UNIVERSITY OF OSLO
Department of Mathematics

On Noncommutative Varieties

Candidata Scientiarum Thesis

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Introduction and Scope

This Thesis is based on the first two sections of the unpublished article "The 4-dimensional Sklyanin Algebra at points of finite order" by S. Paul Smith [Smith 93a]. Our aim is to demonstrate an approach to the study of the projective variety associated with a noncommutative algebra.

In commutative algebra, we have the following fundamental result by Serre: Given an algebra A, the category of finitely generated graded A-modules modulo torsion is equivalent to the category of coherent sheaves on $X = \mathrm{Proj}\,(A)$. The noncommutative analogue to Serre's Theorem (and related results on the cohomology of sheaves) still holds for a class of algebras satisfying a certain condition, among them the AS-regular algebras defined in Chapter 7.

So in the noncommutative case, we are still interested in the structure of the category of finitely generated graded A-modules modulo torsion. Our aim is to show that the set $\mathcal{P}$, which in the commutative case corresponds to the set of closed points in $X = \mathrm{Proj}\,(A)$, can be parametrised by the set $\mathcal{F}$ of finite dimensional simple A-modules, and by the set $\mathcal{C}$ of 1-critical modules that possess such simples as quotients.

We then outline a method of finding the finite dimensional simple quotients of certain 1-critical modules, namely the point modules, and finally demonstrate how this method works in some explicit examples.

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Theory

Chapter 1

Graded Algebras

Let us start by giving the definitions of the objects we want to study, and the restrictions we put on them:

Let k be a field, not necessarily algebraically closed, and A a k-algebra. Given a (not necessarily commutative) semigroup $(G,+)$, the algebra A is ***G*-graded** if we have a direct decomposition of the underlying additive group

$$A = \bigoplus_{n \in G} A_n,$$

such that the ring multiplication maps $A_m \otimes A_n$ into A_{m+n}, $\forall m, n \in G$.
A (left) ***G*-graded module** is a module M over a G-graded algebra A, that can be decomposed into

$$M = \bigoplus_{n \in G} M_n,$$

such that the action of A on M maps $A_m \otimes M_n$ into M_{m+n}, $\forall m, n \in G$. A submodule $N \subset M$ is ***G*-graded** if

$$N = \bigoplus_{n \in G} (N \cap M_n) .$$

A ***graded module homomorphism*** $f : M \to N$ of degree m between two graded A-modules is an A-module homomorphism satisfying $f(M_n) \subseteq N_{m+n}$ for all n.

Elements in A_n and M_n are called homogeneous elements of ***degree*** n.

When $G = \mathbb{Z}_{\geq 0}$ and $A_0 = k$, we say that the graded algebra is ***connected***, and when $dim_k A_n < \infty \ \forall n \in \mathbb{Z}_{\geq 0}$, A is ***locally finite***. For $\mathbb{Z}_{\geq 0}$-graded algebras we also define the ***trivial A-module***: It is the 1-dimensional module A/A^+ where $A^+ = \bigoplus_{n \geq 1} A_n$ is the ***augmentation ideal*** of A. The trivial module is simple, as A is connected: $A/A^+ \cong A_0 = k$.

Assumption 1. *We will restrict ourselves to the case when the (noncommutative) algebra A is*

1. *a $\mathbb{Z}_{\geq 0}$-graded connected k-algebra;*
2. *(left and right) Noetherian;*
3. *finitely presented:*
$$A = k\langle x_0, \ldots, x_r\rangle / I, \tag{1.1}$$
where $x_0, \ldots, x_r$ are of degree 1 and I is a finitely generated graded left ideal of degree s.

Ideals are right or left.

Remark 1.0.1. As A is graded Noetherian and $\dim_k(A_0) = 1 < \infty$, A is locally finite [Stephenson 97b].

Before we state the rest of our assumptions, we need to introduce some more theory.

1.1 A Little Category Theory

A ***class*** is a set of sets (or other mathematical objects) that can be unambiguously defined by a property that all its members share. A ***category*** $\mathcal{C}$ consists of

- a class of ***objects*** $\mathrm{ob}(\mathcal{C})$;
- for each ordered pair $(A, B) \in \mathrm{ob}(\mathcal{C})$ a class of ***morphisms*** $\mathrm{Hom}_{\mathcal{C}}(A, B)$ from A to B, writing $\mathrm{Hom}(\mathcal{C})$ for the class of all such morphisms;
- for each ordered triplet $(A, B, C) \in \mathrm{ob}(\mathcal{C})$, a binary operator
$$\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$$
called the ***composition of morphisms***. The operator $\circ$ is associative, and there exists an ***identity morphism*** $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$, $\forall A \in \mathrm{ob}(\mathcal{C})$, such that $1_B \circ f = f = f \circ 1_A$, $\forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$;

An ***abelian*** category is a category $\mathcal{C}$ in which $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, $\forall A, B \in \mathrm{ob}(\mathcal{C})$, and where kernels and cokernels exist and have nice properties. For a more detailed and rigorous definition, see for example `http://en.wikipedia.org/wiki/Abelian_category`.

A ***subcategory*** $\mathcal{S}$ of $\mathcal{C}$ is a category whose objects are in $\mathcal{C}$ and whose morphisms are morphisms in $\mathcal{C}$. The subcategory is ***full*** if $\forall A, B \in \mathrm{ob}(\mathcal{S})$,
$$\mathrm{Hom}_{\mathcal{S}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B).$$

A ***Serre*** (or ***dense***) subcategory is a non-empty full subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ such that for all short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in $\mathcal{C}$,

$$M \in \mathcal{S} \iff M', M'' \in \mathcal{S}.$$

This means that $\mathcal{S}$ is closed under subobjects, quotients and extensions.
We can then construct the quotient category $\mathcal{C}/\mathcal{S} = \mathcal{C}_\Phi$: the localisation of $\mathcal{C}$ in the class of morphisms

$$\Phi = \{\varphi \in \mathrm{Hom}(\mathcal{C}) \mid \mathrm{Ker}(\varphi), \mathrm{Coker}(\varphi) \in \mathcal{S}\}.$$

Note that the morphisms in Φ are isomorphisms in the quotient category, and the quotient is abelian.

1.1.1 Constructing proj(A)

We write ***grmod(A)*** for the abelian category of finitely generated graded A-modules, with morphisms the graded A-module homomorphisms of degree zero. A graded module M is ***bounded above*** (resp. ***bounded below***) if $M_n = 0$ for $n \gg 0$ (resp. $n \ll 0$). If M is bounded above and below, it has ***finite length***.
Let ***tors(A)*** be the full subcategory of modules of finite length. Then $\mathrm{tors}\,(A)$ is a Serre subcategory [Smith 97, Proposition 2.2], so we may form the quotient category ***proj(A)***:

$$\mathrm{proj}\,(A) := \mathrm{grmod}\,(A) \,/\, \mathrm{tors}\,(A)\,,$$

writing $[M]$ for the equivalence class in $\mathrm{proj}\,(A)$ of a module M in $\mathrm{grmod}\,(A)$.

By the definition of the quotient, if $M, N \in \mathrm{grmod}\,(A)$ and $f : M \longrightarrow N$ is a morphism, then f is an ***isomorphism*** in $\mathrm{proj}\,(A)$ if both the kernel and cokernel of f have finite length.

Two objects M and N in $\mathrm{grmod}\,(A)$ are ***equivalent***, and we write $M \sim N$, if they belong to the same equivalence class in $\mathrm{proj}\,(A)$.

Proposition 1.1.1 ([Smith 97, Proposition 3.6])**.** *M and N are equivalent if and only if they contain submodules $M' \cong N'$ (via an isomorphism of degree 0) such that M/M' and N/N' are torsions, that is, for some n, $M_{\geq n} \cong N_{\geq n}$.*

1.1.2 Equivalences

We have established an equivalence relation for graded modules by passing from $\mathrm{proj}\,(A)$ to $\mathrm{grmod}\,(A)$. Two other equivalences for modules over a graded algebra are shift- and twist equivalence:

If M is a graded module, we write $M[n]$ for the ***shifted module*** where $M[n]_m = M_{m+n}$. Two modules M and N are ***shift equivalent*** if $M[n] \sim N$ for some $n \in \mathbb{Z}$, and we write $M \sim_{sh} N$.

Remark 1.1.2. A graded module homomorphism of degree m from M to N can be identified with a graded module homomorphism of degree 0 from M to $N[d]$. This means that graded module homomorphisms between shift equivalence classes of graded modules always can be assumed to have degree 0.

We have a natural action of $\lambda \in k^*$ as A-automorphisms on A, since A is a $\mathbb{Z}_{\geq 0}$-graded k-algebra:

$$\begin{array}{rcl} \lambda : A_n & \longrightarrow & A_n \\ a & \longmapsto & \lambda^n a \end{array}$$

If M is a (not necessarily graded) A-module, the ***twisted module*** M^λ is M as a k-vector space, but with a new A-action given by

$$a * m = \lambda^n a.m \qquad \forall a \in A_n.$$

Two modules M and N are ***twist-equivalent*** if $\exists \lambda \in k^*$ such that $M^\lambda \cong N$, and we write $M \sim_{tw} N$. If M is a graded module, then $M \cong M^\lambda$, $\forall \lambda \in k^*$.

1.2 Growth of Graded Modules

A useful tool in the study of the objects of proj (A) is the Hilbert series and two of its properties, the GK-dimension and the multiplicity:

Equivalent modules have the same GK-dimension and, if they are not finite dimensional, the same multiplicity. This is useful, as it lets us work on representatives of the equivalence classes in proj (A) instead of the equivalence classes themselves.

Let M be a graded module over a $\mathbb{Z}_{\geq 0}$-graded algebra A. The ***Hilbert function*** f_M of M gives the dimension of each of the graded components:

$$f_M(n) = \dim_k(M_n), \quad n \in \mathbb{Z}_{\geq 0}.$$

If, for $n \gg 0$, $f_M(n)$ equals a polynomial $h_M(n)$, that polynomial is called the ***Hilbert polynomial*** of M.
The formal power series

$$H_M(t) = \sum_{i=0}^{\infty} f_M(i) t^i$$

is called the ***Hilbert series*** (or sometimes the ***Poincaré series***) of M.

1.2.1 Dimensions

From the Hilbert series we can define the ***Gel'fand-Kirillov dimension*** d of M: It is the order of the pole of $H_M(t)$ in 1 (we usually abbreviate to GK-dimension). An equivalent definition is

$$d(M) := \limsup_{n\to\infty} \log_n \left(\sum_{i=0}^{n} f_M(i) \right),$$

so a module of finite GK-dimension has (at most) polynomial growth. With this definition, it is clearer that the GK-dimension is a measure of how fast the dimension of the graded components of the module is growing, but as long as we have a Hilbert series, it is easier to calculate the GK-dimension directly. It is also possible to define the GK-dimension for a non-graded module by looking at ascending chains of subspaces, but that is outside the scope of this paper (see for example [Krause 00] for details).

Remark 1.2.1. An interesting property of the GK-dimension is the ***Bergman Gap***: $d(A) = 0$ if and only if A is a finite dimensional algebra. If not, $d(A) \geq 1$.
Borho and Kraft proved that any real number in the interval $[2, \infty[$ occurs as a GK-dimension, and Bergman showed that there is no algebra A such that $1 < d(A) < 2$ [Lenagan 00]. In short, if the GK-dimension of an algebra A is finite, we have

$$d(A) \in \{0, 1\} \cup [2, \infty[\,.$$

The GK-dimension demonstrates a fundamental difference between commutative and noncommutative algebra:

Proposition 1.2.2 ([McConnell 01, 8.1.15]). *If $R = k[x_1, \ldots, x_r]$ is a commutative polynomial ring, then $d(R) = r$. On the other other hand, if A is the free associative (noncommutative) algebra $k\langle x_1, \ldots, x_r\rangle$ with $r \geq 2$, then $d(A) = \infty$.*

This means that when A is commutative, a module $M \in \operatorname{grmod}(A)$ has finite GK-dimension by default since it is isomorphic to a quotient of a free module.

We have the following relationship between the GK-dimension of a module and its submodules and quotients:

Proposition 1.2.3 ([McConnell 01, 8.3.2(ii)]). *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of A-modules, then*

$$d(M) \geq \max\left\{d(M'), d(M'')\right\}.$$

If the sequence splits, equality holds.

Remark 1.2.4. See [McConnell 01, 8.3.4] for an example where the inequality is strict.

Corollary 1.2.5. *Let N be a proper submodule of the module M. Then*

$$d(M) \geq \max\{d(N), d(M/N)\}.$$

Another dimension is the ***classical Krull dimension*** which is a generalisation of the Krull dimension of commutative algebra: Let ***Spec(A)*** be the set of proper prime ideals in the algebra A. Then the classical Krull dimension of A as a module over itself is defined as

$$\mathrm{cl.\,Kdim}(A) := \sup\{h(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(A)\}$$

where $h(\mathfrak{p})$ is the height of the prime ideal $\mathfrak{p}$, i.e. the supremum of the lengths of the chains of prime ideals contained in $\mathfrak{p}$. However, if we want to study simple Noetherian rings, the classical Krull dimension is not of much use, as these rings all have classical Krull dimension zero.

Proposition 1.2.6 ([Krause 00, Theorem 4.5 (a)]). *When A is a finitely generated commutative algebra, then*

$$d(A) = \mathrm{cl.\,Kdim}(A)$$

We also define the classical Krull dimension for a module: If $M \neq 0$ is an A-module, we define the ***classical Krull dimension*** of M as

$$\mathrm{cl.\,Kdim}(M) := \mathrm{cl.\,Kdim}(A/\,\mathrm{Ann}\,(M)).$$

1.2.2 Multiplicity

Another useful concept which can be derived from the Hilbert series of a module of finite GK-dimension is its ***multiplicity*** (or ***Bernstein degree***) $e(M)$:

$$e(M) = (1-t)^{d(M)}\, H_M(t)\,|_{t=1}\,.$$

While the GK-dimension tells us how fast the module grows, the multiplicity says something about its general size:

Example 1.2.7. Take for example the two modules M and N of GK-dimension 1 with Hilbert polynomials

$$\begin{aligned} & h_M(n) = \dim(M_n) = e(M) = 1 \\ \text{and} \quad & h_N(n) = \dim(N_n) = e(M) = e > 1, \quad \forall n \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

This confirms the intuition that N is larger than M.

The multiplicity is additive over short exact sequences of modules with the same GK-dimension:

Proposition 1.2.8. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of modules with the same GK-dimension, then*

$$e(M) = e(L) + e(N).$$

Proof. The short exact sequence of graded modules induces short exact sequences of their graded components:

$$0 \to L_n \to M_n \to N_n \to 0, \ \forall n,$$

and since the dimension is additive over short exact sequences, so is the Hilbert series:

$$H_M(t) = H_L(t) + H_N(t).$$

If d is the GK-dimension of the modules,

$$(1-t)^d \cdot H_M(t)\Big|_{t=1} = (1-t)^d \cdot H_L(t)\Big|_{t=1} + (1-t)^d \cdot H_N(t)\Big|_{t=1},$$

and consequently $e(M) = e(L) + e(N)$. □

1.2.3 Some Explicit Hilbert Series

For certain families of graded modules we can find a general expression for the Hilbert series. Take for example the family of graded A-modules M that have a Hilbert polynomial h_M. Note that by Hilbert's Theorem, when $A = k[x_0, \dots, x_r]$ and the degree of x_i is 1, $\forall i$, this is true for all modules in $\operatorname{grmod}(A)$.

Proposition 1.2.9. *Let M be a $\mathbb{Z}_{\geq 0}$-graded A-module with Hilbert polynomial*

$$h_M(n) = \sum_{i=0}^{d-1} a_i n^i, \quad a_{d-1} \neq 0.$$

Then the Hilbert series of M is on the form

$$H_M(t) = \frac{Q(t)}{(1-t)^d}$$

with $Q \in \mathbb{Z}[t]$, $Q(1) \neq 0$.

Proof. Consider the $\mathbb{Z}_{\geq 0}$-graded module M with Hilbert polynomial h_M. Since h_M is a numerical polynomial in $\mathbb{Q}[n]$, we can find $c_0, \dots, c_{d-1} \in \mathbb{Z}$ such that

$$h_M(n) = \sum_{i=0}^{d-1} a_i n^i = \sum_{i=0}^{d-1} c_i \binom{n}{i}$$

$\forall n \geq N \gg 0$ [Hartshorne 77, Proposition 7.3, p.49]. We can now calculate the Hilbert series of M:
Consider the Hilbert series

$$H_M(t) = \sum_{n=0}^{\infty} f_M(n)t^i$$

where f_M is the Hilbert function. We assume that for $\forall n \geq N \gg 0$, it equals the Hilbert polynomial h_M, and write

$$H_M(t) = \sum_{n=0}^{N-1} f_M(n)t^n + \sum_{n=N}^{\infty} h_M(n)t^n.$$

Since $H_M^f(t) = \sum_{n=0}^{N-1} f_M(n)t^i$ is an easily determined polynomial in $\mathbb{Z}[t]$, let us look at the rest of the formal power series:

$$H_M^h(t) = H_M(t) - H_M^f(t) = \sum_{n=N}^{\infty} h_M(n)t^i.$$

To simplify calculations, we may assume that $N \geq d-1$:
Then $n \geq N \geq d-1 \geq i \geq 0$ and $\binom{n}{i}$ is well defined, and so is $H_M^h(t)$.

We now have

$$\begin{aligned}
H_M^h(t) &= \sum_{n=N}^{\infty} h_M(n)t^n \\
&= \sum_{n=N}^{\infty} \left(\sum_{i=0}^{d-1} c_i \binom{n}{i} t^n \right) \\
&= \sum_{i=0}^{d-1} c_i \left(\frac{t^i}{i!} \sum_{n=N}^{\infty} \frac{n!}{(n-i)!} t^{n-i} \right) \\
&= \sum_{i=0}^{d-1} c_i \left(\frac{t^i}{i!} \cdot \frac{d^i}{dt^i} \left(\sum_{n=N}^{\infty} t^n \right) \right) \\
&= \sum_{i=0}^{d-1} c_i \left(\frac{t^i}{i!} \cdot \frac{d^i}{dt^i} \left(\frac{t^N}{1-t} \right) \right)
\end{aligned}$$

Using the Leibniz Identity:

$$
\begin{aligned}
&= \sum_{i=0}^{d-1} c_i \left(\frac{t^i}{i!} \cdot \sum_{k=0}^{i} \binom{i}{k} \frac{d^k}{dt^k}\left(t^N\right) \cdot \frac{d^{k-i}}{dt^{k-i}}\left(\frac{1}{1-t}\right)\right) \\
&= \sum_{i=0}^{d-1} c_i \left(\frac{t^i}{i!} \cdot \sum_{k=0}^{i} \frac{i!}{k!\,(i-k)!} \frac{N!}{(N-k)!} t^{N-k} \cdot \frac{(i-k)!}{(1-t)^{i-k+1}} \right) \\
&= \frac{t^N}{(1-t)^d} \cdot \sum_{i=0}^{d-1}\sum_{k=0}^{i} c_i \binom{N}{k} t^{i-k} \left(1-t\right)^{d-1-(i-k)} \\
&= \frac{t^N \cdot P(t)}{(1-t)^d}
\end{aligned}
$$

where

$$
P(t) = \sum_{i=0}^{d-1}\sum_{k=0}^{i} c_i \binom{N}{k} t^{i-k} \left(1-t\right)^{d-1-(i-k)}.
$$

Since $P(t)$ and $H^f_M(t)$ are polynomials in $\mathbb{Z}[t]$, we can finally write the Hilbert series of M as

$$
\begin{aligned}
H_M(t) &= H^f_M(t) + H^h_M(t) \\
&= \frac{1}{(1-t)^d}\left(H^f_M(t)\cdot(1-t)^d + t^N \cdot P(t)\right) \\
&= \frac{Q(t)}{(1-t)^d},
\end{aligned}
$$

with $Q(t) = H^f_M(t)\cdot(1-t)^d + t^N \cdot P(t) \in \mathbb{Z}[t]$.
We have $Q(1) \neq 0$, as

$$
Q(1) = P(1) = c_{d-1} = a_{d-1}\cdot(d-1)! \neq 0,
$$

which gives the proof of the proposition and of the following corollary. □

Corollary 1.2.10. *Keeping the assumptions from the proposition, we have*

$$
d(M) = d = \deg\left(h_M\right) + 1.
$$

and

$$
e(M) = Q(1) = a_{d-1}\cdot(d-1)! \in \mathbb{Z}.
$$

This gives us an equivalent definition of the GK-dimension: $d(M)$ is one more than the degree of the Hilbert polynomial h_M. It also gives the multiplicity as a function of the leading coefficient of the Hilbert polynomial and the GK-dimension.

Question 1.2.11. *Are there any restrictions on the $Q(t)$s? Or if can we obtain any polynomial in $k[t]$ that is non-zero in 1?*

Another class of algebras where we know the Hilbert series the following:

Theorem 1.2.12 (Hilbert-Serre [Atiyah 69, Theorem 11.1])**.** *Let A be a commutative Noetherian graded k-algebra, generated by $x_1, \ldots, x_r$ of degrees $d_1, \ldots, d_r > 0$, and let M be a finitely generated graded A-module.*
Then the Hilbert series of M is on the form

$$H_M(t) = \frac{Q(t)}{\prod_{i=1}^{r}(1 - t^{d_i})},$$

where $Q(t) \in \mathbb{Z}[t]$.

1.3 A Technical Assumption

We make a technical assumption on the algebra A, to be able to use the results from the first two sections of [Smith 93a]:

Assumption 2. *If M is a finitely generated graded A-module, then the Hilbert series of M is on the form*

$$H_M(t) = \frac{q_M(t)}{\prod_{i=1}^{d}(1 - t^{d_i})}$$

where $d_i \in \mathbb{N}$, $q_M(t) \in \mathbb{Z}\left[t, t^{-1}\right]$ satisfies $q_M(1) \neq 0$, and only a finite number of d_i's occur as M varies.

Remark 1.3.1. This might be a stronger restriction than we need for most of the results, but among other things it enables us to prove the existence of a critical composition series for Noetherian rings (Proposition 2.2.2).

An immediate consequence of Assumption 2 is that all finitely generated graded A-modules (including A itself) have finite GK-dimension:

$$d(M) = d < \infty,$$

and we have a particularly simple expression for the multiplicity:

$$e(M) = \frac{q_M(1)}{\prod_{i=1}^{d} d_i}.$$

Corollary 1.3.2. *There exists a strictly positive lower bound e_0 on the multiplicity of modules in* $\operatorname{grmod}(A)$.

Proof from [Smith 93a]. The fact that there are only a finite number of d_i's means that we may find

$$m = \max\left\{\prod_{i=1}^{d} d_i\right\},$$

so

$$e(M) = \frac{q_M(1)}{\prod_{i=1}^{d} d_i} \geq \frac{1}{m} = e_0 > 0.$$

□

By the Hilbert-Serre Theorem 1.2.12, finitely generated commutative Noetherian $\mathbb{Z}_{\geq 0}$-graded k-algebras belong to the family of algebras satisfying Assumption 2. This means that we limit our field of study to algebras with modules that have a Hilbert series similar to that of finitely generated graded modules over commutative Noetherian algebras.

This family also includes all algebras A that admit a Hilbert polynomial (Proposition 1.2.9). In this case $d_i = 1$, $\forall i$.

It also includes algebras A that are quotients of a Noetherian graded algebra R of finite global dimension such that

$$H_R(t) = \frac{g(t)}{\prod_{i=1}^{d}(1 - t^{d_i})}$$

for some $g(t) \in \mathbb{Z}[t]$ and $d_i \in \mathbb{N}$.

Remark 1.3.3 ([Lenagan 00])**.** When A is a Noetherian algebra with finite GK-dimension, the GK-dimension ***separates primes***: for all distinct prime ideals $P \subset Q \subset A$

$$d(A/P) \geq d(A/Q) + 1.$$

Note that this fulfills the second technical assumption of [Smith 93a], that $d(A/P) > d(A/Q)$.

For algebras such that the GK-dimension separates primes, we have an immediate relation between the GK-dimension and the classical Krull dimension:

Proposition 1.3.4. *If A is an algebra such that the GK-dimension separates primes, then*

$$d(A) + 1 \geq \mathrm{cl.\,Kdim}(A)$$

Proof. Consider the chain of prime ideals

$$A \supset P_1 \supset P_2 \supset \cdots \supset P_{m-1} \supset P_m.$$

We have

$$\begin{aligned} d(A) \geq d(A/P_m) \geq d(A/P_{m-1}) + 1 \geq \cdots \\ \cdots \geq d(A/P_2) + (m-2) \geq d(A/P_1) + (m-1), \end{aligned}$$

and since the Krull dimension is the supremum of the lengths of prime chains, this proves the proposition. □

The fact that the GK-dimension separates primes means we can use it to distinguish A/P from A/Q when $P \subset Q$ are distinct prime ideals, and use it to establish isomorphisms. It gives us a strong limitation on the growth of the algebra, and a dimension closer (in terms of properties) to the dimensions we are used to from commutative algebra.

Chapter 2

Critical Modules

We now introduce two concepts that in many ways are similar to that of simple modules. Recall that a module is ***simple*** if it has no non-zero proper submodules. Alternatively, if we regard M as a representation of the algebra A, the representation is ***irreducible*** if it has no invariant subspace.

2.1 Irreducible Objects

First, we define the ***irreducible objects*** in $\operatorname{proj}(A)$ to be the equivalence classes of modules $[M]$ such that

$$N \subseteq M \text{ submodule } \Longrightarrow [N] = [M] \text{ or } [N] = [0] .$$

The second concept is that of *criticality*: A module $M \in \operatorname{grmod}(A)$ is ***d-critical*** if $d(M) = d$ and every proper quotient of M is of strictly smaller GK-dimension.

Proposition 2.1.1 ([Artin 91, Proposition 2.30(vi)])**.** *The annihilator of a critical module is a prime ideal, and is also the annihilator of every non-zero submodule.*

As the chapter heading suggests, we are most interested in *1-critical* modules, that is, modules of GK-dimension 1 that have only finite dimensional quotients.

Theorem 2.1.2. *The equivalence classes of 1-critical modules are irreducible in* $\operatorname{proj}(A)$.

Proof. Let N be a proper submodule of the 1-critical module M. Then by definition $d(M/N) = 0$, in other words $M/N \sim 0$. This means that $M \sim N$, which is what we wanted to prove. □

In some of the examples that have been studied in some detail, the 3-dimensional Artin-Schelter regular algebras and their central extensions, and the 4-dimensional Sklyanin algebra, all the irreducible objects in $\operatorname{proj}(A)$ are equivalence classes of 1-critical modules, namely *points* and *fat points*:

A ***point*** in $\operatorname{proj}(A)$ is an equivalence class of 1-critical modules of multiplicity 1. The equivalence classes of 1-critical modules of multiplicity > 1 are called ***fat points***.

However, in general an irreducible object needs be neither:

Example 2.1.3. Let $A = k[x, y]$ with x and y of degrees 1 and 2, respectively. The A-module A/Ax is irreducible in $\operatorname{proj}(A)$, but $e(A/Ax) = \frac{1}{2}$, and is therefore neither a point nor a fat point.

Irreducible objects can also have GK-dimension > 1. Nevertheless, when A is a ***PI-algebra*** (polynomial identity algebra), i.e. when $\exists f \in A[x_1, \ldots, x_n]$ such that $f(a_1, \ldots, a_n) = 0$, $\forall (a_1, \ldots, a_n) \in A^n$, all the irreducible objects in $\operatorname{proj}(A)$ have GK-dimension 1.

Now let us look at the modules of $\operatorname{grmod}(A)$ whose equivalence classes in $\operatorname{proj}(A)$ are points or fat points:

A $\mathbb{Z}_{\geq 0}$-graded A-module M is a ***point module*** if

(i) M is generated in degree zero,

(ii) $M_0 = k$,

(iii) $\dim M_n = 1$, for all $n \geq 0$.

This means that its Hilbert series is $H_M(t) = \frac{1}{1-t}$, and we have $d(M) = e(M) = 1$. Note that this is the same Hilbert series as the homogeneous coordinate ring of a projective point, hence the name.

Proposition 2.1.4. *If M is a point module, then $[M]$ is a point in* $\operatorname{proj}(A)$.

Proof. As M is generated in degree zero by definition, and $\dim(M_0) = 1$, the module is simple and therefore 1-critical. ☐

A ***truncated point module*** of length $s+1$ is a module M that satisfies (i) and (ii), and whose Hilbert function is

$$\dim M_i = \begin{cases} 1 \text{ if } 0 \leq i \leq s \\ 0 \text{ otherwise,} \end{cases}$$

so its Hilbert series is $H_M(t) = \sum\limits_{i=0}^{s} t^i$, and $d(M) = 0$ and $e(M) = s + 1$.

A point module M can be seen as $M = \bigoplus_{i=0}^{\infty} k$. If we instead look at $M = \bigoplus_{i=0}^{\infty} k^e$, M is an ***e-point module***, and we keep conditions (i) and (ii), and replace (iii) with (iii') $\dim M_i = e$, for all $i \geq 1$.

This gives us the Hilbert series $H_{_M}(t) = \frac{e}{1-t}$, and $d(M) = 1$, $e(M) = e$. When $e > 1$ and M is critical, $[M]$ is a fat point.

2.2 Properties of Critical Modules

Critical modules, and especially 1-critical ones, have several properties analogous to those of simple modules. Take for example Schurs Lemma [Farb 93, Lemma 1.2]. We may formulate a version of the lemma for 1-critical modules:

Lemma 2.2.1. *Any homomorphism between 1-critical modules in* $\operatorname{grmod}(A)$ *is either an isomorphism or the zero homomorphism. Therefore* $\operatorname{End}_A(M)$ *is a division ring when* M *is 1-critical.*

Proof. Given a non-zero homomorphism $\varphi : M \to N$, where $M, N \in \operatorname{grmod}(A)$ are 1-critical A-modules, we want to show that both the kernel and the cokernel of φ are in $\operatorname{tors}(A)$.
By definition $\operatorname{Coker}(\varphi) := N/\operatorname{Im}(\varphi)$, and since N is 1-critical and $\operatorname{Im}(\varphi) \neq 0$, $\operatorname{Coker}(\varphi)$ has finite length. On the other hand $M/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi) \sim N$, so $d(M/\operatorname{Ker}(\varphi)) = d(N) = 1$ and as M is 1-critical, $\operatorname{Ker}(\varphi) = 0$.
It is clear that this means that $\operatorname{End}_A(M)$ is a division ring when M is 1-critical. $\square$

Another illustration of the analogy between critical and simple modules is the existence of a composition series:

Let $M \in \operatorname{grmod}(A)$ be a finitely generated graded A-module of finite GK-dimension. A ***critical composition series*** for M is a finite chain of submodules

$$M = M^0 \supset M^1 \supset \cdots \supset M^l = 0$$

(considered as representatives of equivalence classes in $\operatorname{proj}(A)$) such that each quotient M^i/M^{i+1} is critical, and $d\left(M^i/M^{i+1}\right) = d\left(M^i\right)$. The modules M^i/M^{i+1} are called the ***critical composition factors*** associated with this series. We consider them as objects in $\operatorname{proj}(A)$.

Proposition 2.2.2. *When an algebra A satisfying Assumption 2 is Noetherian, then* $\forall M \in \operatorname{grmod}(A)$ *has a critical composition series.*

Proof. Let $M = M^0 \in \operatorname{grmod}(A)$. As A is Noetherian and M finitely generated, M is Noetherian of finite GK-dimension.
If M is critical,

$$M = M^0 \supset M^1 = 0$$

is a critical composition series, and the proposition is true.
If not, we can find a submodule $M^1 \subset M^0$ such that $d\left(M^0/M^1\right) = d\left(M^0\right)$. Since M is Noetherian we may choose M^1 maximal with this property, so M^0/M^1 is critical because of the correspondence theorem for modules. If M^1 is critical,

$$M = M^0 \supset M^1 \supset M^2 = 0$$

is a critical composition series and the proposition is true.
If not, we repeat the process by choosing a maximal submodule M^2 of M^1 such that $d\left(M^1/M^2\right) = d\left(M^1\right)$ and so on, until we arrive at a critical submodule M^l. This algorithm lets us construct a (possibly infinite) critical composition series for M. It remains to show that $l < \infty$:

We divide the composition series into the parts with constant GK-dimension: Let

$$\cdots \supset M^j \supset \cdots \supset M^k \supset \cdots$$

be the part of the composition series with

$$d\left(M^i\right) = d\left(M^i/M^{i+1}\right) = d \quad i \in [\![j,k]\!]$$

We then have the short exact sequences

$$0 \to M^{i+1} \hookrightarrow M^i \twoheadrightarrow M^i/M^{i+1} \to 0$$

of modules with the same GK-dimension, and the multiplicity is additive:

$$\begin{aligned} e\left(M^j\right) &= e\left(M^{j+1}\right) + e\left(M^j/M^{j+1}\right) \\ &= e\left(M^k\right) + \sum_{i=j}^{k-1} e\left(M^i/M^{i+1}\right). \end{aligned}$$

We know that $e\left(M^j\right) \leq e\left(M\right) < \infty$, and by Corollary 1.3.2 $e\left(M^i/M^{i+1}\right) \geq e_0 > 0$ unless $M^i/M^{i+1} = 0$, so the subchain must have finite length. And since $d\left(M\right) \in \mathbb{N}$ (Assumption 2), we have at most $d\left(M\right) + 1$ such chains, so the critical composition series has finite length. □

We would like to have some sort of uniqueness of composition series:

Conjecture 2.2.3. *Two critical composition series have the same length l, and the composition factors are uniquely determined as objects in* $\operatorname{proj}\left(A\right)$.

Remark 2.2.4. In [Smith 93a] Smith proves this for modules of GK-dimension 1, and that is all that is needed for the proofs in the next chapter. However, it seems probable that it is true for any critical module, possibly by using methods similar to those used in [McConnell 01, 6.2.21] and [Goodearl 04, Theorem 15.9].

Remark 2.2.5. In [McConnell 01, 6.2.19] and [Goodearl 04, p. 261] critical modules and the critical composition series are defined in terms of the Krull dimension instead of the GK-dimension. They go on to prove (in [McConnell 01, 6.2.21] and [Goodearl 04, Theorem 15.9]) that a Noetherian module M has a critical composition series which is unique in the sense that two different critical composition series have the same length, and you can pair the composition factors so that corresponding factors have an isomorphic non-zero submodule.
These proofs do not need Assumption 2, as do the proof of Proposition 2.2.2, because they use the fact that the Krull dimension is, by definition, the deviation on the lattice of submodules of M which, again by definition, means that in a descending chain of submodules of M all but a finite number of submodules have Krull dimension strictly lower than that of M (see [McConnell 01, 6.1.2]). The GK-dimension does not have this property, so we must compensate with Assumption 2 to ensure that the composition series is finite.

CHAPTER 3

Finite Dimensional Simples and 1-critical Modules

In this chapter we will look at the relationship between finite dimensional simple A-modules and 1-critical modules, which leads us to a description of a candidate for a noncommutative projective space and noncommutative projective varieties.

3.1 Modules over Prime Noetherian Algebras

The following results are used in Section 3.3, for proofs see [Smith 93a].

A k-algebra is ***prime*** if the product of any two of its ideals is zero if and only if one of them is the zero ideal.

Proposition 3.1.1 ([Smith 93a, 1.1]). *Suppose A prime and Noetherian, and M and N finitely generated critical graded A-modules with $d(M) = d(N) = d(A)$.*
Then $M[-p]$ embeds in N for some $p \in \mathbb{Z}$.

This helps us prove the following proposition:

Proposition 3.1.2 ([Smith 93a, 1.2]). *Suppose A Noetherian, M and N finitely generated critical A-modules, and that*

$$d(A/\operatorname{Ann}(M)) = d(A/\operatorname{Ann}(N)) = 1.$$

Then

$$\operatorname{Ann}(M) = \operatorname{Ann}(N) \Longleftrightarrow M \sim_{sh} N.$$

Remark 3.1.3. It is possible for non-isomorphic point modules to be equivalent (i.e. give the same object in $\operatorname{proj}(A)$). However a consequence of Proposition 3.1.1 is that this does not occur over a Noetherian PI-ring.

More generally, Proposition 3.1.1 gives us the following condition for when two equivalent point modules are isomorphic:

Proposition 3.1.4 ([Smith 93a])**.** *Let M be a point module over a Noetherian k-algebra A, such that $d\,(A/\operatorname{Ann}(M)) = 1$.*
Then if N is another point module,

$$[N] = [M] \Longrightarrow N \cong M.$$

3.2 The Cone of a Non-trivial Finite Dimensional Module

A useful tool when studying the relationship between non-trivial finite dimensional simples and 1-critical modules is the cone:
Starting with a non-trivial finite dimensional simple A-module S, we define the ***cone*** of S:

$$\widetilde{S} := S \otimes k[t]$$

with $a \in A_n$ acting by $a.(s \otimes t^i) = (a.s) \otimes t^{i+n}$ on $\widetilde{S}$.

Proposition 3.2.1. *Let S be a non-trivial finite dimensional simple A-module.*
Then $\widetilde{S} \in \operatorname{grmod}(A)$, $\widetilde{S}_n = S \otimes kt^n$, and we have $e(\widetilde{S}) = \dim(S)$ and $d(\widetilde{S}) = 1$.

Proof. As S is finite dimensional, $\widetilde{S}$ is a finitely generated graded A-module, and it is obvious that $\widetilde{S}_n := S \otimes kt^n$.
Calculating the Hilbert series of $\widetilde{S}$:

$$h_{\tilde{S}}(i) = \dim \widetilde{S}_i = \dim S$$
$$\Rightarrow H_{\tilde{S}}(t) = \sum_{i=0}^{\infty} \dim S \cdot t^i = \frac{\dim S}{1-t}$$

which gives us

$$e(\widetilde{S}) = \dim(S),\ d(\widetilde{S}) = 1$$

□

At this point, recall that the morphisms of $\operatorname{proj}(A)$ are of degree 0, and that we in Assumption 1 restricted our field of study to $\mathbb{Z}_{\geq 0}$-graded algebras (and therefore $\mathbb{Z}_{\geq 0}$-graded modules).

The passage from a non-trivial finite dimensional simple module to its cone has a universal property identifying

$$\operatorname{Hom}_{A-mod}(M, S) \quad \text{with} \quad \operatorname{Hom}_{\operatorname{grmod}(A)}(M, \widetilde{S})$$

for any module $M \in \operatorname{grmod}(A)$:

Proposition 3.2.2 ([Smith 93a, Proposition 2.2(b)]). *Let $\pi : \widetilde{S} \to S$ be the A-module homomorphism defined by $\pi(s \otimes t^j) = s$. If $M \in \operatorname{grmod}(A)$ and $\psi \in \operatorname{Hom}_{A-mod}(M, S)$, then there exists a unique A-module homomorphism $\widetilde{\psi} \in \operatorname{Hom}_{\operatorname{grmod}(A)}(M, \widetilde{S})$ such that the following diagram is commutative:*

$$\begin{tikzcd} & M \arrow[dl, "\exists!\widetilde{\psi}"'] \arrow[d, "\psi"] \\ \widetilde{S} \arrow[r, "\pi"'] & S \end{tikzcd}$$

Moreover $\widetilde{\psi}(m) = \psi(m) \otimes t^n$, $\forall m \in M_n$.

Corollary 3.2.3 ([Smith 93a, Proposition 2.2(c)]). *If S is a simple quotient of a 1-critical module $M \in \operatorname{grmod}(A)$, then M embeds in $\widetilde{S}$.*

Proof. We consider the image of $\widetilde{\psi}$ when $S = M/N$ is a simple quotient and M is 1-critical:
S is a simple A-module, so it can be written as A/I where I is a maximal ideal. Since S is non-trivial, $\exists a \in A^+ \setminus (A^+ \cap I)$ of degree $n > 0$ which means that $\exists m \in M$ such that $a.\psi(m) = \psi(a.m) \neq 0 \Longrightarrow \widetilde{\psi}(a.m) = a.\widetilde{\psi}(m) \neq 0$. As a sends $\operatorname{Im}(\widetilde{\psi})_i$ into $\operatorname{Im}(\widetilde{\psi})_{i+n}$, we must have $d(\operatorname{Im}(\widetilde{\psi})) \geq 1$. But $\operatorname{Im}(\widetilde{\psi}) \cong M$ or a quotient of M, and since M is 1-critical, all its quotients have GK-dimension < 1, which means that $\operatorname{Im}(\widetilde{\psi}) \cong M$, and M is embedded in $\widetilde{S}$. □

Combining this with Proposition 3.2.1, we get

Corollary 3.2.4. *If M is a 1-critical graded module with quotient S, then $e(M) \leq dim(S)$.*

The A-module homomorphism π is surjective with kernel the submodule $\widetilde{S}(t-1)$, so

$$S \cong \widetilde{S}/\widetilde{S}(t-1).$$

By considering the quotient modules $\widetilde{S}/\widetilde{S}(t-\lambda)$, $\lambda \in k^*$, we find that

Corollary 3.2.5 ([Smith 93a, Proposition 2.2(d)]). *For each $\lambda \in k^*$, S^λ is a quotient of $\widetilde{S}$.*

We have two useful characterisations of the annihilator of $\widetilde{S}$:

Proposition 3.2.6 ([Smith 93a, Proposition 2.2(e)]). $\operatorname{Ann}(\widetilde{S})$ *may be characterised in either of the following ways:*

(i) it is the unique prime graded ideal P such that $d(A/P) = 1$ and $P \subset$ $\operatorname{Ann}(S)$*;*

(ii) it is the largest graded ideal contained in $\operatorname{Ann}(S)$.

Proof. In [Smith 93a, Proposition 2.2], it is shown that $\operatorname{Ann}(\widetilde{S})$ is the sum of all the graded two-sided ideals of A contained in $\operatorname{Ann}(S)$. This is the largest graded ideal contained in $\operatorname{Ann}(S)$, which together with the second technical assumption (that $d(A/P) \neq d(A/Q)$ if $P \subset Q \subset A$ are distinct prime ideals) proves the uniqueness in characterisation (i). □

3.3 Studying the Geometry of the Algebra

Following the reasoning in [Smith 93a], we now establish bijections between three different sets associated with the (Noetherian) algebra A. By studying these sets, we have different approaches to studying the geometry of the algebra.

We start by exploring the connection between finite dimensional simple modules and 1-critical graded modules:

Theorem 3.3.1 ([Smith 93b, Lemma 4.1])**.** *Let A be a Noetherian graded k-algebra with $d(A) \geq 1$, and let S be a finite dimensional simple A-module. Then S is a quotient of some 1-critical graded A-module.*

Proof. If S is the trivial module, it is a quotient of every point module and the result is true. Let us now assume S non-trivial. Then S is a quotient of $\widetilde{S}$. By Proposition 2.2.2 $\widetilde{S}$ has a critical composition series, and since $d(\widetilde{S}) = 1$, either $\widetilde{S}$ is 1-critical, or one or more of its composition factors is. Clearly, if $\widetilde{S}$ is not 1-critical, S is a quotient of one of these composition factors, so S is a quotient of a 1-critical module. □

It is possible that the only simple quotient of a 1-critical module M is the trivial module, however we have the following criterion for when a non-trivial simple quotient exists:

Proposition 3.3.2 ([Smith 93a, Proposition 2.4])**.** *Let M be a 1-critical graded A-module. Then M has a non-trivial (finite dimensional) simple quotient if and only if $d\,(A/\operatorname{Ann}(M)) = 1$.*

Proof from [Smith 93a]. ($\Rightarrow$) Let S be a non-trivial finite dimensional simple quotient of M. Then by Corollary 3.2.3 M embeds in $\widetilde{S}$, so $\operatorname{Ann}(S) \supseteq \operatorname{Ann}(M) \supseteq \operatorname{Ann}(\widetilde{S})$. $\operatorname{Ann}(M)$ is a graded ideal, so by the characterisations of $\operatorname{Ann}(\widetilde{S})$ in Proposition 3.2.6, $\operatorname{Ann}(M) = \operatorname{Ann}(\widetilde{S})$ and $d\,(A/\operatorname{Ann}(M)) = 1$.

($\Leftarrow$) By applying [Smith 93a, Lemma 1.6] to M as an $A/\operatorname{Ann}(M)$-module, M is a free $k[x]$-module of finite rank, so if $0 \neq \nu \in k$ then $M/(x-\nu)M$ is a non-zero finite dimensional quotient of M that must in turn have a simple quotient. □

Now that we have established that non-trivial finite dimensional simple modules are quotients of modules M with $d(A/\operatorname{Ann}(M)) = 1$, we ask ourselves which 1-critical graded modules have non-trivial simple quotients in common:

Proposition 3.3.3 ([Smith 93a, Proposition 2.3])**.** *Two 1-critical graded modules have a common non-trivial simple quotient if and only if they are shift equivalent. In this case they have exactly the same non-trivial simple quotients.*

Elaboration of proof from [Smith 93a]. ($\Rightarrow$) Let S be the common non-trivial simple quotient of the 1-critical graded modules M and N. Then $\operatorname{Ann}(S) \supset \operatorname{Ann}(M) \supset \operatorname{Ann}(\widetilde{S})$ and $\operatorname{Ann}(S) \supset \operatorname{Ann}(N) \supset \operatorname{Ann}(\widetilde{S})$, and since $\operatorname{Ann}(M)$ and $\operatorname{Ann}(N)$ are graded ideals, Proposition 3.2.6(ii) gives us $\operatorname{Ann}(M) = \operatorname{Ann}(N) = \operatorname{Ann}(\widetilde{S})$. By Proposition 3.3.2 $d(A/\operatorname{Ann}(M)) = 1$, so by Proposition 3.1.2 $M{\sim_{sh}}N$.

($\Leftarrow$) If M and M' are two shift equivalent 1-critical graded modules, there is a 1-critical graded module N such that $N[i]$ embeds in M and $N[j]$ in M', for some $i, j \in \mathbb{Z}$. Therefore $N[i]$ and consequently N (respectively $N[j]$ and consequently N) have the same non-trivial simple quotients as M (respectively M'), which means that M and M' have exactly the same non-trivial simple quotients. $\square$

We then look at twist-equivalence classes of non-trivial finite dimensional simple modules. In some cases, all the non-trivial finite dimensional simple modules are twist-equivalent:

Theorem 3.3.4 ([Smith 93a, Proposition 2.5])**.** *Let k be algebraically closed, and suppose that A is prime Noetherian with $d(A) = 1$.*
Then A has only one twist-equivalence class of non-trivial finite dimensional simple modules.

Corollary 3.3.5. *Let M be a 1-critical graded module over a Noetherian A which is an algebra over an algebraically closed field k. If M has a non-trivial finite dimensional simple quotient, then it has only one twist-equivalence class of such quotients.*

The next theorem summarises the above analysis:

Theorem 3.3.6 ([Smith 93a, Corollary 2.6])**.** *Suppose that A is Noetherian, and that k is algebraically closed.*
Then there is a bijection between the following sets:

- $\mathcal{F} :=$ {*twist-equivalence classes of non-trivial finite dimensional simple A-modules*};

- $\mathcal{C} := \{$ *shift-equivalence classes of 1-critical graded modules such that* $d(A/Ann(M)) = 1\}$*;*
- $\mathcal{P} := \{P \in \operatorname{Spec}(A) \mid P$ *is graded and* $d(A/P) = 1\}$.

Proof (based on proof from [Smith 93a]). Proposition 3.3.3 shows that $\mathcal{F}$ is in bijection with the set of shift-equivalence classes of 1-critical graded modules that possess a non-trivial simple quotient. By Proposition 3.3.2, this is actually $\mathcal{C}$.
As the annihilator of a 1-critical module is a prime ideal, and by using Proposition 3.1.2, an object in $\mathcal{C}$ determines a unique prime ideal in $\mathcal{P}$. It now remains to show that a given prime ideal in $\mathcal{P}$ determines a unique shift equivalence class in $\mathcal{C}$, i.e. that it is the annihilator of a 1-critical module. The 1-critical module is unique up to shift-equivalence by Proposition 3.1.2.

Consider the critical composition series of A/P, $P \in \mathcal{P}$:

$$A/P = M^0 \supset M^1 \supset \cdots \supset M^l = 0.$$

As $d\,(A/P) = 1$, either A/P is 1-critical or at least one of its composition factors is. If A/P is 1-critical, it represents a unique shift-equivalence class determined by their common annihilator P.

If A/P is *not* 1-critical, it is clear that P annihilates any 1-critical composition factor M^i/M^{i+1}:

$$P \subseteq \operatorname{Ann}\left(M^i/M^{i+1}\right).$$

By Remark 1.3.3, then if the inclusion is strict,

$$d\,(A/P) \geq d\left(A/\operatorname{Ann}\left(M^i/M^{i+1}\right)\right) + 1$$

The A-module A/P has a non-trivial finite dimensional simple quotient since $\operatorname{Ann}(A/P) = P$ and $d\,(A/P) = 1$ (Proposition 3.3.2). By the same argument as in the proof of Theorem 3.3.1 so does a 1-critical composition factor M^i/M^{i+1}, which means that $d\left(A/\operatorname{Ann}\left(M^i/M^{i+1}\right)\right) = 1$ and we have

$$1 = d\,(A/P) \geq d\left(A/\operatorname{Ann}\left(M^i/M^{i+1}\right)\right) + 1 = 2$$

This contradiction confirms that

$$P = \operatorname{Ann}\left(M^i/M^{i+1}\right),$$

and P is the annihilator of a 1-critical module. This concludes the proof of the theorem. □

Remark 3.3.7. If in addition to the conditions in Theorem 3.3.6 the algebra A is a PI-algebra, then $\mathcal{C}$ consists of all shift-equivalence classes of 1-critical modules.

We may now use properties of the two first sets to gain information about $\mathcal{P}$:

Proposition 3.3.8 ([Smith 93a, Proposition 2.7]). *Suppose that*

$$\bigcap Ann(S) = 0$$

where the intersection is taken over all the finite dimensional simple A-modules S. Then

(a) $\bigcap Ann(C) = \bigcap_{P \in \mathcal{P}} P = 0$ *where this intersection is taken over all 1-critical graded modules C;*

(b) if $d(A) \geq 2$, then $\mathcal{P}$ is infinite;

(c) if $d(A) \geq 2$, then A has infinitely many shift-inequivalent graded 1-critical modules.

Remark 3.3.9. Two cases where the hypothesis in Proposition 3.3.8 is satisfied, is when A is semiprime and satisfies a polynomial identity, and when $d(A) = 2$ and A is a prime Noetherian algebra of countable dimension over an uncountable field.

In the next chapter we will explore a method to determine $\mathcal{F}$ or $\mathcal{C}$ in order to describe $\mathcal{P}$.

Chapter 4

Finite Dimensional Simple Modules

In this chapter we will describe a method for finding the simple quotients of point modules of an algebra A: multilinearisation. Note that this only describes part of the set $\mathcal{F}$ in Theorem 3.3.6, as $\mathcal{F}$ may include quotients of 1-critical modules of multiplicities $\neq 1$.

The first section is based on the section on multilinearisation in [Artin 90]. As this is included mainly to use in the examples, we only state the results needed for AS-regular algebras of dimension 3. For more general results on the multilinearisation, see the original article.

4.1 Multilinearisation

Let $T = k\langle x_0, \ldots, x_r\rangle$ be the free associative k-algebra in $r+1$ variables of degree 1, and write $V = T_1^* \cong k^{r+1}$ for the dual space of T_1. A homogeneous element $f \in T_n = T_1^{\otimes n}$ defines a linear map

$$\widetilde{f} : V \otimes \cdots \otimes V = V^{\otimes n} \to k,$$

or equivalently a multilinear form $\widetilde{f}_* : V \times \cdots \times V = V^n \to k$. We call $\widetilde{f}$ the ***multilinearisation*** of the polynomial f. As $\widetilde{f}$ is homogeneous, its set of zeros $Z(f)$ is in $(\mathbb{P}^r)^n$.

Remark 4.1.1. For two homogeneous polynomials f and g, we have

$$\widetilde{f \cdot g} = \widetilde{f} \cdot \widetilde{g}.$$

We may now do the same for a graded ideal: Given a graded ideal I, we define the ***multilinearised ideal***

$$\widetilde{I} := \bigoplus_{n=0}^{\infty} I_n, \quad \widetilde{I_n} := \left\{ \widetilde{f} \mid f \in I_n \right\}.$$

If we now consider the graded algebra

$$A = k\left\langle x_0, \ldots, x_r \right\rangle / I$$

defined in (1.1), its ***multilinearisation*** is the family $\{I_n\}$. Let

$$\Gamma_n := Z(\widetilde{I_n}) \subseteq (\mathbb{P}^r)^n$$

be the zeros of $\widetilde{I_n}$, and let Γ denote the inverse limit of the sets Γ_n.

Proposition 4.1.2 ([Artin 90, Proposition 3.9])**.** *There is a one-to-one correspondence between points in Γ_n and truncated point modules of length $n+1$.*

Corollary 4.1.3 ([Artin 90, Corollary 3.13])**.** *The points of Γ are in one-to-one correspondence with point modules.*

Proposition 4.1.4. *Let A be an AS-regular algebra of dimension 3, on the form of (1.1). Then*

$$\Gamma \cong \Gamma_s.$$

This means that the point modules of A are parametrised by $\Gamma_s = Z(\widetilde{I_s}) \subseteq (\mathbb{P}^r)^s$. To ease notation, we denote a point P of $\Gamma_s \subseteq (\mathbb{P}^r)^s$ as a matrix:

$$P = (a_{i,j}) \in \mathcal{M}_{r+1 \times s}(k)\,.$$

We note the row vectors $P_i = [a_{i,1}, \ldots, a_{i,s}]$, $i \in [\![0, r]\!]$, and the column vectors $P^j = [a_{0,j}, \ldots, a_{r,j}]^t$, $j \in [\![1, s]\!]$. As noted earlier, the column vectors correspond to points in $\mathbb{P}^r$.

In general, we cannot be sure that Γ is finite dimensional, and a point module is given by an "infinite matrix"

$$P^\infty = \left[P^1, P^2, \ldots\right] = (a_{i,j})\,, i \in [\![0, r]\!], j \in \mathbb{Z}^+.$$

The fact that the point modules are parametrised by $P \in \Gamma_s$ means that we have a map

$$\begin{aligned} \sigma : \qquad \Gamma_s \quad & \longrightarrow \Gamma_s \\ \left(P^1, \ldots, P^s\right) & \longmapsto \left(P^2, \ldots, P^s, P^{s+1}\right) \end{aligned}$$

determining P^{s+1} in terms of $\left\{P^1, \ldots, P^s\right\}$.

Proposition 4.1.5. *The n-dimensional quotients of point modules are given by fix points of σ^n, i.e. they correspond to point modules such that P^∞ is periodic.*

Remark 4.1.6. This is true for point modules of arbitrary algebras, as long as such a σ exists.

4.2 Solving the Multilinear Equations

We start by fixing the notation we will use later in the examples:
Let A be an algebra satisfying Assumption 1 and 2, and $\{f_i | i \in [\![1, m]\!]\}$ the homogeneous polynomials of degree s generating the ideal I in (1.1). We write

$$f = [f_1, \ldots, f_m]^t .$$

Consider the matrix $M \in \mathcal{M}_{m,r+1}(A)$ such that

$$f = M \cdot [x_0, \ldots, x_r]^t .$$

Multilinearising gives us m linear equations in $\{x_{0,s}, \ldots, x_{r,s}\}$:

$$\widetilde{f} = \left[\widetilde{f_1}, \ldots, \widetilde{f_m}\right]^t = \widetilde{M} \cdot [x_{0,s}, \ldots, x_{r,s}]^t = 0.$$

We want to determine the zeros $P \in \Gamma_s \subset (\mathbb{P}^r)^s$ of f. As P^s is a point in $\mathbb{P}^r$, $\Gamma_s \neq \emptyset$ unless the rank of $\widetilde{M}$ is strictly less than s.
So determining Γ_s comes down to finding the points $P = (a_{i,j}) \in (\mathbb{P}^r)^s$ such that

$$\mathrm{rk}\left(\widetilde{M}\,(P)\right) < r + 1 \text{ and } \widetilde{f}(P) = 0.$$

4.3 Simple Quotients of 1-critical Modules

Let ρ be the representation corresponding to the quotient given by $P \in \Gamma$. Then ρ is a s-dimensional representation on the form

$$\rho : x_i \longmapsto \rho_i = \begin{bmatrix} 0 & \cdots & \cdots & 0 & a_{i,s} \\ a_{i,1} & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{i,s-1} & 0 \end{bmatrix},$$

where ρ_i correspond to the row vector P_i.

We now want to determine when ρ is irreducible, i.e. when k^s has no invariant subspaces for the action of ρ. In the case where the relations of the algebra are of degree $s = 2$, this means that none of the ρ_i have eigenvectors in common. This can in fact be checked directly on the matrix P:

Proposition 4.3.1. *Let A be an algebra on the form* (1.1) *with I a graded left ideal of degree* 2*, such that $\Gamma \cong \Gamma_2 \subseteq (\mathbb{P}^r)^2$ parametrise the point modules. Let P be a point in Γ corresponding to a quotient of a point module. Then the quotient associated with P is simple if and only if the rank of P is* 2.

Proof. Let

$$P = \begin{bmatrix} a_{0,1} & a_{0,2} \\ \vdots & \vdots \\ a_{r,1} & a_{r,2} \end{bmatrix} \in \mathcal{M}_{r+1,2}(k)$$

be the point in Γ associated with the representation ρ of A:

$$\rho_i = \begin{bmatrix} 0 & a_{i,2} \\ a_{i,1} & 0 \end{bmatrix}.$$

The only subspaces of k^2 stable for each of the ρ_is are the eigenspaces, so we need to check whether there exists an eigenspace in common for all the ρ_is. Because of the shape of the matrices ρ_i none of the coefficients of the eigenvectors are zero, so we can choose to calculate eigenvectors with first coefficients 1. This means that we can compare eigenvectors instead of eigenspaces, and since the eigenvectors are of dimension 2, we only have to compare the last coefficient. The matrix ρ_i has eigenvalues

$$\lambda_i = \pm\sqrt{a_{i,1} \cdot a_{i,2}},$$

and eigenvectors $[1, y_i]^t$:

$$\begin{bmatrix} 0 & a_{i,2} \\ a_{i,1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y_i \end{bmatrix} = \begin{bmatrix} a_{i,2} \cdot y_i \\ a_{i,1} \end{bmatrix} = \begin{bmatrix} \pm\sqrt{a_{i,1} \cdot a_{i,2}} \\ \pm\sqrt{a_{i,1} \cdot a_{i,2}} \cdot y_i \end{bmatrix}$$

which gives us

$$y_i = \pm\sqrt{\frac{a_{i,1}}{a_{i,2}}}.$$

So we need to check if we can find $j \neq i$ such that

$$\frac{a_{i,1}}{a_{i,2}} \neq \frac{a_{j,1}}{a_{j,2}}$$

that is, whether

$$\det \begin{bmatrix} a_{i,1} & a_{i,2} \\ a_{j,1} & a_{j,2} \end{bmatrix} = 0 \quad \forall j \neq i$$

which is in fact checking if the rank of P is 2.
This proves the proposition. □

In the general case, the comparison is a little more complicated. We now need to compare *linear combinations* of eigenvectors, and this makes the coefficients we compare more complex, even though we still can make the assumption that the first coefficient of the eigenvectors is 1 to simplify the calculations. However, we can still find a criterion for the simplicity of a module.

Remark 4.3.2. For a straightforward calculation of the invariant subspaces for the representation, we depend upon a comparison of the eigenspaces of each of the ρ_i. However, if we can find an $a_{i,j} = 0$, the only eigenvalue of ρ_i is zero and we cannot use eigenspaces to calculate the invariant subspaces. A solution could be to find the invariant subspaces in common for the ρ_i such that none of the $a_{i,j}$ are zero, and then check directly if they are invariant for the rest. In the following, we assume that the point $P \in \Gamma$ is such that $a_{i,j} \neq 0$, $\forall i, j$.

Let us first calculate the eigenvalues of the representation:
When $a_{i,j} \neq 0$, $\forall i, j$, then ρ_i is diagonalisable, with characteristic polynomial

$$X^s - \prod_{j=1}^{s} a_{i,j}$$

which gives us the eigenvalues

$$\lambda_\alpha = \xi^\alpha \left(\prod_{j=1}^{s} a_{i,j} \right)^{1/s} = \xi^\alpha \cdot g_i, \quad \alpha \in [\![1, s]\!]$$

where ξ is a primitive s'th root of unity. Note that

$$g_i = \left(\prod_{j=1}^{s} a_{i,j} \right)^{1/s}$$

is the geometric mean of P_i.

Remark 4.3.3. We choose the same s'th root of unity ξ for all the ρ_i.

We obtain a criterion for the simplicity of a quotient of a point module by considering the following set $\mathcal{S}$:

$$\mathcal{S} := \left\{ S_{m,\beta} = \left(\xi^{\alpha - \alpha j} \cdot \widetilde{a}_{m,j} - \xi^{\beta - \alpha j} \cdot \widetilde{a}_{i,j} \right)_{\alpha, j} \right\}_{\substack{m \in [\![0,r]\!],\ m \neq i \\ \beta \in [\![1,s]\!]}} \subset \mathcal{M}_s(k)$$

Proposition 4.3.4. *A quotient of a point module is simple if and only if for $\forall I \subsetneq [\![1, s]\!]$, when we remove the rows with numbers in I from the matrices $S_{m,\beta} \in \mathcal{S}$, $\exists$ at least $\#I+1$ βs for each m such that the rank of the remainder of the matrix $S_{m,\beta}$ is $s - \#I$.*

Proof. We start by calculating the eigenvector $v_\alpha = [1, v_{\alpha,2}, \ldots, v_{\alpha,s}]$ of ρ_i corresponding to the eigenvalue λ_l:

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 & a_{i,s} \\ a_{i,1} & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{i,s-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ v_{\alpha,2} \\ \vdots \\ \vdots \\ v_{\alpha,s} \end{bmatrix} = \begin{bmatrix} a_{i,s} v_{\alpha,s} \\ a_{i,1} \\ a_{i,2} v_{\alpha,2} \\ \vdots \\ a_{i,s-1} v_{\alpha,s-1} \end{bmatrix} = \lambda_\alpha \cdot \begin{bmatrix} 1 \\ v_{\alpha,2} \\ \vdots \\ \vdots \\ v_{\alpha,s} \end{bmatrix}$$

which gives us the relations

$$v_{\alpha,k} = \frac{\prod_{j=1}^{k-1} a_{i,j}}{\lambda_\alpha^{k-1}}, \quad k \in [\![2, s]\!]$$

The representation ρ is reducible if there exists a subspace of $V = k^s$ that is stable for ρ, that is, if we can find a linear combination of the eigenspaces in common for all the ρ_i. Since ρ_i is diagonalisable $\forall i$, it suffices to show that we can choose $N < s$ eigenvectors of ρ_i such that for $\forall m \neq i$, there exists N eigenvectors w_β of ρ_m that can be written as linear combinations of the N eigenvectors of ρ_i.

Let us start by writing the eigenvector w_β of ρ_m with eigenvalue $\mu_\beta = \xi^\beta \cdot g_m$ as a linear combination of less than s eigenvectors $\{v_\alpha\}$ of ρ_i with eigenvalues $\lambda_\alpha = \xi^\alpha \cdot g_i$, that is:

$$\left(\sum_{\alpha=1}^{s} X_\alpha\right) \cdot w_\beta = \sum_{\alpha=1}^{s} (X_\alpha \cdot v_\alpha)$$

with at least one X_α equal to zero (since all the sets of eigenvectors generate V, w_β is a linear combination of *all* the v_α's). Note that we still choose $w_{\beta,1} = v_{\alpha,1} = 1$, $\forall \alpha$.
For each coefficient this means

$$\left(\sum_{\alpha=1}^{s} X_\alpha\right) \cdot w_{\beta,k} = \sum_{\alpha=1}^{s} (X_\alpha \cdot v_{\alpha,k})$$
$$\iff \left(\sum_{\alpha=1}^{s} X_\alpha\right) \prod_{j=1}^{k-1} \frac{a_{m,j}}{\mu_\beta} = \left(\sum_{\alpha=1}^{s} \frac{X_\alpha}{\lambda_\alpha^{k-1}}\right) \prod_{j=1}^{k-1} a_{i,j}$$

By recursion on k, we find that

$$\left(\sum_{\alpha=1}^{s} \frac{X_\alpha}{\lambda_\alpha^{j-1}}\right) \frac{a_{m,j}}{\mu_\beta} = \left(\sum_{\alpha=1}^{s} \frac{X_\alpha}{\lambda_\alpha^{j}}\right) a_{i,j},$$

which gives us s linear polynomials in $\{X_\alpha\}$:

$$\sum_{\alpha=1}^{s} X_\alpha \cdot \frac{\lambda_\alpha \cdot a_{m,j} - \mu_\beta \cdot a_{i,j}}{\lambda_\alpha^j} = 0.$$

Whether or not we can find $[X_1, \ldots, X_s] \neq 0$ (with at least one $X_\alpha = 0$) satisfying these polynomials depends on the rank of the matrix

$$\left(\frac{\lambda_\alpha \cdot a_{m,j} - \mu_\beta \cdot a_{i,j}}{\lambda_\alpha^j}\right)_{\alpha,j} \in \mathcal{M}_s(k)$$

when one or more rows are removed, corresponding to the eigenvectors *not* used. If the rank is maximal, i.e. if when removing N rows, the rank is $s-N$, then $[X_1,\dots,X_s]=0$ is the only possible solution, and the eigenvector w_β of ρ_m is *not* a linear combination of the selected $s-N$ eigenvectors of ρ_i.
For each selection of $s-N$ eigenvectors of ρ_i, we need to verify that for at least one of the other ρ_ms, at least $N+1$ of the eigenvectors are not linear combinations of the original vectors. If this is true for all selections of $s-N$, $1 \le N < s$, eigenvectors of ρ_i, then the representation ρ is simple.

To simplify the comparison, we start by normalising the row vectors P_i:

$$\widetilde{P}_i = [\widetilde{a}_{i,1},\dots,\widetilde{a}_{i,s}]\,, \quad \text{where} \quad \widetilde{a}_{i,j} = \frac{a_{i,j}}{g_i}$$

where g_i is the geometric mean of P_i. Recall that $\lambda_\alpha = \xi^\alpha \cdot g_i$ and $\mu_\beta = \xi^\beta \cdot g_m$. This gives us

$$\left(\frac{\xi^\alpha \cdot g_i \cdot g_m \cdot \widetilde{a}_{m,j} - \xi^\beta \cdot g_m \cdot g_i \cdot \widetilde{a}_{i,j}}{(\xi^\alpha \cdot g_i)^j}\right)_{\alpha,j}.$$

The rank of the matrix does not change if we multiply each row j with

$$\frac{g_i^{j-1}}{g_m},$$

so we arrive, finally, at the matrix

$$S_{m,\beta} = \left(\xi^{\alpha-\alpha j}\cdot \widetilde{a}_{m,j} - \xi^{\beta-\alpha j}\cdot \widetilde{a}_{i,j}\right)_{\alpha,j} \in \mathcal{M}_s(k), \quad m \neq i,\ \beta \in [\![1,s]\!].$$

$\square$

This comparison process is rather cumbersome to do by hand (we are calculating the ranks of up to $2^s - s$ submatrices of each of the $s \cdot r$ matrices), but it should be straightforward to write a program to check if a given representation ρ is reducible or not.

It is worth noting that the simplicity of the module depends only on the *normalised* row vectors, $\widetilde{P}_i$. This makes sense if we think of multiplying by x_i as a rate of change from A_n to A_{n+1}. Then the geometric mean g_i is the average rate of change, and the nature of the representation is determined by how large each individual rate of change is compared to this average.

Remark 4.3.5. If $P \in \Gamma_s$ corresponds to the representation ρ, then $\lambda P = (\lambda \cdot a_{i,j}) \in \Gamma_s$ corresponds with the twisted representation $\rho^\lambda = \left\{\rho_i^\lambda = \lambda \cdot \rho_i\right\}$. Since the normalised row vectors $\widetilde{\lambda P_i}$ and $\widetilde{P}_i$ are equal, we confirm that twists of finite dimensional simple modules are simple.

Chapter 5

Conclusion

In conclusion let us give a summary of the results in this Thesis:

We have shown that the set $\mathcal{P}$, which can be considered the noncommutative equivalent of the set of closed points in the projective variety associated with an algebra A, is in bijection with the set $\mathcal{F}$ of shift-equivalence classes of 1-critical graded modules such that $d(A/Ann(M)) = 1$, and with the set $\mathcal{C}$ of twist-equivalence classes of non-trivial finite dimensional simple A-modules (Theorem 3.3.6). This means that we can use either of these sets to describe $\mathcal{P}$. We then outlined how multilinearisation of an algebra can be used to parametrise its point modules, which are objects in $\mathcal{C}$.

In the second part of the Thesis, we will consider some examples to illustrate how the theory can be used.

II

Examples

Chapter 6

Commutative Algebras

The noncommutative theory we have developed has been constructed as an extension of the well known properties of commutative algebras. As such, it should coincide with these properties when applied to the commutative case.

Recall that when A is finitely generated and commutative, by Proposition 1.2.6 we have $d(A) = \mathrm{cl.\,Kdim}(A)$, so if M is a 1-critical module with annihilator P, then by definition

$$d(M) = \mathrm{cl.\,Kdim}(M) = \mathrm{cl.\,Kdim}(A/P) = d(A/P) = 1.$$

This means that in the commutative case the set $\mathcal{C}$ in Theorem 3.3.6 includes *all* shift-equivalence classes of 1-critical modules, and there are no fat points, so using the equivalent sets of Theorem 3.3.6 we have a bijection

$$\mathcal{P} \longleftrightarrow \{\text{shift-equivalence classes of point modules}\}.$$

When A is commutative, we define the set $\mathrm{Proj}\,(A)$ to be the set of all homogeneous prime ideals $\mathfrak{p}$ in A that do not contain all of the augmentation ideal A^+. Then $\mathcal{P} \subset \mathrm{Proj}\,(A)$ is the subset of prime ideals of dimension 1. $\mathrm{Proj}\,(A)$ is a scheme whose subscheme of closed points is naturally homeomorphic to the projective variety determined by A, in other words, a graded prime ideal $\mathfrak{p} \in \mathcal{P}$ corresponds to the irreducible subvariety $Y = \mathrm{Proj}\,(A/\mathfrak{p})$ of dimension 0 of $X = \mathrm{Proj}\,(A)$:

$$\mathcal{P} \cong \{\text{Irreducible subvarieties of } X \text{ of dimension } 0\} = \{\text{Closed points in } X\}.$$

So in the commutative case, $\mathcal{P}$ and $\mathcal{C}$ clearly gives us information about the geometry of the algebra. As we will see later in this chapter, $\mathcal{P}$ is either $\mathbb{P}^r$ or a subvariety.

When the algebra is noncommutative, we see $\mathcal{P}$ as an analogue to $\mathbb{P}^r$ and projective varieties, which in turn are parametrised by the finite dimensional simples or the 1-critical modules of the algebra.

6.1 The Commutative Polynomial Ring

Let us study the commutative polynomial ring A in $r+1$ variables:

$$A = k[x_0, \dots, x_r].$$

It is isomorphic to the (noncommutative) algebra

$$k\langle x_0, \dots, x_r\rangle / (f), \quad f = \{x_i x_j - x_j x_i \mid \forall i, j \in [\![0, r]\!]\},$$

and multilinearising gives us

$$(x_{i,1} : x_{i,2}) = (x_{j,1} : x_{j,2}), \quad \forall i, j \in [\![0, r]\!],$$

which means that σ is the identity, and the point P has $P^j = P^1$, $\forall j \in [\![1, s]\!]$. This gives us a bijective correspondence between the point modules and

$$\Gamma \cong \mathbb{P}^r.$$

By Theorem 3.3.6, this gives us

$$\mathcal{P} \cong \mathbb{P}^r.$$

In the commutative case, we can also find this directly: A point module of A is given by a point $(a_0 : \dots : a_r) \in \mathbb{P}^r$:

$$\begin{aligned} \rho : A &\twoheadrightarrow k[t] \\ x_i &\mapsto a_i t \end{aligned}$$

This surjection induces an injective map

$$\rho^* : \operatorname{Spec}(k[t]) \cong \mathbb{A}^1 \hookrightarrow \mathbb{A}^{r+1} \cong \operatorname{Spec}(A).$$

The maximal ideals of $k[t]$ that do not contain all of A^+ are

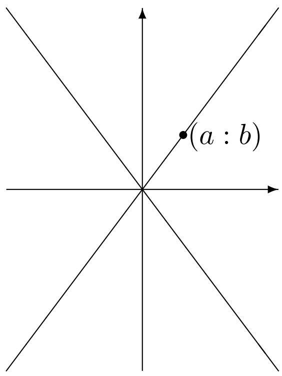
$$\{(t-c), \ c \in k^*\},$$

and

$$\rho^{-1}(t-c) = (x_0 - a_0 c, \dots, x_r - a_r c).$$

So the maximal ideals of A corresponds bijectively to lines $(a_0, \dots, a_r) \cdot c \in \mathbb{A}^{r+1} \cong \operatorname{Spec}(A)$, or equivalently to points $(a_0 : \dots : a_r) \in \mathbb{P}^r \cong \operatorname{Spec}(A)/k^*$. Figure 6.1 shows this graphically for $r = 1$.

Figure 6.1: Point modules of $k[x, y]$ parametrised by the lines in $\mathbb{A}^2$



6.2 Quotients of the Commutative Polynomial Ring

When we want to study a quotient of the commutative polynomial ring, we still have the multilinearised relations

$$(x_{i,1} : x_{i,2}) = (x_{j,1} : x_{j,2}) , \quad \forall i, j \in [\![0, r]\!],$$

so $P^j = P^1$, $\forall j \in [\![1, s]\!]$, and σ is the identity. However the additional relations of A puts restrictions on the set Γ:

If the quotient is given by

$$A = k[x_0, \ldots, x_r]/I$$

where I is a graded ideal, then

$$\Gamma = Z(I)$$

parametrises the point modules of A.

This means that for quotients of the commutative polynomial ring, we have

$$\mathcal{P} \cong Z(I) \subset \mathbb{P}^r.$$

Chapter 7

Artin-Schelter Regular Algebras

An algebra

$$A = k\langle x_0, \dots, x_r\rangle / I$$

is ***Artin-Schelter regular*** (or AS-regular) of dimension d_g if the following conditions are satisfied:

(i) *A has finite global dimension d_g:*
The ***projective dimension*** $\mathrm{p.dim}(M)$ is the length m of the shortest projective resolution

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M. We can then define the ***global dimension*** of an algebra A as

$$\mathrm{gl.dim}(A) := \sup\{\mathrm{p.dim}(M) \,|\, M \text{ } A\text{-module}\}.$$

(ii) *A has finite GK-dimension;*

(iii) *A is **Gorenstein**:*
$\mathrm{Ext}^q_A(k, A) = 0$ if $q \neq d$, and $\mathrm{Ext}^d_A(k, A) \cong k$.

These are algebras with the same Hilbert series as the commutative polynomial rings, and the projective varieties associated with these algebras can therefore be seen as candidates for the noncommutative analogues of $\mathbb{P}^r$.

7.1 AS-regular Algebras of Dimension 2

Let us first cover the AS-regular algebras of dimension 2. Then either

$$A = k\langle x, y\rangle / \left(yx - xy - x^2\right)$$

or A is a quantum plane.

Multilinearising the first algebra gives us

$$y_i x_{i+1} - x_i y_{i+1} - x_i x_{i+1} = 0 \quad \iff \quad (x_{i+1} : y_{i+1}) = (x_i : y_i - x_i) \in \mathbb{P}^1,$$

and the automorphism

$$\sigma = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

on $\mathbb{P}^1$, that sends $(x_i : y_i)$ to $(x_{i+1} : y_{i+1})$. Its only fix point is $(0 : 1)$, and since

$$\sigma^n = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}.$$

there are no other fix points corresponding to quotients of point modules.

Now for the quantum planes:

7.1.1 Quantum Planes

A ***quantum plane*** is an algebra

$$A := k\langle x, y\rangle / (f), \quad f(x,y) = yx - qxy, \quad q \in k^*.$$

Multilinearising the algebra:

$$\begin{aligned} \widetilde{f}(x_i, y_i; x_{i+1}, y_{i+1}) &= y_i x_{i+1} - q x_i y_{i+1} = 0 \\ \iff (x_{i+1} : y_{i+1}) &= (q \cdot x_i : y_i) \in \mathbb{P}^1 \end{aligned}$$

gives us a automorphism

$$\sigma = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}$$

on $\mathbb{P}^1$, that sends $(x_i : y_i)$ to $(x_{i+1} : y_{i+1})$.
This means that Γ_1 and σ completely determines Γ_n, $n > 1$, and

$$\Gamma \cong \mathbb{P}^1$$

parametrises the point modules of A. The finite dimensional quotients of the point modules then correspond to fix points for σ^n in $\mathbb{P}^1$.

- If $q = 1$, all points in $\mathbb{P}^1$ are fixed for σ, so the quotients of the point modules are in one-to-one correspondence with $\mathbb{P}^1$ and

$$\mathcal{P} \supseteq \mathbb{P}^1.$$

- If $q^2 \neq 1$, the only fix points of

$$\sigma^n = \begin{bmatrix} q^n & 0 \\ 0 & 1 \end{bmatrix}$$

in $\mathbb{P}^1$ are $(1:0)$ and $(0:1)$, $\forall n$, so

$$\mathcal{P} \supseteq \{(1:0), (0:1)\}.$$

Remark 7.1.1. Even if $q^n = 1$ for some $n \neq 2$, solving the equation $yx - qxy = 0$ for the representation gives us that $q^2 = 1$, which we have assumed not to be the case.

- If $q = -1$, the fix points of

$$\sigma = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

are $(1:0)$ and $(0:1)$.

The fix points of σ^2, corresponding with the 2-dimensional quotients, are $\mathbb{P}^1 \setminus \{(1:0), (0:1)\}$.

Which of these quotients are simple? We have the set of zeros of $\widetilde{f}$:

$$\Gamma_2 = \left\{ P = \begin{bmatrix} a & -a \\ b & b \end{bmatrix} \right\} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

By Proposition 4.3.1, the representation associated with P is simple if and only if $\det(P) \neq 0$:

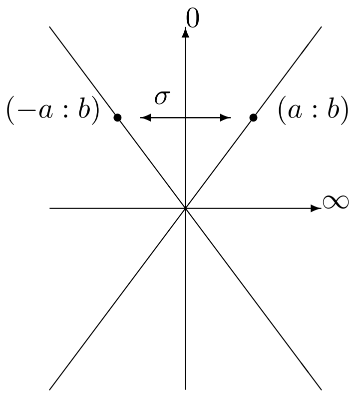
$$\det(P) = 2ab \neq 0.$$

Since $(1:0)$ and $(0:1)$ are the fix points of σ, all the 2-dimensional quotients are simple, and

$$\mathcal{P} \supseteq \mathbb{P}^1.$$

We summarise this in Figure 7.1, identifying the lines containing the two points $(a:b)$ and $\sigma(a:b)$. The axis 0 and ∞ corresponds with the points $(1:0)$ and $(0:1)$ representing the 1-dimensional simples.

Figure 7.1: The finite dimensional simple quotients of point modules of $A = k\langle x, y\rangle / (xy + yx)$, i.e. when $q = -1$, parametrised by pairs of lines in $\mathbb{A}^2$.



7.2 AS-Regular Algebras of Dimension 3

AS-regular algebras of dimension 3 were introduced and partially classified by Artin and Schelter in [Artin 87]. Artin, Tate and Van den Bergh completed the classification for algebras generated by elements of degree 1 in [Artin 90] and [Artin 91], and in [Stephenson 96] and [Stephenson 97a], Stephenson completed the classification for algebras generated by elements of arbitrary degree.

A consequence of these articles is that AS-regular algebras of dimension 3 are Noetherian domains of GK-dimension 3, generated by 2 or 3 elements ([Stephenson 00]).

Let us look at two AS-regular algebras of dimension 3, the enveloping algebra of the Heisenberg algebra, and the 3-dimensional Sklyanin algebra:

7.2.1 The Enveloping Algebra of the Heisenberg Algebra

We define the enveloping algebra of the Heisenberg algebra:

$$A := k\langle x, y\rangle / (f), \quad f(x, y) = \begin{bmatrix} yx^2 - 2xyx + x^2y \\ y^2x - 2yxy + xy^2 \end{bmatrix}$$

Multilinearising the algebra gives us:

$$\begin{aligned}\widetilde{f}\left(x_i, y_i; x_{i+1}, y_{i+1}; x_{i+2}, y_{i+2}\right) &= \begin{bmatrix} y_i x_{i+1} x_{i+2} - 2x_i y_{i+1} x_{i+2} + x_i x_{i+1} y_{i+2} \\ y_i y_{i+1} x_{i+2} - 2y_i x_{i+1} y_{i+2} + x_i y_{i+1} y_{i+2} \end{bmatrix} \\ &= \widetilde{M} \cdot \begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = 0\end{aligned}$$

with

$$\widetilde{M} = \begin{bmatrix} y_i x_{i+1} - 2x_i y_{i+1} & x_i x_{i+1} \\ y_i y_{i+1} & -2y_i x_{i+1} + x_i y_{i+1} \end{bmatrix}.$$

If

$$P = \begin{bmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{bmatrix} \in \Gamma,$$

then the rank of $\widetilde{M}(P)$ is less than 2 if $\det(\widetilde{M}(P)) = 0$:

$$\begin{aligned}\det(\widetilde{M}) &= -\,2\left(b_i a_{i+1} - a_i b_{i+1}\right)^2 = 0 \\ &\Longleftrightarrow (a_{i+1} : b_{i+1}) = (a_i : b_i) \in \mathbb{P}^1,\end{aligned}$$

which means that the automorphism σ is the identity, and we have

$$\Gamma \cong \mathbb{P}^1.$$

This gives us

$$\mathcal{P} \supseteq \mathbb{P}^1.$$

7.2.2 The 3-dimensional Sklyanin Algebra

The 3-dimensional ***Sklyanin algebra*** is defined as follows:

$$\mathrm{Skl}_3\left(\alpha, \beta, \gamma\right) := k\left\langle x, y, z\right\rangle / \left(f\right), \quad f = \begin{bmatrix} \alpha xy + \beta yx + \gamma z^2 \\ \alpha yz + \beta zy + \gamma x^2 \\ \alpha zx + \beta xz + \gamma y^2 \end{bmatrix}$$

where $(\alpha, \beta, \gamma) \in \mathbb{P}^2 \setminus F$, for a (known) finite set F.

Multilinearising gives us

$$\widetilde{f}\left(x_i, y_i, z_i; x_{i+1}, y_{i+1}, z_{i+1}\right) = \widetilde{M} \cdot \begin{bmatrix} x_{i+1} \\ y_{i+1} \\ z_{i+1} \end{bmatrix} = 0,$$

$$\widetilde{M} = \begin{bmatrix} \beta y_i & \alpha x_i & \gamma z_i \\ \gamma x_i & \beta z_i & \alpha y_i \\ \alpha z_i & \gamma y_i & \beta x_i \end{bmatrix}$$

The determinant of $\widetilde{M}$ gives us the elliptic curve

$$E: \ \left(\alpha^3 + \beta^3 + \gamma^3\right) x_i y_i z_i - \alpha\beta\gamma\left(x_i^3 + y_i^3 + z_i^3\right) = 0$$

that parametrises the point modules, and the automorphism

$$\sigma(x,y,z) = \left(\alpha\gamma y^2 - \beta^2 xz : \beta\gamma x^2 - \alpha^2 yz : \alpha\beta z^2 - \gamma^2 xy\right)$$

on E [Verschoren 97].

Choosing $(1,-1,0) \in E$ as the origin for the group law on the elliptic curve,

$$\sigma(1,-1,0) = (\alpha,\beta,\gamma),$$

and σ is actually the translation by the point (α,β,γ) of the points on the elliptic curve E.

This means that

$$\mathcal{P} \supseteq E \cong \Gamma.$$

III

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