Interest rate modelling with non-Gaussian Ornstein-Uhlenbeck processes

by

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Abstract

We study a mean-reverting model for interest rates. The model is an extension of the Vasicek model and is a sum of non-Gaussian Ornstein-Uhlenbeck processes with subordinators, i.e. Lévy processes with only positive jumps, giving variation of the interest rate. The model have the advantage that it gives only positive interest rates, contrary to the Vasicek model. We calculate explicit results for the characteristic function and the autocorrelation function of the interest rate for both general subordinators and the case where the subordinators are compound Poisson. We also find prices of zero-coupon bonds and European options written on these bonds by applying Fourier methods. It seems that the model is simple enough to allow for analytical pricing of bonds and options in addition to capture the characteristics of the interest rate. In the end we demonstrate in a simulation how the model behave with certain values of the variables.

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Chapter 1

Introduction

To model interest rates and price interest rate derivatives on bonds are great and demanding areas in mathematical finance. Interest rate derivatives are instruments whose payoff depends on the level of the interest rate. The volume of trading in interest rate derivatives increased in the 1980s and 1990s. The challenge is to find models capturing the characteristics of the interest rate in a reasonable degree. It's important that they're analytically tractable as well. The Vasicek model is one of the first models of term-structure and is still an attractive class of models because of its analytical properties. However it has the property that the interest rate can be negative. Other models derived are the Cox-Ingersoll-Ross (CIR) model and the Hull and White model, where the latter has time-dependent coefficients. The CIR model is an extension of the Vasicek model, but has the advantage that it only gives positive values. Some important interest rate derivatives are interest rate caps/floors, swap options and bond options. We will only investigate bond options in this thesis.

In this thesis we discuss two models of the short-term interest rate. The first one is the Vasicek model, and the other is an extension of the Vasicek model, proposed in [2] for modelling spot electricity prices. It's motivated from [1]. The model is a sum of Ornstein-Uhlenbeck (OU) processes, each with a pure jump process with only positive jumps. It fits well for modelling spot electricity prices, because they are often dependent of the season. Since the process is a sum of OU processes, it seems reasonable to take one of them to model the seasonality. In our case, with interest rates, it has the advantage that it secure the interest rate to be positive. The model is also simple enough, such that one can calculate analytical expressions for common interest rate derivatives.

Both of the models we consider are mean-reverting. That a model is mean-reverting means that it will eventually pull back to some average level.

The main part of the thesis is to examine the new class of models described above. We find an autocorrelation function of the interest rate given by a sum of weighted exponentials of a constant times the time shift. We want to find out how easy it is to obtain results of zero-coupon bond prices and prices of European options written on these bonds. It turns out that we can easily derive explicit results for the price of zero-coupon bonds by looking at the expression for the interest rate directly. To find prices of European options written on zero-coupon bonds are more complicated than finding prices of European options written on other securities. That is because interest rates are used for discounting as well as for defining the payoff from the option. We find the price by applying inverse Fourier transform, and one can use fast Fourier transform techniques to compute it further.

The thesis is organized as follows: In chapter 2 we give some well-known definitions and results from measure and probability theory. We also introduce stochastic processes like Brownian motion and pure jump processes and state some of their properties. All the results are given without proof. In chapter 3 we consider the Vasicek model and find prices of zero-coupon bonds and bond options. We introduce the new model, the extension of the Vasicek model, in chapter 4. The rest of the thesis is dedicated to the extended Vasicek model. We find the stationary characteristics, characteristic function and the correlation function. In chapter 5 and 6 we derive prices of zero-coupon bonds and European options written on these bonds. We state all our results both in general and for the special case when the Lévy processes are compound Poisson. In chapter 7 we simulate the interest rate and prices of zero-coupon bonds with maturity in one year.

Appendix A contains the matlab files used to simulate the interest rate and the prices of zero-coupon bonds.

Chapter 2

Some Basic Theory

Before we start looking at our problem we need some basic theory. The theory stated in this chapter is well-known and we skip the proofs. More information and proofs can be found in any book in stochastic analysis. First we introduce the notion of a σ-algebra, a probability measure and a probability space. We state some well-known and useful theorems from measure theory. In the end we define a stochastic process and look at Lévy processes and their properties.

2.1 Measure Theory and Probability Theory

Definition 2.1 *If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with*

- $\emptyset \in \mathcal{F}$.
- $F \in \mathcal{F} \;\Rightarrow\; F^c \in \mathcal{F}$, *where* $F^c = \Omega \setminus F$.
- $F_1, F_2, \dots \in \mathcal{F} \;\Rightarrow\; F = \bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a *measurable space.*

Definition 2.2 *A probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0,1]$ such that*

- $P(\emptyset) = 0$, $P(\Omega) = 1$.
- *If $A_1, A_2, \dots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint, i.e. $A_i \cap A_j = \emptyset, i \neq j$, then*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P\left(A_i\right).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space.* We also define the notion of a filtration.

Definition 2.3 *A filtration on a measurable space $(\Omega, \mathcal{F})$ is an increasing family of σ-algebras $\{\mathcal{F}_t\} \in \mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$, where $s \leq t$.*

A filtration is in mathematical finance used to describe the information we got up till today. As time goes by, we know more and more. So it has to be increasing.

We make the following assumption throughout the thesis:

Assumption 2.1 *All our models are modelled directly under the risk-neutral probability measure Q and the probability space we're working in, is $(\Omega, \mathcal{F}, Q)$.*

The next thorem will be used to put the limit outside expectations when computing characteristic functions.

Theorem 2.1 ***Bounded Convergence theorem***
Let $\mu(\Omega) < \infty$. If there exists a $0 < k < \infty$ such that $|f_n| \leq k$ μ-a.e. and $f_n \to f$ μ-a.e. then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu,$$

and

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

Fubini's theorem allows us to change the order of integrals.

Theorem 2.2 ***Fubini's theorem***
Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$, be σ-finite measurable spaces (i.e. there exist a countable collection of sets $A_1^i, A_2^i, \ldots, \in \mathcal{F}_i$ such that $\cup_{n\geq 1} A_n^i = \Omega$ and $\mu_i(A_n^i) < \infty$ for all $n \geq 1$ and $i = 1, 2$) and let $f \in L^1(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$. Then

$$\int_{\Omega_1} \left(\int_{\Omega_2} f d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f d\mu_1 \right) d\mu_2.$$

We now define the notion of a stochastic process.

Definition 2.4 *A stochastic process is a parametrized collection of random variables $\{X_t\}$, $t \in T$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$.*

2.2 Lévy Processes

The Lévy process is an example of a stochastic process. It's named after the mathematician Paul Lévy. A Lévy process L_t has the following properties

- $L_0 = 0$.
- L_t has stationary increments, i.e. the probability distribution of any increment $L_t - L_s$, depends only on the length $t - s$.
- L_t has independent increments, i.e. any two non-overlapping increments are independent of each other.

We will use the characteristic function of a Lévy process, given by the Lévy-Khinchin representation, to obtain several results throughout the thesis.

Theorem 2.3 ***Lévy-Khinchin representation***
Let (X_t) be a Lévy process on $\mathbb{R}$ with characteristic triplet (A, ν, γ). Then the characteristic function of (X_t) is

$$\mathrm{E}\left[e^{izX_t}\right] = e^{\psi(z)t}, \qquad z \in \mathbb{R}^d,$$

where

$$\psi(z) = -\frac{1}{2}z.Az + i\gamma.z + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\boldsymbol{1}_{|x|\leq 1}\right)\nu(x)dx.$$

A is the covariance matrix of a Brownian motion, ν is the Lévy measure and γ is the drift.

2.2.1 Brownian Motion

A Brownian motion B_t is an example of a Lévy process. It was first studied by Robert Brown in 1827. He was studying pollen particles floating in water under the microscope. Brownian motion is often used because it makes computations simple, not because of its accuracy. It is a Lévy process, so it satisfies all the properties above, but it also satisfies

- $B_t - B_s \sim \mathcal{N}(0, t-s)$.

For a Brownian motion, the characteristic triplet is $(1, 0, 0)$, so, from Lévy-Khinchin representation, the characteristic function of a Brownian motion $B(t)$ becomes

$$\mathrm{E}\left[e^{izB(t)}\right] = e^{-\frac{1}{2}z^2 t} \tag{2.1}$$

A simple, but useful tool for solving stochastic differential equations (SDE) is the Itô formula.

Theorem 2.4 ***The one - dimensional Itô formula.***
Let X_t be an Itô process where the dynamics is given by

$$dX_t = udt + vdB_t.$$

Let $f(t, x) \in C^2([0, \infty] \times \mathbb{R})$, i.e. f is two times continuously differentiable on $[0, \infty] \times \mathbb{R}$. Then

$$Y_t = f(t, X_t)$$

is again an Itô process, and

$$dY_t = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2,$$

where

$$f_t(t, X_t) = \frac{\partial}{\partial t}f(t, X_t), \quad f_x(t, X_t) = \frac{\partial}{\partial x}f(t, X_t), \quad f_{xx}(t, X_t) = \frac{\partial^2}{\partial x^2}F(t, X_t),$$

and $(dX_t)^2$ is computed according to the rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt.$$

Proof. For proof, look in Øksendal [4].

Another useful property, which we will use to find the variance of the Vasicek model, is the Itô isometry.

Lemma 2.1 ***Itô isometry.***

$$\mathrm{E}\left[\left(\int_S^T f(t,\omega)dB_t\right)^2\right] = \mathrm{E}\left[\int_S^T f^2(t,\omega)dt\right],$$

for all f in the class of functions

$$g(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

such that

- $(t,\omega) \to g(t,\omega)$ *is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on $[0,\infty)$.*
- $g(t,\omega)$ *is $\mathcal{F}_t$-adapted, i.e. $g(t,\omega)$ is $\mathcal{F}_t$-measurable for all t.*
- $\mathrm{E}\left[\int_S^T g^2(t,\omega)dt\right] < \infty$.

In the section discussion the pricing of European bond options we make use of the Girsanov theorem to change measure.

Theorem 2.5 ***Girsanov's theorem***
Let $B(t)$ be a standard brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that γ_u is a measurable process such that

$$\mathrm{E}_{\mathbb{P}}\left[e^{\int_0^T \gamma_u dB(u) - \frac{1}{2}\int_0^T |\gamma_u|^2 du}\right] = 1. \tag{2.2}$$

Define a probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ by means of the Radon-Nikodým derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{\int_0^T \gamma_u dB(u) - \frac{1}{2}\int_0^T |\gamma_u|^2 du}, \quad \mathbb{P} - a.s.$$

Then the process $\tilde{B}(t)$ given by the formula

$$\tilde{B}(t) = B(t) - \int_0^t \gamma_u du,$$

for all $t \in [0,T]$, follows a standard Brownian motion on the space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$.

A sufficient condition for (2.2) to hold is the Novikov condition:

$$\mathrm{E}_{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T |\gamma_u|^2 du}\right] < \infty.$$

2.2.2 Lévy Processes with Jumps

To find an explicit representation of $r(t)$ in the extended Vasicek model we will need the Itô formula for scalar Lévy processes.

Proposition 2.1 ***Itô formula for scalar Lévy processes***
Let $(X_t)_{t\geq 0}$ be a Lévy process and $f : \mathbb{R} \to \mathbb{R}$ a C^2-function (i.e. 2 times differentiable with continuous derivatives). Then

$$\begin{aligned} f(X_t) &= f(0) + \int_0^t \frac{\sigma^2}{2}\frac{d^2}{dx^2}f(X_s)ds + \int_0^t \frac{d}{dx}f(X_{s-})dX_s \\ &+ \sum_{\substack{0\leq s\leq t \\ \Delta X_s \neq 0}} \left[f(X_{s-} + \Delta X_s) - f(X_{s-}) - \Delta X_s \frac{d}{dx}f(X_{s-})\right]. \end{aligned}$$

We are going to use the generalized case where f also depends on time. Let $(X_t)_{t\geq 0}$ be a Lévy process and let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a $C^{1,2}$-function (i.e 1 time differentiable in the first variable and 2 times differentiable in the second). Then

$$\begin{aligned} f(t, X_t) \;=\; & f(0, X_0) + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-}) dX_s + \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2}(s, X_s) \right] ds \\ & + \sum_{\substack{0 \leq s \leq t \\ \Delta X_s \neq 0}} \left[f(s, X_{s-} + \Delta X_s) - f(s, X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s, X_{s-}) \right]. \end{aligned}$$

The Lévy processes in the extended Vasicek model are subordinators. That is, they are jump processes with only positive jumps. The characteristic triplet is then $(0, \nu, 0)$. From Lévy-Khinchin representation the characteristic function of such processes is

$$\mathrm{E}\left[e^{izL(t)} \right] = e^{\psi(z)t}, \tag{2.3}$$

where $\psi(z) = \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{|x|\leq 1} \right) \nu(x) dx$.

An example of a subordinator is the compound Poisson process and is defined as follows

Definition 2.5 *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process X_t defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where the jump sizes Y_i are independent and identical distributed (i.i.d.) with distribution f, and (N_t) is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$.

Chapter 3

The Vasicek Model

The model analysed by Vasicek in 1977 is one of the first models of term structure. It has some qualities that makes it attractive. It is linear and can therefore be solved explicitly. Its distribution is Gaussian, and zero-coupon bonds and other derivatives are easily obtained. However, a huge drawback is that it allows the interest rate to be negative.

The Vasicek model takes the form

$$dr(t) = (\mu - \alpha r(t))dt + \sigma dB(t). \tag{3.1}$$

It's a mean-reverting Ornstein-Uhlenbeck process where $B(t)$ is a Brownian motion and where μ, α and σ are strictly positive constants. That the process is mean-reverting means that it will eventually pull back towards some long-run average level. That is, if the interest-rate is higher than the expected, it will tend to decrease, and if it is lower, it will tend to increase.

3.1 Solution and Distribution

The solution of the stochastic differential equation above is given by the following proposition.

Proposition 3.1 *The solution of* (3.1) *is given by*

$$r(t) = r(s)e^{-\alpha(t-s)} + \frac{\mu}{\alpha}\left(1 - e^{-\alpha(t-s)}\right) + \sigma \int_s^t e^{-\alpha(t-u)} dB(u), \tag{3.2}$$

where the process starts at time $s \leq t$.

Proof. To prove the proposition we have to use the Itô formula (Theorem 2.4). If we use Itô's formula on $e^{\alpha t} r(t)$ and insert the dynamics of $r(t)$ from (3.1) we get

$$d(e^{\alpha t} r(t)) = \alpha e^{\alpha t} r(t) dt + e^{\alpha t} dr(t) = e^{\alpha t}[\mu dt + \sigma dB(t)].$$

So,

$$e^{\alpha t} r(t) - e^{\alpha s} r(s) = \mu \int_s^t e^{\alpha u} du + \sigma \int_s^t e^{\alpha u} dB(u).$$

And finally we get the solution of $r(t)$:

$$r(t) = r(s)e^{-\alpha(t-s)} + \frac{\mu}{\alpha}\left(1 - e^{-\alpha(t-s)}\right) + \sigma \int_s^t e^{-\alpha(t-u)} dB(u)$$

□

We can now find the distribution of $r(t)$.

Proposition 3.2 *The process $r(t)$, given by (3.2), is Gaussian distributed with expectation*

$$\mathrm{E}\left[r(t)\right] = r(s)e^{-\alpha(t-s)} + \frac{\mu}{\alpha}\left(1 - e^{-\alpha(t-s)}\right)$$

and variance

$$\mathrm{Var}\left[r(t)\right] = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right).$$

And when time goes to infinity we get

$$\lim_{t\to\infty} \mathrm{E}\left[r(t)\right] = \frac{\mu}{\alpha}$$

and

$$\lim_{t\to\infty} \mathrm{Var}\left[r(t)\right] = \frac{\sigma^2}{2\alpha}.$$

Proof. Since $e^{-\alpha(t-u)}$ is deterministic, we get from the properties of Brownian motion and Itô isometry that $\sigma \int_s^t e^{-\alpha(t-u)} dB(u)$ is Gaussian with expected value zero and variance given by

$$\begin{aligned}\mathrm{Var}\left[\sigma \int_s^t e^{-\alpha(t-u)} dB(u)\right] &= \mathrm{E}\left[\left(\sigma \int_s^t e^{-\alpha(t-u)} dB(u)\right)^2\right] - \mathrm{E}\left[\sigma \int_s^t e^{-\alpha(t-u)} dB(u)\right]^2 \\ &= \sigma^2 \int_s^t e^{-2\alpha(t-u)} du = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right).\end{aligned}$$

It follows that the process $r(t)$ is Gaussian distributed with

$$\mathrm{E}\left[r(t)\right] = r(s)e^{-\alpha(t-s)} + \frac{\mu}{\alpha}\left(1 - e^{-\alpha(t-s)}\right)$$

and

$$\mathrm{Var}\left[r(t)\right] = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right).$$

Finding the properties of $r(t)$ when time goes to infinity is straight forward,

$$\lim_{t\to\infty} \mathrm{E}\left[r(t)\right] = \lim_{t\to\infty}\left[r(s)e^{-\alpha(t-s)} + \frac{\mu}{\alpha}\left(1 - e^{-\alpha(t-s)}\right)\right] = \frac{\mu}{\alpha}$$

and

$$\lim_{t\to\infty} \mathrm{Var}\left[r(t)\right] = \lim_{t\to\infty}\left[\frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)\right] = \frac{\sigma^2}{2\alpha}.$$

And the proof is complete.

□

3.2 The Theoretical Autocorrelation Function of $r(t)$

We want to calculate the theoretical autocorrelation function for $r(t)$. The autocorrelation function says something about the degree of similarity between $r(t)$ and a time shifted version of itself. It can take values in the interval $[-1, 1]$, where 1 means perfect positive correlation, and -1 means perfect negative correlation.

The correlation function of r(t) is defined by

$$\begin{aligned} \operatorname{corr}(r(t), r(t+\tau)) &= \frac{\mathrm{E}\left[(r(t) - \mathrm{E}[r(t)])\,(r(t+\tau) - \mathrm{E}[r(t+\tau)])\right]}{\sqrt{\operatorname{Var}[r(t)]\operatorname{Var}[r(t+\tau)]}} \\ &= \frac{\mathrm{E}[r(t)r(t+\tau)] - \mathrm{E}[r(t)]\,\mathrm{E}[r(t+\tau)]}{\sqrt{\operatorname{Var}[r(t)]\operatorname{Var}[r(t+\tau)]}}. \end{aligned}$$

We compute the parts seperately. First look at $\mathrm{E}[r(t)]\,\mathrm{E}[r(t+\tau)]$. We use the expectation derived in the previous section in Proposition (3.2).

$$\begin{aligned} &\mathrm{E}[r(t)]\,\mathrm{E}[r(t+\tau)] \\ &\quad= \left(r(s)e^{-\alpha(t-s)} + \frac{\mu}{\alpha}(1 - e^{-\alpha(t-s)})\right)\left(r(s)e^{-\alpha(t+\tau-s)} + \frac{\mu}{\alpha}(1 - e^{-\alpha(t+\tau-s)})\right) \\ &\quad= B. \end{aligned}$$

Then we look at $\mathrm{E}[r(t)r(t+\tau)]$. To calculate this part, notice that we from the properties of Brownian motion have that $\mathrm{E}\left[\int_s^t e^{-\alpha(t-u)} dB(u)\right]$ is zero, since $e^{-\alpha(t-u)}$ is deterministic. Remember that $r(t)$ is given by (3.2).

$$\begin{aligned} \mathrm{E}[r(t)r(t+\tau)] &= \mathrm{E}\Bigg[\left(r(s)e^{-\alpha(t-s)} + \frac{\mu}{\alpha}(1 - e^{-\alpha(t-s)}) + \sigma\int_s^t e^{-\alpha(t-u)} dB(u)\right) \\ &\qquad \times\left(r(s)e^{-\alpha(t+\tau-s)} + \frac{\mu}{\alpha}(1 - e^{-\alpha(t+\tau-s)}) + \sigma\int_s^{t+\tau} e^{-\alpha(t+\tau-u)} dB(u)\right)\Bigg] \\ &= B + \sigma^2\mathrm{E}\left[\int_s^t e^{-\alpha(t-u)} dB(u) \int_s^{t+\tau} e^{-\alpha(t+\tau-u)} dB(u)\right]. \end{aligned}$$

The expectation in the last term of the above equation can be computed as

$$\begin{aligned} &\mathrm{E}\left[\int_s^t e^{-\alpha(t-u)} dB(u) \int_s^{t+\tau} e^{-\alpha(t+\tau-u)} dB(u)\right] \\ &= \mathrm{E}\left[\int_s^t e^{-\alpha(t-u)} dB(u)\left(\int_s^t e^{-\alpha(t+\tau-u)} dB(u) + \int_t^{t+\tau} e^{-\alpha(t+\tau-u)} dB(u)\right)\right] \\ &= \mathrm{E}\left[e^{-\alpha\tau}\left(\int_s^t e^{-\alpha(t-u)} dB(u)\right)^2\right] + \mathrm{E}\left[\int_s^t e^{-\alpha(t-u)} dB(u)\right]\mathrm{E}\left[\int_t^{t+\tau} e^{-\alpha(t+\tau-u)} dB(u)\right] \\ &= e^{-\alpha\tau}\mathrm{E}\left[\int_s^t e^{-2\alpha(t-u)} du\right] = \frac{1}{2\alpha}e^{-\alpha\tau}(1 - e^{-2\alpha(t-s)}). \end{aligned}$$

Here we used the fact that $\int_s^t e^{-\alpha(t-u)} dB(u)$ and $\int_t^{t+\tau} e^{-\alpha(t+\tau-u)} dB(u)$ are independent of each other, and that the expectation of both of them are zero. We also used Itô isometry.

Let's put it all together. Then we get

$$\begin{aligned}\mathrm{E}\left[r(t)r(t+\tau)\right]-\mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right] &= B+\frac{\sigma^2}{2\alpha}e^{-\alpha\tau}\left(1-e^{-2\alpha(t-s)}\right)-B\\ &= \frac{\sigma^2}{2\alpha}e^{-\alpha\tau}\left(1-e^{-2\alpha(t-s)}\right).\end{aligned}$$

Let's look at $\mathrm{Var}\left[r(t)\right]\mathrm{Var}\left[r(t+\tau)\right]$. We use the variance derived in the previous section in Proposition 3.2

$$\begin{aligned}\mathrm{Var}\left[r(t)\right]\mathrm{Var}\left[r(t+\tau)\right] &= \frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(t-s)}\right)\frac{\sigma^2}{2\alpha}\left(1-e^{-2\alpha(t+\tau-s)}\right)\\ &= \frac{\sigma^4}{4\alpha^2}\left(1-e^{-2\alpha(t-s)}\right)\left(1-e^{-2\alpha(t+\tau-s)}\right).\end{aligned}$$

Proposition 3.3 *The correlation function of $r(t)$ is*

$$\mathrm{corr}\left(r(t),r(t+\tau)\right)=\frac{e^{-\alpha\tau}\left(1-e^{-2\alpha(t-s)}\right)}{\sqrt{\left(1-e^{-2\alpha(t-s)}\right)\left(1-e^{-2\alpha(t+\tau-s)}\right)}}.$$

When time goes to infinity the correlation function of $r(t)$ tends to

$$\lim_{t\to\infty}\mathrm{corr}\left(r(t),r(t+\tau)\right)=e^{-\alpha\tau}.$$

3.3 Zero-Coupon Bond Prices

We are interested in finding the prices of zero-coupon bonds, where we assume that $r(t)$ is modelled directly under the risk-neutral probability measure Q. A zero-coupon bond is a bond paying 1 currency at a future time T with no coupons paid inbetween.

Definition 3.1 *The price of a zero-coupon bond at time $t\le T$ is*

$$P(t,T)=\mathrm{E}_Q\left[e^{-\int_t^T r(s)ds}\mid\mathcal{F}_t\right].$$

To find the price we need to evaluate $-\int_t^T r(s)ds$. As in (3.2), $r(s)$ is given by

$$r(s)=r(t)e^{-\alpha(s-t)}+\frac{\mu}{\alpha}\left(1-e^{-\alpha(s-t)}\right)+\sigma\int_t^s e^{-\alpha(s-u)}dB(u),$$

where the process starts at time $t\le s$.

$$\begin{aligned}-\int_t^T r(s)ds &= -\int_t^T r(t)e^{-\alpha(s-t)}ds-\frac{\mu}{\alpha}\int_t^T\left(1-e^{-\alpha(s-t)}\right)ds\\ &\quad -\sigma\int_t^T\int_t^s e^{-\alpha(s-u)}dB(u)ds\\ &= -I_1-I_2-I_3.\end{aligned}$$

To make it easier, we compute the integrals seperately. We start with I_1 and I_2. It's easy to see that

$$I_1 = r(t)\int_t^T e^{-\alpha(s-t)}ds = r(t)\frac{1}{\alpha}\left(1-e^{-\alpha(T-t)}\right) \tag{3.3}$$

$$I_2 = \frac{\mu}{\alpha}\int_t^T \left(1-e^{-\alpha(s-t)}\right)ds = \frac{\mu}{\alpha}(T-t) - \frac{\mu}{\alpha^2}\left(1-e^{-\alpha(T-t)}\right). \tag{3.4}$$

Then look at the last integral I_3.

$$\begin{aligned} I_3 &= \sigma\int_t^T\int_t^s e^{-\alpha(s-u)}dB(u)\ ds = \sigma\int_t^T e^{\alpha u}\int_u^T e^{-\alpha s}ds\ dB(u) \\ &= \sigma\int_t^T \frac{1}{\alpha}\left(1-e^{-\alpha(T-u)}\right)dB(u). \end{aligned}$$

Here we applied Fubini's theorem to change the order of the integrals. Define now

$$n(t,T) = \frac{1}{\alpha}\left(1-e^{-\alpha(T-t)}\right).$$

Then we get

$$-\int_t^T r(s)ds = -r(t)n(t,T) - \mu\left[\frac{1}{\alpha}(T-t) - \frac{1}{\alpha}n(t,T)\right] - \sigma\int_t^T n(u,T)dB(u).$$

Now calculate

$$\begin{aligned} \int_t^T n(u,T)du &= \int_t^T \frac{1}{\alpha}\left(1-e^{-\alpha(T-u)}\right)du = \frac{1}{\alpha}(T-t) - \frac{1}{\alpha^2}\left(1-e^{-\alpha(T-t)}\right) \\ &= \frac{1}{\alpha}(T-t) - \frac{1}{\alpha}n(t,T), \end{aligned}$$

so we get that

$$-\int_t^T r(s)ds = -r(t)n(t,T) - \mu\int_t^T n(u,T)du - \sigma\int_t^T n(u,T)dB(u).$$

To make the notation easier, let $\xi_T = -\int_t^T r(s)ds$. $n(t,T)$ is deterministic, so we have that $\sigma\int_t^T n(u,T)dB(u)$ is Gaussian with expectation zero and variance $\sigma^2\int_t^T n^2(u,T)du$ by Itô isometry, Lemma 2.1. Also notice that $\int_t^T n(u,T)dB(u)$ is independent of $\mathcal{F}_t$ and that $r(t)$ is $\mathcal{F}_t$-measurable. We therefore have

$$\begin{aligned} P(t,T) &= \mathrm{E}_Q\left[e^{\xi_T}|\mathcal{F}_t\right] = \mathrm{E}_Q\left[e^{\xi_T}\right] \\ &= e^{-r(t)n(t,T)-\mu\int_t^T n(u,T)du}\mathrm{E}_Q\left[e^{-\sigma\int_t^T n(u,T)dB(u)}\right] \\ &= e^{-r(t)n(t,T)-\mu\int_t^T n(u,T)du+\frac{1}{2}\sigma^2\int_t^T n^2(u,T)du}. \end{aligned}$$

We state the result in a proposition.

Proposition 3.4 *The zero-coupon bond price, when $r(t)$ is given by (3.2), is*

$$P(t,T) = e^{m(t,T)-n(t,T)r(t)},$$

where

$$n(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)$$

and

$$m(t,T) = \frac{1}{2}\sigma^2 \int_t^T n^2(u,T)du - \mu \int_t^T n(u,T)du.$$

Chapter 4

Extension of the Vasicek Model with Subordinators

In this chapter, and the rest of the thesis, we are going to investigate an extension of the Vasicek model of the form

$$r(t) = \sum_{k=1}^{n} w_k X_k(t), \tag{4.1}$$

where

$$dX_k(t) = -\alpha_k X_k(t)dt + dL_k(t), \tag{4.2}$$

and

$$X_k(0) = \begin{cases} \frac{r(0)}{w_1} & \text{if } k = 1 \\ 0 & \text{if } k \geq 2 \end{cases}.$$

Here, $L_k(t), k = 1, 2, \ldots, n$ are subordinators. Having several X_k's, rather than just one, gives an opportunity to capture different factors with influence on the interest rate $r(t)$. The model have the advantage that it always gives positive interest rates, something the Vasicek model fails to do. It is, as mentioned before, proposed to model spot electricity prices in [2].

4.1 Solution of $dX_k(t)$ and $r(t)$

To find an explicit solution of (4.2) we use the Itô formula for scalar Lévy processes, Proposition 2.1. Applying Itô's formula on $f(t, X_k(t)) = e^{\alpha_k t} X_k(t)$ we get

$$\begin{aligned} &e^{\alpha_k t} X_k(t) - e^{\alpha_k s} X_k(s) \\ &\quad = \int_s^t e^{\alpha_k u} dX_k(u) + \int_s^t \alpha_k e^{\alpha_k u} X_k(u) du \\ &\quad + \sum_{\substack{s \leq u \leq t \\ \Delta X_k(u) \neq 0}} [e^{\alpha_k u}(X_k(u-) + \Delta X_k(u)) - e^{\alpha_k u} X_k(u-) - \Delta X_k(u) e^{\alpha_k u}] \end{aligned}$$

$$\begin{aligned}
&= \int_s^t e^{\alpha_k u}\left[-\alpha_k X_k(u)du + dL_k(u)\right] + \int_s^t \alpha_k e^{\alpha_k u} X_k(u)du \\
&= \int_s^t e^{\alpha_k u} dL_k(u).
\end{aligned}$$

So an explicit solution of (4.2) becomes

$$X_k(t) = X_k(s)e^{-\alpha_k(t-s)} + \int_s^t e^{-\alpha_k(t-u)} dL_k(u), \tag{4.3}$$

where the process starts at a general time $s \leq t$. We want $r(t)$ to start today, so set $s = 0$. If we put $X_k(t)$ into the expression for $r(t)$ we get

$$\begin{aligned}
r(t) &= \sum_{k=1}^n w_k \left[X_k(0)e^{-\alpha_k t} + \int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \\
&= w_1 \frac{r(0)}{w_1} e^{-\alpha_1 t} + \sum_{k=1}^n w_k \int_0^t e^{-\alpha_k(t-u)} dL_k(u) \\
&= r(0)e^{-\alpha_1 t} + \sum_{k=1}^n w_k \int_0^t e^{-\alpha_k(t-u)} dL_k(u).
\end{aligned}$$

Proposition 4.1 *An explicit solution of* $r(t)$ *starting at time* 0*, is*

$$r(t) = r(0)e^{-\alpha_1 t} + \sum_{k=1}^n w_k \int_0^t e^{-\alpha_k(t-u)} dL_k(u) \tag{4.4}$$

4.2 The Characteristic Function of $r(t)$

We want to find the characteristic function of $r(t)$. The characteristic function defines completely the distribution of any random variable. Generally, the characteristic function of a random variable X is given by

$$\varphi_X(z) = \mathrm{E}\left[e^{izX}\right].$$

So for $r(t)$ we have to compute $\mathrm{E}\left[e^{izr(t)}\right]$;

$$\mathrm{E}\left[e^{izr(t)}\right] = e^{izr(0)e^{-\alpha_1 t}} \mathrm{E}\left[e^{iz\sum_{k=1}^n w_k \int_0^t e^{-\alpha_k(t-u)} dL_k(u)}\right].$$

First we take a look at $\mathrm{E}\left[e^{iz\sum_{k=1}^n w_k \int_0^t e^{-\alpha_k(t-u)} dL_k(u)}\right]$. To do so, let $f_k(u) = e^{-\alpha_k(t-u)}$. The L_k's are independent of each other, so

$$\mathrm{E}\left[e^{iz\sum_{k=1}^n w_k \int_0^t f_k(u) dL_k(u)}\right] = \prod_{k=1}^n \mathrm{E}\left[e^{izw_k \int_0^t f_k(u) dL_k(u)}\right].$$

Let $\{u_j\}_{j=1}^m$ be any partition of the interval $[0, t]$ with $\max_j |u_{j+1} - u_j| < \epsilon$. Then the integral can be written as

$$\int_0^t f_k(u) dL_k(u) = \lim_{\epsilon \to 0} \sum_{j=1}^m f_k(u_j) \Delta L_k(u_j),$$

where $\Delta L_k(u_j) = L_k(u_{j+1}) - L_k(u_j)$ and, since $e^{g(t)}$ is a continuous function, we get

$$\prod_{k=1}^{n} \mathrm{E}\left[e^{izw_k \int_0^t f_k(u)dL_k(u)}\right] = \prod_{k=1}^{n} \mathrm{E}\left[\lim_{\epsilon\to 0} e^{izw_k \sum_{j=1}^{m} f_k(u_j)\Delta L_k(u_j)}\right].$$

The Bounded Convergence Theorem is applied to take the limit outside the expectation. Notice that $\Delta L_k(u_j)$ are independent of $\Delta L_k(u_{j+1})$ for all j since the Lévy processes, $L_k(u)$, have independent increments. Thus

$$\begin{aligned}\prod_{k=1}^{n} \mathrm{E}\left[\lim_{\epsilon\to 0} e^{izw_k \sum_{j=1}^{m} f_k(u_j)\Delta L_k(u_j)}\right] &= \prod_{k=1}^{n} \lim_{\epsilon\to 0} \mathrm{E}\left[e^{izw_k \sum_{j=1}^{m} f_k(u_j)\Delta L_k(u_j)}\right] \\ &= \prod_{k=1}^{n} \lim_{\epsilon\to 0} \prod_{j=1}^{m} \mathrm{E}\left[e^{izw_k f_k(u_j)\Delta L_k(u_j)}\right].\end{aligned}$$

Generally, we have that for a Lévy process $L(t)$, the characteristic function is $\mathrm{E}\left[e^{izL(t)}\right] = e^{\psi(z)t}$, by Lévy-Khinchin representation (Theorem 2.3). It follows that $\mathrm{E}\left[e^{iz\Delta L(t)}\right] = e^{\psi(z)\Delta u}$, where $\psi(z) = \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{|x|\le 1}\right)\nu(x)dx$, since the processes have characteristic triplet $(0,\nu,0)$. Finally

$$\begin{aligned}\prod_{k=1}^{n} \lim_{\epsilon\to 0} \prod_{j=1}^{m} \mathrm{E}\left[e^{izw_k f_k(u_j)\Delta L_k(u_j)}\right] &= \prod_{k=1}^{n} \lim_{\epsilon\to 0} \prod_{j=1}^{m} e^{\psi(zw_k f_k(u_j))\Delta u_j} \\ &= \prod_{k=1}^{n} \lim_{\epsilon\to 0} e^{\sum_{j=1}^{m} \psi(zw_k f_k(u_j))\Delta u_j} \\ &= \prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du} \\ &= e^{\sum_{k=1}^{n} \int_0^t \psi(zw_k f_k(u))du}.\end{aligned}$$

We put it all together in a proposition.

Proposition 4.2 *The characteristic function of $r(t)$ is given by*

$$\mathrm{E}\left[e^{izr(t)}\right] = e^{izr(0)e^{-\alpha_1 t}} e^{\sum_{k=1}^{n} \int_0^t \psi(zw_k f_k(u))du}, \tag{4.5}$$

where

$$\psi(zw_k f_k(u)) = \int_{\mathbb{R}} \left(e^{izw_k f_k(u)x} - 1 - izw_k f_k(u)x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx$$

and $f_k(u) = e^{-\alpha_k(t-u)}$.

Next we find the expectation and variance of $r(t)$. We state the result in a proposition before we prove it.

Proposition 4.3 *The expectation and variance of $r(t)$ are given by*

$$\mathrm{E}\left[r(t)\right] = r(0)e^{-\alpha_1 t} + \sum_{k=1}^{n} \frac{w_k}{\alpha_k}\left(1 - e^{-\alpha_k t}\right)\int_{\mathbb{R}} \left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx, \tag{4.6}$$

and

$$\mathrm{Var}\left[r(t)\right] = \sum_{k=1}^{n} \frac{w_k^2}{2\alpha_k}\left(1 - e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2 \nu(x)dx. \tag{4.7}$$

When time goes to infinity they become

$$\lim_{t\to\infty} \mathrm{E}\left[r(t)\right] = \sum_{k=1}^{n} \frac{w_k}{\alpha_k}\int_{\mathbb{R}} \left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx$$

and

$$\lim_{t\to\infty} \mathrm{Var}\left[r(t)\right] = \sum_{k=1}^{n} \frac{w_k^2}{2\alpha_k}\int_{\mathbb{R}} x^2\nu(x)dx.$$

In the proof we will make use of the following corollary. We state it without proof before proving Proposition 4.3.

Corollary 4.1 *If X is a random variable with characteristic function $\varphi_X(z) = \mathrm{E}[e^{izX}]$ one can find it's n'th moment by using the formula*

$$\mathrm{E}[X^n] = (i)^{-n}\frac{d^n}{dz^n}\mathrm{E}\left[\varphi_X(z)\right]\big|_{z=0}.$$

Proof of Proposition 4.3. We start with the expectation. Notice that L_k is independent of L_j when $k \neq j$. Remember that $r(t)$ is given by (4.4).

$$\begin{aligned}\mathrm{E}\left[r(t)\right] &= \mathrm{E}\left[r(0)e^{-\alpha_1 t} + \sum_{k=1}^{n} w_k\int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right]\\ &= r(0)e^{-\alpha_1 t} + \sum_{k=1}^{n} w_k\mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right].\end{aligned}$$

We have to find $\mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right]$. To do the notation easier, let $f_k(u) = e^{-\alpha_k(t-u)}$. From the calculations leading up to Proposition 4.2, we know that the characteristic function of $\int_0^t f_k(u)dL_k(u)$, with $\psi(zf_k(u)) = \int_{\mathbb{R}}\left(e^{izf_k(u)x} - 1 - izf_k(u)x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx$, is $e^{\int_0^t \psi(zf_k(u))du}$. So by Corollary 4.1

$$\begin{aligned}\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right] &= -i\frac{d}{dz}\,\mathrm{E}\left[e^{iz\int_0^t f_k(u)dL_k(u)}\right]\Bigg|_{z=0}\\ &= -i\frac{d}{dz}\,e^{\int_0^t \psi(zf_k(u))du}\Bigg|_{z=0}\\ &= -i\left[\frac{d}{dz}\int_0^t \psi(zf_k(u))du\right]e^{\int_0^t \psi(zf_k(u))du}\Bigg|_{z=0}\\ &= -i\int_0^t \frac{d}{dz}\psi(zf_k(u))du\; e^{\int_0^t \psi(zf_k(u))du}\Bigg|_{z=0},\end{aligned}$$

with $\psi(zf_k(u))$ as above. We have that $\psi(0) = 0$, so $e^{\int_0^t \psi(zf_k(u))du}|_{z=0} = 1$. Further

$$\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right] = -i\int_0^t \frac{d}{dz}\psi(zf_k(u))du\Bigg|_{z=0} \tag{4.8}$$

We then need to compute $\frac{d}{dz}\psi(zf_k(u))|_{z=0}$.

$$\begin{aligned}\frac{d}{dz}\psi(zf_k(u))\bigg|_{z=0} &= \int_{\mathbb{R}} \left(if_k(u)xe^{izf_k(u)x} - ixf_k(u)\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx\bigg|_{z=0} \\ &= \int_{\mathbb{R}} \left(if_k(u)x - ixf_k(u)\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx.\end{aligned}$$

We get

$$\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right] = \frac{1}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx, \tag{4.9}$$

which is a result of the following computation

$$\begin{aligned}\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right] &= -i\int_0^t \left(\int_{\mathbb{R}}\left(if_k(u)x - ixf_k(u)\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx\right)du \\ &= \int_0^t f_k(u)\left(\int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx\right)du \\ &= \int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx \int_0^t f_k(u)du \\ &= \int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx \int_0^t e^{-\alpha_k(t-u)}du \\ &= \int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx\ \frac{1}{\alpha_k}\left(1-e^{-\alpha_k t}\right).\end{aligned}$$

If we put it all together, we get that

$$\mathrm{E}\left[r(t)\right] = r(0)e^{-\alpha_1 t} + \sum_{k=1}^n \frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx.$$

This proves (4.6). Letting time go to infinity, $\lim_{t\to\infty} e^{-\alpha_k t} = 0$ gives

$$\lim_{t\to\infty}\mathrm{E}\left[r(t)\right] = \sum_{k=1}^n \frac{w_k}{\alpha_k}\int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx.$$

It remains to show that the variance is given by (4.7). Let $f_k(u)$ still be given by $e^{-\alpha_k(t-u)}$. Notice that L_k is independent of L_j when $k \neq j$. We then find

$$\begin{aligned}\mathrm{Var}\left[r(t)\right] &= \mathrm{Var}\left[r(0)e^{-\alpha_1 t} + \sum_{k=1}^n w_k\int_0^t f_k(u)dL_k(u)\right] \\ &= \mathrm{Var}\left[\sum_{k=1}^n w_k\int_0^t f_k(u)dL_k(u)\right] = \sum_{k=1}^n w_k^2 \mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right].\end{aligned}$$

We want to compute $\mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right]$. Generally, we have that $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. So $\mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right] = \mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] - \mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2$. Compute the parts separately. To compute the first part, notice that we from Corollary 4.1 can find the n'th moment of a random variable X by using the formula $\mathrm{E}[X^n] = (i)^{-n}\frac{d^n}{dz^n}\varphi_X(z)|_{z=0}$, where $\varphi_X(z)$ is the characteristic function of X. We have already calculated the characteristic function of $\int_0^t f_k(u)dL_k(u)$, when dealing with the characteristic function of $r(t)$ before Proposition 4.2. Hence

$$\begin{aligned}
\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] &= (i)^{-2}\frac{d^2}{dz^2}\mathrm{E}\left[e^{iz\int_0^t f_k(u)dL_k(u)}\right]\Big|_{z=0} \\
&= -\frac{d^2}{dz^2}e^{\int_0^t \psi(zf_k(u))du}\Big|_{z=0} \\
&= -\frac{d}{dz}\left[\int_0^t \frac{d}{dz}\psi(zf_k(u))du\; e^{\int_0^t \psi(zf_k(u))du}\right]\Big|_{z=0} \\
&= -\Bigg[\left(\int_0^t \frac{d}{dz}\psi(zf_k(u))du\right)^2 e^{\int_0^t \psi(zf_k(u))du} \\
&\quad + \int_0^t \frac{d^2}{dz^2}\psi(zf_k(u))du\; e^{\int_0^t \psi(zf_k(u))du}\Bigg]\Big|_{z=0} \\
&= -\left[\left(\int_0^t \frac{d}{dz}\psi(zf_k(u))du\right)^2 + \int_0^t \frac{d^2}{dz^2}\psi(zf_k(u))du\right]\Big|_{z=0},
\end{aligned}$$

where we used that $e^0 = 1$. We have that $\int_0^t \frac{d}{dz}\psi(zf_k(u))du\big|_{z=0} = i\ \mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]$ from (4.8). Remember that $\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]$ is given by (4.9). We then get

$$\left(\int_0^t \frac{d}{dz}\psi(zf_k(u))du\Big|_{z=0}\right)^2 = -\frac{1}{\alpha_k^2}\left(1-e^{-\alpha_k t}\right)^2\left(\int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx\right)^2$$

So it remains to compute $\int_0^t \frac{d^2}{dz^2}\psi(zf_k(u))du|_{z=0}$. We first compute the expression inside the integral;

$$\begin{aligned}
\frac{d^2}{dz^2}\psi(zf_k(u))\Big|_{z=0} &= \frac{d^2}{dz^2}\int_{\mathbb{R}}\left(e^{izf_k(u)x} - 1 - ixzf_k(u)\mathbf{1}_{|x|\le 1}\right)\nu(x)dx\Big|_{z=0} \\
&= \int_{\mathbb{R}}\frac{d}{dz}\left[\left(if_k(u)xe^{izf_k(u)x} - ixf_k(u)\mathbf{1}_{|x|\le 1}\right)\nu(x)\right]dx\Big|_{z=0} \\
&= -\int_{\mathbb{R}} f_k^2(u)x^2\nu(x)dx.
\end{aligned}$$

Remember that $f_k(u) = e^{-\alpha_k(t-u)}$. It follows that

$$\begin{aligned}
\int_0^t \frac{d^2}{dz^2}\psi(zf_k(u))du\Big|_{z=0} &= -\int_0^t\left(\int_{\mathbb{R}} f_k^2(u)x^2\nu(x)dx\right)du \\
&= -\int_{\mathbb{R}} x^2\nu(x)dx\int_0^t f_k^2(u)du \\
&= -\int_{\mathbb{R}} x^2\nu(x)dx\int_0^t e^{-2\alpha_k(t-u)}du
\end{aligned}$$

$$= -\int_{\mathbb{R}} x^2 \nu(x)dx \frac{1}{2\alpha_k}\left(1 - e^{-2\alpha_k t}\right)$$

Going back to our expression for $\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right]$, and use what we have found, we get

$$\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] = \frac{1}{\alpha_k^2}\left(1 - e^{-\alpha_k t}\right)^2 \left(\int_{\mathbb{R}} \left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx\right)^2 + \frac{1}{2\alpha_k}\left(1 - e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2\nu(x)dx.$$

From (4.9) it follows directly that

$$\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2 = \frac{1}{\alpha_k^2}\left(1 - e^{-\alpha_k t}\right)^2 \left(\int_{\mathbb{R}} \left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx\right)^2.$$

To make the proof of (4.7) complete;

$$\begin{aligned}
\mathrm{Var}\left[r(t)\right] &= \sum_{k=1}^n w_k^2 \mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right] \\
&= \sum_{k=1}^n w_k^2 \left(\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] - \mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2\right) \\
&= \sum_{k=1}^n w_k^2 \Bigg[\frac{1}{\alpha_k^2}\left(1 - e^{-\alpha_k t}\right)^2 \left(\int_{\mathbb{R}} \left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx\right)^2 \\
&\qquad + \frac{1}{2\alpha_k}\left(1 - e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2\nu(x)dx\Bigg] \\
&\quad - \sum_{k=1}^n \frac{w_k^2}{\alpha_k^2}\left(1 - e^{-\alpha_k t}\right)^2 \left(\int_{\mathbb{R}} \left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx\right)^2 \\
&= \sum_{k=1}^n \frac{w_k^2}{2\alpha_k}\left(1 - e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2\nu(x)dx.
\end{aligned}$$

We also have to check what happens when time goes to infinity;

$$\lim_{t\to\infty} \mathrm{Var}\left[r(t)\right] = \sum_{k=1}^n \frac{w_k^2}{2\alpha_k}\int_{\mathbb{R}} x^2\nu(x)dx.$$

And our proof is complete.

□

4.3 Moments of $r(t)$

We want to find the first two moments of $r(t)$. As mentioned before, we can do that by using Corollary 4.1. The first moment is already computed and given by (4.6). We state it again in a proposition.

Proposition 4.4 *The first moment of $r(t)$ is*

$$\mathrm{E}\left[r(t)\right] = r(0)e^{-\alpha_1 t} + \sum_{k=1}^{n} \frac{w_k}{\alpha_k}\left(1 - e^{-\alpha_k t}\right)\int_{\mathbb{R}} \left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx.$$

To find the second moment we look at

$$\mathrm{E}\left[r^2(t)\right] = (i)^{-2}\frac{d^2}{dz^2}\mathrm{E}\left[e^{izr(t)}\right]\Big|_{z=0} = (i)^{-2}\frac{d}{dz}\left[\frac{d}{dz}\mathrm{E}\left[e^{izr(t)}\right]\right]\Big|_{z=0}.$$

Let $f_k(u) = e^{-\alpha_k(t-u)}$ as before. First, let us compute $\frac{d^2}{dz^2}\mathrm{E}\left[e^{izr(t)}\right]$. We use the expression for the characteristic function of $r(t)$ given by Proposition 4.2. Let $b(t) = r(0)e^{-\alpha_1 t}$ and remember that

$$\psi(zw_k f_k(u)) = \int_{\mathbb{R}} \left(e^{izw_k f_k(u)x} - 1 - izw_k f_k(u)x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx. \tag{4.10}$$

Then we get

$$\begin{aligned}\frac{d}{dz}\mathrm{E}\left[e^{izr(t)}\right] &= \frac{d}{dz}\left[e^{izb(t)}e^{\sum_{k=1}^{n}\int_0^t \psi(zw_k f_k(u))du}\right]\\ &= ib(t)e^{izb(t)}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du} + e^{izb(t)}\frac{d}{dz}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du},\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dz^2}\mathrm{E}\left[e^{izr(t)}\right] &= -b^2(t)e^{izb(t)}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du} + ib(t)e^{izb(t)}\frac{d}{dz}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du}\\ &\quad + ib(t)e^{izb(t)}\frac{d}{dz}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du} + e^{izb(t)}\frac{d^2}{dz^2}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du}.\end{aligned}$$

Now, let

$$D_1 = -b^2(t)e^{izb(t)}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du}\Big|_{z=0} \tag{4.11}$$

$$D_2 = ib(t)e^{izb(t)}\frac{d}{dz}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du}\Big|_{z=0} \tag{4.12}$$

$$D_3 = e^{izb(t)}\frac{d^2}{dz^2}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du}\Big|_{z=0}. \tag{4.13}$$

Notice that

$$\mathrm{E}\left[r^2(t)\right] = (i)^{-2}\left[D_1 + 2D_2 + D_3\right]. \tag{4.14}$$

We compute the parts separately, but notice first that

$$e^{\int_0^t \psi(zw_k f_k(u))du}\Big|_{z=0} = 1, \tag{4.15}$$

since $\psi(zw_k f_k(u))\big|_{z=0} = 0$. We start by evaluating D_1. It's fairly easy to see that

$$D_1 = -b^2(t)e^{izb(t)}\prod_{k=1}^{n} e^{\int_0^t \psi(zw_k f_k(u))du}\Bigg|_{z=0} = -b^2(t). \tag{4.16}$$

Before we can evaluate D_2, we need to find $\frac{d}{dz}\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Big|_{z=0}$.

$$\begin{aligned}\frac{d}{dz}\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Big|_{z=0} &= \left[\frac{d}{dz}\sum_{k=1}^n\int_0^t \psi(zw_kf_k(u))du\right]\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Bigg|_{z=0} \\ &= \sum_{k=1}^n\int_0^t \frac{d}{dz}\psi(zw_kf_k(u))du\Bigg|_{z=0},\end{aligned}$$

where the last equality follows from (4.15). From (4.10) we can derive the following for each k;

$$\frac{d}{dz}\psi(zw_kf_k(u)) = \int_{\mathbb{R}}\left(iw_kf_k(u)xe^{izw_kf_k(u)x} - iw_kf_k(u)x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx. \tag{4.17}$$

It the follows that

$$\begin{aligned}\frac{d}{dz}\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Bigg|_{z=0} &= \sum_{k=1}^n\int_0^t f_k(u)du\int_{\mathbb{R}} iw_k\left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx \\ &= i\sum_{k=1}^n\frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx.\end{aligned}$$

We are now ready to derive D_2, given by (4.12). From above we get

$$D_2 = -b(t)\sum_{k=1}^n\frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x - x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx, \tag{4.18}$$

where we used (4.15). It remains to compute D_3.

$$\begin{aligned}D_3 &= e^{izb(t)}\frac{d^2}{dz^2}\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Bigg|_{z=0} = \frac{d^2}{dz^2}\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Bigg|_{z=0} \\ &= \frac{d}{dz}\left(\frac{d}{dz}\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\right)\Bigg|_{z=0} \\ &= \frac{d}{dz}\left(\sum_{k=1}^n\int_0^t\frac{d}{dz}\psi(zw_kf_k(u))du\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\right)\Bigg|_{z=0} \\ &= \sum_{k=1}^n\int_0^t\frac{d^2}{dz^2}\psi(zw_kf_k(u))du\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Bigg|_{z=0} \\ &\quad + \left(\sum_{k=1}^n\int_0^t\frac{d}{dz}\psi(zw_kf_k(u))du\right)^2\prod_{k=1}^n e^{\int_0^t \psi(zw_kf_k(u))du}\Bigg|_{z=0} \\ &= \sum_{k=1}^n\int_0^t\frac{d^2}{dz^2}\psi(zw_kf_k(u))du\Bigg|_{z=0} + \left(\sum_{k=1}^n\int_0^t\frac{d}{dz}\psi(zw_kf_k(u))du\right)^2\Bigg|_{z=0} \\ &= D_3' + D_3''.\end{aligned}$$

So

$$D_3 = D_3' + D_3''. \tag{4.19}$$

To find D_3', notice that we from (4.10) and (4.17) have

$$\frac{d^2}{dz^2}\psi(zw_kf_k(u))\Big|_{z=0} = -\int_{\mathbb{R}} w_k^2 f_k^2(u)x^2\nu(x)dx, \tag{4.20}$$

and

$$\begin{aligned}\int_0^t \frac{d^2}{dz^2}\psi(zw_kf_k(u))du\Big|_{z=0} &= -\int_0^t f_k^2(u)du\int_{\mathbb{R}} w_k^2x^2\nu(x)dx \\ &= -\frac{w_k^2}{2\alpha_k}\left(1-e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2\nu(x)dx.\end{aligned}$$

It therefore follows directly that

$$D_3' = -\sum_{k=1}^{n}\frac{w_k^2}{2\alpha_k}\left(1-e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2\nu(x)dx. \tag{4.21}$$

It still remains to compute D_3''. To do so, remember that from (4.17) it follows that

$$\begin{aligned}\int_0^t \frac{d}{dz}\psi(zw_kf_k(u))du\Big|_{z=0} &= \int_0^t f_k(u)du\int_{\mathbb{R}} iw_k\left(x-x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx \\ &= i\frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x-x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx.\end{aligned}$$

And we can then write

$$D_3'' = -\left(\sum_{k=1}^{n}\frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x-x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx\right)^2. \tag{4.22}$$

We can now derive D_3 from (4.19), (4.21) and (4.22) and it is given by

$$\begin{aligned}D_3 = &-\sum_{k=1}^{n}\frac{w_k^2}{2\alpha_k}\left(1-e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2\nu(x)dx \\ &-\left(\sum_{k=1}^{n}\frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x-x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx\right)^2.\end{aligned}$$

Finally we have computed all we needed to evaluate the second moment of $r(t)$, and we state the result in a proposition. From (4.14), (4.16), (4.18) and from above;

Proposition 4.5 *The second moment of $r(t)$ is*

$$\begin{aligned}\mathrm{E}\left[r^2(t)\right] = &\; b^2(t) + 2b(t)\sum_{k=1}^{n}\frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x-x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx \\ &+\sum_{k=1}^{n}\frac{w_k^2}{2\alpha_k}\left(1-e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2\nu(x)dx \\ &+\left(\sum_{k=1}^{n}\frac{w_k}{\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}}\left(x-x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx\right)^2,\end{aligned}$$

where

$$b(t) = r(0)e^{-\alpha_1 t}$$

and $\nu(x)dx$ is the Lévy measure.

4.4 The Theoretical Autocorrelation Function of $r(t)$

As mentioned before, the autocorrelation function is defined as

$$\operatorname{corr}(r(t), r(t+\tau)) = \frac{\mathrm{E}[r(t)r(t+\tau)] - \mathrm{E}[r(t)]\,\mathrm{E}[r(t+\tau)]}{\sqrt{\operatorname{Var}[r(t)]\operatorname{Var}[r(t+\tau)]}}.$$

To make the computations and notation simpler, we compute the parts separately. We start by considering $\mathrm{E}[r(t)r(t+\tau)] - \mathrm{E}[r(t)]\,\mathrm{E}[r(t+\tau)]$. Remember that $r(t)$ is given by (4.4).

$$\begin{aligned}
&\mathrm{E}[r(t)r(t+\tau)] - \mathrm{E}[r(t)]\,\mathrm{E}[r(t+\tau)] \\
&= \mathrm{E}\Bigg[\left(r(0)e^{-\alpha_1 t} + \sum_{k=1}^{n} w_k \int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right) \\
&\quad \times \left(r(0)e^{-\alpha_1(t+\tau)} + \sum_{k=1}^{n} w_k \int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)}dL_k(u)\right)\Bigg] \\
&\quad - \left(r(0)e^{-\alpha_1 t} + \sum_{k=1}^{n} w_k \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right]\right) \\
&\quad \times \left(r(0)e^{-\alpha_1(t+\tau)} + \sum_{k=1}^{n} w_k \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)}dL_k(u)\right]\right) \\
&= \mathrm{E}\left[\left(\sum_{k=1}^{n} w_k \int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right)\left(\sum_{k=1}^{n} w_k \int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)}dL_k(u)\right)\right] \\
&\quad - \sum_{k=1}^{n} w_k \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right] \times \sum_{k=1}^{n} w_k \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)}dL_k(u)\right] \\
&= B_1 - B_2.
\end{aligned}$$

We examine the expressions further, starting with B_1. We split the two sums into those with equal indexes and those with unequal indexes. Then we get

$$\begin{aligned}
B_1 &= \mathrm{E}\left[\left(\sum_{k=1}^{n} w_k \int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right)\left(\sum_{k=1}^{n} w_k \int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)}dL_k(u)\right)\right] \\
&= \mathrm{E}\left[\sum_{k=1}^{n} w_k^2 \int_0^t e^{-\alpha_k(t-u)}dL_k(u) \int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)}dL_k(u)\right] \\
&\quad + \mathrm{E}\left[\sum_{k=1}^{n-1}\sum_{j=k+1}^{n} w_k w_j \int_0^t e^{-\alpha_k(t-u)}dL_k(u) \int_0^{t+\tau} e^{-\alpha_j(t+\tau-u)}dL_j(u)\right] \\
&\quad + \mathrm{E}\left[\sum_{k=2}^{n}\sum_{j=1}^{k-1} w_k w_j \int_0^t e^{-\alpha_k(t-u)}dL_k(u) \int_0^{t+\tau} e^{-\alpha_j(t+\tau-u)}dL_j(u)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} w_k^2 \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u) \int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)} dL_k(u)\right] \\
&\quad + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} w_k w_j \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_j(t+\tau-u)} dL_j(u)\right] \\
&\quad + \sum_{k=2}^{n} \sum_{j=1}^{k-1} w_k w_j \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_j(t+\tau-u)} dL_j(u)\right],
\end{aligned}
$$

where the last equality follows from the fact that the Lévy processes L_k and L_j are independent of each other when $k \neq j$. Notice also that if we divide the sums in B_2 like we did with B_1, we obtain

$$
\begin{aligned}
B_2 &= \sum_{k=1}^{n} w_k \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \times \sum_{k=1}^{n} w_k \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)} dL_k(u)\right] \\
&= \sum_{k=1}^{n} w_k^2 \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)} dL_k(u)\right] \\
&\quad + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} w_k w_j \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_j(t+\tau-u)} dL_j(u)\right] \\
&\quad + \sum_{k=2}^{n} \sum_{j=1}^{k-1} w_k w_j \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_j(t+\tau-u)} dL_j(u)\right].
\end{aligned}
$$

Considering B_1 and B_2, the expression for $\mathrm{E}\left[r(t)r(t+\tau)\right] - \mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right]$ becomes

$$
\begin{aligned}
&\mathrm{E}\left[r(t)r(t+\tau)\right] - \mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right] = B_1 - B_2 \\
&= \sum_{k=1}^{n} w_k^2 \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u) \int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)} dL_k(u)\right] \\
&\quad - \sum_{k=1}^{n} w_k^2 \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_0^{t+\tau} e^{-\alpha_k(t+\tau-u)} dL_k(u)\right] \\
&= \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\left(\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right)^2\right] \\
&\quad + \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u) \int_t^{t+\tau} e^{-\alpha_k(t-u)} dL_k(u)\right] \\
&\quad - \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right]^2 \\
&\quad - \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_t^{t+\tau} e^{-\alpha_k(t-u)} dL_k(u)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\left(\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right)^2\right] \\
&\quad + \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_t^{t+\tau} e^{-\alpha_k(t-u)} dL_k(u)\right] \\
&\quad - \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right]^2 \\
&\quad - \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{E}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \mathrm{E}\left[\int_t^{t+\tau} e^{-\alpha_k(t-u)} dL_k(u)\right] \\
&= \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{Var}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right].
\end{aligned}
$$

This is almost the same as $\mathrm{Var}\left[r(t)\right]$ which we have evalutated earlier. It is given by (4.7) and is as follows

$$
\mathrm{Var}\left[r(t)\right] = \sum_{k=1}^{n} w_k^2 \mathrm{Var}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] = \sum_{k=1}^{n} \frac{w_k^2}{2\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}} x^2 \nu(x)dx.
$$

Since

$$
\mathrm{E}\left[r(t)r(t+\tau)\right] - \mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right] = \sum_{k=1}^{n} w_k^2 e^{-\alpha_k \tau} \mathrm{Var}\left[\int_0^t e^{-\alpha_k(t-u)} dL_k(u)\right] \tag{4.23}
$$

it follows that

$$
\mathrm{E}\left[r(t)r(t+\tau)\right] - \mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right] = \sum_{k=1}^{n} e^{-\alpha_k \tau}\frac{w_k^2}{2\alpha_k}\left(1-e^{-\alpha_k t}\right)\int_{\mathbb{R}} x^2 \nu(x)dx.
$$

We need to examine $\mathrm{Var}\left[r(t)\right]\mathrm{Var}\left[r(t+\tau)\right]$ to complete the computation of the theoretical autocorrelation function of $r(t)$.

$$
\begin{aligned}
\mathrm{Var}\left[r(t)\right]\mathrm{Var}\left[r(t+\tau)\right] &= \left(\sum_{k=1}^{n} \frac{w_k^2}{2\alpha_k}\left(1-e^{-2\alpha_k t}\right)\int_{\mathbb{R}} x^2 \nu(x)dx\right) \\
&\quad \times \left(\sum_{k=1}^{n} \frac{w_k^2}{2\alpha_k}\left(1-e^{-2\alpha_k (t+\tau)}\right)\int_{\mathbb{R}} x^2 \nu(x)dx\right).
\end{aligned}
$$

Then we are ready to state the result in a proposition.

Proposition 4.6 *The theoretical autocorrelation function of $r(t)$ is given by*

$$
\mathrm{corr}\left(r(t), r(t+\tau)\right) = \frac{\sum_{k=1}^{n} e^{-\alpha_k \tau}\frac{w_k^2}{\alpha_k}\left(1-e^{-\alpha_k t}\right)}{\sqrt{\left(\sum_{j=1}^{n}\frac{w_j^2}{\alpha_j}\left(1-e^{-2\alpha_j t}\right)\right)\left(\sum_{j=1}^{n}\frac{w_j^2}{\alpha_j}\left(1-e^{-2\alpha_j (t+\tau)}\right)\right)}}.
$$

And when time goes to infinity we get

$$
\lim_{t\to\infty} \mathrm{corr}\left(r(t), r(t+\tau)\right) = \sum_{k=1}^{n} \frac{\frac{w_k^2}{\alpha_k}}{\sum_{j=1}^{n}\frac{w_j^2}{\alpha_j}} e^{-\alpha_k \tau}.
$$

4.5 Explicit Results when $L_k(t)$ is Compound Poisson

In this section we consider Lévy processes which are compound Poisson. We want to find explicit results for the characteristic function, the stationary characteristics, the two first moments and the theoretical autocorrelation function of $r(t)$. Let $N_k(t)$ be a Poisson process with intensity λ_k, and let J_i be independent and identically exponential distributed with rate parameter θ_k, independent of $N_k(t)$. Then we know from Definition (2.5) that

$$L_k(t) = \sum_{i=1}^{N_k(t)} J_i$$

is compound Poisson for each k.

4.5.1 Characteristic Function of $r(t)$

To find the characteristic function of $r(t)$ we need to compute the characteristic function of both J_j and $L_k(t)$. For each k, every J_j has density function

$$f(x,\theta_k) = \begin{cases} \theta_k e^{-\theta_k x} & , x \geq 0 \\ 0 & , x < 0 \end{cases} .$$

So the characteristic function of J_j is

$$\mathrm{E}\left[e^{izJ_j}\right] = \int_0^\infty e^{izx}\theta_k e^{-\theta_k x}dx = \int_0^\infty \theta_k e^{-x(\theta_k - iz)}dx.$$

Let $u = x(\theta_k - iz)$, then

$$\mathrm{E}\left[e^{izJ_j}\right] = \theta_k \int_0^\infty e^{-u}\frac{du}{\theta_k - iz} = \frac{\theta_k}{\theta_k - iz}\int_0^\infty e^{-u}du = \frac{\theta_k}{\theta_k - iz}.$$

So the characteristic function of J_j becomes

$$\mathrm{E}\left[e^{izJ_j}\right] = \frac{\theta_k}{\theta_k - iz}. \tag{4.24}$$

Now we can derive the characteristic function of $L_k(t)$. Remember that J_j is independent of $N_k(t)$. For each k we have

$$\begin{aligned}
\mathrm{E}\left[e^{izL_k(t)}\right] &= \mathrm{E}\left[e^{iz\sum_{j=1}^{N_k(t)} J_j}\right] = \mathrm{E}_{N_k(t)}\left[\mathrm{E}\left[e^{iz\sum_{j=1}^{N_k(t)} J_j}\middle|N_k(t)\right]\right] \\
&= \mathrm{E}_{N_k(t)}\left[\mathrm{E}\left[e^{izJ_1}e^{izJ_2}\cdots e^{izJ_{N_k(t)}}\middle|N_k(t)\right]\right] \\
&= \mathrm{E}_{N_k(t)}\left[\prod_{j=1}^{N_k(t)}\mathrm{E}\left[e^{izJ_j}\right]\right] = \mathrm{E}_{N_k(t)}\left[\left(\frac{\theta_k}{\theta_k - iz}\right)^{N_k(t)}\right] \\
&= \sum_{k=0}^{\infty}\left(\frac{\theta_k}{\theta_k - iz}\right)^k e^{-\lambda_k t}\frac{(\lambda_k t)^k}{k!} = e^{-\lambda_k t}\sum_{k=0}^{\infty}\frac{\left(\frac{\theta_k\lambda_k t}{\theta_k - iz}\right)^k}{k!} \\
&= e^{-\lambda_k t}e^{\frac{\theta_k\lambda_k t}{\theta_k - iz}},
\end{aligned}$$

since $N_k(t)$ is a Poisson process for each k. So we find that the characteristic function of $L_k(t)$ is

$$\mathrm{E}\left[e^{izL_k(t)}\right] = e^{\lambda_k t\left(\frac{\theta_k}{\theta_k - iz} - 1\right)}. \qquad (4.25)$$

Finally we are ready to derive the characteristic function of $r(t)$. First, let $b(t) = r(0)e^{-\alpha_1 t}$ and $f_k(u) = e^{-\alpha_k(t-u)}$. Remember that $r(t)$ is given by (4.4). Then

$$\mathrm{E}\left[e^{izr(t)}\right] = e^{izb(t)}\mathrm{E}\left[e^{iz\sum_{k=1}^{n} w_k \int_0^t f_k(u)dL_k(u)}\right].$$

First we compute $\mathrm{E}\left[e^{iz\sum_{k=1}^{n} w_k \int_0^t f_k(u)dL_k(u)}\right]$. Let $\{u_j\}_{j=1}^{m}$ be a partition of the interval $[0,t]$, with $\max_j |u_{j+1} - u_j| < \epsilon$. Then the integral $\int_0^t f_k(u)dL_k(u)$ can be written as $\lim_{\epsilon\to 0}\sum_{j=1}^{m} f_k(u_j)\Delta L_k(u_j)$. It follows from (4.25) that

$$\begin{aligned}
\mathrm{E}\left[e^{iz\sum_{k=1}^{n} w_k \int_0^t f_k(u)dL_k(u)}\right] &= \prod_{k=1}^{n}\lim_{\epsilon\to 0}\prod_{j=1}^{m}\mathrm{E}\left[e^{izw_k f_k(u_j)\Delta L_k(u_j)}\right] \\
&= \prod_{k=1}^{n}\lim_{\epsilon\to 0}\prod_{j=1}^{m} e^{\lambda_k\left(\frac{\theta_k}{\theta_k - izw_k f_k(u_j)} - 1\right)\Delta u_j} \\
&= \prod_{k=1}^{n} e^{\int_0^t \lambda_k\left(\frac{\theta_k}{\theta_k - izw_k f_k(u)} - 1\right)du}.
\end{aligned}$$

We have applied the Bounded Convergence Theorem to take the limit outside the expectation. Notice also that the two products can be taken outside the expectation because L_k is independent of L_j, when $k \neq j$ and that L_k has independent increments for each k. Look further at the integral above.

$$\int_0^t \lambda_k\left(\frac{\theta_k}{\theta_k - izw_k e^{-\alpha_k(t-u)}} - 1\right)du = \lambda_k\int_0^t \frac{1}{1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k t}e^{\alpha_k u}}du - \lambda_k t.$$

It is possible to calculate the integral above explicitly;

Proposition 4.7

$$\int \frac{1}{b + ae^{\lambda x}}dx = \frac{x}{b} - \frac{1}{b\lambda}\ln\left(b + ae^{\lambda x}\right) + C, \qquad b \neq 0.$$

Proof. We show the Proposition above by finding the derivative of $\frac{x}{b} - \frac{1}{b\lambda}\ln\left(b + ae^{\lambda x}\right) + C$ with respect to x. It becomes

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x}{b} - \frac{1}{b\lambda}\ln\left(b + ae^{\lambda x}\right) + C\right] &= \frac{1}{b} - \frac{a\lambda e^{\lambda x}}{b\lambda\left(b + ae^{\lambda x}\right)} \\
&= \frac{b + ae^{\lambda x}}{b\left(b + ae^{\lambda x}\right)} - \frac{ae^{\lambda x}}{b\left(b + ae^{\lambda x}\right)} \\
&= \frac{1}{b + ae^{\lambda x}}.
\end{aligned}$$

□

From Proposition 4.7 above we see that

$$\lambda_k \int_0^t \frac{1}{1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k t}e^{\alpha_k u}} du = \lambda_k \left[u - \frac{1}{\alpha_k} \ln\left(1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k(t-u)}\right)\right]_0^t$$
$$= \lambda_k t - \frac{\lambda_k}{\alpha_k} \ln\left(1 - iz\frac{w_k}{\theta_k}\right) + \frac{\lambda_k}{\alpha_k} \ln\left(1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k t}\right).$$

From the preceding computations;

$$\begin{aligned} \mathrm{E}\left[e^{izr(t)}\right] &= e^{izb(t)} \prod_{k=1}^n e^{\frac{\lambda_k}{\alpha_k}\ln\left(1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k t}\right)} e^{-\frac{\lambda_k}{\alpha_k}\ln\left(1-iz\frac{w_k}{\theta_k}\right)} \\ &= e^{izb(t)} \prod_{k=1}^n \left(1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k t}\right)^{\frac{\lambda_k}{\alpha_k}} \left(1 - iz\frac{w_k}{\theta_k}\right)^{-\frac{\lambda_k}{\alpha_k}} \\ &= e^{izb(t)} \prod_{k=1}^n \left(\frac{1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}. \end{aligned}$$

We state the result in a proposition.

Proposition 4.8 *The characteristic function of $r(t)$ is given by*

$$\mathrm{E}\left[e^{izr(t)}\right] = e^{izb(t)} \prod_{k=1}^n \left(\frac{1 - iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}},$$

where

$$b(t) = r(0)e^{-\alpha_1 t}.$$

We are also interested in the expectation and the variance of $r(t)$ when $L_k(t)$ is compound Poisson.

Proposition 4.9 *$r(t)$ has expectation*

$$\mathrm{E}\left[r(t)\right] = b(t) + \sum_{k=1}^n \frac{\lambda_k}{\theta_k}\frac{w_k}{\alpha_k}\left(1 - e^{-\alpha_k t}\right), \tag{4.26}$$

and variance

$$\mathrm{Var}\left[r(t)\right] = \sum_{k=1}^n \frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1 - e^{-2\alpha_k t}\right), \tag{4.27}$$

where $b(t)$ is given by

$$b(t) = r(0)e^{-\alpha_1 t}.$$

When time goes to infinity we get that

$$\lim_{t\to\infty} \mathrm{E}\left[r(t)\right] = \sum_{k=1}^n \frac{\lambda_k}{\theta_k}\frac{w_k}{\alpha_k}, \tag{4.28}$$

and

$$\lim_{t\to\infty} \mathrm{Var}\left[r(t)\right] = \sum_{k=1}^n \frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}. \tag{4.29}$$

Proof.
We know from Corollary 4.1 that $\mathrm{E}[X^n] = (i)^{-n}\frac{d^n}{dz^n}\varphi_X(z)\big|_{z=0}$ for a stochastic variable X with characteristic function $\varphi_X(z)$. Then

$$\mathrm{E}\left[r(t)\right] = (i)^{-1}\frac{d}{dz}\mathrm{E}\left[e^{izr(t)}\right]\Big|_{z=0}.$$

We already know that the characteristic function of $r(t)$ is given by Proposition 4.8, so

$$\begin{aligned}
\mathrm{E}\left[r(t)\right] &= (i)^{-1}\frac{d}{dz}\left[e^{izb(t)}\prod_{k=1}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\right]\Bigg|_{z=0} \\
&= b(t)e^{izb(t)}\prod_{k=1}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\Bigg|_{z=0} \\
&\quad + (i)^{-1}e^{izb(t)}\frac{d}{dz}\prod_{k=1}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\Bigg|_{z=0} \\
&= D_1 + D_2,
\end{aligned}$$

where $b(t) = r(0)e^{-\alpha_1 t}$ as usual. We compute the parts separately. We start with D_1. First notice that $e^{izb(t)}|_{z=0} = 1$ and that $\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\Big|_{z=0} = 1$, so

$$D_1 = b(t)\prod_{k=1}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\Bigg|_{z=0} = b(t). \tag{4.30}$$

Then consider D_2.

$$\begin{aligned}
D_2 &= (i)^{-1}\frac{d}{dz}\prod_{k=1}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\Bigg|_{z=0} \\
&= (i)^{-1}\frac{d}{dz}\left(\frac{1-iz\frac{w_1}{\theta_1}e^{-\alpha_1 t}}{1-iz\frac{w_1}{\theta_1}}\right)^{\frac{\lambda_1}{\alpha_1}} \times \prod_{k=2}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\Bigg|_{z=0} \\
&\quad + (i)^{-1}\left(\frac{1-iz\frac{w_1}{\theta_1}e^{-\alpha_1 t}}{1-iz\frac{w_1}{\theta_1}}\right)^{\frac{\lambda_1}{\alpha_1}} \times \frac{d}{dz}\prod_{k=2}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\Bigg|_{z=0} \\
&= (i)^{-1}\left[\frac{d}{dz}\left(\frac{1-iz\frac{w_1}{\theta_1}e^{-\alpha_1 t}}{1-iz\frac{w_1}{\theta_1}}\right)^{\frac{\lambda_1}{\alpha_1}} + \frac{d}{dz}\prod_{k=2}^{n}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\right]\Bigg|_{z=0} \\
&= (i)^{-1}\sum_{k=1}^{n}\frac{d}{dz}\left(\frac{1-iz\frac{w_k}{\theta_k}e^{-\alpha_k t}}{1-iz\frac{w_k}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}\Bigg|_{z=0}
\end{aligned}$$

$$= (i)^{-1} \sum_{k=1}^{n} \left[\frac{\lambda_k}{\alpha_k} \left(\frac{1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}} \right)^{\frac{\lambda_k}{\alpha_k} - 1} \frac{d}{dz} \left(\frac{1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}} \right) \right] \Bigg|_{z=0}$$

$$= (i)^{-1} \sum_{k=1}^{n} \frac{\lambda_k}{\alpha_k} \frac{d}{dz} \left(\frac{1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}} \right) \Bigg|_{z=0}.$$

Above we have used that

$$\left(\frac{1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}} \right)^{\frac{\lambda_k}{\alpha_k}} \Bigg|_{z=0} = 1,$$

for all k. Let's make an inbetween computation of $\frac{d}{dz}\left(\frac{1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}} \right)$. First let

$$m = 1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}, \qquad \frac{dm}{dz} = -i\frac{w_k}{\theta_k} e^{-\alpha_k t}$$
$$n = 1 - iz\frac{w_k}{\theta_k}, \qquad \frac{dn}{dz} = -i\frac{w_k}{\theta_k}.$$

Then

$$\frac{d}{dz} \left(\frac{1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}}{1 - iz\frac{w_k}{\theta_k}} \right) = \frac{d}{dz}\frac{m}{n} = \frac{\frac{dm}{dz} n - m\frac{dn}{dz}}{n^2}$$
$$= \frac{\left(-i\frac{w_k}{\theta_k} e^{-\alpha_k t}\right)\left(1 - iz\frac{w_k}{\theta_k}\right) - \left(1 - iz\frac{w_k}{\theta_k} e^{-\alpha_k t}\right)\left(-i\frac{w_k}{\theta_k}\right)}{\left(1 - iz\frac{w_k}{\theta_k}\right)^2}$$
$$= \frac{i\frac{w_k}{\theta_k}\left(1 - e^{-\alpha_k t}\right)}{\left(1 - iz\frac{w_k}{\theta_k}\right)^2}.$$

We then obtain

$$D_2 = (i)^{-1} \sum_{k=1}^{n} \frac{\lambda_k}{\alpha_k} \frac{i\frac{w_k}{\theta_k}\left(1 - e^{-\alpha_k t}\right)}{\left(1 - iz\frac{w_k}{\theta_k}\right)^2} \Bigg|_{z=0} = \sum_{k=1}^{n} \frac{\lambda_k}{\theta_k} \frac{w_k}{\alpha_k} \left(1 - e^{-\alpha_k t}\right). \tag{4.31}$$

We use the results for D_1 and D_2, given by (4.30) and (4.31) respectively, to find that $\mathrm{E}[r(t)]$ is given by

$$\mathrm{E}\left[r(t)\right] = D_1 + D_2 = b(t) + \sum_{k=1}^{n} \frac{\lambda_k}{\theta_k} \frac{w_k}{\alpha_k} \left(1 - e^{-\alpha_k t}\right).$$

Which is what we wanted. Now, if we let the time go to infinity we get

$$\lim_{t\to\infty} \mathrm{E}\left[r(t)\right] = \sum_{k=1}^{n} \frac{\lambda_k}{\theta_k} \frac{w_k}{\alpha_k}.$$

We still have to prove that the variance is given by (4.27) and see what happens to it when time goes to infinity. Let $f_k(u) = e^{-\alpha_k(t-u)}$. It follows directly from (4.4) that

$$\begin{aligned}\mathrm{Var}\,[r(t)] &= \mathrm{Var}\left[\sum_{k=1}^{n} w_k \int_0^t f_k(u)dL_k(u)\right] = \sum_{k=1}^{n} w_k^2 \mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right] \\ &= \sum_{k=1}^{n} w_k^2 \left(\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] - \mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2\right).\end{aligned}$$

First we need to compute $\mathrm{E}\left[e^{iz\int_0^t f_k(u)dL_k(u)}\right]$. Let $\{u_j\}_{j=1}^m$ be a partition of the interval $[0,t]$, with $\max_j |u_{j+1} - u_j| < \epsilon$ as before. We can now write $\int_0^t f_k(u)dL_k(u) = \lim_{\epsilon\to 0}\sum_{j=1}^m f_k(u_j)\Delta L_k(u_j)$, where $\Delta L_k(u_j) = L_k(u_{j+1}) - L_k(u_j)$. So from (4.25) we get for each k

$$\begin{aligned}\mathrm{E}\left[e^{iz\int_0^t f_k(u)dL_k(u)}\right] &= \lim_{\epsilon\to 0}\prod_{j=1}^{m}\mathrm{E}\left[e^{izf_k(u_j)\Delta L_k(u_j)}\right] \\ &= \lim_{\epsilon\to 0}\prod_{j=1}^{m} e^{\lambda_k\left(\frac{\theta_k}{\theta_k - izf_k(u_j)} - 1\right)\Delta u_j} \\ &= e^{\lambda_k\int_0^t\left(\frac{\theta_k}{\theta_k - izf_k(u)} - 1\right)du}.\end{aligned}$$

Again we have used Bounded Convergence Theorem to take the limit outside the expectation. The independence of increments of L_k is used to do the same with the product. To compute the integral in the last term we use Proposition 4.7 again;

$$\begin{aligned}\lambda_k\int_0^t\left(\frac{\theta_k}{\theta_k - izf_k(u)} - 1\right) &= \lambda_k\int_0^t \frac{1}{1 - iz\frac{1}{\theta_k}e^{-\alpha_k t}e^{\alpha_k u}}du - \lambda_k t \\ &= \lambda_k\left[u - \frac{1}{\alpha_k}\ln\left(1 - iz\frac{1}{\theta_k}e^{-\alpha_k(t-u)}\right)\right]_0^t - \lambda_k t \\ &= \lambda_k\left[\frac{1}{\alpha_k}\left[\ln\left(1 - iz\frac{1}{\theta_k}e^{-\alpha_k t}\right) - \ln\left(1 - iz\frac{1}{\theta_k}\right)\right]\right]\end{aligned}$$

So

$$\begin{aligned}\mathrm{E}\left[e^{iz\int_0^t f_k(u)dL_k(u)}\right] &= e^{\frac{\lambda_k}{\alpha_k}\ln\left(1 - iz\frac{1}{\theta_k}e^{-\alpha_k t}\right) - \frac{\lambda_k}{\alpha_k}\ln\left(1 - iz\frac{1}{\theta_k}\right)} \\ &= \left(\frac{1 - iz\frac{1}{\theta_k}e^{-\alpha_k t}}{1 - iz\frac{1}{\theta_k}}\right)^{\frac{\lambda_k}{\alpha_k}}.\end{aligned}$$

Now, let

$$\begin{aligned} m &= 1 - iz\tfrac{1}{\theta_k}e^{-\alpha_k t}, & \tfrac{dm}{dz} &= -i\tfrac{1}{\theta_k}e^{-\alpha_k t} \\ n &= 1 - iz\tfrac{1}{\theta_k}, & \tfrac{dn}{dz} &= -i\tfrac{1}{\theta_k}\end{aligned}.$$

Since $\mathrm{E}\left[e^{iz\int_0^t f_k(u)dL_k(u)}\right] = \left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}}$, we can write

$$\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] = (i)^{-2}\frac{d^2}{dz^2}\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}}\Big|_{z=0}, \tag{4.32}$$

and

$$\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2 = \left((i)^{-1}\frac{d}{dz}\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}}\Big|_{z=0}\right)^2. \tag{4.33}$$

We need $\frac{d}{dz}\frac{m}{n}$ to evaluate the expressions in both (4.32) and (4.33).

$$\begin{aligned}\frac{d}{dz}\frac{m}{n} &= \frac{\left(-i\frac{1}{\theta_k}e^{-\alpha_k t}\right)\left(1-iz\frac{1}{\theta_k}\right)-\left(1-iz\frac{1}{\theta_k}e^{-\alpha_k t}\right)\left(-i\frac{1}{\theta_k}\right)}{\left(1-iz\frac{1}{\theta_k}\right)^2}\\ &= \frac{i\frac{1}{\theta_k}\left(1-e^{-\alpha_k t}\right)}{\left(1-iz\frac{1}{\theta_k}\right)^2}.\end{aligned}$$

Letting $z=0$, it becomes

$$\frac{d}{dz}\frac{m}{n}\bigg|_{z=0} = i\frac{1}{\theta_k}\left(1-e^{-\alpha_k t}\right). \tag{4.34}$$

Before we start computing (4.32) and (4.33), look at $\frac{d^2}{dz^2}\frac{m}{n}$ as well;

$$\begin{aligned}\frac{d^2}{dz^2}\frac{m}{n} &= \frac{d}{dz}\left[\frac{i\frac{1}{\theta_k}\left(1-e^{-\alpha_k t}\right)}{\left(1-iz\frac{1}{\theta_k}\right)^2}\right] = \frac{-2i\frac{1}{\theta_k}\left(1-e^{-\alpha_k t}\right)\left(-i\frac{1}{\theta_k}\right)}{\left(1-iz\frac{1}{\theta_k}\right)^4}\\ &= -2\frac{\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right)}{\left(1-iz\frac{1}{\theta_k}\right)^4}.\end{aligned}$$

and letting $z=0$ yields

$$\frac{d^2}{dz^2}\frac{m}{n}\bigg|_{z=0} = -2\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right). \tag{4.35}$$

Notice also that $\frac{m}{n}\big|_{z=0} = 1$. Now we are ready to compute (4.32) and (4.33), starting with (4.33).

$$\begin{aligned}\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right] &= (i)^{-1}\frac{d}{dz}\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}}\bigg|_{z=0} = (i)^{-1}\frac{\lambda_k}{\alpha_k}\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}-1}\frac{d}{dz}\frac{m}{n}\bigg|_{z=0}\\ &= \frac{\lambda_k}{\alpha_k}\frac{1}{\theta_k}\left(1-e^{-\alpha_k t}\right),\end{aligned}$$

where the last equality follows from (4.34). $\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2$ now becomes

$$\mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2 = \frac{\lambda_k^2}{\alpha_k^2}\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right)^2. \tag{4.36}$$

Let's compute (4.32) as well;

$$\begin{aligned}
&\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] = (i)^{-2}\frac{d^2}{dz^2}\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}}\Big|_{z=0} \\
&= (i)^{-2}\frac{d}{dz}\left[\frac{\lambda_k}{\alpha_k}\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}-1}\frac{d}{dz}\frac{m}{n}\right]\Big|_{z=0} \\
&= (i)^{-2}\frac{\lambda_k}{\alpha_k}\left(\frac{\lambda_k}{\alpha_k}-1\right)\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}-2}\left(\frac{d}{dz}\frac{m}{n}\right)^2\Big|_{z=0} + (i)^{-2}\frac{\lambda_k}{\alpha_k}\left(\frac{m}{n}\right)^{\frac{\lambda_k}{\alpha_k}-1}\frac{d^2}{dz^2}\frac{m}{n}\Big|_{z=0} \\
&= \frac{\lambda_k}{\alpha_k}\left(\frac{\lambda_k}{\alpha_k}-1\right)\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right)^2 + 2\frac{\lambda_k}{\alpha_k}\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right).
\end{aligned}$$

We use the expression for $\mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right]$ derived above and (4.36) to evaluate $\mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right]$;

$$\begin{aligned}
\mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right] &= \mathrm{E}\left[\left(\int_0^t f_k(u)dL_k(u)\right)^2\right] - \mathrm{E}\left[\int_0^t f_k(u)dL_k(u)\right]^2 \\
&= \frac{\lambda_k^2}{\alpha_k^2}\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right)^2 - \frac{\lambda_k}{\alpha_k}\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right)^2 \\
&\quad + 2\frac{\lambda_k}{\alpha_k}\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right) - \frac{\lambda_k^2}{\alpha_k^2}\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right)^2 \\
&= \frac{\lambda_k}{\alpha_k}\frac{1}{\theta_k^2}\left(1-e^{-\alpha_k t}\right)\left[2-\left(1-e^{-\alpha_k t}\right)\right] \\
&= \frac{\lambda_k}{\alpha_k}\frac{1}{\theta_k^2}\left(1-e^{-2\alpha_k t}\right)
\end{aligned}$$

Remember that $\mathrm{Var}\left[r(t)\right] = \sum_{k=1}^n w_k^2 \mathrm{Var}\left[\int_0^t f_k(u)dL_k(u)\right]$. So

$$\mathrm{Var}\left[r(t)\right] = \sum_{k=1}^n \frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1-e^{-2\alpha_k t}\right)$$

and

$$\lim_{t\to\infty}\mathrm{Var}\left[r(t)\right] = \sum_{k=1}^n \frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}.$$

And our proof is complete.

□

4.5.2 The Theoretical Autocorrelation Function of $r(t)$

We also want the theoretical autocorrelation function of $r(t)$ when $L_k(t)$ is compound Poisson. It is as before given by

$$\mathrm{corr}\left(r(t), r(t+\tau)\right) = \frac{\mathrm{E}\left[r(t)r(t+\tau)\right] - \mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right]}{\sqrt{\mathrm{Var}\left[r(t)\right]\mathrm{Var}\left[r(t+\tau)\right]}}.$$

From (4.23) we know that

$$\mathrm{E}\left[r(t)r(t+\tau)\right]-\mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right]=\sum_{k=1}^{n}w_k^2e^{-\alpha_k\tau}\mathrm{Var}\left[\int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right],$$

and from the preceding page we see

$$\mathrm{Var}\left[\int_0^t e^{-\alpha_k(t-u)}dL_k(u)\right]=\frac{\lambda_k}{\alpha_k}\frac{1}{\theta_k^2}\left(1-e^{-2\alpha_k t}\right).$$

Putting these two equations together we get

$$\mathrm{E}\left[r(t)r(t+\tau)\right]-\mathrm{E}\left[r(t)\right]\mathrm{E}\left[r(t+\tau)\right]=\sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}e^{-\alpha_k\tau}\left(1-e^{-2\alpha_k t}\right).$$

From (4.27) we obtain

$$\begin{aligned}\mathrm{Var}\left[r(t)\right] &= \sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1-e^{-2\alpha_k t}\right)\\ \mathrm{Var}\left[r(t+\tau)\right] &= \sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1-e^{-2\alpha_k(t+\tau)}\right),\end{aligned}$$

such that the autocorrelation function becomes

$$\mathrm{corr}\left(r(t),r(t+\tau)\right)=\frac{\sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}e^{-\alpha_k\tau}\left(1-e^{-2\alpha_k t}\right)}{\sqrt{\sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1-e^{-2\alpha_k t}\right)\sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1-e^{-2\alpha_k(t+\tau)}\right)}}.$$

The result can be formulated in a proposition.

Proposition 4.10 *The autocorrelation function of $r(t)$ when $L_k(t)$ is compound Poisson is*

$$\mathrm{corr}\left(r(t),r(t+\tau)\right)=\frac{\sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}e^{-\alpha_k\tau}\left(1-e^{-2\alpha_k t}\right)}{\sqrt{\sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1-e^{-2\alpha_k t}\right)\sum_{k=1}^{n}\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}\left(1-e^{-2\alpha_k(t+\tau)}\right)}},$$

and when time goes to infinity we get

$$\lim_{t\to\infty}\mathrm{corr}\left(r(t),r(t+\tau)\right)=\sum_{k=1}^{n}\frac{\frac{\lambda_k}{\theta_k^2}\frac{w_k^2}{\alpha_k}}{\sum_{j=1}^{n}\frac{\lambda_j}{\theta_j^2}\frac{w_j^2}{\alpha_j}}\;e^{-\alpha_k\tau}.$$

Chapter 5

Zero-Coupon Bond Prices

We are interested in finding the price of a zero-coupon bond for general $L_k(t)$, and when $L_k(t)$ is compound Poisson. We start with the general $L_k(t)$ and use some of the computations derived there to find the price when $L_k(t)$ is compound Poisson. In the next chapter we will evaluate the prices of European options written on these bonds.

5.1 Zero-Coupon Bond Prices for General $L_k(t)$

The price of a zero-coupon bond at time $t \leq T$ is given by Definition 3.1, and is as follows

$$P(t,T) = \mathrm{E}_Q\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t\right],$$

where $\mathcal{F}_t$ is the σ-algebra generated by $L_k(s)$, $s \leq t$. We are interested in finding an expression for $-\int_t^T r(s)ds$. Remember that the process $r(s)$ is given by

$$r(s) = \sum_{k=1}^{n} w_k X_k(s),$$

where

$$dX_k(s) = -\alpha_k X_k(s)ds + dL_k(s).$$

We have found that the solution of $X_k(s)$ is

$$X_k(s) = X_k(t)e^{-\alpha_k(s-t)} + \int_t^s e^{-\alpha_k(s-u)}dL_k(u),$$

where the process starts at time $t \leq s$. $r(s)$ then becomes

$$r(s) = \sum_{k=1}^{n} w_k X_k(t)e^{-\alpha_k(s-t)} + \sum_{k=1}^{n} w_k \int_t^s e^{-\alpha_k(s-u)}dL_k(u). \tag{5.1}$$

Now we can find an expression for $-\int_t^T r(s)ds$.

$$\begin{aligned}
-\int_t^T r(s)ds &= -\int_t^T \sum_{k=1}^{n} w_k X_k(t)e^{-\alpha_k(s-t)}ds - \int_t^T \sum_{k=1}^{n} w_k \int_t^s e^{-\alpha_k(s-u)}dL_k(u)ds \\
&= -I_1 - I_2.
\end{aligned}$$

Compute the parts separately. It's easy to see that

$$I_1 = \sum_{k=1}^{n} w_k X_k(t) \int_t^T e^{-\alpha_k(s-t)} ds = \sum_{k=1}^{n} w_k X_k(t) \frac{1}{\alpha_k} \left(1 - e^{-\alpha_k(T-t)}\right).$$

We now find I_2.

$$\begin{aligned} I_2 &= \sum_{k=1}^{n} w_k \int_t^T \int_t^s e^{-\alpha_k(s-u)} dL_k(u) ds = \sum_{k=1}^{n} w_k \int_t^T \int_u^T e^{-\alpha_k(s-u)} ds dL_k(u) \\ &= \sum_{k=1}^{n} w_k \int_t^T \frac{1}{\alpha_k} \left(1 - e^{-\alpha_k(T-u)}\right) dL_k(u), \end{aligned}$$

where we have used Fubini's Theorem to change the order of the integrals. Let $n_k(t,T) = \frac{1}{\alpha_k}\left(1 - e^{-\alpha_k(T-t)}\right)$. Then we can write I_1 and I_2 as

$$I_1 = \sum_{k=1}^{n} w_k X_k(t) n_k(t,T), \tag{5.2}$$

and

$$I_2 = \sum_{k=1}^{n} w_k \int_t^T n_k(u,T) dL_k(u). \tag{5.3}$$

So we get

$$-\int_t^T r(s)ds = -\sum_{k=1}^{n} w_k X_k(t) n_k(t,T) - \sum_{k=1}^{n} w_k \int_t^T n_k(u,T) dL_k(u). \tag{5.4}$$

Now, call $\xi_T = -\int_t^T r(s)ds$. Notice that $X_k(t)$ is $\mathcal{F}_t$-measurable for all k. And that $n_k(t,T)$ and w_k are deterministic. Notice also that $\int_t^T n_k(u,T)dL_k(u)$ is independent of $\mathcal{F}_t$, since $L_k(u)$ has independent increments for every k, by the properties of Lévy processes. It follows that

$$\begin{aligned} P(t,T) &= \mathrm{E}_Q\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t\right] = \mathrm{E}_Q\left[e^{\xi_T} \middle| \mathcal{F}_t\right] = \mathrm{E}_Q\left[e^{\xi_T}\right] \\ &= e^{-\sum_{k=1}^{n} w_k X_k(t) n_k(t,T)} \mathrm{E}_Q\left[e^{-\sum_{k=1}^{n} w_k \int_t^T n_k(u,T) dL_k(u)}\right]. \end{aligned}$$

So

$$P(t,T) = e^{-\sum_{k=1}^{n} w_k X_k(t) n_k(t,T)} \mathrm{E}_Q\left[e^{-\sum_{k=1}^{n} w_k \int_t^T n_k(u,T) dL_k(u)}\right]. \tag{5.5}$$

We see that we need to find $\mathrm{E}_Q\left[e^{-\sum_{k=1}^{n} w_k \int_t^T n_k(u,T) dL_k(u)}\right]$. Let $\{u_j\}_{j=1}^m$ be any partition of the interval $[t,T]$, with $\max_j |u_{j+1} - u_j| < \epsilon$. Then we already know that we can write $\int_t^T n_k(u,T)dL_k(u) = \lim_{\epsilon \to 0} \sum_{j=1}^m n_k(u_j,T)\Delta L_k(u_j)$, where $\Delta L_k(u_j) = L_k(u_{j+1}) - L_k(u_j)$ as usual. Therefore we have

$$\begin{aligned} \mathrm{E}_Q\left[e^{-\sum_{k=1}^{n} w_k \int_t^T n_k(u,T) dL_k(u)}\right] &= \mathrm{E}_Q\left[e^{-\sum_{k=1}^{n} w_k \lim_{\epsilon\to 0} \sum_{j=1}^m n_k(u_j,T)\Delta L_k(u_j)}\right] \\ &= \prod_{k=1}^{n} \lim_{\epsilon \to 0} \prod_{j=1}^{m} \mathrm{E}_Q\left[e^{-w_k n_k(u_j,T)\Delta L_k(u_j)}\right]. \end{aligned}$$

The Bounded Convergence Theorem is applied to take the limit outside the expectation above. The two products can be taken outside as well because of independence between L_k and L_j, when $k \neq j$, and because of independent increments of the Lévy processes. The characteristic function of a Lévy process $L(t)$ is in general

$$\mathrm{E}\left[e^{izL(t)}\right] = e^{\psi(z)t},$$

where $\psi(z)$ is as in Lévy-Khinchin Representation Theorem (Theorem 2.3). The Lévy process we are studying has characteristic triplet $(0, \nu, 0)$, so in our case we see that $\psi(z) = \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{|x|\leq 1}\right) \nu(x)dx$. We see that

$$\begin{aligned} \mathrm{E}_Q\left[e^{-\sum_{k=1}^n w_k \int_t^T n_k(u,T)dL_k(u)}\right] &= \prod_{k=1}^n \lim_{\epsilon\to 0} \prod_{j=1}^m \mathrm{E}_Q\left[e^{-w_k n_k(u_j,T)\Delta L_k(u_j)}\right] \\ &= \prod_{k=1}^n \lim_{\epsilon\to 0} \prod_{j=1}^m e^{\psi(iw_k n_k(u_j,T))\Delta u_j} \\ &= \prod_{k=1}^n e^{\int_t^T \psi(iw_k n_k(u,T))du}, \end{aligned}$$

where $\psi(iw_k n_k(u,T)) = \int_{\mathbb{R}} \left(e^{-w_k n_k(u,T)x} - 1 + w_k n_k(u,T)x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx$. We are ready to state the result in a proposition.

Proposition 5.1 *The price of a zero-coupon bond at time $t \leq T$ is*

$$P(t,T) = e^{\sum_{k=1}^n \left[\int_t^T \psi(iw_k n_k(u,T))du - w_k X_k(t) n_k(t,T)\right]}, \tag{5.6}$$

where

$$\begin{aligned} n_k(t,T) &= \frac{1}{\alpha_k}\left(1 - e^{-\alpha_k(T-t)}\right) \\ \psi(iw_k n_k(u,T)) &= \int_{\mathbb{R}} \left(e^{-w_k n_k(u,T)x} - 1 + w_k n_k(u,T)x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx. \end{aligned}$$

When we find option prices later we are only interested in the special case when $n = 1$. So we state the result for $n = 1$ as well.

Corollary 5.1 *When $n = 1$ the price of a zero-coupon bond at time $t \leq T$ is*

$$P(t,T) = e^{m(t,T) - n(t,T)r(t)}, \tag{5.7}$$

where

$$\begin{aligned} m(t,T) &= \int_t^T \psi(in(u,T))du \\ n(t,T) &= \frac{1}{\alpha_1}\left(1 - e^{-\alpha_1(T-t)}\right) \\ \psi(in(u,T)) &= \int_{\mathbb{R}} \left(e^{-n(u,T)x} - 1 + n(u,T)x\mathbf{1}_{|x|\leq 1}\right)\nu(x)dx. \end{aligned}$$

Proof. When $n=1$ we get that $r(s)$ becomes

$$r(s) = w_1 X_1(s), \tag{5.8}$$

where

$$X_1(s) = X_1(t)e^{-\alpha_1(s-t)} + \int_t^s e^{-\alpha_1(s-u)} dL_1(u),$$

where $t \leq s$. We set $w_1 = 1$. We then have that $r(t) = X_1(t)$. From Proposition 5.1 we then get with $n = 1$;

$$P(t,T) = e^{\int_t^T \psi(in_1(u,T))du - r(t)n_1(t,T)}.$$

Letting $n(t,T) = n_1(t,T)$ and $m(t,T) = \int_t^T \psi(in(u,T))du$ in $P(t,T)$ above, we have completed the proof.

□

5.2 Zero-Coupon Bond Price when $L_k(t)$ is Compound Poisson

We want to find the explicit result of a zero-coupon bond price when $L_k(t)$ is compound Poisson. As before, the zero-coupon bond price at time $t \leq T$ is given by

$$P(t,T) = \mathrm{E}_Q\left[e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t\right].$$

From (5.5) we have that $P(t,T)$ can be written as

$$P(t,T) = e^{-\sum_{k=1}^n w_k X_k(t) n_k(t,T)} \mathrm{E}_Q\left[e^{-\sum_{k=1} w_k \int_t^T n_k(u,T) dL_k(u)}\right].$$

In this case we know how the characteristic function of $L_k(t)$ looks like, since we know it's compound Poisson. The characteristic function of $L_k(t)$ is then for each k;

$$\mathrm{E}_Q\left[e^{izL_k(t)}\right] = e^{\lambda_k t\left(\frac{\theta_k}{\theta_k - iz} - 1\right)}.$$

Let $\{u_j\}_{j=1}^m$ be any partion of the interval $[t,T]$ with $\max_j |u_{j+1} - u_j| < \epsilon$. Then we can write $\int_t^T n_k(u,T)dL_k(u) = \lim_{\epsilon\to 0} \sum_{j=1}^m n_k(u_j,T)\Delta L_k(u_j)$. So again

$$\begin{aligned}
\mathrm{E}_Q\left[e^{-\sum_{k=1}^n w_k \int_t^T n_k(u,T)dL_k(u)}\right] &= \prod_{k=1}^n \mathrm{E}_Q\left[e^{-w_k \lim_{\epsilon\to 0}\sum_{j=1}^m n_k(u_j,T)\Delta L_k(u_j)}\right] \\
&= \prod_{k=1}^n \lim_{\epsilon\to 0} \prod_{j=1}^m \mathrm{E}_Q\left[e^{-w_k n_k(u_j,T)\Delta L_k(u_j)}\right] \\
&= \prod_{k=1}^n \lim_{\epsilon\to 0} \prod_{j=1}^m e^{\lambda_k\left(\frac{\theta_k}{\theta_k + w_k n_k(u_j,T)} - 1\right)\Delta u_j} \\
&= \prod_{k=1}^n e^{\lambda_k \int_t^T \left(\frac{\theta_k}{\theta_k + w_k n_k(u,T)} - 1\right)du} \\
&= \prod_{k=1}^n e^{\lambda_k I_k - \lambda_k(T-t)},
\end{aligned}$$

where $I_k = \int_t^T \frac{\theta_k}{\theta_k + w_k n_k(u,T)} du$. Bounded Convergence theorem is applied to take the limit outside the expectation above. As noted before the products can be taken outside as well because of independence between L_k and L_j, and independence of increments of a Lévy process. Earlier we have seen an almost similar integral which we solved using Proposition 4.7. So

$$\begin{aligned} I_k &= \int_t^T \frac{1}{1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}\left(1-e^{-\alpha_k(T-u)}\right)} du = \int_t^T \frac{1}{1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k} - \frac{w_k}{\alpha_k}\frac{1}{\theta_k} e^{-\alpha_k T} e^{\alpha_k u}} du \\ &= \left[\frac{u}{1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}} - \frac{1}{\left(1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}\right)\alpha_k} \ln\left(1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}\left(1-e^{-\alpha_k(T-u)}\right)\right)\right]_t^T \\ &= (T-t)\frac{1}{1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}} + \frac{1}{\alpha_k + \frac{w_k}{\theta_k}} \ln\left(1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}\left(1-e^{-\alpha_k(T-t)}\right)\right). \end{aligned}$$

With $n_k(t,T) = \frac{1}{\alpha_k}\left(1-e^{-\alpha_k(T-t)}\right)$;

$$I_k = (T-t)\frac{1}{1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}} + \frac{1}{\alpha_k + \frac{w_k}{\theta_k}} \ln\left(1+\frac{w_k}{\theta_k} n_k(t,T)\right). \tag{5.9}$$

We get

$$\mathrm{E}_Q\left[e^{-\sum_{k=1}^n w_k \int_t^T n_k(u,T) dL_k(u)}\right] = \prod_{k=1}^n e^{\lambda_k I_k - \lambda_k (T-t)},$$

where I_k is given by (5.9). We have finally found the price of a zero-coupon bond when $L_k(t)$ is compound Poisson.

Proposition 5.2 *The price of a zero-coupon bond at time $t \le T$ with $L_k(t)$ compound Poisson is*

$$P(t,T) = e^{\sum_{k=1}^n [\lambda_k I_k - \lambda_k (T-t) - w_k X_k(t) n_k(t,T)]}, \tag{5.10}$$

where

$$\begin{aligned} n_k(t,T) &= \frac{1}{\alpha_k}\left(1-e^{-\alpha_k(T-t)}\right) \\ I_k &= (T-t)\frac{1}{1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}} + \frac{1}{\alpha_k + \frac{w_k}{\theta_k}} \ln\left(1+\frac{w_k}{\theta_k} n_k(t,T)\right). \end{aligned}$$

As in the general case, we will need the result for $n = 1$ when finding prices of European bond options. So let's state the result for the case $n = 1$:

Corollary 5.2 *When $n = 1$, the price of a zero-coupon bond at time $t \le T$ is*

$$P(t,T) = e^{m(t,T) - r(t) n(t,t)},$$

where

$$\begin{aligned} n(t,T) &= \frac{1}{\alpha_1}\left(1-e^{-\alpha_1(T-t)}\right), \\ m(t,T) &= \lambda_1 (T-t)\left(\frac{1}{1+\frac{1}{\alpha_1}\frac{1}{\theta_1}} - 1\right) + \frac{\lambda_1}{\alpha_1 + \frac{1}{\theta_1}} \ln\left(1+\frac{1}{\theta_1} n(t,T)\right). \end{aligned}$$

Chapter 6

European Bond Options

Our goal in this chapter is to find prices of European call and put options written on zero-coupon bonds. First we find prices when $r(t)$ is given by the Vasicek model, (3.1), namely

$$dr(t) = (\mu - \alpha r(t))\,dt + \sigma dB(t),$$

and then we find prices when $r(t)$ is given by the extended Vasicek model given by (4.1), namely

$$r(t) = r(0) + \sum_{k=1}^{n} w_k X_k(t),$$

where

$$dX_k(t) = -\alpha_k X_k(t)dt + dL_k(t).$$

In both cases we only consider $n = 1$.

Definition 6.1 *The price at time $t \leq T$ of an European option written on a zero-coupon bond with price $P(T, S)$, $T \leq S$, is*

$$p(t) = \mathrm{E}_Q\left[e^{-\int_t^T r(s)ds} f(P(T,S))\big|\mathcal{F}_t\right], \tag{6.1}$$

where $f(x)$ is the payoff function. For an European call, $f(x) = max(x - K, 0)$ and an European put has payoff function $f(x) = max(K - x, 0)$. K is the strike price.

6.1 Bond Option Prices with the Vasicek Model

From Proposition 3.4 we already have that the price of a zero-coupon bond is

$$P(t,T) = e^{m(t,T) - n(t,T)r(t)}, \tag{6.2}$$

where

$$n(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) \tag{6.3}$$

$$m(t,T) = \frac{1}{2}\sigma^2 \int_t^T n^2(u,T)du - \mu \int_t^T n(u,T)du. \tag{6.4}$$

When $n(t,T)$ is as in (6.3), $m(t,T)$ becomes

$$\begin{aligned} m(t,T) &= \frac{1}{2}\sigma^2 \int_t^T \frac{1}{\alpha^2}\left(1-e^{-\alpha(T-u)}\right)^2 du - \mu \int_t^T \frac{1}{\alpha}\left(1-e^{-\alpha(T-u)}\right) du \\ &= \frac{1}{2}\sigma^2 \int_t^T \left(1-2e^{-\alpha(T-u)}+e^{-2\alpha(T-u)}\right) du - \mu \int_t^T \frac{1}{\alpha}\left(1-e^{-\alpha(T-u)}\right) du \\ &= \frac{\sigma^2}{2\alpha^2}(T-t) - \frac{\sigma^2}{\alpha^3}\left(1-e^{-\alpha(T-t)}\right) + \frac{\sigma^2}{4\alpha^3}\left(1-e^{-2\alpha(T-t)}\right) \\ &\quad -\frac{\mu}{\alpha}(T-t) + \frac{\mu}{\alpha^2}\left(1-e^{-\alpha(T-t)}\right) \\ &= \left(\frac{\mu}{\alpha}-\frac{\sigma^2}{2\alpha^2}\right)(n(t,T)-(T-t)) - \frac{\sigma^2}{4\alpha}n^2(t,T). \end{aligned}$$

Let's take a look at (6.1). We would like to find a way to take $e^{-\int_t^T r(s)ds}$ outside the expectation. It turns out that if we change measure, we could do so. Define

$$B_t = e^{\int_0^t r(s)ds}.$$

Then the price of the option becomes

$$\begin{aligned} p(t) &= \mathrm{E}_Q\left[e^{-\int_t^T r(s)ds} f(P(T,S))\middle|\mathcal{F}_t\right] = \mathrm{E}_Q\left[B_t B_T^{-1} f(P(T,S))\middle|\mathcal{F}_t\right] \\ &= B_t \mathrm{E}_Q\left[B_T^{-1}\frac{P(0,T)}{P(0,T)} f(P(T,S))\middle|\mathcal{F}_t\right] = B_t \mathrm{E}_Q\left[P(0,T)\eta_T f(P(T,S))\middle|\mathcal{F}_t\right] \\ &= B_t P(0,T)\mathrm{E}_Q\left[\eta_T f(P(T,S))\middle|\mathcal{F}_t\right], \end{aligned}$$

where we have used that B_t is $\mathcal{F}_t$-measurable, and that $\eta_T = B_T^{-1}P^{-1}(0,T) = \frac{d\mathbb{P}_T}{dQ}$. Notice that $P(0,T) = \mathrm{E}_Q[B_T]$, so it can be taken outside the expectation without problems. We are going to use Bayes's rule. We state it in a lemma.

Lemma 6.1 *Let $\mathbb{P}_T$ and Q be two probability measures on a measurable space $(\Omega,\mathcal{F})$ with $d\mathbb{P}_T = \eta_T dQ$. Let X be a random variable on $(\Omega,\mathcal{F})$, then*

$$\mathrm{E}_Q\left[\eta_T X\middle|\mathcal{F}_t\right] = \mathrm{E}_Q\left[\eta_T\middle|\mathcal{F}_t\right]\mathrm{E}_{\mathbb{P}_T}\left[X\middle|\mathcal{F}_t\right].$$

Proof. The proof can be found in Musiela and Rutkowski [5].

Applying this lemma, we get

$$\begin{aligned} p(t) &= B_t P(0,T)\mathrm{E}_Q\left[\eta_T f(P(T,S))\middle|\mathcal{F}_t\right] = B_t P(0,T)\mathrm{E}_{\mathbb{P}_T}\left[f(P(T,S))\middle|\mathcal{F}_t\right]\mathrm{E}_Q\left[\eta_T\middle|\mathcal{F}_t\right] \\ &= B_t \mathrm{E}_{\mathbb{P}_T}\left[f(P(T,S))\middle|\mathcal{F}_t\right]\mathrm{E}_Q\left[\frac{P(0,T)}{B_T P(0,T)}\middle|\mathcal{F}_t\right] = P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[f(P(T,S))\middle|\mathcal{F}_t\right]. \end{aligned}$$

We used that $B_t B_T^{-1} = e^{-\int_t^T r(s)ds}$ and that $P(t,T) = \mathrm{E}_Q\left[B_t B_T^{-1}\middle|\mathcal{F}_t\right] = B_t \mathrm{E}_Q\left[B_T^{-1}\middle|\mathcal{F}_t\right]$. So the price can be written as

$$p(t) = P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[f(P(T,S))\middle|\mathcal{F}_t\right]. \tag{6.5}$$

Since we already have $P(t,T)$, we only need to compute $\mathrm{E}_{\mathbb{P}_T}\left[f(P(T,S))\middle|\mathcal{F}_t\right]$. To do so we need the dynamics of $P(T,S)$ under $\mathbb{P}_T$. Let's first find $dP(t,T)$. Observe that $P(t,T)$ is given by (6.2), where $n(t,T)$ and $m(t,T)$ are as in (6.3) and (6.4). Call $\frac{d}{dt}n(t,T) = n_t(t,T)$ and $\frac{d}{dt}m(t,T) = m_t(t,T)$. Then

$$\begin{aligned} n_t(t,T) &= -e^{-\alpha(T-t)} \\ m_t(t,T) &= -\frac{1}{2}\frac{\sigma^2}{\alpha^2} + \frac{\sigma^2}{\alpha^2}e^{-\alpha(T-t)} - \frac{1}{2}\frac{\sigma^2}{\alpha^2}e^{-2\alpha(T-t)} + \frac{\mu}{\alpha} - \frac{\mu}{\alpha}e^{-\alpha(T-t)}, \end{aligned}$$

so by Itô's formula, Theorem 2.4, we get

$$\begin{aligned} &dP(t,T) \\ &= (m_t(t,T) - n_t(t,T)r(t))\, P(t,T)dt - n(t,T)P(t,T)dr(t) + \frac{1}{2}n^2(t,T)P(t,T)\,(dr(t))^2 \\ &= P(t,T)\Big[-\frac{1}{2}\frac{\sigma^2}{\alpha^2} + \frac{\sigma^2}{\alpha^2}e^{-\alpha(T-t)} - \frac{1}{2}\frac{\sigma^2}{\alpha^2}e^{-2\alpha(T-t)} + \frac{\mu}{\alpha} - \frac{\mu}{\alpha}e^{-\alpha(T-t)} + e^{-\alpha(T-t)}r(t)\Big]dt \\ &\quad -P(t,T)\frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)[(\mu - \alpha r(t))\,dt + \sigma dB(t)] \\ &\quad +P(t,T)\frac{1}{2}\frac{\sigma^2}{\alpha^2}\left(1 - 2e^{-\alpha(T-t)} + e^{-2\alpha(T-t)}\right)dt \\ &= P(t,T)\,[r(t)dt - \sigma n(t,T)dB(t)]\,. \end{aligned}$$

This is the dynamics under Q. Now let $\eta_t = e^{-\sigma\int_0^t n(u,T)dB(u) - \frac{1}{2}\sigma^2\int_0^t n^2(u,T)du}$. From Girsanov Theorem (Theorem 2.5) we know that the process $B^T(t)$ defined as $B^T(t) = B(t) + \sigma\int_0^t n(u,T)du$ is again a Brownian Motion under $\mathbb{P}_T$. To get the dynamics of $P(t,T)$ under $\mathbb{P}_T$, insert $dB(t) = dB^T(t) - \sigma n(t,T)dt$ into the dynamics for $P(t,T)$.

$$\begin{aligned} dP(t,T) &= P(t,T)\,[r(t)dt - \sigma n(t,T)dB(t)] \\ &= P(t,T)\left[r(t)dt - \sigma n(t,T)\left(dB^T(t) - \sigma n(t,T)dt\right)\right] \\ &= P(t,T)\left[\left(r(t) + \sigma^2 n^2(t,T)\right)dt - \sigma n(t,T)dB^T(t)\right]. \end{aligned}$$

To get an expression with just a $dB^T(t)$-term, we consider $F_P(t,S,T) = \frac{P(t,S)}{P(t,T)}$. This makes sense because $F_P(T,S,T) = P(T,S)$. We first find the dynamics of $F_P(T,S,T)$ under Q. Remember that $dP(t,T) = P(t,T)\,(r(t) - \sigma n(t,T)dB(t))$ under Q.

$$\begin{aligned} d\left(\frac{P(t,S)}{P(t,T)}\right) &= \frac{dP(t,S)}{P(t,T)} - \frac{P(t,S)dP(t,T)}{P^2(t,T)} + \frac{P(t,S)\,(dP(t,T))^2}{P^3(t,T)} - \frac{dP(t,S)dP(t,T)}{P^2(t,T)} \\ &= \frac{P(t,S)}{P(t,T)}\,[r(t)dt - \sigma n(t,S)dB(t)] - \frac{P(t,S)}{P(t,T)}\,[r(t)dt - \sigma n(t,T)dB(t)] \\ &\quad + \frac{P(t,S)}{P(t,T)}\sigma^2 n^2(t,T)dt - \frac{P(t,S)}{P(t,T)}\sigma^2 n(t,S)n(t,T)dt \\ &= \frac{P(t,S)}{P(t,T)}\left[\sigma\,(n(t,T) - n(t,S))\,dB(t) + \sigma^2 n(t,T)\,(n(t,T) - n(t,S))\,dt\right]. \end{aligned}$$

And under $\mathbb{P}_T$, we get

$$\begin{aligned} d\left(\frac{P(t,S)}{P(t,T)}\right) &= \frac{P(t,S)}{P(t,T)}\Big[\sigma\,(n(t,T) - n(t,S))\left(dB^T(t) - \sigma n(t,T)dt\right) \\ &\qquad + \sigma^2 n(t,T)\,(n(t,T) - n(t,S))\,dt\Big] \\ &= \frac{P(t,S)}{P(t,T)}\sigma\,(n(t,T) - n(t,S))\,dB^T(t). \end{aligned}$$

The dynamics of $F_P(t,S,T)$ under $\mathbb{P}_T$ becomes

$$dF_P(t,S,T) = F_P(t,S,T)\sigma\left(n(t,T) - n(t,S)\right)dB^T(t), \tag{6.6}$$

and the solution of the above equation is

$$F_P(T,S,T) = F_P(t,S,T)e^{\xi(t,T)-\frac{1}{2}\nu^2(t,T)}. \tag{6.7}$$

The solution of (6.6) is found by applying Itô's formula on $\ln F_P(t,S,T)$. Here we have $\xi(t,T) = \int_t^T \sigma\left(n(u,T) - n(u,S)\right)dB^T(u)$, with expectation equal zero and $\nu^2(t,T) = \mathrm{Var}_{\mathbb{P}_T}\left[\xi(t,T)\right] = \int_t^T \sigma^2\left(n(u,T) - n(u,S)\right)^2 du$. Now we are ready to find the price at time $t \le T$ of an European option written on $P(T,S)$. The price is given by (6.5), so we need to compute $\mathrm{E}_{\mathbb{P}_T}\left[f(P(T,S))\middle|\mathcal{F}_t\right]$. We first find the European call option. The payoff function is then $f(P(T,S)) = \max\left(P(T,S) - K, 0\right)$. Define $D = \{P(T,S) > K\}$, such that

$$\begin{aligned}
p_c(t) &= P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[f(P(T,S))\middle|\mathcal{F}_t\right] \\
&= P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[P(T,S)\mathbf{1}_D - K\mathbf{1}_D\middle|\mathcal{F}_t\right] \\
&= P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[P(T,S)\mathbf{1}_D\middle|\mathcal{F}_t\right] - P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[K\mathbf{1}_D\middle|\mathcal{F}_t\right] \\
&= P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[P(T,S)\mathbf{1}_D\middle|\mathcal{F}_t\right] - P(t,T)K\mathbb{P}_T\left(D\middle|\mathcal{F}_t\right) \\
&= A_1 - A_2.
\end{aligned}$$

We compute the parts separately, starting with A_2. First, look at the set D;

$$\begin{aligned}
D &= \{P(S,T) > K\} = \{F_P(T,S,T) > K\} \\
&= \left\{F_P(t,S,T)e^{\xi(t,T)-\frac{1}{2}\nu^2(t,T)} > K\right\} \\
&= \left\{\xi(t,T) > \ln\left(\frac{K}{F_P(t,S,T)}\right) + \frac{1}{2}\nu^2(t,T)\right\}.
\end{aligned}$$

Notice that $\xi(t,T)$ is independent of $\mathcal{F}_t$ since $B^T(t)$ has independent increments, and that $\nu^2(t,T)$ is deterministic. Notice also that $F_P(t,S,T)$ is $\mathcal{F}_t$-measurable and that $\xi(t,T) \sim \mathcal{N}(0,\nu^2(t,T))$. Then $\nu(t,T)\epsilon$, with $\epsilon \sim \mathcal{N}(0,1)$, have the same distribution. A_2 becomes

$$\begin{aligned}
A_2 &= P(t,T)K\mathbb{P}_T\left(D\middle|\mathcal{F}_t\right) = P(t,T)K\mathbb{P}_T\left(D\right) \\
&= P(t,T)K\mathbb{P}_T\left(\xi(t,T) > \ln\left(\frac{K}{F_P(t,S,T)}\right) + \frac{1}{2}\nu^2(t,T)\right) \\
&= P(t,T)K\mathbb{P}_T\left(\epsilon > \frac{\ln\left(\frac{K}{F_P(t,S,T)}\right) + \frac{1}{2}\nu^2(t,T)}{\nu(t,T)}\right) \\
&= P(t,T)K\mathbb{P}_T\left(\epsilon < \frac{\ln\left(\frac{F_P(t,S,T)}{K}\right) - \frac{1}{2}\nu^2(t,T)}{\nu(t,T)}\right),
\end{aligned}$$

since the standard normal distribution is symmetric around zero. We get

$$A_2 = KP(t,T)N\left(\frac{\ln\left(\frac{F_P(t,S,T)}{K}\right) - \frac{1}{2}\nu^2(t,T)}{\nu(t,T)}\right). \tag{6.8}$$

We want to evaluate A_1 as well. To do so, we change measure again. Let

$$\tilde{\eta}_t = e^{\int_0^t \sigma(n(u,T)-n(u,S))dB^T(u)-\frac{1}{2}\int_0^t \sigma^2(n(u,T)-n(u,S))^2du},$$

and $\tilde{\eta}_T = \frac{d\tilde{\mathbb{P}}_T}{d\mathbb{P}_T}$. Then, from Girsanov's Theorem (Theorem 2.5), $d\tilde{B}^T(t) = dB^T(t) - \sigma\left(n(t,T)-n(t,S)\right)dt$ is again a Brownian Motion under $\tilde{\mathbb{P}}_T$. So

$$\begin{aligned} dF_P(t,S,T) &= F_P(t,S,T)\sigma\left(n(t,T)-n(t,S)\right)dB^T(t) \\ &= F_P(t,S,T)\sigma\left(n(t,T)-n(t,S)\right)d\tilde{B}^T(t) \\ &\quad + F_P(t,S,T)\sigma^2\left(n(t,T)-n(t,S)\right)^2 dt, \end{aligned}$$

and the solution of $F_P(T,S,T)$ under $\tilde{\mathbb{P}}_T$ becomes

$$F_P(T,S,T) = F_P(t,S,T)e^{\tilde{\xi}(t,T)+\frac{1}{2}\nu^2(t,T)},$$

where $\tilde{\xi}(t,T) = \int_t^T \sigma\left(n(u,T)-n(u,S)\right)d\tilde{B}^T(u)$, with $\mathrm{E}_{\tilde{\mathbb{P}}_T}\left[\tilde{\xi}(t,T)\right] = 0$ and $\nu^2(t,T) = \mathrm{Var}_{\tilde{\mathbb{P}}_T}\left[\tilde{\xi}(t,T)\right] = \int_t^T \sigma^2\left(n(u,T)-n(u,S)\right)^2 du$. Let's look at A_1. Notice that $F_P(t,S,T)$ is $\mathcal{F}_t$-measurable and that $\tilde{\eta}_T\tilde{\eta}_t^{-1} = e^{\xi(t,T)-\frac{1}{2}\nu^2(t,T)}$. We will apply Bayes rule (Lemma 6.1). It says that

$$\mathrm{E}_{\tilde{\mathbb{P}}_T}\left[\mathbf{1}_D\middle|\mathcal{F}_t\right] = \frac{\mathrm{E}_{\mathbb{P}_T}\left[\mathbf{1}_D\tilde{\eta}_T\middle|\mathcal{F}_t\right]}{\mathrm{E}_{\mathbb{P}_T}\left[\tilde{\eta}_T\middle|\mathcal{F}_t\right]} = \mathrm{E}_{\mathbb{P}_T}\left[\mathbf{1}_D\tilde{\eta}_T\tilde{\eta}_t^{-1}\middle|\mathcal{F}_t\right],$$

since $\tilde{\eta}_t = \mathrm{E}_{\mathbb{P}_T}\left[\tilde{\eta}_T\middle|\mathcal{F}_t\right]$. Moreover

$$\begin{aligned} A_1 &= P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[F_P(T,S,T)\mathbf{1}_D\middle|\mathcal{F}_t\right] \\ &= P(t,T)\mathrm{E}_{\mathbb{P}_T}\left[F_P(t,S,T)e^{\xi(t,T)-\frac{1}{2}\nu^2(t,T)}\mathbf{1}_D\middle|\mathcal{F}_t\right] \\ &= P(t,T)F_P(t,S,T)\mathrm{E}_{\mathbb{P}_T}\left[e^{\xi(t,T)-\frac{1}{2}\nu^2(t,T)}\mathbf{1}_D\middle|\mathcal{F}_t\right] \\ &= P(t,S)\mathrm{E}_{\mathbb{P}_T}\left[\tilde{\eta}_T\tilde{\eta}_t^{-1}\mathbf{1}_D\middle|\mathcal{F}_t\right] \\ &= P(t,S)\mathrm{E}_{\tilde{\mathbb{P}}_T}\left[\mathbf{1}_D\middle|\mathcal{F}_t\right] \\ &= P(t,S)\tilde{\mathbb{P}}_T\left(D\middle|\mathcal{F}_t\right). \end{aligned}$$

If we look at the set D, with the expression for $F_P(T,S,T)$ under $\tilde{\mathbb{P}}_T$ instead of $\mathbb{P}_T$, we get

$$\begin{aligned} D &= \{P(T,S) > K\} = \{F_P(T,S,T) > K\} \\ &= \left\{F_P(t,S,T)e^{\tilde{\xi}(t,T)+\frac{1}{2}\nu^2(t,T)} > K\right\} \\ &= \left\{\tilde{\xi}(t,T) > \ln\left(\frac{K}{F_P(t,S,T)}\right) - \frac{1}{2}\nu^2(t,T)\right\}. \end{aligned}$$

Notice that we have $\tilde{\xi}(t,T) \sim N\left(0,\nu^2(t,T)\right)$ and that it is independent of $\mathcal{F}_t$. As before we have that $\nu(t,T)\epsilon$, where $\epsilon \sim N(0,1)$, have the same distribution as $\tilde{\xi}(t,T)$. So

$$\begin{aligned} A_1 &= P(t,S)\tilde{\mathbb{P}}_T\left(D\middle|\mathcal{F}_t\right) = P(t,S)\tilde{\mathbb{P}}_T\left(D\right) \\ &= P(t,S)\tilde{\mathbb{P}}_T\left(\tilde{\xi}(t,T) > \ln\left(\frac{K}{F_P(t,S,T)}\right) - \frac{1}{2}\nu^2(t,T)\right) \\ &= P(t,S)\tilde{\mathbb{P}}_T\left(\epsilon > \frac{\ln\left(\frac{K}{F_P(t,S,T)}\right) - \frac{1}{2}\nu^2(t,T)}{\nu(t,T)}\right) \end{aligned}$$

$$
\begin{aligned}
&= P(t,S)\tilde{\mathbb{P}}_T\left(\epsilon < \frac{\ln\left(\frac{F_P(t,S,T)}{K}\right)+\frac{1}{2}\nu^2(t,T)}{\nu(t,T)}\right)\\
&= P(t,S)N\left(\frac{\ln\left(\frac{F_P(t,S,T)}{K}\right)+\frac{1}{2}\nu^2(t,T)}{\nu(t,T)}\right),
\end{aligned}
$$

where we used that the standard normal distribution is symmetric around zero. Finally we are ready to derive the price of an European call option. Remember that the price was $p_c(t) = A_1 - A_2$, where A_1 is as above, and A_2 is given by (6.8).

Proposition 6.1 *The price at time $t \leq T$ of an European call option written on a zero-coupon bond is*

$$
p_c(t) = P(t,S)N(u_1) - KP(t,T)N(u_2), \tag{6.9}
$$

where

$$
u_{1,2} = \frac{\ln\left(\frac{F_P(t,S,T)}{K}\right) \pm \frac{1}{2}\nu^2(t,T)}{\nu(t,T)},
$$

and

$$
\nu^2(t,T) = \int_t^T \sigma^2\left(n(u,T) - n(u,S)\right)^2 du.
$$

We find the price of an European put option by using the Put-Call parity.

Lemma 6.2 *Let C_t and P_t be prices of an European call option and an European put option respectively written on an asset Z_t with strike K. Then the following relationship holds*

$$
C_t - P_t = Z_t - KP(t,T), \tag{6.10}
$$

for all t in the interval $[0,T]$. $P(t,T)$ is the price of a zero-coupon bond. This relationship is called the ***Put-Call parity****.*

We now find the price of an European put option. In our case $Z_t = P(t,S)$, so

$$
\begin{aligned}
p_p(t) &= p_c(t) - P(t,S) + KP(t,T)\\
&= P(t,S)N(u_1) - KP(t,T)N(u_2) - P(t,S) + KP(t,T)\\
&= KP(t,T)\left(1 - N(u_2)\right) - P(t,S)\left(1 - N(u_1)\right)\\
&= KP(t,T)N(-u_2) - P(t,S)N(-u_1).
\end{aligned}
$$

Call $\tilde{u}_1 = -u_1$ and $\tilde{u}_2 = -u_2$. Then

$$
\begin{aligned}
\tilde{u}_1 &= -\frac{\ln\left(\frac{F_P(t,S,T)}{K}\right)+\frac{1}{2}\nu^2(t,T)}{\nu(t,T)} = \frac{\ln\left(\frac{K}{F_P(t,S,T)}\right)-\frac{1}{2}\nu^2(t,T)}{\nu(t,T)}\\
\tilde{u}_2 &= -\frac{\ln\left(\frac{F_P(t,S,T)}{K}\right)-\frac{1}{2}\nu^2(t,T)}{\nu(t,T)} = \frac{\ln\left(\frac{K}{F_P(t,S,T)}\right)+\frac{1}{2}\nu^2(t,T)}{\nu(t,T)}.
\end{aligned}
$$

We state the result in a proposition.

Proposition 6.2 *The price at time $t \leq T$ of an European put option written on a zero-coupon bond is*

$$p_p(t) = KP(t,T)N(\tilde{u}_2) - P(t,S)N(\tilde{u}_1), \tag{6.11}$$

where

$$\tilde{u}_{1,2} = \frac{\ln\left(\frac{K}{F_P(t,S,T)}\right) \mp \frac{1}{2}\nu^2(t,T)}{\nu(t,T)},$$

and

$$\nu^2(t,T) = \int_t^T \sigma^2 \left(n(u,T) - n(u,S)\right)^2 du.$$

6.2 Bond Option Prices with Extended Vasicek Model

The price of an European option written on a zero-coupon bond is given by (6.1). First we are going to derive the price of an European put option and later use the Put-Call parity to find the price of an European call. In the end we are going to state explicit result of prices when $L_k(t)$ is compound Poisson. In both cases we only consider $n = 1$. This gives us

$$r(s) = X_1(s), \tag{6.12}$$

where we have set $w_1 = 1$. $X_1(s)$ is given as before

$$X_1(s) = X_1(t)e^{-\alpha_1(s-t)} + \int_t^s e^{-\alpha_1(s-u)} dL_1(u), \tag{6.13}$$

where $t \leq s$. And

$$r(s) = r(t)e^{-\alpha_1(s-t)} + \int_t^s e^{-\alpha_1(s-u)} dL_1(u). \tag{6.14}$$

6.2.1 Bond Option Prices with Extended Vasicek Model with General $L_k(t)$

From earlier computations we have that the price at time $T \leq S$ of a zero-coupon bond is

$$P(T,S) = e^{m(T,S) - n(T,S)r(T)},$$

with $m(T,S) = \int_T^S \psi(in(u,S))du$ and $n(T,S) = \frac{1}{\alpha_1}\left(1 - e^{-\alpha_1(S-T)}\right)$, where $\psi(in(u,S))$ is given as always. It follows from (5.4), with $n = 1$, $w_1 = 1$ and $X_1(t) = r(t)$, that

$$-\int_t^T r(s)ds = -r(t)n(t,T) - \int_t^T n(u,T)dL_1(u).$$

From all this we can derive the following

$$\begin{aligned} p_p(t) &= \mathrm{E}_Q\left[e^{-\int_t^T r(s)ds} f(P(T,S)) \middle| \mathcal{F}_t\right] \\ &= \mathrm{E}_Q\left[e^{-r(t)n(t,T) - \int_t^T n(u,T)dL_1(u)} \tilde{f}\left(r(T)\right) \middle| \mathcal{F}_t\right], \end{aligned}$$

where $f(x) = \max(K - x, 0)$ and $\tilde{f}(x) = \max\left(K - e^{m(T,S) - n(T,S)x}\right)$. It follows that $f(P(T,S)) = \tilde{f}(r(T))$. As we will see later, we want to apply inverse Fourier transform, Theorem 6.1, to find the price of the option. It is important that the function f is integrable on $\mathbb{R}^n$. A European put option has payoff function as $f(x)$ above and is therefore integrable, but that is not the case for an European call option. We therefore derive the price of an European put option first, and apply the Put-Call parity to obtain results for an European call option as well. From (6.14) we have

$$r(T) = r(t)e^{-\alpha_1(T-t)} + \int_t^T e^{-\alpha_1(T-u)} dL_1(u).$$

Now let

$$Y = \int_t^T n(u,T)dL_1(u) = \int_t^T \frac{1}{\alpha_1}\left(1 - e^{-\alpha_1(T-u)}\right) dL_1(u) \tag{6.15}$$

$$Z = \int_t^T e^{-\alpha_1(T-u)} dL_1(u). \tag{6.16}$$

Notice that $r(t)$ is $\mathcal{F}_t$-measurable and that $n(t,T)$ is deterministic. Y and Z is independent of $\mathcal{F}_t$ since $L_1(t)$ has independent increments. So

$$\begin{aligned} p_p(t) &= \mathrm{E}_Q\left[e^{-r(t)n(t,T)-Y}\tilde{f}\left(r(t)e^{-\alpha_1(T-t)} + Z\right)|\mathcal{F}_t\right] \\ &= e^{-r(t)n(t,T)}\mathrm{E}_Q\left[e^{-Y}g(Z)\right], \end{aligned}$$

where $g(x) = \max\left(K - e^{m(T,S)-n(T,S)\left[r(t)e^{-\alpha_1(T-t)}+x\right]}, 0\right)$. It follows now that $\tilde{f}(r(T)) = g(Z)$.

$$p_p(t) = e^{-r(t)n(t,T)}\mathrm{E}_Q\left[e^{-Y}g(Z)\right]. \tag{6.17}$$

We want to apply inverse Fourier transform to compute $\mathrm{E}_Q\left[e^{-Y}g(Z)\right]$. First we define the Fourier transform of a function f.

Definition 6.2 *The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is*

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{-iy\cdot x}dx. \tag{6.18}$$

One can then recover the function f from it's Fourier transform in the following way.

Theorem 6.1 ***The Fourier Inversion Theorem***
Let $f \in L^1(\mathbb{R}^n)$, with Fourier transform defined by (6.18). *Then*

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(y)e^{iy\cdot x}dy. \tag{6.19}$$

Now define

$$h(y,z) = \begin{cases} e^{-y}g(z) & \text{if } y, z > 0 \\ 0 & \text{if } y \le 0 \text{ or } z \le 0 \end{cases}.$$

Notice that $\mathrm{E}_Q[h(Y,Z)] = \mathrm{E}_Q\left[e^{-Y}g(Z)\right]$. If we can find $\mathrm{E}_Q[h(Y,Z)]$, we can also find the price $p_p(t) = e^{-r(t)n(t,T)}\mathrm{E}_Q\left[e^{-Y}g(Z)\right]$. Remember that Y and Z is given by (6.15) and (6.16) respectively. We can define h the way we do, because $L_1(t)$ is a subordinator and therefore have only positive jumps. This implies that Y and Z is always positive. Thus

$$\hat{h}(u,v)) = \int_{\mathbb{R}^2} h(y,z)e^{-iuy-ivz}dydz,$$

and

$$h(Y,Z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\hat{h}(u,v)e^{iuY+ivZ}dudv.$$

It follows from above that

$$\mathrm{E}_Q\left[h(Y,Z)\right] = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\hat{h}(u,v)\mathrm{E}_Q\left[e^{iuY+ivZ}\right]dudv. \tag{6.20}$$

It remains only to compute the characteristic function of (Y,Z). It is defined to be $\mathrm{E}_Q\left[e^{iuY+ivZ}\right]$. First consider $iuY+ivZ$. Remember that Y and Z are given by (6.15) and (6.16) respectively.

$$\begin{aligned} iuY+ivZ &= iu\int_t^T\frac{1}{\alpha_1}\left(1-e^{-\alpha_1(T-s)}\right)dL_1(s)+iv\int_t^T e^{-\alpha_1(T-s)}dL_1(s) \\ &= i\int_t^T\left[\frac{u}{\alpha_1}\left(1-e^{-\alpha_1(T-s)}\right)+ve^{-\alpha_1(T-s)}\right]dL_1(s) \\ &= i\int_t^T\left[e^{-\alpha_1(T-s)}\left(v-\frac{u}{\alpha_1}\right)+\frac{u}{\alpha_1}\right]dL_1(s). \end{aligned}$$

Let $\{s_j\}_{j=1}^m$ be any partition of the interval $[t,T]$ with $\max_j|s_{j+1}-s_j|<\epsilon$. Let $k(s,T) = e^{-\alpha_1(T-s)}\left(v-\frac{u}{\alpha_1}\right)+\frac{u}{\alpha_1}$. We then have $\int_t^T k(s,T)dL_1(s) = \lim_{\epsilon\to 0}\sum_{j=1}^m k(s_j,T)\Delta L_1(s_j)$. Also remember that the characteristic function of a Lévy process $L(t)$ is $\mathrm{E}\left[e^{izL(t)}\right] = e^{\psi(z)t}$. It follows directly that

$$\begin{aligned} \mathrm{E}_Q\left[e^{iuY+ivZ}\right] &= \lim_{\epsilon\to 0}\prod_{j=1}^m\mathrm{E}_Q\left[e^{ik(s_j,T)\Delta L_1(s_j)}\right] \\ &= \lim_{\epsilon\to 0}\prod_{j=1}^m e^{\psi(k(s_j,T))\Delta s_j} \\ &= e^{\int_t^T\psi(k(s,T))ds}, \end{aligned}$$

where $\psi(k(s,T)) = \int_{\mathbb{R}}\left(e^{ik(s,T)x}-1-ik(s,T)x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx$. The characteristic function of (Y,Z) becomes

$$\mathrm{E}_Q\left[e^{iuY+ivZ}\right] = e^{\int_t^T\psi(k(s,T))ds}. \tag{6.21}$$

Remember that the price $p_c(t)$ is given by (6.17), where $e^{-Y}g(Z) = h(Y,Z)$. Remember also that $\mathrm{E}_Q\left[h(Y,Z)\right]$ is given by (6.20). So from above we get the following result:

Proposition 6.3 *The price at time $t\le T$ of an European put option written on a zero-coupon bond is*

$$p_p(t) = e^{-n(t,T)r(t)}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\hat{h}(u,v)e^{\int_t^T\psi(k(s,T))ds}dudv, \tag{6.22}$$

where

$$\begin{aligned} n(t,T) &= \frac{1}{\alpha_1}\left(1-e^{-\alpha_1(T-t)}\right), \\ k(s,T) &= e^{-\alpha_1(T-s)}\left(v-\frac{u}{\alpha_1}\right)+\frac{u}{\alpha_1}, \\ \psi(k(s,T)) &= \int_{\mathbb{R}}\left(e^{ik(s,T)x}-1-ik(s,T)x\mathbf{1}_{|x|\le 1}\right)\nu(x)dx \end{aligned}$$

and $\hat{h}(u,v)$ *is the Fourier transform of* $h(Y,Z)$.

One can compute $h(Y,Z)$ by using fast Fourier transform algorithm. It's an efficient algorithm to compute the discrete Fourier transform. However, we will not do it in this thesis.

But can we say something more about the Fourier transform of $h(Y,Z)$, $\hat{h}(u,v)$? Yes, indeed

$$\begin{aligned}
\hat{h}(u,v) &= \int_{\mathbb{R}^2} h(y,z)e^{-iuy-ivz}dydz \\
&= \int_{[0,\infty)\times[0,\infty)} e^{-y}g(z)e^{-iuy-ivz}dydz \\
&= \int_0^\infty g(z)e^{-ivz}\left(\int_0^\infty e^{-y(1+iu)}dy\right)dz \\
&= \int_0^\infty g(z)e^{-ivz}\left[-\frac{1}{1+iu}e^{-y(1+iu)}\right]_0^\infty dz \\
&= \frac{1}{1+iu}\int_0^\infty g(z)e^{-ivz}dz,
\end{aligned}$$

with $1 > iu$. We defined $g(z) = \max\left(K - e^{m(T,S)-n(T,S)\left[r(t)e^{-\alpha_1(T-t)}+z\right]}, 0\right)$, so we need K to be bigger than $e^{m(T,S)-n(T,S)\left[r(t)e^{-\alpha_1(T-t)}+z\right]}$ for the integral to be nonzero. That is, when

$$z > \frac{m(T,S) - \ln K - n(T,S)r(t)e^{-\alpha_1(T-t)}}{n(T,S)} = a(t,T,S).$$

It follows that

$$\begin{aligned}
\hat{h}(u,v) &= \frac{1}{1+iu}\int_0^\infty g(z)e^{-ivz}dz \\
&= \frac{1}{1+iu}\int_{a(t,T,S)}^\infty \left(K - e^{m(T,S)-n(T,S)\left[r(t)e^{-\alpha_1(T-t)}+z\right]}\right)e^{-ivz}dz \\
&= \frac{K}{1+iu}\int_{a(t,T,S)}^\infty e^{-ivz}dz - \frac{1}{1+iu}\int_{a(t,T,S)}^\infty e^{m(T,S)-n(T,S)\left[r(t)e^{-\alpha_1(T-t)}+z\right]-ivz}dz \\
&= I_1 - I_2.
\end{aligned}$$

We compute the parts separately. It's easy to see that

$$I_1 = \frac{K}{(1+iu)}\frac{1}{(iv)}e^{-iva(t,T,S)} = \frac{K}{iv-uv}e^{-iva(t,T,S)}. \tag{6.23}$$

It remains to compute I_2.

$$\begin{aligned}
I_2 &= \frac{1}{1+iu}\int_{a(t,T,S)}^\infty e^{m(T,S)-n(T,S)\left[r(t)e^{-\alpha_1(T-t)}+z\right]-ivz}dz \\
&= \frac{1}{1+iu}e^{m(T,S)-n(T,S)r(t)e^{-\alpha_1(T-t)}}\int_{a(t,T,S)}^\infty e^{-z(n(T,S)+iv)}dz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1+iu} e^{m(T,S)-n(T,S)r(t)e^{-\alpha_1(T-t)}} \left[-\frac{1}{n(T,S)+iv} e^{-z(n(T,S)+iv)} \right]_{a(t,T,S)}^{\infty} \\
&= \frac{1}{1+iu} e^{m(T,S)-n(T,S)r(t)e^{-\alpha_1(T-t)}} \frac{1}{n(T,S)+iv} e^{-a(t,T,S)(n(T,S)+iv)} \\
&= \frac{1}{(1+iu)\,(n(T,S)+iv)} e^{m(T,S)-n(T,S)\left[r(t)e^{-\alpha_1(T-t)}+a(t,T,S)\right]} e^{-iva(t,T,S)} \\
&= \frac{K}{(1+iu)\,(n(T,S)+iv)} e^{-iva(t,T,S)}.
\end{aligned}
$$

Here we have used that $a(t,T,S) = \frac{m(T,S)-\ln K - n(T,S)r(t)e^{-\alpha_1(T-t)}}{n(T,S)}$. It follows directly that

$$
\begin{aligned}
\hat{h}(u,v) &= I_1 - I_2 \\
&= \frac{K}{iv-uv} e^{-iva(t,T,S)} - \frac{K}{(1+iu)\,(n(T,S)+iv)} e^{-iva(t,T,S)} \\
&= K e^{-iva(t,T,S)} \left(\frac{1}{iv-uv} - \frac{1}{(1+iu)\,(n(T,S)+iv)} \right).
\end{aligned}
$$

6.2.2 Bond Option Prices with Extended Vasicek Model when $L_k(t)$ is Compound Poisson

From Corollary 5.2 we have that the price of a zero-coupon bond is given by

$$P(t,T) = e^{m(t,T)-n(t,T)r(t)},$$

where

$$m(t,T) = \lambda_1(T-t)\left(\frac{1}{1+\frac{1}{\theta_1}\frac{1}{\alpha_1}} - 1 \right) + \frac{\lambda_1}{\alpha_1+\frac{1}{\theta_1}} \ln\left(1 + \frac{1}{\theta_1} n(t,T) \right),$$

and

$$n(t,T) = \frac{1}{\alpha_1}\left(1 - e^{-\alpha_1(T-t)}\right).$$

It follows from (5.4), with $n=1$, $w_1 = 1$ and $X_1(t) = r(t)$, that

$$-\int_t^T r(s)ds = -r(t)n(t,T) - \int_t^T n(u,T)dL_1(u).$$

The price of an European option is defined by (6.1), so by using what we found in the general case, we can derive the following

$$
\begin{aligned}
p_p(t) &= \mathrm{E}_Q\left[e^{-\int_t^T r(s)ds} f(P(T,S)) \middle| \mathcal{F}_t \right] \\
&= \mathrm{E}_Q\left[e^{-r(t)n(t,T)-\int_t^T n(u,T)dL_1(u)} \tilde{f}(r(T)) \middle| \mathcal{F}_t \right] \\
&= e^{-r(t)n(t,T)} \mathrm{E}_Q\left[e^{-Y} g(Z) \right],
\end{aligned}
$$

where $f(P(T,S)) = \tilde{f}(r(T)) = g(Z)$ as before. Y and Z are the same as in (6.15) and (6.16). We are going to use Fourier transform, so we need to find the characteristic function of (Y,Z). It is defined by

$$\varphi_{Y,Z}(u,v) = \mathrm{E}_Q\left[e^{iuY+ivZ} \right].$$

Once again we start by finding $iuY + ivZ$.

$$\begin{aligned} iuY + ivZ &= iu\int_t^T \frac{1}{\alpha_1}\left(1 - e^{-\alpha_1(T-s)}\right) dL_1(s) + iv\int_t^T e^{-\alpha_1(T-s)} dL_1(s) \\ &= i\int_t^T \left[e^{-\alpha_1(T-s)}\left(v - \frac{u}{\alpha_1}\right) + \frac{u}{\alpha_1}\right] dL_1(s) \\ &= i\int_t^T k(s,T) dL_1(s), \end{aligned}$$

where $k(s,T) = e^{-\alpha_1(T-s)}\left(v - \frac{u}{\alpha_1}\right) + \frac{u}{\alpha_1}$. Remember that the characteristic function of $L_1(t)$, when $L_1(t)$ is compound Poisson, is given by $\mathrm{E}_Q\left[e^{izL_1(t)}\right] = e^{\lambda_1 t\left(\frac{\theta_1}{\theta_1 - iz} - 1\right)}$. Now, as before, let $\{u_j\}_{j=1}^m$ be any partition of the interval $[t,T]$ with $\max_j |u_{j+1} - u_j| < \epsilon$. Then

$$\begin{aligned} \mathrm{E}_Q\left[e^{iuY+ivZ}\right] &= \mathrm{E}_Q\left[e^{i\int_t^T k(s,T)dL_1(s)}\right] = \lim_{\epsilon\to 0}\prod_{j=1}^m \mathrm{E}_Q\left[e^{ik(s_j,T)\Delta L_1(s_j)}\right] \\ &= \lim_{\epsilon\to 0}\prod_{j=1}^m e^{\lambda_1\left(\frac{\theta_1}{\theta_1 - ik(s_j,T)} - 1\right)\Delta s_j} = e^{\lambda_1\int_t^T\left(\frac{\theta_1}{\theta_1 - ik(s,T)} - 1\right)ds} \\ &= e^{\lambda_1 I_1 - \lambda_1(T-t)}. \end{aligned}$$

Bounded Convergence Theorem is used to take the limit outside the expectation, and because L_k has independent increments, we can take the product outside as well. Define

$$h(y,z) = \begin{cases} e^{-y}g(z) & \text{if } y,z > 0 \\ 0 & \text{if } y \le 0 \text{ or } z \le 0 \end{cases}.$$

Let $\tilde{h}(u,v)$ be the Fourier transform of $h(Y,Z)$ defined by (6.18). Then we can find the inverse Fourier transform from (6.19). So

$$\begin{aligned} p_p(t) &= e^{-r(t)n(t,T)}\mathrm{E}_Q\left[e^{-Y}g(Z)\right] \\ &= e^{-r(t)n(t,T)}\mathrm{E}_Q\left[h(Y,Z)\right] \\ &= e^{-r(t)n(t,T)}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\tilde{h}(u,v)\mathrm{E}_Q\left[e^{iuY+ivZ}\right] dudv \\ &= e^{-r(t)n(t,T)}\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\tilde{h}(u,v)e^{\lambda_1 I_1 - \lambda_1(T-t)} dudv. \end{aligned}$$

The integral I_1 in the characteristic function of (Y,Z), can be solved. Remember that $k(s,T) = e^{-\alpha_1(T-s)}\left(v - \frac{u}{\alpha_1}\right) + \frac{u}{\alpha_1}$.

$$\begin{aligned} I_1 &= \int_t^T \frac{\theta_1}{\theta_1 - ik(s,T)} ds = \int_t^T \frac{1}{1 - i\frac{1}{\theta_1}k(s,T)} \\ &= \int_t^T \frac{1}{1 - i\frac{1}{\theta_1}\frac{u}{\alpha_1} - i\frac{1}{\theta_1}\left(v - \frac{u}{\alpha_1}\right)e^{-\alpha_1 T}e^{\alpha_1 s}} ds \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{1}{1 - i\frac{1}{\theta_1}\frac{u}{\alpha_1}} \left(s - \frac{1}{\alpha_1} \ln \left(1 - i\frac{1}{\theta_1}\frac{u}{\alpha_1} - i\frac{1}{\theta_1}\left(v - \frac{u}{\alpha_1} \right) e^{-\alpha_1(T-s)} \right) \right) \right]_t^T \\
&= \frac{T-t}{1 - i\frac{1}{\theta_1}\frac{u}{\alpha_1}} - \frac{1}{\alpha_1 - i\frac{u}{\theta_1}} \ln \left(1 - i\frac{1}{\theta_1}\frac{u}{\alpha_1} - i\frac{1}{\theta_1}\left(v - \frac{u}{\alpha_1} \right) \right) \\
&\quad + \frac{1}{\alpha_1 - i\frac{u}{\theta_1}} \ln \left(1 - i\frac{1}{\theta_1}\frac{u}{\alpha_1} - i\frac{1}{\theta_1}\left(v - \frac{u}{\alpha_1} \right) e^{-\alpha_1(T-t)} \right) \\
&= \frac{T-t}{1 - i\frac{1}{\theta_1}\frac{u}{\alpha_1}} - \frac{1}{\alpha_1 - i\frac{u}{\theta_1}} \ln \left(1 - i\frac{v}{\theta_1} \right) + \frac{1}{\alpha_1 - i\frac{u}{\theta_1}} \ln \left(1 - i\frac{1}{\theta_1}\left(ve^{-\alpha_1(T-t)} + un(t,T) \right) \right)
\end{aligned}
$$

where $n(t,T) = \frac{1}{\alpha_1}\left(1 - e^{-\alpha_1(T-t)}\right)$ as usual.

Proposition 6.4 *The price at time $t \leq T$ of an European put option written on a zero-coupon bond is*

$$
p_p(t) = e^{-r(t)n(t,T)} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \tilde{h}(u,v) e^{\lambda_1 I_1 - \lambda_1(T-t)} du dv,
$$

where I_1 is as above and $\tilde{h}(u,v)$ is the Fourier transform of $h(Y,Z)$ defined by (6.18). *The Lévy process used is compound Poisson.*

An European call option can be found by applying the Put-Call parity relationship (6.10)

Chapter 7

Simulation

In this chapter we simulate the interest rate $r(t)$ and zero-coupon bond prices $P(t,T)$ in both the Vasicek model and the extension of it. The simulation gives us an idea on how accurate the model is. A great deal of the challenge is to give realistic values to all the variables in the model.

7.1 Simulation of the Interest Rate $r(t)$

We start by simulating $r(t)$. We use results obtained earlier in this thesis.

7.1.1 Simulation of $r(t)$ in the Vasicek Model

Remember that the interest rate $r(t)$ in the Vasicek model has dynamics

$$dr(t) = (\mu - \alpha r(t))dt + \sigma dB(t),$$

where μ, α and σ are positive constants and $B(t)$ is a Brownian motion. To simulate $r(t)$ we use the Euler method, i.e. make an Euler scheme. We make $r(t)$ discrete instead of continuous, and get

$$r(t+\Delta t) - r(t) = (\mu - \alpha r(t))\Delta t + \sigma(B(t+\Delta t) - B(t)).$$

Since Brownian motion has stationary increments and is normally distributed, we can write $B(t+\Delta t) - B(t) = B(\Delta t)$ and $\sigma B(\Delta t) \sim \sigma\sqrt{\Delta t}\epsilon$, $\epsilon \sim N(0,1)$.

$$r(t+\Delta t) = r(t) + (\mu - \alpha r(t))\Delta t + \sigma\sqrt{\Delta t}\epsilon,$$

where $\epsilon \sim N(0,1)$. To make the simulating easier, we set $\Delta t = 1$. The values of the variables is as in Table 7.1. See the result of the simulations in Figure 7.1 and Figure 7.2

7.1.2 Simulation of $r(t)$ in the Extended Vasicek Model

Remember that the interest rate $r(t)$ in the extended Vasicek model is

$$r(t) = \sum_{k=1}^{n} w_k X_k(t), \tag{7.1}$$

Table 7.1: The values of the variables in the Vasicek model

μ	α	σ
0.004	0.0693	0.001

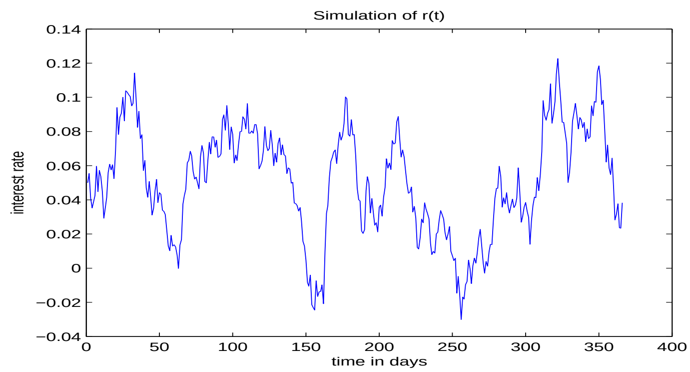

Figure 7.1: Simulation with the Vasicek model

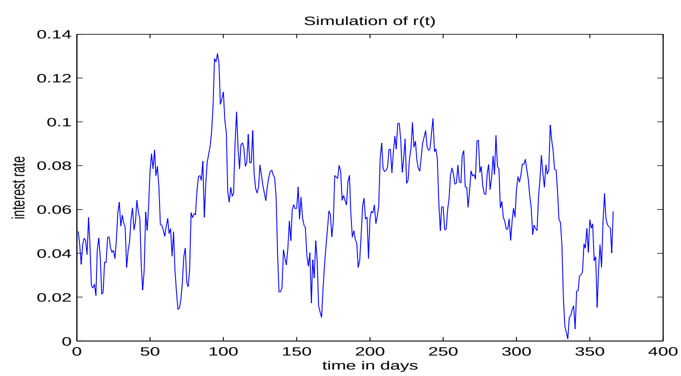

Figure 7.2: Simulation with the Vasicek model

Table 7.2: The values of the variables

k	α_k	λ_k	θ_k
1	0.0231	0.1	100
2	0.198	0.0111	40

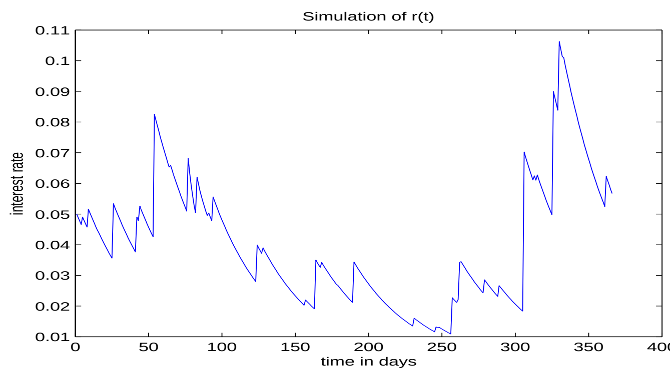

Figure 7.3: Simulation with the extended Vasicek model

where

$$dX_k(t) = -\alpha_k X_k(t)dt + dL_k(t). \tag{7.2}$$

Here are the Lévy processes $L_k(t)$ subordinators. We assume $n = 2$, and let $L_1(t)$ and $L_2(t)$ be compound Poisson. We apply the Euler method again and make $X_k(t)$ discrete. Then we get

$$X_k(t + \Delta t) = X_k(t) - \alpha_k X_k(t)\Delta t + L_k(t + \Delta t) - L_k(t),$$

where

$$L_k(t + \Delta t) - L_k(t) = \sum_{j=N_k(t)}^{N_k(t+\Delta t)} Y_j,$$

and $Y_j \sim exp(\theta_k)$ for $k = 1, 2$. $N_k(t)$ have intensity λ_k, and can have at most one jump in the interval $[t, t + \Delta t]$. To determine if there is a jump in the interval, we let $u \in [0, 1]$ and if $u \leq \lambda_k \Delta t$ there is a jump[1]. The variables are valued as in Table 7.2. Here we let $r(t)$ be a composition of $X_1(t)$ and $X_2(t)$. In our case we let $X_1(t)$ have small jumps and high jump intensity and let $X_2(t)$ have larger jumps, but lower jump intensity. By increasing the jump intensity in $X_1(t)$ we would get a more noisy graph of $r(t)$, i.e. we would get an interest rate with a higher jump intensity. Then we would have to decrease the size of the jumps as well in order for the interest rate not to increase.

We give two simulations of the interest rate $r(t)$, in Figure 7.3 and Figure 7.4.

[1] To see details of the simulation, see Appendix A.

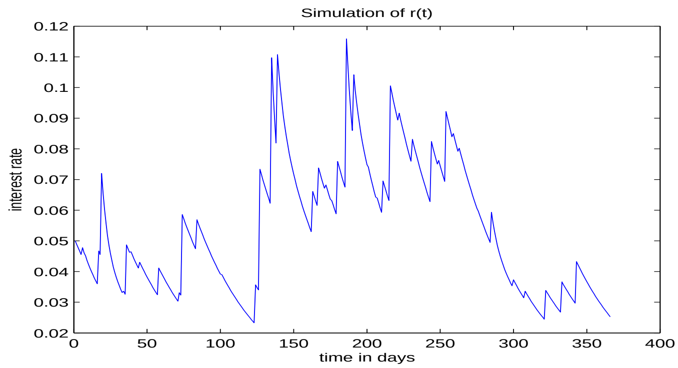

Figure 7.4: Simulation with the extended Vasicek model

7.2 Simulation of Zero-Coupon Bond Prices

We remind the reader that a zero-coupon bond is a bond paying one currency at the time of maturity with no coupons paid inbetween. We will simulate the prices from time 0 to T of such bonds, where T is the time of maturity. We simulate the prices in both the Vasicek model and the extended Vasicek model. Starting with the Vasicek model. We use the interest rates found above, when simulating $r(t)$, since the prices depend on them.

7.2.1 Simulation of Zero-Coupon Bond Prices in the Vasicek Model

Remember that the price at time $t \leq T$ of a zero-coupon bond is

$$P(t,T) = e^{m(t,T)-n(t,T)r(t)},$$

where

$$n(t,T) = \frac{1}{\alpha}\left(1 - e^{-\alpha(T-t)}\right)$$

and with $n(t,T)$ as above;

$$m(t,T) = \left(\frac{\mu}{\alpha} - \frac{\sigma^2}{2\alpha}\right)(n(t,T) - (T-t)) - \frac{\sigma^2}{4\alpha}n^2(t,T).$$

We find the price at all times $t = 0, 1, 2, \ldots, 365$ to see how the price changes as we come closer to the maturity time $T = 365$.[2] We used the values in Table 7.1 for the variables. We did two simulations with the same variables, see Figure 7.5 and Figure 7.6.

7.2.2 Simulation of Zero-Coupon Bond Prices in the Extended Vasicek Model

Remember that the price at time $t \leq T$ of a zero-coupon bond in the extended Vasicek model is

$$P(t,T) = e^{\sum_{k=1}^{n}[\lambda_k I_k - \lambda_k(T-t) - w_k X_k(t) n_k(t,T)]},$$

[2]To see details, take a look at Appendix A

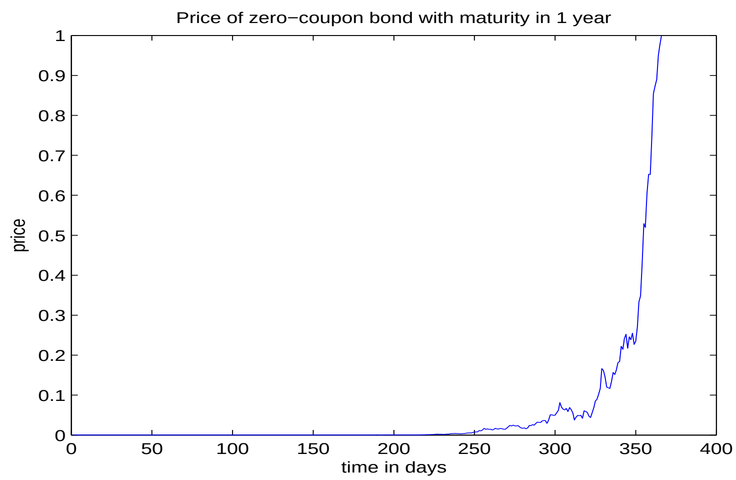

Figure 7.5: Simulation of zero-coupon bond prices with the Vasicek model

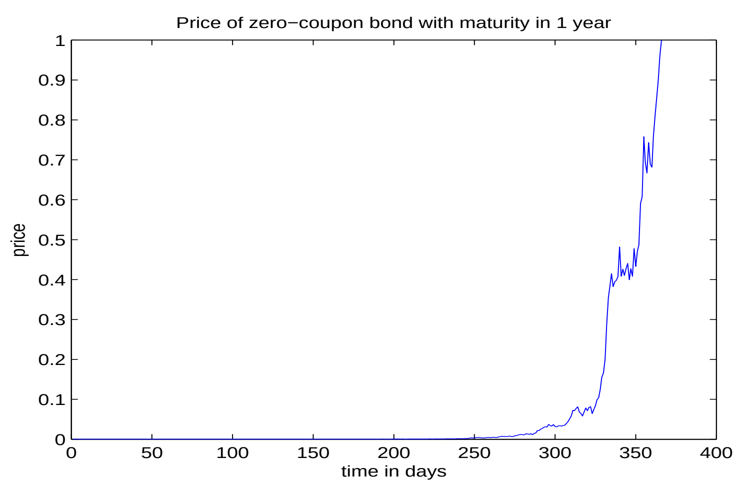

Figure 7.6: Simulation of zero-coupon bond prices with the Vasicek model

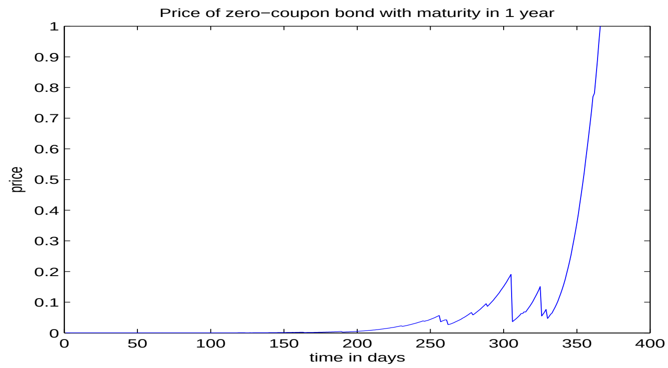

Figure 7.7: Simulation of zero-coupon bond prices with the extended Vasicek model

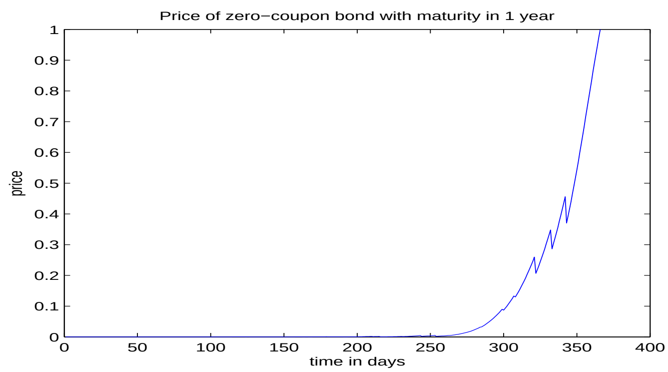

Figure 7.8: Simulation of zero-coupon bond prices with the extended Vasicek model

where

$$n_k(t,T) = \frac{1}{\alpha_k}\left(1 - e^{-\alpha_k(T-t)}\right)$$

and

$$I_k = (T-t)\frac{1}{1+\frac{w_k}{\alpha_k}\frac{1}{\theta_k}} + \frac{1}{\alpha_k + \frac{w_k}{\theta_k}} \ln\left(1 + \frac{w_k}{\theta_k} n_k(t,T)\right).$$

We simplify it a bit, and look only at the case when $n = 2$. We use the same values for the variables as we did when simulating $r(t)$. The prices can be seen in Figure 7.7 and Figure 7.8. If we compare Figure 7.3 with Figure 7.7 we recognise the jumps in $r(t)$ in the price of the zero-coupon bond $P(t,T)$. And we see the same when comparing Figure 7.4 and Figure 7.8.

Generally, we could obtain even more flexibility in the interest rate $r(t)$ by setting it equal to $\sum_{k=1}^{n} w_k X_k(t)$, with $n > 2$. We then get that the interest rate $r(t)$ can be dependent of other factors as well.

Appendix A

MATLAB Files

simulations.m

```
T = 365;
r_0 = 0.05;

%Variables used to simulate the Vasicek model:
alpha = log(2)/10;
mu = 0.004;
sigma = 0.01;

%Variables used to simulate the extended Vasicek model:
w_1 = 1;
w_2 = 1;
alpha_1 = log(2)/30;
alpha_2 = log(2)/3.5;
lambda_1 = 1/10;
lambda_2 = 1/90;
theta_1 = 100;
theta_2 = 40;

r_t_Vas = simulate_r_Vas(T, r_0, mu, alpha, sigma);
p_t_Vas = price_Vas(T, r_t_Vas, mu, alpha, sigma);
[r_t, X_1, X_2] = simulate_r(T, w_1, w_2, r_0, alpha_1, alpha_2, lambda_1,
                  lambda_2, theta_1, theta_2);
p_t = price(T, w_1, w_2, X_1, X_2, alpha_1, alpha_2, lambda_1, lambda_2,
      theta_1, theta_2);

%Plotting the figures:
figure(1)
plot(r_t_Vas)
xlabel('time in days');
ylabel('interest rate');
title('Simulation of r(t)');
```

```
figure(2)
plot(p_t_Vas)
xlabel('time in days');
ylabel('price');
title('Price of zero-coupon bond with maturity in 1 year');

figure(3)
plot(r_t)
xlabel('time in days');
ylabel('interest rate');
title('Simulation of r(t)');

figure(4)
plot(p_t)
xlabel('time in days');
ylabel('price');
title('Price of zero-coupon bond with maturity in 1 year');
```

---

simulate_r_Vas.m

```
%Simulates the Vasicek model
function[r_t_Vas] = simulate_r_Vas(T, r_0, mu, alpha, sigma)
delta = 1;
r_t_Vas = zeros(1, T + 1);
r_t_Vas(1) = r_0;

for t=1:T
    eps = normrnd(0, 1);
    r_t_Vas(t + 1) = r_t_Vas(t) + (mu - alpha*r_t_Vas(t))*delta
                     + sigma*sqrt(delta)*eps;
end
```

---

price_Vas.m

```
%Simulates the price of a zero-coupon bond in the Vasicek model
function[p_t_Vas] = price_Vas(T, r_t_Vas, mu, alpha, sigma)
p_t_Vas = zeros(1, T + 1);
n_t = zeros(1, T + 1);
m_t = zeros(1, T + 1);

for t=1:T+1
    t_t = t - 1;
    n_t(t) = (1/alpha)*(1 - exp(-alpha*(T - t_t)));
```

```
    m_t(t) = ((mu/alpha) - (sigma^2)/(2*(alpha^2)))*(n_t(t) - (T - t_t))
             - ((sigma^2)/(4*alpha))*n_t(t);
    p_t_Vas(t) = exp(m_t(t) - n_t(t)*r_t_Vas(t));
end
```

simulate_r.m

```
%Simulating r(t) from the extended Vasicek model with Euler
%method, when L_1 and L_2 are compound Poisson.
function[r_t, X_1, X_2] = simulate_r(T, w_1, w_2, r_0, alpha_1, alpha_2,
                          lambda_1, lambda_2, theta_1, theta_2)
delta = 1;
X_1 = zeros(1, T + 1);
X_2 = zeros(1, T + 1);
r_t = zeros(1, T + 1);
X_1(1) = r_0/w_1;
X_2(1) = 0;

for t=1:T
    u_1 = rand;
    u_2 = rand;
    if u_1 > lambda_1*delta %If not jump in X_1
        X_1(t + delta) = X_1(t)*(1 - alpha_1)*delta;
    else %If jump in X_1
        Y_1 = exprnd(1/theta_1);
        X_1(t + delta) = X_1(t)*(1 - alpha_1)*delta + Y_1;
    end
    if u_2 > lambda_2*delta %If not jump in X_2
        X_2(t + delta) = X_2(t)*(1 - alpha_2)*delta;
    else %If jump in X_2
        Y_2 = exprnd(1/theta_2);
        X_2(t + delta) = X_2(t)*(1 - alpha_2)*delta + Y_2;
    end
end

for t=1:T+1
    r_t(t) = w_1*X_1(t) + w_2*X_2(t);
end
```

price.m

```
%Pricing of zero-coupon bonds of the extended Vasicek model,
%with L_1 and L_2 compound Poisson.
function[p_t] = price(T, w_1, w_2, X_1, X_2, alpha_1, alpha_2,
```

```
                lambda_1, lambda_2, theta_1, theta_2)
p_t = zeros(1, T + 1);
n_1 = zeros(1, T + 1);
n_2 = zeros(1, T + 1);
I_1 = zeros(1, T + 1);
I_2 = zeros(1, T + 1);

for t=0:T
    n_1(t + 1) = (1/alpha_1)*(1 - exp(-alpha_1*(T - t)));
    n_2(t + 1) = (1/alpha_2)*(1 - exp(-alpha_2*(T - t)));
    I_1(t + 1) = (T - t)/(1 + (w_1/alpha_1)*(1/theta_1))
                 + (1/(alpha_1 + (w_1/theta_1)))
                 *log(1 + (w_1/theta_1)*n_1(t + 1));
    I_2(t + 1) = (T - t)/(1 + (w_2/alpha_2)*(1/theta_2))
                 + (1/(alpha_2 + (w_2/theta_2)))
                 *log(1 + (w_2/theta_2)*n_2(t + 1));
end

for t=0:T
    p_t(t + 1) = exp(- w_1*X_1(t + 1)*n_1(t + 1)
                     + lambda_1*I_1(t + 1)
                     - lambda_1*(T - t)
                     - w_2*X_2(t + 1)*n_2(t + 1)
                     + lambda_2*I_2(t + 1)
                     - lambda_2*(T - t));
end
```

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