DEFORMATION OF VECTOR BUNDLES WITH HOLOMORPHIC CONNECTIONS

An adventure in algebraic deformation

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Introduction

There is a connection between representations of the fundamental group $\pi_1(X)$ of a compact Riemann surface X, and holomorphic vector bundles on X. More specifically, for every representation ρ, there exists an associated holomorphic vector bundle V_ρ. The relation was explored in 1938 by André Weil and later M. S. Narasimhan and C. S. Seshadri in 1965. In particular, any vector bundle that comes from a representation has a *holomorphic connection*, and any bundle with such a connection comes from a representation.

An interesting problem is describing the interactions between this relation and deformation theory: How do the vector bundles associated to deformations of ρ relate to deformations of the associated bundle V_ρ? Providing a complete answer to this question is difficult, but we *can* give a complete description of the deformations of holomorphic vector bundles with holomorphic connections.

The first sections set the scene, giving the necessary definitions and details of vector bundles and the connection to representation theory. Sections 3 and 4 provide some tools of homological algebra and introduce the machinery of deformation theory. In section 5 we construct a double complex, and in section 6 we prove the final result: Deformations of a holomorphic vector bundle V equipped with a holomorphic connection ∇ is given by the first cohomology group $\mathscr{H}^1$ of this double complex, with obstructions in $\mathscr{H}^2$.

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1 Holomorphic vector bundles on compact Riemann surfaces

Definition 1.1. Let X be a compact Riemann surface (or, equivalently, a smooth projective curve over $\mathbb{C}$). A *holomorphic vector bundle on* X of rank n is a pair (V, π), where V is a complex manifold and π is a surjective holomorphic map $V \to X$ such that

(i) For each $x \in X$, $\pi^{-1}(x)$ is a complex vector space of dimension n, and

(ii) there exists an open covering $\{U_i\}_{i \in I}$ of X such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{C}^n \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle pr_1} & \\ U_i & & \end{array}$$

where ϕ_i is a homeomorphism, linear on each fibre of π. The ϕ_i's are called *local trivialisations.*

A vector bundle of rank 1 is called a *line bundle.*

There is an equivalent definition:

Definition 1.2. Let X be as above. A *holomorphic vector bundle on* X of rank n is an open covering $\{U_i\}$ of X together with holomorphic maps $\theta_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})$ such that

(i) $\theta_{ii}(x) = Id$ for all $x \in U_i$, and

(ii) $\theta_{ij}(x)\theta_{jk}(x) = \theta_{ik}(x)$ for all $x \in U_i \cap U_j \cap U_k$.

The θ_{ij}'s are called *transition functions*, and the condition (ii) is called the *cocycle condition.*

So, how are these definitions equivalent? The two definitions are related as follows: The map $\phi_i \circ \phi_j^{-1} : U_i \cap U_j \times \mathbb{C}^n \to U_i \cap U_j \times \mathbb{C}^n$ is given by $(x, v) \mapsto (x, \theta_{ij}(x)(v))$. Thus, given ϕ_i's, the appropriate restriction of $\phi_i \circ \phi_j^{-1}$ gives us the related θ_{ij}'s.

For the converse, assume we are given a cover $\{U_i\}_{i \in I}$ of X and transition functions $\theta_{ij} : U_i \cap U_j \to GL_n(\mathbb{C})$. We now want to produce a vector bundle compatible with the first definition, so let $V := \left(\coprod_{i \in I} U_i \times \mathbb{C}^n\right) / \sim$, where we let $(x, v) \in U_i \times \mathbb{C}^n$ be equivalent to $(x', v') \in U_j \times \mathbb{C}^n$ if $x = x'$ and $v = \theta_{ij}(x)v'$, and we give V the quotient topology. We then have a projection $p : V \to X$ and isomorphisms $V|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^n$, i.e. a vector bundle with local trivialisations.

A vector bundle is a special case of *fibre bundle*, which is a space E, with a surjective map $p : E \to X$, such that the fibres of p all are isomorphic to some given space F, and such that E is locally trivial: each $x \in X$ has a neighbourhood U such that $E|_U \simeq U \times F$. For vector bundles, this space is a vector space, so we may say that a vector bundle V of rank n on X is a fibre bundle with fibre $\mathbb{C}^n$.

Example 1.3 (The trivial bundle). Let X be any compact Riemann surface, then $X \times \mathbb{C}^n$ is the *trivial bundle of rank n*.

Example 1.4 (The Hopf bundle). Let $X = \mathbb{P}^1_{\mathbb{C}}$, the collection of (complex) lines through the origin in $\mathbb{C}^2$. Let $H = \{(x, v) \in X \times \mathbb{C}^2 | v \in x\}$ (when we say $v \in x$, we think of x as a line in $\mathbb{C}^2$). Then H is a vector bundle of rank 1 over X, i.e. a line bundle.

A *morphism* of holomorphic vector bundles (V, π), (V', π') on X is just a holomorphic map $\phi : V \to V'$, linear on each fibre of π, such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & = & X \end{array}$$

An *isomorphism* is a morphism with an inverse.

Let $(X, \mathscr{O}_X)$ be a locally ringed space over $\mathbb{C}$. One can view a vector bundle on X as a locally free sheaf on X, via an equivalence of the categories of vector bundles on X and of locally free coherent $\mathscr{O}_X$-modules. The equivalence goes as follows: Given a vector bundle V on X, the corresponding sheaf $\mathscr{O}(V)$ is the sheaf of its local sections. Conversely, given a locally free sheaf $\mathscr{F}$ of rank n, take for each point $x \in X$ the stalk $\mathscr{F}_x$. We get an n-dimensional $\mathbb{C}$-vector space $\mathscr{F}(x) = \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$, where $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathscr{O}_x$ of the point x. Now let $F := \coprod_{x \in X} \mathscr{F}(x) \to X$, and we give this fibration a structure of vector bundle: let $\{U_i\}_{i \in I}$ be an open covering of X such that $\mathscr{F}|_{U_i} \simeq \mathscr{O}^n_{U_i}$, then this isomorphism induces bijections $\phi_i : F|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{C}^n$, and over $U_i \cap U_j$, $\phi_i \phi_j^{-1}$ provides transition functions.

As X along with $\mathscr{O}_X$, the sheaf of holomorphic functions on X, is a locally ringed space, we can use this. The equivalence is very handy, and in what follows, it will never really matter which perspective we use. Thus, we will apply it with wild abandon, and use the terms 'vector bundle' and 'locally free sheaf' interchangeably. The ability to use sheaf language greatly simplifies some things.

Example 1.5. One can apply the operations of direct sum, tensor product, quotient, dual, exterior product, etc. to produce new vector bundles. Simply speaking, one does this by applying the operation to the fibres over points (i.e. the fibres of $V \oplus W$ are $V_x \oplus W_x$, etc.), and gluing together the vector bundle structures. Or, one simply applies the operation to the associated locally free sheaf.

Example 1.6 (The tangent bundle). The *(holomorphic) tangent bundle* $T(X)$ *of* X is constructed by taking the (holomorphic) tangent space T_pX at each point $p \in X$ as the fibre in p, and gluing together the fibres to create a vector bundle. Alternately, let Ω^1_X be the sheaf of holomorphic 1-forms on X. It is a locally free sheaf of rank 1. Then the tangent bundle $T(X)$ of X is defined as $\mathscr{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)$, i.e. the dual sheaf of Ω^1_X. (Ω^1_X is often called the *cotangent bundle*, being usually constructed as the dual bundle to the tangent bundle.)

One vector bundle of particular interest is the *pullback bundle*:

Example 1.7. Let $f : X \to Y$ be a holomorphic map, and let $p : V \to Y$ be a holomorphic vector bundle. Then the pullback $W := X \times_Y V$, together with the natural projection to X, is a vector bundle on X, called the *pullback bundle*. Its associated locally free sheaf $\mathscr{O}(W)$ is the same as the pullback sheaf $f^*\mathscr{O}(V)$.

We may now define some natural concepts:

Definition 1.8. The *determinant* of a vector bundle V of rank n is defined as the line bundle $\det V = \bigwedge^n V$. It satisfies $\det(V \oplus W) = \det V \otimes \det W$.

Definition 1.9 (Degree of vector bundles). To every line bundle (invertible sheaf) L is associated a divisor $D_L = \sum n_p p$, where the sum is taken over (finitely many) points $p \in X$, and the n_p are integers. We then define the *degree of* L as $\deg L = \sum n_p$. The *degree* of a vector bundle V on X is then defined as $\deg V = \deg(\det V)$. The degree satisfies $\deg(V \oplus W) = \deg V + \deg W$.

2 Some motivation

A reader who has previous experience with vector bundles will know that all of what we have just said works for any kind of vector bundles, with the obvious alterations. For instance, there are such things as *continuous* and *smooth* vector bundles. The definitions are similar, with the requirement of *holomorphic* maps being relaxed to that of *continuous* (resp. *smooth*) maps, and the requirement of *complex manifold* is relaxed to merely *topological space* (resp. *differentiable* manifold). A holomorphic vector bundle is also a continuous vector bundle and a smooth vector bundle, for obvious reasons. One could even talk about *algebraic* vector bundles, where V is an algebraic *variety* and the local trivialisations are *morphisms of varieties* from V to $\mathbb{A}^n$. Nothing in the definitions above depends on the fact that we are on a *curve*, indeed the exact same definitions carry through to the case of vector bundles on higher-dimensional spaces. So, why do we care about *holomorphic* vector bundles? On *curves*?

In any case, we will always mean *holomorphic* vector bundle when we say 'vector bundle'.

2.1 Equivalence of algebraic and analytic structures

First, a little degression into technicalities. There is an equivalence between projective algebraic structures over $\mathbb{C}$ and complex analytic structures, in a way we will sketch below. This will be very useful, because it enables us to apply algebraic methods to solve problems. (Material in this section is taken from [H77], appendix B.)

A *complex analytic space* is defined as a topological space $\mathfrak{X}$, with a sheaf of rings $\mathscr{O}_{\mathfrak{X}}$, that can be covered by open subsets, each isomorphic to a closed subset Y of a polydisc U in $\mathbb{C}^n$, defined by the vanishing of a finite number of holomorphic functions, and the sheaf $\mathscr{O}_Y$ is the appropriate quotient of $\mathscr{O}_U$, the sheaf of holomorphic functions on U.

If X is a scheme of finite type over $\mathbb{C}$, it has an open covering of affine schemes $Y_i = \operatorname{Spec} A_i$, with $A_i \simeq \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_q)$. Viewing the f_i's as holomorphic functions on $\mathbb{C}^n$, this gives us an analytic space $Y_i^{an} \subseteq \mathbb{C}^n$. The Y_i's glue together to form X, and with the same maps, Y_i^{an} can be glued together to form a complex analytic space X^{an}. We call this space the *associated analytic space of* X. This construction is clearly functorial, so we get a functor *an* from the category of schemes of finite type over $\mathbb{C}$ to the category of complex analytic spaces. Given a coherent sheaf $\mathscr{F}$ on X, we can create an associated coherent analytic sheaf $\mathscr{F}^{an}$ on X^{an} in the following way: Any coherent sheaf on $\mathscr{F}$ on X can be written locally as the quotient of free sheaves, $\mathscr{O}_U^m \xrightarrow{\phi} \mathscr{O}_U^n \to \mathscr{F} \to 0$, and by the functor *an*, we get a map $\mathscr{O}_{U^{an}}^m \xrightarrow{\phi^{an}} \mathscr{O}_{U^{an}}^n$, and we define $\mathscr{F}^{an}$ locally as the cokernel of this map.

Now, if we restrict ourselves to *projective* schemes X, the functor *an* induces an equivalence of categories from $\mathfrak{Coh}(X)$ to $\mathfrak{Coh}^{an}(X^{an})$, the category of coherent analytic sheaves on X^{an}, and furthermore, for any $\mathscr{F} \in \mathfrak{Coh}(X)$, we have that the cohomology groups $H^i(X, \mathscr{F}) \simeq H^i(X^{an}, \mathscr{F}^{an})$ are isomorphic.[1]

It is a theorem of Riemann that every compact Riemann surface is the associated analytic space of a projective algebraic curve, so we can view X as a scheme or complex analytic space interchangeably. By the equivalence of $\mathfrak{Coh}(X)$ and $\mathfrak{Coh}^{an}(X^{an})$, we can view a holomorphic vector bundle V as an algebraic one. In most situations it will not matter whether we view a vector bundle V as a holomorphic or algebraic vector bundle. However, at some important places it will matter: the construction of a vector bundle associated to a representation, outlined below, depends crucially on nonalgebraic structures. Thus, when dealing with this correspondence, we need to restrict ourselves to the holomorphic case.

2.2 Connection to representation theory

There is a connection between representations of the fundamental group $\pi := \pi_1(X)$ of X and holomorphic vector bundles on X. (The following is adapted from [V].) Given a representation $\rho : \pi \to GL_n(\mathbb{C})$, we can construct a holomorphic vector bundle in the following way: Let $p : \widetilde{X} \to X$ be the universal covering space of X. This has a natural structure as a holomorphic principal bundle with structure group π (a principal π-bundle on X is a fibre bundle $P \to X$, with fibres homeomorphic to π, with an action by π that acts by permuting elements in the fibres in the natural way). The representation ρ gives an action of π on $\widetilde{X} \times \mathbb{C}^n$, the trivial vector bundle on $\widetilde{X}$, by $\gamma.(\tilde{x}, \lambda) = (\tilde{x}\gamma^{-1}, \rho(\gamma)\lambda)$ for any $\gamma \in \pi$. Let $V_\rho := \widetilde{X} \times_\pi \mathbb{C}^n$ be the orbit space of this action. It is a vector bundle of rank n on X with projection map $\overline{(\tilde{x}, \lambda)} \mapsto p(\tilde{x})$. Thus, we can learn much about representations if we know something about vector bundles. The association $\rho \mapsto V_\rho$ has the following properties:

(i) $V_{\rho_1 \otimes \rho_2} = V_{\rho_1} \otimes V_{\rho_2}$

(ii) $V_{\rho_1 \oplus \rho_2} = V_{\rho_1} \oplus V_{\rho_2}$

(iii) $V_{\bigwedge^r \rho} = \bigwedge^r V_\rho$

(iv) $V_\rho^* = V_{{}^t\rho^{-1}}$, where ${}^t\rho$ is the transpose of ρ and V^* is the dual bundle of V.

(v) Line bundles arising from this construction have degree zero.

[1] See section 3 for definitions of cohomology groups.

Rules (iii) and (v) imply that $\deg V_\rho = 0$. Indeed, by definition, $\deg V_\rho = \deg(\det V_\rho)$, and $\det V_\rho$ is a line bundle. So, an obvious problem raises its hand and waves at us: to characterise those degree zero vector bundles that come from representations.

In a celebrated work of 1938, André Weil gave the answer: The degree zero bundles arising from representations are exactly those whose indecomposable components are all of degree zero. We provide a sketch of a proof of André Weil's theorem, from [V]:

Definition 2.1. V is called *indecomposable* if for all pairs V_1, V_2 of proper subbundles, $V \not\simeq V_1 \oplus V_2$.

By a standard argument, all vector bundles have a decomposition, unique up to permutation of indices, into a direct sum of indecomposable bundles. So, the statement of the theorem:

Theorem 2.2 (André Weil). *A holomorphic vector bundle V on X arises from a representation of π if and only if its indecomposable components all have degree zero.*

It is enough to prove this for indecomposable bundles, as the theorem easily follows from this:

Theorem 2.3. *Let V be an indecomposable vector bundle X. Then $V \simeq V_\rho$ for a representation $\rho : \pi \to GL_n(\mathbb{C})$ if and only if* $\deg V = 0$.

Here we need a definition:

Definition 2.4. A *holomorphic connection* on V is a map

$$\nabla : V \to V \otimes_{\mathscr{O}_X} \Omega^1_X,$$

$\mathbb{C}$-linear as a map of sheaves of complex vector spaces, and satisfying Leibnitz' rule: for each $U \subseteq X$, $f \in \mathscr{O}_X(U)$ and $s \in V(U)$, we have

$$\nabla(f.s) = s \otimes df + f.\nabla(s).$$

Here Ω^1_X is the holomorphic cotangent sheaf, i.e. the sheaf of holomorphic 1-forms.

Equivalently, for any open set $U \subset X$ and holomorphic vector field Y, a holomorphic connection is a $\mathbb{C}$-linear map $\nabla_Y : \Gamma(V|_U) \to \Gamma(V|_U)$, of sheaves of complex vector spaces on U, satisfying:

(i) Leibnitz' rule: $\nabla_Y(f.s) = (Yf).s + f.\nabla_Y s$, for $f \in \mathscr{O}_X(U)$ and $s \in \Gamma(V|_U)$.

(ii) $\mathscr{O}_X$-linearity in Y: $\nabla_{f_1Y_1+f_2Y_2} = f_1\nabla_{Y_1} + f_2\nabla_{Y_2}$ for $f_1, f_2 \in \mathscr{O}_X(U)$.

Remark. On smooth bundles, we have *smooth* connections, like the *holomorphic* connections defined above, with the obvious alterations. Holomorphic connections are obviously viewable as smooth connections on the underlying smooth bundle.

Definition 2.5. Given local vector fields Y_1 and Y_2 on X, the *curvature* of a holomorphic connection ∇ is the map of sheaves of complex vector spaces

$$\begin{aligned} \Omega_\nabla(Y_1, Y_2): \quad V &\to \quad V \\ s &\mapsto \quad \nabla_{Y_1}\nabla_{Y_2}s - \nabla_{Y_2}\nabla_{Y_1}s - \nabla_{[Y_1,Y_2]}s. \end{aligned}$$

This map is $\mathscr{O}_X$-linear, and skew symmetric in Y_1 and Y_2, i.e. $\Omega_\nabla(Y_1, Y_2) = -\Omega_\nabla(Y_2, Y_1)$, so we can think of Ω_∇ as an element in $\Gamma(\mathscr{E}nd\, V \otimes \Omega_X^2)$. Locally, this is a matrix of holomorphic 2-forms. On a compact Riemann surface X, which is our situation, there are no holomorphic 2-forms, so for *any* holomorphic connection ∇ on X, $\Omega_\nabla \equiv 0$. A connection with $\Omega_\nabla \equiv 0$ is called *flat*. It will turn out to be useful that flatness of ∇ is equivalent to having $\nabla^2 = 0$.

Now, why is this map called a *connection*? It connects fibres along curves, or more precisely:

Definition 2.6 (Parallel Transport). Let V be a smooth complex vector bundle equipped with a smooth connection ∇, and a smooth curve $\tau : [0, 1] \to X$ with $\tau(0) = a, \tau(1) = b$, there is an induced $\mathbb{C}$-linear map $P_\tau : V_a \to V_b$ called a *parallel transport operator*. Composition of paths corresponds to composition of the induced parallel transport operators. (We will not give the construction.)

If the connection ∇ is flat, the induced parallel transport operator P_τ is invariant under smooth homotopies of the curve τ, and this gives us a representation $\rho : \pi_1(X, x_0) \to Aut_{\mathbb{C}}(V_{x_0}) \simeq GL_n(\mathbb{C})$ of π, called the *holonomy representation*. Now, given a holomorphic connection ∇ on a compact Riemann surface X, and viewing this as a smooth connection on the underlying smooth bundle, it remains flat. It turns out that $V \simeq V_\rho$ as holomorphic bundles, and conversely, if V comes from a representation $\rho : \pi \to GL_n(\mathbb{C})$, it admits a holomorphic connection. Thus we can reduce our problem to

Theorem 2.7. *An indecomposable vector bundle V on X admits a holomorphic connection if and only if its degree is zero.*

The proof to this theorem is rather long and involved, and not all that interesting for our purposes. The main point we will care about is that there is a correspondence between pairs (V, ∇) of holomorphic vector bundles with holomorphic connections, and representations ρ of π, which will be important later. So, we now know the image of the map $\rho \mapsto V_\rho$. A related question is:

to what degree is this map injective? That is, which representations ρ give isomorphic vector bundles V_ρ? This is a lot harder.

Let us first introduce an equivalence relation: we say that two representations ρ_1, ρ_2 are equivalent, $\rho_1 \sim \rho_2$, if there exists $\phi \in GL_n(\mathbb{C})$ such that $\phi\rho_1(\gamma)\phi^{-1} = \rho_2(\gamma)$ for all $\gamma \in \pi$. Now, let us restrict ourselves to looking at vector bundles of fixed rank n, and consider the map

$$Rep(\pi, GL_n(\mathbb{C})) \xrightarrow{\Psi} V(n, 0),$$

where $V(n,0)$ is the set of isomorphism classes of holomorphic vector bundles of rank n and degree 0, that sends a representation ρ to its associated vector bundle V_ρ. This map is not surjective, because there are bundles on the form $\mathscr{O}(x_0) \oplus \mathscr{O}(-x_0) \oplus \mathscr{O}_X^{n-2}$, for some $x_0 \in X$, which have rank n and degree 0, but not all indecomposable components have degree 0, and thus they do not come from a representation. Furthermore, the map is not injective, even if we restrict the map to equivalence classes of representations, i.e. there exist nonequivalent representations that give isomorphic vector bundles (see [V] for examples). However, if we restrict ourselves to *unitary* representations, we get some nice properties. First, we need a definition:

Definition 2.8. A vector bundle V is called *stable* (resp. *semistable*) if, for every proper subbundle $W \subset V$, we have that $\frac{\deg W}{rkW} < \frac{\deg V}{rkV}$ (resp. $\frac{\deg W}{rkW} \leq \frac{\deg V}{rkV}$).

And now, a theorem, due to M. S. Narasimhan and C. S. Seshadri in 1965:

Theorem 2.9 (Narasimhan-Seshadri). *Let X be Riemann surface of genus $g \geq 2$. A vector bundle V on X of rank n and degree zero is stable if and only if it is isomorphic to V_ρ for an* irreducible unitary *representation ρ of π. The association $\rho \rightsquigarrow V_\rho$ gives an equivalence of categories between the category of irreducible unitary representations of π and the category of stable holomorphic vector bundles of degree zero on X.*

This takes care of the irreducible representations, and what about the rest? Because $U(n)$ is compact, all representations of π are decomposable as a direct sum of irreducible representations, which gives us

Corollary 2.10. *A vector bundle V on X of rank n and degree zero is isomorphic to V_ρ for a unitary representation ρ if and only if V is* polystable, *i.e. $V \simeq \bigoplus V_i$, with each V_i stable of degree zero. Furthermore, we have that for $\rho_1, \rho_2 \in Rep(\pi, U(n))$, $V_{\rho_1} \simeq V_{\rho_2}$ if and only if $\rho_1 \sim \rho_2$.*

For nonunitary representations, or for X with $g < 2$, no similarly detailed description is available.

Later, we will introduce *deformation theory*, the study of slightly altering, or *deforming*, a given structure, and this leads us to an interesting question:

if take a representation ρ, and deform it, what happens to the associated bundle (V, ∇)? This is a rather difficult question to answer in full, due to the difficulty of describing the relation $\rho \mapsto V_\rho$. We could of course consider only unitary representations (and stable bundles associated to them), but unitary representations turn out to never deform nontrivially (that is, only the trivial deformation remains unitary), and deformation of (semi)stable bundles appears to be difficult to describe. Thus, we will focus on the necessary preliminary of characterising deformations of (V, ∇), which is altogether more manageable.

3 A little homological algebra

Before we can embark on solving our problem, we need to introduce the toolbox of *homological algebra*. Homological algebra is tremendously useful, and will be of great value in the next sections. While a full introduction is far beyond the scope of this text, we give a very brief summary of important concepts and terminology, and leave the details to the references.

Definition 3.1. A *complex* $C^\bullet$ of abelian groups is a collection $\{C^i\}_{i\in\mathbb{Z}}$ of abelian groups and group homomorphisms $d^i : C^i \to C^{i+1}$ such that, for all i, $d^{i+1} \circ d^i = 0$. The maps d^i are called *coboundary maps* or *differentials*.

The quotient $\ker d^i/\mathrm{im}\ d^{i-1}$ is called the *i-th cohomology object $h^i(C^\bullet)$ of $C^\bullet$*. The subgroups $\ker d^i$ and $\mathrm{im}\ d^{i-1}$ are called the groups of *i-cocycles* and *i-coboundaries*, respectively. In particular, the i-cocycles are denoted by $Z^i(C^\bullet)$. If the groups C^i are only defined for some i, say, $i \geq 0$, we let all other $C^i = 0$.[2]

The above definition applies to any abelian category, with the obvious alterations. In particular, we will work in the categories of sheaves on X and of modules over k-algebras. The definitions apply to sheaves of abelian groups in general, though we will primarily use coherent sheaves or $\mathscr{O}_X$-modules. The two main cohomology theories we will use are *Hochschild* cohomology of k-algebras, and *derived functor* cohomology of sheaves, along with the equivalent and particularly useful *Čech* cohomology of sheaves. Some other variations will show up, but once the reader has seen these two, she will (hopefully) not be thrown by the rest.

3.1 Some cohomology theories

We are now ready to define Hochschild cohomology of a k-algebra A with coefficients in an A-bimodule M. (For more detail, see [W].)

Definition 3.2. Hochschild cohomology is given by the cohomology objects of the complex $C^n = Hom_k(A^{\otimes n}, M)$ with differential $d^n\phi(a_0, .., a_n) = a_0\phi(a_1, .., a_n) + \sum_{1\leq j\leq n}(-1)^j\phi(a_0, .., a_{j-1}a_j, .., a_n) + (-1)^{n+1}\phi(a_0, .., a_{n-1})a_n$. Here $A^{\otimes n}$ denotes the tensor product (over k) of A with itself n times. By convention, $A^{\otimes 0} = k$ and $A^{\otimes 1} = A$. The Hochschild cohomology groups are usually denoted by $HH^i(A, M)$.

It is worth noting that while Hochschild cohomology is very elegant and easy to do calculations with, its general properties are slightly 'wrong' when compared to other cohomology theories, in the sense that in some general constructions, the indices in the Hochschild situation are slightly different from those in the general situation. For instance, when we later do deformation theory, we will use cohomology groups to classify deformations of the

[2] The *co-* prefix is there for entirely historical reasons. Do not worry about it.

objects under study. The results will be analogous in the various situations we consider, but the index of the groups will be one higher in the Hochschild case, i.e. a result applies to H^{i+1} instead of to H^i. The reason for this misalignment is that the Hochschild complex in a sense is indexed 'wrong'. Other theories have objects analogous to $(n+1)$-simplices as elements in the n'th group, while the Hochschild groups have objects analogous to n-simplices as elements.

Now, to define sheaf cohomology, one must invest in some machinery from category theory. Thus, we will merely state the definitions, somewhat simplified, and leave the details to the reader (see [H77], chap. 3). For this section, X is a topological space, though we will usually think of X as either a complex manifold or a scheme.

An *injective resolution* of a sheaf (of abelian groups) $\mathscr{F}$ on X is a complex $\mathscr{I}^\bullet$ of sheaves on X, with $\mathscr{I}^i = 0$ for $i < 0$, together with a morphism $\epsilon : \mathscr{F} \to \mathscr{I}^0$, such that $\mathscr{I}^i$ is an *injective*[3] sheaf for all i, and the complex

$$0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \cdots$$

is exact.

Now, it so happens that all sheaves have an injective resolution. So, take an injective resolution of $\mathscr{F}$, and apply the global section functor $\Gamma(X, \cdot)$ to it. Thus, we have a complex $I^\bullet$ of abelian groups

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{I}^0) \to \Gamma(X, \mathscr{I}^1) \to \cdots$$

Now, define the *i-th cohomology group of $\mathscr{F}$* to be $H^i(X, \mathscr{F}) := h^i(I^\bullet)$.[4] In particular, $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$. As we are fortunate enough to be working with Riemann surfaces (which are very well-behaved spaces indeed), these groups will turn out to be $\mathbb{C}$-vector spaces.

We say that the cohomology functors $H^i(X, -)$ are the *right derived functors* of the global section functor. Similar constructions can be carried out with other functors, for instance the $Ext^i(\mathscr{F}, -)$ functors are the right derived functors of the $Hom(\mathscr{F}, -)$ functor. This definition of cohomology is due to Grothendieck, and he is also responsible for establishing one very important fact: that $H^k(X, \mathscr{F}) = 0$ for $k > \dim X$.

While elegant and tremendously delicious for theoretical purposes, this definition is practically useless for calculation. This is remedied by introducing a new cohomology theory of sheaves, namely *Čech* cohomology, and then proving that it is equivalent to the original definition (under certain nice circumstances). Again, we simply give the definition, leaving details (and

[3] i.e., $Hom(\cdot, \mathscr{I}^i)$ is an exact functor.

[4] It is a somewhat complicated argument to prove this, but it turns out to not matter which injective resolution we take. See [W] for details.

the all-important proof of equivalence with the original definition) to the references (again, [H77], chap. 3).

Definition 3.3 (Čech cohomology). Let $\mathscr{F}$ be a sheaf (of abelian groups) on X, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Fix a well-ordering $<$ of the index set I, and let $U_{i_0,\cdots,i_p}$ denote $U_{i_0} \cap \cdots \cap U_{i_p}$. Define a complex of abelian groups by

$$\check{C}^p(\mathcal{U}, \mathscr{F}) := \prod_{i_0 < \cdots < i_p} \Gamma(U_{i_0,\cdots,i_p}, \mathscr{F}),$$

with differential given by

$$(\delta^p_{\check{C}} \phi)_{i_0,\ldots,i_p+1} = \sum_{k=0}^{p+1} (-1)^k \phi_{i_0,\cdots,\widehat{i_k},\cdots,i_{p+1}}|_{U_{i_0,\cdots,i_{p+1}}},$$

where the notation $\widehat{i_k}$ means omit i_k from the set of indices. It is tedious but straightforward to check that $\delta_{\check{C}} \circ \delta_{\check{C}} = 0$, thus we have a complex of abelian groups. The *p-th Čech cohomology group* $\check{H}^p(\mathcal{U}, \mathscr{F})$ is defined as the p-th cohomology object of this complex.

The important fact about this theory is that $\check{H}^p(\mathcal{U}, \mathscr{F}) = H^p(X, \mathscr{F})$ when X is a noetherian separated scheme (or paracompact Hausdorff space, in the nonalgebraic situation), $\mathscr{F}$ is a quasi-coherent sheaf, and the covering $\mathcal{U}$ is such that $H^p(U_i, \mathscr{F}) = 0$ for each set U_i in the covering. This in particular includes the case when each U_i is affine (how nice!).[5]

There is also such a thing as a *double complex*, which is in a sense a complex of complexes. A double complex is indexed over $\mathbb{Z}^2$ rather than over $\mathbb{Z}$, with each row and column being a complex as defined above, all the rows and columns going the same way, respectively, and each square commuting. A simple diagram to illustrate:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^{1,0} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
C^{0,0} & \xrightarrow{d} & C^{0,1} & \xrightarrow{d} & \cdots
\end{array}$$

A double complex $C^{\bullet\bullet}$ gives rise to a *total complex* $K^\bullet = Tot(C^{\bullet\bullet})$, where $K^i = \bigoplus_{p+q=i} C^{p,q}$, and the differential is given by $d_{Tot} = d + (-1)^q \delta$.

[5] As a side note, isn't it nice that 'refining' the *sheaf* (by resolutions) gives the same result as refining the *space* (by open covering)?

The alternating sign of one differential is to ensure we get $\partial^{i+1}\partial^i = 0$. Later, when we do deformation with additional constraints, we have much use for double complexes. When talking about the cohomology of the total complex, we will likely abuse terminology and say things like 'the cohomology of the double complex', trusting the reader to not get confused.

3.2 Cup product

A nice property of cohomology with coefficients in a ring (or sheaf of rings) is that they admit a product, called the *cup product*, which enables a structure of graded ring on the collection of cohomology groups. Given a complex $(C^\bullet, \delta)$, we can define the product as a map $\smile\colon C^k \times C^l \to C^{k+l}$, that satisfies

$$\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi.$$

This relation implies that the product of two cocycles is a cocycle, and the product of a cocycle and a coboundary is a coboundary, which gives an induced product $\smile\colon H^k(C^\bullet) \times H^l(C^\bullet) \to H^{k+l}(C^\bullet)$ on cohomology groups. The two cohomology theories we have described so far have the following cup products:

The Hochschild cup product of $f \in C^{k+1}(A, A), g \in C^{l+1}(A, A)$ is given by

$$(f \smile g)(a_0, \dots, a_{k+l}) = \sum_{p=0}^{k} (-1)^p f(a_0, \dots, a_{p-1}, g(a_p, \dots, a_{p+l}), a_{p+l+1}, \dots, a_{k+l}).$$

Notice that this is a map $C^{k+1} \times C^{l+1} \to C^{k+l+1}$, i.e. the indexation appears to be 'wrong', one higher than in general, as noted above.

For Čech cohomology, the cup product is much nicer. For $\phi \in C^k, \psi \in C^l$, the product is simply given by $(\phi \smile \psi)_{i_0,\dots,i_{k+l}} = \phi_{i_0,\dots,a_k} \cdot \psi_{a_k,\dots,a_{k+l}}$.

4 Deformation theory

Let us begin by saying what is meant by a *deformation*. In very general terms, given an object X with some structure (topological, algebraic, analytic, etc.), a *deformation* of X is a family X_t of objects, parametrised by some suitable space, whose structure is obtained by 'deforming' the structure on X in a smooth way as the parameter t varies, and such that $X_0 \simeq X$. First an example (from [Fox]) that will provide some intuition, before we embark on a more formal treatment.

Example 4.1. Let A be a (commutative) k-algebra, given by a multiplication $\phi : A \otimes_k A \to A$, with $(a \otimes b) \mapsto ab$ satisfying certain properties (identity, associativity, commutativity and distributivity over addition). We want to deform the multiplication, letting it vary by some parameter t, while preserving the multiplication properties. We would like to have $A[[t]]$, the $k[[t]]$-algebra of formal power series with coefficients in A, along with a multiplication $\Phi_t : A[[t]] \otimes_{k[[t]]} A[[t]] \to A[[t]]$, given by

$$\Phi_t(a, b) = \phi_0(a, b) + \phi_1(a, b)t + \phi_2(a, b)t^2 + \cdots ,$$

where each ϕ_i is a k-linear map $A \otimes_k A \to A$. Since we want to have $A_0 \simeq A$, we demand that $\phi_0 = \phi$, i.e. $\phi_0(a, b) = ab$. There are obviously many ways to provide a power series extending ϕ, but which ways of extending ϕ to a Φ_t give a deformation, i.e. retain the properties above? Of those properties, the only tricky one is associativity, so we need to find conditions for when Φ_t is associative, that is, when do we have $\Phi_t(\Phi_t(a, b), c) = \Phi_t(a, \Phi_t(b, c))$?

By comparing the coefficients of t^n in the expression $\Phi_t(\Phi_t(a, b), c) = \Phi_t(a, \Phi_t(b, c))$, we get

$$\sum_{i=0}^{n} \phi_i(\phi_{n-i}(a, b), c) = \sum_{i=0}^{n} \phi_i(a, \phi_{n-i}(b, c)).$$

Let ϕ_k be the first non-zero coefficient after ϕ_0 in the expression $\Phi_t = \sum \phi_i t^i$ (called the *infinitesimal* of Φ_t), we then get

$$a\phi_k(b, c) - \phi_k(ab, c) + \phi_k(a, bc) - \phi_k(a, b)c = 0.$$

The left hand side is the Hochschild coboundary $d\phi_k$ of ϕ_k, so we see that ϕ_k is forced by the associativity condition to be a Hochschild-2-cocycle.

Now, suppose we are given $\Phi^n := \phi + \phi_1 t + \cdots + \phi_n t^n$, satisfying the requirement for associativity above. The above implies that the ϕ_i's are cocycles. What might prevent us from extending the deformation by an additional term ϕ_{n+1}? Let us simply add another term $\phi_{n+1}t^{n+1}$, and look at the associativity condition: $(\Phi^n + \phi_{n+1}t^{n+1})(\Phi^n(a, b) + \phi_{n+1}(a, b)t^{n+1}, c) - (\Phi^n + \phi_{n+1}t^{n+1})(a, \Phi^n(b, c) + \phi_{n+1}(b, c)t^{n+1})$. This ugly thing reduces to the associativity condition for Φ^n, which we have assumed is satisfied, plus a coefficient of t^{n+1}. That coefficient is

$$\begin{aligned}\sum_{i=0}^{n+1}&[\phi_i(\phi_{n+1-i}(a,b),c) - \phi_i(a,\phi_{n+1-i}(b,c))] \\ &= \phi_{n+1}(a,b)c - a\phi_{n+1}(b,c) + \phi_{n+1}(ab,c) - \phi_{n+1}(a,bc) \\ &\qquad + \sum_{i=1}^{n}[\phi_i(\phi_{n+1-i}(a,b),c) - \phi_i(a,\phi_{n+1-i}(b,c))].\end{aligned}$$

This is recognizable as $-d\phi_{n+1} + \sum_{i=1}^{n} \phi_i \smile \phi_{n+1-i}$, so we get our condition: the 3-cocycle $\sum_{i=1}^{n} \phi_i \smile \phi_{n+1-i}$ is a 3-coboundary. Thus, the obstruction to extending a deformation of A is an element of $HH^3(A, A)$, and we can extend our deformation if this element is zero.

The general situation will be analogous, where the object under deformation is represented by a power series, with the infinitesimal being a cocycle for some appropriate cohomology theory, and the obstruction to extending a truncated deformation is that some certain cocycle is a coboundary.

What we just did is describing the first-order, or infinitesimal, deformations, which is to say deformations over the dual numbers $D := k[t]/t^2$, and then finding the obstructions for extending this deformation to a larger artinian k-algebra, in this case of the form $k[t]/t^n$. This is how we will proceed in general. Also, this was a *one-parameter* deformation. We could do deformation for multi-parameter situations, i.e. deformation over artinian k-algebras of the form $k[t_1, \dots, t_n]/I$, but for purposes of clarity we restrict ourselves to the one-parameter case. We do not lose much, as the methods used are the same either way.

The group or space parametrising the infinitesimals will be called the *tangent space* of the deformation functor. Some intuitive motivation for this name will be given later. The tangent space contains much interesting information, and has the advantage that it is generally easy to calculate, whereas the full description of the deformation can be very complicated.

Now, let us say what we mean by a deformation of a vector bundle in general, and give a classification. We do this for coherent sheaves generally, and then extract the result about vector bundles as a corollary. This proof, taken from [H04], is not immediately helpful for calculations, but it is very concise and elegant. We will also sketch an alternate proof below, based on [V], which is more fiddly to write out in full, but provides the explicit description we need for calculation.

Definition 4.2. Let X be a scheme over $\mathbb{C}$, $\mathscr{F}$ a coherent sheaf on X. A *deformation of* $\mathscr{F}$ *over* $D := \mathbb{C}[t]/t^2$ is a coherent sheaf $\mathscr{F}'$ on $X' := X \times_{\mathbb{C}} \operatorname{Spec} D$, flat over D, with a sheaf homomorphism $\mathscr{F}' \to \mathscr{F}$ such that the induced map $\mathscr{F}' \otimes_D \mathbb{C} \to \mathscr{F}$ is an isomorphism.

Theorem 4.3. *Let $X, \mathscr{F}$ be as above. There exists a one-to-one correspondence between the set of deformations of $\mathscr{F}$ over D and the group $Ext^1_X(\mathscr{F}, \mathscr{F})$, with the trivial deformation corresponding to the 0 element.*

Proof. The condition that $\mathscr{F}'$ is flat is equivalent to the exactness of the sequence

$$\xi : 0 \to \mathscr{F} \xrightarrow{\cdot t} \mathscr{F}' \to \mathscr{F} \to 0$$

obtained from applying $- \otimes_D \mathscr{F}'$ to the sequence

$$0 \to \mathbb{C} \xrightarrow{\cdot t} D \to \mathbb{C} \to 0.$$

This last sequence splits, inducing a splitting $\mathscr{O}_X \to \mathscr{O}_{X'}$, so the sequence ξ may be viewed as an exact sequence of $\mathscr{O}_X$-modules. ξ corresponds to an element $\xi \in Ext^1_X(\mathscr{F}, \mathscr{F})$. Conversely, an element $\xi \in Ext^1_X(\mathscr{F}, \mathscr{F})$ gives a coherent sheaf $\mathscr{F}'$ as an extension of $\mathscr{F}$ by $\mathscr{F}$ as $\mathscr{O}_X$-modules. We need to supply $\mathscr{F}'$ with a $\mathscr{O}_{X'}$-module structure, by specifying what multiplication with t should do. This can be done in only one way compatible with the sequence and the requirements above, namely by projection $\mathscr{F}' \to \mathscr{F}$ followed by $\mathscr{F} \xrightarrow{\cdot t} \mathscr{F}'$. Thus, we have our correspondence. □

Corollary 4.4. *Let V be a vector bundle on X, then the set of deformations of V over D are in one-to-one correspondence with the cohomology group $H^1(X, \mathscr{E}nd\, V)$.*

Proof. Since V is a locally free $\mathscr{O}_X$-module,

$$Ext^1_X(V, V) \simeq Ext^1_X(\mathscr{O}_X, \mathscr{E}nd\, V) \simeq H^1(X, \mathscr{E}nd\, V)$$

and we are done. □

Now, this was all very algebraic and sheafy, and the reader might ask, could this be stated in more geometric and bundly terms? Yes: Given a (holomorphic) vector bundle V_0 on X, we can look at a family $\{V_t\}_{t \in T}$ of vector bundles on X, parametrised by a complex manifold T, which is to say that $V \to X \times T$ is a holomorphic vector bundle on $X \times T$, such that for a chosen point $t_0 \in T$, we have $V|_{X \times \{t_0\}} \simeq V_0$. We call such a bundle a *deformation of* V_0 *over* T. If $q : X \times T \to X$ is the projection on the first coordinate, we call the pullback bundle $q^* V_0$ the *trivial* deformation of V_0 over T. Now, since for any deformation V, we have $V|_{X \times \{t_0\}} \simeq V_0 \simeq q^* V_0|_{X \times \{t_0\}}$. Can we extend this identity map to an isomorphism on an open set $U \subset T$ around t_0? At least we can always do this locally on X. Using the local trivialisations of V and q^*V_0, we can construct isomorphisms $\psi_i : V|_{U_i \times U} \xrightarrow{\sim} q^*V_0|_{U_i\times U}$, and from these obtain local automorphisms $\tau_{ij} := \psi_i \circ \psi_j^{-1} : q^*V_0|_{U_{ij}\times U} \to q^*V_0|_{U_{ij}\times U}$. Now, how far are these from being trivial?

For simplicity, let us restrict ourselves to letting $\dim T = 1$, i.e. a one-parameter deformation (don't worry, the multi-parameter situation works in the same way, but is more fiddly). We may then assume U to be an open disc in $\mathbb{C}$ around the origin $t_0 = 0$. Then $\tau_{ij} = Id + \xi^1_{ij}t + \xi^2_{ij}t^2 + \cdots$ is locally a lift to V_0 of the identity map over t_0, where the ξ^k_{ij}'s are local endomorphisms of U_{ij}. Since the τ_{ij}'s obviously fulfill the cocycle condition $\tau_{ij} \circ \tau_{jk} = \tau_{ik}$, we can get conditions on the ξ's. Gathering coefficients of t, we see that $\xi^1_{ij} - \xi^1_{ik} + \xi^1_{jk} = 0$, i.e. ξ^1 is a Čech 1-cocycle which is the obstruction to extending the map further, and as 1-coboundaries represent *trivial* extensions, we see that the first-order deformations are parametrised by $H^1(X, \mathscr{E}nd\, V_0)$, as above. Looking at the first-order deformations, which correspond to deformations over D as above, we can get an intuitive idea of why this is called the *tangent space* of the deformation:

Let $\mathsf{T}_{t_0}T$ be the tangent space of T at t_0. We can define a map

$$\eta_{t_0} : \mathsf{T}_{t_0}T \to H^1(X, \mathscr{E}nd\, V_0),$$

called the *Kodaira-Spencer map*, by sending a tangent vector $\mathbf{v} \in \mathsf{T}_{t_0}T$ to the first-order obstruction to extending the identity automorphism along a holomorphic curve in T which has $\mathbf{v}$ as its tangent vector at t_0.

If we consider T as a scheme, this becomes even clearer: Translating the sheaf situation above to vector bundle language, we say that a first-order deformation of V_0 is a vector bundle V_t on $X \times \operatorname{Spec} D$ such that the fibre over the closed point is isomorphic to V_0. A Zariski tangent vector of T at t_0 is a morphism $\mathbf{v} : \operatorname{Spec} D \to T$, which sends the closed point to t_0. Thus, we can view a first-order deformation of V_0 as a pullback $V|_{\mathbf{v}}$ of V_0 along the morphism $Id \times \mathbf{v} : X \times \operatorname{Spec} D \to X \times T$. Now, the Kodaira-Spencer map can be defined as taking $\mathbf{v} \in \mathsf{T}_{t_0}T$ to the isomorphism class of $V|_{\mathbf{v}}$.

Now, given a first-order lift $\tau_{ij} = 1 + \xi_{ij}t$ satisfying the cocycle condition, can we extend this to higher orders of t? We can simply continue what we did above; look at coefficients of higher powers of t in the expression $\tau_{ij}\tau_{jk} - \tau_{ik}$. We find that for this expression to vanish, the coefficients must all be 2-coboundaries. For instance, the coefficient of t^2 is $\xi^2_{ij} + \xi^2_{jk} - \xi^2_{ik} + \xi^1_{ij}\xi^1_{jk}$, which gives the condition $d\xi^2_{ijk} = -\xi^1_{ij}\xi^1_{jk}$, i.e. the 2-cocycle $\{\xi^1_{ij}\xi^1_{jk}\} = \xi^1 \smile \xi^1$ is a coboundary. For higher powers the same thing happens, some sum of cup products of 1-cocycles (which is a 2-cocycle) is forced to be a 2-coboundary. Thus, given some lift of the identity $1 + \xi^1_{ij}t + \ldots + \xi^k_{ij}t^k$, the obstruction to extending it one step further is a 2-cocycle, which gives a class in $H^2(X, \mathscr{E}nd\, V_0)$. Now, in our case, $\dim X = 1$, so $H^2(X, \mathscr{E}nd\, V_0) = 0$, so we can always extend a deformation over D to one of arbitrary order.

4.1 Deformation with additional constraints

One interesting question in deformation theory is, under what conditions does a deformation of a given object preserve certain additional properties of the object? To answer our question about deformation of vector bundles with connections, we will need to know something about this. As an example, we could regard a diagram (say, of A-modules) as a presheaf on a partially ordered set. Then, deforming the presheaf (i.e. deforming each module and map in the diagram), we could ask what conditions had to apply for a certain feature of the diagram to be retained in the deformation. For instance, an exact sequence is a presheaf on the poset $\{2 \geq 1 \geq 0\}$, and the central question would be, when is the deformed diagram an exact sequence? Let us elaborate with an example, and give proper definitions. For details and proofs (where applicable) in what follows, see [S].

Definition 4.5. Let Λ be a partially ordered set, considered as a category with the inclusions $\lambda' \leq \lambda$ as the only morphisms. Let A be a k-algebra and $\mathscr{F}, \mathscr{G} : \Lambda \to \mathfrak{Mod}(A)$ be presheaves of A-modules on Λ. An extension $\mathscr{E}$ of $\mathscr{F}$ by $\mathscr{G}$ is given by a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G}(\lambda) & \longrightarrow & \mathscr{E}(\lambda) & \longrightarrow & \mathscr{F}(\lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{G}(\lambda') & \longrightarrow & \mathscr{E}(\lambda') & \longrightarrow & \mathscr{F}(\lambda') & \longrightarrow & 0
\end{array}$$

of A-modules and A-module homomorphisms, for each pair $\lambda' \leq \lambda$.

The morphism category $\mathfrak{Mor}(\Lambda)$ is the partially ordered set with the morphisms $\lambda' \leq \lambda$ of Λ as objects and morphisms given by $(\lambda' \leq \lambda) \leq (\gamma' \leq \gamma)$ if $\gamma' \leq \lambda' \leq \lambda \leq \gamma$ in Λ, i.e. if the morphism $\gamma' \leq \gamma$ 'factors through' $\lambda' \leq \lambda$. We can now, for $\mathscr{F}, \mathscr{G}$ presheaves of A-modules on Λ as above, define the functor $Hom_k(\mathscr{F}, \mathscr{G}) : \mathfrak{Mor}(\Lambda) \to \mathfrak{Bimod}(A, A)$ by $Hom_k(\mathscr{F}, \mathscr{G})(\lambda' \leq \lambda) = Hom_k(\mathscr{F}(\lambda'), \mathscr{G}(\lambda))$. With this in hand, we get a double complex

$$K^{p,q} = \prod_{\lambda_0 \leq \ldots \leq \lambda_p} C^q(A, Hom_k(\mathscr{F}, \mathscr{G})(\lambda_0 \leq \lambda_p)) \qquad p, q \geq 0$$

where $C^q(A, M)$ is the Hochschild cochain group of A with values in the $A - A$-bimodule M. The double complex is the Hochschild complex in one direction. The differential in the other direction is analogous to the Čech differential in the λ_i's, with the differential being given as an alternating sum in the same way.[6] We denote by $Tot^\bullet(\mathscr{F}, \mathscr{G}) = Tot(K^{\bullet\bullet})$ the total complex of the double complex.

Proposition 4.6. *The set of isomorphism classes of presheaf extensions of $\mathscr{F}$ by $\mathscr{G}$ is in one-to-one correspondence with the elements of $H^1(Tot^\bullet(\mathscr{F}, \mathscr{G}))$.*

[6] This is the *Laudal D-complex*, see [S]

This justifies the notation $Ext^i_\Lambda(\mathscr{F},\mathscr{G}) = H^i(Tot^\bullet(\mathscr{F},\mathscr{G}))$ for $n \geq 0$. By a similar argument to that of 4, this group parametrises the deformations over D. We calculate an example:

Example 4.7. Let M,N and Q be A-modules, with maps $f : M \to Q$, $g : N \to Q$, and form their pullback P:

$$\begin{array}{ccc} P & \xrightarrow{\gamma} & N \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & Q \end{array}$$

We consider this diagram as a presheaf $\mathscr{F}$ as above, and look at its deformations over $k[t]/t^2$, given by $Ext^1_\Lambda(\mathscr{F},\mathscr{F})$. What we want to know is, when is the deformed diagram

$$\begin{array}{ccc} P_t & \xrightarrow{\gamma_t} & N_t \\ {\scriptstyle\phi_t}\downarrow & & \downarrow{\scriptstyle g_t} \\ M_t & \xrightarrow{f_t} & Q_t \end{array}$$

still a pullback-diagram, with P_t a pullback of the rest?

The answer lies in computing the Ext^1 group, i.e. the first cohomology group of the total complex defined above. We could, if we wanted to, have done this for the entire diagram, but since we only care about the cases when the deformation ends up as a pullback, we can restrict ourselves to looking at the restricted diagram $\mathscr{F}'$:

$$\begin{array}{ccc} & & N_t \\ & & \downarrow{\scriptstyle g_t} \\ M_t & \xrightarrow{f_t} & Q_t \end{array}$$

because the pullback is unique. The computation of the cohomology group is somewhat complicated, and the answer is worse: Let H_{MQN} be the quotient of $Hom_k(M,Q) \oplus Hom_k(N,Q)$ by the subgroup generated by maps $(f \circ m - q \circ f,\ g \circ n - q \circ g)$, with m, n and q being endomorphisms of M, N and Q, respectively. (For convenience, we will still refer to elements of H_{MQN} as pairs (ϕ,γ).) Now, $Ext^1_\Lambda(\mathscr{F},\mathscr{F})$ is given by those elements (m,n,q,ϕ,γ) in $Ext^1_A(M,M) \oplus Ext^1_A(N,N) \oplus Ext^1_A(Q,Q) \oplus H_{MQN}$ such that $[-,\phi] = f \circ m - q \circ f$, $[-,\gamma] = g \circ n - q \circ g$.

This example illustrates the unfortunate fact that explicit calculation of the groups parametrising deformations can yield gruesome results. On the other hand, we are fortunate enough to not need explicit descriptions for anything, so we will not have to perform such a feat again.

5 A double complex

So, having introduced some rudiments of deformation theory, and demonstrated some tools, we can get back to our main task of describing the deformation of a vector bundle equipped with a holomorphic connection. The deformation will be given by some cohomology theory, which has to simultaneously describe deformation of our bundle V and deformation of the connection ∇. As we saw in the example of deforming diagrams of A-modules, the deformation was given by a double complex, which in one direction gave the deformation of the main objects (A-modules) and in the other gave the deformation of the additional constraints (the diagrammatic relations), and the required conditions for making these fit together were found in the cross-terms. So, we are looking for a double complex, which in one direction describes deformation of the vector bundle V, and in the other describes deformation of the holomorphic connection ∇. We already know that deformation of vector bundles is given by a Čech-1-cocycle, i.e. we need the Čech complex of $\mathscr{E}nd\, V$, but what about the connection? We'll give the answer first, and see how it all fits together later.

5.1 The generalised de Rham complex

Let V be a vector bundle over X equipped with a connection $\nabla : V \to V \otimes \Omega^1_X$, as defined above. We shall need to assume that ∇ is flat, i.e. $\nabla^2 = 0$. In our case, this is no problem, as we have already seen that all connections on Riemann surfaces are flat.

We can use the connection to construct a generalised de Rham complex

$$C^q(V, \nabla) = Hom_{\mathscr{O}_X}(V, V \otimes_{\mathscr{O}_X} \Omega^q_X)$$

with differential $d_\Omega : C^q(V, \nabla) \to C^{q+1}(V, \nabla)$ given by

$$d_\Omega \phi(s) = (\nabla \wedge 1_{\Omega^q})\phi(s) + (-1)^{q+1}(\phi \wedge 1_{\Omega^1})\nabla(s)$$

We must be a little careful here, as it is not immediately obvious that d_Ω preserves $\mathscr{O}_X$-linearity. If we write $\phi \in Hom_{\mathscr{O}_X}(V, V \otimes_{\mathscr{O}_X} \Omega^p_X)$ as $\phi = \phi_0 \otimes \phi_\Omega$, with ϕ_0 the part that goes into V and ϕ_Ω the part that goes into Ω^p_X, we get, over each $U \subset X$,

$$\begin{aligned}
d_\Omega \phi(f.s) &= (\nabla \wedge 1_{\Omega^p})\phi(f.s) + (-1)^{p+1}(\phi \wedge 1_{\Omega^1})\nabla(f.s) \\
&= (\nabla \wedge 1_{\Omega^p})(f.\phi(s)) + (-1)^{p+1}(\phi \wedge 1_{\Omega^1})(s \otimes df + f.\nabla(s)) \\
&= \phi_0(s) \wedge df \wedge \phi_\Omega(s) + f.(\nabla \wedge 1_{\Omega^p})(\phi(s)) \\
&\quad + (-1)^{p+1}\phi_0(s) \otimes \phi_\Omega(s) \wedge df + (-1)^{p+1} f.(\phi \wedge 1_{\Omega^1})\nabla(s) \\
&= f.(\nabla \wedge 1_{\Omega^p})\phi(s) + (-1)^{p+1} f.(\phi \wedge 1_{\Omega^1})\nabla(s) \\
&\quad + \phi_0(s) \wedge df \wedge \phi_\Omega(s) + (-1)^{p+1}\phi_0(s) \otimes \phi_\Omega(s) \wedge df.
\end{aligned}$$

By the alternating property of the wedge product, $\phi_0(s) \otimes \phi_\Omega(s) \wedge df = (-1)^p \phi_0(s) \otimes df \wedge \phi_\Omega(s)$, i.e. the last terms cancel, and we have $d_\Omega\phi(f.s) = f.d_\Omega\phi(s)$.

Vanishing of ∇^2 implies

$$\begin{aligned} d_\Omega^2\phi(m) &= (\nabla \wedge 1_{\Omega^{q+1}})d_\Omega\phi(m) + (-1)^{q+2}(d_\Omega\phi \wedge 1_{\Omega^1})\nabla(m) \\ &= (\nabla \wedge 1_{\Omega^{q+1}})(\nabla \wedge 1_{\Omega^q})\phi(m) + (-1)^{q+1}(\nabla \wedge 1_{\Omega^{q+1}})(\phi \wedge 1_{\Omega^1})\nabla(m) \\ &\quad +(-1)^{q+2}((\nabla \wedge 1_{\Omega^q})(\phi \wedge 1_{\Omega^1})\nabla(m) \\ &\quad +(-1)^{2q+3}(\phi \wedge 1_{\Omega^2})(\nabla \wedge 1_{\Omega^1})\nabla(m) = 0 \end{aligned}$$

proving that $(C^q(V,\nabla), d_\Omega)$ is a complex.

Now, we have that $C^q(V|_U, \nabla) = \Gamma(U, \mathscr{H}om(V, V \otimes \Omega_X^q)$ for each $U \subset X$, so we may observe that the collection of groups $\{C^q(V|_U, \nabla)\}_{U \subset X}$ forms the sheaf $\mathscr{H}om(V, V \otimes \Omega_X^q)$. If we let $V = \mathscr{O}_X$, and $\nabla = d$, we obtain the usual de Rham complex

$$0 \to \mathscr{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots,$$

thus our complex is a generalisation of the de Rham complex.

The cup product of this complex is the composition, i.e. $\phi \smile \psi = (\phi \wedge 1) \cdot \psi$.

5.2 The Čech-de Rham double complex

Notice that

$$\prod_{i_0 < \cdots < i_p} C^q(V|_{U_{i_0 \ldots i_p}}, \nabla) = \prod_{i_0 < \cdots < i_p} \Gamma(U_{i_0 \ldots i_p}, \mathscr{H}om(V, V \otimes \Omega_X^q))$$

is the p-th Čech group of the Čech-complex $\check{C}^\bullet(\mathcal{U}, \mathscr{H}om(V, V \otimes \Omega_X^q))$ for a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X. After we check that the differentials commute (they do), this means that we can stitch together the two complexes into a double complex:

$$K^{p,q}(V, \nabla) = \check{C}^p(\mathcal{U}, \mathscr{H}om(V, V \otimes \Omega_X^q))$$

with differential $\partial = d_\Omega + (-1)^q \delta_{\check{C}}$. This will turn out to be the double complex we are looking for. The cup product of this double complex is given by $(a_0, \ldots, a_k) \smile (b_0, \ldots, b_l) = (c_0, \ldots, c_{k+l})$, with $c_n = \sum_{i+j=n} (-1)^{j(k+i)} a_i b_j$.

Denote by $\mathscr{H}^n(V, \nabla)$ the cohomology of the double complex.

6 Deformation of vector bundles with connection

So, at last, we are ready to solve our problem. First, a detail of formality: We work in the category of vector bundles on X equipped with a holomorphic connection. Objects of this category are pairs (V, ∇), where V is a vector bundle on X and ∇ is a holomorphic connection on V. In this category, a morphism $(V, \nabla) \to (V', \nabla')$ is a morphism of vector bundles $\phi : V \to V'$ such that $(\phi \otimes 1) \circ \nabla = \nabla' \circ \phi$.

Now we may begin:

A lift of (V, ∇) to $D := \mathbb{C}[t]/t^2$ consists of a lift V_t of V to D, given by extensions $\hat{\theta}_{ij} = \theta_{ij} + \alpha_{ij}t$ of the transition functions, together with a holomorphic connection ∇_t on V_t, given by extensions $\{\nabla + \nu_i t\}$ of ∇ on each $V(U_i)$. We say two lifts $(V_t, \nabla_t), (V'_t, \nabla'_t)$ are equivalent if there exists an isomorphism $\phi : V_t \to V'_t$, and $\nabla'_t \circ \phi = (\phi \otimes 1) \circ \nabla_t$ (i.e. they are isomorphic as bundles-with-connection). The equivalence classes under this equivalence relation are called *deformations of* (V, ∇) *over* D.

Let us make a short digression on this equivalence relation. First, we observe that any lift V_t (which is an $\mathscr{O}_{X \times \operatorname{Spec} D}$-module) is locally isomorphic to $V \oplus tV$, as an $\mathscr{O}_X$-module. Furthermore, any isomorphism $V_t \xrightarrow{\sim} V'_t$ can, by a change of base, be written locally as $(1 + \phi t) : V \oplus tV \to V \oplus tV$, with ϕ being a local endomorphism of V. Thus, saying that V_t is equivalent to V'_t is the same as saying that the following diagram commutes:

$$\begin{array}{ccc} V_t & \xrightarrow{(1+\phi t)} & V'_t \\ \Big\downarrow{\scriptstyle (\nabla+\nu t)} & & \Big\downarrow{\scriptstyle (\nabla+\nu' t)} \\ V_t \otimes \Omega^1_X & \xrightarrow{(1+\phi t)} & V'_t \otimes \Omega^1_X \end{array}$$

Now, if we write this out, we get the following relation: $\nabla\phi + \nu - \phi\nabla = \nu'$, and so any two extensions of ∇ that differ by a term of the form $\nabla\phi - \phi\nabla$ are equivalent.

The lifting V_t is given (as above) by a 1-cocycle $\{\xi_{ij}\} \in \check{Z}^1(X, \mathscr{E}nd\, V)$, that is, a local endomorphism of V on $V|_{U_{ij}}$, obtained from the transition functions $\hat{\theta}_{ij}$: The extensions should obey the cocycle condition $\hat{\theta}_{ij} \circ \hat{\theta}_{jk} = \hat{\theta}_{ik}$, i.e. $(\theta_{ij} + \alpha_{ij}t) \circ (\theta_{jk} + \alpha_{jk}t) - (\theta_{ik} + \alpha_{ik}t) = 0$. The coefficient of t in this expression is $\theta_{ij} \circ \alpha_{jk} + \alpha_{ij} \circ \theta_{jk} - \alpha_{ik}$, an element in $GL_n(\mathbb{C})$. Recall that θ_{ij} is given by $\phi_i \circ \phi_j^{-1}$, so applying $\phi_i^{-1} \cdot - \cdot \phi_k$ to pull this back to V yields $\phi_i^{-1}\alpha_{ij}\phi_j + \phi_j^{-1}\alpha_{jk}\phi_k - \phi_i^{-1}\alpha_{ik}\phi_k$, a Čech cocycle condition. Thus, $\xi_{ij} = \phi_i^{-1}\alpha_{ij}\phi_j$ is our cocycle. (This is, in disguise, the 'geometric' construction from section 4.)

The extensions of ∇ over each U_i must obey Leibnitz' rule, i.e.

$$(\nabla + \nu_i t)(f.s) = s \otimes df + f.(\nabla + \nu_i t)(s),$$

which leads to the condition that ν_i is $\mathscr{O}_X|_{U_i}$-linear. Now, since V is a locally free $\mathscr{O}_X$-module, we have

$$\begin{aligned} Hom_{\mathscr{O}_X|_{U_i}}(V|_{U_i}, V \otimes \Omega^1_X|_{U_i}) &\simeq Hom_{\mathscr{O}_X|_{U_i}}(\mathscr{O}_X|_{U_i}, \mathscr{E}nd\, V \otimes \Omega^1_X|_{U_i}) \\ &\simeq H^0(U_i, \mathscr{E}nd\, V \otimes \Omega^1_X|_{U_i}) = \Gamma(U_i, \mathscr{E}nd\, V \otimes \Omega^1_X), \end{aligned}$$

so the ν_i's form a Čech 0-cochain $\{\nu_i\}_{i\in I} \in \check{C}^0(\mathcal{U}, \mathscr{E}nd\, V \otimes \Omega^1_X)$. We also note that, viewed as elements of the group $K^{0,1}$ in the double complex defined above, the ν_i's are cocycles under the differential of the generalised de Rham complex, since $\Omega^2_X = 0$.
Now, we need to know how $\{\nu_i\}$ and $\{\xi_{ij}\}$ interact. Since a connection is a global map, we would like the extensions of ∇ over each U_i to be equal, modulo some change of base in the t component. This is precisely what the ξ_{ij}'s represent (a change from U_j-base to U_i-base), so if we conjugate the extensions of ∇ with the extension $\tau_{ij} = Id + \xi_{ij}t$ of identity, we would like the result to vanish. The expression we need is then $\nabla_i\tau_{ij} - \tau_{ij}\nabla_j = (\nabla + \nu_i t)(Id + \xi_{ij}t) - (Id + \xi_{ij}t)(\nabla + \nu_j t)$. The zeroth-order terms cancel, and the coefficient of t is $\nabla\xi_{ij} - \xi_{ij}\nabla + \nu_i - \nu_j$, that is, $d_\Omega\xi - \delta_{\check{C}}\nu$. Since the ξ_{ij}'s and the ν_i's are cocycles of the Čech and de Rham-type complexes, respectively, vanishing of this expression is precisely saying that (ξ, ν) is a 1-cocycle of the Čech-de Rham-complex. Thus we see that a lift of (V, ∇) to D is given by a 1-cocycle (ξ, ν) of the total complex. 1-coboundaries in the total complex give trivial extensions of V, as in section 4, and extensions of ∇ of the type $\nabla \circ \phi - (\phi \otimes 1) \circ \nabla$, with ϕ a local endomorphism of V. Since extensions of this kind are equivalent to the zero extension, they are trivial. Thus, dividing out these gives us the equivalence classes, and we can sum all of this up in the following theorem:

Theorem 6.1. *The deformations of* (V, ∇) *over* D *are in one-to-one correspondence with the first cohomology group* $\mathscr{H}^1(V, \nabla)$*, with the zero element corresponding to the trivial deformation.*

6.1 Obstruction/lifting

Now, suppose we have a deformation of (V, ∇) over D, and want to extend it, as in the example with k-algebras. What conditions do we need? We can answer this question in a more general setting. Let us first say what we mean by a deformation over *any* artinian $\mathbb{C}$-algebra, not just D:

Let $\mathfrak{A}$ be the category of artinian $\mathbb{C}$-algebras, and let (V, ∇) be a vector bundle with a holomorphic connection. A lift of (V, ∇) to $R \in \mathfrak{A}$ is a pair (V_R, ∇^R), such that V_R is a lift of V over R, flat over R, and ∇^R is a connection on V_R, right R-linear, with a morphism $(V_R, \nabla^R) \to (V, \nabla)$ such that the induced map $\eta : (V_R, \nabla^R) \otimes_R \mathbb{C} \to (V, \nabla)$ is an isomorphism. We say two such lifts (V_R, ∇^R), (V'_R, ∇'^R) are equivalent if there exists an isomorphism

$\phi : V_R \to V'_R$ such that $\eta' \circ (\phi \otimes_R 1_{\mathbb{C}}) = \eta$, and $\nabla'^R \circ \phi = (\phi \otimes_{\mathbb{C}} 1_{\Omega^1_X}) \circ \nabla^R$. Let

$$Def_{(V,\nabla)}(R) := \{\text{lifts of } (V,\nabla) \text{ to } R\}/\sim,$$

the set of equivalence classes of lifts of (V,∇) to R. We call such equivalence classes *deformations of* (V,∇) *over* R. The construction of $Def_{(V,\nabla)}(R)$ is functorial, giving a covariant functor $Def_{(V,\nabla)} : \mathfrak{A} \to \mathfrak{Set}$. It is easily seen that this definition reduces to our previous definition for $R = D$.

Now, let (V_S, ∇^S) be a deformation of (V,∇) over $S \in \mathfrak{A}$. Given a *small* morphism $u : R \to S$ in $\mathfrak{A}$, i.e. a morphism with $(\ker u)^2 = 0$ in R, can we extend our lift to R along u? We have the following theorem telling us when and how:

Theorem 6.2. *The obstruction to extending a deformation of* (V,∇) *is an element in* $\mathscr{H}^2(V,\nabla)$.

Proof. Let S be an artinian $\mathbb{C}$-algebra, with a map $\pi : S \to \mathbb{C}$ that sends all the indeterminates to zero. Let (V_S, ∇^S) be a deformation of (V,∇) over S, given by local extensions $\tau^S_{ij} = 1 + \xi^S_{ij}(\ker \pi)$ of the identity map of V, and local extensions $\nabla^S_i = \nabla + \nu^S_i(\ker \pi)$ of ∇. Let $u : R \to S$ be a small morphism in $\mathfrak{A}$. If we want to extend our lift to R through u, we consider that $R = S \oplus (\ker u)$ as a $\mathbb{C}$-vector space, so we can lift our extensions over S to R through a section ω of u, add terms for $\ker u$, and proceed as above: Let $\tau^R_{ij} := \tau^S_{ij} \otimes \omega + \xi^R_{ij}(\ker u) = (1 + \xi^S_{ij}(\ker \pi)) \otimes \omega + \xi^R_{ij}(\ker u)$ and $\nabla^R_i := \nabla^S_i \otimes \omega + \nu^R_i(\ker u) = (\nabla + \nu^S_i(\ker \pi)) \otimes \omega + \nu^R_i(\ker u)$ be arbitrary extensions of τ^S, ∇^S to R, and find the obstruction for the vanishing of the expression $\nabla^R_i \tau^R_{ij} - \tau^R_{ij} \nabla^R_j$. Writing this out in full (and abusing some notation), we have

$$\begin{aligned}&\left(\nabla + \nu^S_i(\ker \pi) + \nu^R_i(\ker u)\right)\left(1 + \xi^S_{ij}(\ker \pi) + \xi^R_{ij}(\ker u)\right)\\ &\qquad - \left(1 + \xi^S_{ij}(\ker \pi) + \xi^R_{ij}(\ker u)\right)\left(\nabla + \nu^S_j(\ker \pi) + \nu^R_j(\ker u)\right).\end{aligned}$$

The first-order terms cancel, so this is equal to

$$\begin{aligned}&\left(\nabla \xi^S_{ij} - \xi^S_{ij}\nabla + \nu^S_i - \nu^S_j\right)(\ker \pi) + \left(\nu^S_i \xi^S_{ij} - \xi^S_{ij}\nu^S_j\right)(\ker \pi)^2\\ &\quad + \left(\nabla \xi^R_{ij} - \xi^R_{ij}\nabla + \nu^R_i - \nu^R_j + \nu^S_i \xi^R_{ij} - \xi^S_{ij}\nu^R_j + \nu^R_i \xi^S_{ij} - \xi^R_{ij}\nu^S_j\right)(\ker u).\end{aligned}$$

Now, in R, $(\ker \pi)^2$ has components in both S and $\ker u$, so we can write this as

$$\begin{aligned}&\left(\nabla \xi^S_{ij} - \xi^S_{ij}\nabla + \nu^S_i - \nu^S_j + \left(\nu^S_i \xi^S_{ij} - \xi^S_{ij}\nu^S_j\right)|_{(\ker \pi)}\right)(\ker \pi) + \left(\nabla \xi^R_{ij} - \xi^R_{ij}\nabla + \nu^R_i - \nu^R_j\right.\\ &\quad \left. + \nu^S_i \xi^R_{ij} - \xi^S_{ij}\nu^R_j + \nu^R_i \xi^S_{ij} - \xi^R_{ij}\nu^S_j + (\nu^S_i \xi^S_{ij} - \xi^S_{ij}\nu^S_j)|_{(\ker u)}\right)(\ker u).\end{aligned}$$

Since we already know that (V_S, ∇^S) is a deformation of (V, ∇) over S, we know that $\nabla_i^S \tau_{ij}^S - \tau_{ij}^S \nabla_i^S = 0$, i.e. the $\ker \pi$ term disappears. Thus, we are left with the $\ker u$ term as the obstruction to lifting the deformation to R. Now, looking at this term, we see that its vanishing is the same as

$$\begin{aligned} - d_{Tot}(\xi^R, \nu^R) &= (\nu_i^S \xi_{ij}^S - \xi_{ij}^S \nu_j^S)|_{(\ker u)} + \nu_i^S \xi_{ij}^R - \xi_{ij}^S \nu_j^R + \nu_i^R \xi_{ij}^S - \xi_{ij}^R \nu_j^S \\ &= \big((\xi^S, \nu^S) \smile (\xi^S, \nu^S)\big)\,|_{(\ker u)} + (\xi^S, \nu^S) \smile (\xi^R, \nu^R) + (\xi^R, \nu^R) \smile (\xi^S, \nu^S), \end{aligned}$$

i.e. the right-hand side is a 2-coboundary in the generalised Čech-de Rham total complex $K^{p,q} = \check{C}^p(\mathcal{U}, \mathscr{H}om(V, V \otimes_{\mathbb{C}} (\ker u) \otimes_{\mathscr{O}_X} \Omega_X^q))$. Now, it is obviously possible, and certainly necessary in order to find the exact relations required, to perform any lift to a higher-order $R \in \mathfrak{A}$ by means of successive smaller lifts of first order, i.e. with $\ker u \simeq \mathbb{C}$. Thus, we see that the obstruction to extending any deformation of (V, ∇) is an element in $\mathscr{H}^2(V, \nabla)$. ☐

7 Epilogue

Where can we go from here? We have now solved one part of the problem of describing the relation between deformations of representations of $\pi_1(X)$ and holomorphic vector bundles, respectively, and can embark on the rest. Unfortunately, what we have just done seems to be by far the easy part of that exercise.

We might also be interested in formulating and solving an analogous problem in a purely algebraic situation, i.e. without resorting to 'dirty tricks' involving differential geometry and complex analysis. However, this requires that we invent a new way of producing a bundle associated to a representation, as the construction we used here depends on said non-algebraic methods, and the existing algebraic method only works for abelian groups. As $\pi_1(X)$ is generally nonabelian, this moves us into the realm of noncommutative geometry, which is a challenge of an entirely different order.

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