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Analysis of the power penalty method for American options using viscosity solutions

by

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Preface

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Chapter 1

Introduction

In this thesis we analyze the power penalty method for American options using viscosity solutions.

An option is a contract that is signed between a *seller*, also called *writer*, and a *buyer* (or *holder*) of the option. The value of the option depends on the underlying asset, i.e. stock, interest rate or even some physical commodities like gold or crude oil. Such a contract offers the right, but not the obligation, to buy or sell a certain asset at a specified price, until or at a certain future date. The right to buy is a call option and the right to sell is a put option.

There are many styles of options which differ in their payoff methods. For example, if the holder may exercise his right only at the exercise date, we call such an option *European*. If, however, the option can be exercised at any time before the exercise date, these options are termed *American*. In this project we will focus on American put and call options.

As compared to the European case, one can find it more difficult to evaluate the American option due to the unknown optimal stopping boundary. Such a boundary separates two regions: the stopping region (where the reward function g and the optimal reward $\hat{g}$ are equal) and the continuous region (where $g > \hat{g}$). As a consequence of this lack of information, it is not possible to predict when it will be optimal to exercise the contract. Therefore, after the European option problem was solved, the valuation of American options began to be a prominent problem in the theory of modern finance.

The first approach for determining the price of an American option was done by McKean [19] in 1965. He transformed the initial problem into a *free boundary problem*. This was a starting point for van Moerbeke [22], who studied properties of the optimal stopping boundary. A few years later, Bensoussan [3] and Karatzas [16, 17], applied the above theory to show that the arbitrage free price of the American option is a solution of the optimal stopping problem.

In 1982, Bensoussan and Lions in [4] developed another important technique, *variational inequalities*, which guided Jaillet, Lamberton and Lapeyre [14] while they were working on the problem. Instead of splitting the region into a continuous and a stopping one, they proposed to solve a certain system of inequalities and equalities. In this way one can determine the American option value and propose the numerical approximation. Although easier to be discretized, the new method was not as explicit as the first one.

In 2003, Benth, Karlsen and Reikvam [5] used a *semilinear Black and Scholes type of equations* to show that the American option value is the only viscosity solution for the corresponding semilinear Black and Scholes equation. The term "Black-Scholes" options pricing model first appeared in Merton's paper [20] and has been usually applied when valuing European options. The notion of the viscosity solution is fundamental for what we understand by the solution of the semilinear Black and Scholes equation, and it will also be used later in this thesis. The main advantage of the notion is that it allows discontinuous functions to be solutions of fully nonlinear equations.

In summary, there is no explicit formula for the value of an American option and the three approaches mentioned above lead to numerical approximations. Moreover, they result in different numerical schemes. The simplest numerical algorithm follows from the third formulation and one popular strategy for deriving it is based on the *penalty method.* However, the classical version of the method has recently (2006) been improved by Wang, Yang and Teo [23] and termed as the *power penalty method.* The improvement implies a replacement of a prominent equation in the classical model by a slightly more nonlinear one. It turns out that these new approximations are more accurate. Here, by the approximation to the value of an American option we understand a sequence of solutions which solve (in a viscosity sense) a penalized equation (equation 3.1, Chapter 2), and which converge (in a weak sense) to the value of the American option.

In this thesis we provide a detailed analysis of the power penalty method using the theory of viscosity solutions. A brief introduction to American call and put options together with the main results for pricing of American options can be found in Chapter 2. In Chapter 3, using theory of viscosity solutions for second order partial differential equations, developed by Ishii, Lions and Crandall in [8], we prove the well-possedness (existence, uniqueness, stability) of penalized equation (equation 3.1, Chapter 2). Chapter 4 is devoted to analyzing the convergence of approximate solutions to the

American option value in the space of locally bounded functions. Finally, the numerical results are presented in Chapter 5. Proposed numerical scheme: *power penalty scheme* is set together and compared with two schemes which have appeared in the literature before. The schemes have been implemented in Matlab and results are displayed at the end of the chapter. The Appendix consists of the definitions and theorems omitted in the main text (App. A) and the Matlab codes (App. B).

Chapter 2

Background

2.1 The American Call and Put Option

Although both the option writer and the option holder sign the contract in hope of achieving a profit, their strategies differ. The reward is the payoff of the option. We consider two payoff functions: the call option and the put option.

The call option represents the holder's right (but not the requirement) to buy the underlying asset before the option reaches its expiration date. The seller is obligated to sell such an asset when the buyer decides. The buyer pays a fee for this right. A call option is purchased in hopes that the stock price will rise above the strike price. The seller either expects that it will not occur, or is willing to lose the profit from a price rise in order to have an opportunity for making a gain up to the strike price.

On the other hand, the put option represents the writer's right (but not the requirement) to sell the underlying asset before the option reaches its expiration date. The buyer is obligated to purchase such an asset, if the writer exercises the option. In exchange for having this right, the writer pays the buyer a fee. The holder believes that it is likely that the price of the underlying asset will drop significantly below the strike price. The writer, on the other hand, does not believe it and sells the put to collect the premium.

Assume we start at time t with a price in the market $X(t) = x$. Let K be a strike price. The payoff function denoted by g, becomes

$$g(x) = \begin{cases} (x - K)^+, & \text{call option,} \\ (K - x)^+, & \text{put option.} \end{cases} \tag{2.1}$$

Roughly speaking, it is worthwhile to buy an asset when the stock price is above the strike price K and, conversely, profitable to sell an asset when the price drops below the strike price.

2.2 The pricing of the American Option

In this section, we give a brief overview of the American option valuation theory. The presentation is based on Young and Zhou [24][Section 7.6], Benth, Karlsen and Reikvam [5] and Myneni [21].

Let $(\Omega, \mathcal{F}, Q)$ be a complete probability space. We equip this space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which we assume to be augmented with all $Q-$null measurable sets in $\mathcal{F}$. We consider a market where the price process of a stock $X(s)$ evolves according to the stochastic differential equation:

$$\begin{cases} dX(s) = (r-d)X(s)ds + \sigma X(s)dW(s), & s \in (t,T], \\ X(t) = x. \end{cases} \tag{2.2}$$

Here, T is a fixed expiration time (*maturity*), $W(s)_{s\in[0,T]}$ is a standard Brownian motion, $d \geq 0$ is the constant dividend yield for the stock, $r \geq 0$ is a risk-free interest rate and $\sigma > 0$ is a volatility. The second equation represents an initial condition with $t < T$ as a starting time. The processes $W(s)$, r, d, σ are assumed to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted. We can equivalently say that the price process of a stock $X(s)$ is governed by a geometric Brownian motion.

Starting at the time t with the initial condition given in (2.2), the arbitrage-free value of an American option with expiration at time T, due to Myneni [21][Theorem 3.1], is given by

$$V(t,x) = \sup_{t\leq\tau\leq T} \mathbb{E}^{t,x}[e^{-r(\tau-t)}g(X(\tau))], \tag{2.3}$$

where the supremum is taken over all $\mathcal{F}_t$ stopping times $\tau \in [t,T]$. Here, $\mathbb{E}^{t,x}$ denotes an expectation under the equivalent martingale measure conditioned on $X(t) = x$ and $g : \mathbb{R} \to \mathbb{R}$ is a payoff function given by (2.1).

Let us motivate the notions used in the result. Since the complexity in valuation of American options follows from the freedom to exercise (for the put) or purchase (for the call) it at any point during the life of the contract, the exercise (or purchase) can be justified only by information up to the present day and does not depend on future stock prices. In such a situation one has to deal with stopping times $\tau \in [t,T]$ (Appendix A, definition A.1.5) with respect to filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Filtration can give us useful information, and can be thought of as a history up to the present time. The notion of arbitrage describes a possibility of riskless profit. A simple example of an arbitrage opportunity is the situation when one offers to pay, in addition to the current price of the option, an amount δ, and the owner of the option agrees. Then the profit δ is earned without risk. However, our model excludes arbitrage opportunity and, instead, the notion of martingale (Appendix A, definition A.1.6) is used to characterize this model. We term the price to be arbitrage-free, if it does not allow us to make riskless profit.

There is no explicit formula whose solution could tell us when to stop. Below we present three main approaches for determining the optimal stopping problem. For simplicity of the notation, we use $\mathcal{L}_{BS}$ to denote the linear Black and Scholes differential operator

$$\mathcal{L}_{BS} = \partial_t + (r-d)x\partial_x + \frac{1}{2}\sigma^2 x^2 \partial_x^2. \tag{2.4}$$

2.2.1 The free boundary problem

In the field of the American pricing problem, McKean [19] is claimed to be the first who discovered the relationship between the optimal stopping problem and a certain free boundary problem. In the first formulation we consider the optimal stopping boundary $X^*(t)$, which separates two regions: the stopping region (where the reward function g and the optimal reward $\hat{g}$ are equal) and the continuous region (where $g > \hat{g}$).

The following theorem derives the Riesz decomposition of the American option value.

Theorem 2.2.1. *The value of the American option (2.3), has the following representation*

$$v(t,x) = p(t,x) + e(t,x),$$

where

$$p(t,x) = \mathbb{E}^{t,x}[e^{-r(T-t)}g(X(T))],$$

$$e(t,x) = \mathbb{E}^{t,x}[\int_t^T e^{-r(u-t)} rK1_{X<X^*}du],$$

with $X(t) = x$.

Proof. For proof see, [21][proof of Corollary 3.1]. □

The decomposition is termed as *the early exercise premium representation.* It consists of the European option value $p(t,x)$ and early exercise premium $e(t,x)$. The second term describes the possibility of being able to stop at any time over the period of the option. However, the disadvantage of the method is that it does not determine the optimal stopping boundary $X^*(t)$.

In the new formulation [5], let $x(t)$ denotes the free boundary and we consider the following sets:

$$C(t) = \begin{cases} (0, x(t)), & \text{call option,} \\ (x(t), \infty), & \text{put option,} \end{cases} \quad S(t) = \begin{cases} [x(t), \infty), & \text{call option,} \\ (0, x(t)], & \text{put option.} \end{cases}$$

The free boundary formulation consists of a partial differential equation, two Dirichlet conditions and a Neumann condition. It determines the unknown free boundary $x(t)$ and a function $v(t, x)$ by solving:

$$\begin{cases} \mathcal{L}_{BS}v(t,x) - rv(t,x) = 0, & t \in [0,T), \quad x \in C(t), \\ v(T,x) = g(x), & x \in [0,\infty), \\ v(t,x) = g(x), & t \in [0,T], \quad x \in S(t), \\ \partial_x v(t,x) = \pm 1, & t \in [0,T], \quad x \in S(t), \end{cases}$$

where the sign in the last equation is positive for a call option and negative for the put. Furthermore, the free boundary $x(t)$ possesses the following properties for $t \in [0, T)$:

$$\begin{cases} x(t) \geq \max(\frac{r}{d}K, K) \text{ (call option,)} & x(t) \leq \min(\frac{r}{d}K, K) \text{ (put option),} \\ x \in C(t) \Leftrightarrow v(t,x) > g(x), & \mathcal{L}_{BS}v(t,x) - rv(t,x) = 0, \\ x \in S(t) \Leftrightarrow v(t,x) = g(x), & \mathcal{L}_{BS}v(t,x) - rv(t,x) < 0. \end{cases} \tag{2.5}$$

Observe that if $d = 0$, the American call option is equal to the European call option with the same strike price. On the other hand, if $r = 0$, the American put option is equal to the European one, analogously, with the same price. In both cases we do not need to calculate the free boundary.

2.2.2 The quasi-variational inequalities

The formulation in terms of variational inequalities is based on the work of Bensoussan and Lions [4]. It allows us to treat the domain of the option in the entire region without use of the stopping boundary $X^*(t)$. The American option value (2.3) can be determined by solving the quasi-variational inequality:

$$\begin{cases} \max\big(\mathcal{L}_{BS}v(t,x) - rv(t,x), g(x) - v(t,x)\big) = 0, & (t,x) \in \Omega_T, \\ v(T,x) = g(x), & x \in [0,\infty), \end{cases} \tag{2.6}$$

where Ω_T denotes the time-space cylinder $\Omega_T = (0, T) \times (0, \infty)$. Equivalently, in the view of the article by Jaillet, Lamberton and Lapeyre [14], we can stated (2.6) as:

$$\begin{aligned} \mathcal{L}_{BS}v(t,x) - rv(t,x) &\leq 0, \\ v(t,x) &\geq g(x), \\ v(T,x) &= g(x), \\ (v(t,x) - g(x))(\mathcal{L}_{BS}v(t,x) - rv(t,x)) &= 0. \end{aligned}$$

The quasi-variational inequalities might be studied in the sense of viscosity solutions and we will refer to them in the following chapters.

2.2.3 The semilinear Black and Scholes equation

The new approach was proposed by Benth, Karlsen and Reikvam [5] and it made use of the assumption on the American option value v to be $C^{1,2}(\Omega_T) \cap C(\overline{\Omega}_T)$ regular. For $\alpha, \beta \geq 0$, $C^{\alpha,\beta}(\Omega_T)$ denotes the space of functions defined on Ω_T which are α-times continuously differentiable in t, and β-times continuously differentiable in x.

The following relations were proven to hold for the American option:

$$v \geq g, \quad \mathcal{L}_{BS}v - rv \leq 0, \quad (v - g)(\mathcal{L}_{BS}v - rv) = 0.$$

From this it follows that:

$$\mathcal{L}_{BS}v(t,x) - rv(t,x) = 0, \qquad \text{when } v(t,x) > g(x), \tag{2.7}$$

and

$$\mathcal{L}_{BS}v(t,x) - rv(t,x) \leq 0, \qquad \text{when } v(t,x) = g(x). \tag{2.8}$$

Here, (2.7) corresponds to continuation region and (2.8) to the exercise region. Since $\mathcal{L}_{BS}v - rv$ is nonpositive in the exercise region, the lower bound is required as well, in addition to the upper bound. It has been derived by Benth, Karlsen and Reikvam [5] in the following way. Fix a point (t, x) in the exercise region. From assumption on v to be a function from $C^{1,2}(\Omega_T)$, it follows that v "touches" g from above at (t, x). This means that $v(t, x) = g(x)$ and $v \geq g$ everywhere, and (t, x) is a local minimizer of $g - v$. Since the payoff function cannot be touched by a $C^{1,2}(\Omega_T)$ function at $x = K$ (a kink place), the conclusion is that either $x < K$ or $x > K$.

For the call option, i.e. $g(x) = (x - K)^+$, if $x < K$, then

$$\partial_t v(t,x) = 0, \quad \partial_x v(t,x) = 0, \quad \partial_x^2 v(t,x) \geq 0,$$

and therefore

$$\mathcal{L}_{BS}v(t,x) - rv(t,x) \geq 0. \tag{2.9}$$

Combining (2.8) and (2.9),

$$\mathcal{L}_{BS}v(t,x) - rv(t,x) = 0 \quad \text{when} \quad v(t,x) = g(x) \text{ and } x < K. \tag{2.10}$$

On the other hand, assuming $x > K$ we have

$$\partial_t v(t,x) = 0, \quad \partial_x v(t,x) = 1, \quad \partial_x^2 v(t,x) \geq 0,$$

and therefore

$$\mathcal{L}_{BS}v(t,x) - rv(t,x) \geq -(dx - rK). \tag{2.11}$$

In a view of (2.8) and (2.11), $(dx - rK)$ should not be negative and therefore:

$$\mathcal{L}_{BS}v(t,x) - rv(t,x) \geq -(dx - rK)^+ \quad \text{when } v(t,x) = g(x) \text{ and } x > K.$$

Summarizing, in the exercise region the American call option value satisfies:

$$-(dx - rK)^+ \leq \mathcal{L}_{BS}v(t,x) - rv(t,x) \leq 0, \qquad \text{when } v(t,x) = g(x), \quad (2.12)$$

and similarly, the American put option value satisfies:

$$-(rK - dx)^+ \leq \mathcal{L}_{BS}v(t,x) - rv(t,x) \leq 0, \qquad \text{when } v(t,x) = g(x). \quad (2.13)$$

In fact the left inequalities in (2.12)-(2.13) become equalities, if we take into account the properties (2.5) about the free boundary. Below we present the main result of [5] for determining the value of the American option. Its proof is based on the dynamic programming principle.

Let $c : \mathbb{R} \to [0, \infty)$ be a "cash flow" function define by

$$c(x) = \begin{cases} (dx - rK)^+, & \text{call option,} \\ (rK - dx)^+, & \text{put option.} \end{cases} \quad (2.14)$$

Define also $H : \mathbb{R} \to [0, \infty)$ a Heaviside function:

$$H(\zeta) = \begin{cases} 0, & \zeta < 0, \\ 1, & \zeta \geq 0. \end{cases} \quad (2.15)$$

Theorem 2.2.2. *The function V defined by (2.3) uniqually solves (in viscosity solutions sense) the following semilinear Black and Scholes equation set:*

$$\mathcal{L}_{BS}v(t,x) - rv(t,x) = -q(x,v), \quad (2.16)$$

where $q : \mathbb{R} \times R \to [0, \infty)$ is a nonlinear reaction term given by the formula

$$q(x,v) = c(x)H(g(x) - v), \quad (2.17)$$

and $(t,x) \in \Omega_T$. We augment (2.16) with the terminal condition

$$v(T,x) = g(x). \quad (2.18)$$

Proof. For proof, see [5][proof of Theorem 3.1]. □

Chapter 3

Power Penalty Approach

3.1 Power penalty method

Power penalty method provides an approximation to the American option value, and can be motivated from each of the three approaches described in the section 2.2.

Let $0 < k \leq 1$ be a parameter and let $[a]^+ = \max\{a, 0\}$.

Definition 3.1.1. *We call a sequence $\{v_k\}_k$ a power penalty approach, if the functions v_k solve (in a viscosity sense) sequence of nonlinear partial differential equations*

$$\mathcal{L}_{BS} v_\epsilon - r v_\epsilon = -\frac{1}{\epsilon}[(g(x) - v_\epsilon)^+]^{1/k}, \quad (t, x) \in \Omega_T, \tag{3.1}$$

where $\epsilon > 0$ is the penalty constant, $\mathcal{L}_{BS}$ the same as in (2.4). We augment (3.1) with the boundary and terminal conditions

$$v_\epsilon(t, 0) = 0 \quad t \in (0, T) \quad \text{and} \quad v_\epsilon(T, x) = g(x) \quad x \in [0, \infty). \tag{3.2}$$

When $k = 1$, (3.1) reduces to the classical penalty method. From the article of Benth, Karlsen and Reikvam [5] we learn that for $k = 1$ there exists an unique viscosity solution v_ϵ of (3.1)-(3.2) and from their succeeding article [6] we know that as $\epsilon \to 0$, v_ϵ converges uniformly to the unique viscosity solution v of (2.16)-(2.18) (i.e. the American option value). On the other hand, it is known from Wang, Yang and Teo [23] that the classical penalty method is not very accurate and the converges rate is of order $\mathcal{O}(\epsilon^{1/2})$. One can only achieve an intended accuracy of the approximate solution if ϵ is sufficiently small. It does not have to be the case in computational implementations, where small ϵ leads to numerical errors. The improvement of the classical method was proposed by Wang, Yang and Teo in [23]. In Chapter 4 we prove, using the viscosity solution method, that the convergence rate for the power penalty approach (3.1)-(3.2) is of the order $\mathcal{O}(\epsilon^{k/2})$.

Let us briefly describe the usage of the nonlinear term $\frac{1}{\epsilon}[(g(x)-v_\epsilon)^+]^{1/k}$ on the right hand side of equation (3.1), namely *the penalty term*. If $g(x) \leq v_\epsilon$, then (3.1) reduces to

$$\mathcal{L}_{BS}(t,x) - rv_\epsilon(t,x) = 0.$$

If $g(x) > v_\epsilon$, then

$$g(x) - v_\epsilon = \epsilon^k(-\mathcal{L}_{BS}v(t,x) + rv_\epsilon(t,x))^k.$$

Thus when ϵ is sufficiently small and $(-\mathcal{L}_{BS}v(t,x) + rv_\epsilon(t,x))$ is bounded, then $[g(x) - v_\epsilon]^+ \approx 0$. This way the positive part of $g(x) - v_\epsilon$ is penalized in both cases.

3.2 Viscosity solutions

In this section we clarify what we mean by the viscosity solution of the penalized semilinear Black and Scholes equation (3.1). The following review is based on the presentation in Benth, Karlsen and Reikvam [5, page 288-290] and Crandall and Lions [8, page 9-11,49].

The notion of viscosity solutions was previously introduced for nonlinear first-order PDE's by Crandall and Lions [9] and later extended to a large class of fully nonlinear second-order PDE's. The main idea of the notion is to put derivatives onto a smooth test function, in strength of the maximum principle for semicontinuous functions [7, 8]. This method was studied earlier in Evans' papers [10, 11]. In view of the work by Benth, Karlsen and Reikvam [5], one can combine the theory of viscosity solutions for second order PDE's [8] together with the dynamic programming principle in the optimal stopping theory. In this way, the weak solution for the semilinear Black and Scholes type equation can be interpreted as a viscosity solution.

Let us begin with the notion of the following spaces of semicontinuous functions on $\overline{\Omega}_T = [0,T] \times [0,\infty)$:

$$USC(\overline{\Omega}_T) = \{u : \overline{\Omega}_T \to R \cup \{-\infty\} | \text{ u is upper semicontinuous}\},$$

$$LSC(\overline{\Omega}_T) = \{u : \overline{\Omega}_T \to R \cup \{+\infty\} | \text{ u is lower semicontinuous}\}.$$

As on analogue to Benth, Karlsen and Reikvam [5][Section 4] we introduce the following notions:

Definition 3.2.1. **(i)** *A locally bounded function $u_\epsilon \in USC(\overline{\Omega}_T)$ is a viscosity subsolution of (3.1) if and only if for all $\phi \in C^{1,2}(\overline{\Omega}_T)$ we have:*

$$\begin{cases} \text{for each } (t,x) \in \Omega_T \text{ being a maximizer of } u_\epsilon - \phi, \\ \mathcal{L}_{BS}\phi(t,x) - ru_\epsilon + \frac{1}{\epsilon}[(g(x) - u_\epsilon)^+]^{1/k} \geq 0. \end{cases}$$

If, in addition, $u_\epsilon(t,x) \leq 0$ for $t \in [0,T)$ and $u_\epsilon(T,x) \leq g(x)$ for $x \in [0,\infty)$, then u_ϵ is a viscosity subsolution of (3.1)-(3.2).

(ii) *A locally bounded function $u_\epsilon \in LSC(\overline{\Omega}_T)$ is a viscosity supersolution of (3.1) if and only if for all $\phi \in C^{1,2}(\overline{\Omega}_T)$ we have:*

$$\begin{cases} \text{for each } (t,x) \in \Omega_T \text{ being a minimizer of } u_\epsilon - \phi, \\ \mathcal{L}_{BS}v_\epsilon\phi(t,x) - ru_\epsilon + \frac{1}{\epsilon}[(g(x) - u_\epsilon)^+]^{1/k} \leq 0. \end{cases}$$

If, in addition, $u_\epsilon(t,x) \geq 0$ for $t \in [0,T)$ and $u_\epsilon(T,x) \geq g(x)$ for $x \in [0,\infty)$, then u_ϵ is a viscosity supersolution of (3.1)-(3.2).

(iii) *A function $u_\epsilon \in C(\overline{\Omega}_T)$ is a viscosity solution of (3.1) if, at the same time, it is a sub- and supersolution. If, in addition, $u_\epsilon(t,x) = 0$ for $t \in [0,T)$ and $u_\epsilon(T,x) = g(x)$ for $x \in [0,\infty)$, then u_ϵ is a viscosity solution of (3.1)-(3.2).*

In proving the uniqueness result for viscosity solutions of second order equations, it is convenient to give the equivalent formulations of sub- and supersolutions based on so-called *semijets*.

Definition 3.2.2. **(i)** *For the function $u \in USC(\overline{\Omega}_T)$, $(u \in LSC(\overline{\Omega}_T))$, the second order superjet (subjet) of u at $(t,x) \in \Omega_T$, which is denoted by $\mathcal{P}^{2,+}u(t,x)$ $(\mathcal{P}^{2,-}u(t,x))$, is defined as the set of triples $(a,p,X) \in R^3$ such that*

$$u(s,y) \leq (\geq) u(t,x) + a(s-t) + p(y-x) + \frac{1}{2}X(y-x)^2 \\ + o(|s-t| + |y-x|^2) \text{ as } 0 \ni (s,y) \to (t,x).$$

(ii) *The closure $\overline{\mathcal{P}}^{2,+}u(t,x)$ $(\mathcal{P}^{2,-}u(t,x))$ is the set of triples $(a,p,X) \in R^3$ for which there exists a sequence $(t^n, x^n, p^n, X^n) \in R^4$ such that $(x^n, p^n, X^n) \in \mathcal{P}^{2,+}u(t^n,x^n)$ $(\mathcal{P}^{2,-}u(t^n,x^n))$ and $(t^n,x^n,p^n,X^n) \to (t,x,p,X)$ as $n \uparrow \infty$.*

According to Crandall, Ishii and Lions [8][page 11], the following is true:

$$\mathcal{P}^{2,+}u(t,x) = \{(\partial_t\Phi, \partial_x\Phi, \partial_x^2\Phi) : \Phi \in C^{1,2}(\overline{\Omega}_T), u - \Phi \text{ has a local maximum} \\ \text{at } (t,x), \text{ and } \Phi(t,x) = u(t,x), \partial_t\Phi(t,x) = a, \partial_x\Phi(t,x) = p, \\ \partial_x^2\Phi(t,x) = X\}, \quad (3.3)$$

$$\mathcal{P}^{2,-}u(t,x) = \{(\partial_t\Phi, \partial_x\Phi, \partial_x^2\Phi) : \Phi \in C^{1,2}(\overline{\Omega}_T), u-\Phi \text{ has a local minimum at } (t,x), \text{ and } \Phi(t,x) = u(t,x), \partial_t\Phi(t,x) = a, \partial_x\Phi(t,x) = p, \partial_x^2\Phi(t,x) = X\}. \quad (3.4)$$

In a view of the result we can write the equivalent definitions of sub- and supersolutions based on *semijets*:

Definition 3.2.3. (i) *A locally bounded function $u_\epsilon \in USC(\overline{\Omega}_T)$ is a viscosity subsolution of (3.1), if and only if, for all $(t,x) \in \Omega_T$ and for all $(a,p,X) \in P^{2,+}u(t,x)$ we have:*

$$a + (r-d)xp + \frac{1}{2}\sigma^2x^2X - ru_\epsilon + \frac{1}{\epsilon}[(g(x) - u_\epsilon)^+]^{1/k} \geq 0.$$

If, in addition, $u_\epsilon(t,x) \leq 0$ for $t \in [0,T)$ and $u_\epsilon(T,x) \leq g(x)$ for $x \in [0,\infty)$, then u is a viscosity subsolution of (3.1)-(3.2).

(ii) *A locally bounded function $u_\epsilon \in LSC(\overline{\Omega}_T)$ is a viscosity supersolution of (3.1), if and only if, for all $(t,x) \in \Omega_T$ and for all $(a,p,X) \in P^{2,-}u(t,x)$ we have:*

$$a + (r-d)xp + \frac{1}{2}\sigma^2x^2X - ru_\epsilon + \frac{1}{\epsilon}[(g(x) - u_\epsilon)^+]^{1/k} \leq 0.$$

If, in addition, $u_\epsilon(t,x) \leq 0$ for $t \in [0,T)$ and $u_\epsilon(T,x) \leq g(x)$ for $x \in [0,\infty)$, then u is a viscosity supersolution of (3.1)-(3.2).

For the later purpose we recall also a definition of so called *semicontinuous envelopes.*

Definition 3.2.4. *For any function $u : \Omega_T \to R$ we define $u^* : \overline{\Omega}_T \to R \cup \{+\infty, -\infty\}$ by*

$$u^*(x) = \lim_{r\downarrow 0} \sup\{u(y) : y \in \overline{B}(x;r) \cap \Omega_T\} \quad \textit{for } x \in \overline{\Omega}_T. \quad (3.5)$$

and $u_* : \overline{\Omega}_T \to R \cup \{+\infty, -\infty\}$ by

$$u_*(x) = \lim_{r\downarrow 0} \inf\{u(y) : y \in \overline{B}(x;r) \cap \Omega_T\} \quad \textit{for } x \in \overline{\Omega}_T. \quad (3.6)$$

Functions u^* and u_* are called, respectively, the upper and lower semicontinuous envelopes of u.

Note that $u_* = -(-u)^*$ and $u_* \leq u \leq u^*$, so if u is uppersemicontinuous at $x \in \Omega_T$ then $u^*(x) = u(x)$.

3.3 Comparison principle

The remaining part of this Chapter focuses on establishing the well-possedness of viscosity solutions for (3.1)-(3.2), in view of the theory developed by Ishii, Lions and Crandall [8]. Our discussion is divided into two parts. First we focus on the comparison result which yields uniqueness. The critical step in its proof relies on the maximum principle for semicontinuous functions (see, Theorem 3.3.1). Moreover, once the comparison principle is satisfied, Perron's Method (see, Theorem 3.5.1) provides the existence of viscosity solution.

Let us consider a more general case and think of (3.1)-(3.2) as a terminal-value problem without a boundary condition. Later, for completeness, we shall assume an additional condition. Consider then:

$$\begin{cases} -\partial_t v_\epsilon(t,x) + F_\epsilon(t,x,v_\epsilon(t,x),\partial_x v_\epsilon(t,x),\partial_x^2 v_\epsilon(t,x)) = 0, \ (t,x)\in\Omega_T, \\ \qquad\qquad\qquad\qquad\qquad\qquad v_\epsilon(T,x) = g(x), \ \ x\in(0,\infty), \end{cases} \tag{3.7}$$

where $F_\epsilon : \overline{\Omega}_T \times R \times R \times R \to R$ denotes a second order operator of the form:

$$F_\epsilon(t,x,s,p,X) = -(r-d)xp - \frac{1}{2}\sigma^2 x^2 X + rs - \frac{1}{\epsilon}[(g(x)-s)^+]^{1/k}. \tag{3.8}$$

Corollary 3.3.1. *The operator F_ϵ is continuous and proper.*

Proof. It is clear that F_ϵ is continuous. To show the properness property, let $s, z, X, Y \in \mathbb{R}$ with $s \le z$ and $Y \le X$. Then

$$\begin{aligned} F_\epsilon(t,x,s,p,X) - F_\epsilon(t,x,z,p,Y) &= 1/2\sigma^2x^2(Y-X) + r(s-z) \\ &+ 1/\epsilon[(g(x)-z)^+]^{1/k} - 1/\epsilon[(g(x)-s)^+]^{1/k} \le 0, \end{aligned}$$

where we have used $(g(x)-z)^+ \le (g(x)-s)^+$. □

We seek to prove the comparison result for (3.7)-(3.8) in a class of viscosity sub- and supersolutions which satisfy a natural growth condition. Let us first recall the basic concept of maximum principle. We follow the discussion given in [8, Chapter 3].

In the classical case, where u and v are classical sub- and supersolutions of (3.7)-(3.8), if the function define in the neighborhood of $\hat{y} = (\hat{t},\hat{x}) \in \Omega_T$ by

$$w(y) = w(\hat{y}) + Dw(\hat{y})(y-\hat{y}) + 1/2D^2w(\hat{y})(y-\hat{y})^2 + o(|y-\hat{y}|) \quad y \to \hat{y},$$

is twice differentiable at a local maximum $\hat{y}$, then $\partial_x w(\hat{y}) = 0$, $\partial_t w(\hat{y}) = 0$ and $\partial_x^2 w(\hat{y}) \le 0$. Thus, for $w = u - v$, we would have

$$\partial_x u(\hat{y}) = \partial_x v(\hat{y}), \quad \partial_t u(\hat{y}) = \partial_t v(\hat{y}) \quad \text{and} \quad \partial_x^2 u(\hat{y}) \le \partial_x^2 v(\hat{y}).$$

Furthermore,

$$F_\epsilon(\hat{y}, u(\hat{y}), \partial_y u(\hat{y}), \partial_x^2 u(\hat{y})) \leq 0 \leq F_\epsilon(\hat{y}, v(\hat{y}), \partial_x v(\hat{y}), \partial_x^2 v(\hat{y})) \\ \leq F_\epsilon(\hat{y}, v(\hat{y}), \partial_x u(\hat{y}), \partial_x^2 u(\hat{y})),$$

where the first two inequalities follow from the definition of sub- and super-solution of (3.7)-(3.8), and the last inequality from the degenerate ellipicity of F_ϵ in a last argument. Since F_ϵ is nondecreasing in the second variable, it follows that $u \leq v$ in Ω_T.

On the other hand, in the case where $u \in USC(\overline{\Omega}_T)$ and $v \in LSC(\overline{\Omega}_T)$, the pairs $(Du(\hat{y}), D^2u(\hat{y}))$ and $(Dv(\hat{y}), D^2v(\hat{y}))$ are replaced by the set-valued functions $P^{2,+}u$, $P^{2,-}v$. These sets can be empty at many points, also at the maximum points of $u - v$. Therefore, to overcome the lack of the regularity for u and v, we apply the classical "doubling of variables" device [8]. Roughly speaking, by subtracting extra term which depends on the parameter,

$$\Phi(t, x, y) = u(t, x) - v(t, y) - \alpha\psi(t, x, y), \tag{3.9}$$

we ensure existence of the finite maximum point for $\Phi(t, x, y)$. We approximate maximalization of $u(t, x) - v(t, x)$ over $[0, T) \times [0, \infty)$ by letting $\alpha \to \infty$. To this end, we will apply the following lemma:

Lemma 3.3.1. *Let O be a subset of R, $\Phi \in USC(O)$, $\Psi \in LSC(O)$ and*

$$N_\alpha = \sup_O(\Phi(x) - \alpha\Psi(x)) \text{ for } \alpha > 0. \tag{3.10}$$

Let $-\infty < \lim_{\alpha\to\infty} N_\alpha < \infty$ and $x_\alpha \in 0$ be chosen so that

$$\lim_{\alpha\to\infty}(N_\alpha - (\Phi(x_\alpha) - \alpha\Psi(x_\alpha))) = 0. \tag{3.11}$$

Thus the following holds:

$$\begin{cases} (i) \quad lim_{\alpha\to\infty}\alpha\Psi(x_\alpha) = 0, \\ (ii) \quad \psi(\hat{x}) = 0 \text{ and } lim_{\alpha\to\infty}N_\alpha = \Phi(\hat{x}) = \sup_{\{\Psi(x)=0\}}\Phi(x) \\ \quad\quad \text{whenever } \hat{x} \in O \text{ is a limit point of } x_\alpha \text{ as } \alpha \to \infty. \end{cases} \tag{3.12}$$

Proof. For the proof, see [8, proof of Proposition 3.7]. □

The next theorem, the *maximum principle for semicontinuous functions* applies in several settings, and we refer to its parabolic analogue given in [8, Theorem 8.3]. The theorem is restated here in a form suitable to our application.

Theorem 3.3.1. *Let $u(t,x)$, $-v(t,y)$ be the upper semicontinuous functions in $(0,T)\times R$ and let ϕ be defined on $(0,T)\times R\times R$, such that $(t,x,y)\to\phi(t,x,y)$ is once continuously differentiable in t and twice continuously differentiable in (x,y). Suppose that $\hat{t}\in(0,T)$, $\hat{x},\hat{y}\in R$ and*

$$\omega(t,x,y)\equiv u(t,x)-v(t,y)-\phi(t,x,y)\le\omega(\hat{t},\hat{x},\hat{y}), \tag{3.13}$$

for $0<t<T$ and $x,y\in R$. Assume, moreover, that there is $r>0$ such that for every $M>0$ there is $C>0$ such that

$$\begin{cases} a\le C \text{ whenever } (a,p,X)\in P^{2,+}u(t,x), \\ |x-\hat{x}|+|t-\hat{t}|\le r \text{ and } |u(t,x)|+|p|+|X|\le M, \end{cases}$$

$$\begin{cases} b\le C \text{ whenever } (b,q,Y)\in P^{2,-}v(t,y), \\ |y-\hat{y}|+|t-\hat{t}|\le r \text{ and } |v(t,y)|+|q|+|Y|\le M. \end{cases}$$

Then for each $k>0$ there are two symmetric 2×2 matrices X,Y such that

$$\begin{cases} (i) \quad (a,\partial_x\phi(\hat{t},\hat{x},\hat{y}),X)\in\overline{P}^{2,+}u(\hat{t},\hat{x}) \\ \quad (b,\partial_y\phi(\hat{t},\hat{x},\hat{y}),Y)\in\overline{P}^{2,-}v(\hat{t},\hat{y}) \\ (ii) \quad -(\frac{1}{k}+\|A\|)I\le\begin{pmatrix} X & 0 \\ 0 & -Y\end{pmatrix}\le A+kA^2, \\ (iii) \quad a-b=\phi_t(\hat{t},\hat{x},\hat{y}), \end{cases} \tag{3.14}$$

where $A=(D_x^2\phi)(\hat{t},\hat{x},\hat{y})$ and $\|A\|=sup\{|\langle A\zeta,\zeta\rangle|:\zeta\in R^2,|\zeta|=1\}$.

Proof. For the main idea of the proof, we refer to [8, Theorem 3.2]. □

Theorem 3.3.2 (Comparison principle). *Let $\epsilon\in(0,1)$ and $k\ge 1$ be fixed. Let u be a subsolution of (3.7)-(3.8) and v be a supersolution of (3.7)-(3.8) such that*

$$u(t,x),-v(t,x)\le L(|x|+1) \text{ and } u(T,x)\le v(T,x)$$

for some constant $L>0$. Then $u\le v$ in $\overline{\Omega}_T$.

Proof. The proof follows [8, Theorem 8.2] and [5, Theorem 5.3]. Let $\eta>0$ and $\tilde{u}$ be a function define by

$$\tilde{u}=u+\eta/(T-t) \quad (x\in[0,\infty), 0\le t<T).$$

Observe that by monotonicity of F_ϵ, $\tilde{u}$ is a subsolution of (3.7)-(3.8) and satisfies the parabolic equation in (3.7) with a strict inequality. Indeed, since F_ϵ is nondecreasing in "the third" variable and u is a subsolution of (3.7) we have that

$$-\tilde{u}_t+F_\epsilon(t,x,\tilde{u},\partial_x\tilde{u},\partial_x^2\tilde{u})\le -u_t-\eta/(T-t)^2+F_\epsilon(t,x,u,\partial u,\partial_x^2 u)\le-\eta/(T-t)^2.$$

Note also that as $\eta \downarrow 0$, $\tilde{u} \le v$ implies $u \le v$. It therefore suffices to prove the comparison under the additional assumptions

$$\begin{cases} -u_t + F_\epsilon(t,x,u,Du,D^2u) \le -\eta/T^2 < 0, & \text{in } [0,T) \times [0,\infty), \\ \lim_{t\uparrow T} u(t,x) = \infty & \text{uniformly on } [0,\infty). \end{cases} \tag{3.15}$$

Let us suppose that comparison does not hold and that for some $(\bar{t},\bar{x}) \in [0,T) \times [0,\infty)$

$$u(\bar{t},\bar{x}) - v(\bar{t},\bar{x}) = 2\delta > 0. \tag{3.16}$$

We employ the classical "doubling of variables" device [8] to overcome the lack of regularity of u and v, and we maximize

$$\Phi(t,x,y) = u(t,x) - v(t,y) - \frac{\alpha}{2}|x-y|^2 - \frac{\mu}{2}e^{\lambda(T-t)}(|x|^2 + |y|^2), \tag{3.17}$$

over $[0,T) \times [0,\infty) \times [0,\infty)$; $\mu \in (0,1)$, $\alpha, \lambda > 1$ are parameters. We approximate maximalization of $u(t,x) - v(t,x)$ over $[0,T) \times [0,\infty)$ by letting $\alpha \to \infty$ and $\mu \downarrow 0$. Let

$$M_\alpha = \sup_{[0,T)\times[0,\infty)\times[0,\infty)} \Phi(t,x,y). \tag{3.18}$$

By the assumed linear growth of u and v and from the upper semicontinuity of Φ it follows that $M_\alpha < \infty$ and that there exists $(t_\alpha, x_\alpha, y_\alpha) \in [0,T) \times [0,\infty) \times [0,\infty)$ such that $M_\alpha = \Phi(t_\alpha, x_\alpha, y_\alpha)$. Moreover, if μ is small enough then M_α is a positive number, since

$$M_\alpha \ge u(\bar{t},\bar{x}) - v(\bar{t},\bar{x}) - \mu e^{\lambda(T-t)}\bar{x}^2 \ge \delta > 0. \tag{3.19}$$

This implies:

$$u(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha) \ge \delta, \tag{3.20}$$

for any $\alpha > 1$ and μ sufficiently small. Next, we use $\Phi(T,0,0) \le \Phi(t_\alpha, x_\alpha, y_\alpha)$ and the linear growth of u and v to find that

$$\frac{\mu}{2}(x_\alpha^2 + y_\alpha^2) \le v(T,0) - v(t_\alpha, y_\alpha) + u(t_\alpha, x_\alpha) - u(T,0) \le C + 2L(1 + x_\alpha + y_\alpha). \tag{3.21}$$

Applying Cauchy's inequality [Appendix A.2] with μ, we conclude that there is a finite constant C_μ, which depends on μ, such that $x_\alpha, y_\alpha \le C_\mu$. Moreover, for each fixed μ there exists a subsequence, denoted by $(t_\alpha, x_\alpha, y_\alpha)$, which converges to some $(\hat{t}, \hat{x}, \hat{y}) \in [0,T) \times [0,\infty) \times [0,\infty)$ as $\alpha \uparrow \infty$. Let μ be small and let us apply lemma 3.3.1 via the following correspondence: $(x,y) \to x$, $u(t,x) - v(t,y) \to \Phi(x)$, $(1/2)|x-y|^2 \to \Psi(x)$. Then, for each fixed μ, the maxima $(t_\alpha, x_\alpha, y_\alpha)$ satisfy

$$\begin{cases} x_\alpha - y_\alpha \to 0 \text{ as } \alpha \uparrow \infty, \\ \alpha|x_\alpha - y_\alpha|^2 \to 0 \text{ as } \alpha \uparrow \infty. \end{cases} \tag{3.22}$$

If $\hat{t} = T$, then by (3.19)

$$0 < \delta \leq \Phi(\bar{t}, \bar{x}) \leq M_\alpha \leq u(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha). \tag{3.23}$$

Based on the upper semicontinuity of u, $-v$ we let $\alpha \uparrow \infty$, $\mu \downarrow 0$ and recall that $(u - v)|_{t=T} \leq 0$ on $[0, \infty)$. This contradicts the fact that δ is positive, and therefore we may assume that $\hat{t} \in [0, T)$ and $t_\alpha \in [0, T)$, if α is large. We apply Theorem 3.3.1 at $(t_\alpha, x_\alpha, y_\alpha)$, to get numbers a_α, b_α and X_α, Y_α (if we choose $k = 1/\alpha$ in Theorem 3.3.1) such that

$$(a_\alpha, \alpha(x_\alpha - y_\alpha) + \mu e^{\lambda(T-t_\alpha)} x_\alpha, X_\alpha) \in \overline{P}^{2,+} u(t_\alpha, x_\alpha),$$

$$(b_\alpha, \alpha(x_\alpha - y_\alpha) - \mu e^{\lambda(T-t_\alpha)} y_\alpha, Y_\alpha) \in \overline{P}^{2,-} v(t_\alpha, y_\alpha),$$

and $a_\alpha - b_\alpha = -\frac{\mu}{2}\lambda e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2)$. Moreover, the matrix A in Theorem 3.3.1 takes the form:

$$A = \begin{pmatrix} \alpha + \mu e^{\lambda(T-t_\alpha)} & -\alpha \\ -\alpha & \alpha + \mu e^{\lambda(T-t_\alpha)} \end{pmatrix} \tag{3.24}$$

and thus the following inequalities are satisfied:

$$\begin{aligned} \begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix} &\leq (3\alpha + 2\mu e^{\lambda(T-t)}) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &\quad + (\mu e^{\lambda(T-t_\alpha)} + \frac{\mu^2 e^{2\lambda(T-t_\alpha)}}{\alpha}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.25}$$

By definition of the viscosity sub- and supersolutions,

$$\begin{aligned} &- a_\alpha + F_\epsilon(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), \alpha(x_\alpha - y_\alpha) + \mu e^{\lambda(T-t_\alpha)} x_\alpha, X_\alpha) \leq -\eta/T^2, \\ &\quad - b_\alpha + F_\epsilon(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), \alpha(x_\alpha - y_\alpha) - \mu e^{\lambda(T-t_\alpha)} y_\alpha, Y_\alpha) \geq 0. \end{aligned} \tag{3.26}$$

Combining the above inequalities we obtain:

$$\begin{aligned} 0 < \eta/T^2 \leq &-\frac{\mu}{2}\lambda e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2) \\ &+ F_\epsilon(t_\alpha, y_\alpha, v(t_\alpha, y_\alpha), \alpha(x_\alpha - y_\alpha) - \mu e^{\lambda(T-t_\alpha)} y_\alpha, Y_\alpha) \\ &- F_\epsilon(t_\alpha, x_\alpha, u(t_\alpha, x_\alpha), \alpha(x_\alpha - y_\alpha) + \mu e^{\lambda(T-t_\alpha)} x_\alpha, X_\alpha) \\ &= -\frac{\mu}{2}\lambda e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2) + \Delta(\alpha). \end{aligned}$$

We obtain a contradiction to (3.16) if we show that the right-hand remains negative when $\alpha \to \infty$ and $\mu \downarrow 0$. Observe that

$$\limsup_{\alpha \uparrow \infty} -\frac{\mu}{2}\lambda e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2) = -\mu\lambda e^{\lambda(T-\hat{t})}\hat{x}^2. \tag{3.27}$$

Let us estimate the remainder term $\Delta(\alpha)$:

$$\begin{aligned}\Delta(\alpha) = (r-d)\big(\alpha(x_\alpha^2 - y_\alpha^2) + \mu e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2)\big)\\ + 1/2\sigma^2\big(x_\alpha^2 X_\alpha - y_\alpha^2 Y_\alpha\big) + r\big(v(t_\alpha, y_\alpha) - u(t_\alpha, x_\alpha)\big)\\ + 1/\epsilon\big[\big(g(x_\alpha) - u(t_\alpha, x_\alpha)\big)^+\big]^{\frac{1}{k}} - 1/\epsilon\big[\big(g(y_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}}\\ := \Delta_1(\alpha) + \Delta_2(\alpha) + \Delta_3(\alpha),\end{aligned}$$

where $\Delta_i(\alpha)$ corresponds to i-raw. It follows by (3.22) that

$$\limsup_{\alpha\uparrow\infty} \Delta_1(\alpha) = (r-d)2\mu e^{\lambda(T-\hat{t})}\hat{x}^2. \tag{3.28}$$

In view of (3.20), $r(v(t_\alpha, y_\alpha) - u(t_\alpha, x_\alpha)) \le -\delta r$ and the second term in $\Delta_2(\alpha)$ can be estimate using the matrix inequality (3.25):

$$\begin{aligned}&\limsup_{\alpha\uparrow\infty}\Big[\begin{pmatrix} X_\alpha & 0 \\ 0 & -Y_\alpha \end{pmatrix}\begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} \times \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix}\Big]\frac{1}{2}\sigma^2\\ &\le \limsup_{\alpha\uparrow\infty}\big[(3\alpha + 2\mu e^{\lambda(T-t)})|x_\alpha - y_\alpha|^2 + (\mu e^{\lambda(T-t_\alpha)} + \frac{\mu^2 e^{2\lambda(T-t_\alpha)}}{\alpha})(x_\alpha^2 + y_\alpha^2)\big]\frac{1}{2}\sigma^2\\ &= \mu e^{\lambda(T-\hat{t})}\hat{x}^2\sigma^2.\end{aligned} \tag{3.29}$$

It remains to estimate $\Delta_3(\alpha)$:

$$\begin{aligned}\Delta_3(\alpha) = \frac{1}{\epsilon}\big[\big(g(x_\alpha) - u(t_\alpha, x_\alpha)\big)^+\big]^{\frac{1}{k}} - \frac{1}{\epsilon}\big[\big(g(x_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}}\\ + \frac{1}{\epsilon}\big[\big(g(x_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}} - \frac{1}{\epsilon}\big[\big(g(y_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}}.\end{aligned}$$

In a view of (3.20) it follows

$$-u(t_\alpha, x_\alpha) \le -\delta - v(t_\alpha, y_\alpha), \tag{3.30}$$

and thus

$$\begin{aligned}&\frac{1}{\epsilon}\big[\big(g(x_\alpha) - u(t_\alpha, x_\alpha)\big)^+\big]^{\frac{1}{k}} - \frac{1}{\epsilon}\big[\big(g(x_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}}\\ &\le \frac{1}{\epsilon}\big[\big(g(x_\alpha) - v(t_\alpha, y_\alpha) - \delta\big)^+\big]^{\frac{1}{k}} - \frac{1}{\epsilon}\big[\big(g(x_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}} \le 0.\end{aligned}$$

Moreover, in view of (3.22) and by applying a Taylor expansion to the function $x \to \frac{1}{\epsilon}\big[\big(g(x) - v(t, y)\big)^+\big]^{\frac{1}{k}}$ we get:

$$\begin{aligned}&\frac{1}{\epsilon}\big[\big(g(x_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}} - \frac{1}{\epsilon}\big[\big(g(y_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}}\\ &\le \frac{k}{\epsilon}\big[\big(g(x_\alpha) - v(t_\alpha, y_\alpha)\big)^+ - \big(g(y_\alpha) - v(t_\alpha, y_\alpha)\big)^+\big]^{\frac{1}{k}}.\\ &\le \frac{k}{\epsilon}\big[\big(g(x_\alpha) - g(y_\alpha)\big)^+\big]^{\frac{1}{k}} \le C|x_\alpha - y_\alpha|^{\frac{1}{k}} \to 0 \quad \text{as } \alpha \to \infty.\end{aligned}$$

Combining the above estimates we obtain:

$$0 < \eta/T^2 \le -\mu\lambda e^{\lambda(T-\hat{t})}\hat{x}^2 + (r-d)2\mu e^{\lambda(T-\hat{t})}\hat{x}^2 + \mu\lambda e^{\lambda(T-\hat{t})}\hat{x}^2\sigma^2 - r\delta \le 0, \tag{3.31}$$

if λ is sufficiently large. The contradiction is caused by the assumption (3.16). □

3.4 Comparison with more regularity

More generally, consider a terminal-value problem (3.7) with F_ϵ replaced by

$$F_\epsilon(t,x,s,p,X) = F(t,x,s,p,X) - h_\epsilon(x,s), \tag{3.32}$$

where

$$F(t,x,s,p,X) = -(r-d)x^\theta p - \frac{1}{2}\sigma^2 x^2 X + rs, \tag{3.33}$$

$$h_\epsilon(x,s) = \frac{1}{\epsilon}[(g(x)-s)^+]^{1/k}. \tag{3.34}$$

Since the Lipschitz continuous F is weakened to be Hölder continuous with exponent $\theta \in (0,1]$, the proof may need to be modified. Roughly speaking, one can find it difficult to estimate the Δ_1 term in the previous proof since

$$\alpha(x_\alpha - y_\alpha)(x_\alpha^\theta - y_\alpha^\theta) \le \frac{\alpha}{\theta}(x_\alpha - y_\alpha)^{\theta+1} \tag{3.35}$$

will not necessarily converge to zero as $\alpha \to \infty$. Following the advice of Crandall, Ishii, and Lions [8, Chapter 5.A], we assume more regularity for u to be $C^\gamma(\Omega_T)$, i.e. a Hölder continuous with exponent $\gamma \in (0,1]$. If $\gamma > 2-(1+\theta) = 1-\theta$, then the proof will remain unchanged. In more details, with the notation of the previous proof

$$\begin{aligned} u(t,x) - v(t,y) - \frac{\alpha}{2}|x-y|^2 - \frac{\mu}{2}e^{\lambda(T-t)}(|x|^2+|y|^2) \\ \le u(t_\alpha,x_\alpha) - v(t_\alpha,y_\alpha) - \frac{\alpha}{2}|x_\alpha-y_\alpha|^2 - \frac{\mu}{2}e^{\lambda(T-t_\alpha)}(|x_\alpha|^2+|y_\alpha|^2), \end{aligned}$$

and by putting $x = y = y_\alpha$ and $t = t_\alpha$, we simplified

$$\frac{\mu}{2}e^{\lambda(T-t_\alpha)}(|x_\alpha|^2-|y_\alpha|^2)+\alpha|x_\alpha-y_\alpha|^2 \le 2(u(t_\alpha,x_\alpha)-u(t_\alpha,y_\alpha)) \le C|x_\alpha-y_\alpha|^\gamma.$$

It follows that if we choose μ to be sufficiently small, $\alpha|x_\alpha - y_\alpha|^\sigma \to 0$ provided by $\sigma > 2-\gamma$, in particular $\sigma = 1+\theta$.

However, the above argumentation fails while estimating $\alpha|x_\alpha - y_\alpha|$, and we need to adapt the strategy to obtain the result. The new idea [8, Chapter 5.A], relies on upgrowth of regularity for u to become Hölder continuously differentiable, i.e. $Du \in C^\gamma(\Omega_T)$. When $\gamma = 0$, Du is continuous and when $\gamma = 1$, Du is Lipschitz continuous. By upgrading the regularity for u, the following lemma applies.

Lemma 3.4.1. *Let $K \subset \Omega_T$ be a compact subset Ω_T. If $Du \in C^\gamma(\Omega_T)$ then there exists a sequence ψ_n in $C(\overline{\Omega}_T) \cap C^2(K)$ with the properties*

(i) $u - \psi_n \to 0$ *and* $\partial_x u - \partial_x \psi_n \to 0$ *uniformly on* K*,*

(ii) $|\partial_t \psi_n(t,x) - \partial_t \psi_n(t,y)| \leq C|x-y|^\gamma$ *on* K*,*

(iii) $|\partial_x \psi_n(t,x) - \partial_x \psi_n(t,y)| \leq C|x-y|^\gamma$ *on* K*,*

Proof. The ψ_n can be constructed by mollification of u near K and then extended. Let $\eta \in C_0^\infty(\mathbb{R})$ be a nonnegative function satisfying

$$\eta(z) = \eta(-z), \quad \eta(z) \equiv 0 \text{ for } |z| \geq 1, \quad \int_{\mathbb{R}} \eta(z)dz = 1.$$

For each $n > 0$, let $\eta_n(z) = \frac{1}{n}\eta(\frac{z}{n})$ and $I_n(t,x) = \eta_n(t)\eta_n(x)$. We define an approximation

$$\psi_n(t,x) = (I_n * u)(t,x) = \int_{[0,T]} \int_{\mathbb{R}} I_n(s,z)u(t-s,x-z)dzds.$$

(i) Let $V_T \subset\subset W_T \subset\subset \Omega_T$, $u \in C(\Omega_T)$. Then u is uniformly continuous on W_T. Let $x \in V_T$, $x = (t,y)$. For each $n > 0$ define an open ball $V_T \supseteq B(x,n) = \{(s,z) \in V_T : \quad |t-s| < n, |z-y| < n\}$.

$$\begin{aligned}|u(x) - \psi_n(x)| &\leq \frac{1}{n^2}\int_{B(x,n)} |I(\frac{x_1}{n})||u(x) - u(x-x_1)|dx_1 \\ &\leq \max_{x_2 \in B(x,n)} |u(x) - u(x-x_2)|1/n^2 \int_{B(x,n)} |I(\frac{x_1}{n})|dx_1.\end{aligned}$$

The right-hand side converges to zero uniformly as $n \downarrow 0$. Therefore $\psi_n \to u$ uniformly on V_T. Let us now calculate the partial derivative of the ψ_n.

$$\begin{aligned}\frac{\psi_n(x+he_i) - \psi_n(x)}{h} &= \frac{1}{n^2}\int_{V_T} \frac{1}{h}[I(\frac{x+he_i-z}{n}) - I(\frac{x-z}{n})]u(z)dz \\ &= \frac{1}{n^2}\int_{V_T} \frac{1}{h}[u(x+he_i-z) - u(x-z)]I(\frac{z}{n})dz,\end{aligned}$$

where $V_T \subset\subset \Omega_T$. Since

$$\frac{1}{h}[u(x+he_i-z) - u(x-z)] \to \frac{\partial u}{\partial x_i}(x-z)$$

$$\frac{1}{h}[I(\frac{x+he_i-z}{n}) - I(\frac{x-z}{n})] \to \frac{\partial I}{\partial x_i}(\frac{x-z}{n})$$

uniformly on V_T, we calculate the partial derivative of ψ_n as

$$\frac{\partial \psi_n}{\partial x_i}(x) = \int_{V_T} \frac{\partial u}{\partial x_i}(x-z)I_n(z)dz = \int_{V_T} \frac{\partial I_n}{\partial x_i}(x-z)u(z)dz. \tag{3.36}$$

Let us take $x \in V_T$, $x = (t, y)$,

$$|\partial_y u(x) - \partial_y \psi_n(x)| \leq 1/n^2 \int_{B(x,n)} |\partial_y u(x) - \partial_y u(x-z)||I(\frac{z}{n})|dz$$
$$\leq \max_{z_1 \in B(x,n)} |\partial_y u(x) - \partial_y u(x-z_1)| 1/n^2 \int_{B(x,n)} |I(\frac{z}{n})|dz.$$

The right-hand side converges to zero uniformly as $n \downarrow 0$. Therefore $\partial_y \psi_n \to \partial_y u$ uniformly on V_T.
(ii) Let $z_1, z_2 \in \Omega_T$,

$$|\partial_t \psi(t, z_1) - \partial_t \psi(t, z_2)| \leq \int_{\mathbb{R}} |I_n(z)||\partial_t u(z_1 - z) - \partial_t u(z_2 - z)|dz$$
$$\leq C|z_1 - z_2|^\gamma \int_{\mathbb{R}} |I_n(z)|dz \leq C|z_1 - z_2|^\gamma.$$

(iii)Let $z_1, z_2 \in \Omega_T$,

$$|\partial_x \psi(t, z_1) - \partial_x \psi(t, z_2)| \leq \int_{\mathbb{R}} |I_n(z)||\partial_x u(z_1 - z) - \partial_x u(z_2 - z)|dz$$
$$\leq C|z_1 - z_2|^\gamma \int_{\mathbb{R}} |I_n(z)|dz \leq C|z_1 - z_2|^\gamma.$$

□

Let us briefly discuss the modification in the comparison proof under an additional assumption: u is Hölder continuously differentiable. Let $(t_\alpha, x_\alpha, y_\alpha)$ be a maximum point of the function

$$\Phi(t, x, y) = (u(t, x) - \psi_n(t, x)) - (v(t, y) - \psi_n(t, y)) - \alpha/2|x - y|^2 - \frac{\mu}{2} e^{\lambda(T-t)}(x^2 + y^2), \quad (3.37)$$

defined in $[0, T) \times [0, \infty) \times [0, \infty)$; $\mu \in (0, 1)$, $\alpha > 1$ are parameters. As in the proof of Theorem 3.3.2, for each fixed μ, there exists a subsequence, denoted by $(t_\alpha, x_\alpha, y_\alpha)$, which converges to some $(\hat{t}, \hat{x}, \hat{y}) \in [0, T) \times [0, \infty) \times [0, \infty)$ as $\alpha \uparrow \infty$. For small μ, we apply Lemma 3.3.1 via the following correspondence: $(x, y) \to x$, $(u(t, x) - \psi_n(t, x)) - (v(t, y) - \psi_n(t, y)) \to \Phi(x)$, $(1/2)|x - y|^2 \to \Psi(x)$. Then, for each fixed μ, the maxima $(t_\alpha, x_\alpha, y_\alpha)$ satisfies

$$\begin{cases} x_\alpha - y_\alpha \to 0 \text{ as } \alpha \uparrow \infty, \\ \alpha|x_\alpha - y_\alpha|^2 \to 0 \text{ as } \alpha \uparrow \infty. \end{cases} \quad (3.38)$$

Moreover, applying Lemma 3.4.1(iii), it follows that

$$\alpha(x_\alpha - y_\alpha) = \partial_x u(t_\alpha, x_\alpha) - \partial_x \psi_n(t_\alpha, x_\alpha) \to 0 \text{ as } \alpha \uparrow \infty. \quad (3.39)$$

3.5 Existence of unique solution

Theorem 3.5.1 (Perron's Method). *Suppose that the comparison holds for (3.7)-(3.8), $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution of (3.7)-(3.8). Then*

$$W(x) = sup\{w(x) : \underline{u} \leq w \leq \overline{u} \text{ and } w \text{ is a subsolution of (3.7)-(3.8)}\}$$

is a solution of (3.7)-(3.8).

Proof. By Theorem 3.3.2 the comparison holds for (3.7)-(3.8) for any fixed k. The result follows from [8, Theorem 4.1]. □

Lemma 3.5.1. *The value of an European call/put option is a subsolution of (3.7)-(3.8).*

Let us recall that in Black and Scholes environment, the European option price is given by

$$p(t,x) = \mathbb{E}^{t,x}[e^{-r(T-t)}g(X(T))].$$

For finding the price of European option we can equivalently solve a certain partial differential equation. More precisely the following theorem applies:

Theorem 3.5.2. *Let u be a $C^{1,2}(\Omega_T)$ function with a bounded derivative with respect to x. If u satisfies*

$$-\mathcal{L}_{BS}u(t,x) + ru(t,x) = 0 \text{ for } (t,x) \in \overline{\Omega}_T, \tag{3.40}$$

$$u(T,x) = g(x) \text{ for } x \in [0,\infty), \tag{3.41}$$

then $\forall (t,x) \in \overline{\Omega}_T$ $u(t,x) = \mathbb{E}^{t,x}\left(e^{-\int_t^T r(s,X(s))ds}g(X(T))\right)$, where $X(s)$ is a solutions of (2.2).

Proof. For proof see, [18][proof of Theorem 5.1.7.]. □

Proof of Lemma 3.5.1. By plugging the function $p(t,x)$ into (3.7)-(3.8) we get

$$-\mathcal{L}_{BS}p(t,x)+rp(t,x)-\frac{1}{\epsilon}[(g(x)-p(t,x))^+]^{1/k} = -\frac{1}{\epsilon}[(g(x)-p(t,x))^+]^{1/k} \leq 0. \tag{3.42}$$

Moreover, the terminal condition is satisfied by (3.41). □

Lemma 3.5.2. *Let $\hat{u}$ be a function defined by*

$$\hat{u} = \begin{cases} x^+, & \text{call option,} \\ K, & \text{put option.} \end{cases} \tag{3.43}$$

Then $\hat{u}$ is a supersolution of (3.7)-(3.8).

Proof. Let us first prove lemma for the call option $g(x) = (x - K)^+$. By plugging $\hat{u}$ into (3.7)-(3.8) we get

$$-\mathcal{L}_{BS}\hat{u} + r\hat{u} - \frac{1}{\epsilon}[(g(x) - \hat{u})^+]^{1/k} = -(r - d)x + rx = dx > 0. \qquad (3.44)$$

Moreover,

$$\hat{u}(T, x) = x \geq g(x).$$

Now, let us prove the lemma for the put option $g(x) = (K - x)^+$. By plugging $\hat{u}$ into (3.7)-(3.8) we get

$$-\mathcal{L}_{BS}\hat{u} + r\hat{u} - \frac{1}{\epsilon}[(g(x) - \hat{u})^+]^{1/k} = rK > 0. \qquad (3.45)$$

Moreover,

$$\hat{u}(T, x) = K \geq g(x).$$

□

We conclude the chapter with the main theorem which provides the well-possedness for (3.7)-(3.8).

Theorem 3.5.3. *For fixed values ϵ and k, there exists at most one viscosity solution $v_\epsilon : \overline{\Omega}_T \to R$ of the terminal value problem (3.7)-(3.8), that satisfies*

$$0 \leq v_\epsilon(t, x) \leq C_1 + C_2 x \quad (t, x) \in \overline{\Omega}_T,$$

where $C_1 = 0$ and $C_2 = 1$ for the call option, and $C_1 = K$ and $C_2 = 0$ for the put option.

Proof. *Existence*. Put $\underline{u} = p(t, x)$ and $\overline{u} = \hat{u}$. By lemma 3.5.1, $p(t, x)$ is a subsolution of (3.7)-(3.8) and by lemma 3.5.2 $\hat{u}$ is a supersolution of (3.7)-(3.8). The existence of the solution of (3.7)-(3.8) follows from Theorem 3.5.1.

Uniqueness. Let us suppose that v_1 and v_2 are two viscosity solutions satisfying (3.7)-(3.8), i.e. v_1 and v_2 are simultaneously viscosity sub- and supersolutions. By Theorem 3.3.2 $v_1 \leq v_2$ and $v_1 \geq v_2$. Hence $v_1 \equiv v_2$.

□

In the next chapter we will show that the sequence of approximate solutions v_ϵ converge to the American option value.

Chapter 4

Convergence of approximate solutions

4.1 Half-relaxed weak limit method

The aim of this section is to prove, using the so called half-relaxed (weak) limit method of Crandall, Ishii and Lions [8], that for fixed $k > 0$ the power penalty approximations $(v_\epsilon)_\epsilon$ converge uniformly to the American option value, as $\epsilon \to 0$. In the literature one can find many examples where the weak limit method is a tool for showing the convergence of approximate solutions of fully nonlinear second order partial differential equations [2, 8, 13]. For inspiration to our work we have chosen Benth, Karlsen and Reikvam [6], where approximations solve (3.7)-(3.8) with $k = 1$ (classical penalty method).

Let us recall the definitions and basic properties of the upper and lower weak limits.

Definition 4.1.1. *Suppose v_ϵ is locally uniformly bounded.*

The upper weak limit of v_ϵ, denoted by $\overline{v}$, is defined as

$$\begin{aligned}\overline{v}(t,x) &= \lim_{\epsilon\downarrow 0}\sup_{\overline{\Omega}_T\ni(s,y)\to(t,x)} v_\epsilon(s,y)\\ &= \lim_{\delta\downarrow 0}\sup\{v_\epsilon(s,y) \mid \quad (s,y)\in\overline{\Omega}_T, |t-s|, |x-y|\le\delta, 0<\epsilon\le\delta\}.\end{aligned}$$

The lower weak limit of v_ϵ, denoted by $\underline{v}$, is defined as

$$\begin{aligned}\underline{v}(t,x) &= \lim_{\epsilon\downarrow 0}\inf_{\overline{\Omega}_T\ni(s,y)\to(t,x)} v_\epsilon(s,y)\\ &= \lim_{\delta\downarrow 0}\inf\{v_\epsilon(s,y) \mid \quad (s,y)\in\overline{\Omega}_T, |t-s|, |x-y|\le\delta, 0<\epsilon\le\delta\}.\end{aligned}$$

Since v_ϵ is locally uniformly bounded, the weak limits $\overline{v}$ and $\underline{v}$ are finite.

Lemma 4.1.1. *(i) The upper weak limit* $\overline{v}$ *belongs to* $USC(\overline{\Omega}_T)$ *and the lower weak limit* $\underline{v}$ *belongs to* $LSC(\overline{\Omega}_T)$.

(ii) If $\overline{v} = \underline{v} = v$ *on compact subsets of* $\overline{\Omega}_T$*, then* v *is continuous and* $v_\epsilon \to v$ *in* L^∞ *(i.e., uniformly) on this set as* $\epsilon \to 0$.

(iii) Let $v_\epsilon \in USC(\overline{\Omega}_T)$*(resp.* $LSC(\overline{\Omega}_T)$*) be locally uniformly (in* ϵ*) bounded. Let* $(t,x) \in \overline{\Omega}_T$ *be a strict local maximizer of* $\overline{v} - \phi$ *(resp. minimizer of* $\underline{v} - \phi$*),* $\phi \in C^{1,2}(\overline{\Omega}_T)$*. Then there exists the subsequence, which we do not relabel,* $(t_\epsilon, x_\epsilon) \to (t,x)$ *and* $v_\epsilon(t_\epsilon, x_\epsilon) \to \overline{v}(t,x)$*(resp.* $\underline{v}(t,x)$*) as* $\epsilon \downarrow 0$ *such that* (t_ϵ, x_ϵ) *is a local maximizer (resp. minimizer) of* $v_\epsilon - \phi$ *for each* $\epsilon > 0$.

Proof. (i) Using the definition of weak limits we observe that

$$\overline{v}(t,x) = \lim_{\epsilon\downarrow 0} \sup{}_{\overline{\Omega}_T \ni (s,y)\to(t,x)} v_\epsilon(s,y) \geq \lim_{\overline{\Omega}_T \ni (s,y)\to(t,x)} \overline{v}(s,y),$$

$$\underline{v}(t,x) = \lim_{\epsilon\downarrow 0} \inf{}_{\overline{\Omega}_T \ni (s,y)\to(t,x)} v_\epsilon(s,y) \leq \lim_{\overline{\Omega}_T \ni (s,y)\to(t,x)} \underline{v}(s,y).$$

(ii) As being both upper and lower semicontinuous, the function v is continuous. Moreover, for all $\epsilon > 0$, v_ϵ are continuous:

$$\forall_{\eta>0} \forall_{(t,x),(t,y)\in\overline{\Omega}_T} \exists_{\delta(\epsilon,\eta)>0} \text{ such that } |x-y| < \delta(\epsilon,\eta) \Rightarrow |v_\epsilon(t,x) - v_\epsilon(t,y)| < \eta. \quad (4.1)$$

Following the lines of [8, Remark 6.4], we claim that on compact subsets the sequence v_ϵ converges uniformly to v as $\epsilon \downarrow 0$. If this is not the case and the uniform convergence fails on some compact subset K, then there is an $\eta > 0$ and a sequence $\epsilon_j \to 0$, $(t_j,x_j) \in K$ such that

$$v_{\epsilon_j}(t_j,x_j) - v(t_j,x_j) \geq \eta \quad \text{or} \quad v_{\epsilon_j}(t_j,x_j) - v(t_j,x_j) \leq -\eta. \quad (4.2)$$

Let us assume that $(t_j, x_j) \to (t,x)$ and recall that v is continuous. Then by (4.2)

$$|v(t,x) - v(t,x)| \geq \eta,$$

which is a contradiction. Therefore, $v_\epsilon \to v$ uniformly on compact subsets of $\overline{\Omega}_T$ as $\epsilon \downarrow 0$.

(iii) Let us assume that $v_\epsilon \in USC(\overline{\Omega}_T)$. In view of (3.3)-(3.4) we can equivalently prove the following result:

Proposition 4.1.1. *Let* $v_\epsilon \in USC(\overline{\Omega}_T)$ *be locally uniformly (in* ϵ*) bounded. If* $(p,X) \in J^{2,+}\overline{v}(t,x)$*, then there exist sequences, which we do not relabel*

$$(t_\epsilon, x_\epsilon) \in \overline{\Omega}_T, \quad (p_\epsilon, X_\epsilon) \in J^{2,+} v_\epsilon(t_\epsilon, x_\epsilon), \quad (4.3)$$

such that

$$(t_\epsilon, x_\epsilon, v_\epsilon(t_\epsilon, x_\epsilon), p_\epsilon, X_\epsilon) \to (t, x, \overline{v}(t,x), p, X) \ \textit{as } \epsilon \downarrow 0. \tag{4.4}$$

Proof of proposition 4.1.1. We are following the lines of [8][proof of Lemma 6.1]. Without a loss of generality we can assume that $t = x = 0$. By the definition of the weak limit, there exist sequences such that

$$(s_\epsilon, z_\epsilon) \to (0,0) \quad \text{and} \quad v_\epsilon(s_\epsilon, z_\epsilon) \to v(0,0). \tag{4.5}$$

Since $v_\epsilon \leq \overline{v}$ and $\overline{v} \in USC(\overline{\Omega}_T)$, for any sequence $(t'_\epsilon, x'_\epsilon)$ which converges to $(0,0)$ the following is true

$$\text{lim sup}_{\epsilon\downarrow 0} u_\epsilon(t'_\epsilon, x'_\epsilon) \leq \text{lim sup}_{\epsilon\downarrow 0} \overline{v}(t'_\epsilon, x'_\epsilon) \leq \overline{v}(0,0). \tag{4.6}$$

Moreover, Ω_T is locally compact and therefore for every $\delta > 0$ there is $r > 0$ such that a closed ball $\overline{B}_r \subset \Omega_T$ with the center in the point $(0,0)$ is a compact set and

$$\overline{v}(s,z) \leq \overline{v}(0,0) + \langle p, (s,z)\rangle + 1/2\langle X(s,z), (s,z)\rangle + \delta|(s,z)|, \tag{4.7}$$

for $(s,z) \in \overline{B}_r$. Let $(t_\epsilon, x_\epsilon) \in \overline{B}_r$ be a maximum point of the function

$v_\epsilon(s,z) - (\langle p, (s,z)\rangle + 1/2\langle X(s,z),(s,z)\rangle + 2\delta|(s,z)|^2)$ over the ball $\overline{B}_r$,

so that

$$\begin{aligned} &v_\epsilon(s,z) \leq v_\epsilon(t_\epsilon, x_\epsilon) + \langle p, (s - t_\epsilon, z - x_\epsilon)\rangle \\ &+1/2(\langle X(s,z),(s,z)\rangle - \langle X(t_\epsilon, x_\epsilon),(t_\epsilon, x_\epsilon)\rangle) + 2\delta(|(s,z)|^2 - |(t_\epsilon, x_\epsilon)|^2). \end{aligned} \tag{4.8}$$

Suppose that $(t_\epsilon, x_\epsilon) \to (\hat{t}, \hat{x})$ as $\epsilon \downarrow 0$. By putting $(s,z) = (s_\epsilon, z_\epsilon)$ and taking a limit inferior as $\epsilon \downarrow 0$ in (4.8), we get

$$\begin{aligned} \overline{v}(0,0) \leq \liminf_{\epsilon\downarrow 0} v_\epsilon(t_\epsilon, x_\epsilon) - \langle p, (\hat{t}, \hat{x})\rangle - \langle X(\hat{t}, \hat{x}), (\hat{t}, \hat{x})\rangle \\ - 2\delta|(\hat{t}, \hat{x})|^2. \end{aligned} \tag{4.9}$$

By uppersemicontinuity of $\overline{v}$ it follows that $\liminf_{\epsilon\downarrow 0} v_\epsilon(t_\epsilon, x_\epsilon) \leq \overline{v}(\hat{t}, \hat{x})$ and by (4.7) it follows that

$$\begin{aligned} \overline{v}(\hat{t}, \hat{x}) - \langle p, (\hat{t}, \hat{x})\rangle - \langle X(\hat{t}, \hat{x}), (\hat{t}, \hat{x})\rangle - 2\delta|(\hat{t}, \hat{x})|^2 \leq \overline{v}(0,0) \\ - \delta|(\hat{t}, \hat{x})|^2. \end{aligned} \tag{4.10}$$

We combine (4.9) and (4.10) to get

$$\overline{v}(0,0) \leq \overline{v}(\hat{t},\hat{x}) - \langle p, (\hat{t},\hat{x})\rangle - \langle X(\hat{t},\hat{x}), (\hat{t},\hat{x})\rangle - 2\delta|(\hat{t},\hat{x})|^2$$
$$\leq \overline{v}(0,0) - \delta|(\hat{t},\hat{x})|^2. \quad (4.11)$$

This proves that $\hat{t} = \hat{x} = 0$, and therefore

$$(t_\epsilon, x_\epsilon) \to (0,0) \quad \text{and} \quad v(0,0) = \lim_{\epsilon\downarrow 0} v_\epsilon(t_\epsilon, x_\epsilon). \quad (4.12)$$

Next we observe that

$$v_\epsilon(s,z) \leq v_\epsilon(t_\epsilon,x_\epsilon) + \langle p, (s-t_\epsilon, z-x_\epsilon)\rangle + 1/2(\langle X(s,z),(s,z)\rangle$$
$$- \langle X(t_\epsilon,x_\epsilon),(t_\epsilon,x_\epsilon)\rangle) + 2\delta(|(s,z)|^2 - |(t_\epsilon,x_\epsilon)|^2)$$
$$= v_\epsilon(t_\epsilon,x_\epsilon) + \langle p + 4\delta(t_\epsilon,x_\epsilon) + X(t_\epsilon,x_\epsilon), (s-t_\epsilon, z-x_\epsilon)\rangle$$
$$+1/2\langle X(s-t_\epsilon,z-x_\epsilon),(s-t_\epsilon,z-x_\epsilon)\rangle + \langle 2\delta I(s-t_\epsilon,z-x_\epsilon),(s-t_\epsilon,z-x_\epsilon)\rangle,$$

so

$$(p + 4\delta(t_\epsilon,x_\epsilon) + X(t_\epsilon,x_\epsilon), X + 4\delta I) \in J^{2,+}v_\epsilon(t_\epsilon,x_\epsilon)$$

for ϵ small enough. We conclude that the set of (q,Y), such that there exists $(s_\epsilon,z_\epsilon) \in \Omega_T$ and $(p_\epsilon,X_\epsilon) \in J^{2,+}v_\epsilon(s_\epsilon,z_\epsilon)$

$$(s_\epsilon, z_\epsilon, v_\epsilon(s_\epsilon,z_\epsilon), p_\epsilon, X_\epsilon) \to (0,0,\overline{v}(0,0),q,Y),$$

is closed and contains $(p, X + 4\delta I)$ for $\delta > 0$. □

The case when $v_\epsilon \in LSC(\overline{\Omega}_T)$ and $(t,x) \in \overline{\Omega}_T$ is a strict local minimizer of $\underline{v} - \phi)$, can be proved similarly. □

The next theorem provides a uniform convergence of the power penalty approximations to the American option value.

Theorem 4.1.1. *Let v be the unique viscosity solution of (2.6) (Appendix A, definition A.3.1). For each $\epsilon > 0$, let v_ϵ be the unique viscosity solution of (3.7)-(3.8). Then $v_\epsilon \to v$ in $L^\infty_{loc}(\overline{\Omega}_T)$ as $\epsilon \to 0$.*

Proof. We are following the lines of [6, proof of Theorem 3.1] Let $\overline{v}$ and $\underline{v}$ be the upper and lower limits of v_ϵ. Then $0 \leq \overline{v}, \underline{v} \leq C(1+x)$ on $\overline{\Omega}_T$ and by lemma 4.1.1(i) $\overline{v} \in USC(\overline{\Omega}_T)$ and $\underline{v} \in LSC(\overline{\Omega}_T)$.

It suffices to show that $\underline{v}$ is a viscosity supersolution and $\overline{v}$ is a viscosity subsolution of (2.6). Once it is done, Theorem 3.3.2 (Comparison Principle) implies $\overline{v} \leq \underline{v}$ and hence $\overline{v} = v = \underline{v}$. The conclusion of the proof will follow from Lemma 4.1.1(ii).

Let us first show that $\underline{v}$ is a viscosity supersolution of (2.6). We will work towards a contradiction and suppose that $\underline{v}$ fails to satisfy the terminal condition, at some point (T, y), $y \in [0, \infty)$. Then there exists $\delta > 0$ such that

$$\underline{v}(T, y) \leq g(y) - \delta. \tag{4.13}$$

Let us pick sequences

$$(t_\epsilon, x_\epsilon) \to (T, y) \quad \text{and} \quad v_\epsilon(t_\epsilon, x_\epsilon) \to \underline{v}(T, y) \text{ as } \epsilon \downarrow 0.$$

In view of the terminal condition in (3.7)-(3.8), there exists $\epsilon_0 > 0$ such that $t_\epsilon < T$ for all $\epsilon \leq \epsilon_0$. Next, choose a function $\hat{g} \in C^2([0, \infty))$ such that

$$\begin{cases} \hat{g} \leq g & \text{on } [0, \infty), \\ \hat{g}(y) = g(y) - \delta/2, & \\ \hat{g} = \text{const} & \text{on } [\hat{K}, \infty) \quad \text{whenever } \hat{K} > y. \end{cases} \tag{4.14}$$

We use $\hat{g}$ to define a function

$$G = -C(T - t) + \hat{g},$$

where $C > 0$ is a constant to be chosen later. By (4.14) and definition of G it follows that $G < g$ on Ω_T. Moreover,

$$\begin{aligned} \partial_t G(t, x) + (r - d) x \partial_x G(t, x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 G(t, x) - rG(t, x) + \frac{1}{\epsilon}[((g(x) - G(t, x))^+]^{1/k} \\ \geq C + (r - d) x \partial_x \hat{g}(x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 \hat{g}(x) - r\hat{g}(x), \end{aligned}$$

so G is a subsolution if we choose

$$C \geq -\min_{x \in [0, \infty)} \{(r - d) x \partial_x \hat{g}(x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 \hat{g}(x) - r\hat{g}(x)\}.$$

The minimum is finite since $\hat{g}$ is constant on $[\hat{K}, \infty)$ and therefore the minimum is achieved on the compact set $[0, \hat{K}]$. By Theorem 3.3.2 (Comparison Principle) subsolution G and the supersolution v_ϵ satisfy $G \leq v_\epsilon$ on $\overline{\Omega}_T$ for any $\epsilon \in (0, \epsilon_0]$. By letting $\epsilon \downarrow 0$, we get $G \leq \underline{v}$ on $\overline{\Omega}_T$. In particular,

$$\underline{v}(T, y) \geq G(T, y) = \hat{g}(y) = g(y) - \delta/2.$$

Hence $\underline{v}(T, y) > g(y) - \delta$ which is a contradiction to (4.13). Therefore, $\underline{v}$ satisfies the terminal condition $\underline{v}|_{t=T} \geq g$ on $[0, \infty)$.

Let us now show that $\underline{v}$ is a supersolution for $t \in [0, T)$. Choose $\phi \in C^{1,2}(\overline{\Omega}_T)$ as a test function and let $(t, x) \in \Omega_T$ be a strict local minimizer of $\underline{v} - \phi$. We claim that

$$\underline{v}(t, x) \geq g(x). \tag{4.15}$$

Suppose that there exists $\delta > 0$ such that $\underline{v}(t,x) \leq g(x) - \delta$. By lemma 4.1.1(iii), there exist sequences such that

$$(t_\epsilon, x_\epsilon) \to (t,x) \quad \text{and} \quad v_\epsilon(t_\epsilon, x_\epsilon) \to \underline{v}(t,x) \text{ as } \epsilon \downarrow 0.$$

Moreover, (t_ϵ, x_ϵ) is a local minimizer of $\underline{v} - \phi$ for each ϵ. In the neighborhood of (t,x), we have:

$$\exists_{\epsilon(\delta)>0} \forall_{\epsilon \leq \epsilon(\delta)} \quad \frac{\delta}{2} \leq g(x_\epsilon) - v_\epsilon(t_\epsilon, x_\epsilon).$$

In view of this and since v_ϵ is a supersolution of (3.7)-(3.8), we have

$$\begin{aligned}&\partial_t \phi(t_\epsilon, x_\epsilon) + (r-d)x_\epsilon \partial_x \phi(t_\epsilon, x_\epsilon) + \frac{1}{2}\sigma^2 x_\epsilon^2 \partial_x^2 \phi(t_\epsilon, x_\epsilon) - r v_\epsilon(t_\epsilon, x_\epsilon) \\ &\leq -\frac{1}{\epsilon}[(g(x_\epsilon) - v_\epsilon(t_\epsilon, x_\epsilon))^+]^{1/k} = -\frac{1}{\epsilon}[g(x_\epsilon) - v_\epsilon(t_\epsilon, x_\epsilon)]^{1/k} \leq -\frac{1}{\epsilon}\left(\frac{\delta}{2}\right)^{1/k},\end{aligned}$$

for $\epsilon \leq \epsilon(\delta)$. Letting $\epsilon \downarrow 0$ in this inequality we get a contradiction since the left-hand side converges to a finite number while the right-hand side converges to $-\infty$. The property (4.15) has just been proved and, with this information in hand, we let $\epsilon \downarrow 0$ in the inequality

$$\begin{aligned}&\partial_t \phi(t_\epsilon, x_\epsilon) + (r-d)x_\epsilon \partial_x \phi(t_\epsilon, x_\epsilon) + \frac{1}{2}\sigma^2 x_\epsilon^2 \partial_x^2 \phi(t_\epsilon, x_\epsilon) - r v_\epsilon(t_\epsilon, x_\epsilon) \\ &\qquad\qquad \leq -\frac{1}{\epsilon}[(g(x_\epsilon) - v_\epsilon(t_\epsilon, x_\epsilon))^+]^{1/k} \leq 0.\end{aligned}$$

Therefore,

$$\max\left(\mathcal{L}_{BS}\phi(t,x) - r\underline{v}(t,x), g(x) - \underline{v}(t,x)\right) \leq 0.$$

This concludes the proof of the supersolution property of $\underline{v}$ at (t,x).

Let us now show that $\overline{v}$ is a viscosity subsolution of (2.6). We will work towards a contradiction and suppose that $\overline{v}$ fails to satisfy the terminal condition at some point (T,y), $y \in [0,\infty)$. Then there exists $\delta > 0$ such that

$$\overline{v}(T,y) \geq g(y) + \delta. \tag{4.16}$$

Let us pick sequences

$$(t_\epsilon, x_\epsilon) \to (T,y) \quad \text{and} \quad v_\epsilon(t_\epsilon, x_\epsilon) \to \overline{v}(T,y) \text{ as } \epsilon \downarrow 0.$$

In view of the terminal condition in (3.7)-(3.8), there exists $\epsilon_0 > 0$ such that $t_\epsilon < T$ for all $\epsilon \leq \epsilon_0$. Choose a function $\hat{g} \in C^2([0,\infty))$ such that

$$\begin{cases} \hat{g} \geq g & \text{on } [0,\infty), \\ \hat{g}(y) = g(y) + \delta/2, & \\ \hat{g} = g & \text{on } [\hat{K},\infty) \quad \text{with } \hat{K} > \max(y,K). \end{cases} \tag{4.17}$$

We use $\hat{g}$ to define a function

$$G = C(T-t) + \hat{g},$$

and $C > 0$ is a constant to be chosen later. By (4.17) and the definition of G it follows that $G \geq g$ on Ω_T. Moreover,

$$\partial_t G(t,x) + (r-d)x\partial_x G(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 G(t,x) - rG(t,x) + \frac{1}{\epsilon}[(g(x)-G(t,x))^+]^{1/k}$$
$$= -C + (r-d)x\partial_x \hat{g}(x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 \hat{g}(x) - r\hat{g}(x),$$

so G is a subsolution if we choose

$$C \geq \max_{x\in[0,\infty)} \{(r-d)x\partial_x \hat{g}(x) + \frac{1}{2}\sigma^2 x^2 \partial_x^2 \hat{g}(x) - r\hat{g}(x)\}. \tag{4.18}$$

We claim that the maximum is a finite number. Indeed, for a compact subset $[0, \hat{K}]$ the maximum is finite. For $x > \hat{K}$ we have that $g = \hat{g}$. Let us treat the call and the put case separately. For the call option, $\hat{g} = x - K$ when $x > \hat{K} \geq K$. The expression inside the curly brackets in (4.18) is bounded by

$$(r-d)x - r(x-K) = rK - dx \leq rK - d\hat{K} < \infty.$$

For the put option $\hat{g} = 0$ when $x > \hat{K}$.

By Theorem 3.3.2 (Comparison Principle) supersolution G and a subsolution v_ϵ satisfy $v_\epsilon \leq G$ on $\overline{\Omega}_T$ for any $\epsilon \in (0, \epsilon_0]$. By letting $\epsilon \downarrow 0$, we get $\overline{v} \leq G$ on $\overline{\Omega}_T$. In particular,

$$\overline{v}(T,y) \leq G(T,y) = \hat{g}(y) = g(y) + \delta/2.$$

Hence $\overline{v}(T,y) < g(y) + \delta$ which is a contradiction to (4.1). Therefore $\overline{v}$ satisfies the terminal condition $\overline{v}|_{t=T} \leq g$ on $[0,\infty)$.

It remains to show that $\overline{v}$ is a subsolution for $t \in [0,T)$. Let us choose $\phi \in C^{1,2}(\overline{\Omega}_T)$ as a test function and let $(t,x) \in \Omega_T$ be a strict local maximizer of $\overline{v} - \phi$. Similarly like in the supersolution case, Lemma 4.1.1 (iii) implies the existence of a sequence such that

$$(t_\epsilon, x_\epsilon) \to (t,x) \quad \text{and} \quad v_\epsilon(t_\epsilon, x_\epsilon) \to \overline{v}(t,x) \text{ as } \epsilon \to 0.$$

Moreover, (t_ϵ, x_ϵ) is a local maximizer of $\overline{v} - \phi$ for each ϵ. Observe that the maximum between zero and any number will be nonnegative. This explains why g is a subsolution of (2.6). By (4.15) and Theorem 3.3.2 (Comparison Principle) we can conclude that

$$g(x) \leq \underline{v}(t,x) \leq \overline{v}(t,x). \tag{4.19}$$

If $\overline{v}(t,x) = g(x)$, then $g - \phi$ has a local maximum at (t,x). Since g is a subsolution of (2.6), $\overline{v}$ is a subsolution as well. On the other hand, if $\overline{v}(t,x) > g(x)$, then $v_\epsilon(t_\epsilon, x_\epsilon) > g(x_\epsilon)$ for any ϵ small enough. In the view of this and since v_ϵ is a subsolution of (3.7)-(3.8), we have

$$\partial_t \phi(t_\epsilon, x_\epsilon) + (r-d)x_\epsilon \partial_x \phi(t_\epsilon, x_\epsilon) + \frac{1}{2}\sigma^2 x_\epsilon^2 \partial_x^2 \phi(t_\epsilon, x_\epsilon) - r v_\epsilon(t_\epsilon, x_\epsilon) \\ \geq -\frac{1}{\epsilon}[(g(x_\epsilon) - v_\epsilon(t_\epsilon, x_\epsilon))^+]^{1/k} = 0.$$

Let $\epsilon \downarrow 0$ in this inequality and observe that

$$\max\left(\mathcal{L}_{BS}\phi(t,x) - r\overline{v}(t,x), g(x) - \overline{v}(t,x)\right) \geq 0.$$

This concludes the proof of the subsolution property of $\overline{v}$ at (t,x). □

4.2 The rate of convergence of penalty approach

In this section we will derive error bounds for the convergence of the power penalty approach v_k, i.e. solution of (3.1)-(3.2), and a solution of the quasi-variational inequality formulation (2.6), i.e. American option value u. We will use comparison arguments to derive such a bound. We are following the presentation given in [15][Section 2].

Let us rewrite the equations (3.1)-(3.2) and (2.6) in a form suitable for the application of this section. Consider therefore the problem (2.6) with the opposite signs and with the maximum replaced by the minimum:

$$\begin{cases} \min\{-\partial_t u + F(t,x,u,\partial_x u, \partial_x^2 u), u - g\} = 0 & (t,x) \in \Omega_T, \\ u(T,x) = g(x) & x \in [0,\infty). \end{cases} \tag{4.20}$$

The second order operator $F : \overline{\Omega}_T \times R \times R \times R \to R$ is defined by

$$F(t,x,s,p,X) = -(r-d)xp - \frac{1}{2}\sigma^2 x^2 X + rs. \tag{4.21}$$

The natural definition of viscosity sub- and supersolution of the equation (4.20) is given in the Appendix (Definition A.3.1). Similarly, we change the signs in (3.1)-(3.2) to get:

$$\begin{cases} -\partial_t v_\epsilon + F(t,x,v_\epsilon,\partial_x v_\epsilon, \partial_x^2 v_\epsilon) = \frac{1}{\epsilon}[(g(x) - v_\epsilon)^+]^{1/k}, & (t,x) \in \Omega_T, \\ v_\epsilon(T,x) = g(x), & x \in [0,\infty). \end{cases} \tag{4.22}$$

Theorem 4.2.1. *Let u be a subsolution of (4.20) and v be a supersolution of (4.20) such that*

$$u(t,x), -v(t,x) \leq L(|x|+1) \text{ and } u(T,x) \leq v(T,x)$$

for some constant $L > 0$. Then $u \leq v$ in $\overline{\Omega}_T$.

Proof. The proof is given in the Appendix A. □

In the previous section we have proven that the power penalty approximations $(v_\epsilon)_\epsilon$ converge uniformly to the American option value, as $\epsilon \to 0$. The convergence takes place in the space of locally bounded functions with supremum norm. Preceding towards error estimates, we observe that:

(A1)(Comparison) By Theorem 3.3.2 and Theorem 4.2.1, (4.20) and (4.22) satisfy the comparison principle in the class of viscosity solutions.

(A2)(Monotonicity) If $z \leq s$ and $X \leq Y$ then

$$F(t, x, z, p, X) \leq F(t, x, s, p, Y).$$

This follows by simple calculations

$$|F(t, x, z, p, X) - F(t, x, s, p, Y)| \leq 1/2\sigma^2 x^2 (Y - X) + r(z - s) \leq 0.$$

(A3)(Regularity) For all $(t, x) \in B(0, R) \subset \Omega_T$ and $\Phi \in C^2(\mathbb{R}^2)$

$$|\Phi| + |\partial_x \Phi| + |\partial_x^2 \Phi| \leq R \Rightarrow F(t, x, \Phi, \partial_x \Phi, \partial_x^2 \Phi) \leq C_R.$$

Indeed, fix $(t, x) \in B(0, R)$ and let $\Phi \in C^2(\mathbb{R}^2)$.Then

$$\begin{aligned} F(t, x, \Phi, \partial_x \Phi, \partial_x^2 \Phi) &= -(r - d)x\partial_x \Phi - \frac{1}{2}\sigma^2 x^2 \partial_x^2 \Phi + r\Phi \\ &\leq C(|\Phi| + |\partial_x \Phi| + |\partial_x^2 \Phi|) \leq CR =: C_R. \end{aligned}$$

The main result in this section gives the rate of convergence for the penalization problem.

Theorem 4.2.2. *Assume (A1)-(A2) hold and u and v_ϵ are solutions of (4.20) and (4.22). Then $\|u - v_\epsilon\|_{L^\infty} \leq C\epsilon^{k/2}$, where the constant C depends only on g.*

Proof. We are following Jakobsen [15][proof of Theorem 2.1]. We will gradually prove a series of lemmas to obtain the main result. We begin with the preliminary estimate:

Lemma 4.2.1. *Assume (A1) and (A2) hold. Let u and v_ϵ be a solution of (4.20) and (4.22) respectively. Then*

$$|u - v_\epsilon| \leq |(g - v_\epsilon)^+| \text{ in } \Omega_T.$$

Proof. By monotonicity (A2), $v_\epsilon + |(g - v_\epsilon)^+|$ is a viscosity supersolution of (4.20). Let $(a, p, X) \in \mathcal{P}^{2,-}(v_\epsilon + |(g - v_\epsilon)^+|)$, then

$$\begin{aligned}\min\{-a + F(t, x, v_\epsilon + |(g - v_\epsilon)^+|, p, X), v_\epsilon + |(g - v_\epsilon)^+| - g\} \\ \geq \min\{-a + F(t, x, v_\epsilon, p, X), (v_\epsilon - g) + |(g - v_\epsilon)^+|\} \\ \geq \min\{\frac{1}{\epsilon}[(g - v_\epsilon)^+]^{1/k}, (v_\epsilon - g) + |(g - v_\epsilon)^+|\} = 0.\end{aligned}$$

The comparison principle for (4.20) implies

$$u \leq v_\epsilon + |(g - v_\epsilon)^+|. \tag{4.23}$$

Similarly, we observe that by monotonicity (A2), $v_\epsilon - |(g - v_\epsilon)^+|$ is a viscosity subsolution of (4.20). Let $(a, p, X) \in \mathcal{P}^{2,+}(v_\epsilon - |(g - v_\epsilon)^+|)$, then

$$\begin{aligned}\min\{-a + F(t, x, v_\epsilon - |(g - v_\epsilon)^+|, p, X), v_\epsilon - |(g - v_\epsilon)^+| - g\} \\ \leq \min\{-a + F(t, x, v_\epsilon, p, X), (v_\epsilon - g) - |(g - v_\epsilon)^+|\} \\ \leq \min\{\frac{1}{\epsilon}[(g - v_\epsilon)^+]^{1/k}, v_\epsilon - g\} \leq 0.\end{aligned}$$

The comparison principle for (4.20) implies

$$v_\epsilon - |(g - v_\epsilon)^+| \leq u. \tag{4.24}$$

The Lemma 4.2.1 follows from (4.23) and (4.24). □

Let us now estimate the right-most term $|(g - v_\epsilon)^+|$ in Lemma 4.2.1.

Lemma 4.2.2. *Assume (A1)-(A2) hold. Let v_ϵ be the solution of (4.22). $K_1 = C_R$ and $R = |g| + 1$. Then*

$$g - v_\epsilon \leq \epsilon^k K_1^k.$$

Proof. Observe that $g - \epsilon^k K_1^k$ is a viscosity subsolution of (4.22). Let $(a, p, X) \in \mathcal{P}^{2,+}(g - \epsilon^k K_1^k)$, then

$$\begin{aligned}- a + F(t, x, g - \epsilon^k K_1^k, p, X) - \frac{1}{\epsilon}[\epsilon^k K_1^k]^{1/k} \\ \leq -a + F(t, x, g, p, X) - K_1 \leq -a + C_R - C_R \leq 0.\end{aligned} \tag{4.25}$$

Comparison principle for (4.22) implies that $g - u \leq \epsilon^k K_1^k$. □

Since g is not a smooth function, we shall approximate the result. Consider therefore η_δ, a standard mollifier:

$$\eta_\delta(x) = 1/\delta \eta(x/\delta),$$

where $\eta \in C_0^\infty(\mathbb{R})$ a nonnegative function satisfying:

$$\eta(z) = \eta(-z), \quad \eta(z) \equiv 0 \text{ for } |z| \geq 1, \quad \int_{\mathbb{R}} \eta(z)dz = 1.$$

We define an approximation $g_\delta = (\eta \star g)$. Denote by u^δ and v_ϵ^δ the solutions of (4.20) and (4.21) when g is replaced by g^δ, i.e.

$$\begin{cases} \min\{-\partial_t u^\delta + F(t,x,u^\delta,\partial_x u^\delta,\partial_x^2 u^\delta); u^\delta - g^\delta\} = 0 & (t,x) \in \Omega_T, \\ u^\delta(T,x) = g^\delta(x) & x \in [0,\infty), \end{cases} \tag{4.26}$$

$$\begin{cases} -\partial_t v_\epsilon^\delta + F(t,x,v_\epsilon^\delta,\partial_x v_\epsilon^\delta,\partial_x^2 v_\epsilon^\delta) = \frac{1}{\epsilon}[(g^\delta - v_\epsilon^\delta)^+]^{1/k}, & (t,x) \in \Omega_T, \\ v_\epsilon^\delta(T,x) = g^\delta(x), & x \in [0,\infty). \end{cases} \tag{4.27}$$

Let us prove the following bounds on $u - u^\delta$ and $v - v^\delta$:

Lemma 4.2.3. *Assume (A1)-(A3), and let u, u^δ, v_ϵ and v_ϵ^δ be solutions of (4.20), (4.22), (4.26), and (4.27). Then*

$$|u - u^\delta| + |v_\epsilon - v_\epsilon^\delta| \leq 2|g - g^\delta|.$$

Proof. Since $v_\epsilon \to u$ in $L^\infty(\Omega_T)$, it is sufficient to prove the lemma for v_ϵ and v_ϵ^δ. The result for u and u^δ can be obtained by going to the limit in the v_ϵ and v_ϵ^δ- result. Let $L := |g - g^\delta|$ and define

$$w^\pm = v_\epsilon^\delta \pm L.$$

Observe that w^+ is a supersolution of (4.27) and w^- is a subsolution of (4.27):

$$\begin{aligned} &-\partial_t v_\epsilon^\delta + F_\epsilon(t,x,v_\epsilon^\delta + L,\partial_x v_\epsilon^\delta,\partial_x^2 v_\epsilon^\delta) - \frac{1}{\epsilon}[(g - v_\epsilon^\delta - L)^+]^{1/k} \\ &\qquad \geq -\partial_t v_\epsilon^\delta + F_\epsilon(t,x,v_\epsilon^\delta,\partial_x v_\epsilon^\delta,\partial_x^2 v_\epsilon^\delta) - \frac{1}{\epsilon}[(g - v_\epsilon^\delta)^+]^{1/k} \geq 0. \end{aligned} \tag{4.28}$$

$$\begin{aligned} &-\partial_t v_\epsilon^\delta + F_\epsilon(t,x,v_\epsilon^\delta - L,\partial_x v_\epsilon^\delta,\partial_x^2 v_\epsilon^\delta) - \frac{1}{\epsilon}[(g - v_\epsilon^\delta + L)^+]^{1/k} \\ &\qquad \leq -\partial_t v_\epsilon^\delta + F_\epsilon(t,x,v_\epsilon^\delta,\partial_x v_\epsilon^\delta,\partial_x^2 v_\epsilon^\delta) - \frac{1}{\epsilon}[(g - v_\epsilon^\delta)^+]^{1/k} \leq 0. \end{aligned} \tag{4.29}$$

The comparison principle implies that $|v_\epsilon - v_\epsilon^\delta| \leq |g - g^\delta|$. □

Now we are ready to give the proof of the Theorem 4.2.2. Let u^δ and v_ϵ^δ be solutions of (4.26) and (4.27). By Lemmas 4.2.1 and 4.2.2 we have

$$|u^\delta - v_\epsilon^\delta| \leq |(g - v_\epsilon^\delta)^+| \leq K_1^k \epsilon^k, \tag{4.30}$$

where K_1 is defined in Lemma 4.2.2. Note that $u - v_\epsilon = (u - u^\delta) + (u^\delta - v_\epsilon^\delta) + (v_\epsilon^\delta - v_\epsilon)$. By the triangle inequality, (4.30), Lemma 4.2.3 and Lipschitz continuity of g we get the estimates:

$$|u - v_\epsilon| \le |u - u^\delta| + |v_\epsilon^\delta - v_\epsilon| + K_1^k \epsilon^k \le 2|g - g^\delta| + K_1^k \epsilon^k \le 2|g|\delta + K_1^k \epsilon^k.$$

Put $K_1 = C\delta^{-1/k}$ and minimize

$$f(\delta) = 2|g|\delta + K_1^k \epsilon^k,$$

with respect to δ. Since $f'(\delta) = 0$ if $\delta = \epsilon^{k/2}$, we obtain the following estimate:

$$|u - v_\epsilon| \le 2|g|\epsilon^{k/2} + C\epsilon^k \epsilon^{-k/2} = C\epsilon^{k/2},$$

and the following rate of convergence:

$$\|u - v_\epsilon\|_{L^\infty(\Omega_T)} = \text{ess} \sup_{(t,x)\in\Omega_T} |u - v_\epsilon| \le C\epsilon^{k/2}.$$

□

To achieve a given accuracy, we do not need ϵ to be very small, when k is large. The order of convergence rate is the same as the one calculated by Wang, Yang and Teo [23], but in the different norm. Due to their work

$$\|u - v_\epsilon\|_{L^\infty(0,T;L^2((0,\infty)))} + \|u - v_\epsilon\|_{L^2(0,T;H^1_{0,\omega}((0,\infty)))} \le C\epsilon^{k/2},$$

where $L^p(0, T; H^1_{0,\omega}((0, \infty))$ denotes the space defined by

$$L^p(0,T;H^1_{0,\omega}((0,\infty)) = \{v(\cdot,t) \in H((0,\infty)) \text{ a.e. in } (0,T); \\ \|v(\cdot,t)\|_H \in L^2((0,T))\},$$

with the norm

$$\|v\|_{L^2(0,T;H^1_{0,\omega}((0,\infty))} = (\int_0^T \|v(\cdot,t)\|_H^p dt)^{1/p}.$$

Moreover, H is a Hilbert space, $1 \le p \le \infty$, $H^1_{0,\omega}((0,\infty)))$ is a weighted Sobolev space defined by:

$$H^1_{0,\omega}((0,\infty))) = \{v : v \in L^2((0,\infty)), v' \in L^2_\omega((0,\infty)) \text{ and } v(X) = 0\}.$$

Here $L^2_\omega((0,\infty))$ is a space of all weighted square-integrable functions. In the next chapter we will present numerical results to confirm the theory developed so far.

Chapter 5

Numerical schemes

In this chapter we compare three numerical schemes: a power-penalty scheme, a predictor-corrector scheme and the Brennan and Schwartz algorithm. The numerical results and the payoff function g defined by (2.1), are set together and compared. The aim of this section is to show that by choosing reasonable parameters ϵ and k in the power-penalty scheme, we will obtain the most accurate numerical approximation.

5.1 Power-penalty scheme

The following scheme discretizes the power penalty equation (3.1). Let us start with the truncation of the infinite domain $[0, \infty)$, to a finite domain $[0, L)$ where $0 < L < \infty$ is fixed. Later, we will also provide a suitable boundary condition at $x = L$ (see (5.7)). The advantage of the truncation technique [1], is that the choice of the boundary condition at $x = L$, does not affect the theoretical analysis of convergence. This can be achieved by sending $L \uparrow \infty$ and $\Delta x \downarrow 0$.

Below we discretizate the spatial and temporal domains. Let $\Delta x > 0$ be a spatial discretization parameter and choose an integer J, such that $J\Delta x = L$. We divide $[0, L]$ into grid cells:

$$I_j = [x_j, x_{j+1}), \quad j = 0, 1, ..., J-2, \tag{5.1}$$

where $x_l = l\Delta x$ for $l = 0, 1, ..., J$, and we set $I_{J-1} = [x_{J-1}, x_J]$. Similarly, let $\Delta t > 0$ be a temporal discretization parameter and let N be an integer such that $N\Delta t = T$. We divide the time interval $[0, T]$, into time strips

$$I^n = [t^n, t^{n+1}) \quad n = 0, 1, ..., N-2, \tag{5.2}$$

where $t^n = n\Delta t$, $n = 0, ..., N$. Furthermore, set $I^{N-1} = [t^{N-1}, t^N]$. Denote by Ω_T^L a rectangle $[0, T] \times [0, L]$ which, by the aforementioned discretization procedure, is divided to $N \times J$ rectangles $R_j^n = I^n \times I_j$. By v_j^n, denote an approximation associated with the point (t^n, x_j), for $j = 0, 1, ..., J$ and $n = 0, 1, ..., N$. Let us introduce a simple difference scheme for the penalized equation (3.1). For simplicity, set $\lambda = \Delta t/\Delta x$ and $\mu = \Delta t/(\Delta x)^2$. Denote by Δ_+ and Δ_- the difference operators in x direction:

$$\Delta_+ v_j^n = v_{j+1}^n - v_j^n, \quad \Delta_- v_j^n = v_j^n - v_{j-1}^n.$$

Define the *upwind numerical flux function* [6] $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by:

$$F(a, b) = \begin{cases} b & \text{when } r - d > 0, \\ a & \text{when } r - d < 0. \end{cases} \tag{5.3}$$

The power-penalty scheme takes the following form for $j = 0, ..., J - 1$ and $n = N - 1, ..., 0$:

$$v_j^n = v_j^{n+1} + \lambda(r-d)x_j\Delta_- F(v_j^{n+1}, v_{j+1}^{n+1}) + \mu\frac{1}{2}\sigma^2 x_j^2 \Delta_-\Delta_+ v_j^{n+1} - r\Delta t v_j^{n+1} + \frac{1}{\epsilon}\Delta t[(g(x_j) - v_j^{n+1})^+]^{1/k}. \tag{5.4}$$

For simplicity, let us assume that $r - d > 0$, and therefore (5.4) becomes:

$$v_j^n = v_j^{n+1} + (r-d)x_j\lambda(v_{j+1}^{n+1} - v_j^{n+1}) + \frac{1}{2}\sigma^2 x_j^2 \mu(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) - r\Delta t v_j^{n+1} + \frac{1}{\epsilon}\Delta t[(g(x_j) - v_j^{n+1})^+]^{1/k}. \tag{5.5}$$

We begin a backward iteration (5.5), by setting terminal conditions:

$$v_j^N = g(x_j), \quad j = 0, .., J, \tag{5.6}$$

and boundary conditions at $x = 0$ and $x = L$:

$$v_0^n = g(0), \quad v_J^n = g(L), \quad n = N - 1, ..., 0. \tag{5.7}$$

5.1.1 Stability of the power-penalty scheme

Numerical examples shows that the power-penalty scheme is not stable, unless we ensure that the approximate solution lies above the payoff function g. The improvement turns the scheme (5.5) into:

$$\begin{cases} v_j^{n+1/2} = v_j^{n+1} + (r-d)x_j\lambda(v_{j+1}^{n+1} - v_j^{n+1}) + \frac{1}{2}\sigma^2 x_j^2 \mu(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) \\ \qquad -r\Delta t v_j^{n+1} + \frac{1}{\epsilon}\Delta t[(g(x_j) - v_j^{n+1})^+]^{1/k}, \\ v_j^n = \max(g(x_j), v_j^{n+1/2}). \end{cases} \tag{5.8}$$

Let us assume that the following *parabolic CFL condition* hold:

$$\lambda|r-d|L+\mu\sigma^2L^2+\Delta tr+\frac{1}{\epsilon k}\max\{K,L\}^{1/k-1}\leq 1. \tag{5.9}$$

Note that when $\Delta x\downarrow 0$ and $\epsilon\downarrow 0$, then also $\Delta t\downarrow 0$ by the above condition. The following lemma shows that the power-penalty approximation v_j^n lies above g and has at most linear growth as $x\uparrow\infty$. This implies that the the power-penalty scheme (5.8) is L^∞_{loc} stable.

Lemma 5.1.1. *Suppose the parabolic CFL condition (5.9) hold. Then the power-penalty solution v_j^n defined by (5.8) satisfies*

$$v_j^n\geq g(x_j),\quad j=0,...,L,\quad n=N,N-1,...,0. \tag{5.10}$$

Furthermore, there exist finite constants C_1 and C_2, such that

$$v_j^n\leq C_1+C_2x_j,\quad j=0,...,L,\quad n=N,N-1,...,0. \tag{5.11}$$

For the call option, $C_1=0$ and $C_2=1$. For the put option, $C_1=K$ and $C_2=0$.

Proof. We are following [6][proof of Lemma 4.1]. By additional conditions we trivially have that $v_j^n\geq g(j)$ for all j and n. The proof of (5.11) is inductive. By boundary and terminal conditions (5.6)-(5.7), the statement holds for $(j=0,...,J,n=N,N-1,...,0)$, $(j=J,n=N,N-1,...,0)$, and $(j=0,...,J,n=N)$.

Let us start with a put option and assume that (5.11) holds at time level $n+1$ and we seek to prove that it holds at time level n. To this end, we introduce a function S defined by:

$$S(x_j,v_{j-1}^{n+1},v_j^{n+1},v_{j+1}^{n+1})=v_j^{n+1}+(r-d)x_j\lambda(v_{j+1}^{n+1}-v_j^{n+1})$$
$$+\frac{1}{2}\sigma^2x_j^2\mu(v_{j+1}^{n+1}-2v_j^{n+1}+v_{j-1}^{n+1})-r\Delta tv_j^{n+1}+\frac{1}{\epsilon}\Delta t[(g(x_j)-v_j^{n+1})^+]^{1/k}. \tag{5.12}$$

We can therefore rewrite (5.5) as

$$S(x_j,v_{j-1}^{n+1},v_j^{n+1},v_{j+1}^{n+1})-v_j^n=0. \tag{5.13}$$

Observe that

$$\partial S/\partial v_{j-1}^{n+1}=1/2\sigma^2x_j^2\geq 0,\quad \partial S/\partial v_{j+1}^{n+1}=(r-d)x_j\lambda+1/2\sigma^2x_j^2\mu\geq 0. \tag{5.14}$$

In a view of the parabolic CFL condition (5.9):

$$\partial S/\partial v_j^{n+1}=1-(r-d)x_j\lambda-\sigma^2x_j^2\mu-r\Delta t-\Delta t\frac{1}{\epsilon k}[(g(x_j)-v_j^{n+1})^+]^{1/k-1}$$
$$\geq 1-(r-d)L\lambda-\sigma^2L^2\mu-r\Delta t-\Delta t\frac{1}{\epsilon k}[\max(L,K)]^{1/k-1}\geq 0. \tag{5.15}$$

This implies that the scheme is *monotone*. It follows from the monotonicity of S that for a put option

$$v_j^{n+1/2} \leq K(1 - r\Delta t) + \Delta t \frac{1}{\epsilon}[(g(x_j) - K)^+]^{1/k} = K(1 - r\Delta t). \tag{5.16}$$

Hence,

$$v_j^n = \max(g(x_j), v_j^{n+1/2}) \leq K \quad \text{for all j.} \tag{5.17}$$

For the call, we similarly assume that (5.11) holds at time level $n + 1$ and we seek to prove that it holds at time level n. By monotonicity of S, we get

$$v_j^{n+1/2} \leq x_j(1 - d\Delta t) + \Delta t \frac{1}{\epsilon}[(g(x_j) - x_j)^+]^{1/k} = x_j(1 - d\Delta t). \tag{5.18}$$

Hence,

$$v_j^n = \max(g(x_j), v_j^{n+1/2}) \leq x_j \quad \text{for all j.} \tag{5.19}$$

□

5.1.2 Numerical implementation

The main algorithm for the put option.

1. Denote by x, *gput* vectors with coordinates corresponding, respectively, to discrete points x_l, values of payoff function $g(x_l)$ in this points, $l = 0, ..., J$.

2. Denote by *wput* and *vput* vectors with coordinates corresponding to approximate solutions in $n + 1$ and n time step, respectively, $n = N - 1, ..., 0$.

3. *wput* ← *gput*. %terminal conditions

4. for $n = 1 : N$
   vput ← $PDEput$(*wput*).
   wput ← *vput*.
   end.

Function PDEput(u)

1. Denote by *temp* the return vector.

2. *temp*(0) ← *gput*(0). %boundary conditions
   temp(j) ← *gput*(L).

3.for $n = 2 : J$

$$y \leftarrow u(j)+\lambda(r-d)x(j)(u(j+1)-u(j))+\mu 0.5\sigma^2 x^2(j)(u(j+1)-2u(j) \\ + u(j-1)) - r\Delta t u(j) + \Delta t(1/\epsilon)[(gput(j) - u(j))^+]^{1/k}.$$

$temp(j) \leftarrow \max(gput(j), y)$.
end.

Moreover, by replacing "put" with "call" in the above algorithm, one can obtain the main algorithm for the call option and a function PDEcall(u).

5.2 The predictor-corrector scheme

The following scheme discretizes the Black and Scholes equation (2.16). The discretization of the spatial and temporal domains is similar to the one in the power-penalty scheme, and we will be rather brief here. Let $\Delta x > 0$ be a spatial discretization parameter and we choose an integer J, such that $J\Delta x = L$. We divide $[0, L]$ into grid cells:

$$I_j = [x_{j-1/2}, x_{j+1/2}), \quad j = 1, ..., J-1, \tag{5.20}$$

where $x_l = l\Delta x$ for $l = 0, 1/2, 1, ..., J-1, J-1/2$. We set $I_0 = [0, x_{1/2})$ and $I_J = [x_{J-1/2}, x_J]$. Let $\Delta t > 0$ be a temporal discretization parameter, N be an integer such that $N\Delta t = T$. We divide the time interval $[0, T]$, into time strips

$$I^n = [t^n, t^{n+1}) \quad n = 0, 1, ..., N-2, \tag{5.21}$$

where $t^n = n\Delta t$, $n = 0, ..., N$. Moreover, set $I^{N-1} = [t^{N-1}, t^N]$. With the same notation as before: Ω_T^L is divided into rectangles $R_j^n = I^n \times I_j$, v_j^n denotes an approximation associated with the point (t^n, x_j), $\lambda = \Delta t/\Delta x$ and $\mu = \Delta t/(\Delta x)^2$. The predictor-corrector scheme [6] for the semilinear Black and Scholes equation (2.15)-(2.16), takes the following form for $j = 0, ..., J-1$ and $n = N-1, ..., 0$:

Predictor step:

$$v_j^{n+1/2} = v_j^{n+1} + \lambda(r-d)x_j\Delta_- F(v_j^{n+1}, v_{j+1}^{n+1}) + \mu\frac{1}{2}\sigma^2 x_j^2 \Delta_-\Delta_+ v_j^{n+1} \\ - r\Delta t v_j^{n+1}. \tag{5.22}$$

Corrector step:

$$v_j^n = v_j^{n+1/2} + \Delta t c(x_j) H(g(x_j) - v_j^{n+1/2}). \tag{5.23}$$

Assume for simplicity $r - d \geq 0$. Observe that $v_j^{n+1/2} \leq g(x_j)$, then (5.22)-(5.23) reduces to

$$v_j^n = v_j^{n+1} + (r-d)x_j\lambda(v_{j+1}^{n+1} - v_j^{n+1}) + \frac{1}{2}\sigma^2 x_j^2 \mu(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) - r\Delta t v_j^{n+1} + \Delta t c(x_j). \quad (5.24)$$

Otherwise, (5.22)-(5.23) reduces to

$$v_j^n = v_j^{n+1} + (r-d)x_j\lambda(v_{j+1}^{n+1} - v_j^{n+1}) + \frac{1}{2}\sigma^2 x_j^2 \mu(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) - r\Delta t v_j^{n+1}. \quad (5.25)$$

We recall, that

$$c(x_j) = \begin{cases} (dx_j - rK)^+ & \text{call option,} \\ (rK - dx_j)^+, & \text{put option.} \end{cases} \quad (5.26)$$

We begin a backward iteration (5.24)-(5.25), by setting terminal conditions:

$$v_j^N = g(x_j), \quad j = 0, .., J, \quad (5.27)$$

and boundary conditions at $x = 0$ and $x = L$:

$$v_0^{n+1/2} = v_0^n = g(0), \quad v_J^{n+1/2} = v_J^n = g(L), \quad n = N-1, ..., 0. \quad (5.28)$$

5.2.1 Stability of the predictor-corrector scheme

The following lemma shows that the predictor-corrector approximation v_j^n lies above g and has at most linear growth as $x \uparrow \infty$. This implies that the the predictor-corrector scheme (5.24)-(5.25) is L_{loc}^∞ stable.

Lemma 5.2.1. *Let us assume that the following parabolic CFL' condition holds:*

$$\lambda |r - d| L + \mu\sigma^2 L^2 + r\Delta t \leq 1. \quad (5.29)$$

Then the predictor-corrector solution v_j^n defined by (5.24)-(5.25) satisfies

$$v_j^n \geq g(x_j), \quad j = 0, ..., L, \quad n = N, N-1, ..., 0.$$

Furthermore, there exist finite constants C_1 and C_2, such that

$$v_j^n \leq C_1 + C_2 x_j, \quad j = 0, ..., L, \quad n = N, N-1, ..., 0.$$

For the call option, $C_1 = 0$ and $C_2 = 1$. For the put option, $C_1 = K$ and $C_2 = 0$.

Proof. For the proof, see [6][proof of Lemma 4.1]. □

5.2.2 Numerical implementation

The main algorithm for the put option.

1. Denote by x, *gput* vectors with coordinates corresponding, respectively, to discrete points x_l, values of payoff function $g(x_l)$ in this points, $l = 0, ..., J$.

2. Denote by *Pwput* and *Pvput* vectors with coordinates corresponding to approximate solutions in $n + 1$ and n time step, respectively, $n = N - 1, ..., 0$.

3. $Pwput \leftarrow gput$. %terminal conditions

4. for $n = 1 : N$
   $Pvput \leftarrow PDEput(Pwput)$.
   $Pwput \leftarrow Pvput$.
   end.

Function PDEput(u)

1. Denote by *temp* the return vector.

2. $temp(0) \leftarrow gput(0)$. %boundary conditions
   $temp(j) \leftarrow gput(L)$.

3. for $n = 2 : J$

$$y \leftarrow u(j)+\lambda(r-d)x(j)(u(j+1)-u(j))+\mu 0.5\sigma^2 x^2(j)(u(j+1)-2u(j) + u(j-1)) - r\Delta t u(j).$$

   if $gput(j) > y$ then
   $temp(j) \leftarrow y + \Delta t(\max(rK - dx(j), 0))$.

   else
   $temp(j) \leftarrow y$.
   end.
   end.

Moreover, by replacing "put" with "call" in the above algorithm, one can obtain the main algorithm for the call option. Additionally, inside the function PDEcall(u), exchange also $\max(rK - dx(j), 0)$ with $\max(dx(j) - rK, 0)$.

5.3 The Brennan and Schwartz algorithm

By applying a predictor-corrector discretization to the penalized equation (3.1), we obtain a numerical scheme that is similar to (5.22)-(5.23), but with the following corrector step:

$$v_j^n = v_j^{n+1/2} + \frac{\Delta t}{\epsilon}[(g(x_j) - v_j^{n+1/2})^+]^{1/k}. \tag{5.30}$$

Put $k = 1$ and $\epsilon = \Delta t$. This results in the following scheme:

$$\begin{cases} v_j^{n+1/2} = v_j^{n+1} + (r-d)x_j\lambda(v_{j+1}^{n+1} - v_j^{n+1}) + \frac{1}{2}\sigma^2 x_j^2 \mu(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) \\ \qquad\qquad -r\Delta t v_j^{n+1}, \\ v_j^n = \max(v_j^{n+1/2}, g(x_j)). \end{cases} \tag{5.31}$$

We begin the backward iteration (5.31), by setting terminal conditions:

$$v_j^N = g(x_j), \quad j = 0, .., J, \tag{5.32}$$

and boundary conditions at $x = 0$ and $x = L$:

$$v_0^{n+1/2} = v_0^n = g(0), \quad v_J^{n+1/2} = v_J^n = g(L), \quad n = N-1, ..., 0. \tag{5.33}$$

5.3.1 Stability of the Brennan and Schwartz algorithm

The following lemma shows that the Brennan and Schwartz approximation v_j^n lies above g and has at most linear growth as $x \uparrow \infty$. This implies that the the Brennan and Schwartz scheme (5.31) is L^∞_{loc} stable.

Lemma 5.3.1. *Assume that the parabolic CFL' condition (5.29) hold. Then the Brennan and Schwartz solution v_j^n defined by (5.31) satisfies*

$$v_j^n \geq g(x_j), \quad j = 0, ..., L, \quad n = N, N-1, ..., 0.$$

Furthermore, there exist finite constants C_1 and C_2, such that

$$v_j^n \leq C_1 + C_2 x_j, \quad j = 0, ..., L, \quad n = N, N-1, ..., 0.$$

For the call option, $C_1 = 0$ and $C_2 = 1$. For the put option, $C_1 = K$ and $C_2 = 0$.

Proof. For the proof, see [6][proof of Theorem 5.1]. □

5.3.2 Numerical implementation

The main algorithm for the put option.

1. Denote by x, *gput* vectors with coordinates corresponding, respectively, to discrete points x_l, values of payoff function $g(x_l)$ in this points, $l = 0, ..., J$.

2. Denote by *Bwput* and *Bvput* vectors with coordinates corresponding to approximate solutions in $n+1$ and n time step, respectively, $n = N-1, ..., 0$.

3. $Bwput \leftarrow gput$. %terminal conditions

4. for $n = 1 : N$
   $Bvput \leftarrow PDEput(Bwput)$.
   $Bwput \leftarrow Bvput$.
   end.

Function PDEput(u)

1. Denote by *temp* the return vector.

2. $temp(0) \leftarrow gput(0)$. %boundary conditions
   $temp(j) \leftarrow gput(L)$.

3. for $n = 2 : J$

$$y \leftarrow u(j)+\lambda(r-d)x(j)(u(j+1)-u(j))+\mu 0.5\sigma^2 x^2(j)(u(j+1)-2u(j) + u(j-1)) - r\Delta t u(j).$$

   $temp(j) \leftarrow \max(gput, y)$.
   end.
   end.

Moreover, by replacing "put" with "call" in the above algorithm, one can obtain the main algorithm for the call option and a function PDEcall(u).

5.4 A numerical experiment

In this section we test schemes defines by (5.8), (5.24)-(5.25) and (5.31). All schemes has been implemented in Matlab. We have chosen the following parameters:

$$r = 0.1, \quad d = 0 \quad \sigma = 0.2, \quad K = 1, \quad L = 4, \quad T = 1, \quad J = 76 \quad L = 650.$$

A spatial parameter Δx has been calculated to be equal to 0.057. We have chosen $\Delta t = 0.00154$ according to CFL' conditions (5.29), i.e.

$$\Delta t \leq \frac{(\Delta x)^2}{\Delta x |r-d| L + \sigma^2 L^2 + (\Delta x)^2 r}. \tag{5.34}$$

Fixing k to be equal to 10^3 we have specified $\epsilon = 6 \cdot 10^{-4}$ in view of CFL condition (5.9). The same set of discretization parameters has been used for all schemes. Additionally, the "exact solution" was computed by the power-penalty scheme on a very fine grid. The "exact" and approximate solutions, for the power-penalty, predictor-corrector and Brennan and Schwartz algorithms are displayed, respectively, in the Figures: 5.1, 5.2 and 5.3. We are presenting the results for the American put option, the results for the call option are analogies.

In the "visual norm" the schemes produce solutions of more or less the same quality. The difference is hardly visible. To confirm that the power-penalty scheme produce better results, we compare the power-penalty and predictor-corrector schemes. The "zoom in plots" can be viewed on the Figures 5.4-5.5. According to them, the power-penalty scheme gives a slightly better result than the corrector-predictor scheme, and hence, in view of Benth, Karlsen and Reikvam [6][Section 6], better then the Brennan and Schwartz algorithm.

5.5 Conclusions

The numerical examples have confirmed that by improving the classical penalty method to power-penalty method, one can obtain better results. The numerical experiments concludes also the theoretical analysis of convergence done in the previous chapters and illustrates the effectiveness and usefulness of the method. It remains to show that under CFL conditions (5.9) and in a view of the Lemma 4.1.1, the solution of the power penalty scheme converge in $L^\infty_{loc}(\Omega_T)$ to a unique viscosity solution of (2.6), i.e. American option value, as $\Delta x \downarrow 0$.

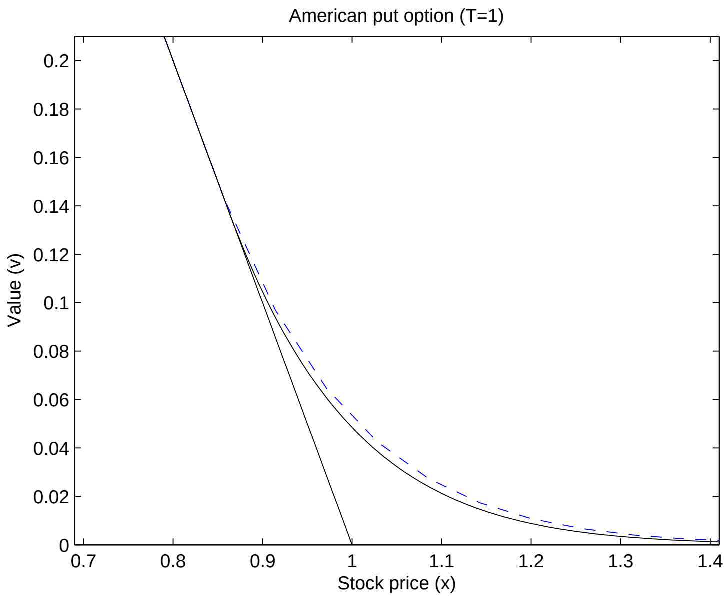

Figure 5.1: *The price of the American put option with expiration time T=1: the exact solution with the payoff function (solid line) and the power-penalty solution (dashed line).*

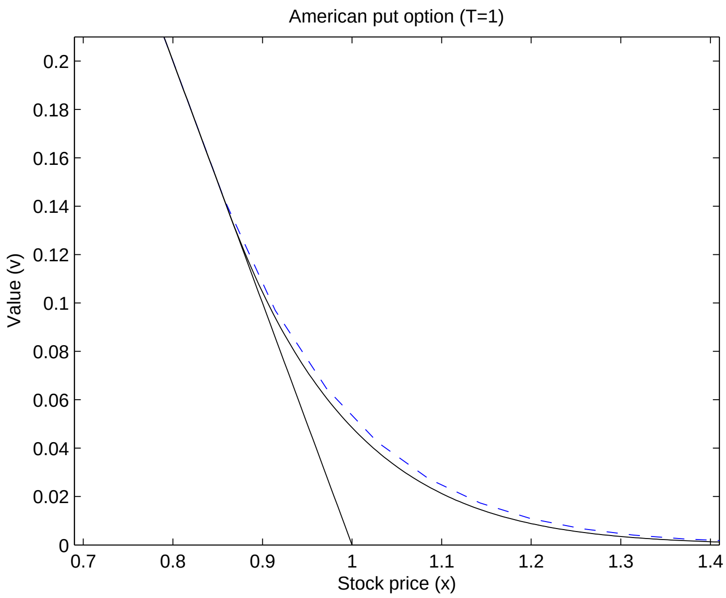

Figure 5.2: *The price of the American put option with expiration time T=1: the exact solution with the payoff function (solid line) and the predictor-corrector solution (dashed line).*

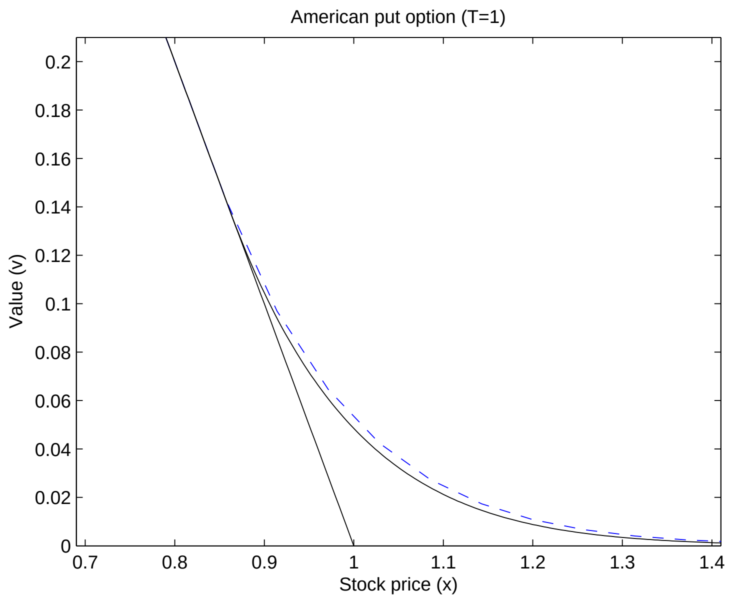

Figure 5.3: *The price of the American put option with expiration time T=1: the exact solution with the payoff function (solid line) and the Brennan and Schwartz solution (dashed line).*

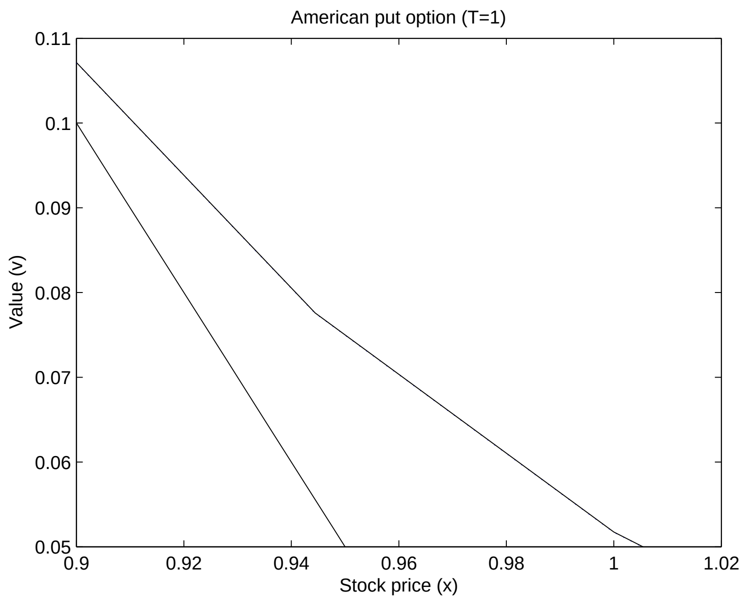

Figure 5.4: *"Zoom-in plots" of the power-penalty solution and the predictor corrector solution (solid upper line) with the exact solution (solid lower line).*

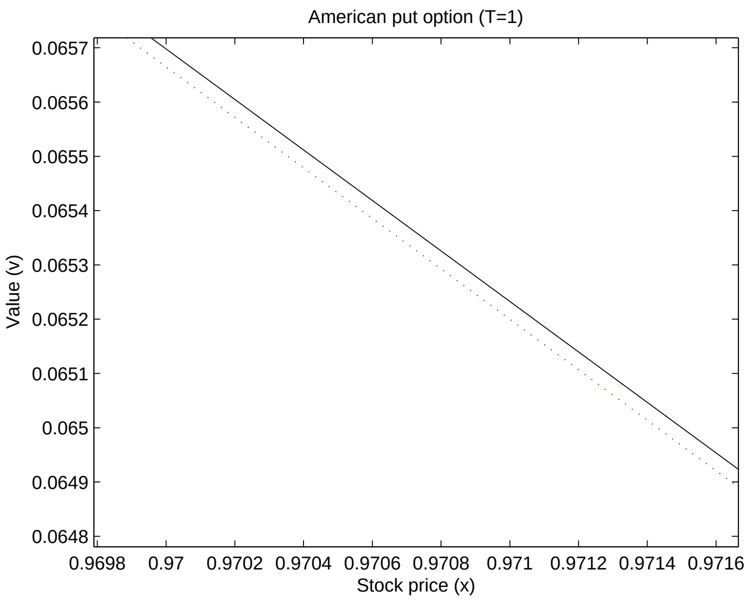

Figure 5.5: *"Zoom-in plots" of the power-penalty solution (dotted line) and the predictor corrector solution (solid line).*

Appendix A

Definitions and Theorems

A.1 Chapter 1

Definition A.1.1. *A probability space is a triple* $(\Omega, \mathcal{F}, P)$*, where* Ω *is a set,* $\mathcal{F}$ *is a* σ*-field of subsets of* Ω *and* P *is a non-negative measure with total mass 1.*

Definition A.1.2. *A filtration with time index* T *is (an increasing) family* $(\mathcal{F}_t, t \in T)$ *of* σ*-algebras of* $\mathcal{F}$ *such that* $\mathcal{F}_s \subset \mathcal{F}_t$ *for all* $s < t$ *in* T*.*

Definition A.1.3. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space, and let* $(E, \mathcal{E})$ *be a measurable space. A mapping* $X : T \times \Omega \to E$*, where* T *is a subset of the extended positive real line* $\overline{R}^+$*, is called a stochastic process if for every* $t \in T$*,* X_t *is an* E*-valued measurable function.*

Definition A.1.4. *A stochastic process* $(X_t)_{t \in T}$ *is called* $\mathcal{F}_t$*-adapted if for every* $t \in T$*, the mapping* $\omega \to X_t(\omega)$ *is* $\mathcal{F}_t$*-measurable.*

Definition A.1.5. *A* T*-valued random variable* τ *is a stopping time with respect to* $\mathcal{F}_t$ *if for every* $t \in T$ *the set* $\{\tau \leq t\}$ *belongs to* $\mathcal{F}_t$*. It can be interpreted that the process knows whether* τ *has happened by time* t *from the information available in* $\mathcal{F}_t$*.*

Definition A.1.6. *A real valued stochastic process* $(X_t)_{t \in T}$*, is said to be a supermartingale, (resp. a submartingale), on* $(\Omega, \mathcal{F}, P)$ *with respect to the filtration* $(\mathcal{F}_t, t \in T)$ *if each* X_t *is integrable,* $\mathcal{F}_t$*-measurable and*

$$E[X_t|\mathcal{F}_s] \leq (\geq) X_s \quad P - a.s. \text{ for } s \leq t \text{ in } T.$$

If the process X *is both a supermartingale and a submartingale than it is said to be martingale.*

Definition A.1.7. *A Brownian motion is a real-valued, continuous stochastic process* $(X_t)_{t \geq 0}$*, with independent and stationary increments. In other words:*

(i) *continuity: P-almost surely the map $s \to X_s(\omega)$ is continuous.*

(ii) *independent increments: if $s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s$.*

(iii) *stationary increments: if $s \leq t$, $X_t - X_s$ and $X_{t-s} - X_0$ have the same probability law.*

A.2 Chapter 3

Cauchy's inequality with ϵ

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon} \quad (a, b > 0, \epsilon > 0).$$

Proof. For proof, see [12]. □

A.3 Chapter 4

Theorem A.3.1 (Arzeli-Ascoli compactness criterion for uniform convergence). *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of real-valued functions defined on R^n, such that*

1. *The sequence $\{f_k\}_{k=1}^{\infty}$ is uniformly bounded by some constant $M > 0$,*
2. *$\{f_k\}_{k=1}^{\infty}$ are uniformly equicontinuous, meaning*

$$\forall_{\epsilon>0} \exists_{\delta>0} \text{ such that } |x - y| < \delta \Rightarrow |f_k(x) - f_k(y)| < \epsilon,$$

for $x, y \in R^n$, $k = 1, 2, ...$

Then there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subseteq \{f_k\}_{k=1}^{\infty}$ and a continuous function f, such that

$$f_{k_j} \to f \text{ uniformly on compact subsets of } R^n.$$

Proof. For proof, see for example [12]. □

Definition A.3.1. *A locally bounded function $u \in USC(\overline{\Omega}_T)$ ($LSC(\overline{\Omega}_T)$) is a viscosity subsolution (supersolution) of (4.20), if and only if, for all $(t, x) \in \Omega_T$ and for all $(a, p, X) \in P^{2,+}u(t, x)$ ($P^{2,-}u(t, x)$) we have:*

$$\begin{cases} min\{-a + (r - d)xp + \frac{1}{2}\sigma^2 x^2 X - ru_\epsilon, u_\epsilon - g\} \leq (\geq) 0, \\ u_\epsilon(T, x) \leq (\geq) g(x) \text{ for } x \in [0, \infty). \end{cases}$$

Theorem A.3.2 (Theorem 4.2.1). *Let u be a subsolution of (4.20) and v be a supersolution of (4.20) such that*

$$u(t,x), -v(t,x) \le L(|x|+1) \text{ and } u(T,x) \le v(T,x)$$

for some constant $L > 0$. Then $u \le v$ in $\overline{\Omega}_T$.

Proof of Theorem 4.2.1 . The proof follows the proof of Theorem 3.3.2 and we will be rather brief here. With the same notations as before it is suffices to prove the comparison under additional assumption

$$\begin{cases} \min\{-\partial_t u + F(t,x,u,\partial_x u, \partial_x^2 u), u - g\} < -\eta/T^2, & \text{in } [0,T) \times [0,\infty) \\ \lim_{t \uparrow T} u(t,x) = \infty, & \text{uniformly on } [0,\infty). \end{cases}$$

Let us suppose that comparison does not hold and that for some $(\bar{t},\bar{x}) \in [0,T) \times [0,\infty)$

$$u(\bar{t},\bar{x}) - v(\bar{t},\bar{x}) = 2\delta > 0. \tag{A.1}$$

With the notation of the previous proof, (A.1) implies:

$$u(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha) \ge \delta. \tag{A.2}$$

The proof remains unchanged including the matrix inequalities (3.25). Let us explain how to get the contradiction to (A.1). By definition of the viscosity sub- and supersolutions,

$$\begin{aligned} &\min\{-a + F(t_\alpha, x_\alpha, u(t_\alpha,x_\alpha), \alpha(x_\alpha - y_\alpha) + \mu e^{\lambda(T-t_\alpha)} x_\alpha, X_\alpha), \\ &\qquad\qquad u(t_\alpha,x_\alpha) - g(x_\alpha)\} \le -\eta/T^2, \\ &\min\{-b + F(t_\alpha, y_\alpha, v(t_\alpha,y_\alpha), \alpha(x_\alpha - y_\alpha) - \mu e^{\lambda(T-t_\alpha)} y_\alpha, Y_\alpha), v(t_\alpha,y_\alpha) - g(y_\alpha)\} \ge 0. \end{aligned}$$

Combining the above inequalities we obtain:

$$\begin{aligned} &\min\{-a + F(t_\alpha, x_\alpha, u(t_\alpha,x_\alpha), \alpha(x_\alpha - y_\alpha) + \mu e^{\lambda(T-t_\alpha)} x_\alpha, X_\alpha), u(t_\alpha,x_\alpha) - g(x_\alpha)\} \\ &-\min\{-b + F(t_\alpha, y_\alpha, v(t_\alpha,y_\alpha), \alpha(x_\alpha - y_\alpha) - \mu e^{\lambda(T-t_\alpha)} y_\alpha, Y_\alpha), v(t_\alpha,y_\alpha) - g(y_\alpha)\} \\ &\qquad\qquad \le -\eta/T^2 < 0. \end{aligned}$$

Since $\min\{a,b\} - \min\{c,d\} < 0$ implies either $a - c < 0$ or $b - d < 0$, we have that either

$$\begin{aligned} \Delta_\alpha := b - a + F(t_\alpha, x_\alpha, u(t_\alpha,x_\alpha), \alpha(x_\alpha - y_\alpha) + \mu e^{\lambda(T-t_\alpha)} x_\alpha, X_\alpha) \\ - F(t_\alpha, y_\alpha, v(t_\alpha,y_\alpha), \alpha(x_\alpha - y_\alpha) - \mu e^{\lambda(T-t_\alpha)} y_\alpha, Y_\alpha) < 0, \end{aligned} \tag{A.3}$$

or

$$u(t_\alpha,x_\alpha) - g(x_\alpha) - v(t_\alpha,y_\alpha) + g(y_\alpha) < 0. \tag{A.4}$$

Choose α so large that $|x_\alpha - y_\alpha| \le \frac{\delta}{2}$. In a view of (A.2)

$$\begin{aligned} g(y_\alpha) - v(t_\alpha, y_\alpha) &= g(x_\alpha) - v(t_\alpha, y_\alpha) + (g(y_\alpha) - g(x_\alpha)) \\ &\ge g(x_\alpha) - v(t_\alpha, y_\alpha) - \frac{\delta}{2} \ge g(x_\alpha) - u(t_\alpha, x_\alpha) + \frac{\delta}{2}. \end{aligned}$$

This implies

$$u(t_\alpha, x_\alpha) - g(x_\alpha) - v(t_\alpha, y_\alpha) + g(y_\alpha) \ge \frac{\delta}{2} > 0,$$

which is a contradiction to (A.4).
On the other hand,

$$\begin{aligned} \Delta_\alpha = \frac{\mu}{2}\lambda e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2) - (r-d)\big(\alpha(x_\alpha^2 - y_\alpha^2) + \mu e^{\lambda(T-t_\alpha)}(x_\alpha^2 + y_\alpha^2)\big) \\ - 1/2\sigma^2\big(x_\alpha^2 X_\alpha - y_\alpha^2 Y_\alpha\big) + r\big(u(t_\alpha, x_\alpha) - v(t_\alpha, y_\alpha)\big). \end{aligned}$$

Similarly to the previous proof, by sending $\alpha \to \infty$ and in a view of (A.3) we obtain

$$\begin{aligned} 0 < -\limsup_{\alpha\uparrow\infty} \Delta_\alpha = -\mu\lambda e^{\lambda(T-\hat{t})}\hat{x}^2 + (r-d)2\mu e^{\lambda(T-\hat{t})}\hat{x}^2 + \mu\lambda e^{\lambda(T-\hat{t})}\hat{x}^2\sigma^2 \\ - r\delta \le 0, \quad \text{(A.5)} \end{aligned}$$

if λ is sufficiently large. This is also a contradiction and the proof of the comparison is finished. □

Appendix B

Numerical code

B.1 The power-penalty scheme

```
function [gcall, gput, x, wcall, wput] = PowerPenalty(r,d,sigma,epsilon,
k,K,L,T,J,N)

  deltax = L/J;
  deltat = T/N;

  x=zeros(J+1);
  gcall=zeros(J+1);
  gput=zeros(J+1);

  for j=1:(J+1)
    x(j)=(j-1)*deltax;
    gput(j)= max(K-x(j),0);
    gcall(j)= max(x(j)-K,0);
  end
  wcall=gcall;
  wput =gput;

  %algorithm
  for n=1:N
    vcall = PDEcall(wcall,gcall,r,d,sigma,epsilon,k,K,L,J,deltat);
    wcall = vcall;
  end

  for n=1:N
    vput = PDEput(wput,gput,r,d,sigma,epsilon,k,K,L,J,deltat);
    wput = vput;
  end
```

```
end

%Functions
function y=max(x1, x2)
  if x1>x2
    y=x1;
  else
    y=x2;
  end
    end


function temp = PDEcall(u,gcall,r,d,sigma,epsilon,k,K,L,J,deltat)
  temp(1)=0;
  temp(J+1)=max(L-K,0);
  for j=2:J
    m=max(gcall(j)-u(j), 0);
    y= u(j)+(r-d)*(j-1)*deltat*(u(j+1)-u(j))+0.5*sigma*sigma*(j-1)*(j-1)
*deltat*(u(j+1)-2*u(j)+u(j-1))-r*deltat*u(j)+deltat*(1 / epsilon)*(m^(1/k));
    temp(j)=max(gcall(j),y);
  end
    end

function temp = PDEput(u,gput,r,d,sigma,epsilon,k,K,L,J,deltat)
  temp(1)=K;
  temp(J+1)=max(K-L,0);
  for j=2:J
    m=max(gput(j)-u(j), 0);
    y = u(j)+(r-d)*(j-1)*deltat*(u(j+1)-u(j))+0.5*sigma*sigma*(j-1)*(j-1)
*deltat*(u(j+1)-2*u(j)+u(j-1))-r*deltat*u(j)+deltat*(1 / epsilon)*(m^(1/k));
    temp(j)=max(gput(j),y);
  end
    end
```

B.2 The predictor-corrector scheme

```
function [gcall,gput, x, Pwcall, Pwput] = predictorCorrector(r,d,sigma,K,L,T,J,N)

  deltax = L/J;
  deltat = T/N;

  x=zeros(J+1);
  gcall=zeros(J+1);
```

```
  gput=zeros(J+1);

  for j=1:(J+1)
    x(j)=(j-1)*deltax;
    gput(j)= max(K-x(j),0);
    gcall(j)= max(x(j)-K,0);
  end

  Pwcall=gcall;
  Pwput =gput;

  %algorithm
  for n=1:N
    vcall = PDEcall(Pwcall,gcall,x,r,d,sigma,K,L,J,deltat);
    Pwcall = vcall;
  end

  for n=1:N
    vput = PDEput(Pwput,gput,x,r,d,sigma,K,L,J,deltat);
    Pwput = vput;
  end
end


%Functions
function y=max(x1, x2)
  if x1>x2
    y=x1;
  else
    y=x2;
  end
  end

function temp = PDEcall(u,gcall,x,r,d,sigma,K,L,J,deltat)
  temp(1)=0;
  temp(J+1)=max(L-K,0);
  for j=2:J
    y= u(j)+(r-d)*(j-1)*deltat*(u(j+1)-u(j))+0.5*sigma*sigma*(j-1)*(j-1)
*deltat*(u(j+1)-2*u(j)+u(j-1))-r*deltat*u(j);
    if gcall(j)>y
      temp(j)= y+deltat*(max(d*x(j)-r*K,0));
    else
      temp(j)= y;
    end
```

```
  end
    end

function temp = PDEput(u,gput,x,r,d,sigma,K,L,J,deltat)
  temp(1)=K;
  temp(J+1)=max(K-L,0);
  for j=2:J
    y= u(j)+(r-d)*(j-1)*deltat*(u(j+1)-u(j))+0.5*sigma*sigma*(j-1)*(j-1)
*deltat*(u(j+1)-2*u(j)+u(j-1))-r*deltat*u(j);
    if gput(j)>y
      temp(j)= y+deltat*(max(r*K-d*x(j),0));
    else
      temp(j)= y;
    end
  end
end
```

B.3 The Brennan and Schwartz algorithm

```
function [gcall,gput,x,Bwcall,Bwput] = BrennanSchwartz(r,d,sigma,K,L,T,J,N)

  deltax = L/J;
  deltat = T/N;

  x=zeros(J+1);
  gcall=zeros(J+1);
  gput=zeros(J+1);

    for j=1:(J+1)
      x(j)=(j-1)*deltax;
      gput(j)= max(K-x(j),0);
      gcall(j)= max(x(j)-K,0);
    end
    Bwcall=gcall;
    Bwput =gput;

    %algorithm
    for n=1:N
      Bvcall = PDEcall(Bwcall,gcall,r,d,sigma,deltat,J);
      Bwcall = Bvcall;
    end
```

```
    for n=1:N
      Bvput = PDEput(Bwput,gput,r,d,sigma,deltat,J);
      Bwput = Bvput;
    end
end

%Functions
function y=max(x1, x2)
  if x1>x2
    y=x1;
  else
    y=x2;
  end
end


function temp = PDEcall(u,gcall,r,d,sigma,deltat,J)
  temp(1)=gcall(1);
  temp(J+1)=gcall(J+1);
  for j=2:J
    y = u(j)+(r-d)*(j-1)*deltat*(u(j+1)-u(j))+0.5*sigma*sigma*(j-1)*(j-1)
*deltat*(u(j+1)-2*u(j)+u(j-1))-r*deltat*u(j);
    temp(j) = max(gcall(j),y);
  end
end

function temp = PDEput(u,gput,r,d,sigma,deltat,J)
  temp(1)=gput(1);
  temp(J+1)=gput(J+1);
  for j=2:J
    y = u(j)+(r-d)*(j-1)*deltat*(u(j+1)-u(j))+0.5*sigma*sigma*(j-1)*(j-1)
*deltat*(u(j+1)-2*u(j)+u(j-1))-r*deltat*u(j);
    temp(j) = max(gput(j),y);
  end
end
```

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