Topics in Computational Finance: The Barndorff-Nielsen & Shephard Stochastic Volatility Model

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Oslo, March 2007 Martin Groth

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AN INTRODUCTORY NOTE

In an expanding financial world it is necessary to analyse and understand the methods used and the models they rely on. For an investor to stay competitive and safeguard against failure the need for thorough and careful examination from a mathematical perspective is immense. A pure mathematical dissection is of considerable value, but with more complicated models, which are increasingly involved and technically demanding, the search for an analytical answer to pricing and hedging problems could be futile and the only possibility is to resort to numerics.

This thesis is centered around numerical methods applied to problems in mathematical finance. While being in the same field, the problems differ substantially from each other. The articles cover many of the big questions in finance: option pricing, hedging, price sensitivities, Value-at-Risk, implied volatility and risk aversion. The numerical methods are varying; finite difference methods for partial differential equations, Monte Carlo and quasi-Monte Carlo, the fast Fourier transform and numerical search methods are all used where applicable. This is not a thesis where new theory is developed in numerical mathematics and neither in finance, but rather in the borderland in between, in applied mathematical finance. It adds to the understanding of stock price models with jump processes, in particular the Barndorff-Nielsen and Shephard stochastic volatility model.

The purpose of the introductory chapter is to give a brief presentation of the theory behind the material presented in the articles. Even though the aim was to make it self-contained it requires basic knowledge of finance theory, stochastic analysis and also some background in mathematical analysis. Numerous references are given for those interested in the original research in mathematical finance. Interested readers seeking a way into the subject should consider the following books: Björk [21] *Arbitrage theory in continuous time*, Cont and Tankov [35] *Financial modelling with jump processes*, Glasserman [64] *Monte Carlo methods in financial engineering*, Hull [78] *Options, futures and other derivatives* and Wilmott, Dewynne and Howison [116] *Option pricing*, which together well cover the material needed to indulge in this thesis.

1. Lévy processes

Lévy processes have a central role in this thesis, although the focus is not on the processes themselves, but as building blocks. The financial models studied are driven by Lévy processes and to understand how they are used some background material is needed. This said, the Lévy processes are not studied from a theoretical point; no new properties are derived, nor are any new insights about Lévy processes brought to the table. In a sense this thesis is about the use of Lévy processes in mathematical finance, from a computational and applied view. For the coherence of the introduction, a brief summary of the theory needed to understand Lévy processes and how they are treated in the sequels is provided here.

A Lévy process is a stochastic process with stationary independent increments. That is, pick a series of times with a fixed time step, measure the process at those times and calculate the change between times, then these numbers will have the same distribution and be independent of each other. To be formal, a Lévy process $\{X_t, t > 0\}$ is a càdlàg process (*i.e.* right continuous with left limits) with $X_0 = 0$, *a.s.* having the properties

- For any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
- The distribution of $X_{s+t} - X_s$ does not depend on s.
- X_t is stochastic continuous, *i.e.* $\forall\, \epsilon > 0, t \geq 0, \lim_{s \to t} \mathbb{P}(|X_s - X_t| > \epsilon) = 0$.

Stochastic continuity is not the same as the sample paths being continuous. A Lévy process may have jumps in the path but the probability that the process exhibit a jump at any given time is zero.

If μ is a probability measure on $\mathbb{R}^d$ and then μ^n be the n-fold convolution with itself

$$\mu^n = \mu * \cdots * \mu.$$

A probability measure μ is said to be *infinitely divisible* if for any positive integer n there is a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^n$. This implies that for any infinitely divisible distribution μ and any positive integer n there exist n random variables such that the sum of the variables have distribution μ. This resembles quite a lot the first point in the definition of a Lévy process and indeed, denoting the distribution of X by P_X the following result holds true

Theorem 1.1 (Theorem 7.10 Sato [110])**.**

- *If $\{X_t, t \geq 0\}$ is a Lévy process in law on $\mathbb{R}^d$ then, for any $t \geq 0$, P_{X_t} is infinitely divisible and, letting $P_{X_1} = \mu$, we have $P_{X_t} = \mu^t$.*
- *Conversely, if μ is an infinitely divisible distribution on $\mathbb{R}^d$, then there is a Lévy process in law $\{X_t, t \geq 0\}$ such that $P_{X_1} = \mu$.*
- *If $\{X_t\}$ and $\{X'_t\}$ are Lévy processes in law on $\mathbb{R}^d$ such that $P_{X_t} = P_{X'_t}$ then $\{X_t\}$ and $\{X'_t\}$ are identical in law.*

Here a *Lévy process in law* is defined similar to a Lévy process but without the càdlàg property. Examples of distributions which are infinitely divisible includes the Gaussian, Cauchy, Poisson, compound Poisson, exponential, inverse Gaussian, normal inverse Gaussian and the generalised version of the last two.

Letting $\langle x, y \rangle$ denote the inner product on $\mathbb{R}^d$, the characteristic function of a Lévy process can be written as

$$\mathbb{E}\left[e^{i\langle z, X_t\rangle}\right] = e^{t\phi(z)}, \quad z \in \mathbb{R}^d.$$

The continuous function ϕ, called the characteristic exponent, is the cumulant generating function of X_1. The dependence on t is linear so the law of X_t is determined by the knowledge of the law of X_1. The form of the characteristic exponent for all infinitely divisible distributions is given by the *Lévy-Khintchine representation*, an important result about Lévy processes. Given a Lévy process X_t on $\mathbb{R}^d$ then ϕ has the representation

$$\phi(z) = -\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}\langle z, x\rangle} - 1 - \mathrm{i}\langle z, x\rangle 1_{|x|\leq 1}(x)\right) \nu(\mathrm{d}x), \quad z \in \mathbb{R}^d, \tag{1.1}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ a vector and ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x| \wedge 1)\nu(\mathrm{d}x) < \infty.$$

The three parts (A, ν, γ) are called the generating triplet for X_t and are uniquely determined by the distribution of X_1. A is called the *Gaussian covariance matrix* and

ν the *Lévy measure*. For a subset $A \in \mathcal{B}(\mathbb{R}^d)$ the Lévy measure $\nu(A)$ can be interpreted as the expected number of jumps with jump size in A per unit time. The triplet is unique, however the representation (1.1) is not. Other functions than $1_{|x|\leq 1}$ can be used to truncate the larger jumps in the integrand. This effects γ so one should clearly state the truncating function considered if it differs from the one in (1.1).

The second important result is the Lévy-Itô decomposition which says that a Lévy process can be expressed as the sum of two independent parts, a continuous part and a part expressible as a compensated sum of independent jumps. Here the version from Cont and Tankov [35] is given, which is slightly more accessible than Sato's version. To begin with, observe that it is possible to define a measure on $[0,\infty) \times \mathbb{R}^d$ counting the jumps of X_t in $[t_1, t_2]$ with jump size B

$$J_X([t_1,t_2] \times B) = \#\{(t \in [t_1,t_2], X_t - X_{t-}) \in B\}$$

for any measurable set $[t_1,t_2] \times B \subset [0,\infty) \times \mathbb{R}^d$. It will be required that the jump measure J_X of a Lévy process X is a Poisson random measure, see Cont and Tankov for the definition. The Lévy-Itô decomposition then states

Theorem 1.2 (Prop. 3.7 Cont and Tankov [35])**.** *For a Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$, where X_1 has the generating triplet (A, ν, γ), the following holds*

- *ν is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ and verifies*
$$\int_{|x|\leq 1} |x|^2 \nu(\mathrm{d}x) < \infty, \qquad \int_{|x|\geq 1} \nu(\mathrm{d}x) < \infty.$$
- *The jump measure of X_t, denoted by J_X, is a Poisson random measure on $[0,\infty) \times \mathbb{R}^d$ with intensity measure $\nu(\mathrm{d}x)\mathrm{d}t$.*
- *There exists a d-dimensional Brownian motion $\{B_t, t \geq 0\}$ with covariance matrix A such that*
$$X_t = \gamma t + B_t + X_t^l + \lim_{\epsilon \downarrow 0} \widetilde{X}_t^\epsilon, \quad \textit{where}$$
$$X_t^l = \int_{|x|\geq 1, s\in[0,t]} x J_X(\mathrm{d}s \times \mathrm{d}x) \quad \textit{and}$$
$$\widetilde{X}_t^\epsilon = \int_{\epsilon \leq |x| < 1, s\in[0,t]} x\{J_X(\mathrm{d}s \times \mathrm{d}x) - \nu(\mathrm{d}x)\mathrm{d}s\}.$$

All parts of the decomposition are independent and the convergence is almost sure and uniform in t on any bounded interval.

The first two terms of the decomposition together form a Gaussian Lévy process, which is the continuous part. The two last terms form the discontinuous jump part. The condition that the Lévy measure has finite mass for $|x| \geq 1$ makes X_t^l into a compound Poisson process with almost surely finite number of jumps. The last term is a compensated jump integral for the small jumps, enabling processes with infinite jump activity, *i.e.* processes with infinitely many small jumps. It can be noticed that without passing to the limit, the last term will also form a compound Poisson process. An arbitrary Lévy process can therefore be approximated by a jump-diffusion, the sum of a Brownian motion with drift and a compound Poisson process.

The last concept needed to be defined is a subordinator, a Lévy process with almost surely nondecreasing sample paths. Hence a subordinator $\{X_t, t \geq 0\}$ is increasing

such that $X_t \geq 0$ *a.s.* for every $t > 0$. For a Lévy process on $\mathbb{R}$ to be increasing the characteristic triplet needs to satisfy $A = 0$,

$$\int_{(-\infty,0)} \nu(\mathrm{d}x) = 0, \qquad \int_{(0,1]} x\nu(\mathrm{d}x) < \infty$$

and

$$\gamma_0 := \gamma - \int_{|x|\leq 1} x\nu(\mathrm{d}x) \geq 0.$$

The variable γ_0 is called the drift and the integral in the definition of γ_0 is finite, otherwise there would be infinitely many small jumps with positive jump size at any time. Hence a subordinator always has finite variation (no Brownian motion and finite jump activity).

2. Arbitrage pricing and Martingale measures

In order to trade with claims there has to be a way to attribute a price in a manner excluding possibilities to make money out of nothing. To make a profit without risking any loss is called *arbitrage* and in a working theory for financial derivatives it is necessary that there are no arbitrage opportunities. The idea of arbitrage is fundamental in finance and the quest is to find conditions such that the market model is arbitrage-free. As will be showed later, absence of arbitrage is closely connected to the existence of *equivalent martingale measures* which will make the (discounted) price process of a claim into a martingale, concepts which will be defined below.

In the Black & Scholes framework martingale pricing comes naturally from arbitrage considerations but for more complicated models this is not the case. The martingale approach started with Harrison and Kreps [70] and Harrison and Pliska [71]. They originally considered trading strategies which only allowed for *simple predictable integrands*. This constraint ruled out unfavorable trading strategies such as the "doubling strategy" but was still too restrictive. Delbaen and Schachermeyer [42] replaced *No arbitrage* with the concept of *No Free Lunch with Vanishing Risk* (NFLVR). The difference between the concepts is a question of functional analysis definitions, *i.e.* choosing space to work in, and is left to the reader to find out from the references. Instead of considering only simple predictable integrands the NFLVR-concept opened up for the possibility to include a larger group of strategies, restricted to be *admissible*.

Consider a market consisting of n traded risky assets whose evolutions are strictly positive and described by a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$. A real adapted process $\{X_t, t \geq 0\}$ is a *martingale* if for all t

$$\begin{aligned} &\mathbb{E}[|X_t|] < \infty, \\ &\mathbb{E}[X_t|\mathcal{F}_s] = X_s \quad \forall \quad 0 \leq s \leq t \leq \infty. \end{aligned} \tag{2.1}$$

If there exists a nondecreasing sequence of stopping times $\{\tau_k\}$ of the filtration $\{\mathcal{F}_t\}$ such that $X_{t\wedge\tau_k}$ is a martingale for all k, then X_t is called a *local martingale*.

Let X denote a contingent claim with maturity T, referred to as a T-claim. Assume that the risky asset prices $S(t) = [S_0(t) \cdots S_n(t)]$ develop according to some underlying stochastics. In the Black & Scholes market the assets follow stochastic differential equations driven by Wiener processes, but for the general martingale pricing the stochastics are allowed to be semimartingales, see Protter [105]. S_0 is often thought of as the risk-free asset in the market, a bank account with short rate r. In the general theory the only assumption is that $S_0(t) > 0$ $\mathbb{P} - a.s.$ for all $t \geq 0$.

Instead of looking at the price vector process $S(t)$, consider the normalised market with price vector process

$$Z(t) = [Z_1(t), \dots, Z_n(t)] = \left[\frac{S_1(t)}{S_0(t)}, \dots, \frac{S_n(t)}{S_0(t)}\right]. \tag{2.2}$$

Here S_0 is used as the numeraire and in the Z-economy $Z_0(t) = 1$ is a risk-free asset, a money account with zero interest rate.

Let $\theta(t) = [\theta_0(t), \dots, \theta_n(t)]$ be a portfolio, where $\theta_i(t)$ represents the number of units held of the ith asset at time t. Since a trading strategy can only depend on the information available at the current time it must be assumed that $\theta(t)$ is adapted (or even predictable). The value of the portfolio at any time t is given by the value process

$$V(t;\theta) = \sum_{i=0}^{n} \theta_i(t) S_i(t).$$

The value process can equally well be defined using the normalised market, giving the *Z-value process*

$$V^Z(t;\theta) = \sum_{i=0}^{n} \theta_i(t) Z_i(t).$$

It is necessary to narrow down the class of strategies to avoid cases such as the doubling strategy. One common way is to require the portfolio to be admissible in the sense that it is limited from below: An adapted process $\theta_Z = [\theta_1, \dots, \theta_n]$ is called *admissible* if there exists a nonnegative real number α such that

$$\int_0^t \theta_Z(u)\, \mathrm{d}Z(u) \geq -\alpha \quad \text{for all} \quad t \in [0, T].$$

A process $\theta(t) = [\theta_0(t)\ \theta_Z(t)]$ is called an *admissible portfolio process* if θ_Z is admissible. The value process should reflect the actual rise and fall of the assets, *i.e.* there is no flow of funds in or out of the portfolio. It should be *self-financing*: An admissible portfolio is said to be Z-self-financing if

$$\mathrm{d}V^Z(t;\theta) = \sum_{i=0}^{n} \theta_i(t)\, \mathrm{d}Z_i(t).$$

The choice of numeraire is not crucial for the concept of self-financing portfolios as it can be proved that a portfolio θ is S-self-financing if and only if it is Z-self-financing. Adding to this, a contingent claim X is said to be *reachable* if there exists a portfolio θ such that $V(T, \theta) = X$. This extends straightforwardly to definitions of S-reachable and Z-reachable claims.

Arbitrage is the possibility to make a positive amount of money while starting with nothing. Such a possibility can not exist over time in a sound market as it will be exploited by investors making a fortune without taking any risk. A mathematical definition of arbitrage can be given using the value function: A self-financing trading strategy $\theta(t)$ is called an *arbitrage* if either

$$\begin{aligned} V(0;\theta) &< 0, \\ \mathbb{P}(V(T;\theta) \geq 0) &= 1, \end{aligned}$$

or

$$\begin{aligned} V(0;\theta) &= 0, \\ \mathbb{P}(V(T;\theta) \geq 0) &= 1, \\ \mathbb{P}(V(T;\theta) > 0) &> 0. \end{aligned}$$

The concept of arbitrage-free markets is closely related to the existence of probability measures under which asset dynamics of the normalised market are martingales. Two separate probability measures $\mathbb{P}$ and $\mathbb{Q}$ on a measurable space $(X, \mathcal{F})$ are said to be *equivalent* ($\sim$) if they define the same set of events as impossible, *i.e.*

$$\mathbb{P} \sim \mathbb{Q} : \forall A \in \mathcal{F} \quad \mathbb{Q}(A) = 0 \Longleftrightarrow \mathbb{P}(A) = 0.$$

This is important since it will be shown that pricing takes place under measures equivalent to the historical measure. If this is not the case events which are impossible under the pricing measure could have positive probability under the historical measure, which could lead to arbitrage.

A probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ is called an *equivalent martingale measure* for the market model given by $Z(t)$, the numeraire S_0 and the time interval $[0, T]$ if it has the following properties:

- $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$.
- All price processes $Z_0, Z_1, \ldots, Z_n$ are martingales under $\mathbb{Q}$ on the time interval $[0, T]$.

If $Z_0, Z_1, \ldots, Z_n$ are local martingales under $\mathbb{Q}$ it is called a *local martingale measure*.

Theorem 2.1 (First fundamental theorem of asset pricing)**.** *Consider the market model $S_0, S_1, \ldots, S_n$ where it is assumed that $S_0(t) > 0$ P-a.s. for all $t \geq 0$. Assume furthermore that $S_0, S_1, \ldots, S_n$ are locally bounded. Then the following conditions are equivalent:*

- *The model satisfies NFLVR.*
- *There exists a measure $\mathbb{Q} \sim \mathbb{P}$ such that the processes $Z_0, Z_1, \ldots, Z_n$ defined in* (2.2) *are local martingales under $\mathbb{Q}$.*

See Delbaen and Schachermeyer [42] for a proof in the case of bounded price processes.

The second fundamental theorem of asset pricing states that, presuming the market is free of arbitrage, then the market is complete, *i.e.* all contingent claims are reachable, if and only if the equivalent martingale measure is unique. Few of the markets studied in this thesis will be complete, and it is questioned whether market completeness is a financially realistic property. Completeness will therefore not play a significant role in the following.

Having a T-claim X, what is a reasonable price process $\Lambda(t; X)$? It is clear from the first fundamental theorem that the price has to be consistent with the market $S(t)$ and that including the claim in the market can not give rise to any arbitrage possibilities. For the extended market $\{\Lambda(t; X), S_0, \ldots, S_n\}$ there must then exist a local martingale measure $\mathbb{Q}$. Using the definition of a martingale (2.1), the first fundamental theorem states that the price process divided by the numeraire is a martingale, hence

$$\frac{\Lambda(t;X)}{S_0(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\Lambda(T;X)}{S_0(T)}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right].$$

This gives the result:

Theorem 2.2 (General pricing formula)**.** *The arbitrage-free price process for the T-claim X is given by*

$$\Lambda(t;X) = S_0(t)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right],$$

where $\mathbb{Q}$ is a local martingale measure for the a priori given market $S_0, S_1, \ldots, S_n$ with S_0 as the numeraire.

Assuming that there exists a short rate $r(t)$, the price process is given by the risk neutral pricing formula

$$\Lambda(t;X) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,\mathrm{d}s}\,X\middle|\mathcal{F}_t\right], \tag{2.3}$$

with the money account $S_0(t) = S_0(0)\,e^{\int_0^t r(s)\,\mathrm{d}s}$ as the numeraire. Left to determine are the claim X and the dynamics of the underlying assets, and some way to sample paths for the assets. Below is discussed different approaches proposed to model the dynamics of asset prices; models driven by Lévy processes and stochastic volatility models.

This concise exposition of the theory for derivative pricing is on no account a full treatment of the subject; that is a task left to writers of textbooks such as Benth [9], Björk [21], Duffie [46] or Musiela and Rutkowski [94]. Those interested in reading some of the original work in the field of arbitrage pricing or seeking proofs of the theory should look up the following articles: Black and Scholes [22], Delbaen and Schachermeyer [42, 43], Harrison and Kreps [70], Harrison and Pliska [71] and Merton [92].

2.1. **Equivalent martingale measures.** The first fundamental theorem of asset pricing states that there is a unique correspondence between the existence of an equivalent martingale measure and the absence of arbitrage. If the market is complete, like the Black & Scholes market, the martingale measure is unique. In incomplete markets this is not true, instead there exists a range of different martingale measures which are all equivalent to the historical measure. To price a contingent claim involves choosing under which of these martingale measures to work. Market incompleteness arises of several reasons; adding transaction costs, jumps in the asset dynamics or stochastic volatility, all of these make a market incomplete. If the market model contains a Lévy process with jumps, the class of equivalent martingale measures is surprisingly large, the precise formulation of equivalence of measures for Lévy processes is found in Sato [110]. It turns out that there is a large freedom to change the Lévy measure but unless there is a diffusion component present the drift can not be changed. In general one also has more freedom to change the distribution of the large jumps than the small ones.

Presuming the market is incomplete one must decide what equivalent martingale measure to use, for Lévy processes there exist several different approaches. Raible [106] considers exponential Lévy models and suggests using the Esscher transform. This is an analogue to the drift change for the geometric Brownian motion. If X is a Lévy process, under suitable regularity conditions, the Esscher transform is a change of measure from the historical measure $\mathbb{P}$ to a local equivalent measure $\mathbb{Q}$ with transform density process

$$Z_t = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{e^{\theta X_t}}{\mathbb{E}\left[e^{\theta X_t}\right]},$$

where $\theta \in \mathbb{R}$. Let r be the interest rate and assume that the Lévy process is neither almost surely decreasing nor almost surely increasing. Then there exists a real constant θ which, through the Esscher transform, ensures the existence of a locally equivalent measure $\mathbb{Q}$ under which the discounted asset price $\exp(-rt)S_t = S_0 \exp(X_t)$ is a martingale. Clearly the market will be free of arbitrage since $\mathbb{Q}$ is an equivalent martingale measure.

Another possibility is to choose the equivalent martingale measure $\mathbb{Q}$ that is closest to the historical measure $\mathbb{P}$ in some sense. Examples of ways to measure the distance from $\mathbb{P}$ are the *quadratic distance*

$$\mathbb{E}^{\mathbb{P}}\left[\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^2\right]$$

and the *relative entropy*

$$H(\mathbb{Q},\mathbb{P}) = \begin{cases} \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\ln\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] & \mathbb{Q} \ll \mathbb{P}, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

The measure $\mathbb{Q}^{\mathrm{ME}}$ which minimise the distance in the entropy sense is called the *minimal entropy martingale measure* (MEMM), *i.e.*

$$H(\mathbb{Q}^{\mathrm{ME}},\mathbb{P}) = \min_{\mathbb{Q}\in\mathcal{M}} H(\mathbb{Q},\mathbb{P})$$

where $\mathcal{M}$ is the set of equivalent martingale measures. Cont and Tankov [35] claim this can be interpreted in an information theoretic setting: minimising relative entropy corresponds to choosing a martingale measure by adding the least amount of information to the prior model. Frittelli [62] studies the minimal entropy martingale measure in a general context of incomplete markets and proves that if there exists an equivalent martingale measure $\mathbb{Q}$ with $H(\mathbb{Q},\mathbb{P}) < \infty$, then $\mathbb{Q}^{\mathrm{ME}}$ exists, is unique and is equivalent to $\mathbb{P}$. A similar result is proved in Grandits and Rheinländer [67], using the same assumption as Frittelli: If there exists a measure $\mathbb{Q} \in \mathcal{M}$ s.t. $H(\mathbb{Q},\mathbb{P}) < \infty$, the density of $\mathbb{Q}^{\mathrm{ME}}$ can be written as

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = c\exp\left(\int_0^T \eta_t \,\mathrm{d}S_t\right) \tag{2.5}$$

where c is a constant and η is a predictable process such that the integral is a $\mathbb{Q}^{\mathrm{ME}}$-martingale, *i.e.*

$$\mathbb{E}^{\mathbb{Q}^{\mathrm{ME}}}\left[\int_0^T \eta_t \,\mathrm{d}S_t\right] = 0.$$

There is not a unique measure with the representation (2.5) so the opposite need not be true; a measure with this representation is not necessarily $\mathbb{Q}^{\mathrm{ME}}$. To verify that a measure with this form is indeed the minimal entropy martingale measure an additional verification result discussed in Rheinländer [107] is needed.

Two different methods to find $\mathbb{Q}^{\mathrm{ME}}$ in a stochastic volatility model are presented by Benth and Karlsen [15] and Rheinländer [107], the first via a solution of a semi-linear partial differential equation and the second by a duality method. The latter is stated in a general semimartingale setting with examples using the Stein-Stein model. The specific form of the MEMM in the Barndorff-Nielsen and Shephard model is discussed in connection with the introduction of the model in Section 4.3. The minimal entropy martingale measure is also studied in Fujiwara and Miyahara [63] for exponential Lévy

processes, Benth and Meyer-Brandis [17] and Hobson [75] for stochastic volatility models. The minimal entropy measure is closely related to utility indifference pricing in the risk aversion limit, see Section 3.

3. Utility indifference pricing

There is something strikingly intuitive about the concept of arbitrage pricing in the Black & Scholes market. Taking positions in the option and the underlying asset, forming a locally riskless portfolio, determines the price if no arbitrage exists in the market. A short, non-technical argument gives the main idea in a few lines. It is just as easy to understand why the concept fails. The possibility to make a perfect replication of the option by trading in the underlying is of fundamental importance in arbitrage pricing. In the Black & Scholes market there are several conditions to ensure this is possible, which all are simplifications of the real world. The theory assumes that there are no transaction costs, continuous trading is possible and that any fraction of a stock can be bought. Without these assumptions a perfect hedge is not achievable, and arbitrage pricing fails. It is a bit paradoxical that only the contracts possible to replicate perfectly are possible to price, something which makes them redundant in a sense. Market completeness implies that all options are replicable, and hence redundant. It is argued that the mere fact that options are traded implies that market completeness is not a financially justified property.

In an incomplete market there is no longer a single arbitrage-free price, neither a unique perfect hedge of an option, and therefore it is an unavoidable risk associated with trading. Instead of trying to find the one arbitrage-free price one tries to measure the risk to hedge and price the claim. Other strategies are needed in incomplete markets, such as superhedging [54], quadratic hedging, both mean-variance [23] and (local) risk-minimisation [58], and utility indifference pricing [76]. Superhedging is a conservative approach that tries to eliminate all risk associated with the option, quadratic hedging is a strategy minimising some quadratic function of the hedging error while utility indifference pricing, which is discussed below, builds on the old idea of expected utility maximisation.

Hodges and Neuberger [76] study a Black & Scholes market with transaction costs. By removing the assumption that the market is friction-free it is made incomplete, so instead of arbitrage pricing they suggest an approach based on utility indifference. Let the market consist of a risky asset S_t and a bond R_t and let the investor have the possibility to issue an option on the risky asset. Hodges and Neuberger's main idea is that the utility indifference price of a claim is the price at which the investor is indifferent between entering into the market directly, or to first issue a claim and then enter into the market with the incremented wealth. Let the investor have an initial wealth x at time t and a utility function $u(x)$, a concave increasing function with $u(0) = 0$ that depends on a risk aversion parameter γ. Assuming that $\mathcal{A}$ is the set of *admissible trading strategies* then $\pi_t \in \mathcal{A}$ is the fraction of the wealth invested in the risky asset at time t. The value function when no claim is issued can be defined as

$$V^0(t, x, S) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}\left[u(X_T^\pi)\right]$$

where X_T^π is the wealth dynamics at time T given π. The form of the wealth dynamics depends on the specific model chosen. If the investor issues a claim with payoff function

$f(S_t)$ then the value function will be

$$V^c(t,x,S) = \sup_{\pi_t \in \mathcal{A}} \mathbb{E}\left[u(X_T^\pi - f(S_T))\right].$$

The utility indifference price defined by Hodges and Neuberger for a given risk aversion γ is the price $\Lambda^{(\gamma)}$ s.t.

$$V^0(t,x,S) = V^c(t,x+\Lambda^{(\gamma)},S).$$

Then $\Lambda^{(\gamma)}$ is the price which provides the same utility in both cases: the investor is indifferent whether to issue a claim or not.

The utility indifference price depends for most choices of the utility function on the initial wealth. Two investors with the same utility function but different amounts to invest could therefore disagree on the price of an option. The important exception is the exponential utility function,

$$u(x) = 1 - \exp(-\gamma x)$$

leading to a price independent of the initial wealth. The exponential utility has been extensively studied because of the connection between utility indifference pricing and certain hedging and pricing strategies. Using exponential utility and letting $\gamma \to \infty$ the utility indifference price will tend to the superhedging price, which in general is considered to be too high. More interesting is letting $\gamma \to 0$. Several authors [6, 41, 55, 113] have noticed that there is a duality between the utility indifference price in the risk aversion limit and the price under the minimal entropy martingale measure. Assume the price process S_t is a semimartingale and X_t^π the wealth process with self-financing strategy π and initial wealth x. For a contingent claim with payoff $f(S_T)$ one tries to maximise the utility over all π in a suitable class Θ

$$\max_{\pi \in \Theta} \mathbb{E}^{\mathbb{P}}\left[1 - \exp(-\gamma(X_T^\pi - f(S_T)))\right].$$

In a general semimartingale framework Delbaen *et.al.*[41] gives different choices of Θ and shows that there is a dual problem where the relative entropy minus a correction is minimised

$$\min_{\mathbb{Q} \in \mathcal{M}} 1 - \exp\left(-H(\mathbb{Q},\mathbb{P}) - \gamma x + \gamma \mathbb{E}^{\mathbb{Q}}\left[f(S_T)\right]\right)$$

over a suitable class $\mathcal{M}$ of local martingale measures $\mathbb{Q}$ for S_t. Hence

$$\sup_{\pi \in \Theta} \mathbb{E}\left[1 - \exp(-\gamma(X_T^\pi - f(S_T)))\right] = 1 - \exp\left(\inf_{\mathbb{Q} \in \mathcal{M}} \left(-H(\mathbb{Q},\mathbb{P}) - \gamma x + \gamma \mathbb{E}^{\mathbb{Q}}\left[f(S_T)\right]\right)\right)$$

for $\gamma > 0$. Becherer [6] shows that when taking the risk aversion limit $\gamma \to 0$, the utility optimisation problem coincides with pricing under the minimal entropy martingale measure. That is,

$$\Lambda^{(\gamma)} = \sup_{\mathbb{Q} \in \mathcal{M}} \left\{ \mathbb{E}^{\mathbb{Q}}[f(S_T)] - \frac{1}{\gamma}\left(H(\mathbb{Q},\mathbb{P}) - H(\mathbb{Q}^{\mathrm{ME}},\mathbb{P})\right) \right\}$$

and taking the limit it holds that

$$\lim_{\gamma \downarrow 0} \Lambda^{(\gamma)} = \mathbb{E}^{\mathbb{Q}^{\mathrm{ME}}}[f(S_T)].$$

The measure $\mathbb{Q}^{\mathrm{ME}}$ for a general continuous semimartingale is derived through duality in the method developed by Rheinländer [107], as discussed in Section 2.1. For the stochastic volatility market proposed by Barndorff-Nielsen and Shephard, see section 4.3, the connection between $\mathbb{Q}^{\mathrm{ME}}$ and the risk-aversion limit of the utility indifference price under exponential utility appears in papers by Benth and Meyer-Brandis [17]

and Rheinländer and Steiger [108]. In the first paper a representation of the minimal entropy martingale measure is developed for the Barndorff-Nielsen and Shephard model without leverage, which is generalised in the second paper. For this model the representation of the utility indifference price as the solution of a semi-linear partial differential equation is also discussed in Section 4.3.

4. Exponential Lévy and Stochastic volatility models

Even before the Chicago Board Options Exchange opened as the first stock option exchange there was an interest in modelling the erratical behaviour of the stock movement in order to price derivatives. The pioneer was Louis Bachelier with his thesis from 1900, followed by Samuelson [109] who introduced the geometric Brownian motion, and Mandelbrot [89] who preferred "L-stable" probability laws and multifractals. Not until Fisher Black and Myron Scholes [22] together with Robert C. Merton [92] developed the theory nowadays bearing the names of the two first mentioned, there existed a consistent way to handle options. Black and Scholes built on Samuelson's work, where the stock price dynamics is a geometric Brownian motion:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t$$

adding a risk-free money account with rate of return r. The Black & Scholes framework has been the industry standard, mainly because it is simple, clear and easy to use. Explicit formulas exist for the price of vanilla contracts and, because of the widespread use, the model is well understood. However, the Black & Scholes model has some drawbacks noticed by market traders throughout the years. Apart from the simplifications made with regards to transaction costs, short selling and dividends, one major disadvantage is the Black & Scholes theory's inability to explain the volatility smile.

It was well known before the 1987 crash that the implied Black & Scholes volatilities of market prices gave rise to a smile, i.e. the volatilities implied by the Black & Scholes formula were higher for in-the-money and out-of-the-money options than options with strikes around the spot price. Empirical work clearly show that the implied volatilities of market prices are not constant but vary with strike price and time to maturity. After the 1987 crash a more frequent appearance of skewness was noticed in the implied volatilities, resulting in more of a smirk or sneer than a smile, see Dumas *et.al.*[47]. The common explanation is that investors became more aware of the risk for large downward movements in the market. Neither the smile nor the smirk are possible to explain within the Black & Scholes framework, as both indicate that the market emphasises the risk associated with large stock price movements more than the theory does. Empirical work also clearly indicates that stock price log-returns on a short time horizon exhibit a distribution with heavier tails than expected from the Black & Scholes model, and also jumps in the paths.

A stream of new models have been proposed to replace the Black & Scholes model, all of them with the intention to model the market prices, and hence the implied volatilities, in a better way. Depending on the focus of the research different aspects have been considered important to capture in the modelling: the heavy tails of the returns, the jumps in the paths of asset prices, volatility clustering and/or dependence structures. Shortly after Black and Scholes proposed their model Merton [93] suggested to add a jump term in the stock price dynamics to incorporate jumps with unpriced

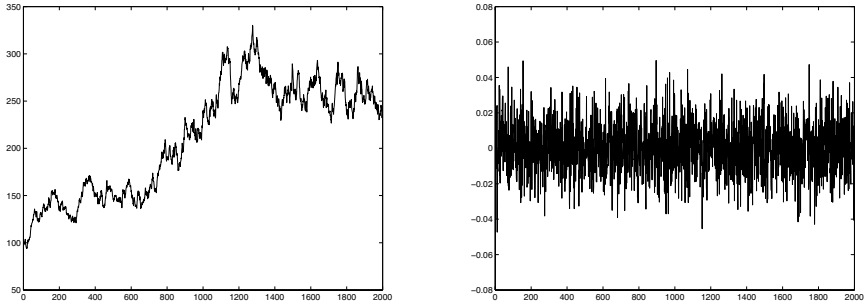

FIGURE 1. *Left:* Stock price path from the Black & Scholes model with mean $6.4 * 10^{-4}$ and variance $2.21 * 10^{-4}$. The mean and variance are equal to the mean and variance of the Lévy process used in Benth, Groth and Kettler [11]. *Right:* The log-marginal returns from the stock price. The use of a Brownian motion gives marginal returns being normal distributed.

risk:

$$S_t = S_0 \exp\left[\mu t + \sigma B_t + \sum_{i=0}^{N_t} Y_i\right],$$

where N_t is a Poisson process with intensity λ independent of the Brownian motion B_t and $Y_i \sim N(\alpha, \delta^2)$ are i.i.d. random variables independent from B_t and N_t. The pricing approach Merton devises assumes that the risk associated with the jumps is possible to diversify away and that hedging only takes the average effect of jumps into account. Simple as it is, the assumption that the individual jumps can be ignored because the investor diversifies leaves the position exposed to the jump risk, which in many cases is an unwanted situation.

Three decades later two large classes of models can be distinguished from the literature; Firstly models built on replacing the geometric Brownian motion with some other exponential model, lately a lot of research has been done on exponential Lévy models. Secondly stochastic volatility models, where the constant volatility is replaced by some stochastic process. A third approach exists, the *local volatility models*, where the volatility depends on the price and time through a deterministic function

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma(t, S_t) S_t \,\mathrm{d}W_t.$$

Local volatility models and fitting of the local volatility surface will not be discussed further, the interested reader finds more information in Derman and Kani [45] and Dupire [49].

4.1. **Exponential Lévy models.** Adding jumps can be accomplished by replacing the Brownian motion with a Lévy process, so called *exponential Lévy models*

$$S_t = S_0 \exp(\mu t + L_t),$$

where L_t is a Lévy process with characteristic triplet (σ^2, ν, γ). An equivalent approach is to write down the dynamics directly

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}L_t.$$

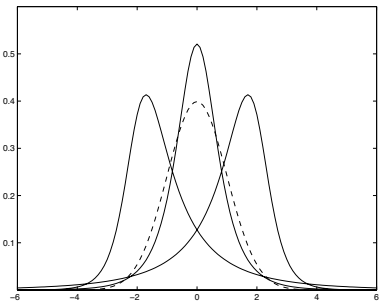

FIGURE 2. The normal inverse Gaussian density with three different parameter sets, $(1, 0.75, -2, 1), (1, 0, 0, 1)$ and $(1, -0.75, 2, 1)$. The dashed line is the standard normal distribution density.

Exponential Lévy models can be built with marginal log-returns in a range of different distributions, with heavier tails to better fit log-return data. This is actually what Merton did, with a jump-diffusion process as the driving noise. Models built around Lévy processes goes back to Mandelbrot in the 1960's but resurged in the late 1990s. Madan *et.al.*[87, 88] used the variance-gamma process, Carr *et.al.*[27, 28] the *CGMY*-process, a subclass of tempered stable processes, Barndorff-Nielsen [3] introduced the *normal inverse Gaussian process* while the use of the *hyperbolic Lévy process* was proposed by Eberlein and Keller [51]. The latter two are both subclasses of the family of generalised hyperbolic Lévy processes, for more information about applications to finance see [50, 52, 53, 104, 106].

The class of hyperbolic Lévy processes, especially the normal inverse Gaussian Lévy process, requires some special attention. Beginning with the inverse Gaussian process $IG(\delta, \gamma)$, a subordinator, having probability density

$$p^{\mathrm{IG}}(x; \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} x^{-3/2} \exp\left\{ -\frac{\gamma^2}{2x} \left(x + \frac{\delta}{\gamma} \right)^2 \right\}, \quad x > 0.$$

One way to interpret $p^{\mathrm{IG}}(x; \delta, \gamma)$ is as the distribution of the time it takes for a Brownian motion to reach a fixed distance. The mean and variance of an $IG(\delta, \gamma)$-distribution are δ/γ and δ/γ^3. The distribution in itself is interesting because it is one possible choice for the stationary distribution of the volatility process in the Barndorff-Nielsen and Shephard model below. The IG-Lévy process is a subordinator, a process with nondecreasing paths. As a such it can be used to stochastically time change other processes, *i.e. subordinate* other processes. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Lévy process $\{X_t, t \geq 0\}$ with cumulant generating function $\phi(u)$. If $\{S_t, t \geq 0\}$ is a subordinator with Laplace exponent $l(u)$ then the process $\{Y_t, t \geq 0\}$ defined by $Y(t, \omega) = X(S(t, \omega), \omega)$ for each $\omega \in \Omega$ is a Lévy process with characteristic function

$$\mathbb{E}\left[e^{iuY_t}\right] = e^{tl(\phi(u))}.$$

The process Y_t is said to be subordinate to X_t and in effect S_t is used to change the clock of X_t.

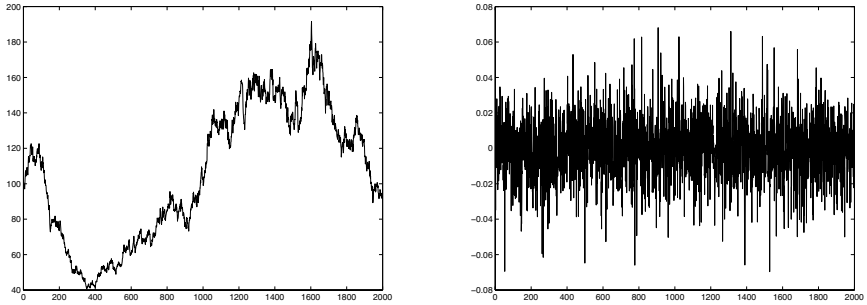

FIGURE 3. *Left:* Stock price path from an exponential Lévy model with the normal inverse Gaussian Lévy process having parameters $\alpha = 136.29, \beta = -15.197, \delta = 0.0295, \mu = 0.00395$. The parameter set is used in Benth, Groth and Kettler [11]. *Right:* The log-marginal returns from the stock price. The use of the normal inverse Gaussian Lévy process gives marginal returns with a more peaked look than expected from the normal distribution due to the heavier tails.

Using the inverse Gaussian subordinator to time change a Brownian motion results in the *normal inverse Gaussian (NIG)* process. The NIG distribution was proposed by Barndorff-Nielsen [2] in the context of wind-borne sand and is a normal variance-mean mixture distribution. If $\sigma^2 \sim IG(\delta, \gamma)$ and $\epsilon \sim N(0,1)$ then $x = \mu + \beta\sigma^2 + \sigma\epsilon$ have a $NIG(\alpha, \beta, \mu, \delta)$ distribution with density function

$$p^{\text{NIG}}(x; \alpha, \beta, \mu, \delta) = \frac{\delta\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{K_1(\alpha q(x - \mu))}{q(x - \mu)}$$

where

$$q(x) = \sqrt{\delta^2 + x^2}$$

and

$$x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \delta > 0, \quad 0 \leq |\beta| \leq \alpha.$$

K_1 is the modified Bessel function of third kind with index 1 and α is given as $\alpha = \sqrt{\gamma^2 + \beta^2}$. The parameters of the distribution have interpretations with the shape of the density: increasing α gives a steeper density, increasing β gives an increasingly asymmetric distribution, δ scales the distribution and μ translates it, see Figure 2. The density will be asymmetric unless $\beta = 0$. The moments κ_i of the distribution are easily calculated from the moment generating function with mean and variance given as

$$\kappa_1 = \mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}},$$
$$\kappa_2 = \frac{\delta\alpha^2}{\left(\sqrt{\alpha^2 - \beta^2}\right)^3}.$$

The asymptotic behaviour of the distribution is

$$g(x; \alpha, \beta, \mu, \delta) \sim c|x|^{-3/2} \exp(-\alpha|x| + \beta x) \quad \text{as} \quad x \to \pm\infty$$

giving the distribution semi-heavy tails. The inverse Gaussian distribution can be generalised by adding a parameter λ, resulting in the *generalised inverse Gaussian (GIG)*. A normal mean-variance mixture with GIG gives the family of *generalised hyperbolic (GH)* distributions, of which NIG is a special case. GH distributions are studied by Eberlein and Keller [51] in relation to financial modelling. Figure 3 shows an example path of an exponential NIG-Lévy model and its log-marginal returns, using parameters relevant for daily observed stock prices.

Exponential Lévy models share a considerable part of the quantitative properties observed in asset returns. They make it possible to model heavy or semiheavy (exponential) tails, the increments are independent, there are jumps in the paths and the distributions can be modeled to be asymmetric to capture differences in the behaviour of upward and downward movements. For a model to exhibit marginal returns with these stylised facts it needs to have a distribution of the returns with four parameters: a location parameter, a scale (volatility) parameter, a parameter describing the decay of the tails and an asymmetry parameter for the right and left tail to differ. The family of generalised hyberbolic distributions, including the normal inverse Gaussian distribution, is fulfilling this requirement as shown above. The choice of distribution becomes not a question of one fitting better than another but which one is the easiest to handle for the purpose and in the circumstances considered.

Not all quantitative features of returns are possible to capture with an exponential Lévy model. Volatility clustering and correlation in volatility are observed in the market but not exhibited by exponential Lévy models. It is possible to include these features in a stochastic volatility model as discussed below. However, the presence of heavy tails makes the realised volatility have "stochastic volatility"-like behaviour, with high variability. Nor are exponential Lévy models able to handle leverage effects, an observed correlation between negative price movements and increasing volatility.

As for the Black & Scholes model there is a partial differential equation governing the price of an option in an exponential Lévy model. Let S_t be given by a stock price model of the exponential Lévy type, driven by Lévy process L_t having characteristic triplet (A, ν, γ) under $\mathbb{Q}$. Consider an option with payoff function $f(S_t)$, and assume that the option price can be expressed as a function of the log forward price $X_t = \ln\left(e^{r(T-t)}S_t\right)$. The price of the option under the martingale measure $\mathbb{Q}$ is then

$$\Lambda(x,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[f\left(e^{x+L_{T-t}}\right)\right].$$

Assuming sufficient differentiability conditions of the payoff function and regularity of the Lévy measure the option price satisfies the following integro-partial differential equation

$$\frac{\partial\Lambda}{\partial t} + \gamma\frac{\partial\Lambda}{\partial x} + \frac{A}{2}\frac{\partial^2\Lambda}{\partial x^2} - r\Lambda + \int_{\mathbb{R}}\left(\Lambda(x+z,t) - \Lambda(x,t) - z1_{|z|<1}\frac{\partial\Lambda}{\partial x}\right)\nu(\mathrm{d}z) = 0 \tag{4.1}$$

with $x \in \mathbb{R}, t \in (0,T)$ and terminal condition $\Lambda(x,T) = f(e^x)$. The introduction of the non-local integral term makes the pde harder to solve than the Black & Scholes equation, both analytically and numerically. One can especially notice that if restricting (4.1) to a finite grid the integral term needs to be extended beyond the boundary to make sense. Integro-partial differential equations and other aspects of exponential Lévy models in finance are discussed extensively in Cant and Tankov [35].

4.2. **Stochastic volatility models.** Instead of replacing the Brownian motion as the driving source one could instead add another random process, making the volatility

non-constant:

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma(Y_t)S_t\,\mathrm{d}B_t$$

where B_t is a Brownian motion but σ now is function depending on another stochastic process Y_t, modelling the random volatility. Common driving processes for the volatility are the geometric Brownian motion, the Ornstein-Uhlenbeck process

$$\mathrm{d}Y_t = \alpha(\eta - Y_t)\,\mathrm{d}t + \beta\,\mathrm{d}W_t$$

and the Cox-Ingerson-Ross (CIR) process

$$\mathrm{d}Y_t = \kappa(\eta - Y_t)\,\mathrm{d}t + v\sqrt{Y_t}\,\mathrm{d}W_t.$$

The process W_t is another Brownian motion, correlated or uncorrelated to the Brownian motion in the stock price dynamics. However for the Ornstein-Uhlenbeck process there are also models where the second process is a Lévy process, as shown in the next section.

Introducing stochastic volatility makes it possible to capture volatility clustering and dependence structures, at the same time as the models can replicate implied volatility smiles. Adding a jump term to the price dynamics also make the models realistic on a short-term scale when it comes to jumps in the paths. The drawback is the extra dimension that is added which has the effect that the stock price is no longer a Markov process. Instead it is necessary to consider a two-dimensional process. The complications it means for numerical methods to have a second dimension accounts for a lot of the hesitation shown towards the use of stochastic volatility models. Though, in recent years there has been an increasing interest from practitioners in these models, mainly in the model suggested by Heston [72]. The volatility process in the Heston model is a Cox-Ingersoll-Ross process with a Brownian motion correlated to the Brownian motion driving the stock price, *i.e.*

$$\begin{aligned}\mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sqrt{Y_t}S_t\,\mathrm{d}B_t,\\ \mathrm{d}Y_t &= \kappa(\eta - Y_t)\,\mathrm{d}t + v\sqrt{Y_t}\,\mathrm{d}W_t,\end{aligned}$$

with the correlation between the two Brownian motions given as

$$\mathrm{d}B_t\,\mathrm{d}W_t = \rho\,\mathrm{d}t.$$

A common feature for many of the suggested models is that the volatility process is mean reverting, like the mentioned Cox-Ingersoll-Ross process and Ornstein-Uhlenbeck process. This is thought to be a realistic feature observed in market data, new information perceived by the traders makes the activity jump up suddenly and then revert back towards a steady state.

Assuming that the stochastic volatility model is of the Ornstein-Uhlenbeck class with dynamics

$$\begin{aligned}\mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sigma(Y_t)S_t\,\mathrm{d}B_t,\\ \mathrm{d}Y_t &= \alpha(m - Y_t)\,\mathrm{d}t + \beta\,\mathrm{d}W_t,\end{aligned}$$

for some function $\sigma(y)$, Fouque *et.al.*[59] derive a pricing partial differential equation similar to the Black & Scholes pde. Denoting the instantaneous correlation coefficient between the two Brownian motions by ρ, the price of an European derivative with

payoff function $f(x)$ is given by

$$\begin{aligned}(4.2)\qquad &\frac{\partial\Lambda}{\partial t}+\frac{1}{2}\sigma^2(y)s^2\frac{\partial^2\Lambda}{\partial s^2}+\rho\beta s\sigma(y)\frac{\partial^2\Lambda}{\partial s\partial y}+\frac{1}{2}\beta^2\frac{\partial^2\Lambda}{\partial y^2}\\ &\quad+r\left(s\frac{\partial\Lambda}{\partial s}-\Lambda\right)+\left(\alpha(m-y)-\beta\left(\rho\frac{\mu-r}{\sigma(y)}+\gamma(t,x,y)\sqrt{1-\rho^2}\right)\right)\frac{\partial\Lambda}{\partial y}=0\end{aligned}$$

with the condition $\Lambda(T,x,y)=f(x)$. Here r is the interest rate and $\gamma(t,x,y)$ is an arbitrary function representing the risk premium factor from W_t. In the perfectly correlated case this factor does not appear. Otherwise it is the market price of risk which needs to be selected, an issue of great debate, see [59].

Models where the second random process is another Brownian motion also include the models by Hull-White [79] and Stein-Stein [114]. Scott [111] uses a Gaussian Ornstein-Uhlenbeck process but adds normal distributed jumps with exponential distributed arrival times, while Bates [5] adds a compound Poisson process to the stock price dynamics in the Heston model. The next chapter will contain a more detailed examination of a model where the second added process is not a Brownian motion but a Lévy process. Several books contain sections about stochastic volatility models and their usage. Nice overviews of the different stochastic volatility models and their properties can be found in Cont and Tankov [35], while Fouque, Papanicolaou and Sircar [59] and Lewis [85] concentrate around models without jumps.

4.3. **The Barndorff-Nielsen - Shephard model.** The returns predicted by most models suggested will by a central limit theorem tend towards a Gaussian distribution if sampled with low frequency. For long time horizons the Black & Scholes model could therefore seem like a reasonable choice, while on a short or moderate time scale the observed returns are typically heavy tailed, with volatility clustering and skewness. Barndorff-Nielsen and Shephard suggested in an inspiring paper [4] a model constructed to handle the short term aspects. The stock price dynamics is driven by a Brownian motion with drift

$$\mathrm{d}S_t=(\mu+\beta\sigma^2(t))S_t\,\mathrm{d}t+\sigma(t)S_t\,\mathrm{d}B_t, \tag{4.3}$$

but the volatility is assumed to be a stochastic process. Instead of a Brownian motion driving the volatility process a Lévy process with only positive jumps, a subordinator, is the driving source in a process of Ornstein-Uhlenbeck type

$$\mathrm{d}\sigma^2(t)=-\lambda\sigma^2(t)\,\mathrm{d}t+\mathrm{d}L(\lambda t). \tag{4.4}$$

The process $L(\lambda t)$ is termed the *background driving Lévy process* (BDLP) and the volatility process is said to be a non-Gaussian Ornstein-Uhlenbeck process. Like the Gaussian Ornstein-Uhlenbeck process it is a mean-revering process, however, because the subordinator only has positive jumps the volatility jumps up and reverts down. The subordinator will assure the positivity of the process $\sigma^2(t)$, something which is required from the squared volatility. The unusual timing $L(\lambda t)$ is to decouple the modelling of the marginal distribution of the stock's log-returns and the autocorrelation structure. Whatever value of λ the marginal distribution of $\sigma^2(t)$ will be unchanged. A generalised model is achieved by adding a leverage term $\rho\,\mathrm{d}L(\lambda t)$ to the stock price dynamics, which accounts for empirical studies showing that large downward moves in prices are associated with upward moves in volatility. The generalised model will not be considered here.

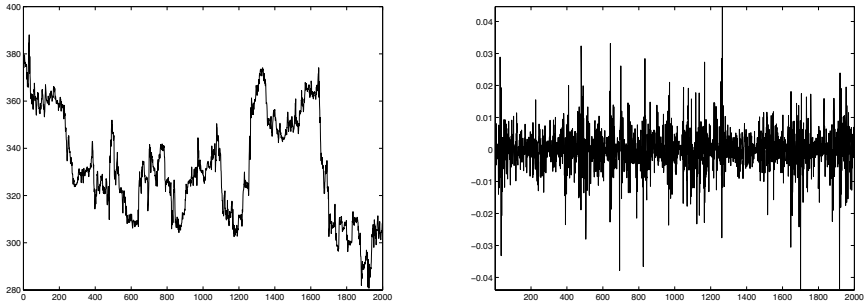

FIGURE 4. *Left:* Stock price path from the Barndorff-Nielsen and Shephard model without leverage simulated with parameters $\delta = 0.0116, \gamma = 54.2, \mu = 0.000621, \beta = 0.5$ and $\lambda = 0.83$. *Right:* The log-marginal returns from the stock price. The peaked structure is clearly visible together with a pattern of volatility clustering.

Barndorff-Nielsen and Shephard [4] proposed to use a superposition of Ornstein-Uhlenbeck processes $Y_k(t)$, with different speed of mean-reversion λ_k, to obtain a more general correlation pattern in the volatility structure. Let the volatility follow a weighted sum, with positive weights w_k adding up to one,

$$\sigma^2(t) = \sum_{k=1}^{m} w_k Y_k(t)$$

where

$$\mathrm{d}Y_k(t) = -\lambda_k Y_k(t)\,\mathrm{d}t + \mathrm{d}L_k(\lambda_k t) \tag{4.5}$$

and $L_k(\lambda_k t)$ are assumed to be independent but not necessarily identically distributed subordinators with Lévy measures $\ell_k(\mathrm{d}z)$. The autocorrelation function for the stationary $\sigma^2(t)$ then becomes

$$r(u) = \sum_{k=1}^{m} \widetilde{w}_k \exp(-\lambda_k |u|)$$

where the weights $\widetilde{w}_k$ are proportional to $w_k \mathrm{Var}(L_k)$. Letting some of the components represent short term and others long term movements both long-range and quasi-long-range dependence in the logreturns can be modeled. Below we will sometimes use the notation $\alpha(y) = (\mu + \beta y)$, $\sigma(y) = \sqrt{y}$ for the parameter functions in (4.3), assuming the volatility is given by one function $Y(t)$ of the form in (4.5).

The choice of Ornstein-Uhlenbeck processes driving the volatility lead to some interesting aspects for the model. From a modelling perspective one can choose any self-decomposable distribution D and find a stationary process of Ornstein-Uhlenbeck type which has one-dimensional marginal law D. A self-decomposable distribution has the property that for any $\zeta \in \mathbb{R}$ and $c \in (0, 1)$ the characteristic function of the distribution can be written as

$$\phi(\zeta) = \phi(c\zeta)\phi_c(\zeta)$$

where ϕ_c is another characteristic function. Two ways to approach the modelling of the Ornstein-Uhlenbeck process exist. Either write down the specific parametric form of the distribution D and calculate the implied behaviour of the BDLP. Otherwise, instead of starting out with the distribution, pick $L(\lambda t)$ and construct the Ornstein-Uhlenbeck process based on it. Some restrictions apply to what Lévy process can be used to get a self-decomposable distribution, more specifically, a necessary and sufficient condition for (4.4) to have a stationary solution is that

$$\mathbb{E}\left[\log(1+|L(1)|)\right] < \infty.$$

From the point of option pricing it is essential that the model is arbitrage-free. Barndorff-Nielsen and Shephard [4] use Esscher transforms to show that this is the case. Hence, there exist equivalent martingale measures under which $\exp(S_t)$ is a martingale. Since the model is a stochastic volatility model, including a jump process, the model is incomplete and more than one equivalent martingale measure exist. Pricing becomes a question under which measure to work, for which there are several strategies as mentioned in Section 3. Nicolato and Venardos [96] investigate option pricing under *structure preserving measures*, *i.e.* measure under which the price dynamics remains of the Barndorff-Nielsen and Shephard type (4.3)-(4.4). The tractability makes it possible to price derivatives in closed form under structure preserving measures, especially the Laplace transform of log-prices has a simple form. The cumulant function of the log-price at time t, $\ln[\psi(\theta)] = \ln[\mathbb{E}^{\mathbb{Q}}[\exp(\mathrm{i}\theta S_T)]$, under structure preserving measures, assuming the stationary law of σ_t^2 is inverse Gaussian $IG(\delta,\gamma)$, is given as

$$\begin{aligned}\ln[\psi(\theta)] =& \mathrm{i}\theta(S_t + r(T-t)) - \frac{\theta^2+\mathrm{i}\theta}{2\lambda}[1-\exp\{-\lambda(T-t)\}]\sigma^2(t) + \frac{\delta\sqrt{f_1}}{\lambda} \\ &- \frac{\delta\gamma}{\lambda} - \frac{\delta(\theta^2+\mathrm{i}\theta)}{\lambda^2\sqrt{f_2}}\left[\tan^{-1}\left(\frac{\gamma}{\sqrt{f_2}}\right) - \tan^{-1}\left(\frac{\sqrt{f_1}}{\sqrt{f_2}}\right)\right]\end{aligned}$$

where

$$\begin{aligned} f_1 &= \gamma^2 + \frac{\theta^2+\mathrm{i}\theta}{\lambda}[1-\exp\{-\lambda(T-t)\}], \\ f_2 &= -\gamma^2 - \frac{\theta^2+\mathrm{i}\theta}{\lambda}. \end{aligned}$$

Given the characteristic function it is feasible to use numerical inversion techniques to price options in the Barndorff-Nielsen and Shephard model under structure preserving measures, see Groth [68] and also Nicolato and Venardos contribution to the discussion in Barndorff-Nielsen and Shephard [4].

Another choice is to use the measure which minimises the relative entropy (2.4), the minimal entropy martingale measure $\mathbb{Q}^{\mathrm{ME}}$. Benth and Meyer-Brandis [17] studies the minimal entropy martingale measure in the Barndorff-Nielsen and Shephard model and derives the density function. Commencing with the utility maximisation problem as described above and going to the risk aversion limit, using verification theorems from Grandits and Rheinländer [67], they can identify the density. With the stochastic exponents

$$\begin{aligned} Z_t' &= \exp\left(-\int_0^t \frac{\alpha(Y_s)}{\sigma(Y_s)}\,\mathrm{d}B_s - \int_0^t \frac{1}{2}\frac{\alpha^2(Y_s)}{\sigma^2(Y_s)}\,\mathrm{d}s\right) \\ Z_t'' &= \exp\left(\int_0^t\int_0^\infty \ln\delta(s,Y_s,z)N(\mathrm{d}z,\mathrm{d}s) + \int_0^t\int_0^\infty (1-\delta(s,Y_s,z))\,\nu(\mathrm{d}z)\,\mathrm{d}s\right) \end{aligned}$$

the density process of $\mathbb{Q}^{\mathrm{ME}}$ will, under sufficient conditions, be given as

$$Z_t := Z_t' Z_t''.$$

Here $\delta(t, y, z)$ is the function

$$\delta(t, y, z) = \frac{H(t, y + z)}{H(t, y)}$$

and $H(t, y)$ is a function associated with the utility optimisation problem in the case when the investor is not issuing a claim. H can be represented as

$$H(t, y) = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{\alpha^2(Y_s)}{\sigma^2(Y_s)}\,\mathrm{d}s\right)\middle| Y_t = y\right], \quad (t, y) \in [0, T] \times \mathbb{R}_+,$$

and it was devised by Benth and Meyer-Brandis that $H(t, y)$ is governed by the partial differential equation

$$\frac{\partial H}{\partial t} - \frac{\alpha^2(y)}{2\sigma^2(y)}H - \lambda y\frac{\partial H}{\partial y} + \lambda\int_0^\infty (H(t, y + z) - H(t, y))\,\nu(\mathrm{d}z) = 0,$$

given that $H(T, y) = 1, (t, y) \in [0, T) \times \mathbb{R}_+$. The minimal entropy martingale measure for the generalised Barndorff-Nielsen and Shephard model, including the leverage term, is studied in Steiger [113].

The minimal entropy martingale measure is, as mentioned in Section 2.1, equivalent to the historical measure which makes it suitable for option pricing. The utility indifference pricing setting considered when the density Z_t is identified also leads to an integro-pde governing the price Λ of the option the investor can issue. Using a dynamic programming approach Benth and Meyer-Brandis derives the Hamilton-Jacobi-Bellman (HJB) equation associated with the value process of the investor under $\mathbb{Q}^{\mathrm{ME}}$:

$$\begin{aligned}\frac{\partial \Lambda}{\partial t} + rs\frac{\partial \Lambda}{\partial s} + \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 \Lambda}{\partial s^2} - \lambda y\frac{\partial \Lambda}{\partial y} \\ + \lambda\int_0^\infty (\Lambda(t, y + z, s) - \Lambda(t, y, s))\frac{H(t, y + z)}{H(t, y)}\nu(\mathrm{d}z) = r\Lambda\end{aligned} \tag{4.6}$$

with $(t, y, s) \in [0, T) \times \mathbb{R}_+^2$ and terminal condition $\Lambda(T, y, s) = f(s)$. Under the minimal entropy martingale measure the subordinator $L(\lambda t)$ is changed into a pure jump Markov process $\widetilde{L}(\lambda t)$ with jump measure

$$\widetilde{\nu}(\omega, \mathrm{d}z, \mathrm{d}t) = \frac{H(t, \widetilde{Y}_t(\omega) + z)}{H(t, \widetilde{Y}_t(\omega))}\nu(\mathrm{d}z)\,\mathrm{d}t \tag{4.7}$$

where the stochastic process $\widetilde{Y}_t$ is given as

$$\mathrm{d}\widetilde{Y}_t = -\lambda\widetilde{Y}_t\,\mathrm{d}t + \mathrm{d}\widetilde{L}(\lambda t).$$

An equation similar to (4.6), with only some sign changes, can be derived for the buyer of the claim, illustrating the problem of pricing in an incomplete market. It is known that the price under the minimal entropy martingale measure is the highest price the buyer can accept at the same time as it is the lowest price the seller will agree to. If the market prices deviate from this then the market will be in favour of one part. Notice that the function $\delta(s, y, z)$ appears as a measure change in (4.7) and also in the partial differential equation (4.6). The time and state-dependent ratio re-distribute the jump measure under the $\mathbb{Q}^{\mathrm{ME}}$, rescaling the jumps. The integro-PDE (4.6) is

studied numerically in Benth and Groth [10], Paper II, using finite difference methods to calculate option prices in the Barndorff-Nielsen and Shephard model.

A related equation, for a general risk aversion parameter γ, is derived in Benth and Meyer-Brandis [16], giving again a pde governing the option price $\Lambda^{(\gamma)}$ for the issuer of the claim

$$\frac{\partial \Lambda^{(\gamma)}}{\partial t} + rs\frac{\partial \Lambda^{(\gamma)}}{\partial s} + \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 \Lambda^{(\gamma)}}{\partial s^2} - \lambda y\frac{\partial \Lambda^{(\gamma)}}{\partial y} + \lambda\int_0^\infty \frac{1}{\gamma}\left\{\exp(\gamma(\Lambda^{(\gamma)}(t,y+z,s) - \Lambda^{(\gamma)}(t,y,s))) - 1\right\}\frac{H(t,y+z)}{H(t,y)}\nu(\mathrm{d}z) = r\Lambda^{(\gamma)}$$

with $\Lambda^{(\gamma)}(T,y,s) = f(s)$, for $(t,y,s) \in [0,T)\times\mathbb{R}^2_+$. Using the change of variable

$$\Lambda^{(\gamma)}(t,y,s) = \frac{1}{\gamma}\ln h^{(\gamma)}(t,y,s)$$

removes the exponential term in the integrand but instead introduces a non-linearity in the pde

$$\frac{\partial h^{(\gamma)}}{\partial t} + rs\frac{\partial h^{(\gamma)}}{\partial s} + \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2 h^{(\gamma)}}{\partial s^2} - \frac{1}{2}ys^2\frac{1}{h^{(\gamma)}}\left(\frac{\partial h^{(\gamma)}}{\partial s}\right)^2 - \lambda y\frac{\partial h^{(\gamma)}}{\partial y} + \lambda\int_0^\infty (h^{(\gamma)}(t,y+z,s) - h^{(\gamma)}(t,y,s))\frac{H(t,y+z)}{H(t,y)}\nu(\mathrm{d}z) = rh^{(\gamma)}. \tag{4.8}$$

After the change of variable the terminal condition is $h^{(\gamma)}(T,y,s) = \exp(\gamma f(s))$. The numerical solution of (4.8) is used in Benth, Groth and Lindberg [13], Paper IV, together with a root-finding algorithm to find the investors' implied risk aversion from actual traded options, assuming the underlying model is the stochastic volatility model by Barndorff-Nielsen and Shephard.

5. Numerical methods

5.1. **Monte Carlo and quasi-Monte Carlo methods.** Monte Carlo methods have over the years become indispensable tools in many areas, including financial engineering, and are perhaps the most flexible and applicable numerical methods available. Based on random sampling the elementary application is numerical integration, but there is a broad field of problems where Monte Carlo methods can be used. Assume that your problem can be cast as an integration over some measure, for which you know how to generate suitable random numbers. Then Monte Carlo integration is the easy task of sampling sequences of random numbers and using these to evaluate the integrand. The sample mean gives a probabilistic approximation of the integral and, when it is not possible to get an analytic solution, this probabilistic approach may prove to be very useful. But Monte Carlo methods have several drawbacks, the main thing being the slow convergence which makes them reliant on computational power and time.

The commonly used introduction problem is Monte Carlo integration: Let $f(x)$ be a function integrated over the unit interval

$$\int_0^1 f(x)\,\mathrm{d}x.$$

Assuming the integration is over the Lebesgue measure the evaluation of the integral can be represented as the approximative calculation of an expectation $\mathbb{E}[f(U)]$ over

the interval $U \sim \text{Unif}[0,1]$. This expectation can be estimated by sampling points uniformly from the interval, resulting in the sequence $a_1, \ldots, a_n$, and then taking the sample mean over these points

$$\mathbb{E}[f(U)] \approx \frac{1}{n} \sum_{i=1}^{n} f(a_i).$$

The strong law of large numbers guarantees that this estimate converges almost surely.

One of the disadvantages with Monte Carlo is that the error introduced by replacing the expectation with the sample mean is only a probabilistic measure. If f is square integrable then the standard error in the Monte Carlo estimate is approximately normal distributed with mean zero and standard deviation $\sigma(f)/\sqrt{n}$. Hence, the Monte Carlo integration yields a probabilistic error bound of order $\mathcal{O}(n^{-1/2})$. This error is not depending on the dimension, which makes Monte Carlo integration more attractive in higher dimensions. Conducting Monte Carlo integration also depends on the ability to sample from the underlying distribution, which could be difficult. Together with the probabilistic error bounds these are the main drawbacks of Monte Carlo integration according to Niederreiter [98].

For complicated financial derivatives, or models with other types of driving noise than Brownian motion, where no analytic answer can be obtained, a numerical method may be the only choice. A Monte Carlo method is an instrument which is incredibly flexible and usable under such premises. If it is known how to generate random numbers from the desired distribution, then it requires, in its basic form, little extra analytic work to get started. In the limit it will, due to the law of large numbers, give a correct answer. The key to use Monte Carlo simulation in finance is that one may write the price of an option as the expectation of the payoff depending on the stochastic development of the asset price. For many financial problems Monte Carlo simulations are especially suitable since the dimension turns out to be high or even infinite, for example when valuing a large portfolio consisting of several different types of assets. Other numerical approaches, such as solving partial differential equations, become hard to handle when the problem has more than a few dimensions. Monte Carlo methods, on the contrary, are not significantly harder to work with in higher dimensions than in a few. One of the Achilles tendons for the use in finance has otherwise been American options. For a long time Monte Carlo methods were considered incapable of handling pricing problems involving options with American exercise but since then both Broadie and Glasserman [26] and Tilley [115] have proposed methods to handle American options.

Since Monte Carlo methods sample randomly, the points can in the short run be concentrated in a small part of the interval sampled from. If instead the interval is divided according to a Cartesian grid with n points and the points are sampled randomly in any order the convergence can be increased. This procedure is disregarded on the basis that it requires the number of points to be known in advance to form the grid. Using a Cartesian grid rules out the possibility to sample until a terminal condition is met, for example some convergence requirement.

The concept behind *quasi-Monte Carlo methods* and *Low-discrepancy sequences* is a formalisation of the idea of how to be able to sample a sequence of deterministic numbers which fill the interval or space in an evenly distributed way. In contrast to a Cartesian grid, if sampling repeatedly from a low-discrepancy sequence the points retain an even distribution in the sense of discrepancy, a notion of uniformity described

below. Because these sequences do not try to mimic randomness, as the pseudo-random sequences used in Monte Carlo methods, the error when using low-discrepancy sequences in numerical integration is deterministic. The notion of *low-discrepancy* is reserved for sequences with a convergence rate of order $\mathcal{O}(\log(n)^d n^{-1})$ in d-dimensions and with sufficiently regular integrands. In low dimensions this is clearly better than the Monte Carlo error bound, and it has the extra benefit that the bound is deterministic. In higher dimensions the advantage over Monte Carlo methods is not as prominent since the error bound is depending on the dimension. But, as pointed out by Glasserman [64], for some problems in finance these methods are still more effective even in dimensions up to 150.

Discrepancy is the measure used to describe how our point set is distributed compared to a uniform distribution and hence, it is a measure of deviation from uniformity. Given a nonempty family of Lebesgue-measurable subsets $\mathcal{B} \in I^d$ and a point set $P = \{x_1, \ldots, x_n\}$, the discrepancy of P is given as

$$D(P; \mathcal{B}) = \sup_{B \in \mathcal{B}} \left| \frac{\sum_{i=1}^{n} \chi(x_i; B)}{n} - \lambda_d(B) \right|$$

where λ_d denotes the d-dimensional Lebesgue-measure and χ the characteristic function. It is clear that $0 \leq D(P; \mathcal{B}) \leq 1$ always. There are a few different notions of discrepancy where the *star discrepancy* $D^*(P)$ and the *extreme discrepancy* $D(P)$ are the most important. The difference is the choice of subsets $\mathcal{B}$ considered, see Niederreiter [98]. It is, according to Niederreiter [98] *widely believed* that the star discrepancy of any d-dimensional point set P consisting of n points satisfies

$$D^*(P) \geq c_d \frac{\log(n)^{d-1}}{n}$$

for some constant c_d. It is therefore usual to refer to sequences as low-discrepancy sequences if they have star discrepancy in order of $\mathcal{O}(\log(n)^d/n)$. Although the $\log(n)^d$ becomes insignificant to the n^{-1} term as the number of points increases this might not be relevant for manageable point sets if d is large. Quasi-Monte Carlo has therefore traditionally been considered inferior to Monte Carlo in higher dimensions. Sequences used for financial applications include *Faure* [57], *Halton* [69], *Niederreiter* [97] and *Sobol* [112] sequences. The construction of low-discrepancy sequences is out of the scope of this text, see Glasserman [64] and the references in there for more information.

Discrepancy plays a vital role in the *Koksma-Hlawka inequality*. This explains much of the great interest put into finding low-discrepancy sequences, while discrepancy itself is a rather theoretical concept. The Koksma-Hlawka inequality is a classic result providing a bound on the error introduced when substituting the integral with a sum and evaluating the integrand over a low-discrepancy sequence. The result builds on a one-dimensional result by Jürjen Koksma from 1942 which was extended by Edmund Hlawka in 1961.

Theorem 5.1 (The Koksma-Hlawka inequality)**.** *If f has bounded variation $V(f)$ in the sense of Hardy-Krause on the closed hypercube $\bar{I}^d = [0,1]^d$, then for any set of points $x_1, \ldots, x_n \in I^d$ it holds that*

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \int_{I^d} f(u)\, \mathrm{d}u \right| \leq V(f) D^*(x_1, \ldots, x_n) \tag{5.1}$$

where $D^*(x_1, \ldots, x_n)$ is the star discrepancy.

This error bound provides a strict deterministic bound on the integration error but is merely of theoretical value since it often grossly overestimates the error and both the Hardy-Krause variation and the star discrepancy are difficult to compute. The Koksma-Hlawka bound (5.1) is stated only for the unit hypercube and the Lebesgue measure but using slightly different definitions Kainhofer [81] provides a Koksma-Hlawka bound for general measures and domains. Kainhofer also studies problems on unbounded domains, which appears frequently in finance, and uses the Hlawka-Mück method [74] for option pricing. The method enables generation of low-discrepancy sequences from arbitrary distributions, provided the distribution function is known. This is discussed in Benth, Groth and Kettler [11], Paper I, for the normal inverse Gaussian distribution.

Starting with Boyle [24] in 1977, the research on Monte Carlo methods in finance has increased rapidly. Boyle *et.al.*[25] contains references to some of the applications of Monte Carlo in finance during the eighties and nineties including variance reduction techniques and low-discrepancy sequences. A short and comprehensive summary can also be found in Lehoczky [83]. The use of low-discrepancy sequences in finance started surprisingly late, with the first articles on the subject not appearing until the mid-nineties. Joy *et.al.*[80] use Faure sequences to price a variety of options including vanilla calls and Asian options. Faure sequences is also the choice of low-discrepancy sequence when Papageorgiou and Paskov [100] estimates Value-at-Risk for portfolios of stocks and mortgage obligations. The results from quasi-Monte Carlo in their study are superior compared to Monte Carlo, see also Papageorgiou and Traube [101], Paskov [102] and Paskov and Traube [103]. Glasserman [64] is an excellent source for information on Monte Carlo and quasi-Monte Carlo methods in finance, including a long list of the most important references.

5.2. **Fast Fourier transform.** The fast Fourier transform (FFT) is a computationally very fast and reliable method to calculate the discrete Fourier transform of a function $g_n = g(n\Delta u)$ for a range of parameter values $x_k = k\Delta x$, $k = 0, \ldots, N-1$ simultaneously. Here $\Delta x = {}^{2\pi}/_{N\Delta u}$ and for the FFT to be most efficient N has to be an integer power of 2. The algorithm takes N complex numbers as input and returns N complex numbers

$$G_k = \sum_{n=0}^{N-1} e^{-2\pi \mathrm{i} \frac{nk}{N}} g_n, \quad k = -{}^{N}/_{2}, \ldots, {}^{N}/_{2}.$$

In the nineties research surfaced where Fourier analysis and Laplace analysis were used for transform-based methods to price options in extensions of the Black & Scholes model, see Bakshi and Chen [1], Bates [5], Chen and Scott [34], Heston [72] and Scott [111]. The models include stochastic volatility elements and jumps to give better correspondence to observed asset prices as well as interest rate options. However, the approaches of these authors could not utilise the computational power of the fast Fourier transform.

Carr and Madan [32] propose a method able to price options when the characteristic function of the return is known analytically. The foundation for Carr and Madan's use of the fast Fourier transform is the following: Assume one wants to know the price of an European option with maturity T. The payoff depends on the terminal spot price S_T of the underlying asset. Denoting the logarithm of the spot price by s_T, it is necessary to know analytically the characteristic function of s_T, defined as

$$\phi_T(u) = \mathbb{E}[\exp(\mathrm{i}us_T)].$$

Denote the logarithm of the strike price by k, and let $C_T(k)$ be the value of a call option with strike $\exp(k)$. If $q_T(s)$ is the risk-neutral density of the log-price then the characteristic function of q_T is

$$\phi_T(u) = \int_{-\infty}^{\infty} e^{\mathrm{i}us} q_T(s)\,\mathrm{d}s.$$

The value of the call can be described as an integral over this density, *i.e.*

$$C_T(k) = \mathbb{E}[f(s_T, k)] = \int_k^{\infty} e^{-rT}(e^s - e^k) q_T(s)\,\mathrm{d}s.$$

To kill out the option price as $k \to -\infty$, and get a square integrable function, Carr and Madan consider the modified call price $c_T(k)$

$$c_T(k) = \exp(\alpha k) C_T(k), \quad \alpha > 0$$

and give suggestions for appropriate choices of the parameter α, the damping parameter. Now, the Fourier transform of $c_T(k)$ is defined as

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{\mathrm{i}vk} c_T(k)\,\mathrm{d}k,$$

and Carr and Madan's idea is to get an analytical value of ψ_T in terms of ϕ_T and then use the inverse Fourier transform to obtain option prices. The option price is given by the equation

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-\mathrm{i}vk} \psi_T(v)\,\mathrm{d}v \tag{5.2}$$

since $C_T(k)$ is real. The analytic expression for $\psi_T(v)$ is determined as

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^{\infty} e^{\mathrm{i}vk} \int_k^{\infty} e^{\alpha k} e^{-rT}(e^s - e^k) q_T(s)\,\mathrm{d}s\,\mathrm{d}k \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \int_{-\infty}^{s} (e^{s+\alpha k} - e^{(1+\alpha)k}) e^{\mathrm{i}vk}\,\mathrm{d}k\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \left(\frac{e^{(\alpha+1+\mathrm{i}v)s}}{\alpha + \mathrm{i}v} - \frac{e^{(\alpha+1+\mathrm{i}v)s}}{\alpha + 1 + \mathrm{i}v} \right) \mathrm{d}s \\
&= \frac{e^{-rT} \phi_T(v - (\alpha+1)\mathrm{i})}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v}.
\end{aligned}$$

After discretisation and introduction of Simpson's rule weights the option prices can be represented as

$$C(k_u) = \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^{N} e^{-\mathrm{i}\frac{2\pi}{N}(j-1)(u-1)} e^{\mathrm{i}bv_j} \psi(v_j) \frac{\eta}{3} [3 + (-1)^j - \delta_{j-1}] \tag{5.3}$$

where δ_n is the Kronecker delta function which is one for $n = 0$ and zero otherwise. Carr and Madan use this approach for the variance gamma model, which assumes that the log-price obeys a one-dimensional pure jump Markov process with stationary independent increments.

The Carr-Madan method is both fast and reliable but has its limitations. One is that it requires the analytical form of the characteristic function, but the probably severest is that the method is quite restricted in what kind of option types it can handle. In specific, it is unable to handle path dependent options, such as Asian options. The method was generalised to include other options by Raible [106] who uses Fourier and

bilateral Laplace transforms and Lewis [86] who uses generalised Fourier transforms consistently. Carr and Madan consider a Fourier transformation in the strike price but as showed in Groth [68] it is equivalent and natural to Fourier transform using the spot price.

5.3. **PDE-methods: Finite differences and Finite elements.** If the asset price is driven by a geometric Brownian motion there is a direct connection between solving the risk neutral pricing problem and solving a bounded value problem formulated with a parabolic partial differential equation. Following the original Black & Scholes analysis one can derive the Black & Scholes partial differential equation from Itô's formula. Assume sufficient regularity and that the asset S is given by the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}B_t.$$

From the Feynman-Kac formula it follows that the derivative Λ, written on the underlying asset S, solves the partial differential equation

$$\begin{aligned} \frac{\partial \Lambda}{\partial t} + rs\frac{\partial \Lambda}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 \Lambda}{\partial s^2} - r\Lambda = 0 \\ \Lambda(T, s) = f(s), \end{aligned} \tag{5.4}$$

where $f(s)$ is the payoff function and r is the interest rate.

For all models discussed above the price of an option has a representation as the solution to a partial differential equation. The price in the Black & Scholes model solves the one-dimensional pde in equation (5.4). If the stochastic volatility is driven by a Brownian motion the equation is the two-dimensional linear pde (4.2). As shown in Section 4 it is possible to derive an integro-pde representing the price of a contingent claim, in both exponential Lévy and stochastic volatility models including a Lévy process.

Solving an integro-pde numerically is naturally a more involved task than solving an ordinary pde. The integral term is non-local, depending on the whole solution and not only on the variables in a small neighbourhood. The use of standard techniques to solve the equation includes finding a suitable way to represent the integral on the possibly infinite domain, either with the information at hand or by approximation. This can prove to be cumbersome and introduce severe numerical problems if not treated carefully. If the Lévy process driving the model has infinite activity the measure is singular at zero, causing additional problems in the implementation. Benth and Groth [10], Paper II, discuss how to solve the integro-pde in equation (4.6) using the finite difference method, while Benth, Groth and Lindberg [13], Paper IV, consider equation (4.8).

The standard techniques for solving partial differential equations are the finite difference and the finite element methods. The foundation of these methods will not be discussed here, as the methods are used only as tools. The interested reader is referred to standard textbooks on numerical solutions of partial differential equations. The main reference for pde-methods in finance is Wilmott, Dewynne and Howison [116], but coming to age the book lacks any treatment of models with jumps or stochastic volatility and focuses mainly on finite difference methods. Cont and Tankov [35] includes a chapter, with numerous references, about integro-pdes in exponential Lévy markets based partly on Cont, Tankov and Voltchkova [36] and Cont and Voltchkova [37, 38]. Important research on finite element methods in finance is done by the group around Schwab [73, 90, 91].

6. Option sensitivities

Numerous research articles focus on the question of how to price options and other derivatives. Equally many, if not more, instead ask the related question on how to hedge the positions. Financial institutions need to know how to manage the risk their portfolios face from changes in the market. The classic Black & Scholes analysis depends on the possibility to set up a risk-free portfolio, with the rate of return equal to the interest rate, consisting of a short position in the option and a long position in Δ shares of the underlying. This quantity Δ is the sensitivity of the options to changes in the price of the underlying asset, *i.e.*

$$\Delta = \frac{\partial \Lambda}{\partial s}.$$

This is called the *delta* of the option and it is one of the option sensitivities often grouped together under the name *the Greeks*. They are all measures of how sensitive the option price is to changes in one parameter in the model of the underlying asset. Common ones are *rho* ρ, *theta* Θ, *vega* $\mathcal{V}$ and *gamma* Γ, which measure in order the sensitivity to the interest rate, the passage of time, the volatility and the second derivative with respect to the price of the underlying. The primary one is clearly delta because of the connection with the Black & Scholes analysis and the concept of delta-hedging. Holding the portfolio described above the option owner is instantaneously secured against any changes in the price of the asset as the gain (loss) in the price of the option is offset by a similar fall (rise) in the price of the position in the stock. Maintaining a delta-neutral portfolio enables traders to manage the risk from asset price changes. This holds true in theory only though, since delta-hedging is a dynamic hedging strategy that needs continuously rebalancing of the portfolio, incurring prohibiting large transaction costs. Similarly, investors can aim to keep a portfolio gamma-neutral to reduce the risk from the curvature of the option price which is not covered by the delta-hedge. See Hull [78] for a more extensive introduction.

While the price of liquid options are observable in the market the sensitivities are not and need to be calculated, which in reality means estimating a derivative. For certain models and simple option types, for example European options in the Black & Scholes model, it is possible to derive analytical expressions and there is no need to involve in simulations. For more complicated contracts in advance models this is not feasible and one needs to resort to numerical approximations. This section is concentrated solely on Monte Carlo simulations of the sensitivities, with a brief covering of three different methods, these being the finite difference, the pathwise differentiation and the likelihood ratio methods, and finally a more in depth cover of the Malliavin method.

Suppose the price of the option is represented as a discounted expectation similar to (2.3), with payoff function f and asset price S_t depending on a parameter θ. Assume for clarity that the interest rate is constant. The sensitivity of the price with respect to θ is then the derivative

$$\alpha(\theta) = \frac{\partial}{\partial \theta} \mathbb{E}\left[e^{-rT} f(S_T(\theta))\right].$$

The obvious approach to simulate α is to use a finite difference approximation of the derivative. Simulate n independent replications of $S_T(\theta)$ and $S_T(\theta + h)$, take the averages $\widehat{f}$ over the two sets of paths and let the estimate $\widehat{\alpha}$ be

$$\widehat{\alpha}(\theta, h) = e^{-rT} \frac{\widehat{f}(S_T(\theta + h)) - \widehat{f}(S_T(\theta))}{h}.$$

There are some obvious drawbacks with the finite difference approach. To begin with it has a bias dependent on the value of h but the variance is proportional to h^{-2}. While the bias is reduced by taking a smaller h this has to be weighted against the effect on the variance. Using a forward difference and independent random numbers for the two sequences the best convergence rate is typically $\mathcal{O}(n^{-1/4})$. The convergence rate can be improved to $\mathcal{O}(n^{-1/2})$ by taking central differences and by using common random numbers, as suggested by Glasserman and Yao [65], which is the best that can be expected from Monte Carlo simulations. Then however, the convergence rate can be sensitive to the smoothness of the payoff function, leading to poor performance for options with discontinuous payoffs like binary options.

To achieve a better convergence rate than with the finite difference method Broadie and Glasserman [26] investigate two different methods, the pathwise method and the likelihood ratio method. Instead of taking the derivative of the expectation the pathwise method assumes α can be represented as

$$\alpha(\theta) = \frac{\partial}{\partial\theta}\mathbb{E}\left[e^{-rT}f(S_T(\theta))\right] = \mathbb{E}\left[e^{-rT}\frac{\partial}{\partial\theta}f(S_T(\theta))\right].$$

The last part can be considered as a pathwise derivative of the payoff function and sufficient regularity of the payoff function is assumed to be able to interchange differentiation and expectation. According to Glasserman [64] this method has usually much less variance than the finite difference and the likelihood ratio method. To yield an unbiased estimator the pathwise method requires that the differentiation can be moved inside the expectation, which in general demands that the payoff is pathwise continuous with respect to θ. Binary options are not continuous with respect to the price of the underlying so the pathwise method is not applicable for the Greeks of a binary option. Neither are barrier options and for the same reasons the pathwise method is unable to handle the gamma of an ordinary call option.

The likelihood ratio method assumes that the distribution of the underlying asset S_t has a density $p(S_t)$ with θ being a parameter of the density. Again assume there is enough regularity to change the order of expectation and differentiation. Using the density, the sensitivity can be written as

$$\alpha(\theta) = \mathbb{E}\left[e^{-rT}\frac{\partial}{\partial\theta}f(S_T(\theta))\right] = \int_{\mathbb{R}} f(x)\frac{\partial}{\partial\theta}p(x)\,\mathrm{d}x.$$

Since smoothness is rarely a problem for densities the likelihood ratio method is applicable for a wider range of options than the pathwise method. Dividing with $p(x)$ and rewriting the integrand leaves

$$\alpha(\theta) = \mathbb{E}\left[e^{-rT}f(S_T)\frac{\partial \log\, p(S_T)}{\partial\theta}\right].$$

Here $\partial \log p(S_T)/\partial\theta$ works as a weight function multiplying the payoff function. The product is an unbiased estimator of the derivative when applicable but the weight often produces large variance, limiting the use of the method. The main limitation is nevertheless the need for explicit knowledge of the density, which in turn needs to depend on the parameter θ. An example where the density of the marginal log-returns is not explicitly known is the Barndorff-Nielsen and Shephard model when the stationary distribution of the volatility process is inverse Gaussian.

The likelihood ratio method is interesting because the derivation is not applied to the expectation or the payoff function, instead the payoff is multiplied by a weight function.

In a sense this can be viewed as a derivative-free calculation of the Greeks. The derivative is in the weight function which can cause high variance of the simulations. Taking this idea further Fournié *et.al.*[61] used in an inspiring paper Malliavin calculus to derive weight functions.

6.1. **Malliavin calculus and Greeks.** The drawbacks of the pathwise and the likelihood ratio methods make it hard to estimate option sensitivities for more complicated contracts and in markets where the option price density is unknown. At the same time the finite difference method is prone to large bias and large variance, especially for options with discontinuous payoff functions.

A method capable of handling the contracts the pathwise and likelihood methods struggle with, while still producing unbiased results with low variance, is the Malliavin method proposed by Fournié *et.al.*[61]. The idea is to use variational stochastic calculus to derive a derivative-free method of calculating the Greeks in the Black & Scholes market. The method relies on the theory called Malliavin calculus, especially the integration-by-parts formula, to devise weights multiplying the payoff. In this way it is possible to avoid taking the derivative of the payoff function, similar to the way it is avoided in the likelihood ratio method. What follows is a short primer to Malliavin calculus, for a full account of the theory see Nualart [99].

Let $W_t, t \in \mathbb{R}_+$ be a d-dimensional Brownian motion, and let $\mathcal{C}$ denote the space of random variables F of the form

$$F = f\left(\int_0^\infty h_1(t)\,\mathrm{d}W_t, \ldots, \int_0^\infty h_n(t)\,\mathrm{d}W_t\right), \quad f \in \mathcal{S}(\mathbb{R}^n),$$

$h_1, \ldots, h_n \in L^2(\mathbb{R}_+)$, where $\mathcal{S}(\mathbb{R}^n)$ is the space of rapidly decreasing $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$. For a given $F \in \mathcal{C}$ the Malliavin derivative $D_t F$ of F is the process $D_t F, t \in \mathbb{R}_+$ in $L^2(\Omega \times \mathbb{R}_+)$ defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^\infty h_1(t)\,\mathrm{d}W_t, \ldots, \int_0^\infty h_n(t)\,\mathrm{d}W_t\right) h_i(t), \quad t \in \mathbb{R}_+, \quad a.s.$$

Define the norm on $\mathcal{C}$ by

$$\|F\|_{1,2} = (\mathbb{E}[F^2])^{1/2} + \left(\mathbb{E}\left[\int_0^\infty |D_t F|^2\,\mathrm{d}t\right]\right)^{1/2}, \; F \in \mathcal{C}.$$

Let $\mathbb{D}^{1,2}$ denote the Banach space which is the completion of $\mathcal{C}$ with respect to the norm $\|\cdot\|_{1,2}$. The derivative operator D is a closed linear mapping defined on $\mathbb{D}^{1,2}$ with values in $L^2(\Omega \times \mathbb{R}_+)$.

The derivative operator has a chain-rule for derivation, *i.e.* if $\psi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and $F = (F_1, \ldots, F_n)$ a random vector whose components belong to $\mathbb{D}^{1,2}$, then $\psi(F) \in \mathbb{D}^{1,2}$ and

$$D_t \psi(F) = \sum_{i=1}^n \frac{\partial \psi}{\partial x_i}(F) D_t F_i, \quad t \in \mathbb{R}_+, \quad a.s.$$

The divergence operator δ, also called the Skorohod integral, exists and is the adjoint of D. Assuming u is a stochastic process in $L^2(\Omega \times \mathbb{R}_+)$ then $u \in \mathrm{Dom}(\delta)$ if and only if for all $F \in \mathbb{D}^{1,2}$ it holds that

$$\mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] := \mathbb{E}\left[\int_0^\infty D_t F u(t)\,\mathrm{d}t\right] \leq K(u)\|F\|_{1,2}\,,$$

where $K(u)$ is a constant independent of F. If $u \in \mathrm{Dom}(\delta)$, then $\delta(u)$ is defined by the following integration-by-parts formula

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}], \quad \forall F \in \mathbb{D}^{1,2}.$$

The domain of δ contains all adapted processes which belong to $L^2(\Omega \times \mathbb{R}_+)$, and for such processes the Skorohod integral coincides with the Itô integral. That is, for an adapted process $u \in L^2(\Omega \times \mathbb{R}_+)$

$$\delta(u) = \int_0^\infty u(t)\,\mathrm{d}W_t.$$

Also, if $F \in \mathbb{D}^{1,2}$ then for all $u \in \mathrm{Dom}(\delta)$ such that $F\delta(u) - \int_0^T D_t F u(t)\,\mathrm{d}t \in L^2(\Omega)$ it holds that

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u(t)\,\mathrm{d}t.$$

The main result for computation of sensitivities with Malliavin calculus is the following: Let $(F^\alpha)_\alpha$ be a family of random variables, continuously differentiable in $\mathrm{Dom}(D)$ with respect to the parameter α and let $u(t), t \in [0,T]$ be a process in $L^2(\Omega \times \mathbb{R}_+)$. Assuming that $\langle DF^\alpha, u\rangle_{L^2(\mathbb{R}_+)} \neq 0$, *a.s.* then

$$\frac{\partial}{\partial\alpha}\mathbb{E}\left[f(F^\alpha)\right] = \mathbb{E}\left[f(F^\alpha)\delta\left(u\frac{{}^{\partial F^\alpha}/_{\partial\alpha}}{\langle DF^\alpha, u\rangle_{L^2(\mathbb{R}_+)}}\right)\right] \tag{6.1}$$

for all functions f such that $f(F^\alpha) \in L^2(\Omega)$. Using (6.1) one can compute Malliavin weights assuming it is allowed to interchange differentiation and expectation. u is a weighting function which can be chosen to get an optimal tuning for specific contracts. The Malliavin weights produce unbiased estimates and do not rely on an explicit knowledge of the stock price density, as the likelihood ratio method does. The result in Fournié *et.al.* [61] suggests that the method gives significantly lower variance for options with discontinuous payoffs.

The research literally exploded after the first article, with the same analysis done for other type of contracts, with other weighting functions and in other models, see [7, 8, 18, 19, 20, 60, 66]. As noticed in Kohatsu-Higa and Montero [82] the likelihood ratio method is similar to the Malliavin method if the density is known. It was also shown by Chen and Glasserman [33] that taking a time-step approximation using Euler schemes, applying the likelihood ratio method and then passing to the continuous-time limit results in the same weights as in the Malliavin method for several important cases, *i.e.* delta, rho and vega.

The Malliavin method sprung the interest in doing similar research on methods including jumps. Except the pure-jump setting examined by El-Khatib and Privault [56] the main idea has been to consider the derivative in the direction of the Wiener process. León *et.al.*[84] was first to consider *simple Lévy processes*, a linear combination of a Brownian motion and several Poisson processes with fixed jump size. Developing a Malliavin calculus for simple Lévy processes they showed that the analysis can be made on the Wiener space and the formulas from the pure Wiener case can be used. A similar approach is considered by Davis and Johansson [39] while Debelley and Privault [40] extend the idea to cover general jump-diffusions. The directional derivative approach is also applied on the Barndorff-Nielsen and Shephard model in Benth, Groth and Wallin [14], Paper V.

7. Volatility derivatives

The volatility is the easiest measure of the uncertainty attached to a financial asset. It was considered as a constant quantity in the Black & Scholes theory, something which has been disputed because the volatility is known to change over time. How it changes and how it can be modeled are discussed in Section 4. Given a stochastic model for the dynamics of the volatility it is a short leap to the idea of constructing contracts written on realised volatility, and trade these contracts to hedge against the changes.

Calculating *the Greeks* of an option can tell investors about the exposure they face from changes in underlying parameters in the models, but not how to hedge it away. Trading in the underlying asset can help the investor reduce the risk associated with changes in the price, the *delta*-exposure. In the Black & Scholes market this is the only risk perceived since it is assumed that all other parameters are constant under the time horizon considered. The inability of the Black & Scholes model to capture the implied volatility rises the question about the risk associated with changes in the volatility of the underlying asset. A change in the volatility will influence the price of the option, possibly without changing the price of the underlying asset, but this change is not possible to hedge by the usual delta-hedging approach. The exposure to volatility is measured in vega ($\mathcal{V}$), the sensitivity to changes in the volatility parameter in the Black & Scholes model. An investor managing a large portfolio might find that his vega-exposure is high and wish to hedge away this risk. The market has met this demand by offering derivatives written on realised variance and volatility. In 1993 the Chicago Board Option Exchange (CBOE) introduced a volatility index (VIX) which became the benchmark for stock market volatility. It measures the market expectation on the 30-day volatility based on S&P 500 index option prices with a range of strike prices. Accompanying the VIX there exists a family of derivative products written with the VIX as the underlying, including futures and options.

The structure of a volatility contract is in principle not different from contracts on other underlying assets. Let the realised volatility $\sigma_R(T)$ over a period $[0, T]$ be defined as

$$\sigma_R(T) = \sqrt{\frac{1}{T}\int_0^T \sigma^2(s)\,\mathrm{d}s}.$$

The process $\sigma^2(s)$ depends on the model, from constant in the Black & Scholes model to a non-Gaussian Ornstein-Uhlenbeck process in the Barndorff-Nielsen and Shephard model. A volatility swap is the simplest contract, paying at time T the amount

$$N(\sigma_R(T) - \Sigma)$$

where Σ is the strike, a predefined level of volatility, and N is a notional, turning the volatility difference into money. The strike Σ is chosen such that the swap is entered into at zero cost. A variance swap is similarly defined as

$$N(\sigma_R^2(T) - \Sigma_2).$$

The extension to options on realised volatility or variance is obvious. In effect the buyer swaps a fixed volatility against the actual realised volatility. Under the risk-neutral probability measure $\mathbb{Q}$ the fixed level of volatility, sometimes referred to as the price of the swap, can be expressed as

$$\Sigma(t, T) = \mathbb{E}^{\mathbb{Q}}[\sigma_R(T)|\mathcal{F}_t]$$

and the price of the variance swap as

$$\Sigma_2(t,T) = \mathbb{E}^{\mathbb{Q}}[\sigma_R^2(T)|\mathcal{F}_t].$$

The fair price of variance can be calculated directly by calculating the risk neutral expectation of a variance swap, something that would enforce to specify a model for the variance, see for example Benth, Groth and Kufakunesu [12], Paper III, who price swaps in the Barndorff-Nielsen and Shephard model. Much of the interest has rather been focused on how to replicate swaps on realised variance and volatility. Early work by Derman *et.al.*[44], Dupire [48] and Neuberger [95] shows that a continuously sampled variance swap in a diffusive market is possible to replicate by trading in the asset and its options. Assume that the price S_t of the asset has dynamics

$$\mathrm{d}S_t = \mu(t)S_t\,\mathrm{d}t + \sigma(t)S_t\,\mathrm{d}W_t \tag{7.1}$$

where the drift $\mu(t)$ and the continuously sampled volatility $\sigma(t)$ are arbitrary functions of time and other parameters. Applying Itô's lemma to $\log S_t$ and subtracting from (7.1) then

$$\mathrm{d}S_t - S_t\,\mathrm{d}(\log S_t) = \frac{1}{2}\sigma^2(t)\,\mathrm{d}t$$

and hence

$$\sigma_R^2 = \frac{2}{T}\left[\int_0^T \frac{\mathrm{d}S_t}{S_t} - \log\frac{S_T}{S_0}\right].$$

Taking the conditional expectation gives the price of the variance swap. For replication one can notice that the first part inside the brackets can be considered as the continuously rebalanced position of being long $1/S_t$ shares. The second term represents the static short position in a claim on $\log S_T/S_0$. The problem of trading on the logarithmic contract can be solved by synthesizing it with liquid options on the asset. If an arbitrary put-call separator $\kappa > 0$ is picked then the log-payoff can be decomposed such that

$$-\log\frac{S_T}{S_t} = -\frac{S_T-\kappa}{\kappa} + \int_0^\kappa \frac{1}{K^2}(K-S_T,0)^+\,\mathrm{d}K + \int_\kappa^\infty \frac{1}{K^2}(S_T-K,0)^+\,\mathrm{d}K.$$

This suggests that in addition to the $1/S_t$ shares held one should hold a short position in $1/\kappa$ forward contracts struck at κ, a long position in $1/K^2$ put options at K for all strikes from 0 to κ and a similar position in call options for all strikes from κ to ∞, all contracts expiring at T. The fair price of the swap follows from the initial value of each part.

Swaps and options written on volatility are known to be more difficult to price and hedge than their variance counterparts. Naively, the price of a volatility swap could be thought to be the square root of the variance swap. By Jensen's inequality it is easy to see that this might not be the case, *i.e.*

$$\mathbb{E}[\sigma_R(T)|\mathcal{F}_t] \leq \sqrt{\mathbb{E}\left[\sigma_R^2(T)|\mathcal{F}_t\right]}.$$

The common knowledge was that the replication strategy for volatility swaps was highly model-dependent, something which was challenged in recent papers by Carr and Lee [30, 31]. Trading dynamically in the underlying together with positions in European options, call, puts and straddles, Carr and Lee generate a synthetic volatility swap, without specifying a model for the volatility. The replication strategy is more involved than for variance derivatives but holds under a general assumption about correlation between the stock and volatility. For pricing of volatility options Carr and

Lee assume the time-t conditional distribution of the volatility is a displaced lognormal, and derive explicit formulas. Trading variance and volatility swaps they show how to hedge options, however the formulas are rather complex.

The market interest in volatility derivative contracts pushes the academic research interest. Except from the references mentioned work is done by Windcliff, Forsyth and Vetzal [117] for a model with jumps in the asset price dynamics while Howison, Rafailidis and Rasmussen [77] study a stochastic volatility model with a mean-reverting lognormal volatility dynamics. Also notable in the field is the paper by Carr *et.al.*[29] which studies properties of the volatility in a model driven by pure jump processes, preferably the class of $CGMY$ processes.

References

[1] G. Bakshi and Z. Chen. An alternative valuation model for contingent claims. *J. Financial Econ.*, 44:123–165, 1997.

[2] O. E. Barndorff-Nielsen. Exponentially decreasing distributions for the logarithm of particle size. *Proc. Royal society London A*, 353:401–419, 1977.

[3] O. E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2:41–68, 1998.

[4] O. E. Barndorff-Nielsen and N. Shepard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. Royal Stat. Society*, 63:167–241, 2001.

[5] D. A. Bates. Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche mark options. *Review of Financial Studies*, 9(1):69–107, 1996.

[6] D. Becherer. *Rational hedging and valuation with utility-based preferences*. PhD thesis, Technischen Universität Berlin, 2001.

[7] E. Benhamou. Smart monte carlo: various tricks using Malliavin calculus. *Quant. Finance*, 2(5):329–336, 2002.

[8] E. Benhamou. Optimal Malliavin weighting function for the computation of the Greeks. *Mathematical Finance*, 13(1), 2003.

[9] F. E. Benth. *An introduction to mathematical finance*. Springer-Verlag, Berlin, 2004.

[10] F. E. Benth and M. Groth. The minimal entropy martingale measure and numerical option pricing for the Barndorff-Nielsen and Shephard stochastic volatility model. *Submitted*, 2006.

[11] F. E. Benth, M. Groth, and P. C. Kettler. A quasi-Monte Carlo algorithm for the normal inverse Gaussian distribution and valuation of financial derivatives. *Int. J. Theor. Applied Finance*, 9(6), 2006.

[12] F. E. Benth, M. Groth, and R. Kufakunesu. Valuing volatility and variance swaps for a non-Gaussian Ornstein-Uhlenbeck stochastic volatility model. *Submitted*, 2006.

[13] F. E. Benth, M. Groth, and C. Lindberg. The implied risk aversion from utility indifference option pricing in a stochastic volatility model. *Submitted*, 2007.

[14] F. E. Benth, M. Groth, and O. Wallin. Derivative-free Greeks for the Barndorff-Nielsen and Shephard stochastic volatility model. *Submitted*, 2007.

[15] F. E. Benth and K. H. Karlsen. A PDE-representation of the density of the minimal entropy martingale measure in stochastic volatility markets. *Stoch. Stoch. Rep.*, 77(2):109–137, 2005.

[16] F. E. Benth and T. Meyer-Brandis. Indifference pricing and the minimal entropy martingale measure in a stochastic volatility model with jumps. *Preprint Pure Math. Univ. of Oslo*, 3, 2004.

[17] F. E. Benth and T. Meyer-Brandis. The density process of the minimal entropy martingale measure in a stochastic volatility model with jumps. *Finance and Stochastics*, 9(4), 2005.

[18] H.-P. Bermin. A general approach to hedging options: applications to barrier and partial barrier options. *Mathematical Finance*, 12:199–218, 2002.

[19] H.-P. Bermin. Hedging options: The Malliavin calculus approach versus the δ-hedging approach. *Mathematical Finance*, 13(1), 2003.

[20] G. Bernis, E. Gobet, and A. Kohatsu-Higa. Monte carlo evaluation of Greels for multidimensional barrier and lookback options. *Mathematical Finance*, 13(1), 2003.

[21] T. Björk. *Arbitrage theory in continuous time*. Oxford University Press, Oxford, 1998.

[22] F. Black and M. Scholes. The pricing of options and corporate liabilities. *J. Political Econ.*, 81:637–659, 1973.

[23] N. Bouleau and D. Lamberton. Residual risks and hedging strategies in Markovian markets. *Stoch. Proc. Appl.*, 33:131–150, 1989.

[24] P. Boyle. Options: A Monte Carlo approach. *J. Financial Econ.*, 4:323–338, 1977.

[25] P. Boyle, M. Broadie, and P. Glasserman. Monte Carlo methods for security pricing. *J. Econ. Dynamics Control*, 21:1267–1321, 1997.

[26] M. Broadie and P. Glasserman. Estimating security price derivatives by simulation. *Management Science*, 42:269–285, 1996.

[27] P. Carr, H. Geman, D. B. Madan, and M. Yor. The fine structure of asset returns: An empirical investigation. *J. Business*, 75, 2002.

[28] P. Carr, H. Geman, D. B. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13:345–382, 2003.
[29] P. Carr, H. Geman, D. B. Madan, and M. Yor. Pricing options on realized variance. *Finance and Stochastics*, 9:453–475, 2005.
[30] P. Carr and R. Lee. Pricing and hedging options on realized volatility and variance. *Preprint*, 2006.
[31] P. Carr and R. Lee. Robust replication of volatility derivatives. *Preprint*, 2006.
[32] P. Carr and D. B. Madan. Option valuation using the fast Fourier transform. *J. Computational Finance*, 2:61–73, 1998.
[33] N. Chen and P. Glasserman. Malliavin Greeks without Malliavin calculus. *Preprint*, 2006.
[34] R.-R. Chen and L. Scott. Pricing interest rate options in a two-factor Cox-Ingersoll-Ross model of the term structure. *Review of Financial Studies*, 5(4):613–636, 1992.
[35] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC, Boca Raton, Florida, 2005.
[36] R. Cont, P. Tankov, and E. Voltchkova. Option pricing models with jumps: Integro-differential equations and inverse problems. In P. Neittaanmäki, T. Rossi, E. Korotov, J. Periaux, and D. Knörzer, editors, *Proceedings 4th European Congress*. ECCOMAS, 2004.
[37] R. Cont and E. Voltchkova. A finite difference scheme for option pricing in jump diffusion and exponential Lévy models. In P. Neittaanmäki, T. Rossi, E. Korotov, J. Periaux, and D. Knörzer, editors, *Proceedings 4th European Congress*. ECCOMAS, 2004.
[38] R. Cont and E. Voltchkova. Integro-differential equations for option prices in exponential Lévy models. *Finance and Stochastics*, 9:299–325, 2005.
[39] M. H. A. Davis and M. Johansson. Malliavin Monte Carlo Greeks for jump-diffusions. *Stoch. Proc. Appl.*, 116(1):101–129, 2006.
[40] V. Debelley and N. Privault. Sensitivity analysis of European options in jump-diffusion models via the Malliavin calculus on the Wiener space. *Preprint*, 2004.
[41] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker. Exponential hedging and entropic penalties. *Mathematical Finance*, 12(2):99–123, 2002.
[42] F. Delbaen and W. Schachermeyer. A general version of the fundamental theorem of asset pricing. *Mathematishe Annalen*, 300:463–520, 1994.
[43] F. Delbaen and W. Schachermeyer. The fundamental theorem for unbounded processes. *Mathematishe Annalen*, 312:215–250, 1998.
[44] K. Demeterfi, E. Derman, M. Kamal, and J. Zou. More than you ever wanted to know about volatility swaps. Goldman Sashs, March 1999.
[45] E. Derman and I. Kani. Riding on a smile. *RISK*, 7(2):32–39, 1994.
[46] D. Duffie. *Dynamic asset pricing theory*. Princeton university press, Princeton, New Jersey, 2001.
[47] B. Dumas, J. Fleming, and R. E. Whaley. Implied volatility functions; Empirical tests. *J. Finance*, 53(6):2059–2106, 1998.
[48] B. Dupire. Model art. *RISK*, September:118–120, 1993.
[49] B. Dupire. Pricing with a smile. *RISK*, 7(1), 1994.
[50] E. Eberlein. Application of generalized hyperbolic Lévy motions to finance. In O. E. Barndorff-Nielsen, T. Mikosch, and S. I. Resnick, editors, *Lévy processes, Theory and Applications*, pages 319–336. Birkhäuser, Boston, 2001.
[51] E. Eberlein and U. Keller. Hyperbolic distributions in finance. *Bernoulli*, 1(3):281–299, 1995.
[52] E. Eberlein and K. Prause. The generalized hyperbolic model: financial derivatives and risk measures. In *Mathematical Finance - Bachelier Congress 2000*, pages 245 – 267. Springer-Verlag, 1998.
[53] E. Eberlein and S. Raible. Term structure models driven by general Lévy processes. *Mathematical Finance*, 9(1):31–53, 1999.
[54] N. El Karoui and M. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.*, 33:29–66, 1995.
[55] N. El Karoui and R. Rouge. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.
[56] Y. El-Khatib and N. Privault. Computations of Greeks in a market with jumps via the Malliavin calculus. *Finance and Stochastics*, 8:161–179, 2004.

[57] H. Faure. Discrépence de suites associée à un systéme de numèration (en dimension s). *Acta Arithmetica*, 43:337–351, 1982.
[58] H. Föllmer and M. Schweizer. Hedging of contingent claims under incomplete information. In M. Davis and R. Elliot, editors, *Applied stochastic analysis*, pages 389–414. Gordon and Breach, London, 1991.
[59] J.-P. Fouque, G. Papanicolaou, and R. Sircar. *Derivatives in financial markets with stochastic volatility*. Cambridge University Press, Cambridge, 2000.
[60] E. Fournié, J.-M. Lasry, J. Lebuchoux, and P.-L. Lions. Applications of Malliavin calculus to Monte Carlo methods in finance ii. *Finance and Stochastics*, 5:201–236, 2001.
[61] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance. *Finance and Stochastics*, 3:391–412, 1999.
[62] M. Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, 10(1):39–52, 2000.
[63] T. Fujiwara and Y. Miyahara. The minimal entropy martingale measure for geometric Lévy processes. *Finance and Stochastics*, 7:509–531, 2003.
[64] P. Glasserman. *Monte Carlo methods in financial engineering*. Springer-Verlag, New York, 2004.
[65] P. Glasserman and D. Yao. Some guidelines and guarantees for common random numbers. *Management Science*, 38(6):884–908, 1992.
[66] E. Gobet and A. Kohatsu-Higa. Computation of Greeks for barrier and lookback options using Malliavin calculus. *Electronic Communications in Probability*, 8:51–62, 2003.
[67] P. Grandits and T. Rheinländer. On the minimal entropy martingale measure. *Annals of Prob.*, 30:1003–1038, 2002.
[68] M. Groth. Simulation in financial mathematics. Licentiate Thesis, Växjö University, 2005.
[69] J. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. *Numerische Mathematik*, 2:84–90, 1960.
[70] J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod securities markets. *J. Economical Theory*, 20:381–408, 1979.
[71] J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Proc. Appl.*, 11:215–260, 1981.
[72] S. L. Heston. A closed-form solution for options with stochastic volatility, with applications to bond and currency options. *Review Financial Studies*, 6(2):327–343, 1993.
[73] N. Hilber, A. Matache, and C. Schwab. Sparse wavelet methods for option pricing under stochastic volatility. *J. Computational Finance*, 8(4), 2005.
[74] E. Hlawka and R. Mück. A transformation of equidistributed sequences. In *Applications of number theory to numerical analysis*, pages 371–388. Academic Press, 1972.
[75] D. Hobson. Stochastic volatility models, correlation, and the q-optimal measure. *Mathematical Finance*, 14(4):537–556, 2004.
[76] S. Hodges and A. Neuberger. Optimal replication of contingent claims under transaction costs. *Review Futures Markets*, 8:222–239, 1989.
[77] S. Howison, A. Rafailidis, and H. Rasmussen. On the pricing and hedging of volatility derivatives. *Applied Math. Finance*, 11(4):317–346, 2004.
[78] J. C. Hull. *Option, futures and other derivatives; Fifth edition*. Prentice Hall, Upper Saddle River, New Jersey, 2003.
[79] J. C. Hull and A. White. The pricing of options on assets with stochastic volatility. *J. Finance*, 42:281–300, 1987.
[80] C. Joy, P. P. Boyle, and K. S. Tan. Quasi-Monte Carlo methods in numerical finance. *Management Science*, 42(6):926–939, 1996.
[81] R. Kainhofer. *Quasi-Monte Carlo algorithms with applications in numerical analysis and finance*. PhD thesis, Technischen Universität Graz, 2003.
[82] A. Kohatsu-Higa and M. Montero. Malliavin calculus in finance. In S. T. Rachev and G. A. Anastassiou, editors, *Handbook of computational and numerical methods in finance*, pages 111–174. Springer-Verlag, Berlin, 2004.
[83] J. P. Lehoczky. Simulation methods for option pricing. In M. A. Dempster and S. R. Pliska, editors, *Mathematics of Derivative Securities*, pages 528–544. Cambridge University Press, Cambridge, 1997.

[84] J. A. León, J. L. Solé, F. Utzet, and J. Vives. On Lévy processes, Malliavin calculus and market models with jumps. *Finance and Stochastics*, 6:197–225, 2002.
[85] A. L. Lewis. *Option valuation under stochastic volatility*. Finance Press, Newport Beach, California, 2000.
[86] A. L. Lewis. A simple option formula for general jump-diffusions and other exponential Lévy processes. *Envision Financial Systems*, 2001.
[87] D. B. Madan and F. Milne. Option pricing with variance gamma martingale components. *Mathematical Finance*, 1:39–55, 1991.
[88] D. B. Madan and E. Seneta. The variance gamma (V.G.) model for share market returns. *J. Business*, 63(4):511–524, 1990.
[89] B. B. Mandelbrot. *Fractals and scaling in finance; discontinuity, concentration, risk*. Springer-Verlag, New York, 1997.
[90] A.-M. Matache, P. Nitsche, and C. Schwab. Wavelet Galerkin pricing of American options on Lévy driven assets. *Quant. Finance*, 5(4):403–424, 2005.
[91] A.-M. Matache, T. von Petersdorff, and C. Schwab. Fast deterministic pricing of options on Lévy driven assets. *Math. Modelling Num. Analysis*, 38(1):37–72, 2004.
[92] R. C. Merton. Theory of rational option pricing. *Bell J. Econ. Manag. Sciences*, 4:141–183, 1973.
[93] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *J. Financial Econ.*, 3:125–144, 1976.
[94] M. Musiela and M. Rutkowski. *Martingale methods in financial modelling*. Springer-Verlag, Berlin, 1997.
[95] A. Neuberger. The log contract. *J. Portfolio Management*, 20(2), 1994.
[96] E. Nicolato and E. Venardos. Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Mathematical Finance*, 13(4):445–466, 2003.
[97] H. Niederreiter. Low-discrepancy and low-discrepancy sequences. *J. Number Theory*, 30:51–70, 1988.
[98] H. Niederreiter. *Random number generation and quasi-Monte Carlo methods*. SIAM, Philadelphia, Pennsylvania, 1992.
[99] D. Nualart. *Malliavin calculus and related topics*. Springer-Verlag, Berlin, 1995.
[100] A. Papageorgiou and S. Paskov. Deterministic simulation for risk management. *J. Portfolio Management*, 25th anniversary issue:122–127, 1999.
[101] A. Papageorgiou and J. Traube. Beating Monte Carlo. *RISK*, 9:63–65, 1996.
[102] S. Paskov. New methodologies for valuing derivatives. In S. Pliska and M. Dempster, editors, *Mathematics of Derivative Securities*, pages 545–582. Cambridge University Press, Cambridge, 1997.
[103] S. Paskov and J. Traube. Faster valuation of financial derivatives. *J. Risk Management*, 22:113–120, 1995.
[104] K. Prause. *The generalized hyperbolic model: Estimation, financial derivatives, and risk measures*. PhD thesis, Albert-Ludwigs-Universität Frieburg, 1999.
[105] P. Protter. *Stochastic integration and differential equations*. Springer-Verlag, New York, 2003.
[106] S. Raible. *Lévy processes in finance: Theory, numerics, and empirical facts*. PhD thesis, Albert-Ludwigs-Universität Frieburg, 2000.
[107] T. Rheinländer. An entropy approach to the Stein/Stein model with correlation. *Finance and Stochastics*, 9(3), 2005.
[108] T. Rheinländer and G. Steiger. The minimal martingale measure for general Barndorff-Nielsen/Shephard models. *Annals Applied Prob.*, 16(3), 2006.
[109] P. Samuelson. Rational theory of warrant pricing. *Industrial management review*, 6:13–32, 1965.
[110] K.-I. Sato. *Lévy Processes and infinitely divisible distributions*. Cambridge University Press, Cambridge, 1999.
[111] L. O. Scott. Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Applications of Fourier inversion methods. *Mathematical Finance*, 7(4):413–424, 1997.
[112] I. Sobol. The distribubtion of points in a cube and the approximate evaluation of integrals. *USSR Computational Math. and Math. Physics*, 7(4):86–112, 1967.
[113] G. Steiger. *The optimal martingale measure for investors with exponential utility function*. PhD thesis, Swiss federal institute of Technology, 2005.

[114] E. Stein and J. Stein. Stock price distributions with stochastic volatility: An analytical approach. *Review Financial Studies*, 4:727–752, 1991.
[115] J. Tilley. Valuing American options in a path simulation model. *Transactions Society of Actuaries*, 43:83–104, 1993.
[116] P. Wilmott, J. Dewynne, and S. Howison. *Option pricing, mathematical models and computation.* Oxford Financial Press, Oxford, 1993.
[117] H. Windcliff, P. A. Forsyth, and K. Vetzal. Pricing methods and hedging strategies for volatility derivatives. *J. Banking Finance*, 30:409–431, 2006.