big since $D' = (2\Gamma_1 + 3\Gamma_2) + 2(\Gamma_1 + \Gamma_2)$ and has $h^0(X, D') = 21$. Hence by multiplying a base of $H^0(X, D')$ by y we get 21 linearly independent sections in $H^0(X, D)$. Now we add the section v^2, which cannot be a linear combination of the other monomials since

these are all divisible by y. By switching the roles of Γ_1 and Γ_2 we also prove it for the divisor $6\Gamma_1 + 4\Gamma_2$.

We now proceed with the induction. Let $D = a\Gamma_1 + b\Gamma_2$ be an effective divisor class. As before we may assume D to be nef (and hence big).

Case 1: $a = b = n$, $n \geq 4$. Here $D = nH$. Note that the arithmetic genus $p_a(H) = \frac{1}{2}(H^2 + 2) = 2$ so the general member $C \in |H|$ is a hyperelliptic curve. We now recall a theorem from classical curve theory:

Lemma 5.10 (Noether's theorem). *Let C be a smooth curve of genus g and let $R_C = \bigoplus_{n\geq 0} H^0(C, nK_C)$ be its canonical ring.*

1. *If C is not hyperelliptic, the R_C is generated in degree 1.*
2. *If $g = 2$, and C is hyperelliptic, then R_C is generated by elements of degree 1, and by 1 element of degree 3.*
3. *If $g \geq 3$, and C is hyperelliptic, then R_C is generated by elements of degree 1, and by $g - 2$ elements of degree 2.*

Let C be a curve in $|H|$. Note that we have the exact sequence

$$0 \to H^0(X, (n-1)H) \to H^0(X, nH) \to H^0(C, nH|_C) \to 0.$$

Here the right-exactness follows from Kodaira Vanishing and ampleness of H. Also, $H|_C = K_X(H)|_C = K_C$ by adjunction, so $H^0(C, nH|_C) = H^0(C, nK|_C)$. Now, since $H^0(X, 3H) \to H^0(C, 3K|_C)$ is surjective, we choose a set of sections from $H^0(X, 3H)$ mapping isomorphically to a basis for $H^0(C, 3K|_C)$. By the lemma the elements of $H^0(C, nK|_H)$ are polynomials in sections from $H^0(C, 3K|_H)$. As vector spaces, we have a splitting

$$H^0(X, nH) \cong H^0(C, nK_C) \oplus H^0(X, (n-1)H).$$

Hence if $n \geq 4$, it follows that sections in $H^0(X, nH)$ are polynomials in sections of lower degree.

Case 2: $a > b$. We now need a lemma where we apply methods from Koszul cohomology:

Lemma 5.11. *Let $H = \Gamma_1 + \Gamma_2$, and suppose D is an effective divisor class such that i) $H^1(D - 2H) = H^1(D - H) = 0$, ii) $D - 3H$ is effective. Then the multiplication map*

$$H^0(X, H) \otimes H^0(X, D - H) \to H^0(X, D)$$

is surjective.

Proof. Let $\mathscr{K}_{0,0}(X, D, H)$ denote the homology of the following complex

$$\bigwedge^1 H^0(X, H) \otimes H^0(X, D - H) \to \bigwedge^0 H^0(X, H) \otimes H^0(X, D) \to 0$$

Proving the lemma is equivalent to showing that $\mathscr{K}_{0,0}(X, D, H) = 0$. Now, the assumption i) and the base-point freeness of $|H|$ ensures us that we are in position to apply Green's Duality theorem of [MG84], which states that in these circumstances,

$$\mathscr{K}_{0,0}(X,D,H) \cong \mathscr{K}_{r-n,n+1}(X,K_X-D,H)^*$$

where $r = h^0(X,H)-1 = 3$ and $n = \dim X = 2$. In this case, of course $\mathscr{K}_{r-n,n+1}(X,K_X-D,H) \cong \mathscr{K}_{0,3}(X,-D,H)$ is the homology of the complex

$$\bigwedge^1 H^0(X,H)\otimes H^0(X,-D+2H) \to \bigwedge^0 H^0(X,H)\otimes H^0(X,-D+3H) \to 0.$$

But by assumption $D-3H$ is effective, hence $H^0(X,-D+3H)=0$, and the homology of the complex is zero. This proves the lemma. □

Note that all nef divisors except the "special divisors" satisfy that $D-3H$ is effective (see Figure 5.1).

Now for the induction part. Write for simplicity $N = 3\Gamma_1 + 2\Gamma_2$, and note that in this case (where $a > b$), D can be written uniquely in the from

$$D = mN + nH \quad m \geq 1, n \geq 0$$

We use induction on n. If $n = 0$, we choose an irreducible curve $C \in |N|$. Since C has genus 6, it follows from Lemma 5.10 that the algebra $\bigoplus_{r\geq 0} H^0(C, nK_C)$ is generated in degrees ≤ 2. We proceed as before and use the exact sequence

$$0 \to H^0(X,(n-1)N) \to H^0(X,nN) \to H^0(C,nN|_C) \to 0..$$

to conclude that sections in $H^0(X,nN)$ are polynomials in sections of lower degree, for all $n \geq 3$. Hence the result follows by induction.

If $n = 1$, then $D = mN+H$ where $H = \Gamma_1+\Gamma_2$. We check the assumptions of Lemma 5.11 to ensure that we have a surjection $H^0(X,H)\otimes H^0(X,D-H) \to H^0(X,D)$, then the result will follow by induction on the degree.

First, $H^1(X,D-H) = H^1(X,mN) = 0$ by nef and bigness of N. Now, $D-2H$ is not nef (it has Γ_1 as a fixed component), but we will verify that $H^1(X,D-2H) = 0$. Note that $D-2H = mN-(\Gamma_1+\Gamma_2) = (m-1)N+2\Gamma_1+\Gamma_2$, and that the long exact sequence of cohomology applied to the sequence

$$0 \to \mathscr{O}_X(D-2H-\Gamma_1) \to \mathscr{O}_X(D-2H-\Gamma_1) \to \mathscr{O}_{\Gamma_1}(D-2H-\Gamma_1) \to 0$$

gives $H^1(X,D-2H)=0$ by exactness, since $\deg((m-1)N+2\Gamma_1+\Gamma_2|_{\Gamma_1}) = -1$, and so $H^1(\Gamma_1,(m-1)N+2\Gamma_1+\Gamma_2) = H^1(\mathbf{P}^1,\mathscr{O}(-1)) = 0$ and since $D-2H-\Gamma_1 = (m-1)N+\Gamma_1+\Gamma_2 = (m-1)N+H$ is nef.

If $n \geq 2$, then both $D-H$ and $D-2H$ are nef, so the criteria are satisfied.

Case 3: $a < b$. The argument is completely analogous to that of Case 2, by switching the roles of Γ_1 and Γ_2. □

Remark. This approach can with little modification be used in tackling K3 surfaces with higher d .

5.3.2 Relations

By using the Reynolds operator and Noether's theorem, we find that the polynomials invariant under σ are exactly the polynomials in z_1, z_2, xy, $vx + wy$ and vw, i.e

$$k[x, y, z_1, z_2, v, w]^{\langle\sigma\rangle} = k[z_1, z_2, xy, vx + wy, vw].$$

Consider the the expression $xv + yw$. Since it is invariant under σ, we can write it as a polynomials in pullbacks of sections from $H^0(\mathbf{P}^2, \mathscr{O}(3))$, that is, in terms of z_1, z_2, xy, and hence we have a relation of the form

$$g := xv + yw - \beta(xy, z_1, z_2) = 0.$$

A quick check reveals that there are exactly 12 monomials in $k[x, y, z_1, z_2, v]$ of degree $D = 3H$: 10 of these come from $\mathrm{Sym}^3(H^0(X, H))$, and we have in addition the sections xv, yw. Since $h^0(X, D) = 11$, this shows g is the only relation in degree D and so $I_D = (g)_D$.

Similarly, note that vw is an σ-invariant section of degree $5H$ in $\mathrm{Cox}(X)$, and thus can be written as a linear combination of pullbacks of sections from $H^0(\mathbf{P}^2, \mathscr{O}(5))$. That means we have a relation of the form

$$f := vw - \alpha(xy, z_1, z_2) = 0$$

where α_5 is a degree 5 polynomial.

Now, there are 34 monomials in degree $D = 5H$, and $h^0(X, 5H) = 27$. Note that since there are 6 monomials in $k[x, y, z_1, z_2, v]_{2H}$, we must have $\dim_k(g)_{5H} = 6$ (since g has degree $3H$), and hence there should be exactly $34 - 27 - 6 = 1$ new relation of degree $5H$, namely f. This shows that $I_{5H} = (f, g)_{5H}$:

We denote the ideal generated by f and g by J. Since J has codimension 2, it is reasonable to expect that $\mathrm{Cox}(X) \cong k[x, y, z_1, z_2, v]/J$.

Lemma 5.12. *The elimination ideal $k[x, y, z_1, z_2, v] \cap J = (h)$ where $h = yf - vg$ is the resultant of f and g with respect to the variable w. Mutatis mutandis for the ideal $k[x, y, z_1, z_2, w] \cap J$.*

Proof. Write $R = k[x, y, z_1, z_2, v]$. Note that $h \in R \cap J$, while it is not so clear that it is a generator for the elimination ideal. However, let $P = pf - qg$ be an arbitrary element in $R \cap J$, where $p = \sum_{k=0}^n a_k w^k$, $q = \sum_{k=0}^n b_k w^k$ are considered as elements in $R[w]$.

Claim: We may assume $n = 0$. Suppose $n > 0$. Since the terms in P involving w^n must cancel we must have $a_n v = b_n y$, and consequently there is an $r \in R$ such that $a_n = yr$ and $b_n = vr$. Hence

$$P = pf - qg = (p - rgw^{n-1})f - (q - rfw^{n-1})$$

Now $p - rgw^{n-1} = (a_{n-1} - xvr + \beta r)w^{n-1} + \ldots$, and $q - rfw^{n-1} = (b_{n-1} - r\alpha)w^{n-1} + \ldots$ are polynomials in w of degrees $< n$, so by iterating this process, we eliminate successive powers of n.

For $n = 0$, the problem is trivial, since

$$P = pf - qg = w(pv - qy) + \text{"terms not containing } w\text{"}$$

Since $pv - qy$ must vanish, there is an $r \in R$ such that $p = yr, q = vr$, hence

$$P = yr \cdot f - vr \cdot g = r(yf - vg) \in (h).$$

□

Theorem 5.13. *Let X be a degree 2 K3 surface with Picard number 2. Then the Cox ring* $\mathrm{Cox}(X)$ *is isomorphic to a quotient of* $k[x, y, z_1, z_2, v, w]$ *by* $J = (f, g)$*. That is,*

$$\mathrm{Cox}(X) \cong k[x, y, z_1, z_2, v, w]/J$$

Proof. Combinatorics galore. Let $D = a\Gamma_1 + b\Gamma_2$ be a nef divisor class. Then we must show that

$$\dim_k \left(k[x, y, z_1, z_2, v, w]/J\right)_D = \dim_k \mathrm{Cox}(X)_D = 3ab - a^2 - b^2 + 2.$$

Let $x^{i_1} y^{i_2} z_1^{i_3} z_2^{i_4} v^{i_5} w^{i_6}$ a monomial of $k[x, y, z_1, z_2, v, w]$. Note that using the relations f and g we may modulo J remove all terms containing vw and yw, and hence we may decompose $k[x, y, z_1, z_2, v, w]/J$ as a vector space

$$k[x, y, z_1, z_2, v, w]/J \cong \bigoplus_{n>0} k[x, z_1, z_2]w^n \oplus k[x, y, z_1, z_2, v]/(h)$$

where $h = yf - vg$ is the generator for the elimination ideal $J' = k[x, y, z_1, z_2, v] \cap J$. Our job is now to calculate the dimensions of these two vector spaces in degree D separately. We may assume for the moment that $a \geq b$. The case where $a \leq b$ is completely analogous, and is obtained by switching the roles of v and w above.

$\dim_k \left(\bigoplus_{n>0} k[x, z_1, z_2]w^n\right)_D$. Note that we are looking for the number of monomials m in $k[x, z_1, z_2]$ such that $\deg mw^k = D$ for some $k \in \mathbf{N}$. By looking at these monomials' degrees, we find that this problem is equivalent to the following counting problem: Find the number of non-negative integer solutions to the system

$$\begin{aligned} a_1 + a_2 + a_3 + 3a_4 &= a \\ a_2 + a_3 + 2a_4 &= b \end{aligned} \tag{5.1}$$

Write this as

$$\begin{aligned} a_1 + a_2 + a_3 &= a - 3a_4 \\ a_2 + a_3 &= b - 2a_4 \end{aligned} \tag{5.2}$$

and note that given a solution to the 2nd equation uniquely determines a_1 as $a_1 = a - b - a_4$. Of course, $a - b \geq 0$ by assumption, so its clear that we must restrict ourselves to values of a_4 in the range $0 \leq a_4 \leq a - b$ to ensure non-negativity of a_1. Note that in this case we have

$$b - 2a_4 \geq b - 2(a - b) = 3b - 2a = D.\Gamma_1 \geq 0$$

where the last inequality is precisely ensured by the nef condition on D (!). Now we find the number of solutions to (5.2) by counting: for every $0 \leq a_4 \leq a - b$, we seek the number of ways of writing $b - 2a_4$ as a sum of two non-negative integers a_2, a_3, which is $b - 2a_4 + 1$, hence the total number of solutions to the system is given by

$$\sum_{i=1}^{a-b}(b-2i+1) = b(a-b) - (a-b+1)(a-b) + (a-b) \\ = 3ab - 2b^2 - a^2.$$

$\dim_k \left(k[y, z_1, z_2, v]/(h)\right)_D$. Write $S = k[y, z_1, z_2, v]$ and let $\chi(a, b)$ be the number of monomials in $S_{a\Gamma_1 + b\Gamma_2}$. Note that h has degree $5\Gamma_1 + 6f_2$. By the exact sequence

$$0 \to S(-5\Gamma_1 - 6\Gamma_2) \to S \to S/(h) \to 0$$

we get $\dim_k (S/(h))_D = \chi(a, b) - \chi(a-5, b-6)$. As before the dimension count reduces to the combinatorial problem of finding the number of solutions $\chi(a, b)$, to

$$\begin{aligned} a_1 + a_3 + a_4 + 2a_5 &= a \\ a_2 + a_3 + a_4 + 3a_5 &= b \end{aligned} \tag{5.3}$$

and our goal is to get an expression for $\chi(a, b) - \chi(a-5, b-6)$. Note that any solution to the last equation gives a_1 uniquely determined as $a_1 = a - b + a_2 + a_5$, hence as long as $a \geq b$, we need only find the number of solutions to the 2nd equation. Of course, the number of non-negative integer solutions to $a_2 + a_3 + a_4 + 3a_5 = b$ appears as the coefficient of x^b in the expression $(1 + x + x^2 + \ldots)^3 \cdot (1 + x^3 + x^6 + \ldots) = \frac{1}{(1-x)^3(1-x^3)}$. Hence $\chi(a, b) - \chi(a-5, b-6)$ is equal to the coefficient of x^b in the following expression:

$$\begin{aligned} \frac{1}{(1-x)^3(1-x^3)} - \frac{x^6}{(1-x)^3(1-x^3)} &= \frac{1-x^6}{(1-x)^3(1-x^3)} \\ &= \frac{1+x^3}{(1-x)^3} \\ &= 1 + \sum_{n=1}^{\infty}\left(\binom{n+2}{2} + \binom{n-1}{2}\right)x^n \\ &= 1 + \sum_{n=1}^{\infty}(n^2+2)x^n \end{aligned}$$

This shows that $\chi(a, b) - \chi(a-5, b-6) = b^2 + 2$

In all we have that

$$\begin{aligned} \dim_k \left(k[x, y, z_1, z_2, v, w]/J\right)_D &= (3ab - 2b^2 - a^2) + (b^2 + 2) \\ &= 3ab - a^2 - b^2 + 2 = h^0(X, a\Gamma_1 + b\Gamma_2). \end{aligned}$$

This finishes the proof. □

Cox(X) is always singular when X is such a special rank 2 K3 surface. For example, by looking at the jacobian of f, g we find that the singular locus of Cox(X) contains the codimension 1 subvariety $Z(x = z_1 = z_2 = w = 0)$. This differs of course from the generic K3 surface, since in this case the Cox ring is isomorphic to the (smooth) coordinate ring of X.

5.4 K3 surfaces with a rational curve and a elliptic curve

In this section we consider the case where $N^1(X)$ is generated by the classes of two curves, say, Γ_1 and Γ_2, where $\Gamma_1^2 = -2$ and $\Gamma_2^2 = 0$. Let $d = \Gamma_1.\Gamma_2$ be the number of points they intersect taken with multiplicity. The intersection matrix is then given by

$$\begin{pmatrix} -2 & d \\ d & 0 \end{pmatrix}$$

Lemma 5.14. *The effective cone $NE^1(X)$ is generated by Γ_1 and Γ_2. Also, these generate the monoid of effective divisor classes $NE^1(X, \mathbf{Z})$.*

If $d = 2n$ the nef monoid is generated by the classes $a\Gamma_1 + \Gamma_2$ for $a = 1, \dots, n$ and if $d = 2n + 1$ we need also the divisor class $d\Gamma_1 + 2\Gamma_2$.

Theorem 5.15. *The Cox ring of X is finitely generated, and any generating set of sections contains at least $\frac{d^2}{4} + 3$ elements if d is even and $\frac{d^2-1}{4} + 4$ elements if d is odd.*

The proofs of these results are similar to those of Lemma 5.6 and 5.7. We end this section with an example calculation.

5.4.1 A Quartic Surface with a line

In this section we investigate the Cox ring of a certain smooth quartic surface X in $\mathbf{P}^3$, which is a classical example of a K3 surface. This surface was studied thoroughly in [GM00], where the authors refer to it as the *Mori quartic.*

The surface X contains a line Γ_1 and a very ample divisor H, such that $H^2 = 4$. The divisor class $H - \Gamma_1$ is effective and its linear system contains an irreducible elliptic curve Γ_2. The intersection matrix here given by $\begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix}$. By the above theorem, the effective cone is generated by Γ_1, Γ_2 and $\mathrm{Cox}(X)$ needs generators in degrees $\Gamma_1, \Gamma_2, \Gamma_1 + \Gamma_2, 3\Gamma_1 + 2\Gamma_2$. It turns out that this example is similar to the double cover example: there are two minimal relations in degree $3H$.

Theorem 5.16. *Let X be a quartic K3 surface with a line. Then the Cox ring of X is isomorphic to*

$$k[l, s_1, s_2, t_1, t_2, u]/I$$

where $\deg(l) = \Gamma_1, \deg s_i = \Gamma_2, \deg t_i = \Gamma_1 + \Gamma_2, \deg u = \Gamma_1 + 2\Gamma_2$. *The ideal is generated by two relations of degree* $3\Gamma_1 + 3\Gamma_2$.

Proof. Since the method of proof is similar to that of the K3 surface in Section 5.3, we provide only a sketch of the proof. As before we find generators of $\mathrm{Cox}(X)$ by looking in low degree nef classes. Note that Γ_2 is an extremal ray in both the nef cone and the effective cone. We need a section $l \in H^0(X, \Gamma_1)$, and two basis elements s_1, s_2 from $H^0(X, \Gamma_2)$. Also we find that we need two sections t_1, t_2 from $H^0(X, H)$ and one additional element u from $H^0(X, \Gamma_1 + 2\Gamma_2)$, giving the generators above.

The rest is a direct checking using the fact that Γ_2 moves in a pencil and the base-point free pencil trick, and Koszul cohomology for the remaining divisors.

The relations arise by noting there are exactly 22 monomials of degree $3H$: 20 forming a basis for $\mathrm{Sym}^3(H^0(X,H))$ plus the monomials us_1, us_2. Hence we have two relations of the form

$$us_i = f_i(l, s_1, s_2, t_1, t_2)$$

The argument to show that these relations generate the ideal is done by a (slightly shorter) combinatorial argument as in Theorem 5.13. □

5.5 K3 Surfaces with two elliptic curves

Consider the case where the Picard group $\mathrm{Pic}(X)$ is generated by classes of elliptic curves Γ_1 and Γ_2. By the adjunction formula we have $\Gamma^2 = 0$. Let d be the number of intersection points taken with multiplicity. This means that X has the following intersection matrix:

$$\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$$

The divisor classes of Γ_1, Γ_2 form a $\mathbf{Z}$-basis for $NE^1(X)$ and since $\Gamma_i.\Gamma_j \geq 0$ for $1 \leq i, j \leq 2$, this shows that also the effective cone is generated by these curves and equals the nef cone in this basis. This means that every effective divisor $a\Gamma_1 + b\Gamma_2$ is nef for $a, b \geq 0$, and ample as long as $a, b \geq 1$, by the Nakai-Moishezon criterion. In this case the Riemann-Roch theorem gives the following formula:

$$h^0(X, a\Gamma_1 + b\Gamma_2) = \frac{1}{2}(a\Gamma_1 + b\Gamma_2)^2 + 2 = abd + 2.$$

In particular, this implies that the linear systems $|\Gamma_i|$ are pencils.

Write for simplicity $H = \Gamma_1 + \Gamma_2$. Note that H is an ample divisor on X, and that $H^2 = 2d$.

Note that since Γ_1 moves in a pencil, we need two generators x_1, x_2 for $H^0(X, \Gamma_1)$, and similarly two generators y_1, y_2 for $H^0(X, \Gamma_2)$.

Proposition 5.17. *Let X_d be a K3 surface with intersection matrix* $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$.

- *If $d = 2$, $\mathrm{Cox}(X)$ is generated by x_1, x_2, y_1, y_2, z where $\deg z = 2H$.*
- *If $d \geq 3$, $\mathrm{Cox}(X)$ is generated by $x_1, x_2, y_1, y_2, z_1, \ldots, z_{d-2}$, where $\deg z_i = H$.*

Proof. We have that $h^0(X, H) = d + 2$, so we need $d - 2$ new generators in degree H. Let $D = a\Gamma_1 + b\Gamma_2$ be an effective (hence nef) divisor class. We may suppose $a \geq b$. Note that -conveniently- the Γ_i are base-point free pencils, so the base-point free pencil trick gives us a surjection

$$H^0(X, D - \Gamma_1) \otimes H^0(X, \Gamma_1) \to H^0(X, D)$$

provided that $H^1(D - 2\Gamma_1) = 0$, which is the case for all divisors $D = a\Gamma_1 + b\Gamma_2$ with $a > 2$ or $(a, b) = (2, 1), (1, 2)$. The remaining divisor classes give the sections above. It follows that we reduce to checking degree $2H$. We apply the trick from before by using Noether's theorem

$$0 \to H^0(X,H) \to H^0(X,2H) \to H^0(C,2K|_C) \to 0$$

This follows by Lemma 5.10 since H is non-hyperelliptic for $d \geq 3$ (by Lemma 5.3) and hyperelliptic for $d = 2$, since $p_a(H) = d$ by the genus formula. □

Remark. For $d = 2$, X can be realized as a double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ ramified over a curve of bidegree $(4, 4)$.

5.5.1 Relations

It is remarkable the we are able to describe the Cox ring in this case. This is much owed to the facts that Γ_1, Γ_2 are pencils, and the Koszul sequence from the proof of the base-point free pencil trick, that is,

$$0 \to H^0(X, D - 2\Gamma_1) \xrightarrow{a} H^0(X, D - \Gamma_1) \otimes H^0(X, \Gamma_1) \xrightarrow{b} H^0(X, D) \to 0.$$

The maps here are as follows: $a(s) = sx_1 \otimes x_2 - sx_2 \otimes x_1$ and $b(t \otimes x_i) = tx_i$ is the contraction. The main observation is that all monomials of degree $n\Gamma_1 + n\Gamma_2$ are divisible by x_1 or x_2 except the ones that are products of z_i's. This easy observation will be sufficient in proving that the ideal of relations is generated in degree $(2, 2)$.

Theorem 5.18. *Let X be a a K3 surface with intersection matrix $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ and let $H = \Gamma_1 + \Gamma_2$. If $d = 2$, the Cox ring is isomorphic to a quotient $k[x_1, x_2, y_1, y_2, z]/(z^2 - F)$, where F is a polynomial of degree $4H$.*

If $d \geq 3$, the Cox ring is a quotient

$$k[x_1, x_2, y_1, y_2, z_1, \ldots, z_{d-2}]/I_d,$$

where the ideal is generated by $\binom{d-1}{2} - 1$ relations of degree $2H$.

Proof. Suppose first that $d \geq 3$. First we claim that there are $\binom{d-1}{2} - 1$ quadrics in I_{2H}. Riemann-Roch gives $h^0(X, 2H) = 4d+2$. Now the monomials $z_i z_j$ give $\binom{d-1}{2}$ monomials in degree $2H$, and we need $4 \cdot (d-2)$ monomials of the form $x_i y_j z_j$ and $x_i x_j y_k y_l$ give 9 monomials. In all

$$\dim_k I_{2H} = \binom{d-1}{2} + 4 \cdot (d-2) + 9 - (4d+2) = \binom{d-1}{2} - 1.$$

Note that any relation $f \in I_d$ in degree $2H$ must involve some $z_i z_j$ terms, since otherwise we may write $f = x_1 P + x_Q x_2 = 0$, and by the UFD property we have that x_1 divides Q and x_2 divides P. f is then a product of $x_1 x_2$ and terms of degree $2\Gamma_2$. But there are no relations in $H^0(X, 2\Gamma_2) = \langle y_1^2, y_1 y_2, y_2^2 \rangle$. Note that the number of monomials $z_i z_j$ is exactly one more than the number of relations. By Gaussian elimination, it follows we have minimal relations of the form

$$z_i z_j = P_{ij} x_1 + x_2 Q_{ij} + c_{ij} z_m z_n \qquad P_{ij}, Q_{ij}, c_{ij} \in k[x_i, y_i, z_i], \text{ for all } i \neq j \tag{5.4}$$

for some fixed $1 \le m, n \le d-2$. Denote their ideal by J.

Note that when $d = 3$, we have a single relation of the form $z^2 = Px_1 + Qx_2$ where $P, Q \in k[x_1, x_2, y_1, y_2, z]$.

Suppose $D = a\Gamma_1 + b\Gamma_2, a \ge b$ is the class of an effective divisor. Let $A = R/J$ and consider the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & \ker p & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\psi & \longrightarrow & A_{D-\Gamma_1} \otimes A_{\Gamma_1} & \xrightarrow{\psi} & A_D & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow p & & \\
0 & \longrightarrow & H^0(X, D-2\Gamma_1) & \longrightarrow & H^0(X, D-\Gamma_1) \otimes H^0(X, \Gamma_1) & \longrightarrow & H^0(X, D) & \longrightarrow & 0
\end{array}$$

where the middle vertical map is an isomorphism by induction on the degree $H.D \ge 0$, and the bottom sequence is exact by the base-point free pencil trick. We claim that the middle sequence is also exact, i.e

Claim: The map $\psi : A_{D-\Gamma_1} \otimes A_{\Gamma_1} \to A_D$ is surjective.

To see why this implies the result, note that $A_{D-2\Gamma_1} \subseteq \ker\psi$ maps surjectively to $H^0(X, D-2\Gamma_1)$. By the snake lemma and exactness we have that $\ker p = 0$, and so $A_D \cong H^0(X, D)$.

Proof of Claim: We show that we may modulo the relations (5.4) write any monomial as a sum of terms divisible by either x_1 or x_2. For $d = 3$, this is immediate since we may use the relation $z^2 - F$ to reduce the monomial $x_1^{i_1} x_1^{i_2} y_1^{j_1} y_1^{j_2} z^n$ to a linear combination of terms with lower exponents in z, and by the multigrading these terms must be divisible by either x_1 or x_2. For $d \ge 4$, the same argument and the equations (5.4) are *almost* enough to ensure the surjection. We need more information about the relations. We first use Proposition 5.4 to conclude the ideal of a K3 surface is generated by quadrics if $d \ge 4$. In particular, there are no minimal relations in degree $3H$, and $A_{3H} = H^0(X, 3H)$. Consider then the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{3H-2\Gamma_1} & \longrightarrow & A_{3H-\Gamma_1} \otimes A_{\Gamma_1} & \xrightarrow{\psi} & A_{3H} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & H^0(X, 3H-2\Gamma_1) & \longrightarrow & H^0(X, 3H-\Gamma_1) \otimes H^0(X, \Gamma_1) & \longrightarrow & H^0(X, 3H) & \longrightarrow & 0
\end{array}$$

Since the bottom right map is surjective and $z_i z_j z_k \in A_{3H}$ we have that $z_i z_j z_k \in x_1 R_{3H-\Gamma_1} + x_2 R_{3H-\Gamma_1}$ modulo J for all $1 \le i, j, k \le d-2$. Note that this gives $\binom{d}{3}$ relations, one for each monomial $z_i z_j z_k$

Now the surjection $A_{D-\Gamma_1} \oplus A_{D-\Gamma_1} \to A_D$ is clear. Indeed, if $D \ne nH$, then any monomial of degree D must be divisible by x_1 or x_2 (since the z_i all have degree H) and the map is surjective. Now, if $D = nH$, and $n \ge 3$, then a monomial $z_1^{n_1} \cdots z_{d-2}^{n_{d-2}}$ may, by chopping off three z_i's in an arbitrary manner, be written as a linear combinations of terms divisible by x_1 or x_2 modulo the relations above. This proves the theorem for $d \ge 3$.

If $d = 2$, By Riemann-Roch, $h^0(X, 4H) = 34$, while there are 35 monomials of degree $4H$: z^2 and 34 monomials from $\mathrm{Sym}^2 H^0(X, 2H)$. This means that we have a relation in degree $4H$ $z^2 - F$ where $F \in R$ is a polynomial of degree $4H$. Notice that all the terms of the polynomial F must have x_i's in them (z^2 is the only term of degree $4H$ without x_1 or x_2). This means that $F = x_1 f + x_2 g$ and we may use the relation $z^2 - F$ and the argument above to get a surjection $A_{D-\Gamma_1} \otimes A_{\Gamma_1} \to A_D$. □

Note that some of the cases above may be tackled relatively easily by combinatorics. For example, if $d = 2$, we find that

$$\begin{aligned}\dim_k A_{a\Gamma_1+b\Gamma_2} &= \dim_k k[x_1, x_2, y_1, y_2, z]/z^2 \\ &= \dim k[x_1, x_2, y_1, y_2]_{a\Gamma_1+b\Gamma_2} \oplus k[x_1, x_2, y_1, y_2]_{(a-2)\Gamma_1+(b-2)\Gamma_2} z.\end{aligned}$$

Hence the dimension in degree $a\Gamma_1 + b\Gamma_2$ is $(a+1)(b+1) + (a-1)(b-1) = 2ab + 2$, which is exactly what Riemann Roch gives for $H^0(X, a\Gamma_1 + b\Gamma_2)$.

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