NON-HAUSDORFF ÉTALE GROUPOIDS AND C*-ALGEBRAS OF LEFT CANCELLATIVE MONOIDS

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Abstract. We study the question whether the representations defined by a dense subset of the unit space of a locally compact étale groupoid are enough to determine the reduced norm on the groupoid C*-algebra. We present sufficient conditions for either conclusion, giving a complete answer when the isotropy groups are torsion-free. As an application we consider the groupoid $\mathcal{G}(S)$ associated to a left cancellative monoid S by Spielberg and formulate a sufficient condition, which we call C*-regularity, for the canonical map $C^*_r(\mathcal{G}(S)) \to C^*_r(S)$ to be an isomorphism, in which case S has a well-defined full semigroup C*-algebra $C^*(S) = C^*(\mathcal{G}(S))$. We give two related examples of left cancellative monoids S and T such that both are not finitely aligned and have non-Hausdorff associated étale groupoids, but S is C*-regular, while T is not.

Introduction

The C*-algebras of non-Hausdorff locally compact groupoids were introduced by Connes in [Con82], where the main examples were given by the holonomy groupoids of foliations. It is known that some of the basic properties of groupoid C*-algebras of Hausdorff groupoids can fail in the non-Hausdorff case. One of such properties is that to compute the reduced norm it suffices to consider the representations $\rho_x\colon C_c(\mathcal{G}) \to B(L^2(\mathcal{G}_x))$ for x running through any dense subset $Y \subset \mathcal{G}^{(0)}$. A simple counterexample is provided by the line with a double point. The first systematic study of which extra conditions on Y one needs was carried out by Khoshkam and Skandalis [KS02]. Our starting point is the simple observation, which can be viewed as a reformulation of a result in [KS02], that for étale groupoids it suffices to require that for every point $x \in \mathcal{G}^{(0)} \setminus Y$ there is a net in Y converging to x and having no other accumulation points in $\mathcal{G}^x_x$. As we show, this condition is in general not necessary, but it becomes so if the isotropy groups $\mathcal{G}^x_x$ do not have too many finite subgroups, in particular, if they are torsion-free.

Our motivation for studying these questions comes from the problem of defining a full semigroup C*-algebra of a left cancellative monoid. Every such monoid S has a regular representation on $\ell^2(S)$ and hence a well-defined reduced C*-algebra $C^*_r(S)$. It is natural to try to define the full semigroup C*-algebra as a universal C*-algebra generated by isometries v_s, $s \in S$, such that $v_s v_t = v_{st}$, but one quickly sees that more relations are needed to get an algebra that is not unreasonably bigger than $C^*_r(S)$. A major progress in this old problem was made by Li [Li12], who realized that in $C^*_r(S)$ there are extra relations coming from the action of S on the constructible ideals of S, which are right ideals of the form $s_1^{-1}t_1 \dots s_n^{-1}t_n S$. Soon afterwards Norling [Nor14] observed that this has an interpretation in terms of the left inverse hull $I_\ell(S)$ of S: the C*-algebra $C^*_r(S)$ is obtained by reducing the reduced C*-algebra of the inverse semigroup $I_\ell(S)$ to an invariant subspace of its regular representation, and so the new relations in $C^*_r(S)$ arise from those in $C^*_r(I_\ell(S))$. Since the representations of inverse semigroups are a well-studied subject and the corresponding C*-algebras have groupoid models defined by Paterson [Pat99], this opened the possibility to defining $C^*(S)$ as a groupoid C*-algebra.

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Specifically (see Section 2 for details), the subrepresentation of the regular representation of $I_\ell(S)$ defining $C^*_r(S)$ gives rise to a reduction $\mathcal{G}_P(S)$ of the Paterson groupoid of $I_\ell(S)$ and to a surjective homomorphism $C^*_r(\mathcal{G}_P(S)) \to C^*_r(S)$. When this map is an isomorphism, it is natural to define $C^*(S)$ as $C^*(\mathcal{G}_P(S))$. This C*-algebra can be described in terms of generators and relations, since there is such a description for $C^*(I_\ell(S))$, and simultaneously its definition as a groupoid C*-algebra subsumes a number of results on (partial) crossed product decompositions of semigroup C*-algebras. The trouble, however, is that this does not work for all S, the map $C^*_r(\mathcal{G}_P(S)) \to C^*_r(S)$ is not always an isomorphism.

In [Spi20] Spielberg introduced, in a more general context of left cancellative small categories, a quotient $\mathcal{G}(S)$ of $\mathcal{G}_P(S)$ that kills some "obvious" elements in the kernel of $C^*_r(\mathcal{G}_P(S)) \to C^*_r(S)$ (see Proposition 2.13). But as he showed, the canonical homomorphism $C^*_r(\mathcal{G}(S)) \to C^*_r(S)$ can still have a nontrivial kernel. It should be said that the kernel of $C^*_r(\mathcal{G}(S)) \to C^*_r(S)$ is small: under rather general assumptions (for example, for all countable S with trivial group of units) $C^*_r(S)$ can be identified with the essential groupoid C*-algebra $C^*_{\text{ess}}(\mathcal{G}(S))$ of $\mathcal{G}(S)$, as defined by Kwaśniewski and Meyer [KM21]. Still, we do not think that this is enough to call $C^*(\mathcal{G}(S))$ the full semigroup C*-algebra of S when $C^*_r(\mathcal{G}(S)) \ne C^*_r(S)$.

Spielberg showed that there are two sufficient conditions for the equality $C^*_r(\mathcal{G}(S)) = C^*_r(S)$, one is that $\mathcal{G}(S)$ is Hausdorff, the other is that S is finitely aligned, which is equivalent to saying that every constructible ideal of S is finitely generated. Already the first condition covers, for example, all group embeddable monoids. For such monoids we have $\mathcal{G}(S) = \mathcal{G}_P(S)$, and the corresponding full semigroup C*-algebras $C^*(S) = C^*(\mathcal{G}(S)) = C^*(\mathcal{G}_P(S))$ have been recently comprehensively studied by Laca and Sehnem [LS22].

For general S, the question whether $C^*_r(\mathcal{G}(S)) \to C^*_r(S)$ is an isomorphism is exactly the type of question we started with: can the reduced norm on $C_c(\mathcal{G}(S))$ be computed using certain dense subset $Y = \{\chi_s | s \in S\}$ of $\mathcal{G}(S)^{(0)}$? In this formulation it is immediate that the answer is "yes" when $\mathcal{G}(S)$ is Hausdorff. In the non-Hausdorff case we can try to use our general results to arrive to either conclusion. This leads to a simple (to formulate, but in general not to check) sufficient condition for the equality $C^*_r(\mathcal{G}(S)) = C^*_r(S)$ that we call C*-regularity. We give an example of a C*-regular monoid S that is not finitely aligned and such that the groupoid $\mathcal{G}(S)$ is non-Hausdorff. A small modification of S gives a monoid T with $C^*_r(\mathcal{G}(T)) \ne C^*_r(T)$. It is interesting that the kernel of $C^*_r(\mathcal{G}(T)) \to C^*_r(T)$ has nonzero elements already in the $*$-algebra generated by the canonical elements v_t, $t \in T$, so in some sense $\mathcal{G}(T)$ is a wrong groupoid model for the semigroup $*$-algebra of T already at the purely algebraic level.

Let us finally mention that an interesting related problem is to find groupoid models for boundary quotients of semigroup C*-algebras, but we are not going to touch it in the present paper.

1. C*-algebras of non-Hausdorff étale groupoids

Assume $\mathcal{G}$ is a locally compact, not necessarily Hausdorff, étale groupoid. By this we mean that $\mathcal{G}$ is a groupoid endowed with a locally compact topology such that

- the groupoid operations are continuous;
- the unit space $\mathcal{G}^{(0)}$ is a locally compact Hausdorff space in the relative topology;
- the range map $r\colon \mathcal{G} \to \mathcal{G}^{(0)}$ and the source map $s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms.

For an open Hausdorff subset $V \subset \mathcal{G}$, consider the usual space $C_c(V)$ of continuous compactly supported functions on V. Every such function can be extended by zero to $\mathcal{G}$; in general this extension is not a continuous function on $\mathcal{G}$. This way we can view $C_c(V)$ as a subspace of the space of functions $\text{Func}(\mathcal{G})$ on $\mathcal{G}$. For arbitrary open subsets $U \subset \mathcal{G}$ we denote by $C_c(U) \subset \text{Func}(\mathcal{G})$ the linear span of the subspaces $C_c(V) \subset \text{Func}(\mathcal{G})$ for all open Hausdorff subsets $V \subset U$. Instead of all possible V it suffices to take a collection of open bisections covering U.

The space $C_c(\mathcal{G})$ is a $*$-algebra with the convolution product

$$(f_1 * f_2)(g) := \sum_{h\in\mathcal{G}^{r(g)}} f_1(h)f_2(h^{-1}g) \quad \text{for} \quad g \in \mathcal{G},$$

and involution $f^*(g) = \overline{f(g^{-1})}$, where $\mathcal{G}^x = r^{-1}(x)$. The full groupoid C*-algebra $C^*(\mathcal{G})$ is defined as the C*-enveloping algebra of $C_c(\mathcal{G})$.

For every $x \in \mathcal{G}^{(0)}$, define a representation $\rho_x\colon C_c(\mathcal{G}) \to B(\ell^2(\mathcal{G}_x))$, where $\mathcal{G}_x = s^{-1}(x)$, by

$$(\rho_x(f)\xi)(g) = \sum_{h\in\mathcal{G}^{r(g)}} f(h)\xi(h^{-1}g).$$

Then the reduced C*-algebra $C^*_r(\mathcal{G})$ is defined as the completion of $C_c(\mathcal{G})$ with respect to the norm

$$\|f\|_r = \sup_{x\in\mathcal{G}^{(0)}} \|\rho_x(f)\|.$$

Recall (see, e.g., [Exe08, Section 3]) that for all $f \in C_c(\mathcal{G})$ we have the inequalities $\|f\|_\infty \le \|f\|_r \le \|f\|$, where $\|\cdot\|_\infty$ denotes the supremum-norm, and if $f \in C_c(U)$ for an open bisection U, then

$$\|f\| = \|f\|_r = \|f\|_\infty.$$

For a closed (in $\mathcal{G}^{(0)}$) invariant subset $X \subset \mathcal{G}^{(0)}$, denote by $\mathcal{G}_X$ the subgroupoid $r^{-1}(X) = s^{-1}(X) \subset \mathcal{G}$. In the second countable case the next result and the subsequent corollary follow easily from Renault's disintegration theorem, cf. [Ren91, Remark 4.10]. The case of étale groupoids allows for the following elementary proof without any extra assumptions on $\mathcal{G}$.

Proposition 1.1. *Assume $\mathcal{G}$ is a locally compact étale groupoid and $X \subset \mathcal{G}^{(0)}$ is a closed invariant subset. Then the following sets coincide:*

1. *the kernel of the $*$-homomorphism $C^*(\mathcal{G}) \to C^*(\mathcal{G}_X)$, $C_c(\mathcal{G}) \ni f \mapsto f|_{\mathcal{G}_X}$;*
2. *the closure of $C_c(\mathcal{G} \setminus \mathcal{G}_X)$ in $C^*(\mathcal{G})$;*
3. *the closed ideal of $C^*(\mathcal{G})$ generated by $C_0(\mathcal{G}^{(0)} \setminus X) \subset C_0(\mathcal{G}^{(0)})$.*

Proof. The sets in (2) and (3) coincide, since $C_c(\mathcal{G} \setminus \mathcal{G}_X)$ is an ideal in $C_c(\mathcal{G})$ (with respect to the convolution product) and for every $f \in C_c(G\setminus\mathcal{G}_X)$ we can find $f' \in C_c(\mathcal{G}^{(0)}\setminus X)$ such that $f * f' = f$. It is also clear that $C_c(\mathcal{G} \setminus \mathcal{G}_X)$ is contained in the kernel of the $*$-homomorphism $C^*(\mathcal{G}) \to C^*(\mathcal{G}_X)$. It follows that in order to prove the proposition it suffices to show that every representation of $C_c(\mathcal{G})$ on a Hilbert space that vanishes on $C_c(\mathcal{G} \setminus \mathcal{G}_X)$ factors through $C_c(\mathcal{G}_X)$. For this, in turn, it suffices to prove that $C_c(\mathcal{G} \setminus \mathcal{G}_X)$ is dense, with respect to the norm on $C^*(\mathcal{G})$, in the space of functions $f \in C_c(\mathcal{G})$ such that $f|_{\mathcal{G}_X} = 0$.

Let us first prove the following claim. Assume $f = \sum_{i=1}^n f_i \in C_c(\mathcal{G})$ satisfies $\|f|_{\mathcal{G}_X}\|_\infty < \varepsilon$ for some $\varepsilon > 0$, $f_i \in C_c(U_i)$ and open bisections U_i. Then there exist functions $\tilde f_i \in C_c(U_i)$ such that

$$f - \sum_{i=1}^n \tilde f_i \in C_c(\mathcal{G} \setminus \mathcal{G}_X) \quad \text{and} \quad \|\tilde f_i\|_\infty < 2^n\varepsilon \quad \text{for} \quad i = 1, \dots, n.$$

The proof is by induction on n. As the base of induction we take $n = 0$, meaning that $f = 0$. In this case there is nothing to prove. So assume the claim is true for some $n \ge 0$. For the induction step assume $f \in C_c(\mathcal{G})$ satisfies $\|f|_{\mathcal{G}_X}\|_\infty < \varepsilon$ and we can write $f = \sum_{i=1}^{n+1} f_i$ for some $f_i \in C_c(U_i)$ and open bisections U_i. Let $K_{n+1} \subset U_{n+1}$ be the support of $f_{n+1}|_{U_{n+1}}$. Consider the set $K = K_{n+1} \setminus \bigcup_{i=1}^n U_i$. As $f = f_{n+1}$ on K, we have $\|f_{n+1}|_{K\cap\mathcal{G}_X}\|_\infty < \varepsilon$. Hence there exists an open neighbourhood U of $K \cap \mathcal{G}_X$ in U_{n+1} such that $\|f_{n+1}|_U\|_\infty < \varepsilon$. Let V be an open neighbourhood of $K \setminus U$ in U_{n+1} such that $\bar V \cap U_{n+1} \cap \mathcal{G}_X = \emptyset$. Then the open sets $U_1 \cap U_{n+1}, \dots, U_n \cap U_{n+1}, U, V$

cover K_{n+1}. Hence we can find functions $\rho_1,\dots,\rho_n,\rho_U,\rho_V\in C_c(U_{n+1})$ taking values in the interval $[0,1]$ such that $\operatorname{supp}\rho_i\subset U_i\cap U_{n+1}$, $\operatorname{supp}\rho_U\subset U$, $\operatorname{supp}\rho_V\subset V$ and

$$\sum_{i=1}^n \rho_i(g)+\rho_U(g)+\rho_V(g)=1 \quad \text{for all} \quad g\in K_{n+1}.$$

Define $f_i'=f_i+\rho_i f_{n+1}$ (pointwise product) for $i=1,\dots,n$, $f'=\sum_{i=1}^n f_i'$ and $\tilde f_{n+1}=\rho_U f_{n+1}$. Then $f_i'\in C_c(U_i)$, $\tilde f_{n+1}\in C_c(U_{n+1})$ and we have

$$f-f'-\tilde f_{n+1}=\rho_V f_{n+1}\in C_c(\mathcal{G}\setminus\mathcal{G}_X).$$

We also have $\|\tilde f_{n+1}\|_\infty\le\|f_{n+1}|_U\|_\infty<\varepsilon<2^{n+1}\varepsilon$. It follows that

$$\|f'|_{\mathcal{G}_X}\|_\infty=\|(f-\tilde f_{n+1})|_{\mathcal{G}_X}\|_\infty\le\|f|_{\mathcal{G}_X}\|_\infty+\|\tilde f_{n+1}|_{\mathcal{G}_X}\|_\infty<2\varepsilon.$$

We can therefore apply the inductive hypothesis to f' and 2ε and find functions $\tilde f_i\in C_c(U_i)$, $i=1,\dots,n$, such that

$$f'-\sum_{i=1}^n\tilde f_i\in C_c(\mathcal{G}\setminus\mathcal{G}_X) \quad\text{and}\quad \|\tilde f_i\|_\infty<2^n2\varepsilon=2^{n+1}\varepsilon \quad\text{for}\quad i=1,\dots,n.$$

Then the functions $\tilde f_1,\dots,\tilde f_{n+1}$ have the required properties.

Now, if $f\in C_c(\mathcal{G})$ satisfies $f|_{\mathcal{G}_X}=0$, we write $f=\sum_{i=1}^n f_i$ for some $f_i\in C_c(U_i)$ and open bisections U_i and apply the above claim to an arbitrarily small $\varepsilon>0$. Recalling that the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ coincide on $C_c(U)$ for any open bisection U, we conclude that there is a function $\tilde f=\sum_{i=1}^n\tilde f_i\in C_c(\mathcal{G})$ such that $f-\tilde f\in C_c(\mathcal{G}\setminus\mathcal{G}_X)$ and $\|\tilde f\|\le n2^n\varepsilon$. Hence f lies in the closure of $C_c(\mathcal{G}\setminus\mathcal{G}_X)$. □

Corollary 1.2. *We have a short exact sequence*

$$0\to C^*(\mathcal{G}\setminus\mathcal{G}_X)\to C^*(\mathcal{G})\to C^*(\mathcal{G}_X)\to 0.$$

Proof. Since the restriction map $C_c(\mathcal{G})\to C_c(\mathcal{G}_X)$ is surjective, the fact that the sets in (1) and (2) coincide implies that we have an exact sequence $C^*(\mathcal{G}\setminus\mathcal{G}_X)\to C^*(\mathcal{G})\to C^*(\mathcal{G}_X)\to 0$. Therefore we only need to explain why the map $C^*(\mathcal{G}\setminus\mathcal{G}_X)\to C^*(\mathcal{G})$ is injective. For this it suffices to show that any nondegenerate representation $\pi\colon C^*(\mathcal{G}\setminus\mathcal{G}_X)\to B(H)$ extends to $C^*(\mathcal{G})$. Since $C_c(\mathcal{G}\setminus\mathcal{G}_X)$ is an ideal of $C_c(\mathcal{G})$, we can define a representation $\tilde\pi$ of $C_c(\mathcal{G})$ on $\pi(C_c(\mathcal{G}\setminus\mathcal{G}_X))H$ by possibly unbounded operators in the standard way: for $f\in C_c(\mathcal{G})$, put $\tilde\pi(f)\pi(f')\xi=\pi(f*f')\xi$. On $C_c(\mathcal{G}^{(0)})$ this agrees with the unique extension of $\pi|_{C_0(\mathcal{G}^{(0)}\setminus X)}$ to a representation of $C_0(\mathcal{G}^{(0)})$. Hence $\|\tilde\pi(f)\|\le\|f\|_\infty$ for $f\in C_c(\mathcal{G}^{(0)})$, and then $\|\tilde\pi(f)\|\le\|f\|$ for any open bisection U and $f\in C_c(U)$, as $f^**f\in C_c(\mathcal{G}^{(0)})$. □

Remark 1.3 (cf. [CN22, Remark 2.9]). Since the ideal $C_c(\mathcal{G}\setminus\mathcal{G}_X)\subset C_c(\mathcal{G})$ is dense with respect to the norm on $C^*(\mathcal{G})$ in the space of functions $f\in C_c(\mathcal{G})$ such that $f|_{\mathcal{G}_X}=0$, it is also dense with respect to the reduced norm. It follows that there is a C*-norm on $C_c(\mathcal{G}_X)$ dominating the reduced norm such that for the corresponding completion $C^*_e(\mathcal{G}_X)$ the sequence

$$0\to C^*_r(\mathcal{G}\setminus\mathcal{G}_X)\to C^*_r(\mathcal{G})\to C^*_e(\mathcal{G}_X)\to 0$$

is exact. ♢

We now turn to the question when a set of representations ρ_y, $y\in Y\subset\mathcal{G}^{(0)}$, determines the reduced norm on $C_c(\mathcal{G})$. It is easy to see that if Y is $\mathcal{G}$-invariant, which we may always assume since the equivalence class of ρ_x depends only on the orbit of x, a necessary condition is that Y is dense in $\mathcal{G}^{(0)}$. But this is not enough in the non-Hausdorff case. We start with the following sufficient condition.

Proposition 1.4. *Let $\mathcal{G}$ be a locally compact étale groupoid, $Y \subset \mathcal{G}^{(0)}$ and $x \in \mathcal{G}^{(0)} \setminus Y$. Assume there is a net $(y_i)_i$ in Y such that x is the only accumulation point of $(y_i)_i$ in $\mathcal{G}^x_x = \mathcal{G}_x \cap \mathcal{G}^x$. Then the representation ρ_x of $C^*_r(\mathcal{G})$ is weakly contained in $\bigoplus_{y\in Y} \rho_y$.*

Proof. We may assume that $y_i \to x$. For every $g \in \mathcal{G}_x$ we then choose a net $(g_i)_i$ converging to g as follows. Let U be an open bisection containing g. Then for all i large enough we have $y_i \in s(U)$, and for every such i we take the unique point $g_i \in U \cap \mathcal{G}_{y_i}$. For all other indices i we put $g_i = y_i$.

Take $g, h \in \mathcal{G}_x$. Observe that by our assumptions if $g \neq h$, then $g_i \neq h_i$ for all i large enough, since otherwise we could first conclude that $r(g) = r(h)$ and then that $g^{-1}h \in \mathcal{G}^x_x$ is an accumulation point of $(y_i)_i$.

Next, take an open bisection V and $f \in C_c(V)$. Then

$$(\rho_x(f)\delta_g, \delta_h) = f(hg^{-1}), \qquad (\rho_{y_i}(f)\delta_{g_i}, \delta_{h_i}) = f(h_i g_i^{-1}).$$

These equalities and the observation above imply that in order to prove the proposition it suffices to show that $f(h_i g_i^{-1}) \to f(hg^{-1})$.

Assume first that $hg^{-1} \in V$. As V is open and $h_i g_i^{-1} \to hg^{-1}$, it follows that eventually $h_i g_i^{-1} \in V$. But then $f(h_i g_i^{-1}) \to f(hg^{-1})$ by the continuity of f on V.

Assume next that $hg^{-1} \notin V$. It is then enough to show that eventually $h_i g_i^{-1}$ does not lie in the support K of $f|_V$. Suppose this is not the case. Then by passing to a subnet we may assume that $h_i g_i^{-1} \to w$ for some $w \in K$. Since we also have $h_i g_i^{-1} \to hg^{-1}$, we must have $r(w) = r(h)$ and $s(w) = r(g)$. Then $h^{-1}wg \in \mathcal{G}^x_x$, $h^{-1}wg \neq x$ and $y_i = h_i^{-1}(h_i g_i^{-1})g_i \to h^{-1}wg$, which contradicts our assumptions. □

Remark 1.5. The above proposition can also be deduced from results in [KS02, Section 2]. In order to make the connection to [KS02] more transparent, let us reformulate the assumptions of Proposition 1.4 as follows. The functions $f|_{\mathcal{G}^{(0)}}$ for $f \in C_c(\mathcal{G})$ generate a C*-subalgebra B of the algebra of bounded Borel functions on $\mathcal{G}^{(0)}$ equipped with the supremum-norm. Let Z be the spectrum of B. As every point of $\mathcal{G}^{(0)}$ defines a character of B, we have an injective Borel map $i\colon \mathcal{G}^{(0)} \to Z$ with dense image. We claim that a net as in Proposition 1.4 exists if and only if $i(x) \in \overline{i(Y)}$.

In order to show this, assume first that $(y_i)_i$ is a net in Y converging to x and having no other accumulation points in $\mathcal{G}^x_x$. We claim that then $i(y_i) \to i(x)$. It suffices to show that $f(y_i) \to f(x)$ for every open bisection U and $f \in C_c(U)$. If $x \in U$, this is true by continuity of $f|_U$. If $x \notin U$, then the net $(y_i)_i$ does not have any accumulation points in U and therefore it eventually lies outside the support of $f|_U$, so again $f(y_i) \to f(x)$. Conversely, assume we have a net $(y_i)_i$ in Y such that $i(y_i) \to i(x)$. Then obviously $y_i \to x$. Take $g \in \mathcal{G}^x_x \setminus \{x\}$, an open bisection U containing g and $f \in C_c(U)$ such that $f(g) \neq 0$. As $f(y_i) \to f(x) = 0$ by assumption, we conclude that g cannot be an accumulation point of $(y_i)_i$. ♢

If $x \in (\mathcal{G}^{(0)} \cap \bar{Y}) \setminus Y$, then nonexistence of a net as in Proposition 1.4 is equivalent to the following property:

Condition 1.6. For some $n \geq 1$, there are elements $g_1, \dots, g_n \in \mathcal{G}^x_x \setminus \{x\}$, open bisections $U_1, \dots, U_n$ such that $g_k \in U_k$ and a neighbourhood U of x in $\mathcal{G}^{(0)}$ satisfying $Y \cap U \subset U_1 \cup \dots \cup U_n$.

Indeed, if this condition is satisfied, then any net in Y converging to x has one of the elements $g_1, \dots, g_n$ as its accumulation point. Conversely, assume Condition 1.6 is not satisfied. For every $g \in \mathcal{G}^x_x \setminus \{x\}$ choose an open bisection U_g containing g. Then for every finite set $F = \{g_1, \dots, g_n\} \subset \mathcal{G}^x_x \setminus \{x\}$ and every neighbourhood U of x in $\mathcal{G}^{(0)}$ we can find $y_{F,U} \in (Y \cap U) \setminus (U_{g_1} \cup \dots \cup U_{g_n})$. Then $(y_{F,U})_{F,U}$, with the obvious partial order defined by inclusion of F's and containment of U's, is the required net.

Remark 1.7. Following the terminology of [KM21], a point $x \in \mathcal{G}^{(0)}$ is called dangerous if there is a net in $\mathcal{G}^{(0)}$ converging to x and to a point in $\mathcal{G}_x^x \setminus \{x\}$. Therefore the set of points $x \in (\mathcal{G}^{(0)} \cap \bar{Y}) \setminus Y$ satisfying Condition 1.6 is a subset of dangerous points. As a consequence, if Y is dense in $\mathcal{G}^{(0)}$ and $\mathcal{G}$ can be covered by countably many open bisections, then by [KM21, Lemma 7.15] the set of points $x \in \mathcal{G}^{(0)} \setminus Y$ satisfying Condition 1.6 is meager in $\mathcal{G}^{(0)}$. ♢

If Y is $\mathcal{G}$-invariant and Condition 1.6 is satisfied for $n = 1$, then ρ_x is not weakly contained in $\bigoplus_{y\in Y} \rho_y$. In order to see this, take an open neighbourhood $V \subset U$ of x such that $V \subset r(U_1) \cap s(U_1)$ and a function $f \in C_c(V)$ such that $f(x) \neq 0$. Then it is easy to check that $0 \neq f * (\mathbb{1}_{U_1} - \mathbb{1}_U) * f \in \ker \rho_y$ for all $y \in Y$. A simple example of such a situation is the real line with a double point at 0, cf. [KS02, Example 2.5].

But in general, as we will see soon, Condition 1.6 is not enough to conclude that ρ_x is not weakly contained in $\bigoplus_{y\in Y} \rho_y$. A sufficient extra condition is given by the following proposition.

Proposition 1.8. *Assume $\mathcal{G}$ is a locally compact étale groupoid, $Y \subset \mathcal{G}^{(0)}$ is a $\mathcal{G}$-invariant subset and $x \in \mathcal{G}^{(0)} \setminus Y$ is a point satisfying Condition 1.6 such that*

$$\sum_{k=1}^{n} \frac{1}{\operatorname{ord}(g_k)} < 1,$$

where $\operatorname{ord}(g_k)$ is the order of g_k in $\mathcal{G}_x^x$. Then ρ_x is not weakly contained in $\bigoplus_{y\in Y} \rho_y$.

For the proof we need the following simple lemma.

Lemma 1.9. *Let $A = C^*(a)$ be a C^*-algebra generated by a contraction a. Assume that for some $m \in \{2, 3, \dots, +\infty\}$ we have a $*$-homomorphism $\pi\colon A \to C^*(\mathbb{Z}/m\mathbb{Z})$ such that $\pi(a) = u$, where u is the unitary generator of $C^*(\mathbb{Z}/m\mathbb{Z})$. Take numbers $\alpha > 0$ and $\varepsilon > 0$ and denote by $\Omega_{\alpha,\varepsilon}$ the convex set of states φ on A such that*

$$\varphi \geq \alpha \sum_{l=1}^{p} \lambda_l \chi_l$$

for some $p \geq 1$, $\lambda_1, \dots, \lambda_p \geq 0$, $\sum_{l=1}^{p} \lambda_l = 1$, and characters $\chi_1, \dots, \chi_p\colon A \to \mathbb{C}$ such that $|1 - \chi_l(a)| < \varepsilon$ for all l. Then, for every $\alpha > 1/m$, there is $\varepsilon > 0$ depending only on m and α such that $\tau \circ \pi$ does not belong to the weak* closure of $\Omega_{\alpha,\varepsilon}$, where τ is the canonical trace on $C^*(\mathbb{Z}/m\mathbb{Z})$.

Here we use the convention $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}$ for $m = +\infty$.

Proof. Assume first that m is finite. Consider the positive element $b \in A$ defined by

$$b = \frac{1}{m^2} \sum_{k,l=1}^{m} (a^k)^* a^l.$$

Then $\tau(\pi(b)) = 1/m$. On the other hand, if χ is a character on A such that $|1 - \chi(a)| < \varepsilon$, then

$$|1 - \chi(a)^k| \leq k|1 - \chi(a)| < m\varepsilon \quad \text{for all} \quad 1 \leq k \leq m,$$

hence, assuming $m\varepsilon < 1$, we have

$$\chi(b) = \Big| \frac{1}{m} \sum_{k=1}^{m} \chi(a)^k \Big|^2 > (1 - m\varepsilon)^2$$

and therefore

$$\varphi(b) \geq \alpha(1 - m\varepsilon)^2 \quad \text{for all} \quad \varphi \in \Omega_{\alpha,\varepsilon}.$$

It follows that $\tau \circ \pi \notin \bar{\Omega}_{\alpha,\varepsilon}$ as long as ε is small enough so that $\alpha(1 - m\varepsilon)^2 > 1/m$.

Assume now that $m = +\infty$. Choose $m' \geq 1$ such that $1/m' < \alpha$. Then the same arguments as above with m replaced by m' show that $\tau \circ \pi \notin \bar{\Omega}_{\alpha,\varepsilon}$ as long as $1 - m'\varepsilon > 1/\sqrt{m'\alpha}$. ◻

Proof of Proposition 1.8. Let $g_1, \dots, g_n$ be as in the formulation of the proposition and $U, U_1, \dots, U_n$ be given by Condition 1.6. Choose functions $f_k \in C_c(U_k)$ such that $0 \le f_k \le 1$ and $f_k(g_k) = 1$. Consider the C*-subalgebras A_k of $C^*_r(\mathcal{G})$ generated by f_k.

Consider the restriction map $C_c(\mathcal{G}) \to C_c(\mathcal{G}^x_x)$, $f \mapsto f|_{\mathcal{G}^x_x}$. It extends to a completely positive contraction $\vartheta_{x,r}\colon C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}^x_x)$, with the elements f_k contained in its multiplicative domain, see [CN22, Lemmas 1.2 and 1.4]. By restricting $\vartheta_{x,r}$ to A_k we therefore get $*$-homomorphisms $\pi_k\colon A_k \to C^*_r(\mathcal{G}^x_x)$. The image of π_k is $C^*(G_k) \subset C^*_r(\mathcal{G}^x_x)$, where G_k is the subgroup of $\mathcal{G}^x_x$ generated by g_k. Therefore if we let $m_k = \operatorname{ord}(g_k)$, then we can view each π_k as a $*$-homomorphism $A_k \to C^*(\mathbb{Z}/m_k\mathbb{Z})$.

Choose numbers $\alpha_k > 1/m_k$ such that $\sum_{k=1}^n \alpha_k < 1$. Let $\varepsilon_k > 0$ be given by Lemma 1.9 for the homomorphism $\pi_k\colon A_k \to C^*(\mathbb{Z}/m_k\mathbb{Z})$, $\alpha = \alpha_k$ and $a = f_k$. Put $\varepsilon = \min_{1\le k\le n} \varepsilon_k$. Choose an open neighbourhood V of x in $\mathcal{G}^{(0)}$ such that $V \subset U$ and

$$f_k(g) > 1 - \varepsilon \quad \text{for all} \quad g \in r^{-1}(V) \cap U_k \quad \text{and} \quad 1 \le k \le n. \tag{1.1}$$

Let $f \in C_c(V)$ be such that $0 \le f \le 1$ and $f(x) = 1$.

Now, denote $\bigoplus_{y\in Y} \rho_y$ by ρ_Y and assume that ρ_x is weakly contained in ρ_Y. Denoting the canonical trace on $C^*_r(\mathcal{G}^x_x)$ by τ, it follows that $\tau \circ \vartheta_{x,r} = (\rho_x(\cdot)\delta_x, \delta_x)$ lies in the weak* closure of the states φ of the form

$$\varphi = \sum_{i=1}^N (\rho_Y(\cdot)\xi_i, \xi_i), \tag{1.2}$$

where ξ_i are finitely supported functions on $s^{-1}(Y)$ such that

$$\sum_{i=1}^N \|\xi_i\|^2 = \sum_{i=1}^N \sum_{g\in s^{-1}(Y)} |\xi_i(g)|^2 = 1.$$

As $\tau(\vartheta_{x,r}(f)) = f(x) = 1$, it suffices to consider states such that

$$\varphi(f) > \sum_{k=1}^n \alpha_k.$$

Since

$$\varphi(f) = \sum_{i=1}^N \sum_{g\in s^{-1}(Y)} f(r(g))|\xi_i(g)|^2 = \sum_{i=1}^N \sum_{g\in r^{-1}(Y)} f(r(g))|\xi_i(g)|^2,$$

where we used that $r^{-1}(Y) = s^{-1}(Y)$ by the invariance of Y, and f is zero outside V, this implies that

$$\sum_{i=1}^N \sum_{g\in r^{-1}(Y\cap V)} |\xi_i(g)|^2 > \sum_{k=1}^n \alpha_k. \tag{1.3}$$

We claim that then for some k we must have $\varphi|_{A_k} \in \Omega^{(k)}_{\alpha_k,\varepsilon}$, where $\Omega^{(k)}_{\alpha_k,\varepsilon}$ is defined as in Lemma 1.9.

Denote by X_k the finite set of pairs (i, g), $1 \le i \le N$, $g \in r^{-1}(Y \cap V)$, such that $\xi_i(g) \ne 0$ and $r(g) \in U_k$. As $Y \cap V \subset U_1 \cup \dots \cup U_n$ by assumption, the inequality (1.3) implies

$$\sum_{k=1}^n \sum_{(i,g)\in X_k} |\xi_i(g)|^2 > \sum_{k=1}^n \alpha_k.$$

It follows that for some k we have

$$\sum_{(i,g)\in X_k} |\xi_i(g)|^2 > \alpha_k. \tag{1.4}$$

If $(i,g)\in X_k$, then

$$\rho_Y(f_k)\delta_g=\rho_Y(f_k^*)\delta_g=f(r(g))\delta_g.$$

Therefore every such point (i,g) defines a one-dimensional subrepresentation of $\rho_Y|_{A_k}$ and a character $\chi_{i,g}\colon A_k\to\mathbb{C}$ satisfying $\chi_{i,g}(f_k)>1-\varepsilon$ by (1.1). Then on A_k we have

$$(\rho_Y(\cdot)\xi_i,\xi_i)=(\rho_Y(\cdot)\tilde\xi_i,\tilde\xi_i)+\sum_{g:(i,g)\in X_k}|\xi_i(g)|^2\chi_{i,g},$$

where $\tilde\xi_i(g)=\xi_i(g)$ if $(i,g)\notin X_k$ and $\tilde\xi_i(g)=0$ otherwise. By (1.4) this implies that $\varphi|_{A_k}\in\Omega^{(k)}_{\alpha_k,\varepsilon}$, proving our claim.

It follows that if there is a net of states of the form (1.2) that converges weakly* to $\tau\circ\vartheta_{x,r}$, then by passing to a subnet $(\varphi_j)_j$ we can find an index k, $1\le k\le n$, such that $\varphi_j|_{A_k}\in\Omega^{(k)}_{\alpha_k,\varepsilon}$ for all j. This contradicts Lemma 1.9. □

By combining this with Proposition 1.4 we get the following criterion.

Corollary 1.10. *Let $\mathcal{G}$ be a locally compact étale groupoid, $Y\subset\mathcal{G}^{(0)}$ a $\mathcal{G}$-invariant subset and $x\in(\mathcal{G}^{(0)}\cap\bar Y)\setminus Y$. Assume that for every finite set of distinct cyclic nontrivial subgroups $G_1,\dots,G_n$ of $\mathcal{G}^x_x$ we have*

$$\sum_{k=1}^n\frac{1}{|G_k|}<1. \tag{1.5}$$

Then ρ_x is weakly contained in $\bigoplus_{y\in Y}\rho_y$ if and only if Condition 1.6 is not satisfied (equivalently, if and only if there is a net in Y such that x is its unique accumulation point in $\mathcal{G}^x_x$).

Proof. The "only if" part follows from Proposition 1.8 by observing that if Condition 1.6 is satisfied for $g_1,\dots,g_n$ and some elements g_k and g_l $(k\ne l)$ generate the same subgroup, then Condition 1.6 is still satisfied if we omit g_k or g_l. Therefore in Condition 1.6 we may assume in addition that the elements $g_1,\dots,g_n$ generate different subgroups. □

Condition (1.5) is most probably not optimal, but as the following example shows, some assumptions are needed for the conclusion of the corollary to be true.

Example 1.11. Let X be the disjoint union of countably many copies of $\{0,1\}$. Consider the involutive map $S\colon X\to X$ that acts as a flip on every copy of $\{0,1\}$. Take three copies X_1,X_2,X_3 of X and let X^+ be the one-point compactification of the discrete set $X_1\sqcup X_2\sqcup X_3$. Define an action of $\Gamma=(\mathbb{Z}/2\mathbb{Z})^2$ on X^+ as follows: the element $(1,0)$ acts by S on X_1 and X_2 and trivially on X_3 and ∞, the element $(0,1)$ acts by S on X_2 and X_3 and trivially on X_1 and ∞. Consider the corresponding groupoid of germs $\mathcal{G}$, so $\mathcal{G}$ is the quotient of the transformation groupoid $\Gamma\ltimes X^+$ by the equivalence relation defined by $(h,x)\sim(g,x)$ iff $hy=gy$ for all y in a neighbourhood of x. Thus, if we ignore the topology on $\mathcal{G}$, our groupoid is the disjoint union of $\Gamma\times\{\infty\}\cong\Gamma$ and three copies of $(\mathbb{Z}/2\mathbb{Z})\ltimes_S X$.

Consider the set $Y=X^+\setminus\{\infty\}$ and the point $x=\infty$. Condition 1.6 is satisfied for $n=3$, since Y is discrete and for every point $y\in Y$ there is $g\in\Gamma\setminus\{0\}$ that acts trivially on y. We claim that nevertheless ρ_x is weakly contained in $\bigoplus_{y\in Y}\rho_y$.

We have a short exact sequence

$$0\to C^*_r(\mathcal{G}\setminus\mathcal{G}^x_x)\to C^*_r(\mathcal{G})\xrightarrow{\rho_x}C^*(\Gamma)\to 0.$$

It has a canonical splitting $\psi\colon C^*(\Gamma)\to C^*_r(\mathcal{G})$ defined as follows. For every $g\in\Gamma$ consider the image U_g of the set $\{(g,x^+)\mid x^+\in X^+\}\subset\Gamma\ltimes X^+$ in $\mathcal{G}$. The sets $U_g\subset\mathcal{G}$ are bisections and their characteristic functions span a copy of $C^*(\Gamma)$ in $C^*_r(\mathcal{G})$. We define $\psi(\lambda_g)=\mathbb{1}_{U_g}$.

For $i = 1, 2, 3$, let x_{in} be 0 in the nth copy of $\{0, 1\}$ in X_i. Then every $f \in C_c(\mathcal{G} \setminus \mathcal{G}^x_x)$ is contained in $\ker \rho_{x_{in}}$ for all n sufficiently large. On the other hand, $\rho_{x_{in}} \circ \psi$ is equivalent to the representation λ_i obtained by composing the regular representation of $C^*(\mathbb{Z}/2\mathbb{Z})$ with the homomorphism $C^*(\Gamma) \to C^*(\mathbb{Z}/2\mathbb{Z})$ defined by one of the three nontrivial homomorphisms $\Gamma \to \mathbb{Z}/2\mathbb{Z}$. Namely, for $i = 1$ we get the homomorphism that maps $(1, 0)$ and $(1, 1)$ into $1 \in \mathbb{Z}/2\mathbb{Z}$, for $i = 2$ it maps $(1, 0)$ and $(0, 1)$ into 1, and for $i = 3$ it maps $(0, 1)$ and $(1, 1)$ into 1. As $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ is a faithful representation of $C^*(\Gamma)$ and $C_c(\mathcal{G} \setminus \mathcal{G}^x_x) + \psi(C^*(\Gamma))$ is dense in $C^*_r(\mathcal{G})$, it follows that ρ_x is weakly contained in $\bigoplus_{n=1}^\infty \bigoplus_{i=1}^3 \rho_{x_{in}}$. ♢

We finish the section with a short discussion of essential groupoid C*-algebras. Following [KM21], define

$$J_{\rm sing} = \{a \in C^*_r(\mathcal{G}) \mid \text{ the set of } x \in \mathcal{G}^{(0)} \text{ such that } \rho_x(a)\delta_x \neq 0 \text{ is meager}\}.$$

This is a closed ideal in $C^*_r(\mathcal{G})$; in order to see that it is a right ideal, note that if $U \subset \mathcal{G}$ is an open bisection and $f \in C_c(U)$, then for all $x \in s(U)$ we have

$$\|\rho_x(a * f)\delta_x\| = |f(g_x)| \|\rho_{T(x)}(a)\delta_{T(x)}\|, \tag{1.6}$$

where g_x is the unique element in $U \cap \mathcal{G}_x$ and $T\colon s(U) \to r(U)$ is the homeomorphism defined by $T(x) = r(g_x)$. The essential groupoid C*-algebra of $\mathcal{G}$ is defined by

$$C^*_{\rm ess}(\mathcal{G}) = C^*_r(\mathcal{G})/J_{\rm sing}.$$

Proposition 1.12 (cf. [KM21, Proposition 7.18])**.** *Assume $\mathcal{G}$ is a locally compact étale groupoid that can be covered by countably many open bisections. Let $D_0 \subset \mathcal{G}^{(0)}$ be the set of points $x \in \mathcal{G}^{(0)}$ satisfying the following property: there exist elements $g_1, \dots, g_n \in \mathcal{G}^x_x \setminus \{x\}$, open bisections $U_1, \dots, U_n$ and an open neighbourhood U of x in $\mathcal{G}^{(0)}$ such that $g_k \in U_k$ for all k and $U \setminus (U_1 \cup \dots \cup U_n)$ has empty interior. Let Y be a dense subset of $\mathcal{G}^{(0)} \setminus D_0$. Then*

$$J_{\rm sing} = \{a \in C^*_r(\mathcal{G}) \mid \text{ the set of } x \in \mathcal{G}^{(0)} \text{ such that } \rho_x(a) \neq 0 \text{ is meager}\} = \bigcap_{y \in Y} \ker \rho_y.$$

In particular, if $D_0 = \emptyset$, then $J_{\rm sing} = 0$.

Before we turn to the proof, let us make the connection to [KM21] more explicit. Let $D \subset \mathcal{G}^{(0)}$ be the set of dangerous points [KM21], that is, points $x \in \mathcal{G}^{(0)}$ such that there is a net in $\mathcal{G}^{(0)}$ converging to x and to an element of $\mathcal{G}^x_x \setminus \{x\}$. It is easy to see then that $D_0 \subset D$. (Should the points of D_0 be called extremely dangerous?)

Proof of Proposition 1.12. Let $(U_n)_n$ be a sequence of open bisections covering $\mathcal{G}$. For every n, let $T_n\colon s(U_n) \to r(U_n)$ be the homeomorphism defined by U_n and $g_n\colon s(U_n) \to U_n$ be the inverse of $s\colon U_n \to s(U_n)$, so $T_n(x) = r(g_n(x))$. Similarly to (1.6), for all $a \in C^*_r(\mathcal{G})$ and $x \in s(U_n)$, we have

$$\|\rho_x(a)\delta_{g_n(x)}\| = \|\rho_{T_n(x)}(a)\delta_{T_n(x)}\|.$$

It follows that if $a \in J_{\rm sing}$, then the set A_n of points $x \in s(U_n)$ such that $\rho_x(a)\delta_{g_n(x)} \neq 0$ is meager in $\mathcal{G}^{(0)}$. Then the set $\cup_n A_n$ is meager as well. Since it coincides with the set of points $x \in \mathcal{G}^{(0)}$ such that $\rho_x(a) \neq 0$, this proves the first equality of the proposition.

For the second equality, observe first that if $x \in \mathcal{G}^{(0)} \setminus D$ and $\rho_x(a) \neq 0$ for some $a \in C^*_r(\mathcal{G})$, then $\rho_z(a) \neq 0$ for all z close to x. This follows from [KM21, Lemma 7.15] or our Proposition 1.4, since otherwise we could find a net (x_i) converging to x such that $\rho_{x_i}(a) = 0$ for all i and then conclude that $\rho_x(a) = 0$, as ρ_x is weakly contained in $\bigoplus_i \rho_{x_i}$.

The observation implies that if $a \in \cap_{y \in Y} \ker \rho_y$, then $\rho_x(a) = 0$ for all $x \in \bar{Y} \setminus D$. The set D is meager by [KM21, Lemma 7.15]. As $D_0 \subset D$, it follows that Y is dense in $\mathcal{G}^{(0)}$. Therefore if $a \in \cap_{y \in Y} \ker \rho_y$, then $\rho_x(a)$ can be nonzero only for elements x of the meager set D, hence $a \in J_{\rm sing}$.

Conversely, assume $a \in J_{\rm sing}$. Then the observation above implies that $\rho_x(a) = 0$ for all $x \in \mathcal{G}^{(0)} \setminus D$. Therefore to finish the proof it suffices to show that $\rho_x(a) = 0$ for all $x \in D \setminus D_0$. By Proposition 1.4, for this, in turn, it suffices to show that for every $x \in D \setminus D_0$ Condition 1.6 is not satisfied for $Y = \mathcal{G}^{(0)} \setminus D$. Assume this condition is satisfied for some $x \in D$, that is, there exist elements $g_1, \dots, g_n \in \mathcal{G}^x_x \setminus \{x\}$, open bisections $U_1, \dots, U_n$ such that $g_k \in U_k$ and a neighbourhood U of x in $\mathcal{G}^{(0)}$ satisfying $U \setminus D \subset U_1 \cup \dots \cup U_n$. As the set D is meager, this implies that $x \in D_0$. $\square$

2. C*-algebras associated with left cancellative monoids

Let S be a left cancellative monoid with identity element e. Consider its left regular representation

$$\lambda\colon S \to B(\ell^2(S)), \quad \lambda_s \delta_t = \delta_{st}.$$

The reduced C*-algebra $C^*_r(S)$ of S is defined as the C*-algebra generated by the operators λ_s, $s \in S$.

Consider the left inverse hull $I_\ell(S)$ of S, that is, the inverse semigroup of partial bijections on S generated by the left translations $S \to S$. Whenever convenient we view S as a subset of $I_\ell(S)$ by identifying s with the left translation by s. For $s \in S$, we denote by $s^{-1} \in I_\ell(S)$ the bijection $sS \to S$ inverse to the bijection $S \to sS$, $t \mapsto st$. If the map with the empty domain is present in $I_\ell(S)$, we denote it by 0.

Let $E(S)$ be the abelian semigroup of idempotents in $I_\ell(S)$. Every element of $E(S)$ is the identity map on its domain of definition $X \subset S$, which is a right ideal in S of the form

$$X = s_1^{-1} t_1 \dots s_n^{-1} t_n S$$

for some $s_1, \dots, s_n, t_1, \dots, t_n \in S$. Such right ideals are called constructible [Li12]. We denote by $\mathcal{J}(S)$ the collection of all right constructible ideals. It is a semigroup under the operation of intersection, and we have an isomorphism $E(S) \cong \mathcal{J}(S)$. Denote by $p_X \in E(S)$ the idempotent corresponding to $X \in \mathcal{J}(S)$.

Denote by $\widehat{E(S)}$ the collection of semi-characters of $E(S)$, that is, semigroup homomorphisms $E(S) \to \{0,1\}$ that are not identically zero, where $\{0,1\}$ is considered as a semigroup under multiplication. Note that every semi-character $\chi \in \widehat{E(S)}$ must satisfy $\chi(p_S) = 1$. If $0 \in I_\ell(S)$, then denote by χ_0 the semi-character that is identically one. This is the unique semi-character satisfying $\chi_0(0) = 1$. The set $\widehat{E(S)}$ is compact Hausdorff in the topology of pointwise convergence.

Consider the Paterson groupoid $\mathcal{G}(I_\ell(S))$ associated with $I_\ell(S)$ [Pat99]:

$$\mathcal{G}(I_\ell(S)) = \Sigma/{\sim_P}, \quad \text{where} \quad \Sigma = \{(g,\chi) \in I_\ell(S) \times \widehat{E(S)} \mid \chi(g^{-1}g) = 1\}$$

and the equivalence relation $\sim_P$ is defined by declaring (g_1, χ_1) and (g_2, χ_2) to be equivalent if and only if

$$\chi_1 = \chi_2 \ \text{ and there exists } \ p \in E(S) \ \text{ such that } \ g_1 p = g_2 p \ \text{ and } \ \chi_1(p) = 1.$$

We denote by $[g,\chi]$ the class of $(g,\chi) \in \Sigma$ in $\mathcal{G}(I_\ell(S))$. The product is defined by

$$[g,\chi]\,[h,\psi] = [gh,\psi] \quad \text{if} \quad \chi = \psi(h^{-1} \cdot h).$$

In particular, the unit space $\mathcal{G}(I_\ell(S))^{(0)}$ can be identified with $\widehat{E(S)}$ via the map $\widehat{E(S)} \to \mathcal{G}(I_\ell(S))$, $\chi \mapsto [p_S, \chi]$, the source and range maps are given by

$$s([g,\chi]) = \chi, \qquad r([g,\chi]) = \chi(g^{-1} \cdot g),$$

while the inverse is given by $[g,\chi]^{-1} = [g^{-1}, \chi(g^{-1} \cdot g)]$.

For a subset U of $\widehat{E(S)}$, define

$$D(g,U) = \{[g,\chi] \in \mathcal{G}(I_\ell(S)) \mid \chi \in U\}.$$

Then the topology on $\mathcal{G}(I_\ell(S))$ is defined by taking as a basis the sets $D(g,U)$, where $g \in I_\ell(S)$ and U is an open subset of the clopen set $\{\chi \in \widehat{E(S)} \mid \chi(g^{-1}g) = 1\}$. This turns $\mathcal{G}(I_\ell(S))$ into a locally compact, but not necessarily Hausdorff, étale groupoid.

For every $s \in S$ define a semi-character $\chi_s \in \widehat{E(S)}$ by

$$\chi_s(p_X) = \mathbb{1}_X(s).$$

The following lemma is a groupoid version of the observation of Norling [Nor14, Section 3] on a connection between the regular representations of S and $I_\ell(S)$. A closely related result was also proved by Spielberg [Spi20, Proposition 11.4].

Lemma 2.1. *Put $\mathcal{G} = \mathcal{G}(I_\ell(S))$ and $Z = \mathcal{G}^{(0)} = \widehat{E(S)}$. Then the map $S \to \mathcal{G}_{\chi_e}$, $s \mapsto [s, \chi_e]$, is a bijection. If we identify S with $\mathcal{G}_{\chi_e}$ using this map, so that the representation ρ_{χ_e} of $C_r^*(\mathcal{G})$ is viewed as a representation on $\ell^2(S)$, then*

$$\rho_{\chi_e}(C_r^*(\mathcal{G})) = C_r^*(S) \quad \textit{and} \quad \rho_{\chi_e}(\mathbb{1}_{D(s,Z)}) = \lambda_s \quad \textit{for all} \quad s \in S.$$

Proof. Since $\chi_e(p_J) = 0$ for every constructible ideal J different from S, we have $[s, \chi_e] = [t, \chi_e]$ only if $s = t$. This shows that the map $S \to \mathcal{G}_{\chi_e}$, $s \mapsto [s, \chi_e]$, is injective. In order to prove that it is surjective, assume that $(g, \chi_e) \in \Sigma$, that is, $\chi_e(g^{-1}g) = 1$, for some $g \in I_\ell(S)$. This means that the domain of definition of g contains e, and since this domain is a right ideal, it must coincide with S. But then if $s \in S$ is the image of e under the action of g, we must have $g(t) = g(e)t = st$ for all $t \in S$, so $g = s$, which proves the surjectivity.

Next, as $\chi_e(t^{-1} \cdot t) = \chi_t$ and $[s, \chi_t]\,[t, \chi_e] = [st, \chi_e]$, we immediately get that $\rho_{\chi_e}(\mathbb{1}_{D(s,Z)}) = \lambda_s$ for all $s \in S$. It is not difficult to see that the C*-algebra $C_r^*(\mathcal{G})$ is generated by the elements $\mathbb{1}_{D(s,Z)}$. (One can also refer to [Pat99, Theorem 4.4.2] that shows that the C*-algebra $C_r^*(\mathcal{G})$ is the reduced C*-algebra of the inverse semigroup $I_\ell(S)$, which is generated by S.) Hence $\rho_{\chi_e}(C_r^*(\mathcal{G}))$ is exactly $C_r^*(S)$. □

This lemma leads naturally to a candidate for a groupoid model for $C_r^*(S)$: define

$$\mathcal{G}_P(S) := \mathcal{G}(I_\ell(S))_{\Omega(S)},$$

where $\Omega(S) \subset \mathcal{G}(I_\ell(S))^{(0)}$ is the closure of the $\mathcal{G}(I_\ell(S))$-orbit of χ_e. As $\chi_e(s^{-1} \cdot s) = \chi_s$, by Lemma 2.1 this orbit is exactly the set of semi-characters χ_s, $s \in S$. The closure of this set is known and easy to find, cf. [CELY17, Corollary 5.6.26]: the set $\Omega(S) = \overline{\{\chi_s \mid s \in S\}} \subset \widehat{E(S)}$ consists of the semi-characters χ satisfying the properties

- (i) if $0 \in I_\ell(S)$, then $\chi \neq \chi_0$ (equivalently, $\chi(0) = 0$);
- (ii) if $\chi(p_X) = 1$ and $X = X_1 \cup \cdots \cup X_n$ for some $X, X_1, \ldots, X_n \in \mathcal{J}(S)$, then $\chi(p_{X_i}) = 1$ for at least one index i.

The groupoid $\mathcal{G}_P(S)$ is denoted by $I_l \ltimes \Omega$ in [Li21].

We are now in the setting of Section 1, with $\mathcal{G} = \mathcal{G}_P(S)$ and $Y = \{\chi_s \mid s \in S\}$ a dense invariant subset of $\mathcal{G}^{(0)}$. Negation of Condition 1.6 leads to the following definition.

Definition 2.2. We say that S is **strongly C*-regular** if, given elements $h_1, \ldots, h_n \in I_\ell(S)$ and constructible ideals $X, X_1, \ldots, X_m \in \mathcal{J}(S)$ satisfying

$$\emptyset \neq X \setminus \bigcup_{i=1}^{m} X_i \subset \bigcup_{k=1}^{n} \{s \in S : h_k s = s\}, \tag{2.1}$$

there are constructible ideals $Y_1,\dots,Y_l\in\mathcal{J}(S)$ and indices $1\le k_j\le n$ $(j=1,\dots,l)$ such that

$$X\setminus\bigcup_{i=1}^m X_i\subset\bigcup_{j=1}^l Y_j\qquad\text{and}\qquad h_{k_j}p_{Y_j}=p_{Y_j}\quad\text{for all}\quad 1\le j\le l. \tag{2.2}$$

Lemma 2.3. *Condition 1.6 is not satisfied for $\mathcal{G}=\mathcal{G}_P(S)$, $Y=\{\chi_s\mid s\in S\}$ and every $x\in\mathcal{G}^{(0)}\setminus Y$ if and only if S is strongly C*-regular.*

Proof. Assume first that S is strongly C*-regular. Suppose there is $\chi\in\mathcal{G}^{(0)}\setminus Y$ such that Condition 1.6 is satisfied for $x=\chi$, and let $g_k=[h_k,\chi]$, U_k $(1\le k\le n)$ and U be as in that condition. We may assume that $U_k=D(h_k,\Omega(S))$ and

$$U=\{\eta\in\Omega(S)\mid \eta(p_X)=1,\ \eta(p_{X_i})=0\text{ for }i=1,\dots,m\} \tag{2.3}$$

for some $X,X_1,\dots,X_m\in\mathcal{J}(S)$. Then Condition 1.6 says that for every $s\in X\setminus\cup_{i=1}^m X_i$ there is k such that $\chi_s\in U_k$, that is, $h_ks=s$. By the strong C*-regularity we can find $Y_1,\dots,Y_l\in\mathcal{J}(S)$ satisfying (2.2). As $\chi\in\Omega(S)$, there must exist j such that $\chi(p_{Y_j})=1$. But then $g_{k_j}=[h_{k_j},\chi]=\chi$, which contradicts the assumption that $g_1,\dots,g_n$ are nontrivial elements of the isotropy group $\mathcal{G}^\chi_\chi$.

Assume now that S is not strongly C*-regular, so there are elements $h_1,\dots,h_n\in I_\ell(S)$ and constructible ideals $X,X_1,\dots,X_m\in\mathcal{J}(S)$ such that (2.1) holds but (2.2) doesn't for any choice of Y_j and k_j. In other words, if we consider the set $\mathcal{F}$ of all constructible ideals J such that there is k (depending on J) satisfying $h_kp_J=p_J$, then for any finite set $F\subset\mathcal{F}$ we have

$$X\setminus\Big(\bigcup_{i=1}^m X_i\cup\bigcup_{J\in F}J\Big)\ne\emptyset.$$

Pick a point s_F in the above set and consider a cluster point χ of the net $(\chi_{s_F})_F$, where F's are partially ordered by inclusion. Then χ lies in the set U defined by (2.3), and $\chi(p_J)=0$ for all $J\in\mathcal{F}$. The semi-character χ cannot be of the form χ_s, since otherwise we must have $s\in X\setminus\cup_{i=1}^m X_i$, and then $sS\in\mathcal{F}$ and $\chi(p_{sS})=\chi_s(p_{sS})=1$, which is a contradiction.

We claim that it is possible to replace U by a smaller neighbourhood of χ and discard some of the elements h_k in such a way that Condition 1.6 gets satisfied for $x=\chi$, $g_k=[h_k,\chi]$ and $U_k=D(h_k,\Omega(S))$. Namely, if $(h_k,\chi)\notin\Sigma$ for some k, then we add $\operatorname{dom}h_k$ to the collection $\{X_1,\dots,X_m\}$ and discard such h_k. If $(h_k,\chi)\in\Sigma$ but $\chi(h_k^{-1}\cdot h_k)\ne\chi$, then $\chi_s(h_k^{-1}\cdot h_k)\ne\chi_s$ for all χ_s close χ, so by replacing X by a smaller ideal and adding more constructible ideals to $\{X_1,\dots,X_m\}$ we may assume that $\chi_s(h_k^{-1}\cdot h_k)\ne\chi_s$ for all $\chi_s\in U$ and again discard such h_k. For the remaining elements h_k and the new U we have that for every $s\in S$ such that $\chi_s\in U$ there is an index k satisfying $h_ks=s$. Then, in order to show that Condition 1.6 is satisfied, it remains to check that the elements $g_k=[h_k,\chi]$ of $\mathcal{G}^\chi_\chi$ are nontrivial. But this is clearly true, since $\chi(p_J)=0$ for every $J\in\mathcal{J}(S)$ such that $h_kp_J=p_J$. □

Remark 2.4. From the last part of the proof we see that in Definition 2.2 we may assume in addition that $X\subset\operatorname{dom}h_k$ for all k. More directly this can be seen as follows. Assume (2.1) is satisfied. Consider the nonempty subsets $F\subset\{1,\dots,n\}$ such that

$$X_F:=X\cap\Big(\bigcap_{k\in F}\operatorname{dom}h_k\Big)\not\subset\bigcup_{i=1}^m X_i\cup\bigcup_{k\notin F}\operatorname{dom}h_k.$$

Then (2.1) is satisfied for X_F, $\{X_1,\dots,X_m,\operatorname{dom}h_k\ (k\notin F)\}$ and $\{h_k\ (k\in F)\}$ in place of X, $\{X_1,\dots,X_m\}$ and $\{h_k\ (1\le k\le n)\}$. Since

$$X\setminus\bigcup_{i=1}^m X_i\subset\bigcup_F\left(X_F\setminus\Big(\bigcup_{i=1}^m X_i\cup\bigcup_{k\notin F}\operatorname{dom}h_k\Big)\right),$$

we conclude that if for every F condition (2.2) can be satisfied for X_F, $\{X_1,\dots,X_m,\operatorname{dom} h_k\ (k\notin F)\}$ and $\{h_k\ (k\in F)\}$, then it can be satisfied for X, $\{X_1,\dots,X_m\}$ and $\{h_k\ (1\le k\le n)\}$ as well. $\diamondsuit$

Since the points χ_s, $s\in S$, lie on the same $\mathcal{G}_P(S)$-orbit, the corresponding representations ρ_{χ_s} of $C^*_r(\mathcal{G}_P(S))$ are mutually equivalent. Thus, by Lemma 2.1 and Proposition 1.4, we get the following result.

Proposition 2.5. *If S is a strongly C*-regular left cancellative monoid, then the representation ρ_{χ_e} of $C^*_r(\mathcal{G}_P(S))$ defines an isomorphism $C^*_r(\mathcal{G}_P(S))\cong C^*_r(S)$.*

Therefore if S is strongly C*-regular, it is natural to define the full semigroup C*-algebra of S by

$$C^*(S):=C^*(\mathcal{G}_P(S)).$$

As $C^*(\mathcal{G}(I_\ell(S)))$ has a known description in terms of generators and relations [Pat99], we can quickly obtain such a description for $C^*(\mathcal{G}_P(S))$ as well.

Proposition 2.6 (cf. [Spi20, Theorem 9.4],[LS22, Definition 3.6])**.** *Assume S is a countable left cancellative monoid. Consider the elements $v_s=\mathbb{1}_{D(s,\Omega(S))}\in C^*(\mathcal{G}_P(S))$, $s\in S$. Then $C^*(\mathcal{G}_P(S))$ is a universal unital C*-algebra generated by the elements v_s, $s\in S$, satisfying the following relations:*

- (R1) *$v_e=1$;*
- (R2) *for every $g=s_1^{-1}t_1\dots s_n^{-1}t_n\in I_\ell(S)$, the element $v_g:=v_{s_1}^*v_{t_1}\dots v_{s_n}^*v_{t_n}$ is independent of the presentation of g;*
- (R3) *if $0\in I_\ell(S)$, then $v_0=0$;*
- (R4) *if $X=X_1\cup\dots\cup X_n$ for some $X,X_1,\dots,X_n\in\mathcal{J}(S)$, then*

$$\prod_{i=1}^n(v_{p_X}-v_{p_{X_i}})=0.$$

Note that relation (R4) is unambiguous, since relation (R2) implies that the elements v_{p_X}, $X\in\mathcal{J}(S)$, are mutually commuting projections.

Proof. Consider a universal unital C*-algebra with generators v_s, $s\in S$, satisfying relations (R1) and (R2). This is nothing else than the full C*-algebra $C^*(I_\ell(S))$ of the inverse semigroup $I_\ell(S)$, which is by definition generated by elements v_g, $g\in I_\ell(S)$, satisfying the relations

$$v_gv_h=v_{gh},\qquad v_g^*=v_{g^{-1}}.$$

By [Pat99, Theorem 4.4.1], we can identify $C^*(I_\ell(S))$ with $C^*(\mathcal{G}(I_\ell(S)))$. As $\mathcal{G}_P(S)=\mathcal{G}(I_\ell(S))_{\Omega(S)}$, it follows that in order to prove the proposition it remains to show that relations (R3) and (R4) describe the quotient $C^*(\mathcal{G}(I_\ell(S))_{\Omega(S)})$ of $C^*(\mathcal{G}(I_\ell(S)))$. By Proposition 1.1, the kernel of the map $C^*(\mathcal{G}(I_\ell(S)))\to C^*(\mathcal{G}(I_\ell(S))_{\Omega(S)})$ is generated as a closed ideal by the functions

$$f\in C(\mathcal{G}(I_\ell(S))^{(0)})=C(\widehat{E(S)})$$

vanishing on $\Omega(S)$. The C*-algebra $C(\widehat{E(S)})$ is a universal C*-algebra generated by the projections $e_X:=\mathbb{1}_{U_X}$, $X\in\mathcal{J}(S)$, where $U_X=\{\eta\in\widehat{E(S)}:\eta(p_X)=1\}$, satisfying the relations $e_Xe_Y=e_{X\cap Y}$. By the definition of $\Omega(S)$, relations (R3) and (R4), with e_X instead of v_{p_X}, describe the quotient $C(\Omega(S))$ of $C(\widehat{E(S)})$. This gives the result. $\square$

Remark 2.7. The assumption of countability of S is certainly not needed in the above proposition, we added it to be able to formally apply results of [Pat99].

Remark 2.8. Relation (R2) can be slightly relaxed, cf. [LS22, Definition 3.6]: it suffices to require that the elements v_g are well-defined only for $g=p_X$, $X\in\mathcal{J}(S)$. Indeed, then, given $g=a_1^{-1}b_1\dots a_n^{-1}b_n=c_1^{-1}d_1\dots c_m^{-1}d_m\in I_\ell(S)$, for the elements $v=v_{a_1}^*v_{b_1}\dots v_{a_n}^*v_{b_n}$ and $w=v_{c_1}^*v_{d_1}\dots v_{c_m}^*v_{d_m}$ we have $v^*v=v^*w=w^*v=w^*w$, hence $(v-w)^*(v-w)=0$ and $v=w$. $\diamondsuit$

Let us next give a few sufficient conditions for strong C*-regularity. Recall that a left cancellative monoid S is called finitely aligned [Spi20], or right (ideal) Howson [ES18], if the right ideal $sS \cap tS$ is finitely generated for all $s, t \in S$. If S is finitely aligned, then by induction on n one can see that the constructible ideals $s_1^{-1}t_1 \dots s_n^{-1}t_n S$ are finitely generated.

Proposition 2.9. *A left cancellative monoid S is strongly C*-regular if either of the following conditions is satisfied:*

1. *the groupoid $\mathcal{G}_P(S)$ is Hausdorff;*
2. *the monoid S is group embeddable;*
3. *the monoid S is finitely aligned.*

We remark that apart from the easy implication (2) $\Rightarrow$ (1), which will be explained shortly, there are no relations between conditions (1)–(3). For example, certain Baumslag–Solitar monoids are group embeddable but are not finitely aligned [Spi12, Lemma 2.12]. Examples of finitely aligned monoids with non-Hausdorff $\mathcal{G}_P(S)$ can be found among the Zappa–Szép products $G \bowtie X^*$ defined by self-similar actions of groups on free monoids with infinite number of generators [Law08]; see Remark 3.9 for a related in spirit example with trivial group of units. Nevertheless, if S is finitely aligned and right cancellative, then $\mathcal{G}_P(S)$ is Hausdorff, see [Spi20, Lemma 7.1] or [Li21, Remark 4.3].

Proof of Proposition 2.9. Condition (1) is obviously sufficient for strong C*-regularity, since Condition 1.6 can be satisfied only for non-Hausdorff groupoids.

Condition (2) is known, and is easily seen, to be stronger than (1): if S is a submonoid of a group G, then every nonzero element g of $I_\ell(S)$ acts by the left translation by an element $h_g \in G$, and we have either $h_g = e$ and $D(g, \Omega(S)) \subset \mathcal{G}_P(S)^{(0)}$ or $h_g \neq e$ and $D(g, \Omega(S)) \cap \mathcal{G}_P(S)^{(0)} = \emptyset$.

Assume now that S is finitely aligned and (2.1) is satisfied. Choose a finite set of generators of the right ideal X. Let $s_1, \dots, s_l$ be those generators that do not lie in $X_1 \cup \dots \cup X_m$. By assumption, for every $1 \leq j \leq l$ we can find $1 \leq k_j \leq n$ such that $h_{k_j} s_j = s_j$. Then (2.2) is satisfied for $Y_j = s_j S$. ☐

Remark 2.10. By [Li21, Lemma 4.1], the groupoid $\mathcal{G}_P(S)$ is Hausdorff if and only if whenever $g \in I_\ell(S)$ and $\{s \in S \mid gs = s\} \neq \emptyset$, there are $Y_1, \dots, Y_l \in \mathcal{J}(S)$ such that $\{s \in S \mid gs = s\} = Y_1 \cup \dots \cup Y_l$. Using this characterization one can easily see that (1) implies strong C*-regularity without relying on Lemma 2.3. ♢

In addition to $\mathcal{G}_P(S)$ there is another closely related groupoid associated to S, which was introduced by Spielberg [Spi20] and which we will now turn to.

Consider the collection $\bar{\mathcal{J}}(S)$ of subsets of S obtained by adding to $\mathcal{J}(S)$ all sets of the form $X \setminus \cup_{i=1}^m X_i$ for $X, X_1, \dots, X_m \in \mathcal{J}(S)$. It is a semigroup under intersection. When we want to view $E(S)$ as its subsemigroup, we will write p_X instead of X for the elements of $\bar{\mathcal{J}}(S)$. Every $\chi \in \Omega(S)$ extends to a semi-character on $\bar{\mathcal{J}}(S)$ by letting $\chi(p_{X \setminus \cup_{i=1}^m X_i}) = 1$ ($X, X_1, \dots, X_m \in \mathcal{J}(S)$) if $\chi(p_X) = 1$ and $\chi(p_{X_i}) = 0$ for all i, and $\chi(p_{X \setminus \cup_{i=1}^m X_i}) = 0$ otherwise.

Now, define an equivalence relation $\sim$ on $\mathcal{G}_P(S)$ by declaring $[g, \chi] \sim [h, \chi]$ iff there exists $X \in \bar{\mathcal{J}}(S)$ such that $\chi(p_X) = 1$ and $g|_X = h|_X$. Consider the quotient groupoid

$$\mathcal{G}(S) := \mathcal{G}_P(S)/\sim .$$

This groupoid is denoted by $G_2(S)$ in [Spi20] and by $I_l \bar{\ltimes} \Omega$ in [Li21].

Similarly to Lemma 2.1, the representation ρ_{χ_e} of $C_r^*(\mathcal{G}(S))$ defines a surjective $*$-homomorphism $C_r^*(\mathcal{G}(S)) \to C_r^*(S)$. Negation of Condition 1.6 for $\mathcal{G} = \mathcal{G}(S)$, $Y = \{\chi_s \mid s \in S\}$ and all $x \in \mathcal{G}^{(0)} \setminus Y$ leads to the following definition.

Definition 2.11. We say that S is **C*-regular** if, given $h_1, \dots, h_n \in I_\ell(S)$ and $X \in \bar{\mathcal{J}}(S)$ satisfying

$$\emptyset \neq X \subset \bigcup_{k=1}^{n} \{s \in S : h_k s = s\},$$

there are sets $Y_1, \dots, Y_l \in \bar{\mathcal{J}}(S)$ and indices $1 \le k_j \le n$ $(j = 1, \dots, l)$ such that

$$X \subset \bigcup_{j=1}^{l} Y_j \qquad \text{and} \qquad h_{k_j}|_{Y_j} = \mathrm{id} \quad \text{for all} \quad 1 \le j \le l.$$

Note that by the same argument as in Remark 2.4, in order to check C*-regularity it suffices to consider $X = X_0 \setminus (X_1 \cup \dots \cup X_m)$ $(X_i \in \mathcal{J}(S))$ and h_k such that $X_0 \subset \operatorname{dom} h_k$ for all k. Note also that the only difference between C*-regularity and strong C*-regularity is that the sets Y_j are required to be in $\bar{\mathcal{J}}(S)$ in the first case and in $\mathcal{J}(S)$ in the second. In particular, strong C*-regularity implies C*-regularity.

Similarly to Proposition 2.5 we get the following result.

Proposition 2.12. *If S is a C*-regular left cancellative monoid, then the representation ρ_{χ_e} of $C_r^*(\mathcal{G}(S))$ defines an isomorphism $C_r^*(\mathcal{G}(S)) \cong C_r^*(S)$.*

Thus, if S is C*-regular, we can define, following [Spi20], the full semigroup C*-algebra of S by

$$C^*(S) := C^*(\mathcal{G}(S)).$$

A presentation of $C^*(\mathcal{G}(S))$ in terms of generators and relations is given in [Spi20, Theorem 9.4].

We therefore have two candidates for a groupoid model of $C_r^*(S)$, and hence two potentially different definitions of full semigroup C*-algebras associated with S. As the following result shows, it is $\mathcal{G}(S)$ which is the preferred model and we have only one candidate for $C^*(S)$.

Proposition 2.13. *Assume S is a left cancellative monoid such that ρ_{χ_e} defines an isomorphism $C_r^*(\mathcal{G}_P(S)) \cong C_r^*(S)$. Then $\mathcal{G}_P(S) = \mathcal{G}(S)$.*

Proof. Assume $\mathcal{G}_P(S) \neq \mathcal{G}(S)$. Then there exists $[g, \chi] \in \mathcal{G}_P(S)$ such that $[g, \chi] \neq \chi$ but $[g, \chi] \sim \chi$. Let $X, X_1, \dots, X_m \in \mathcal{J}(S)$ be such that $\chi(p_X) = 1$, $\chi(p_{X_i}) = 0$ for all i and $gs = s$ for all $s \in X \setminus \cup_{i=1}^m X_i$. Consider the clopen set $U \subset \mathcal{G}_P(S)^{(0)}$ defined by (2.3) and the function $f = \mathbb{1}_{D(g,U)} - \mathbb{1}_U$ on $\mathcal{G}_P(S)$. Then $f \neq 0$, but if $g = s_1^{-1} t_1 \dots s_n^{-1} t_n$ and we identify $\ell^2(\mathcal{G}_P(S)_{\chi_e})$ with $\ell^2(S)$, then

$$\rho_{\chi_e}(f) = (\lambda_{s_1}^* \lambda_{t_1} \dots \lambda_{s_n}^* \lambda_{t_n} - 1) \mathbb{1}_{X \setminus \cup_{i=1}^m X_i} = 0.$$

Therefore $\rho_{\chi_e} \colon C_r^*(\mathcal{G}_P(S)) \to C_r^*(S)$ has a nontrivial kernel. ☐

Corollary 2.14. *If S is strongly C*-regular, then S is C*-regular and $\mathcal{G}_P(S) = \mathcal{G}(S)$.*

Proof. The first statement follows from the definitions, as was already observed after Definition 2.11. The equality $\mathcal{G}_P(S) = \mathcal{G}(S)$ follows from Propositions 2.5 and 2.13. ☐

In particular, by Proposition 2.9, if $\mathcal{G}_P(S)$ is Hausdorff or S is finitely aligned, then $\mathcal{G}_P(S) = \mathcal{G}(S)$. This has been already known, see [Li21, Lemma 3.2].

The equality $\mathcal{G}_P(S) = \mathcal{G}(S)$ in the strong C*-regular case is also an immediate consequence of the following criterion.

Lemma 2.15. *For every left cancellative monoid S, we have $\mathcal{G}_P(S) = \mathcal{G}(S)$ if and only if S has the following property: given $g \in I_\ell(S)$ and $X, X_1, \dots, X_m \in \mathcal{J}(S)$ such that $gs = s$ for all $s \in X \setminus \cup_{i=1}^m X_i \neq \emptyset$, there are $Y_1, \dots, Y_l \in \mathcal{J}(S)$ such that*

$$X \setminus \bigcup_{i=1}^{m} X_i \subset \bigcup_{j=1}^{l} Y_j \qquad \text{and} \qquad g p_{Y_j} = p_{Y_j} \quad \text{for all} \quad 1 \le j \le l.$$

Proof. Assume first that the condition in the formulation of the lemma is satisfied. In order to prove that $\mathcal{G}_P(S) = \mathcal{G}(S)$, we have to show that if $[g,\chi] \sim \chi$ for some $g \in I_\ell(S)$ and $\chi \in \Omega(S)$, then $[g,\chi] = \chi$. Let $X, X_1, \dots, X_m \in \mathcal{J}(S)$ be such that $\chi(p_X) = 1$, $\chi(p_{X_i}) = 0$ for all i and $gs = s$ for all $s \in X \setminus \cup_{i=1}^m X_i$. By assumption, there are $Y_1, \dots, Y_l \in \mathcal{J}(S)$ such that $X \setminus \bigcup_{i=1}^m X_i \subset \bigcup_{j=1}^l Y_j$ and $gp_{Y_j} = p_{Y_j}$ for all $1 \le j \le l$. But then $\chi(p_{Y_j}) = 1$ for some j, hence $[g,\chi] = \chi$.

Assume now that the condition in the formulation is not satisfied. Then, similarly to the proof of Lemma 2.3, we can find $g \in I_\ell(S)$, $X, X_1, \dots, X_m \in \mathcal{J}(S)$ and $\chi \in \Omega(S)$ such that $\chi(p_X) = 1$, $\chi(p_{X_i}) = 0$ for all i, $\chi(p_J) = 0$ for all $J \in \mathcal{J}(S)$ satisfying $gp_J = p_J$, and $gs = s$ for all $s \in X \setminus \cup_{i=1}^m X_i$. Then $[g,\chi] \sim \chi$ and $[g,\chi] \ne \chi$, so $\mathcal{G}_P(S) \ne \mathcal{G}(S)$. ☐

We finish this section by showing that under rather general assumptions $C_r^*(S)$ coincides with the essential groupoid C*-algebras of $\mathcal{G}_P(S)$ and $\mathcal{G}(S)$.

Proposition 2.16. *Let S be a countable left cancellative monoid. Assume that for any nontrivial units $u_1, \dots, u_k \in S^* \setminus \{e\}$ and every $X \in \bar{\mathcal{J}}(S)$ containing e, there is $Z \in \bar{\mathcal{J}}(S)$ such that $\emptyset \ne Z \subset X$ and $u_i s \ne s$ for all $s \in Z$ and $i = 1, \dots, k$. Then the maps $\rho_{\chi_e} \colon C_r^*(\mathcal{G}_P(S)) \to C_r^*(S)$ and $\rho_{\chi_e} \colon C_r^*(\mathcal{G}(S)) \to C_r^*(S)$ define isomorphisms*

$$C^*_{\mathrm{ess}}(\mathcal{G}_P(S)) \cong C_r^*(S) \cong C^*_{\mathrm{ess}}(\mathcal{G}(S)).$$

Proof. Consider the groupoid $\mathcal{G}_P(S)$. Let $Y = \{\chi_s | s \in S\}$ and D_0 be the set defined in Proposition 1.12. As Y is dense in $\Omega(S)$ and $\ker \rho_{\chi_s}$ is independent of s, by Proposition 1.12 in order to prove the first isomorphism it suffices to show that $D_0 \cap Y = \emptyset$.

Since both sets D_0 and Y are invariant, it is enough to show that $\chi_e \notin D_0$. By Lemma 2.1, the isotropy group $\mathcal{G}_P(S)^{\chi_e}_{\chi_e}$ consists of the elements $[u, \chi_e]$, $u \in S^*$. Therefore we need to show that if $u_1, \dots, u_k \in S^* \setminus \{e\}$ and U is a neighbourhood of χ_e in $\Omega(S)$, then the set $U \setminus \cup_{i=1}^k D(u_i, \Omega(S))$ has nonempty interior. By the definition of the topology on $\Omega(S)$, we can find $X \in \bar{\mathcal{J}}(S)$ such $e \in X$ and $\{\chi \mid \chi(p_X) = 1\} \subset U$. Let $Z \in \bar{\mathcal{J}}(S)$ be as in the formulation of the proposition. Then the clopen set $V = \{\chi \mid \chi(p_Z) = 1\}$ is contained in U. We claim that it does not intersect $D(u_i, \Omega(S))$ for all i.

Assume $\chi \in V \cap D(u_i, \Omega(S))$ for some i. This means that $[u_i, \chi] = [e, \chi]$, that is, there is $W \in \mathcal{J}(S)$ such that $\chi(p_W) = 1$ and $u_i s = s$ for all $s \in W$. But then $\chi(p_{Z \cap W}) = 1$, so $Z \cap W$ is nonempty, contradicting the property $u_i s \ne s$ for all $s \in Z$.

This proves the proposition for $\mathcal{G}_P(S)$, the proof for $\mathcal{G}(S)$ is essentially the same. ☐

3. Example of a regular monoid

Consider the monoid S given by the monoid presentation

$$S = \langle a, b, x_n, y_n \ (n \in \mathbb{Z}) : abx_n = bx_n,\ aby_n = by_{n+1} \ (n \in \mathbb{Z}) \rangle. \tag{3.1}$$

Our goal is to prove the following.

Proposition 3.1. *The monoid S defined by* (3.1) *is left cancellative and strongly C*-regular. It is not finitely aligned and the groupoid $\mathcal{G}_P(S) = \mathcal{G}(S)$ is not Hausdorff.*

The proof is divided into several lemmas. But first we need to introduce some notation. Consider the set $\mathbb{S}$ of finite words (including the empty word) in the alphabet $\{a, b, x_n, y_n \ (n \in \mathbb{Z})\}$. We say that two words are equivalent if they represent the same element of S. Let $\tau \subset \mathbb{S} \times \mathbb{S}$ be the symmetric set of relations defining S, so

$$\tau = \{(abx_n, bx_n), (bx_n, abx_n), (aby_n, by_{n+1}), (by_{n+1}, aby_n) \ (n \in \mathbb{Z})\}.$$

By a τ-*sequence* we mean a finite sequence $s_0, \dots, s_n$ of words such that for every $i = 1, \dots, n$ we can write $z_{i-1} = c_i p_i d_i$ and $z_i = c_i q_i d_i$ with $(p_i, q_i) \in \tau$. Then by definition two words s and t are equivalent if and only if there is a τ-sequence $s_0, \dots, s_n$ with $s_0 = s$ and $s_n = t$.

Lemma 3.2. *The monoid S is left cancellative.*

Proof. It will be convenient to use the following notation. For words s and t, let us write $s \perp_0 t$ if every word abx_n, bx_n, aby_n, by_n ($n \in \mathbb{Z}$) in st that begins in s ends in s. We write $s \perp t$ if for all words s' and t' such that $s \sim s'$ and $t \sim t'$, we have $s' \perp_0 t'$. We will repeatedly use that if $s \perp t$ and $st \sim w$ for a word w, then $w = s't'$ for some $s' \sim s$ and $t' \sim t$.

In order to prove the lemma, it suffices to show that for all letters x and words w, w', the equivalence $xw \sim xw'$ implies that $w \sim w'$.

Case $x = x_n, y_n$:
The only word equivalent to x is x itself, and we have $x \perp w$, so the equivalence $xw \sim xw'$ furnishes the equivalence $w \sim w'$.

Case $x = a$:
Write $xw = a^k v$ ($k \geq 1$), with v not starting with a. Consider several subcases.

Assume v is empty or starts with x_n or y_n. Then $a^k \perp v$. The only word equivalent to a^k is a^k itself. It follows that $xw' = a^k v'$, with $v' \sim v$, and therefore $w = a^{k-1}v \sim a^{k-1}v' = w'$.

Assume now that v starts with b and write $xw = a^k bu$. Assume u is empty or starts with a or b. Then $a^k b \perp u$. The only word equivalent to $a^k b$ is $a^k b$ itself. It follows that $xw' = a^k bu'$, with $u' \sim u$, hence $w \sim w'$.

Assume next that u starts with x_n and write $xw = a^k bx_n z$. Then $a^k bx_n \perp z$. The words equivalent to $a^k bx_n$ are $a^m bx_n$ ($m \geq 0$). It follows that $xw' = a^m bx_n z'$ for some $m \geq 1$ and $z' \sim z$, hence $w \sim w'$.

Similarly, if u starts with y_n, write $xw = a^k by_n z$. Then $a^k by_n \perp z$. The words equivalent to $a^k by_n$ are $a^m by_{n+m-k}$ ($m \geq 0$). It follows that $xw' = a^m by_{n+m-k} z'$ for some $m \geq 1$ and $z' \sim z$, hence $w \sim w'$.

Case $x = b$:
If w is empty or starts with a or b, then $x \perp w$ and we are done similarly to the first case above.

If w starts with x_n, write $xw = bx_n v$. Then $bx_n \perp v$. The words equivalent to bx_n are $a^m bx_n$ ($m \geq 0$), and the only one among them beginning with b is bx_n itself. It follows that $xw' = bx_n v'$, with $v' \sim v$, hence $w \sim w'$. The case when w starts with y_n is similar, since the only word beginning with b that is equivalent to by_n is by_n itself. ∎

Next we want to describe the constructible ideals of S. We start with the following lemma.

Lemma 3.3. *We have:*

1. *if $x \in \{x_n, y_n\ (n \in \mathbb{Z})\}$, $y \in \{a, b, x_n, y_n\ (n \in \mathbb{Z})\}$ and $x \neq y$, then*
$$xS \cap yS = \emptyset;$$
2. *for all $k \geq 1$,*
$$bS \cap a^k S = bS \cap a^k bS = \bigcup_{n \in \mathbb{Z}} bx_n S \cup \bigcup_{n \in \mathbb{Z}} by_n S.$$

Proof. (1) None of the words occurring in the defining relations of S begins with x_n or y_n. From this it follows that a word beginning with x_n, resp., y_n, can only be equivalent to a word beginning with x_n, resp., y_n. Thus, $xS \cap yS = \emptyset$.

(2) Since we have $bx_n = a^k bx_n$ and $by_n = a^k by_{n-k}$ for all n and k, it is clear that
$$\bigcup_n bx_n S \cup \bigcup_n by_n S \subset bS \cap a^k bS \subset bS \cap a^k S.$$

To prove the opposite inclusions, assume $s \in bS \cap a^k S$. Take words w and w' in $\mathbb{S}$ such that s is represented by bw and $a^k w'$. There is a τ-sequence $z_0, \dots, z_m$ such that $bw = z_0$ and $a^k w' = z_m$. We have $z_{i-1} = c_i p_i d_i$ and $z_i = c_i q_i d_i$, with $(p_i, q_i) \in \tau$. There must be an index i such that $c_i = \emptyset$ and p_i starts with b. But then $p_i = bx_n$ or $p_i = by_n$ for some n, since these are the only words in the defining relations of S that begin with b. Therefore s lies in bx_nS or in by_nS. $\square$

This lemma already implies that S is not finitely aligned, since $b^{-1}aS = \bigcup_n x_nS \cup \bigcup_n y_nS$ by (2) and the sets x_nS, y_mS are disjoint for all n and m by (1), so the right ideal $b^{-1}aS$ is not finitely generated.

Lemma 3.4. *The constructible ideals of S are*

$$\emptyset, \quad sS, \quad \bigcup_{n\in\mathbb{Z}} sx_nS \cup \bigcup_{n\in\mathbb{Z}} sy_nS \quad (s \in S). \tag{3.2}$$

Proof. By Lemma 3.3, the ideals in (3.2) are constructible. In order to prove the lemma it is then enough to show that for every $x \in \{a, b, x_n, y_n \ (n \in \mathbb{Z})\}$ and $s \in S$, the right ideals $x^{-1}sS$ and $\bigcup_n x^{-1}sx_nS \cup \bigcup_n x^{-1}sy_nS$ are again of the form (3.2). This is obviously true when $s \in xS$. By Lemma 3.3(1) this is also true if $s = e$. So from now on we assume that $s \notin xS$ and $s \neq e$.

<u>Case $x = x_n, y_n$:</u>
In this case, from Lemma 3.3(1) we see that the sets $x^{-1}sS$ and $\bigcup_n x^{-1}sx_nS \cup \bigcup_n x^{-1}sy_nS$ are empty.

<u>Case $x = a$:</u>
Again, Lemma 3.3(1) tells us that the sets $a^{-1}sS$ and $\bigcup_n a^{-1}sx_nS \cup \bigcup_n a^{-1}sy_nS$ are empty if $s \in x_mS$ or $s \in y_mS$ for some m. As $s \notin aS$ and $s \neq e$, we may therefore assume that $s \in bS$. Consider several subcases.

Assume $s = b$. Then, by Lemma 3.3(2), $aS \cap bS = \bigcup_n bx_nS \cup \bigcup_n by_nS$. As a maps this set onto itself, we conclude that both $a^{-1}bS$ and $\bigcup_n a^{-1}bx_nS \cup \bigcup_n a^{-1}by_nS$ are equal to $\bigcup_n bx_nS \cup \bigcup_n by_nS$.

Next, assume $s \in baS$ or $s \in b^2S$. From the defining relations we see that every word in $\mathbb{S}$ that starts with ba or b^2 can only be equivalent to a word that again starts with ba or b^2. Hence the sets $a^{-1}baS$ and $a^{-1}b^2S$ are empty, and therefore $a^{-1}sS$ and $\bigcup_n a^{-1}sx_nS \cup \bigcup_n a^{-1}sy_nS$ are empty as well.

It remains to consider the subcase when $s \in bx_mS$ or $s \in by_mS$. But then $s \in aS$, which contradicts our assumption on s.

<u>Case $x = b$:</u>
Similarly to the previous case, we may assume that $s \in aS$. Write $s = a^k t$ for some $k \geq 1$ and $t \in S$ such that t can be represented by a word not starting with a. Consider several subcases.

Assume $t = e$. Then, using Lemma 3.3(2), we get

$$b^{-1}a^kS = b^{-1}(bS \cap a^kS) = b^{-1}\Big(\bigcup_n bx_nS \cup \bigcup_n by_nS\Big) = \bigcup_n x_nS \cup \bigcup_n y_nS.$$

Every word in $\mathbb{S}$ that starts with a^kx_n can only be equivalent to a word that again starts with a^kx_n. The same is true for y_n in place of x_n. Hence

$$\bigcup_n b^{-1}a^kx_nS \cup \bigcup_n b^{-1}a^ky_nS = \emptyset. \tag{3.3}$$

Next, assume $t \in x_mS$ or $t \in y_mS$. Then (3.3) implies that both $b^{-1}a^ktS$ and $\bigcup_n b^{-1}a^ktx_nS \cup \bigcup_n b^{-1}a^kty_nS$ are empty.

It remains to consider the subcase $t \in bS$. This splits into several subsubcases.

If $t = b$, then using Lemma 3.3(2) again,

$$b^{-1}a^kbS = b^{-1}\Big(bS \cap a^kbS\Big) = b^{-1}\Big(\bigcup_n bx_nS \cup \bigcup_n by_nS\Big) = \bigcup_n x_nS \cup \bigcup_n y_nS.$$

As a maps $\bigcup_n bx_nS \cup \bigcup_n by_nS$ onto itself, we also have

$$\bigcup_n b^{-1}a^kbx_nS \cup \bigcup_n b^{-1}a^kby_nS = \bigcup_n x_nS \cup \bigcup_n y_nS.$$

Assume next that $t \in baS$ or $t \in b^2S$. Every word in $\mathbb{S}$ that starts with a^kba or a^kb^2 can only be equivalent to a word that again starts with a^kba or a^kb^2 . Hence the sets $b^{-1}a^kbaS$ and $b^{-1}a^kb^2S$ are empty, and therefore $b^{-1}a^ktS$ and $\bigcup_n b^{-1}a^ktx_nS \cup \bigcup_n b^{-1}a^kty_nS$ are empty as well.

Finally, assume $t \in bx_mS$ or $t \in by_mS$. But then $s = a^kt \in bS$, which contradicts our assumption on s. □

We now look at the topology on $\mathcal{G}_P(S)$. We will need the following lemma.

Lemma 3.5. *The constructible ideals that contain at least two elements bx_n are*

$$S, \quad a^kbS \quad (k \ge 0), \quad \bigcup_{n\in\mathbb{Z}} bx_nS \cup \bigcup_{n\in\mathbb{Z}} by_nS. \tag{3.4}$$

Proof. It is clear that the ideals in (3.4) contain bx_n for all $n \in \mathbb{Z}$. Assume $X \in \mathcal{J}(S)$ contains bx_l and bx_m for some $l \ne m$. Observe that the words in $\mathbb{S}$ equivalent to bx_l for a fixed l are a^kbx_l, $k \ge 0$. It follows that if $bx_l \in sS$ for some s, then any word in $\mathbb{S}$ representing s has the form a^k, a^kb or a^kbx_l for some $k \ge 0$. Consider two cases.

Assume $X = sS$. By the above observation, the only possibilities for s to have $bx_l, bx_m \in sS$ are $s = a^k$ or $s = a^kb$ for some $k \ge 0$, so X has the required form.

Assume $X = \bigcup_n sx_nS \cup \bigcup_n sy_nS$. If $bx_l \in sx_nS$ for some n, then by the above observation $n = l$ and $s = a^kb$ for some $k \ge 0$, so X has the required form. Otherwise we must have $bx_l \in sy_nS$ for some n, but this is not possible, again by the above observation. □

This lemma implies that the semi-characters χ_{bx_n} on $E(S)$ converge as $n \to \pm\infty$ to a semi-character χ such that $\chi(p_X) = 1$ for all X in (3.4) and $\chi(p_X) = 0$ for all other constructible ideals.

Lemma 3.6. *The semi-characters χ_{bx_n} converge in $\mathcal{G}_P(S)$ to the different elements χ and $[a, \chi]$, so $\mathcal{G}_P(S)$ is not Hausdorff.*

Proof. The convergence $\chi_{bx_n} \to [a, \chi]$ follows from the fact that $[a, \chi_{bx_n}] = [e, \chi_{bx_n}]$ for all n, because a fixes bx_n. The element $[a, \chi] \in \mathcal{G}_P(S)^\chi_\chi$ is nontrivial, since by Lemma 3.5 if $X \in \mathcal{J}(S)$ and $\chi(p_X) = 1$, then X contains the elements by_n that a does not fix. □

It remains to show that S is strongly C*-regular. Recall that by Definition 2.2 and Remark 2.4 this means that, given elements $h_1, \dots, h_N \in I_\ell(S)$ and constructible ideals $X, X_1, \dots, X_M \in \mathcal{J}(S)$ satisfying

$$X \subset \bigcap_{k=1}^N \operatorname{dom} h_k, \qquad \emptyset \ne X \setminus \bigcup_{i=1}^M X_i \subset \bigcup_{k=1}^N \{s \in S : h_ks = s\},$$

we need to show that there are constructible ideals $Y_1, \dots, Y_L \in \mathcal{J}(S)$ and indices $1 \le k_j \le N$ $(j = 1, \dots, L)$ such that

$$X \setminus \bigcup_{i=1}^M X_i \subset \bigcup_{j=1}^L Y_j \qquad \text{and} \qquad h_{k_j}p_{Y_j} = p_{Y_j} \quad \text{for all} \quad 1 \le j \le L.$$

We will actually show more: there is an index k such that $h_kp_X = p_X$.

This is obviously true when X is a principal right ideal, cf. Proposition 2.9(3). Therefore we need only to consider $X = \bigcup_n sx_nS \cup \bigcup_n sy_nS$. By replacing X, X_i and h_k by $s^{-1}X$, $s^{-1}X_i$ and $s^{-1}h_ks$ we may assume that

$$X = \bigcup_{n\in\mathbb{Z}} x_nS \cup \bigcup_{n\in\mathbb{Z}} y_nS.$$

Lemma 3.7. *The only constructible ideals that contain y_n for a fixed n are S, y_nS and X.*

Proof. A principal ideal sS contains y_n only if $s = e$ or $s = y_n$, since the only word in $\mathbb{S}$ that is equivalent to y_n is y_n itself. For the same reason if an ideal $\bigcup_k sx_kS \cup \bigcup_k sy_kS$ contains y_n, then we must have $s = e$, so the ideal is X. □

Since by assumption $X\setminus\cup_{i=1}^M X_i \neq \emptyset$, it follows that every ideal X_i contains at most one element y_n. Therefore $X \setminus \cup_{i=1}^M X_i$ contains y_n for all but finitely many n's. In particular, there are indices m and k such that $h_ky_m = y_m$. To finish the proof of strong C*-regularity it is now enough to establish the following.

Lemma 3.8. *If $h \in I_\ell(S)$ fixes y_m for some m and satisfies $\operatorname{dom} h \supset X = \bigcup_n x_nS \cup \bigcup_n y_nS$, then $h = p_S = e$ or $h = p_X$.*

Proof. For $k \geq 1$, consider the element $g_k = a^{-k}b \in I_\ell(S)$. Note that by Lemma 3.3(2) we have $\operatorname{dom} g_k = X$ and $g_kX = bX$. For $k \in \mathbb{Z}\setminus\{0\}$, consider $g'_k = b^{-1}a^kb \in I_\ell(S)$. The domain and range of g'_k is X.

We will show that if $h \in I_\ell(S)$ satisfies $\operatorname{dom} h \supset X$, then h must be of the form

$$sg_kp_J, \quad sg'_kp_J \quad \text{or} \quad sp_J \quad (s \in S,\ J \in \mathcal{J}(S)). \tag{3.5}$$

This will yield the proposition. Indeed, first of all, we then have $J = S$ or $J = X$, since by Lemma 3.7 these are the only constructible ideals containing X. Then $hy_m = sby_{m+k}$, $hy_m = sy_{m+k}$ or $hy_m = sy_m$, resp., and this equals y_m only in the third case with $s = e$.

Take $h \in I_\ell(X)$ that is not of the form (3.5). We will show that $\operatorname{dom} h \not\supset X$. Begin by writing h as a word in the generators of S and their inverses. We may assume that there are no occurrences of $x^{-1}x$ or xx^{-1} in this word for each generator x. Indeed, $x^{-1}x = e$ can be omitted, while $xx^{-1} = p_{xS}$ and if $h = h_1p_{xS}h_2$, then $h = h_1h_2p_{h_2^{-1}xS}$, so instead of h we can consider h_1h_2. We may also assume that there are no occurrences of a^kbx_n and a^kby_n for $k \in \mathbb{Z}\setminus\{0\}$ and $n \in \mathbb{Z}$, as these can be simplified to bx_n and by_{n+k}, resp. We then say that h (or, more precisely, our word for h) is reduced.

The assumption that h is not of the form (3.5) implies that our word for h has the form $h_1x^{-1}wg$, where h_1 is a word in the generators and their inverses, x is one of the generators, w is a word in the generators and we have one of the following options for g: (i) $g = g'_k$; (ii) $g = g_k$ and either w is not trivial or $x \neq a, b$; (iii) $g = e$ and neither ($x = a$ and $w = b$) nor ($x = b$ and $w = a^mb$ for some $m \geq 1$). Without loss of generality we may assume that h_1 is trivial. We are not going to distinguish between the word $w \in \mathbb{S}$ and the element of S it represents.

<u>Case $x = x_n, y_n$:</u>
By the proof of Lemma 3.4, for every word v in the generators, we have $x^{-1}vS = \emptyset$ unless v is empty or starts with x. Since w does not start with x and we have $g_kX = bX$, it follows that $x^{-1}wgX = \emptyset$ unless w is empty and either $g = g'_k$ or $g = e$. In order to deal with the remaining cases we have to show that $g'_kX = X \not\subset xS$, which is clearly true by Lemma 3.3(1).

<u>Case $x = a$:</u>
By the proof of Lemma 3.4, for every word v in the generators, we have $a^{-1}vS = \emptyset$ unless v is empty, $v = b$ or v starts with a, bx_n or by_n. Since w cannot start with a, bx_n or by_n, it follows that $a^{-1}wgX = \emptyset$ unless we have one of the following: (i) w is empty and $g = g_k$; (ii) $w = b$ and $g = g'_k$;

(iii) $w = b$ and $g = e$. Cases (i) and (iii) are not possible by our assumptions on x, w and g. Case (ii) is not possible either, as the word $bg'_k = bb^{-1}a^k b$ is not reduced.

Case $x = b$:
By the proof of Lemma 3.4, for every word v in the generators, we have $b^{-1}vS = \emptyset$ unless v is empty, $v = a^m$, $v = a^m b$ or v starts with b, $a^m bx_n$ or $a^m by_n$ ($m \geq 1$, $n \in \mathbb{Z}$). Since w cannot start with b, $a^m bx_n$ or $a^m by_n$, it follows that $b^{-1}wgX = \emptyset$ unless we have one of the following: (i) w is empty and $g = g_k$; (ii) $w = a^m$ and $g = g_k$; (iii) $w = a^m b$ and $g = g'_k$; (iv) $w = a^m b$ and $g = e$. Cases (i) and (iv) are not possible by our assumptions on x, w and g. Cases (ii) and (iii) are not possible either, as the words $a^m g_k = a^m a^{-k} b$ and $a^m bg'_k = a^m bb^{-1}a^k b$ are not reduced. □

This finishes the proof of Proposition 3.1.

Remark 3.9. If we drop the generator b and consider

$$\tilde{S} = \langle a, x_n, y_n \ (n \in \mathbb{Z}) : ax_n = x_n,\ ay_n = y_{n+1} \ (n \in \mathbb{Z})\rangle,$$

then we get a left cancellative right LCM monoid, meaning that every constructible ideal is either empty or principal. Similarly to Lemma 3.6, the semi-characters χ_{x_n} converge to different elements χ and $[a, \chi]$, so the groupoid $\mathcal{G}_P(\tilde{S}) = \mathcal{G}(\tilde{S})$ is not Hausdorff.

4. Example of a nonregular monoid

Consider a small modification T of the monoid S from the previous section:

$$\begin{aligned} T = \langle a, b, c, x_n, y_n \ (n \in \mathbb{Z}) :\ & abx_n = bx_n,\ aby_n = by_{n+1}, \\ & cbx_n = bx_{n+1},\ cby_n = by_n \ (n \in \mathbb{Z})\rangle. \end{aligned} \tag{4.1}$$

For this monoid we have the following result.

Proposition 4.1. *The monoid T defined by* (4.1) *is left cancellative. It is not C^*-regular, furthermore, the homomorphism $\rho_{\chi_e}\colon C^*_r(\mathcal{G}(T)) \to C^*_r(T)$ has nontrivial kernel.*

A large part of the analysis of T is similar to that of S, so we will omit most of it.

The claim that T is left cancellative is proved similarly to Lemma 3.2, the main difference being that powers of a get replaced by products of a and c. The description of the constructible ideals is exactly the same as for S (Lemma 3.4):

Lemma 4.2. *The constructible ideals of T are*

$$\emptyset, \quad tT, \quad \bigcup_{n\in\mathbb{Z}} tx_nT \cup \bigcup_{n\in\mathbb{Z}} ty_nT \quad (t \in T).$$

The next result is similar to Lemma 3.5.

Lemma 4.3. *The constructible ideals of T that contain at least two elements bx_n or at least two elements by_n are*

$$T, \quad tbT \ \ (t \text{ is a product of } a, c), \quad \bigcup_{n\in\mathbb{Z}} bx_nT \cup \bigcup_{n\in\mathbb{Z}} by_nT. \tag{4.2}$$

This lemma implies that the semi-characters χ_{bx_n} converge as $n \to \pm\infty$ to a semi-character χ such that $\chi(p_X) = 1$ for all X in (4.2) and $\chi(p_X) = 0$ for all other $X \in \mathcal{J}(T)$. The semi-characters χ_{by_n} converge to χ as well. The following lemma finishes the proof of Proposition 4.1.

Lemma 4.4. *The representation ρ_χ of $C^*_r(\mathcal{G}(T))$ is not weakly contained in ρ_{χ_e}, hence*

$$\rho_{\chi_e}\colon C^*_r(\mathcal{G}(T)) \to C^*_r(T)$$

has nontrivial kernel.

Proof. Consider the constructible ideal $X = \bigcup_n bx_nT \cup \bigcup_n by_nT$, the neighbourhood $U = \{\eta \in \Omega(T) : \eta(p_X) = 1\}$ of χ and the elements $g_1 = [a, \chi]$ and $g_2 = [c, \chi]$ of $\mathcal{G}(T)^\chi_\chi$. That g_1 and g_2 are indeed nontrivial elements of the isotropy group is proved as in Lemma 3.6; furthermore, we can see that these are elements of infinite order generating a copy of $\mathbb{Z}^2$. Now, if $\chi_t \in U$ for some $t \in T$, then $t \in X$ and hence t is fixed by a or by c. It follows that either $\chi_t \in D(a, U)$ or $\chi_t \in D(c, U)$. We thus see that $x = \chi$ satisfies Condition 1.6 for $Y = \{\chi_t : t \in T\}$. By Proposition 1.8 we conclude that ρ_χ is not weakly contained in ρ_{χ_e}. □

At this point it is actually not difficult to exhibit an explicit nonzero element of $\ker \rho_{\chi_e}$: consider, for example, the function

$$(\mathbb{1}_{D(a,\Omega(T))} - \mathbb{1}_{\Omega(T)}) * (\mathbb{1}_{D(c,\Omega(T))} - \mathbb{1}_{\Omega(T)}) * \mathbb{1}_U \in C_c(\mathcal{G}(T)).$$

Its restriction to $\mathcal{G}(T)^\chi_\chi$ is nonzero, since g_1 and g_2 are nontrivial elements of $\mathcal{G}(T)^\chi_\chi$. It lies in the kernel of ρ_{χ_e}, since

$$(\lambda_a - 1)(\lambda_c - 1)\mathbb{1}_X = 0 \quad \text{on} \quad \ell^2(T).$$

Therefore $\mathcal{G}(T)$ does not provide a groupoid model for $C^*_r(T)$. Since the unit group of T is trivial, by Proposition 2.16 we nevertheless have

$$C^*_r(T) \cong C^*_{\mathrm{ess}}(\mathcal{G}(T)).$$

Remark 4.5. An argument similar to the proof of Lemma 3.8 shows that if $h \in I_\ell(T)$ fixes x_k and y_m for some k and m and satisfies $\operatorname{dom} h \supset X = \bigcup_n x_nT \cup \bigcup_n y_nT$, then $h = p_T$ or $h = p_X$. Using this it is not difficult to show that $\mathcal{G}_P(T) = \mathcal{G}(T)$.

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