Calculations of tropical Welschinger numbers via broccoli curves

by

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Contents

Introduction

The background for this thesis is the work on broccoli curves by Gathmann, Markwig and Schroeter [GMS11]. In that paper, broccoli curves are introduced in order to give a tropical proof of the invariance of tropical Welschinger numbers. As a by-product, the broccoli curves are used by Gathmann, Markwig and Schroeter to find recursive formulæ sufficient to compute all Welschinger invariants of the projective plane. In the present thesis, focus will be on these calculations, generalising the formulæ to $\mathbb{P}^2$ blown up at one or two points.

In Section 1, we take a brief look at the classical background of the enumerative questions addressed by Welschinger invariants.

Section 2 explains how tropical methods are employed to compute these invariants. A key result in this respect is Shustin's theorem (Theorem 2.29) on the tropical count of the Welschinger invariants when the collection of points $\mathscr{P}$ contains non-real points. In this theorem, Shustin identifies tropical curves with some specific properties, which are then counted with suitable multiplicities to find the Welschinger invariants.

Oriented (r, s)-marked curves are introduced in Section 3. A special class of such curves are the Welschinger curves. They are parametrised versions of the curves Shustin uses to compute the Welschinger invariants, but proving the invariance of their count directly is tricky. That is why Gathmann, Markwig and Schroeter introduce broccoli curves, another kind of oriented (r, s)-marked curves. The count of these curves equals the corresponding count for Welschinger curves and is locally invariant in the moduli space. Gathmann, Markwig and Schroeter then show that the count of broccoli curves is independent of the collection of points. Hence, the Welschinger numbers are invariant, see Corollary 3.53 of the present thesis.

With the equivalence of Welschinger and broccoli numbers in mind, Section 4 shows how broccoli curves are used by Gathmann, Markwig and Schroeter to find explicit recursive formulæ which are sufficient to compute all Welschinger invariants of the projective plane. The output from these recursions up to degree 5 is presented, the program code is available upon request.

In Sections 5 and 6, we look at extensions of these recursive formulæ to $\mathbb{P}^2$ blown up in one and two points, respectively. Formulæ sufficient to compute the Welschinger invariants of F_1, $\mathbb{P}^2_2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are presented.

A note on the generality of the results seems to be in order. Until Section 5, we state our results only for the projective plane. However, all the underlying results (such as Shustin's theorem and the equivalence of broccoli and Welschinger numbers) hold for any tropical degree corresponding to a real unnodal toric Del Pezzo surface.

I want to use this opportunity to express my gratitude to Kristian Ranestad for being a fantastic supervisor.

1 Real enumerative geometry

Suppose we are given a set of points $\mathscr{P}$ in $\mathbb{P}^2$ and want to know how many nodal algebraic curves of degree d and genus g pass through the given points. The space of such curves has dimension $3d-1+g$, so if $\mathscr{P}$ consists of $3d-1+g$ points in general position, the number of curves should be finite. These numbers, denoted $N_{\mathbb{P}^2}(d,g)$, are referred to as the Gromov-Witten invariants of the projective plane.

In the early 1990's Kontsevich [KM94] gave the elegant recursion formula

$$N_{\mathbb{P}^2}(d,0) = \sum_{a+b=d} N_{\mathbb{P}^2}(a,0)\cdot N_{\mathbb{P}^2}(b,0)\cdot\left(a^2b^2\binom{3d-4}{3a-2} - a^3b\binom{3d-4}{3a-1}\right),$$

starting from the observation that there exists a unique line through any two distinct points. Caporaso and Harris [CH98] settled the question by giving a recursive formula for the numbers $N_{\mathbb{P}^2}(d,g)$ of nodal curves through $3d-1+g$ points in general position for any genus g.

The situation becomes a lot more complicated when shifting to the count of real curves; the number of curves may depend on the chosen points. Hence, the best we can hope for is upper and lower bounds. Does there exist any such curves at all?

Let $\mathbb{R}\mathcal{C}\,(d,g,\mathscr{P})$ denote the space of all real curves of degree d and genus g through the configuration of points $\mathscr{P}$. Since every real curve is a complex curve, we immediately see that $\#(\mathbb{R}\mathcal{C}(d,g,\mathscr{P})) \leq N_{\mathbb{P}^2}(d,g)$. An important point to note is that if a real curve goes through P, it necessarily passes through $\overline{P}$. Thus, we must distinguish between real and non-real points of $\mathscr{P}$.

The question then is:

Problem 1.1. *Given r "real" and s "complex" points in general position in $\mathbb{C}^2$ (satisfying $r+2s = 3d-1+g$), how many real curves of degree d and genus g pass through the given points?*

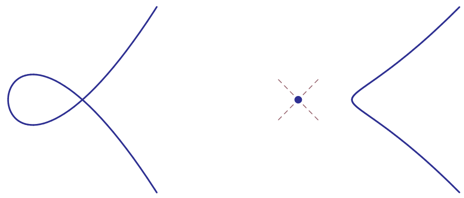
A rational curve of degree $d > 2$ is necessarily singular, a typical curve has $\binom{d-1}{2}$ double points. There are three distinct types of double points for real curves:

Nodes Two real branches of the curve intersect transversally. Locally, this looks like the variety given by $x^2 - y^2 = 0$.

Solitary points Two complex conjugate branches intersect transversally. Locally, this looks like the variety given by $x^2 + y^2 = 0$.

Complex ordinary double points A pair of complex conjugate double points.

The picture below shows these three possibilities (modified from [Sot11]). Note that the pair of complex conjugate double points will not be visible in $\mathbb{R}^2$.



A node   A solitary double point   Invisible double points

Let C be a real algebraic curve. The *multiplicity* of C, denoted m_C, is its number of solitary double points. Let $\mathscr{P}$ be a generic configuration of $3d-1+g-s$ points in $\mathbb{P}^2$, of which there are exactly s non-real points. We define

$$W(d,g,s,\mathscr{P}) := \sum_{C \in \mathbb{R}\mathcal{C}(d,g,\mathscr{P})} (-1)^{m_C},$$

where the sum runs over all real algebraic curves C of genus g and degree d passing through $\mathscr{P}$.

Theorem 1.2 (Welschinger, [Wel03, Wel05]). *For any $d \geq 1$ and $s \geq 0$, the number $W(d,0,s,\mathscr{P})$ does not depend on the collection $\mathscr{P}$ of points in general position.*

Unfortunately, the analogous statement is no longer true if we let g be non-zero.

Since the number $W(d,0,s,\mathscr{P})$ does not depend on the choice of $\mathscr{P}$, we will denote it by $W_{\mathbb{P}^2}(d,3d-2s-1,s)$. These numbers are referred to as the *Welschinger invariants* of the projective plane.

In our case, the main interest of the invariants $W_{\mathbb{P}^2}(d,3d-2s-1,s)$ is that they give lower bounds for the cardinality of $\mathbb{R}\mathcal{C}(d,0,\mathscr{P})$; for any $d \geq 1$ and any generic configuration $\mathscr{P}$ of $3d-s-1$ points in general position in $\mathbb{P}^2$ of which exactly s are non-real,

$$|W_{\mathbb{P}^2}(d,3d-2s-1,s)| \leq \#(\mathbb{R}\mathcal{C}(d,0,\mathscr{P})).$$

Although Welschinger showed that the numbers $W(d,0,s,\mathscr{P})$ do not depend on the choice of conditions in general position, the question of whether there exist *any* real rational curves through the given points was still open (Welschinger only showed that $W_{\mathbb{P}^2}(d,0,3d-1,0)$ was nonzero for $d \leq 5$). Welschinger's theorem does not in itself guarantee the existence of any real rational curve through $\mathscr{P}$. When $\mathscr{P}$ consists only of real points, $\mathbb{R}\mathcal{C}(d,g,\mathscr{P})$ is congruent to $N_{\mathbb{P}^2}(d,0)$ modulo 2, and the latter is even for $d \geq 3$ by the formula of Kontsevich. Hence, $\mathbb{R}\mathcal{C}(d,g,\mathscr{P})$ is an even number and could very well be 0.

2 Tropical geometry

The roots of tropical geometry date back at least to the work of Bergman on limits of algebraic sets under the logarithm in the early 1970s [Ber71]. In tropical geometry, algebraic-geometric objects are replaced by piecewise affine-linear objects. For example, tropical plane curves are graphs in $\mathbb{R}^2$ whose edges are straight with rational slopes. Still, there are some surprising results connecting tropical curves to the seemingly very different algebraic curves.

This section gives a quick introduction to the geometry of plane tropical curves before we come to results tying tropical curves to the classical enumerative problems discussed in the previous section.

2.1 Set-up

One description of tropical plane curves is through "tropical" polynomials. First, we define two new binary operations on $\mathbb{R}$;

$$a \oplus b := \max\{a, b\},$$
$$a \otimes b := a + b.$$

We denote $\underbrace{a \otimes a \otimes \cdots \otimes a}_{n \text{ times}}$ by a^n.

Definition 2.1 (Tropical polynomial)**.** Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. A tropical polynomial p in two variables with support $\mathcal{A}$ is an expression of the form $p = \bigoplus_{(i,j)\in\mathcal{A}} a_{ij} \otimes x^i \otimes y^j$. We denote the support of p by $\mathcal{A}_p$.

When it is clear from the context that we are referring to a tropical polynomial, we will use juxtaposition instead of $\otimes$.

Definition 2.2 (A tropical curve as a set)**.** Let p be a tropical polynomial. We define the function

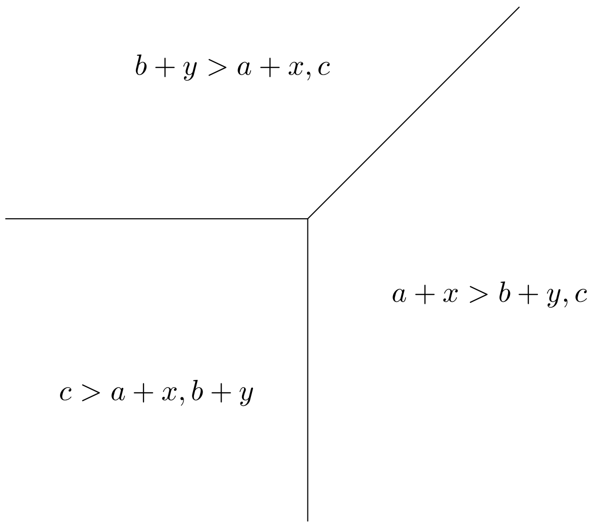
$$f_p(x, y) := \max_{(i,j)\in\mathcal{A}_p} \{a_{ij} + ix + jy\}.$$

With the arithmetic definitions of $\oplus$ and $\otimes$ above, this is just how we would evaluate a tropical polynomial at (x, y). f_p is a convex, piecewise affine-linear function. The *underlying set* of the tropical curve corresponding to p is the corner locus

$$T_p := \{(x, y) \in \mathbb{R}^2 \mid f_p \text{ is not locally affine at } (x, y)\}.$$

The set $\Gamma_p := \{(x, y, z) \in \mathbb{R}^3 \mid z = f_p(x, y)\}$ is a polyhedral surface in $\mathbb{R}^3$. T_p consists of the projections of the edges of Γ_p to $\mathbb{R}^2$, i.e. the points in $\mathbb{R}^2$ where more than one monomial assume the maximum.

Example 2.3 (A tropical line)**.** Let p be given by $p = ax \oplus by \oplus c$. T_p divides the plane into three regions, a south-western region where the term c is largest, a north-western region where the term $b + y$ is largest, and a south-eastern region where $a + x$ is largest.



In general, T_p will be a graph in $\mathbb{R}^2$. Here T_p consists of three unbounded edges (half-rays in $\mathbb{R}^2$) emanating from the same vertex. Its unique vertex is where the three terms are all equal; $(c - a, c - b)$.

Definition 2.4 (Weight of an edge)**.** Let σ be an edge of T_p corresponding to the edge $\hat{\sigma}$ of Γ_p. Then there are two faces of Γ_p meeting along $\hat{\sigma}$. One face looks like the graph of the function $(x, y) \mapsto a_{i_1 j_1} + i_1 x + j_1 y$, the other looks like the graph of $(x, y) \mapsto a_{i_2 j_2} + i_2 x + j_2 y$. Then we define the *weight* of σ, denoted $\omega(\sigma)$, as the integral lattice length of $(i_2 - i_1, j_2 - j_1)$.

Definition 2.5 (The tropical curve corresponding to a tropical polynomial)**.** Let p be a tropical polynomial. The set T_p with the weights $\omega(\sigma)$ of all edges is called the *plane tropical curve* associated to p.

Example 2.6. In Example 2.3 we saw the underlying set of a tropical line. All of its edges have weight 1.

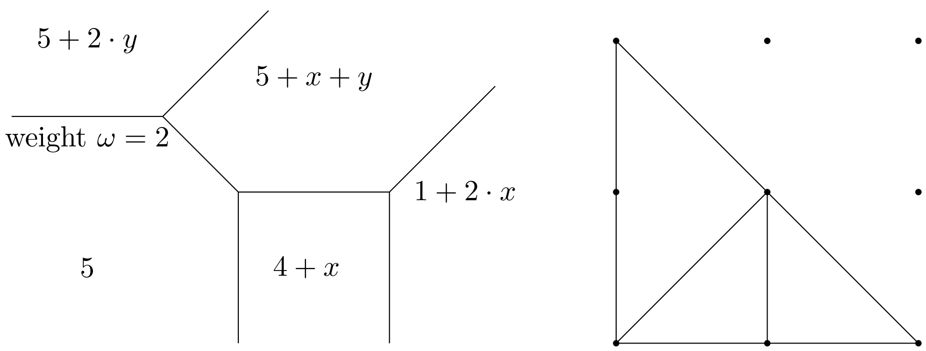
Definition 2.7 (The Newton polygon Newt_p)**.** Let p be a tropical polynomial with support $\mathcal{A}_p$. The *Newton polygon* corresponding to p is the convex hull of $\mathcal{A}_p$ when viewed as a subset of $\mathbb{R}^2$. We denote the Newton polygon corresponding to p by Newt_p.

Definition 2.8 (The subdivision Subdiv_p)**.** The *regular subdivision* of Newt_p associated to a tropical polynomial p is constructed in the following manner:

- Plot the points $d_{ij} = (x_i, y_j, a_{ij}) \in \mathbb{R}^3$ for $(i,j) \in \mathcal{A}_p$.
- Construct the convex hull of the d_{ij}'s.
- Consider the faces of this polytope with an outer normal pointing upward. Project these faces to the xy-plane by deleting the last coordinate. These polygons are called *cells* of the regular subdivision of Newt_p.

We denote the regular subdivision of Newt_p by Subdiv_p.

Example 2.9 (A tropical conic)**.** Let $p = 5 \oplus 4x \oplus 1x^2 \oplus 5xy \oplus 5y^2$ (that is, $p = 5 \oplus 4 \otimes x \oplus 1 \otimes x^2 \oplus 5 \otimes x \otimes y \oplus 5 \otimes y^2$). Its curve is shown with its Newton subdivision below. In each region of $\mathbb{R}^2 \setminus T_p$, the dominating term is shown.



The curve consists of two bounded edges and five half-rays. It has one edge of weight 2, the other six are of weight 1. T_p has three vertices. One is at $(0,0)$, this is where the terms 5, $5 \otimes x \otimes y$ and $5 \otimes x^2$ are greater than all other terms. The other vertices are at $(1,-1)$ (where $5 = 4 + x = 5 + x + y$) and $(3,-1)$ (where $4 + x = 5 + x + y = 1 + 2 \cdot x$).

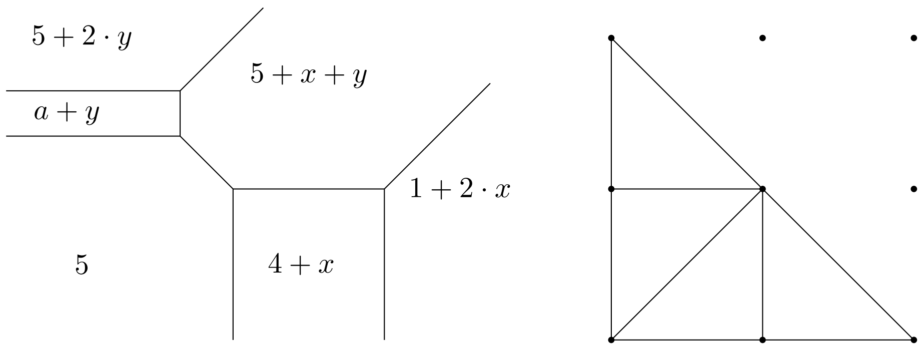
For both this tropical curve and the tropical line of Example 2.3, each monomial of the defining polynomial p corresponded to a unique connected component of $\mathbb{R}^2 \setminus T_p$. This is not true in general; let us see what happens if we add a monomial $a \otimes y$ to the tropical polynomial p of the last example.

Example 2.10 (A family of tropical conics)**.** We let p be the polynomial $p = 5 \oplus 4x \oplus 1x^2 \oplus 5xy \oplus 5y^2$ as in the previous example and let p_a denote the tropical polynomial

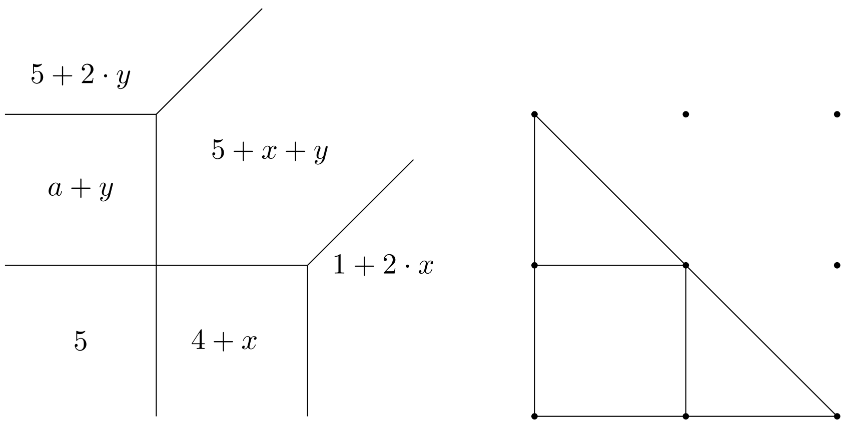
$$p_a := 5 \oplus 4x \oplus ay \oplus 1x^2 \oplus 5xy \oplus 5y^2.$$

First, if $a \leq 5$, then $a + y$ is never greater than *both* 5 and $5 + 2y$, so adding the monomial ay to p does not alter f_p. Hence, T_{p_a} equals T_p and we can check that $\mathrm{Subdiv}_{p_a} = \mathrm{Subdiv}_p$.

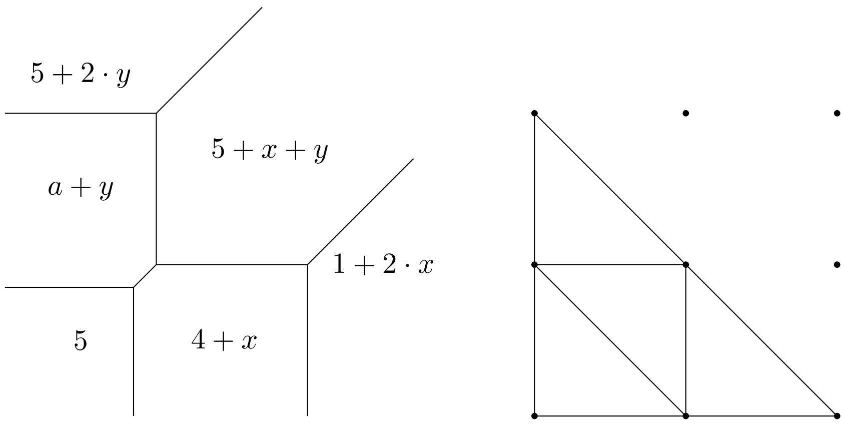
For $5 < a < 6$, adding the monomial $a \otimes y$ to p *does* alter f_p; there exists a region $(-\infty, a-5) \times (5-a, a-5)$ where $a + y$ is strictly greater than all other monomials. Such a curve is shown below with its corresponding Newton subdivision.



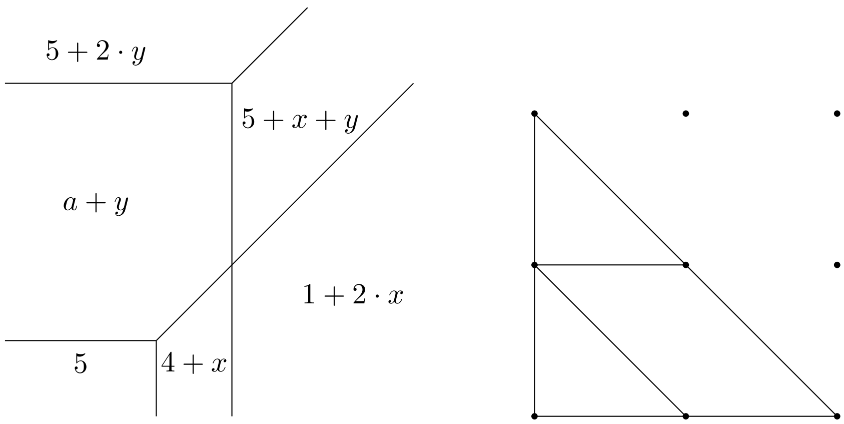
When $a = 6$, the curve is *reducible*; it is the union of two tropical lines:



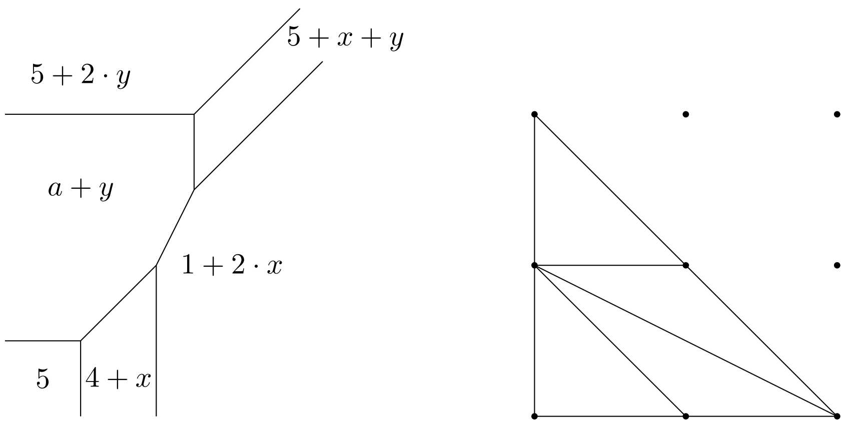
As a increases, the next combinatorial type ("shape") of the tropical curves in the family is shown below.



When $a = 8$ we get a new example of a reducible curve:



For $a > 8$ the curves T_{p_a} are of the same combinatorial type as the curve shown below:

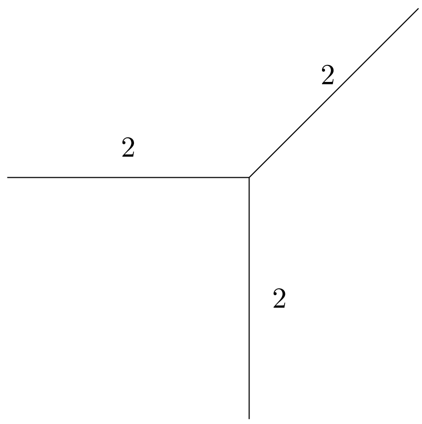


We called the curves in the last example conics, because their defining tropical polynomials were of degree 2. In general, we define the degree of a tropical polynomial as follows.

Definition 2.11 (Degree of a tropical curve)**.** A tropical curve with underlying set T_p corresponding to a tropical polynomial p is said to be of *degree* d if it has d ends in the western direction, d ends in the southern direction and d ends in the north-eastern direction (counted with their weights). This is equivalent to requiring that Newt_p (after a suitable translation in $\mathbb{Z}^2$) is the triangle with vertices $(0,0),(d,0)$ and $(0,d)$

As we did with some of the conics above, we say a tropical curve C is *reducible* if it is the "union" of two strictly smaller tropical curves. Otherwise, we say the curve is *irreducible.*

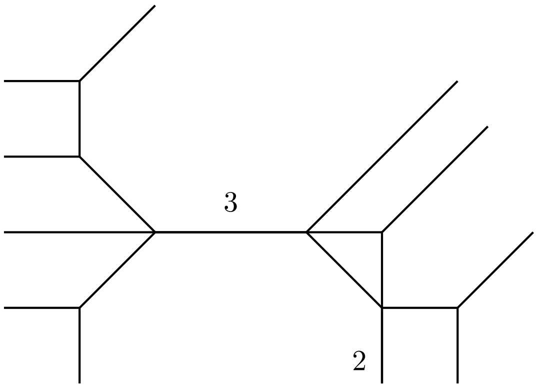
Example 2.12. An edge of a tropical curve with weight ω may, depending on circumstances, be viewed as a collection of edges whose weights sum to ω. Let p be the tropical polynomial given by $p = x^2 \oplus y^2 \oplus 1$. Its tropical curve is shown below.



Its underlying set is exactly like that of a tropical line. We will interpret this curve as the union of two lines with the same underlying set. Hence, we view this as a reducible conic.

A tropical curve C is reducible if and only if it is the tropical curve corresponding to the tropical product of two polynomials, none of which is a single monomial.

Below is a reducible quartic.



It is a union of a tropical line and a tropical cubic.

A plane irreducible tropical curve C is said to be *nodal* if all of its vertices are either 3-valent or locally the transversal intersection of two (classical) lines.

Example 2.13. All irreducible curves of Example 2.9 and Example 2.10 are nodal tropical curves of degree 2.

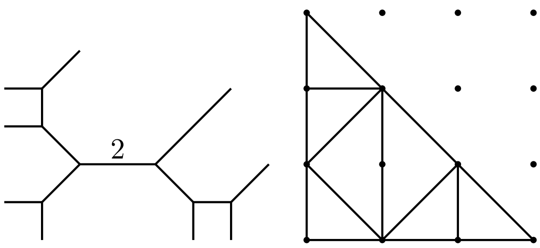
Definition 2.14 (Genus of a nodal curve)**.** The genus g of an irreducible nodal plane tropical curve with N unbounded rays and δ 3-valent vertices is given by

$$g(C) := \frac{\delta - N + 2}{2}.$$

Example 2.15. The tropical curve corresponding to the tropical polynomial

$$p := 10 \oplus 10x \oplus 10y \oplus 6x^2 \oplus 9xy \oplus 8y^2 \oplus 1x^3 \oplus 6x^2y \oplus 8xy^2 \oplus 5y^3$$

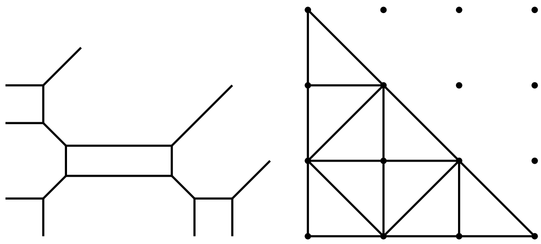
is a nodal curve of degree 3 and genus 0. It is shown below with its corresponding Newton subdivision.



The curve has 7 3-valent vertices, one for each cell of Subdiv_p. It has 15 edges, each corresponding to an edge of Subdiv_p. An edge and its dual edge in Subdiv_p are perpendicular to each other, and the weight of an edge is the integral lattice length of the dual edge. Each connected component of $\mathbb{R}^2 \setminus T_p$ corresponds to a vertex of Subdiv_p.

Note that $9+x+y$ will never be greater than both $10+x$ and $8+x+2\cdot y$, so removing the term $9xy$ from the polynomial would not alter f_p or T_p.

We can actually see this directly from Subdiv_p. Notice that $(1,1)$ (corresponding to the xy-term of p) is *not* a vertex of the Subdivision. As we will see in Theorem 2.16 below, this means that removing this term would not alter the curve. If the coefficient were to be slightly greater, however, we would get a nodal curve of degree 3 whose genus is 1, such as the curve below:



Now, $(1,1)$ is a vertex of the subdivision, so there exists a region where the xy-term is greatest. The curve has 9 unbounded edges and 9 3-valent vertices, so its genus is indeed 1.

Although there exist situations in which not all monomials of p contribute to f_p, we have seen that a lot of information about T_p can be found in the corresponding Newton subdivision.

Theorem 2.16 (Duality)**.** *Let p be a tropical polynomial.*

- *There is a bijection between vertices of* Subdiv_p *and connected components of* $\mathbb{R}^2 \setminus T_p$.

- *There is a bijection between edges of* Subdiv_p *and edges of T_p. Their direction vectors are perpendicular to each other and the weight of an edge of T_p is equal to the lattice length of the corresponding edge of* Subdiv_p. *In addition, if E is an edge of T_p connecting two vertices V_1 and V_2, the edge dual to E is the edge between the cells dual to V_1 and V_2 in the subdivision.*

- *There is a bijection between cells of* Subdiv_p *and vertices of T_p.*

Definition 2.17 (Tropical curves as weighted graphs)**.** A *parametrised plane tropical curve* is a pair (Γ, h) where Γ is a metric graph and h is a continuous map from Γ to $\mathbb{R}^2$ such that

(a) Every vertex of Γ has valence at least 3.

(b) Every edge E is given a positive integer weight $\omega(E)$.

(c) On each edge E of Γ, the map h is an embedding. Its image is contained in a line with rational slope.

(d) Letting $v(E,V)$ denote the primitive integral direction vector of the edge E away from V, the *balancing condition*

$$\sum_{E:V\in\partial E} \omega(E)v(E,V) = 0$$

is satisfied at each vertex V.

Proposition 2.18. *For any parametrised plane tropical curve $C = (\Gamma, h)$ there exists a tropical polynomial p such that $h(\Gamma) = T_p$ and such that the weights of the edges of T_p correspond to those of Γ. Conversely, any tropical curve corresponding to a tropical polynomial p can be parametrised by some Γ and h.*

Remark 2.19. With this correspondence in mind, we speak of the Newton polygon (or subdivision) dual to T_p also as the Newton polygon (or subdivision) dual to $h(\Gamma)$. This polygon (or subdivision) is defined uniquely up to translations in $\mathbb{Z}^2$.

2.2 Correspondence theorems

Compared to classical curves, the piecewise linear nature of tropical curves simplifies their count considerably. A theorem of Mikhalkin states that we can count real or complex curves by counting their tropical counterparts with suitable multiplicities.

Definition 2.20 (The complex vertex multiplicity $\mathrm{mult}(V)$)**.** Let V be a 3-valent vertex of a tropical curve C and let its three adjacent edges E_1, E_2, E_3 have primitive integral direction vectors v_1, v_2, v_3 and weights $\omega_1, \omega_2, \omega_3$. Then the *complex vertex multiplicity* $\mathrm{mult}(V)$ is given by

$$\mathrm{mult}(V) := \omega_1 \cdot \omega_2 \cdot |\det(v_1, v_2)|.$$

Remark 2.21. By the balancing condition, the multiplicity $\mathrm{mult}(V)$ does not depend on the numbering E_1, E_2, E_3 of the edges;

$$\omega_1 \cdot \omega_2 \cdot |\det(v_1, v_2)| = \omega_1 \cdot \omega_3 \cdot |\det(v_1, v_3)| = \omega_2 \cdot \omega_3 \cdot |\det(v_2, v_3)|.$$

There is an equivalent definition of the vertex multiplicity, looking only at the corresponding cell of the Newton subdivision corresponding to the curve. The multiplicity of a 3-valent vertex V is the lattice area of the corresponding triangle in the Newton subdivision.

Definition 2.22 (Complex curve multiplicity)**.** Let C be a nodal plane tropical curve. The "complex" multiplicity of C is defined as

$$\mathrm{mult}_{\mathbb{C}}(C) := \prod_{V} \mathrm{mult}(V),$$

where the product runs over all 3-valent vertices of C.

Theorem 2.23 (Mikhalkin [Mik05])**.** *Let $\mathscr{P} = (P_1, \dots, P_{3d-1+g}) \subseteq \mathbb{R}^2$ be a collection of points in general position. Then*

$$N_{\mathbb{P}^2}(d, g) = \sum_{C} \mathrm{mult}_{\mathbb{C}}(C),$$

where the sum runs over all nodal tropical curves of degree d and genus g through the points $P_1, \dots, P_{3d-1+g}$.

It turns out that the Welschinger invariants can be computed in a similar way; one just has to change the multiplicity slightly.

Definition 2.24 (Real curve multiplicity)**.** Let C be a nodal plane tropical curve. The "real" multiplicity of C is defined as

$$\mathrm{mult}_{\mathbb{R}}(C) := \begin{cases} 0 & \text{if } \mathrm{mult}_{\mathbb{C}}(C) \text{ is even,} \\ 1 & \text{if } \mathrm{mult}_{\mathbb{C}}(C) \cong 1 \text{ modulo } 4, \\ -1 & \text{if } \mathrm{mult}_{\mathbb{C}}(C) \cong 3 \text{ modulo } 4. \end{cases}$$

Theorem 2.25 (Mikhalkin [Mik05])**.** *Let $\mathscr{P} = (P_1, \dots, P_{3d-1}) \subseteq \mathbb{R}^2$ be a collection of points in general position. Then*

$$W_{\mathbb{P}^2}(d, 3d-1, 0) = \sum_{C} \mathrm{mult}_{\mathbb{R}}(C),$$

where the sum runs over all rational nodal tropical curves of degree d through the points $P_1, \dots, P_{3d-1}$.

Using these tropical methods, Itenberg, Kharlamov and Shustin [IKS04] computed the Welschinger invariants for small d, see table 1.

Di Francesco and Itszykson [DFI94] had shown that

$$\lim_{d\to\infty} \frac{\log N_{\mathbb{P}^2}(0, d)}{d \ln d} = 3,$$

and in [IKS04], Itenberg, Kharlamov and Shustin showed that

$$\lim_{d\to\infty} \frac{\log W_{\mathbb{P}^2}(d, 3d-1, 0)}{d \log d} = \lim_{d\to\infty} \frac{\log N_{\mathbb{P}^2}(0, d)}{d \log d}.$$

So, the number of real curves through a given set of real points grows almost as fast as the number of complex curves. In particular, they are all non-zero.

d	$W_{\mathbb{P}^2}(d, 3d-1, 0)$	$N_{\mathbb{P}^2}(0, d)$
1	1	1
2	1	1
3	8	12
4	240	620
5	18264	67304
6	2845440	26312976

Table 1: The first values of $W_{\mathbb{P}^2}(d, 3d-1, 0)$ and $N_{\mathbb{P}^2}(0, d)$

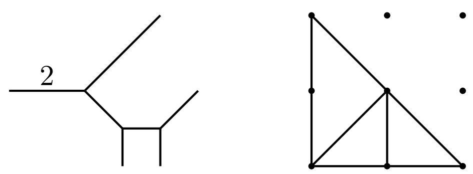
When we let $\mathscr{P}$ contain non-real points, a more refined method is required; a small change in the multiplicities is not sufficient. Still, Shustin found a class of tropical curves and a suitable multiplicity such that their count equals the Welschinger invariants in this case as well.

Definition 2.26 (Shustin multiplicity)**.** Let C be a nodal tropical curve corresponding to a Newton subdivision Subdiv. Denote by a the number of integral points in the interior of triangles in Subdiv, let b be the number of triangles in Subdiv such that all sides have even lattice lengths, and let c be the number of triangles of Subdiv whose lattice area is even. The *Shustin multiplicity* of C is

$$\text{mult}_S(C) := (-1)^{a+b} 2^{-c} \prod_V \text{mult}(V),$$

where the product runs over all vertices V of C such that the dual triangle has even lattice area.

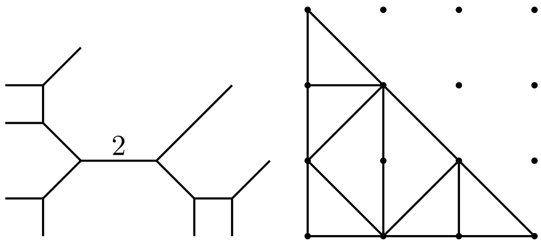
Example 2.27. Recall the curve of Example 2.9 and its corresponding Newton subdivision as shown below.



There are no integral lattice points in the interiors of any triangles of the subdivision and no triangles have sides all of whose lenghts are even. The number of triangles whose lattice area is even is 1. Thus, the Shustin multipliticy of C is given by

$$\text{mult}_S(C) = (-1)^{0+0} 2^{-1} \prod_V \text{mult}(V) = 1.$$

Example 2.28. We will determine $\text{mult}_S(C)$ for the first curve of Example 2.15.



There are no integral lattice points in the interiors of any triangles and no triangles have sides all of whose lenghts are even. The number of triangles whose lattice area is even is 2. Thus, the Shustin multiplicity of C is given by

$$\text{mult}_S(C) = (-1)^{0+0} 2^{-2} \prod_V \text{mult}(V) = 1.$$

Theorem 2.29 (Shustin [Shu06, Theorem 3.1])**.** *Let $r + 2s = 3d - 1$ and let $\mathscr{P} = (P_1, \ldots, P_{r+s})$ be a collection of points in general position in $\mathbb{R}^2$, exactly s of which are "complex". Then*

$$W_{\mathbb{P}^2}(d, r, s) = \sum_C \text{mult}_S(C),$$

where the sum runs over some of the rational nodal tropical curves C of degree d through the given points; the r "real" points must be at edges of odd weight and the s "complex" points must be either at trivalent vertices of C or at edges of even weight. In addition, there is a condition on the collection of edges of C of even weight which must be satisfied (see Definition 3.26 for this condition in the setting of (r, s)-marked curves).

By the Welschinger theorem, these numbers do not depend on the choice of points, but Gathmann, Markwig and Schroeter [GMS11] sought a proof of this invariance within tropical geometry. This is the content of the next section.

3 Oriented marked curves

We will change our objects of study again, this time to the combinatorial objects called *oriented (r, s)-marked curves.* The curves counted by Shustin in Theorem 2.29 then correspond to a special type of oriented (r, s)-marked curves. Proving the invariance of their count directly is tricky - the count is not locally invariant in the moduli space. Gathmann, Markwig and

Schroeter introduce a similar type of oriented curves, *broccoli curves*, whose count *is* locally invariant in the moduli space. The count of these curves is then shown to be equivalent to the similar count for Welschinger curves. This leads to the invariance of Welschinger numbers, Corollary 3.53.

Note. As opposed to the last section, all curves considered from now on will be connected of genus 0. We therefore include this restriction in the definition of the curves we consider.

3.1 (r, s)-marked curves

Our (r, s)-marked plane tropical curves will be parametrised plane tropical curves in the sense of Definition 2.17 with the added data of some "real" and "complex" markings (marked ends). These markings will be ends of the underlying graph of our new curves which are sent to a point under the map to $\mathbb{R}^2$. Instead of looking at curves through a collection of r "real" and s "complex" given points in the plane, we look at the space of all (r, s)-marked curves whose marked ends are sent to the given collection of points. By using this definition, the conditions imposed by Shustin on the curves counted in Theorem 2.29 become inherent properties of the curve itself.

Definition 3.1 (Metric graphs)**.** Let $I_1, \ldots, I_n \subseteq \mathbb{R}$ be a finite set of closed (bounded or half-bounded) real intervals. Pick some (not neccessarily distinct) boundary points $P_1, \ldots, P_k, Q_1, \ldots, Q_k$ in the disjoint union of these intervals, $I_1 \coprod \cdots \coprod I_n$. The topological space Γ obtained by identifying P_i with Q_i in $I_1 \coprod \cdots \coprod I_n$ for all $i \in 1, \ldots, k$, is called a *metric graph.*

(a) The boundary points of the intervals $I_1, \ldots, I_n$ are called *flags*, their image points in Γ are called *vertices* of Γ. For a vertex V, the number of flags F such that V is the image point of F is called the *valence* of V, denoted $\mathrm{val}(V)$.

(b) The intervals $I_1, \ldots, I_n$ are called the *edges* of Γ. They are all closed subsets of Γ. An edge will be called *bounded* if its corresponding interval is, otherwise it is called *unbounded.* The unbounded edges will be referred to as the *ends* of Γ.

(c) The *genus* of a connected graph Γ is its first Betti number. If Γ is connected of genus 0, we say it is *rational.*

Definition 3.2 (Abstract tropical curve)**.** A (rational, abstract) *tropical curve* is a connected, rational metric graph Γ, all of whose vertices have valence at least 3.

Definition 3.3 (Abstract (r, s)-marked curve)**.** Let r, s be non-negative integers. An (abstract) (r, s)-*marked tropical curve* is a tuple of the form $(\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n)$ for some $n \in \mathbb{N}$, where Γ is a tropical curve and $x_1, \ldots, x_{r+s}, y_1, \ldots, y_n$ is a labelling of all distinct unbounded edges of Γ.

Two abstract (r,s)-marked curves $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n)$ and $C' = (\Gamma', x'_1, \dots, x'_{r+s}, y'_1, \dots, y'_n)$ will be called *isomorphic* if there exists an isometric isomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi(x_i) = x'_i$ for $i = 1, \dots, r+s$ and $\phi(y_j) = y'_j$ for all $j = 1, \dots, n$.

Definition 3.4 (Marked curves)**.** An (r,s)*-marked plane tropical curve* is a tuple $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h)$ for some $n \in \mathbb{N}$ such that:

(a) $(\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n)$ is an abstract (r,s)-marked tropical curve.

(b) **Affine linear map:** h is a continuous map from Γ to $\mathbb{R}^2$. If we parametrise an edge E of Γ starting at V, h is locally of the form $h|_E(t) = a + t \cdot v$ for some $a \in \mathbb{R}^2$ and $v \in \mathbb{Z}^2$ ("h is integer affine linear at the edges"). The integral vector v in this equation will be called the direction vector of E starting at V, and denoted $v_{E,V}$. When the edge is an end, we let v_E denote $v_{E,V}$ for its unique adjacent vertex V.

(c) **Balancing condition:** At every vertex V,

$$\sum_{E : V \in \partial E} v_{E,V} = 0$$

is satisfied.

(d) **Marked and unmarked ends:** Each of the *marked ends*, the unbounded edges $x_1, \dots, x_{r+s}$, is mapped to a point in $\mathbb{R}^2$ by h. Therefore $v_{E,V} = 0$ for its unique adjacent vertex V. We say that ends with $v_{E,V} = 0$ are *contracted*. The contracted ends $x_1, \dots, x_r$ will be referred to as *real markings*, the contracted ends $x_{r+1}, \dots, x_{r+s}$ will be called *complex markings*. The ends $y_1, \dots, y_n$ are called *unmarked ends*.

(e) **Degree:** The collection $\Delta = (v(y_1), \dots, v(y_n))$ is called the *degree* of C. In general, we say that a finite, non-empty collection Δ of vectors in $\mathbb{Z}^2 \setminus \{(0,0)\}$ is a *tropical degree* if there exists a tropical curve whose degree is Δ, i.e. if the sum of the vectors in Δ is $(0,0)$.

We say a curve is of degree d if its degree consists of d times each of the vectors $(0,-1)$, $(-1,0)$ and $(1,1)$. These will be the curves corresponding to classical curves of degree d in $\mathbb{P}^2$.

We will call two (r,s)-marked curves $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h)$ and $C' = (\Gamma', x'_1, \dots, x'_{r+s}, y'_1, \dots, y'_n, h')$ *isomorphic* if there exists an isomorphism ϕ from $(\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n)$ to $(\Gamma', x'_1, \dots, x'_{r+s}, y'_1, \dots, y'_n)$ of the underlying abstract (r,s)-marked curves, such that $h' \circ \phi = h$. We will identify isomorphic curves.

We denote the space of all isomorphism classes of (r, s)-marked curves of degree Δ by $M_{(r,s)}(\Delta)$, but will refer to an isomorphism class by picking a representative.

Definition 3.5 (Even and odd edges, weights)**.** Let an (r, s)-marked plane tropical curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ be given.

(a) A vector in $\mathbb{Z}^2$ is called *even* if both coordinates are even, otherwise it is called *odd*.

(b) For every edge E of C, $v_{E,V}$ can be written as a non-negative multiple $\omega(E)$ of a primitive integral vector. $\omega(E)$ is called the *weight* of E. E will be called *even* if its weight is even and *odd* otherwise.

Definition 3.6 (Combinatorial type)**.** Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ be an (r, s)-marked plane tropical curve. The combinatorial type of C consists of the information of $\big(|\Gamma|, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, (v_{E,V})_{V\in\partial E}\big)$, i.e. the underlying non-metric graph, the labelling of the unbounded edges and the direction vectors of all edges.

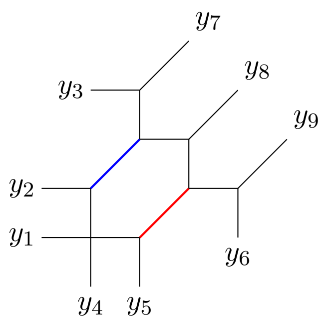
Let α be the combinatorial type of a curve C. We denote by $M^{\alpha}_{(r,s)}(\Delta)$ the subspace of $M_{(r,s)}(\Delta)$ consisting of all curves of combinatorial type α. The curves in $M^{\alpha}_{(r,s)}(\Delta)$ differ only in the lengths of their bounded edges and the image of a given root vertex in $\mathbb{R}^2$.

Definition 3.7 (Subdivision dual to an (r, s)-marked plane tropical curve C)**.** Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ be an (r, s)-marked tropical curve. By Proposition 2.18, $h(\Gamma)$ is a tropical curve corresponding to some tropical polynomial p. As in Remark 2.19 we will refer to Subdiv_p also as the subdivision dual to C or the subdivision dual to $h(\Gamma)$, depending on circumstances.

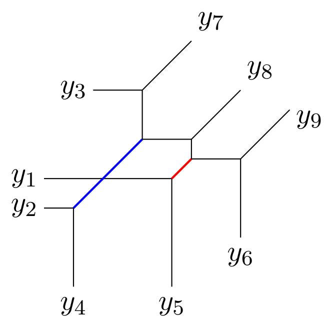
Example 3.8 (Subdivisions may vary within one combinatorial type of curve)**.** Let Δ be given by

$$\Delta = ((-1, 0), (-1, 0), (-1, 0), (0, -1), (0, -1), (0, -1), (1, 1), (1, 1), (1, 1)) .$$

Below are the images in $\mathbb{R}^2$ of two $(0, 0)$-marked plane tropical curves $C = (\Gamma, y_1, \ldots, y_9, h)$ and $C' = (\Gamma', y'_1, \ldots, y'_9, h')$ of degree Δ.

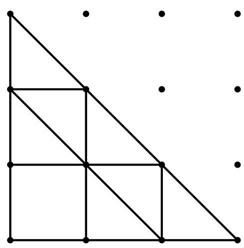


$h(\Gamma)$

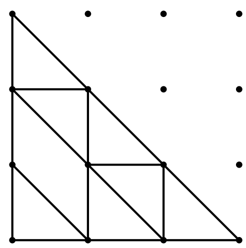


$h'(\Gamma')$

The curves C and C' are of the same combinatorial type, the difference lies in the lengths of the bounded edges. We have stretched one edge (the blue) and shrunk another (the red). Even though these curves are of the same combinatorial type, they are dual to distinct Newton subdivisions of the triangle with vertices $(0,0), (3,0)$ and $(0,3)$:



Subdivision corresponding to $h(\Gamma)$



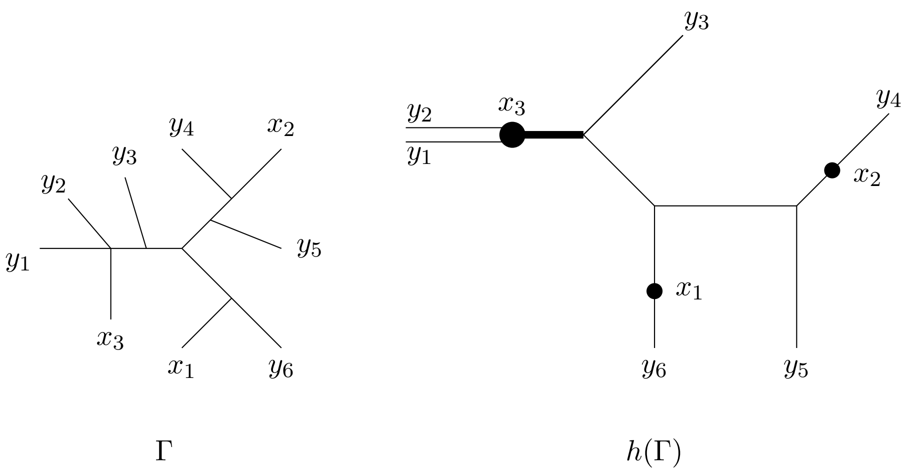
Subdivision corresponding to $h'(\Gamma')$

The multiplicities (real, complex and Shustin) of $h(\Gamma)$ and $h'(\Gamma')$ of the curves are equal, though.

Convention 3.9 (Curve drawing). Given an (r, s)-marked plane tropical curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ we will draw the curve as the image $h(\Gamma) \subseteq \mathbb{R}^2$ as in the example above. The image points of the marked ends will be shown as small dots for real markings and big dots for complex markings. Other edges will be displayed as thin lines if their weight is odd and thick lines if their weight is even. To ease visualisation, parallel edges will be shown as distinct edges in our pictures, even though this is a feature of the abstract graph and would not be visible in $\mathbb{R}^2$.

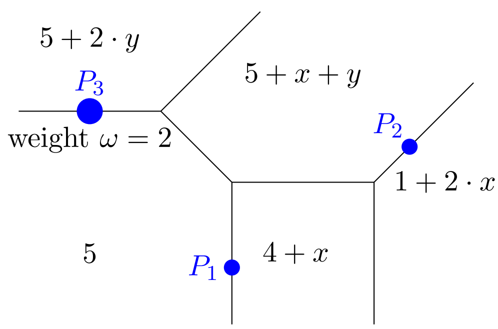
Example 3.10 (A $(2,1)$-marked plane tropical curve)**.** The picture below shows a $(2,1)$-marked plane tropical curve $C = (\Gamma, x_1, x_2, x_3, y_1, \ldots, y_6, h)$

(that is, it has two "real" markings and one "complex" marking) of degree $\Delta = ((-1,0),(-1,0),(1,1),(1,1),(0,-1),(0,-1))$. It has five 3-valent vertices and one 4-valent vertex (recall that the markings are themselves edges of the underlying graph Γ).

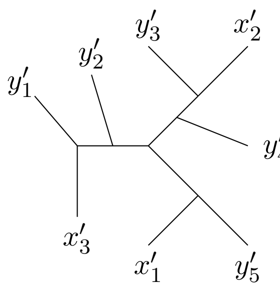


The unmarked ends y_1 and y_2 are sent to the same line by h, but as explained in Convention 3.9 they are drawn as distinct edges in our image.

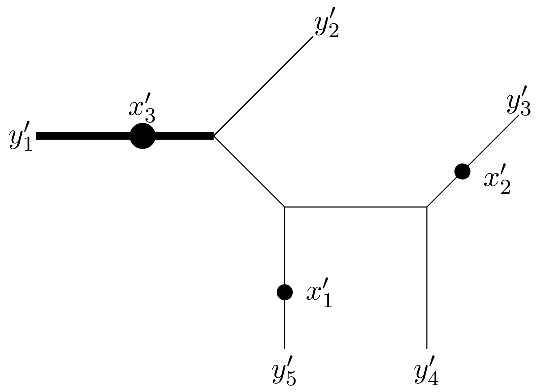
We may think of C as a parametrisation of the curve corresponding to the polynomial p of Example 2.9 in the following way. Suppose we were looking at the family of tropical conics of degree 2 through the "real" points $(1,-2.2)$ and $(3.5,-0.5)$ and the "complex point" $(-1,0)$. Then the curve corresponding to $p = 5 \oplus 4x \oplus 1x^2 \oplus 5xy \oplus 5y^2$ would pass through these points as seen in the picture below.



In the setting of (r,s)-marked curves, we do not look at curves through the given points. Instead, we look at curves such that their marked ends are sent to the given points. The natural curve to consider as an (r,s)-marked analogue to the tropical curve corresponding to p would be the curve C' below.



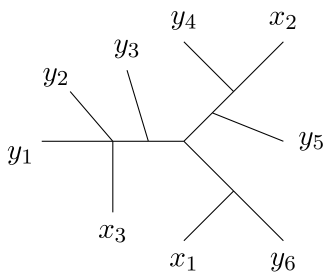
Γ'



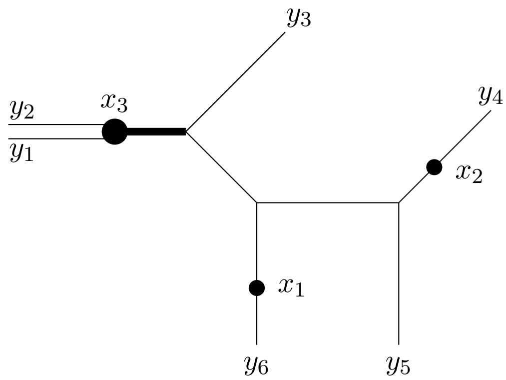
$h'(\Gamma')$

Now, $C' = (\Gamma', x'_1, x'_2, x'_3, y'_1, \ldots, y'_5, h')$ is a $(1,2)$-marked curve of degree $\Delta' = ((-2,0),(1,1),(1,1),(0,-1),(0,-1))$ such that $h(\Gamma) = T_p$, the weights of the edges of Γ equal the weights of the corresponding edge of T_p and $h'(x'_1) = P_1$, $h'(x'_2) = P_2$ and $h'(x'_3) = P_3$.

We said that an (r,s)-marked curve would be of degree d if its degree consists of d times each of the vectors $(0,-1)$, $(-1,0)$ and $(1,1)$. Hence, we parametrise the end y'_1 of weight 2 as two parallel ends of weight 1 to get the curve $C = (\Gamma, x_1, x_2, x_3, y_1, \ldots, y_6, h)$.



Γ



$h(\Gamma)$

Definition 3.11 (Abstract polyhedral complexes)**.** Let $X_1, \ldots, X_n$ be open convex polyhedra in a real vector space. A *polyhedral complex* with cells $X_1, \ldots, X_n$ is a topological space X together with continuous inclusion maps

$i_k : X_k \to X$ such that X is the disjoint union of the sets $i_k(X_k)$.

The dimension of a polyhedral complex X is the maximum of the dimensions of its cells, denoted $\dim X$. X is said to be of *pure dimension* $\dim X$ if every cell of X is contained in the closure of a cell of dimension $\dim X$.

We will focus on certain spaces of curves and these spaces turn out to be polyhedral complexes in a natural way. The different cells of these complexes correspond to different combinatorial types of curves. First, we will explain how to give the space $M_{(r,s)}(\Delta)$ the structure of a polyhedral complex. The first thing to note is that the number of cells is finite:

Proposition 3.12 (Finite number of combinatorial types [GM08, Lemma 2.10])**.** *For all $r, s \geq 0$ and tropical degrees Δ, there are only finitely many combinatorial types α such that $M^{\alpha}_{(r,s)}(\Delta)$ is a subspace of $M_{(r,s)}(\Delta)$.*

Proposition 3.13. *A 3-valent (abstract) tropical curve Γ with N unbounded edges has exactly $N - 3$ bounded edges.*

Every vertex of a tropical curve must (by definition) have valence at least 3, so if a tropical curve Γ has N unbounded edges, $N - 3$ is an upper bound for the number of bounded edges. Any 4-valent vertex of a graph can be resolved by two 3-valent vertices connected by a bounded edge as in the figure below.

Similarly, a 5-valent vertex could be resolved by a 3-valent vertex and a 4-valent vertex connected by a bounded edge and so forth. Hence, if we have vertices which are not 3-valent, the number of bounded edges of the graph is diminished by $\sum_V(\text{Val } V - 3)$ compared to a "similar" graph, all of whose vertices are 3-valent.

Proposition 3.14. *A plane tropical (r, s)-marked curve C in $M_{(r,s)}(\Delta)$ has $|\Delta| + r + s$ ends and $|\Delta| + r + s - 3 - \sum_V(\text{Val } V - 3)$ bounded edges.*

The curves in $M^{\alpha}_{(r,s)}(\Delta)$ are parametrised by the image of a vertex in $\mathbb{R}^2$ and the lengths of their non-contracted bounded edges, all of which can be any positive real number.

Proposition 3.15 ($M^{\alpha}_{(r,s)}(\Delta)$ as a polyhedron [GM08, Proposition 2.11 2.10])**.** *Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h)$ be a plane tropical curve of combinatorial type α. Then the space $M^{\alpha}_{(r,s)}(\Delta)$ is an open convex polyhedron (possibly unbounded) in a real vector space, its dimension is given by*

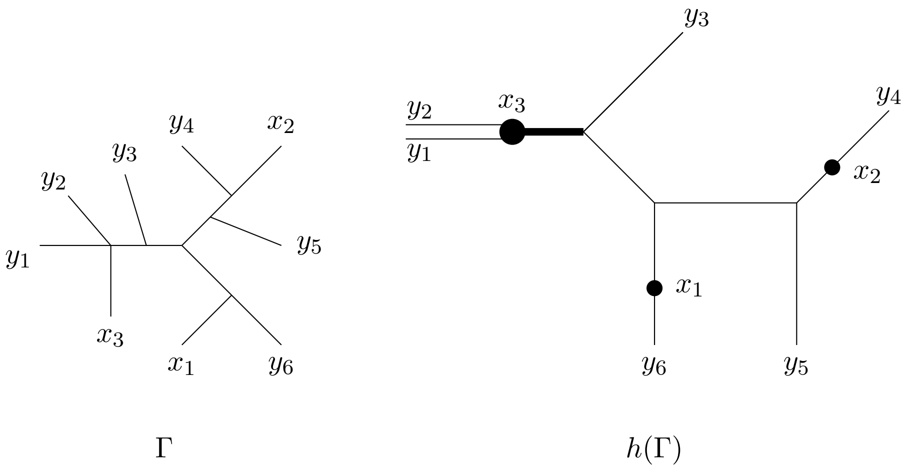
$$\dim M^{\alpha}_{(r,s)}(\Delta) = |\Delta| + r + s - 1 - \sum_V (\text{Val } V - 3).$$

Elements of the boundary of the cell $M^{\alpha}_{(r,s)}(\Delta)$ correspond to limits of tropical curves of combinatorial type α where one or more of the lengths of the bounded edges tend to zero, and every cell of lower dimension is in the boundary of some cell of maximal dimension in this sense (see Example 3.19). Combining this observation with Propositions 3.12 and 3.14 we get the following result.

Proposition 3.16 ($M_{(r,s)}$ as a polyhedral complex)**.** *$M_{(r,s)}(\Delta)$ is a polyhedral complex of pure dimension $|\Delta| + r + s - 1$. Its cells correspond to the spaces $M^{\alpha}_{(r,s)}(\Delta)$ for different α.*

Proposition 3.17 (Classification of maximal cells of $M_{(r,s)}(\Delta)$)**.** *Let $r, s \geq 0$ and let Δ be a tropical degree. A curve $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h)$ is in a cell of maximal dimension in $M_{(r,s)}(\Delta)$ if and only if every vertex of Γ is 3-valent.*

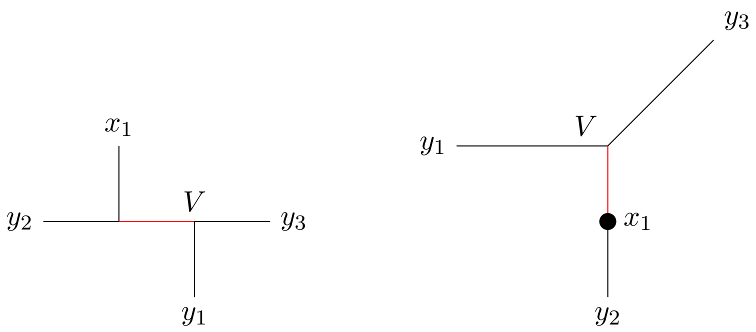
Example 3.18 (The cell corresponding to a combinatorial type)**.** Recall the $(2,1)$ marked plane tropical curve $C = (\Gamma, x_1, x_2, x_3, y_1, \dots, y_6, h)$ from Example 3.10 and denote its combinatorial type by α.



The distinct curves of this combinatorial type differ in the lengths of their bounded edges and their position in $\mathbb{R}^2$ (a translation of h by a vector in $\mathbb{R}^2$ would produce a curve of the same combinatorial type). Since C has 4 bounded edges, the space of all curves of the same combinatorial type as C has dimension 6.

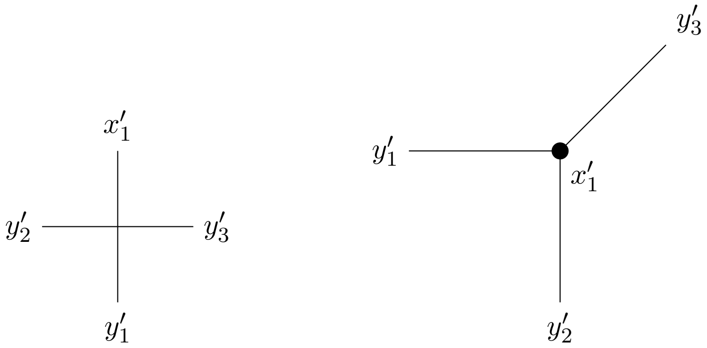
Example 3.19 (The polyhedral complex $M_{(1,0)}(\Delta)$)**.** Let us take a look at $M_{(1,0)}(\Delta)$, where $\Delta = ((-1,0),(0,-1),(1,1))$. That is, we look at the space of tropical lines with one real marking. In this case, the corollary states that $M_{(1,0)}(\Delta)$ is a polyhedral complex where every cell is contained in the closure of a cell of dimension $|\Delta| + r + s - 1 = 3 + 1 + 0 - 1 = 3$.

There are three possible combinatorial types α such that the cell $M^{\alpha}_{(1,0)}(\Delta)$ has dimension 3; below is the image of one such curve $C = (\Gamma, x_1, y_1, y_2, y_3, h) \in M^{\alpha_1}_{(1,0)}(\Delta)$ and a sketch of its underlying abstract $(1,0)$-marked curve.



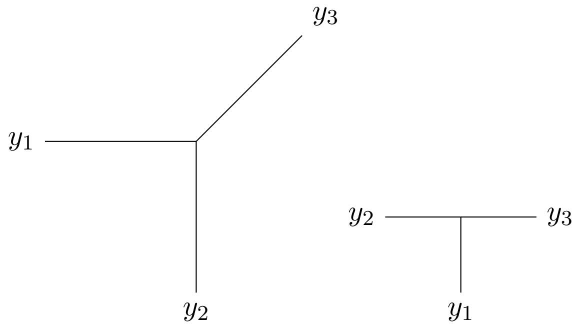
Γ $\qquad\qquad$ $h(\Gamma)$

Let us see what happens as the length of the single bounded edge of Γ shrinks towards 0. The limit is a new curve $C' = (\Gamma', x_1', y_1', y_2', y_3', h')$ of combinatorial type α_2 with no unbounded edges. Then C' lies in the boundary of $M^{\alpha_1}_{(1,0)}(\Delta)$ and $M^{\alpha_2}_{(1,0)}(\Delta)$ is a cell of $M_{(1,0)}(\Delta)$ of dimension 2 (two parameters describe possible translations in $\mathbb{R}^2$). The curve C' is pictured next to its underlying abstract graph below.



Γ' $\qquad\qquad$ $h'(\Gamma')$

We are now ready to describe $M_{(1,0)}(\Delta)$. The isomorphism class of any curve $C = (\Gamma, x_1, y_1, y_2, y_3, h) \in M_{(1,0)}(\Delta)$ is uniquely determined by $h(V)$ (where V is the unique vertex not adjacent to x_1 if such a vertex exists and the unique vertex of Γ otherwise) and $h(x_1)$. For any choice of $P \in \mathbb{R}^2$, the subspace of $M_{(1,0)}(\Delta)$ consisting of curves such that $h(V) = P$ can be identified with the image of the unique $(0,0)$-marked curve of degree Δ such that its unique vertex is mapped to P:



In fact, $M_{(1,0)}(\Delta)$ is not merely a polyhedral complex, but corresponds to a tropical variety in a natural way. Let p be the polynomial given by $p(x, y, z, w) = yz \oplus xw \oplus xy$. Generalising Definition 2.5 to hypersurfaces in spaces of higher dimension, we let T_p be the set of points in $\mathbb{R}^4$ such that the maximum is assumed by more than one monomial of p. First, fix a point $P = (a, b) \in \mathbb{R}^2$, and see which points of the form (a, b, z, w) belong to $V(f)$. This is exactly the line in the zw-plane corresponding to $bz \oplus aw \oplus ab$, which has its unique vertex at $((a + b) - b, (a + b) - a) = (a, b)$.

This observation is true in a more general setting, see [GKM09, Proposition 4.7].

Definition 3.20 (Morphism of polyhedral complexes)**.** A *morphism* from a polyhedral complex X to a polyhedral complex Y is a continuous map $f : X \to Y$ such that for each cell X_i of X, its image $f(X_i)$ is contained in a cell Y_j of Y and the restriction of f to X_i is a linear map of polyhedra.

We employ these constructions in order to shift our attention to curves satisfying some fixed conditions; in particular we want to look at curves passing through a collection of given points with some of the unmarked ends sent to fixed lines in $\mathbb{R}^2$.

Definition 3.21 (The evaluation map ev_F)**.** Let $r, s \geq 0$, let $\Delta = (v_1, \dots v_n)$ be a tropical degree, and let F be a subset of $\{1, \dots, n\}$.

We define the *evaluation map* ev_F to be

$$\mathrm{ev}_F \; : \; M_{(r,s)}(\Delta) \longrightarrow (\mathbb{R}^2)^{r+s} \times \prod_{i \in F} (\mathbb{R}^2 / \langle v_i \rangle) \cong \mathbb{R}^{2(r+s)+|F|}$$
$$(\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h) \longmapsto ((h(x_1), \dots, h(x_{r+s})), (h(y_i) \; : \; i \in F)).$$

That is, the image of a curve C under ev_F is the tuple of images of its contracted ends under h and the lines in $\mathbb{R}^2$ into which the unmarked ends y_i of C with $i \in F$ are mapped by h. ev_F is a morphism of polyhedral complexes. We call the unmarked ends y_i with $i \in F$ *fixed ends*.

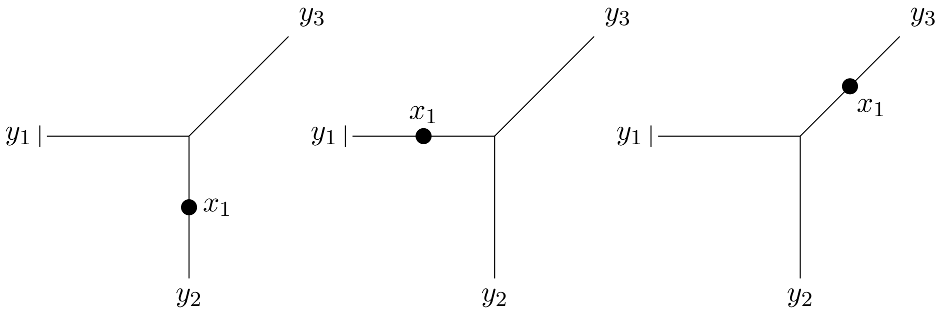
In our pictures, we will show that an end is “fixed” by drawing a perpendicular bar at its unbounded side, see Example 3.24.

Definition 3.22 (General and special position of points)**.** Let $f : M \to \mathbb{R}^N$ be a map of polyhedral complexes. Then $\mathscr{P} \in \mathbb{R}^N$ is said to be in *special position* for f if it is in the image of some cell of M whose image is not of dimension N. Otherwise, it is said to be in *general position*.

The locus of points of $\mathbb{R}^N$ in general position for f is the complement of a finite union of subsets of $\mathbb{R}^N$, each of which is of dimension at most $N-1$. Hence, the locus of points in general position is a dense open subset.

Proposition 3.23. *Let $M \subseteq M_{(r,s)}(\Delta)$ be a polyhedral subcomplex and let $F \subseteq (1,\dots,|\Delta|)$. Then a collection $\mathscr{P} \in \mathbb{R}^{2(r+s)} \times \prod_{i\in F}(\mathbb{R}^2/\langle v_i\rangle) \cong \mathbb{R}^{2(r+s)+|F|}$ is in general position for the morphism $\mathrm{ev}_F|_M$ if and only if for each curve C in $M \cap \mathrm{ev}_F^{-1}(\mathscr{P})$ and every small perturbation $\mathscr{P}'$ of $\mathscr{P}$ it is possible to find a curve C' of the same combinatorial type as C in $\mathrm{ev}_F^{-1}(\mathscr{P}')$.*

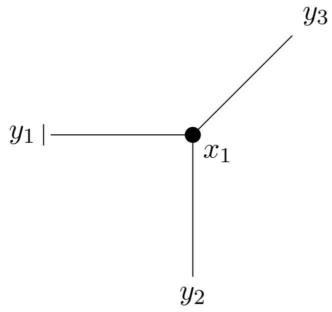
Example 3.24. Let $\Delta = ((-1,0),(0,-1),(1,1))$, $F = \{1\}$. Recall from Example 3.19 that $M_{(1,0)}(\Delta)$ has 4 cells, three of which are of dimension 3. These cells correspond to the combinatorial types of the curves shown below:



As previously promised, we have drawn a perpendicular bar at the end of each end y_i with $i \in F$. Here, ev_F sends a curve C to the tuple (P, L), where P is the image of x_1 under h and L is the horizontal line into which y_1 is mapped.

The images of the cells corresponding to the first and third combinatorial types are both 3-dimensional. For any curve of the same combinatorial type as the middle curve, however, we must have $h(x_1) \in h(y_1)$. Thus, the image of the cell corresponding to the curve in the middle consists of all $\mathscr{P} = (P, L)$ such that the horizontal line L contains P. The collection of such conditions is a 2-dimensional space.

The same is true if we look at the cell of $M_{(1,0)}(\Delta)$ with dimension 2:



The image of this cell is also the set of $\mathscr{P} = (P, L)$ such that the horizontal line L contains P.

Hence, a collection of conditions $\mathscr{P} = (P, L)$ for the evaluation map ev_F is in special position if and only if $P \in L$, otherwise $\mathscr{P}$ is in general position.

We will now describe the conditions imposed by Shustin on the curves counted in theorem 2.29 in the setting of (r, s)-marked curves. Curves with these properties will be called unoriented Welschinger curves.

Definition 3.25 (Γ_{even}, roots)**.** Let $r, s \geq 0$ and let Δ be a tropical degree. Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M_{(r,s)}(\Delta)$ and let F be a subset of $\{1, \ldots, |\Delta|\}$. Γ_{even} is the subgraph of Γ consisting of

- all even edges,
- all *double ends* (a pair of unmarked, ends y_i and y_j of the same weight and direction adjacent to the same 4-valent vertex such that $i, j \notin F$),
- all marked ends.

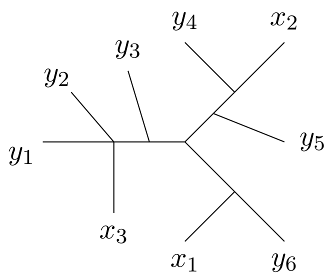
Vertices of $\Gamma_{\text{even}} \cap \overline{\Gamma \setminus \Gamma_{\text{even}}}$ and unmarked non-fixed even ends of Γ_{even} are called *roots* of Γ_{even}.

Definition 3.26 (Unoriented Welschinger curves)**.** Let $r, s \geq 0$, let Δ be a tropical degree and let F be a subset of $\{1, \ldots, |\Delta|\}$. We say that a curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M_{(r,s)}(\Delta)$ is an *unoriented Welschinger curve* with set of fixed ends F if

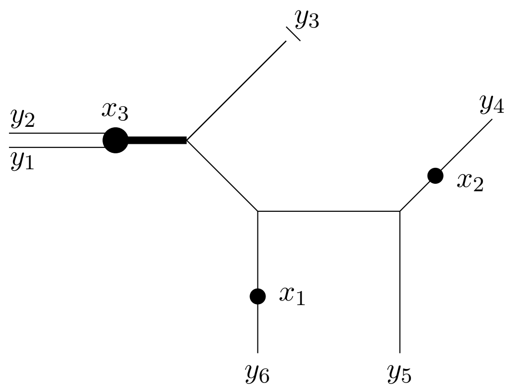
- complex markings are adjacent to 4-valent vertices of Γ or non-isolated in Γ_{even},
- real markings are adjacent to 3-valent vertices of Γ and isolated in Γ_{even}, and
- each connected component of Γ_{even} has a unique root.

Example 3.27 (An unoriented Welschinger curve)**.** Let Δ be the tropical degree given by $\Delta = ((-1,0),(-1,0),(1,1),(1,1),(0,-1),(0,-1))$ and let $F = \{3\}$. ev_F is then the map sending a curve $C = (\Gamma, x_1, x_2, x_3, y_1, \ldots, y_6, h)$ to the tuple consisting of the points $h(x_1), h(x_2)$ and $h(x_3)$ and the line into which h maps y_3.

Recall our unoriented $(2,1)$-marked curve $C = (\Gamma, x_1, x_2, x_3, y_1, \ldots, y_6, h)$ of Example 3.10.

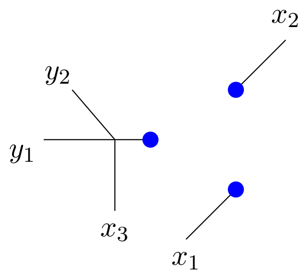


Γ

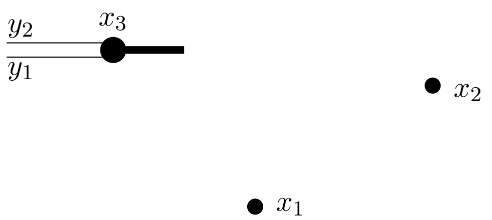


$h(\Gamma)$

C has one even edge, one double end (y_1 and y_2) and three markings. The subgraph Γ_{even} consisting of even edges, double ends and marked ends is the unconnected graph with six edges shown below. The *roots* of Γ_{even} are shown as blue dots.



Γ_{even}



$h(\Gamma_{\text{even}})$

Each connected component of Γ_{even} has a unique root, and the unique complex marking of C is non-isolated in Γ_{even}. Hence, the curve C is an unoriented Welschinger curve.

The next result is a cornerstone in the approach taken by Gathmann, Markwig and Schroeter. It allows us to orientate the edges of a curve C satisfying certain conditions. With these orientations, being an unoriented Welschinger curve may be checked locally - it only depends on the specific types of the curve's vertices.

Lemma 3.28 ([GMS11, 2.13]). *Let $M \subseteq M_{(r,s)}(\Delta)$ be a polyhedral subcomplex, and let $\mathscr{P}$ be a collection of conditions in general position for the evaluation map $\mathrm{ev}_F : M \to \mathbb{R}^{2(r+s)+|F|}$. Consider a curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in \mathrm{ev}_F^{-1}(\mathscr{P})$. Then:*

(a) Each connected component of $\Gamma \setminus (x_1 \cup \ldots \cup x_{r+1})$ has at least one unmarked end y_i with $i \notin F$.

(b) If the combinatorial type of C has dimension $2(r+s)+|F|$ and every vertex of C that is not adjacent to a marking is 3-valent, then every connected component of $\Gamma \setminus (x_1 \cup \ldots \cup x_{r+s})$ as in (a) has exactly one unmarked end y_i with $i \notin F$.

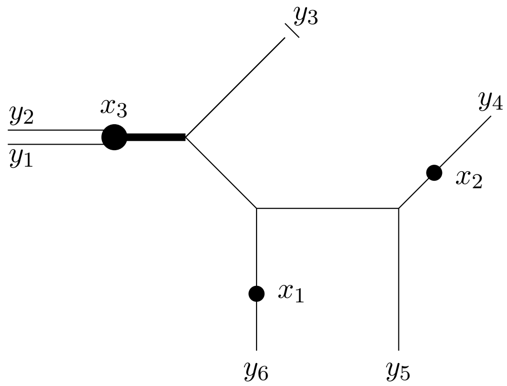
3.2 Oriented (r, s)-marked curves

Let C be an (r, s)-marked curve, and let E be an unmarked edge. Then E is contained in a connected component of $\Gamma \setminus \{x_1, \ldots, x_{r+s}\}$. Lemma 3.28 (if applicable) implies that there is a unique way to orientate each edge E so that it points towards the unique unmarked non-fixed end of the component, see Example 3.30. We will *always* refer to this orientation when we talk about "the oriented version" of an unoriented curve.

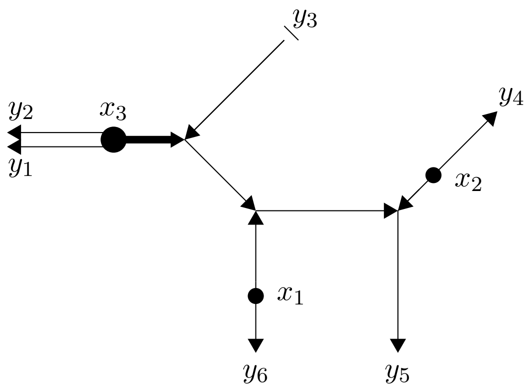
Definition 3.29 (Oriented marked curves). An *oriented (r, s)-marked curve* C is an (r, s)-marked curve in which each unbounded edge of Γ is equipped with an orientation. Let Δ denote the degree of C. The subset $F \subseteq \{1, \ldots, |\Delta|\}$ of all i such that the unmarked end y_i is oriented inwards is called the set of *fixed ends* of C.

In our pictures, we will draw orientations as arrows.

Example 3.30 (Orienting an unoriented curve). Recall the $(2, 1)$-marked plane tropical curve $C = (\Gamma, x_1, x_2, x_3, y_1, \ldots, y_6, h)$ of Example 3.10. In Example 3.27 it was shown that this curve was an unoriented Welschinger curve when $F = \{3\}$.



The space of all curves of this combinatorial type has dimension 7 (it has 5 bounded edges in addition to possible translations in $\mathbb{R}^2$), i.e. $2(r+s)+|F|$. In addition, $\mathscr{P} = \mathrm{ev}_F(C)$ is in general position for ev_F restricted to the closure of the space of all unoriented Welschinger curves (this is a polyhedral subcomplex of $M_{(r,s)}(\Delta)$), hence all conditions of Lemma 3.28 (b) are satisfied; the corresponding oriented curve is shown below:



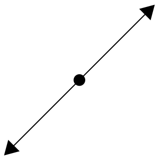
The fixed ends of C will always be oriented inwards, while the non-fixed ends of C will point outwards.

The space of all oriented (r,s)-marked curves of a given degree Δ and set of fixed ends F will be denoted by $M^{\mathrm{or}}_{(r,s)}(\Delta, F)$; it is a polyhedral complex. If $F = \emptyset$ we also write $M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ as $M^{\mathrm{or}}_{(r,s)}(\Delta)$. The *forgetful map* is the map $\mathrm{ft} : M^{\mathrm{or}}_{(r,s)}(\Delta, F) \to M_{(r,s)}(\Delta)$ that ignores the orientations of the edges. The forgetful map ft is a surjective morphism of polyhedral complexes; it is injective on each cell of $M^{\mathrm{or}}_{(r,s)}(\Delta, F)$. There are evaluation maps from $M^{\mathrm{or}}_{(r,s)}(\Delta, F)$ to $\mathbb{R}^{2(r+s)+|F|}$ defined by the composition $\mathrm{ev}_F \circ \mathrm{ft}$. By abuse of notation we will refer to this composition as ev_F as well.

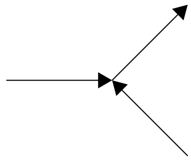
Definition 3.31. The *combinatorial type* of an oriented (r, s)-marked curve is the information of the combinatorial type of the underlying (r, s)-marked curve together with the data of the orientations of all edges.

In the remainder of this thesis, we will focus on oriented (r, s)-marked curves satisfying conditions in general position. Assume $\mathscr{P}$ is sufficiently general and look at curves in $\mathrm{ev}_F^{-1}(\mathscr{P})$. If C is an unoriented curve satisfying these conditions, Lemma 3.28 states that we can orientate its edges so that the orientation of every fixed end is inward and every non-fixed end is oriented outwards. The property of being the oriented version of an unoriented Welschinger curve turns out to be a local property (in the sense that it can be checked at the vertices). These oriented curves will be referred to as oriented Welschinger curves (see Definition 3.36). We will also look at another type of oriented (r, s)-marked curves similar to the oriented Welschinger curves, the broccoli curves. The property of being a broccoli curve is also defined by whether or not the vertices of the curve are of certain types. Hence, even though there are many possible vertex types for (r, s)-marked plane tropical curves, only a few of them will be relevant for our discussion.

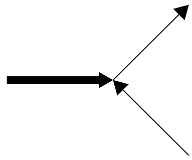
Definition 3.32 (Vertex types and multiplicities)**.** Let V be a vertex of an oriented (r, s)-marked curve. We distinguish such vertices by the parities and orientations of their adjacent edges. In the rest of this thesis, all vertices will be only of the 8 types shown below. We give each vertex a multiplicity, where a denotes the complex vertex multiplicity of Definition 2.20. These multiplicities are then used in Definition 3.33 to give each (r, s)-marked curve with only these types of vertices a multiplicity. They turn out to be exactly what we need in order to tie our count and the count of Shustin together (Proposition 3.46).



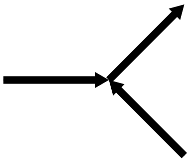
(1)
$m_V = 1$



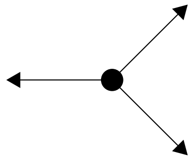
(2)
$m_V = i^{a-1}$



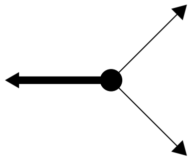
(3)
$m_V = a \cdot i^{a-1}$



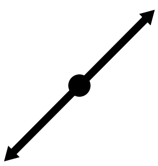
(4)
$m_V = a \cdot i^{a-1} = a \cdot i^{-1}$



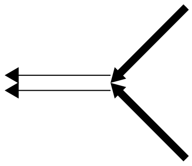
(5)
$m_V = a \cdot i^{a-1}$



(6)
$m_V = i^{a-1}$

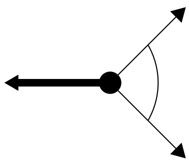


(7)
$m_V = 1$



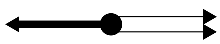
(8)
$m_V = -a$

For all vertices, the edges can be in any direction, except in case (8), where the odd edges are assumed to be non-fixed unmarked ends of the same weight and direction. We also distinguish between two subtypes of (6). If the two odd edges adjacent to the vertex are unmarked ends of the same weight and direction, we say it is a vertex of type (6b). In all other cases, we say it is a vertex of type (6a) and indicate this by an arc as below:



(6a)

$m_V = i^{a-1}$



(6b)

$m_V = i^{a-1} = i^{-1}$

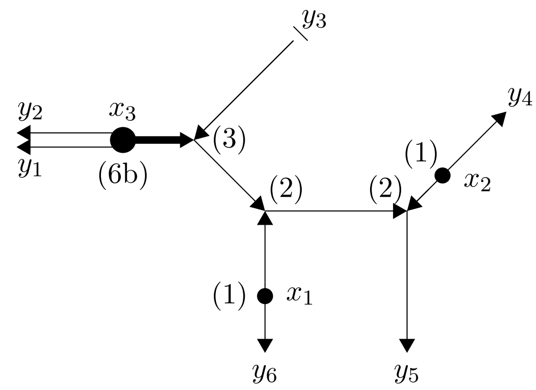
Definition 3.33 (Multiplicity of an (r, s)-marked plane tropical curve)**.** Let $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h)$ be an oriented (r, s)-marked curve, all of whose vertices are of the types listed in Definition 3.32. We define the *multiplicity* of C by

$$m_C := \prod_{k=1}^{n} i^{\omega(y_k)-1} \cdot \prod_{V} m_V,$$

where the last product runs over all vertices of C.

Note that two curves C and D of the same combinatorial α type have the same multiplicity.

Example 3.34. The oriented $(2, 1)$-marked curve in Example 3.30 has two vertices of type (1), two vertices of type (2), one vertex of type (3) and one vertex of type (6b).



All of its unmarked ends have weight 1. Its multiplicity is given by

$$m_C = \left(i^{1-1}\right)^6 \cdot \left(1 \cdot 1 \cdot i^{1-1} \cdot i^{1-1} \cdot (2 \cdot i^{2-1}) \cdot i^{-1}\right) = 2i \cdot i^{-1} = 2.$$

Proposition 3.35. *Every oriented (r, s)-marked curve C, all of whose vertices are of the types in Definition 3.32, has a real multiplicity.*

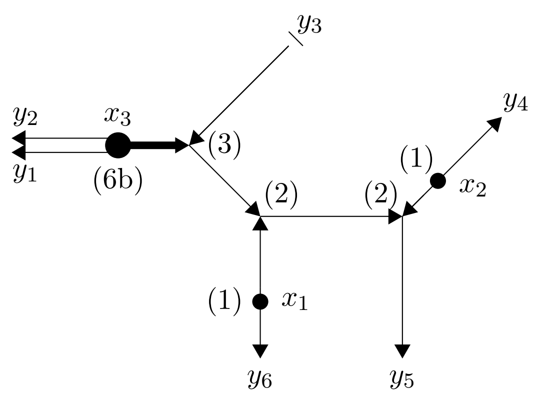
3.3 Oriented Welschinger curves

In Definition 3.32 we introduced several vertex types of oriented (r, s)-marked curves and their multiplicities after vaguely implying that these vertex types are the "important" ones for the curves we want to count.

Recall that Shustin counted plane tropical curves through a given set of "real" and "complex" points. The complex points had to be at either 3-valent vertices (those will be 4-valent now that the markings are edges) or at edges of even weight. In Definition 3.26 we described this in the setting of (r,s)-marked curves. If our conditions are in general position and we orient a curve through them as described after Lemma 3.28, we get a curve all of whose vertices are of some specific types. Conversely, any oriented (r,s)-marked curve through the given points, all of whose vertices are of the specified types, corresponds precisely to an unoriented Welschinger curve (just forgetting the orientations gives the bijection).

Definition 3.36 (Oriented Welschinger curves)**.** An oriented (r,s)-marked curve C, all of whose vertices are of types (1) to (5), (6b), (7) or (8) of Definition 3.32, is called a *Welschinger curve.*

Example 3.37. In Example 3.27, we saw an unoriented Welschinger curve C. Looking at the oriented version of C we found in 3.30, we note that the oriented curve has two vertices of type (1), two vertices of type (2), one vertex of type (3) and one vertex of type (6b):



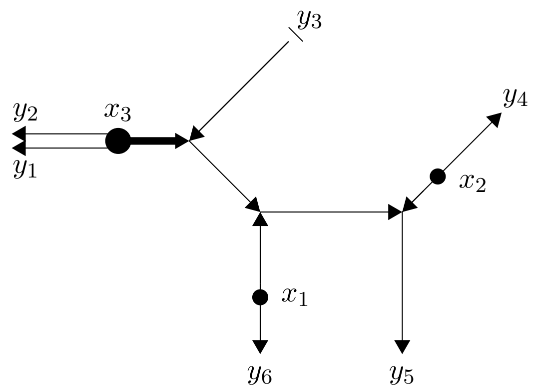
So in this case, the oriented version of an unoriented Welschinger curve was an oriented Welschinger curve. The wonderful thing is that this is true in general; when $\mathscr{P}$ is in general position, the unoriented Welschinger curves correspond exactly to the oriented Welschinger curves.

Proposition 3.38 (Equivalence of oriented and unoriented Welschinger curves [GMS11, Proposition 4.10])**.** *Let $r,s \geq 0$, let Δ be a tropical degree, and let F be a subset of $\{1,\ldots,|\Delta|\}$ such that $r+2s+|F| = |\Delta|-1$. Let $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ be a collection of conditions in general position for the evaluation map $\mathrm{ev}_F : M_{(r,s)}(\Delta) \to \mathbb{R}^{2(r+s)+|F|}$. Then the forgetful map* ft *that disregards the orientations of the edges gives a bijection between oriented* *and unoriented Welschinger curves of degree Δ, with set of fixed ends F and which are mapped to $\mathscr{P}$ by the evaluation maps.*

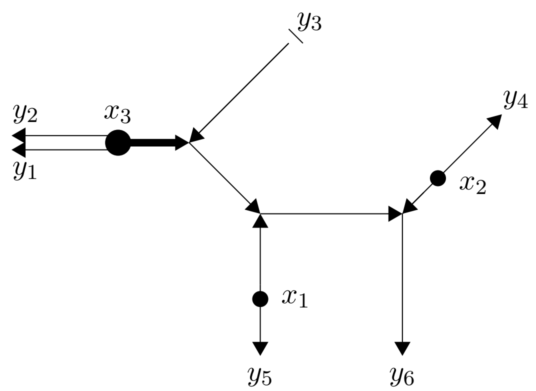
The proposition allows us to shift our full attention to the oriented case. From now on, when speaking of a Welschinger curve, we will always refer to an oriented Welschinger curve.

Let $r+2s+|F| = |\Delta|-1$, denote by $M^W_{(r,s)}(\Delta, F)$ the closure of the space of all Welschinger curves in $M^{\text{or}}_{(r,s)}(\Delta, F)$. $M^W_{(r,s)}(\Delta, F)$ is a polyhedral subcomplex of $M^{\text{or}}_{(r,s)}(\Delta, F)$ of pure dimension $2(r+s)+|F|$. Its maximal open cells correspond exactly to the combinatorial types of Welschinger curves in $M^W_{(r,s)}(\Delta, F)$.

Example 3.39. In Example 3.37, we looked at the $(2,1)$-marked plane tropical curve $C = (\Gamma, x_1, x_2, x_3, y_1, \ldots, y_6, h)$ below.



It has degree $\Delta = ((-1,0),(-1,0),(1,1),(1,1),(0,-1),(0,-1))$ and set of fixed ends $F = \{3\}$. If we count $(2,1)$-marked plane tropical curves C' of this degree such that $\text{ev}_F(C') = (h(x_1), h(x_2), h(x_3), h(y_3))$, we will also count curves such as the one below:



The curves differ only in the labelling of their unmarked ends. They are of the same degree and both satisfy the given conditions. In Theorem 2.29, Shustin counts *unparametrised* curves without this labelling, so to compensate for the overcounting, we must divide by the number of such relabellings of the non-fixed ends.

Definition 3.40 ($G(\Delta, F)$)**.** Let $\mathbb{S}_n$ denote the group of all permutations of $\{1, \dots, n\}$. Then $G(\Delta, F)$ is the subgroup of all permutations σ in $\mathbb{S}_n$ such that $\sigma(i) = i$ for all $i \in F$ and $v_{\sigma(i)} = v_i$ for all $i = 1, \dots, n$. If $F = \emptyset$, we denote ev_F by ev and $G(\Delta, F)$ by $G(\Delta)$.

Definition 3.41 (Welschinger numbers)**.** Let $r, s \geq 0$, let Δ be a tropical degree and let F be a subset of $\{1, \dots, |\Delta|\}$ satisfying $r + 2s + |F| = |\Delta| - 1$. Let $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ be in general position for the evaluation map ev_F restricted to $M^{\mathrm{W}}_{(r,s)}(\Delta, F)$. Then the *Welschinger number* corresponding to Δ, F and $\mathscr{P}$ is

$$N^{\mathrm{W}}_{(r,s)}(\Delta, F, \mathscr{P}) := \frac{1}{|G(\Delta, F)|} \sum_C m_C,$$

where the sum runs over all Welschinger curves C of degree Δ and set of fixed ends F such that $\mathrm{ev}_F(C) = \mathscr{P}$.

Remark 3.42. Interchanging the labelling y_1 and y_2 in Example 3.39 above produces the same curve, so our discounting factor is slightly too big. Luckily, the missing factor will be exactly the difference between the multiplicity defined in Definition 3.32 and the Shustin multiplicity of the corresponding curve (this is Lemma 3.44). This is the summed up in Proposition 3.46 below.

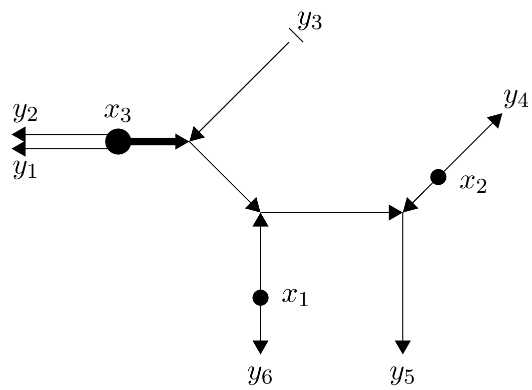
Definition 3.43 (Shustin multiplicity)**.** Let $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h)$ be a Welschinger curve curve such that the Newton subdivision dual to $h(\Gamma)$ consists only of triangles and parallellograms. Then the *Shustin multiplicity* of C is

$$\mathrm{mult}_S(C) := (-1)^{a+b} \cdot 2^{-c} \cdot \prod_V \mathrm{mult}(V),$$

where a is the number of lattice points inside triangles of this subdivision, b is the number of triangles such that all sides have even lattice length, c is the number of triangles whose lattice area is even, and the product runs over all triangles with even lattice area or dual to vertices with a complex marking.

Lemma 3.44 (Multiplicity and Shustin multiplicity [GMS11, Lemma 4.19])**.** *Let $C = (\Gamma, x_1, \dots, x_{r+s}, y_1, \dots, y_n, h)$ be a Welschinger-curve of degree Δ with no fixed ends, such that $\omega(y_i) = 1$ for all $i = 1, \dots, n$ and passing through points in general position. Then the multiplicity m_C and the Shustin multiplicity $\mathrm{mult}_S(C)$ are related by $m_C = 2^k \cdot \mathrm{mult}_S(C)$, where k is the number of double ends of C.*

Example 3.45. Recall the oriented Welschinger curve from Example 3.37.



Now, $\text{mult}_S(C) = 1$ by Example 2.27 while $m_C = 2$ by Example 3.34. The curve has one pair of double ends, so this is consistent with Lemma 3.44.

Let Δ be the tropical degree consisting of d times each of the vectors $(-1, 0)$, $(0, -1)$ and $(1, 1)$ and let F be empty. Combining Lemma 3.44 with Remark 3.42, we see that the Welschinger numbers equal the Welschinger invariants:

Proposition 3.46. *Let Δ be the tropical degree consisting of d times each of the vectors $(-1, 0)$, $(0, -1)$ and $(1, 1)$. Let r and s be non-negative integers such that $r + 2s = 3d - 1$. Then*

$$W_{\mathbb{P}^2}(d, r, s) = N^W_{r,s}(\Delta).$$

Hence, when Δ consists of d times each of the vectors $(0, -1)$, $(-1, 0)$ and $(1, 1)$ and F is empty, the Welschinger numbers $N^W_{(r,s)}(\Delta)$ are independent of the collection $\mathscr{P}$ in general position by Shustin's theorem (Theorem 2.29). Proving this invariance directly turns out to be difficult.

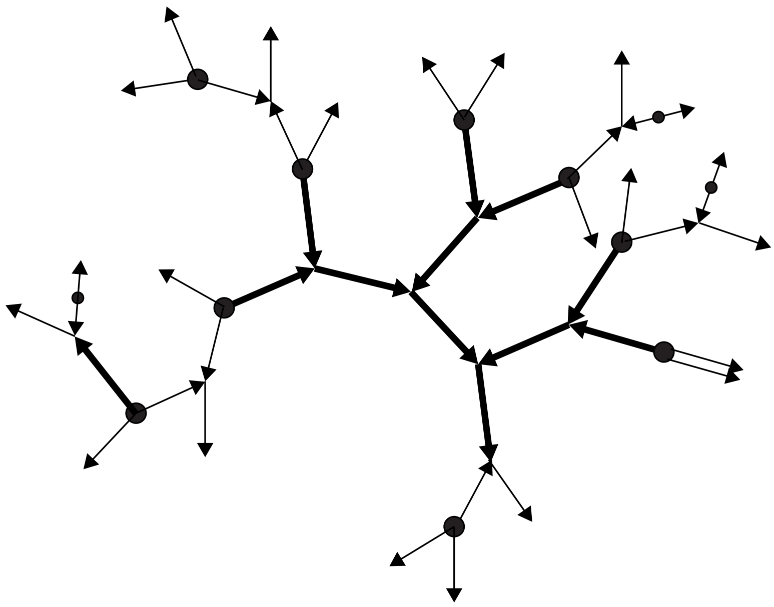
There is, however, a type of curve whose count is locally invariant in the moduli space. Better still, their count equals the count of Welschinger curves. These curves are the broccoli curves.

3.4 Broccoli curves

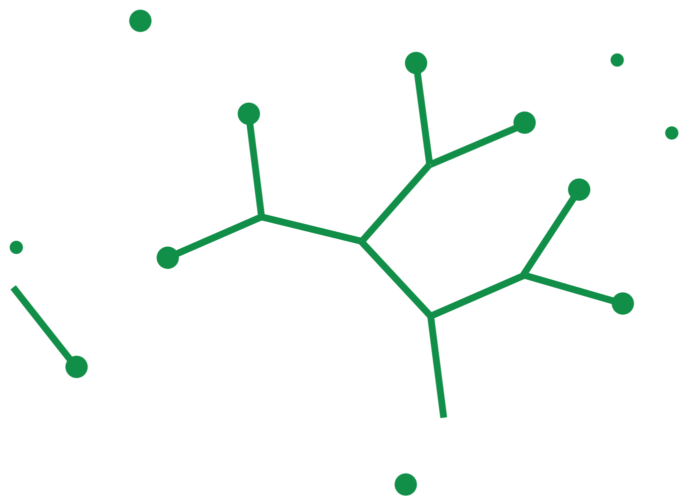
Definition 3.47 (Broccoli curves)**.** An oriented plane tropical curve C, all of whose vertices are of types (1) to (6) of Definition 3.32, is called a *broccoli curve*.

Example 3.48. The oriented Welschinger-curve C we met in Example 3.37 and Example 3.45 is also a broccoli curve.

Example 3.49. Consider the oriented $(3, 9)$-marked curve C below. It is a broccoli curve in which all vertex types allowed for broccoli curves appear.



The name stems from the unoriented version of such curves; forget the orientations of the edges and consider the subgraph of Γ consisting of all even edges and all marked ends. Its image under h is shown in green below.



Denote by $M^B_{(r,s)}(\Delta, F)$ the closure of the space of all broccoli curves in $M^{\text{or}}_{(r,s)}(\Delta, F)$. It is a polyhedral subcomplex of $M^{\text{or}}_{(r,s)}(\Delta, F)$. $M^B_{(r,s)}(\Delta, F)$ is non-empty if and only if $r + 2s + |F| = |\Delta| - 1$.

Let $r + 2s + |F| = |\Delta| - 1$. Then $M^B_{(r,s)}(\Delta, F)$ is of pure dimension $2(r+s)+|F|$. Its maximal open cells correspond exactly to the combinatorial types of broccoli curves in $M^B_{(r,s)}(\Delta, F)$.

Definition 3.50 (Broccoli numbers)**.** Let $r, s \geq 0$, let Δ be a tropical degree and let F be a subset of $\{1, \ldots, |\Delta|\}$ satisfying $r + 2s + |F| = |\Delta| - 1$. Let $\mathscr{P} \in \mathbb{R}^{2(r+s)+|F|}$ be in general position for the evaluation map ev_F restricted to $M^{\mathrm{B}}_{(r,s)}(\Delta, F)$. Then the *broccoli number* corresponding to Δ, F and $\mathscr{P}$ is

$$N^{\mathrm{B}}_{(r,s)}(\Delta, F, \mathscr{P}) := \frac{1}{|G(\Delta, F)|} \sum_C m_C,$$

where the sum runs over all broccoli curves C of degree Δ and set of fixed ends F such that $\mathrm{ev}_F(C) = \mathscr{P}$.

The function $\mathscr{P} \mapsto N^B_{(r,s)}(\Delta, F, \mathscr{P})$ is locally constant on the open subset of $\mathbb{R}^{2(r+s)+|F|}$ of points in general position for broccoli curves and may vary only at the image under ev_F of the boundary of cells of $M^B_{(r,s)}(\Delta, F)$ of maximal dimension. The image of these boundaries is a union of polyhedra in $\mathbb{R}^{2(r+s)+|F|}$ of positive codimension. Gathmann, Markwig and Schroeter prove that $\mathscr{P} \mapsto N^B_{(r,s)}(\Delta, F, \mathscr{P})$ is locally constant around a cell of codimension 1 (in $\mathbb{R}^{2(r+s)+|F|}$) in this image. Since any cells of maximal dimension can be connected through codimension-1 cells, this completes the proof of the following theorem.

Theorem 3.51 (Invariance of broccoli numbers [GMS11, Theorem 3.6])**.** *Let $r, s \geq 0$, let Δ be a tropical degree and let F be a subset of $\{1, \ldots, |\Delta|\}$ satisfying $r + 2s + |F| = |\Delta| - 1$. Then $N^B_{(r,s)}(\Delta, F, \mathscr{P})$ is independent of the conditions $\mathscr{P}$.*

Since the broccoli numbers are independent of $\mathscr{P}$ when Δ is as above, we write them as $N^{\mathrm{B}}_{(r,s)}(\Delta, F)$ (or $N^{\mathrm{B}}_{(r,s)}(\Delta)$ if F is empty).

Gathmann, Markwig and Schroeter proceed to show the equivalence of broccoli and Welschinger numbers for some specific degrees Δ:

Theorem 3.52 (Welschinger numbers equal broccoli numbers [GMS11, Corollary 5.16])**.** *Let $r, s \geq 0$, let Δ be a degree consisting of d times each of the vectors $(-1, 0)$, $(0, -1)$ and $(1, 1)$, and let F be a subset of $\{1, \ldots, |\Delta|\}$ satisfying $r+2s+|F| = |\Delta|-1$. For any configuration $\mathscr{P}$ in general position for Welschinger curves,*

$$N^W_{(r,s)}(\Delta, F, \mathscr{P}) = N^B_{(r,s)}(\Delta, F, \mathscr{P}).$$

Combining Theorem 3.51 and Theorem 3.52 we get the invariance of the Welschinger numbers.

Corollary 3.53 (The Welschinger numbers do not depend on $\mathscr{P}$)**.** *Let $r, s \geq 0$, let Δ be a degree consisting of d times each of the vectors $(-1,0)$, $(0,-1)$ and $(1,1)$, and let F be a subset of $\{1,\ldots,|\Delta|\}$ satisfying $r+2s+|F| = |\Delta|-1$. Then $N^{W}_{(r,s)}(\Delta, F, \mathscr{P})$ is independent of the conditions $\mathscr{P}$.*

Since the Welschinger numbers are independent of $\mathscr{P}$ when Δ is as above, we write them as $N^{\mathrm{W}}_{(r,s)}(\Delta, F)$ (or $N^{\mathrm{W}}_{(r,s)}(\Delta)$ if F is empty). Even though the broccoli numbers are independent of $\mathscr{P}$ for any tropical degree, the same is not true for Welschinger curves (see Section 7.2 (Figure 2) of [ABLdM11]. The same example is given with our notation in [GMS11, Example 4.25]).

4 Calculations in $\mathbb{P}^2$

By Proposition 3.46, the Welschinger invariants for degree d of the projective plane equal the Welschinger numbers of degrees consisting of d times each of the vectors $(0,-1)$, $(-1,0)$ and $(1,1)$ with no fixed ends. In turn, the Welschinger numbers equal the corresponding broccoli numbers by Theorem 3.52.

Gathmann, Markwig and Schroeter found a pair of recursive formulæ sufficient to compute all broccoli numbers for these degrees and therefore all Welschinger invariants of the projective plane. The derivation of these formulæ will be the topic of this section.

In Section 5 and Section 6 we discuss generalisations of the formulæ to tropical degrees corresponding to some other surfaces.

4.1 Notation

We start by introducing some notation to simplify the discussion.

Let $\mathbb{N}_0$ be the set of non-negative integers. The set of sequences in $\mathbb{N}_0$ with only finitely many nonzero terms will be denoted by $\mathbb{N}_0^\infty$.

For α in $\mathbb{N}_0^\infty$, we denote its ith coordinate by $(\alpha)_i$. The vector in $\mathbb{N}_0^\infty$ whose only nonzero coordinate is $(\alpha)_i = 1$ will be denoted by e_i. Given two sequences α and β in $\mathbb{N}_0^\infty$, we write $\alpha \geq \beta$ if $(\alpha)_i \geq (\beta)_i$ for all i. If the inequality is strict for at least one i, we write $\alpha > \beta$.

$$|\alpha| := \sum_{i=1}^{\infty} (\alpha)_i.$$

$$I\alpha := \sum_{i=1}^{\infty} i(\alpha)_i.$$

To add two sequences, we sum componentwise;

$$\alpha + \beta := ((\alpha)_1 + (\beta)_1, (\alpha)_2 + (\beta)_2, \ldots).$$

If a and b are two non-negative integers such that $a \geq b$, $\binom{a}{b}$ denotes the binomial coefficient. If a and $b_1, \ldots, b_k$ are non-negative integers such that $a \geq b_1 + \ldots + b_k$, $\binom{a}{b_1,\ldots,b_k}$ denotes the multinomial coefficient;

$$\binom{a}{b_1, \ldots, b_k} := \frac{a!}{(a-b_1)!(a-b_2)! \cdots (a-b_k)!(a-b_1-b_2-\ldots-b_k)!}.$$

For sequences $\alpha, \alpha^1, \alpha^2, \ldots, \alpha^k \in N_0^\infty$, we define a generalised multinomial coefficient;

$$\binom{\alpha}{\alpha^1, \alpha^2, \ldots, \alpha^k} := \prod_i \binom{(\alpha)_i}{(\alpha^1)_i, (\alpha^2)_i, \ldots, (\alpha^k)_i}.$$

For a positive integer k, we let

$$M_k := \begin{cases} k & \text{if } k \text{ is odd,} \\ -1 & \text{if } k \text{ is even,} \end{cases} \quad \text{and} \quad \tilde{M}_k := \begin{cases} k & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

Definition 4.1 (Broccoli curves of type (α, β))**.** Let $d > 0$ and let α and β be sequences such that $I\alpha + I\beta = d$. Let $\Delta(\alpha, \beta)$ be a degree consisting of d times the vector $(0, -1)$, d times the vector $(1, 1)$, and $(\alpha)_i + (\beta)_i$ times $(-i, 0)$ for all i (in any fixed order). Let $F(\alpha, \beta) \subseteq \{1, \ldots, |\Delta(\alpha, \beta)|\}$ be a fixed subset with $|\alpha|$ elements such that the entries of $\Delta(\alpha, \beta)$ with index in F are $(\alpha)_i$ times $(-i, 0)$ for all i.

A broccoli curve in $M^{\mathrm{B}}_{(r,s)}(\Delta(\alpha, \beta), F(\alpha, \beta))$ will be called a *curve of type* (α, β). Its unmarked ends with directions $(-i, 0)$ will be referred to as *left ends.*

Example 4.2 (A broccoli curve of type $((1), (2))$)**.** Let Δ be the tropical degree given by

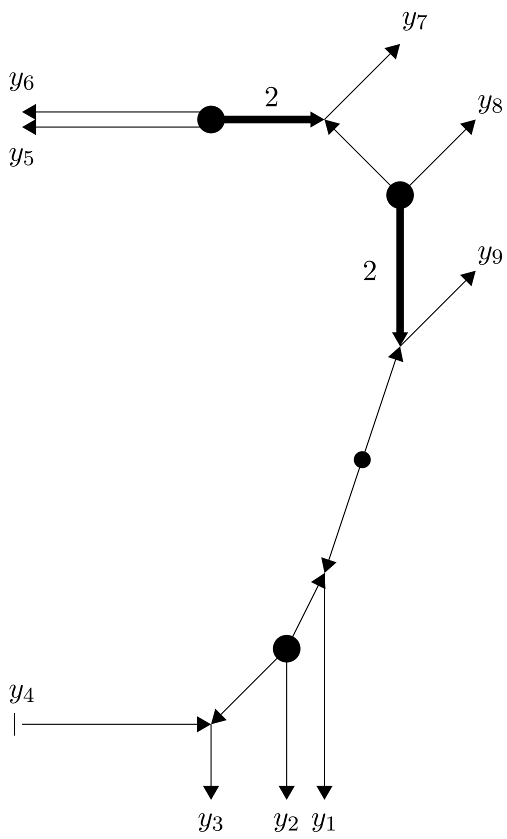
$$\Delta = ((0, -1), (0, -1), (0, -1), (-1, 0), (-1, 0), (-1, 0), (1, 1), (1, 1), (1, 1))$$

and let $F = \{4\}$. Then Δ consists of 3 times the vectors $(0, -1)$, $(1, 1)$ and $(1, 1)$. F consists of a single element, corresponding to a left end of weight 1. The curves in $M^{\mathrm{B}}_{(r,s)}(\Delta, F)$ are curves of type $((1), (2))$.

The curve shown below is an example of such a curve with one real and three complex markings. Note that F could have been $\{5\}$ or $\{6\}$ instead (this would just amount to a relabelling of our unmarked ends). The main point is that the curve has two non-fixed left ends of weight 1 and one fixed left end of weight 1 (in addition to three non-fixed unmarked ends of direction $(1, 1)$ and three non-fixed unmarked ends of direction $(0, -1)$). Likewise, the exact ordering of the directions of the ends in Δ does not matter; we could just as well have been considering broccoli curves of degree

$$\Delta = ((-1, 0), (1, 1), (1, 1), (1, 1), (0, -1), (0, -1), (-1, 0), (-1, 0), (0, -1))$$

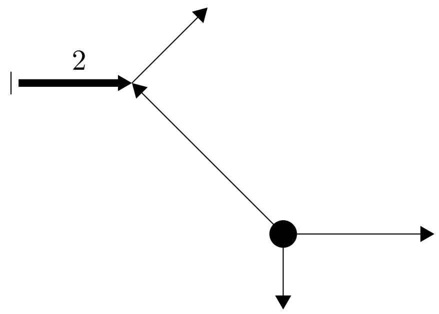
with $F = \{8\}$.



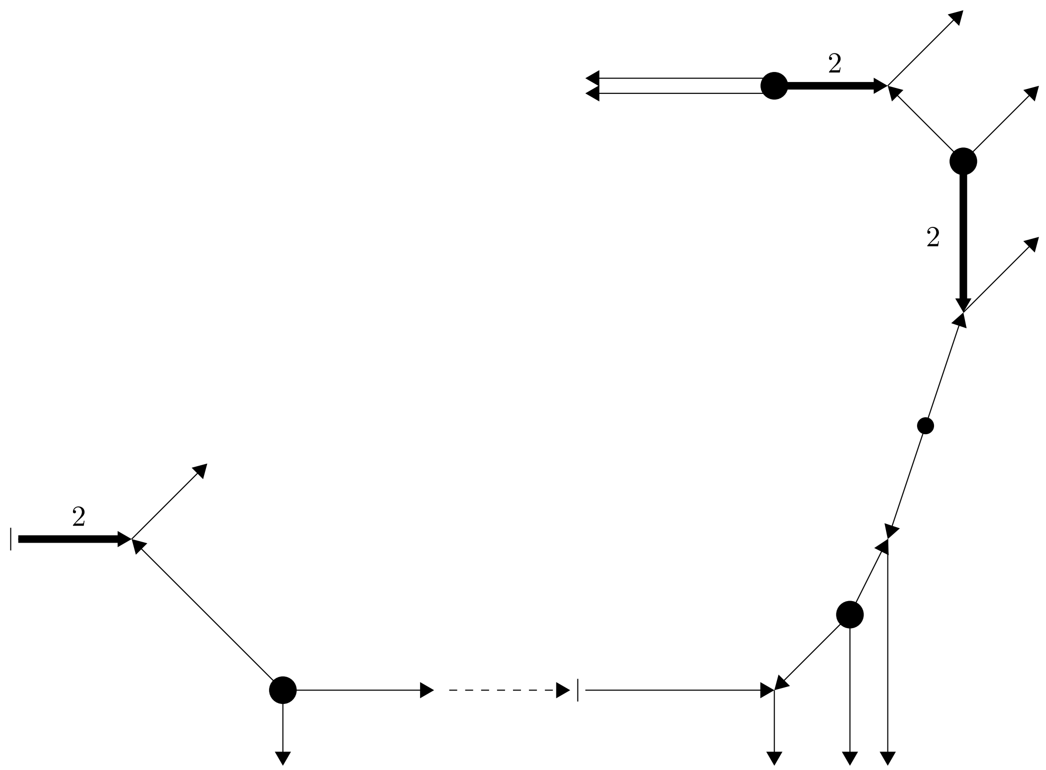
As in this example, the labelling of the unmarked ends will not be relevant to our discussion, only their directions and orientations. We will therefore draw curves without the labellings of their unmarked ends.

Definition 4.3 (Relative broccoli invariants)**.** Let $\Delta(\alpha,\beta)$ and $F(\alpha,\beta)$ be as in Definition 4.1 and let $r,s \geq 0$ such that $|\Delta| - 1 - |F| = 2d + |\beta| - 1 = r + 2s$. We then use $N^d(\alpha,\beta,s)$ as a shorthand expression for $N^{\mathrm{B}}_{(r,s)}(\Delta(\alpha,\beta), F(\alpha,\beta))$.

Example 4.4. One reason we distinguish between orientations of the curves is the possibility of "gluing" and decomposing of curves. Suppose we see this curve:



Its right end is non-fixed, so we can combine this curve with the one from Example 4.2 to give:



The new curve is a curve of type $((0,1),(2))$ as it has one fixed left end of weight 2 and two non-fixed left ends of weight 1.

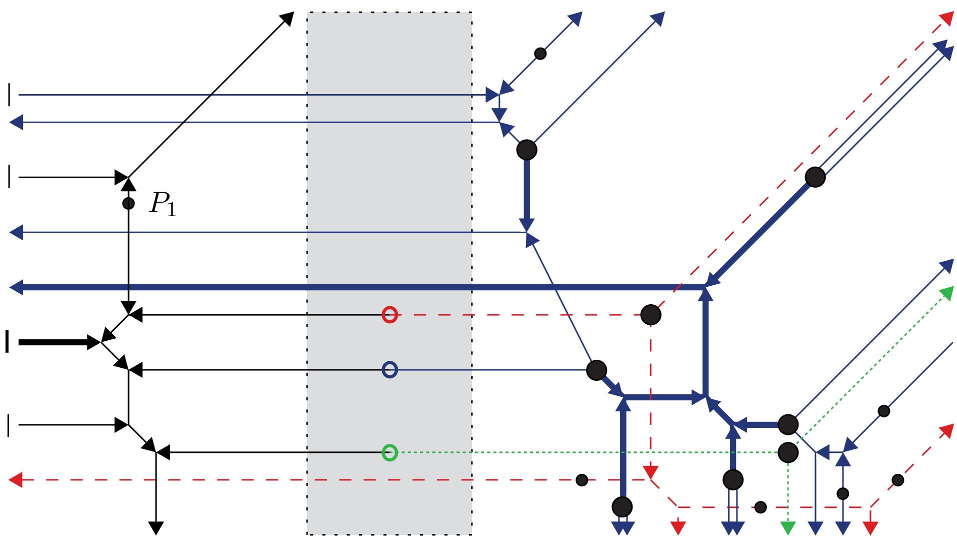
To find recursive relations among the relative broccoli invariants, we reverse this procedure. Given a collection $\mathscr{P}$ (of points and y-coordinates for the fixed left ends) in general position, we move one of the points, say P_1, to the far left. Then, the curve may be decomposed into two parts:

Lemma 4.5 (Decomposing a curve into a left and a right part [GM07, GMS11])**.** *Let $\Delta = \Delta(\alpha, \beta)$ and $F = F(\alpha, \beta)$ be as in Definition 4.1 and let $2d + |\beta| - 1 = r + 2s$. Fix a small number ϵ and a large number $N > 0$. Choose $r + s$ points $P_1, \ldots, P_{r+s}$ and $|\alpha|$ y-coordinates for the fixed left ends such that these points are in general position and*

- *the y-coordinates of all P_i and the fixed ends are in the open interval $(-\epsilon, \epsilon)$,*
- *the x-coordinates of $P_2, \ldots, P_{r+s}$ are in $(-\epsilon, \epsilon)$,*
- *the x-coordinate of P_1 is smaller than $-N$.*

Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M^B_{(r,s)}(\Delta, F)$ be a broccoli curve satisfying these conditions. Then no vertex of C can have its y-coordinate below $-\epsilon$ or above ϵ, and there exists a rectangle $R = [a, b] \times [-\epsilon, \epsilon]$ (with $a \geq -N$, $b \leq -\epsilon$ depending only on d) such that $R \cap h(\Gamma)$ contains only horizontal edges of C.

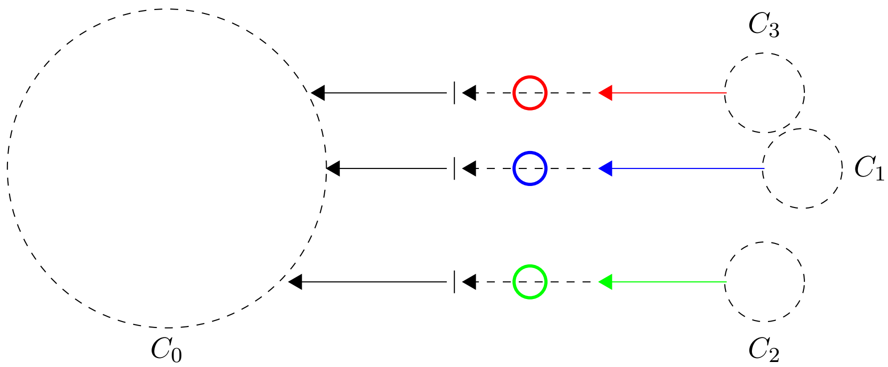
Example 4.6 (Splitting on horizontal edges)**.** The decomposition of a concrete curve after moving a real point to the far left is depicted in the figure below (modified from [GMS11]).



Here, C is a curve of type $((3, 1), (3, 1))$ with 7 real and 9 complex marked ends. We cut C at each bounded edge through the rectangle R (in grey), the cutting points are shown by red, blue and green circles.

Denote the component passing through P_1 (the left part) by C_0 and the union of the other connected components (the right part) by $\tilde{C}$. At each of the cutting points, the edge is replaced by an end of the same direction and orientation. The connected components of $\tilde{C}$ correspond exactly to the *right ends* of C_0, i.e. the ends of direction $(k_i, 0)$ for some positive integer k_i.

With this particular curve, all of the cutting points are at edges oriented towards the left part. Hence, we split up these edges as in the figure below.



Now, C_0 has 3 fixed left ends of weight 1 and one fixed left end of weight 2. It has one end of direction $(1,1)$ and one end of direction $(0,-1)$. In addition, C_0 has three right ends, all of weight 1 (these are the ones we just created through the splitting process). These right ends of C_0 are all bounded edges of C passing through the rectangle R. In this case, they are all fixed. Each right end of C_0 corresponds to a unique connected component of $\tilde{C}$. The blue solid curve, denote it by C_1, is a curve of type $((1),(3,1))$ (the edge connecting C_1 to C_0 is now an end of C_1). The green dotted curve, C_2, is a curve of type $((0),(1))$ and the red dashed curve, C_3 is a curve of type $((0),(2))$.

Thus, when one of the points in $\mathscr{P}$ is moved to the far left, the curves in $\text{ev}^{-1}(\mathscr{P})$ may be decomposed into two parts. All we have to do to find the invariants we seek is to sum the contributions over all possible decompositions.

4.2 Examples

In this subsection, we will compute some relative broccoli invariants explicitly by looking at the possible decompositions of curves as one of the marked points is moved to the far left.

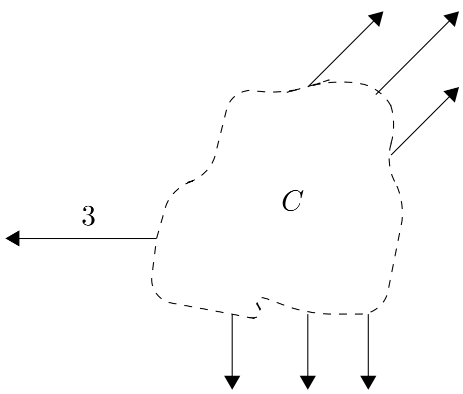
For easy reference (we will use this result frequently in this and the subsequent section), we state a key lemma once more:

Lemma 3.28 ([GMS11, 2.13]). *Let $M \subseteq M_{(r,s)}(\Delta)$ be a polyhedral subcomplex, and let $\mathscr{P}$ be a collection of conditions in general position for the evaluation map $\mathrm{ev}_F : M \to \mathbb{R}^{2(r+s)+|F|}$. Consider a curve $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in \mathrm{ev}_F^{-1}(\mathscr{P})$. Then:*

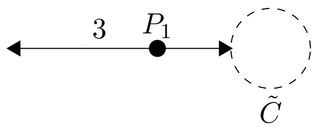
(a) Each connected component of $\Gamma \setminus (x_1 \cup \ldots \cup x_{r+1})$ has at least one unmarked end y_i with $i \notin F$.

(b) If the combinatorial type of C has dimension $2(r+s)+|F|$ and every vertex of C that is not adjacent to a marking is 3-valent, then every connected component of $\Gamma \setminus (x_1 \cup \ldots \cup x_{r+s})$ as in (a) has exactly one unmarked end y_i with $i \notin F$.

Example 4.7. First we will compute $N^3((0),(0,0,1),1)$. That is, we will look at broccoli curves through 4 "real" and 1 "complex" point in the plane with three non-fixed ends of direction $(0,-1)$, three non-fixed ends of direction $(1,1)$, and one non-fixed end of direction $(-3,0)$. How many curves, counted with multiplicity, fit in the picture below?



i) We start by seeing what happens as we move a "real" point to the far left. By Lemma 4.5, a curve C satisfying the given conditions may be decomposed into two parts. One such possible decomposition is shown below:



Here, $\tilde{C}$ consists of a single component; a curve of type $((0,0,1),(0))$ (it has exactly one left end, which is a fixed end of weight 3) through one complex point and three real points.

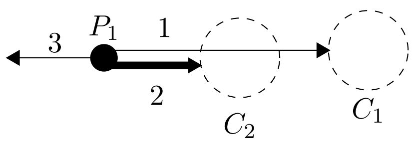
In fact, this is the only possible decomposition! Recall that a real marking *must* be adjacent to a vertex of type (1) for a broccoli curve. If C_0 were to be contained in a horizontal line and have more than one vertex, the length of the bounded edges would not be fixed by our conditions (the marking mapped to P_1 is the only marked end of C_0). But if C_0 is not contained in a horizontal line, the balancing condition ensures that C_0 contains ends of directions $(1,1)$ and $(0,-1)$. These are both non-fixed, and so is the left end of weight 3. However, $C_0 \setminus \{x_1\}$ (where x_1 is the marking such that $h(x_1) = P_1$) consists of two connected components, each of which has a unique non-fixed end by Lemma 3.28. This is a contradiction. Hence, no other decompositions than the one shown above may exist.

The curves of type $((0),(0,0,1))$ with one complex marking correspond exactly to the curves of type $((0,0,1),(0))$ with one complex marking. The vertex multiplicity of the unique vertex of C_0 is 1. C has one end which is not an end of $\tilde{C}$ (the left non-fixed end of weight 3), while $\tilde{C}$ has one end which is not an end of C (the edge connecting C_0 to $\tilde{C}$ of weight 3). The contributions to the curve multiplicities (of C and $\tilde{C}$, respectively) of these ends are both i^{3-1}, so they outweigh each other; $m_C = 1 \cdot i^{3-1} \cdot \frac{m_{\tilde{C}}}{i^{3-1}}$. Summing up over all curves of type $((0,0,1),(0))$ with one complex marking, we see that

$$N^3\left((0),(0,0,1),1\right) = N^3\left((0,0,1),(0),1\right).$$

ii) Now, suppose that we moved a complex point to the left instead. The marking must then be adjacent to a vertex of type (5) or (6), there are three possible decompositions of C.

Firstly, C_0 could be contained in a horizontal line as depicted below:

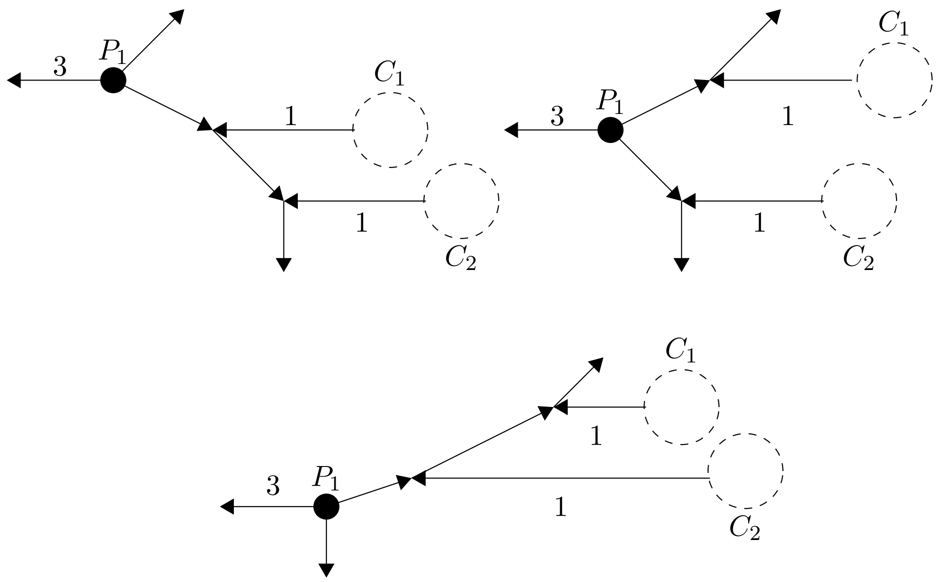


C_1 and C_2 must be curves of types $((1),(0))$ and $((0,1),(0))$ with 1 and 3 real markings, respectively. There are 4 possible ways to distribute the real markings in such a way among C_1 and C_2. Hence, the contribution from this sort of decomposition becomes

$$4 \cdot N^1((1),(0),0) \cdot N^2((0,1),(0),0).$$

This is the only possible decomposition such that the image of C_0 is contained in a line; the type of the curve C specifies that it has exactly one non-fixed left end, while the balancing condition ensures that the right ends of C_0 must be of weight 1 and 2.

Secondly, C_0 could contain bounded edges. Then, the complex marking which is sent to P_1 must be adjacent to a left end of C_0. $\tilde{C}$ could consist of 2 connected components:

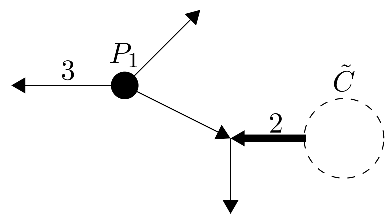


Only one of these decompositions will be possible; which one depends on the specific configuration $\mathscr{P}$. In the rest of this thesis, we will stick to one representative for each such group of decompositions, keeping this in mind.

As a consequence of Lemma 3.28, all right ends of C_0 must be fixed. Both C_1 and C_2 must therefore be curves of type $((0),(1))$. To find the multiplicity of C we first take the product of m_{C_1} and m_{C_2}. In addition, we get a factor $3i^2$ from the vertex adjacent to the complex marking, and a factor i^2 appears since the left end of C is not a left end of $\tilde{C}$. The four real markings can be distributed in 6 ways among the two curves C_1 and C_2 so that they have 2 each. It is impossible to distinguish between C_1 and C_2, so we have overcounted by a factor of 2. Hence, the contribution from this kind of decomposition equals

$$\frac{1}{2} \cdot 3 \cdot 6 \cdot N^1\left((0),(1),0\right) \cdot N^1\left((0),(1),0\right).$$

Similarly, $\tilde{C}$ could consist of a single connected component:



In this case, we get a factor of $3i^2$ from the vertex adjacent to the marking and a factor $2i$ from the vertex not adjacent to the marking. In addition, we must multiply by i^2 since the left end of C is not a left end of $\tilde{C}$ and divide by i since the edge of weight 2 is an end of $\tilde{C}$ but not an end of C. Hence, the contribution from this kind of decomposition equals

$$3 \cdot 2 \cdot N^2((0),(0,1),0).$$

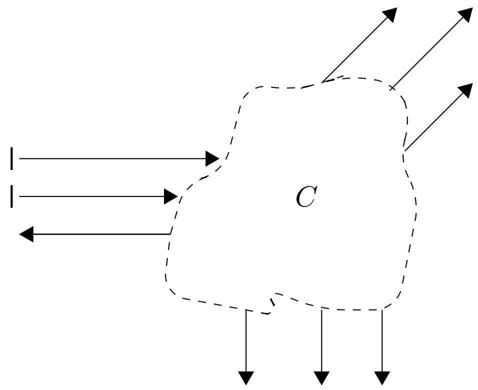
One could also ask if the complex marking could not be adjacent to a right end or not adjacent to an end at all. But in this case, we would have three non-fixed ends of C_0, with only 2 connected components of $C_0 \setminus \{x_1\}$ (where x_1 is the marked end whose image is P_1), contradicting Lemma 3.28.

Concluding, we see that

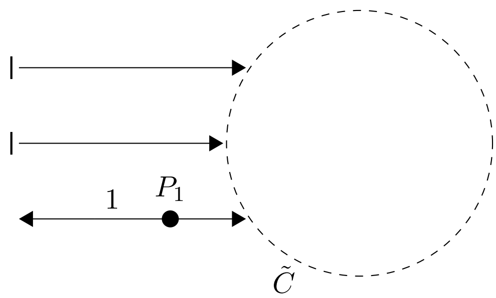
$$\begin{aligned} N^3\left((0),(0,0,1),1\right) = 4 \cdot N^1((1),(0),0) \cdot N^2((0,1),(0),0) \\ + 9 \cdot N^1\left((0),(1),0\right) \cdot N^1\left((0),(1),0\right) \\ + 6 \cdot N^2((0),(0,1),0). \end{aligned}$$

Example 4.8. We will compute $N^3((2),(1),1)$.

This is the number of broccoli curves with three non-fixed ends of direction $(1,1)$, three non-fixed ends of direction $(0,-1)$, two fixed left ends of weight 1 and one non-fixed left end of weight 1.



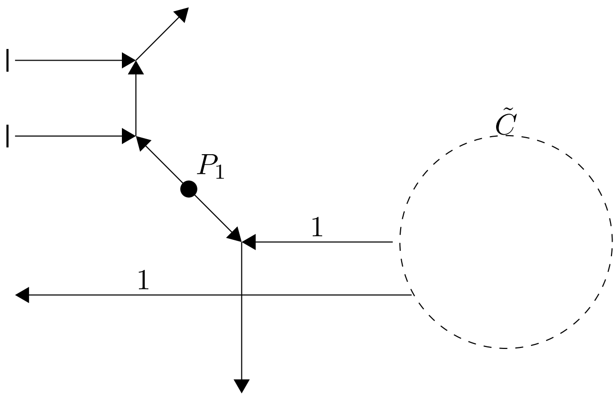
i) First, the real marking may be adjacent to the unique non-fixed left end of C as we saw in the case of our computation of $N^3\left((0),(0,0,1),1\right)$:



The curve $\tilde{C}$ is then a curve of type $((3),(0))$. The multiplicity of the vertex adjacent to P_1 is 1 and the ends of C have the same weights as the ends of $\tilde{C}$, so $m_C = m_{\tilde{C}}$. Thus, the contribution to $N^3((2),(1),1)$ from this sort of decomposition is

$$N^3((3),(0),1).$$

Secondly, C_0 could contain a bounded edge:



All left and right ends of C_0 must be fixed, since Lemma 3.28 guarantees the existence of a *unique* non-fixed end of each connected component of $C_0 \setminus \{x_1\}$ (there are already two such - the ends with directions $(0,-1)$ and $(1,1)$).

The multiplicity of C is given by the multiplicity of $\tilde{C}$. Since we can choose which of the two non-fixed left ends of $\tilde{C}$ to use as the "connecting edge" between $\tilde{C}$ and C_0, we multiply this multiplicity by a factor of 2. The contribution from this sort of decomposition is then given by

$$N^2\left((0),(2),1\right).$$

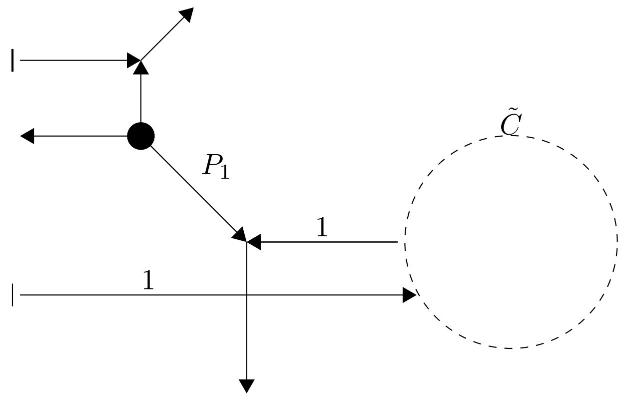
Summing up the two cases, we see that

$$N^3\left((2),(1),1\right) = N^3\left((3),(0),1\right) + 2 \cdot N^2\left((0),(2),1\right).$$

ii) Now, we move to computing $N^3((2),(1),1)$ by moving a complex marking to the far left instead.

First, note that the image of C_0 can *not* be contained in a line. This would require that the complex marking and the unique non-fixed end of C were adjacent to the same vertex of type (5) or (6). By the balancing condition, the other two edges would have to be horizontal edges whose direction vectors $(k_1,0)$ and $(k_2,0)$ sum to $(1,0)$, which is impossible.

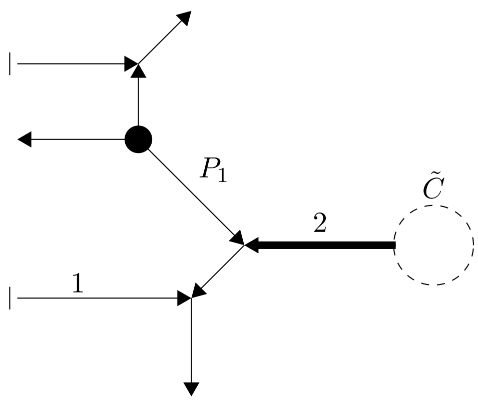
Secondly, the complex marking could still be adjacent to the non-fixed left end. Then, if only one of the fixed left ends of C is an end of C_0, we get the curve shown below:



We must distinguish between the fixed left end of C which is an end of C_0 and the fixed left end of C which is an end of $\tilde{C}$, so there are two decompositions of this type. In both cases $m_C = m_{\tilde{C}}$ since all vertices of C_0 have multiplicity 1 and the factors corresponding to the extra ends of C_0 are 1. Hence, the contribution from this kind of decomposition is

$$2 \cdot N^2((1),(1),0).$$

Thirdly, both fixed left ends of C could be left ends of C_0. Then $\tilde{C}$ could have one or two components. First, if $\tilde{C}$ has a single connected component, it must be connected to C_0 by an edge of weight 2:

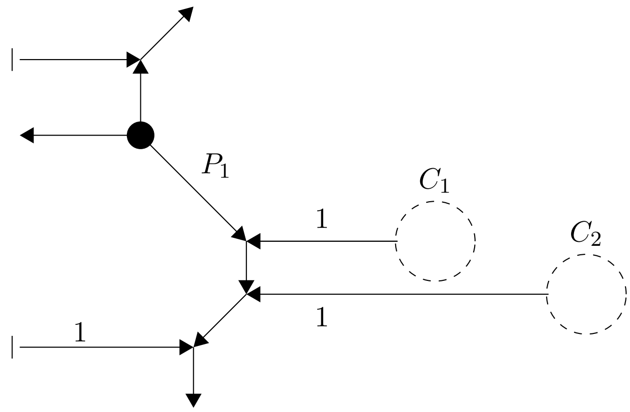


$\tilde{C}$ is then a curve of type $((0),(0,1))$. The vertex adjacent to the edge connecting C and $\tilde{C}$ has multiplicity $2i$, while we get a factor i extra when computing $m_{\tilde{C}}$ instead of m_C (the edge connecting C_0 and $\tilde{C}$ is an end of $\tilde{C}$ but not of C), so $m_C = 2m_{\tilde{C}}$.

The contribution from this kind of decomposition is

$$2 \cdot N^2\left((0),(0,1),0\right).$$

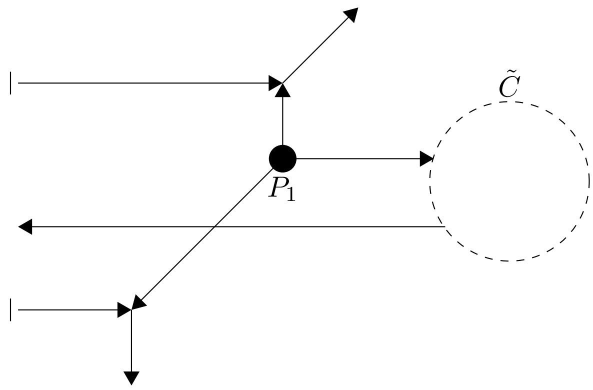
If $\tilde{C}$ consists of two components, the connecting edges must both be of weight 1:



C_1 and C_2 are both curves of type $((0),(1))$. There are 6 ways to distribute the 4 real markings among these curves so that each gets 2. We overcount by a factor of 2 as a result of the labelling of C_1 and C_2, so the contribution from this sort of decomposition is

$$3 \cdot N^1\left((0),(1),0\right) \cdot N^1\left((0),(1),0\right).$$

The marking could also be adjacent to a right end of C_0. In this case, all left ends of C_0 must be fixed and both fixed left ends of C must be left ends of C_0 by the balancing condition:



$\tilde{C}$ is a curve of type $((1),(1))$ with multiplicity equal to that of C. The contribution from this kind of decomposition is:

$$N^2((1),(1),0).$$

Summing up,

$$\begin{aligned} N^3((2),(1),1) =& 2\cdot N^2((1),(1),0) + 2\cdot N^2((0),(0,1),0) \\ &+ 3\cdot N^1((0),(1),0)\cdot N^1((0),(1),0) + N^2((1),(1),0). \end{aligned}$$

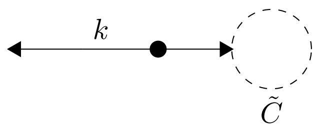
4.3 The general case

We have seen some possible decompositions in the previous examples, and these show almost all possible shapes of the left part of a decomposed broccoli curve after one point is moved to the far left. In the following, our pictures will just be sketches of possible decompositions.

We first take a look at the case when P_1 is real.

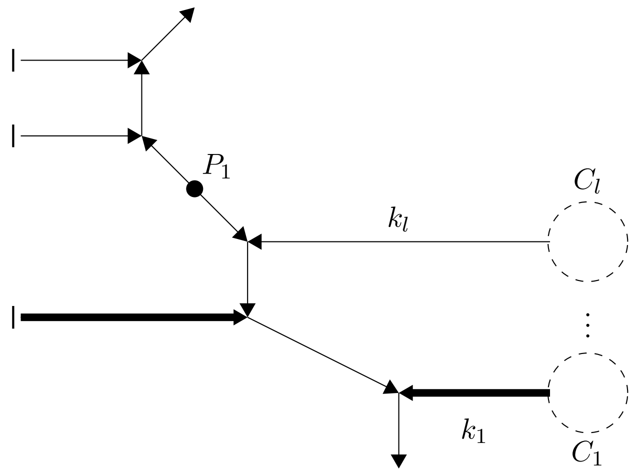
Proposition 4.9 (Possible shapes of C_0 when moving a real point to the left)**.** *Assume we have decomposed a curve of type (α,β) after moving a real point to the left. Below is a list of all possible shapes of its left part C_0.*

(A): If C_0 has no bounded edges, it looks like the picture below.



In this case, the curve $\tilde{C}$ is an irreducible curve of type $(\alpha+e_k,\beta-e_k)$, where k is the weight of the left non-fixed end in the picture. The multiplicity of C equals the multiplicity of $\tilde{C}$.

(B): If C_0 has bounded edges, it is similar to the curve below. C_0 has a number of left ends, all of which are fixed, while $\tilde{C}$ consists of zero or more connected components. The edge connecting C_0 to C_j may have any weight k_j, and is a fixed right end of C_0 and a non-fixed left end of C_j. All the bounded edges of C_0 must be odd.



Each component C_j of $\tilde{C}$ is an irreducible curve of type (α^j, β^j). We must have $\sum_{j=1}^{l} d_j = d - 1$ and $\sum_{j=1}^{l} s_j = s$. The left ends of C_0 are fixed ends of C without being left ends of $C_1, \ldots, C_l$, while C_0 has at least one fixed left end, hence $\sum \alpha^l < \alpha$. The left ends of $\tilde{C}$ which are not connections to C_0 correspond exactly to the left ends of C, so $\sum(\beta^j - e_{k_j}) = \beta$.

So how do we compute the multiplicity of C? First, every vertex of $\tilde{C}$ is a vertex of C, so we start by taking the product of m_{C_i} for i from 1 to l. Now, note that the fixed left ends of C_0 are not left ends of $\tilde{C}$, so we must multiply by a factor i^{k-1} for every such end of weight k. Every such fixed end is also adjacent to a vertex of C_0. This vertex has multiplicity $k \cdot i^{k-1}$ if k is even (vertex type (3)) and i^{k-1} if k is odd (vertex type (2)). Combining this factor with the factor from the adjacent end, we have to multiply by $\prod_{m \text{ even}} (-m)^{(\alpha')_m}$, where $\alpha' := \alpha - \sum \alpha^j$.

Similarly, the edge connecting C_j to C_0 contributes a factor $i^{k_j - 1}$ in the multiplicity of C_j that we do not need for the multiplicity of C. The vertex of C_0 adjacent to the edge has a multiplicity of $k_j \cdot i^{k_j-1}$ if k_j is even (vertex type 3) and i^{k_j-1} if k_j is odd (vertex type 2). Hence, we multiply by $\prod_{k_j \text{ even}} k_j$.

The multiplicity of C is thus given by

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m} \cdot \prod_{k_j \text{ even}} k_j.$$

Theorem 4.11 summarises these results to give a recursive formula for relative broccoli invariants in the case when r is non-zero.

Convention 4.10. Given α, β, s, we define r by

$$r := 2d + |\beta| - 2s - 1.$$

Similarly, the number of real markings of the component C_i is

$$r_i := 2d_i + |\beta^i| - 2s_i - 1.$$

In later formulæ, $N^d(\alpha, \beta, s)$ will be interpreted as 0 if

- $r < 0$,
- $s < 0$,
- $(\alpha)_i < 0$ for some i,
- $(\beta)_i < 0$ for some i, or
- $I\alpha + I\beta \neq d$.

The sequence of numbers of fixed left ends of C of weight i which are not left ends of $\tilde{C}$ is denoted by

$$(\alpha')_i := (\alpha)_i - \sum_{j=1}^{l} (\alpha^j)_i.$$

Theorem 4.11 (Gathmann, Markwig and Schroeter [GMS11, Theorem 6.10 (a)])**.** *Let $r > 0$ and $I\alpha + I\beta = d$. To find $N^d(\alpha, \beta, s)$, all we have to do is sum over all possible decompositions after moving a real point to the left.*

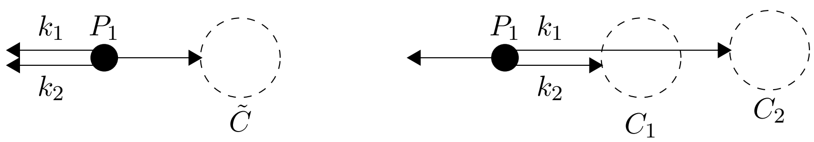
$$\begin{aligned} N^d(\alpha,\beta,s) = & \sum_{k \text{ odd}} N^d(\alpha + e_k, \beta - e_k, s) \\ & + \sum_I \frac{1}{l!} \binom{s}{s_1, \ldots, s_l} \binom{r-1}{r_1, \ldots, r_l} \binom{\alpha}{\alpha^1, \ldots, \alpha^l} \prod_{m \text{ even}} (-m)^{(\alpha')_m} \\ & \cdot \prod_{\substack{j=1 \\ k_j \text{ even}}}^{l} k_j \cdot \prod_{j=1}^{l} \left((\beta^j)_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \right), \end{aligned}$$

where the index set I runs over all $l \geq 0$ and all $\alpha^j, \beta^j, k_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j < \alpha$,
- $\sum_{j=1}^{l} \left(\beta^j - e_{k_j}\right) = \beta$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s$.

The multinomial coefficients in the last sum are just a way to keep count of the number of possible distributions of the real and complex markings, as well as the fixed left ends, among the C_j. For the component C_j, there are $(\beta^j)_{k_j}$ ways in which to chose which of its non-fixed left ends to use for the "connection" to C_0.

Proposition 4.12 (Possible shapes of C_0 when moving a complex point to the left)**.** *Assume we have decomposed a curve of type (α, β) after moving a complex point to the far left. Then there are two "elevator cases":*



In both cases, at least one of k_1 and k_2 have to be even by the balancing condition.

(C): In the first case, $\tilde{C}$ is a curve of type $(\alpha + e_{k_1+k_2}, \beta - e_{k_1} - e_{k_2})$. When computing m_C by taking the product of m_{C_1} and m_{C_1}, we lose factors of i^{k_1-1} and i^{k_2-1} since the left ends adjacent to the complex marking are ends of C without being ends of C_1 or C_2. Instead we have gained an extra factor $i^{k_1+k_2-1}$ for the left end of $\tilde{C}$ which is not an end of C. For the unique vertex of C_0 we get a factor of i^{-1}. Hence,

$$m_C = i^{k_1-1} \cdot i^{k_2-1} \cdot i^{-1} \cdot \frac{m_{\tilde{C}}}{i^{k_1+k_2-1}} = -m_{\tilde{C}}.$$

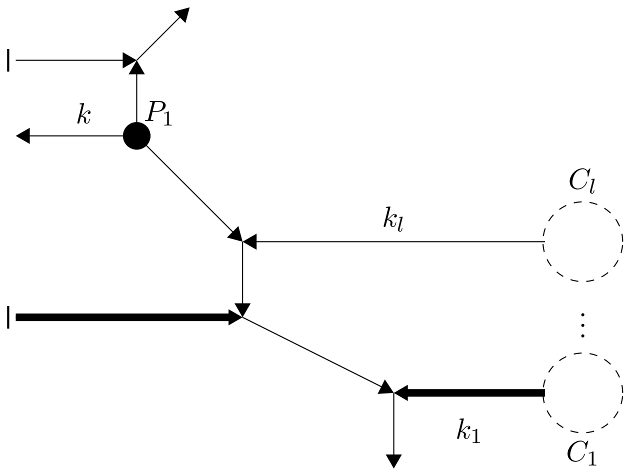
(D): In the second case, C_1 is a curve of type $(\alpha^1 + e_{k_1}, \beta^1)$ and C_2 is a curve of type $(\alpha^2 + e_{k_2}, \beta^2)$. Then $\alpha^1 + \alpha^2 = \alpha$ and $\beta^1 + \beta^2 = \beta$.

The left end contributes a factor $i^{k_1+k_2-1}$ and the vertex of C_0 a factor i^{-1}. This is compensated by the contributions from the left ends of $\tilde{C}$ which are not ends of C, contributing i^{k_1-1} and i^{k_2-1} to $m_{C_1} \cdot m_{C_2}$, so

$$m_C = i^{k_1+k_2-1} \cdot i^{-1} \cdot \frac{m_{C_1} m_{C_2}}{i^{k_1-1} i^{k_2-1}} = m_{C_1} \cdot m_{C_2}.$$

In addition there are two "floor cases":

(E): First, the complex marking could be adjacent to a non-fixed left end of C:



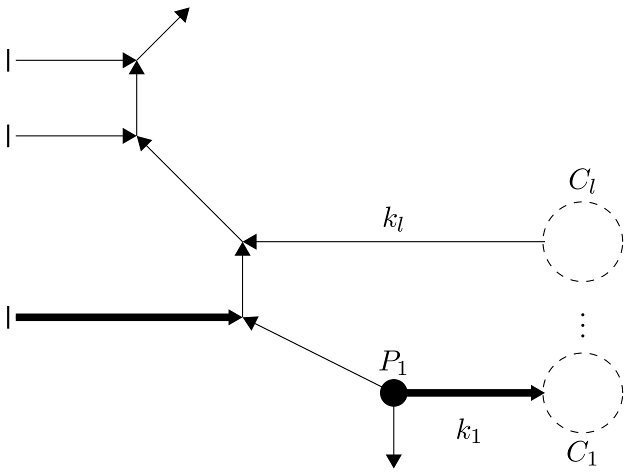
All other left and right ends of C_0 must be fixed. Now $\tilde{C}$ consists of l connected components, each of type (α^j, β^j), where l can be any non-negative integer. The edge connecting C_j to C_0 must be a non-fixed left end of C_j, but all other non-fixed left ends of $\tilde{C}$ are non-fixed ends of C. The only non-fixed left end of C_0 is the one adjacent to P_1. Hence, the condition $\sum(\beta^j - e_{k_j}) = \beta - k$ must be satisfied, where k is the weight of the left end adjacent to P_1. Every fixed left end of $\tilde{C}$ is a fixed end of C, so we must have $\sum \alpha^j \leq \alpha$.

The computation of m_C is similar to that of case (B) in Proposition 4.9. We only have to note that the vertex adjacent to the left non-fixed end of C of weight k has multiplicity i^{k-1} if k is even (vertex type (6)) and $k \cdot i^{k-1}$ if k is odd (vertex type (5)). The non-fixed end itself contributes a factor i^{k-1}, yielding a total contribution of -1 if k is even and k if k is odd. In combination, these factors contribute $M_k = \begin{cases} k & \text{if } k \text{ is odd,} \\ -1 & \text{if } k \text{ is even.} \end{cases}$

The multiplicity of C is thus given by

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m} \cdot \prod_{k_j \text{ even}} k_j \cdot M_k.$$

(F): The complex marking could also be adjacent to a non-fixed right end of C_0:



All other left and right ends of C_0 must be fixed. $\tilde{C}$ consists of a number of connected components $C_1, \ldots, C_l$. One of them, C_1 is connected to C_0 by an edge which is a non-fixed right end of C_0 of weight k_1. With the exception of the edge connecting C_0 to C_1, all right and left ends of C_0 are fixed. We let α^j and β^j be sequences such that C_j is a curve of type (α^j, β^j) for $j = 2, \ldots, l$, and C_1 is a curve of type $(\alpha^1 + e_k, \beta)$. Then, the condition on the fixed left ends will be $\sum_{j=1}^{l} \alpha^j < \alpha$ since C_0 must have at least one fixed left end.

Computing the multiplicity of C, we first take the product of the m_{C_j}'s. The vertex adjacent to P_1 has multiplicity i^{k_1-1} if k_1 is even (vertex type (6)) or $k_1 \cdot i^{k_1-1}$ if k_1 is odd (vertex type (5)). The edge connecting C_0 to C_1 contributes a factor i^{k_1-1} to m_{C_1} which is not a factor of m_C. Combined, we must multiply by a factor of $\tilde{M}_{k_1} = \begin{cases} k_1 & \text{if } k_1 \text{ is odd,} \\ 1 & \text{if } k_1 \text{ is even.} \end{cases}$

We then look at the components C_j for $j \geq 2$. The vertex of C_0 adjacent to the edge connecting C_0 to C_j has multiplicity $k_j \cdot i^{k_j-1}$ if k_j is even (vertex type 3) and i^{k_j-1} if k_j is odd (vertex type (2)). The connecting edge is an end of C_j, so the contribution i^{k_j-1} has already been accounted for. The factor from these vertices is then given by the product of all even k_j as j ranges from 2 to l.

Each vertex adjacent to a fixed left end of C_0 of weight ω contributes a factor $\omega \cdot i^{\omega-1}$ if ω is even (vertex type (3)) and $i^{\omega-1}$ if ω is odd (vertex type (2)), while the end itself contributes a factor $i^{\omega-1}$. Combined, these give a factor $\prod_{m \text{ even}} (-m)^{(\alpha')_m}$.

Summing up,

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \tilde{M}_{k_1} \cdot \prod_{\substack{j=2 \\ k_j \text{ even}}}^{l} k_j \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m}.$$

These observations are summarised in the following theorem.

Theorem 4.13 (Gathmann, Markwig and Schroeter [GMS11, Theorem 6.10 (b)])**.** *Let $s > 0$ and $I\alpha + I\beta = d$. To find $N^d(\alpha, \beta, s)$, all we have to do is sum over all possible decompositions after moving a complex point to the left.*

$$\begin{aligned} N^d(\alpha,\beta,s) = &\sum_{I_1} -\frac{1}{2} N^d(\alpha + e_{k_1+k_2}, \beta - e_{k_1} - e_{k_2}, s-1) \\ &+ \sum_{I_2} \frac{1}{2} \binom{s-1}{s_1, s_2} \binom{r}{r_1, r_2} \binom{\alpha}{\alpha^1, \alpha^2} \prod_{j=1}^{2} N^{d_j}(\alpha^j + e_{k_j}, \beta^j, s_j) \\ &+ \sum_{I_3} \frac{1}{l!} \binom{s-1}{s_1, \dots s_l} \binom{r}{r_1, \dots, r_l} \binom{\alpha}{\alpha^1, \dots, \alpha^l} M_k \prod_{m \textit{ even}} (-m)^{(\alpha')_m} \\ &\cdot \prod_{\substack{j=1 \\ k_j \textit{ even}}}^{l} k_j \cdot \prod_{j=1}^{l} \Big((\beta^j)_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \Big) \\ &+ \sum_{I_4} \frac{1}{(l-1)!} \binom{s-1}{s_1, \dots s_l} \binom{r}{r_1, \dots, r_l} \binom{\alpha}{\alpha^1, \dots, \alpha^l} \tilde{M}_{k_1} \prod_{m \textit{ even}} (-m)^{(\alpha')_m} \\ &\cdot \prod_{\substack{j=2 \\ k_j \textit{ even}}}^{l} k_j \cdot N^{d_1}(\alpha^1 + e_k, \beta^1, s_1) \prod_{j=2}^{l} \Big((\beta^j)_{k_j} N^{d_j}(\alpha^j, \beta^j, s_j) \Big). \end{aligned}$$

I_1 consists of $k_1, k_2 \geq 1$ such that at least one of them is odd.
I_2 consists of all $\alpha^1, \alpha^2, \beta^1, \beta^2, k_1 \geq 1, k_2 \geq 1, s_1 \geq 0, s_2 \geq 0$ such that

- *at least one of k_1, k_2 is odd,*
- $\alpha^1 + \alpha^2 = \alpha$,
- $\beta^1 + \beta^2 = \beta - e_{k_1+k_2}$,
- $d_1 + d_2 = d$,
- $s_1 + s_2 = s - 1$.

I_3 consists of all $l \geq 0$ and all $\alpha^j, \beta^j, k \geq 1, k_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j \leq \alpha$,
- $\sum_{j=1}^{l} \left(\beta^j - e_{k_j}\right) = \beta - e_k$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$.

I_4 consists of all $l > 0$ and all $\alpha^j, \beta^j, k_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j < \alpha$,
- $\beta^1 + \sum_{j=2}^{l} \left(\beta^j - e_{k_j}\right) = \beta$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$.

As in the corresponding real formula, the multinomial coefficients in the last three sums are just a way to keep count of the number of possible distributions of the real and complex markings, as well as the fixed left ends, among the C_j. The factors $\left(\beta^j\right)_{k_j}$ are there to enumerate the ways in which we can choose a non-fixed left end of C_j to use as a "connection edge" to C_0.

Note that in the case of $N^1\left((0),(1),1\right)$, the index sets I_1, I_2 and I_4 are all empty, while I_3 consists of a single element; $l = 0$, $k = 1$. Hence, $N^1\left((0),(1),1\right) = 1$. Similarly, when computing $N^1\left((0),(1),0\right)$ by the formula in Theorem 4.11, the first index set is empty, while the second consists of a single element; $l = 0$. Thus, $N^1\left((0),(1),0\right)$ is also equal to 1. Combining these observations, we get the following corollary.

Corollary 4.14. *The formulæ of Theorems 4.11 and 4.13 are sufficient to compute all relative broccoli invariants of the projective plane (and therefore all Welschinger invariants of the projective plane).*

The two distinct ways of computing the numbers when r and s are both non-zero may be quite useful. The expressions for a given invariant from the two formulæ may differ quite substantially.

As an example, we saw in the first part of Example 4.8 that

$$N^3\left((2),(1),1\right) = N^3\left((3),(0),1\right) + 2 \cdot N^2\left((0),(2),1\right).$$

In the second part of the same example, we noted that

$$\begin{aligned}N^3((2),(1),1) =&2\cdot N^2((1),(1),0)+2\cdot N^2((0),(0,1),0)\\&+3\cdot N^1((0),(1),0)\cdot N^1((0),(1),0)+N^2((1),(1),0).\end{aligned}$$

Combining these observations, we see that

$$\begin{aligned}N^3\left((3),(0),1\right)+2\cdot N^2\left((0),(2),1\right) =&2\cdot N^2((1),(1),0)\\&+2\cdot N^2((0),(0,1),0)\\&+3\cdot N^1((0),(1),0)\cdot N^1((0),(1),0)\\&+N^2((1),(1),0),\end{aligned}$$

which is not at all obvious at a first glance.

4.4 Computed values

Tables 2, 3, 4, 5 and 6 show the computed relative broccoli invariants for degrees 1 to 5. The program code used for the computation is available upon request.

Table 2: Relative broccoli invariants for degree 1.

α,β	$s=0$	$s=1$
(0), (1)	1	1
(1), (0)	1	

Table 3: Relative broccoli invariants for degree 2.

α,β	$s=0$	$s=1$	$s=2$
(0), (0, 1)	0	0	−1
(0), (2)	1	1	1
(0, 1), (0)	−2	−2	
(1), (1)	1	1	1
(2), (0)	1	1	

We find our beloved Welschinger invariants in the rows with $\alpha=(0)$ and $\beta=(d)$. These rows are also shown in table 7. The computed invariants are consistent with the ones computed by means of floor diagrams by Arroyo, Brugallé and López de Medrano in [ABLdM11].

Table 4: Relative broccoli invariants for degree 3.

α, β	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$
$(0),(0,0,1)$	3	1	-1	-3	
$(0),(1,1)$	0	0	0	0	
$(0),(3)$	8	6	4	2	0
$(0,0,1),(0)$	3	1	-1		
$(0,1),(1)$	-12	-8	-4	0	
$(1),(0,1)$	0	0	0	0	
$(1),(2)$	8	6	4	2	
$(1,1),(0)$	-8	-4	0		
$(2),(1)$	8	6	4	2	
$(3),(0)$	6	4	2		

Table 5: Relative broccoli invariants for degree 4.

α, β	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
$(0),(0,0,0,1)$	0	0	0	0	8	
$(0),(0,2)$	0	0	0	0	0	
$(0),(1,0,1)$	108	44	12	-4	-20	
$(0),(2,1)$	0	0	0	0	0	16
$(0),(4)$	240	144	80	40	16	0
$(0,0,0,1),(0)$	-72	-16	8	32		
$(0,0,1),(1)$	75	33	11	1	-5	
$(0,1),(0,1)$	0	0	0	0	-16	
$(0,1),(2)$	-288	-160	-80	-32	0	
$(0,2),(0)$	120	48	8	-32		
$(1),(0,0,1)$	33	11	1	-5	-15	
$(1),(1,1)$	0	0	0	0	0	
$(1),(3)$	240	144	80	40	16	0
$(1,0,1),(0)$	33	11	1	-5		
$(1,1),(1)$	-240	-124	-56	-20	0	
$(2),(0,1)$	0	0	0	0	0	
$(2),(2)$	240	144	80	40	16	
$(2,1),(0)$	-124	-56	-20	0		
$(3),(1)$	216	126	68	34	16	
$(4),(0)$	126	68	34	16		

5 Calculations in F_1

Although we have only discussed the projective plane so far, nothing in our tropical set-up prevents us from doing the same for other toric surfaces; all we have to change is the tropical degrees we consider. Welschinger's theorem (Theorem 1.2) holds in much greater generality and Shustin's Theorem

Table 6: Relative broccoli invariants for degree 5.

α, β	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$
$(0),(0,0,0,0,1)$	189	33	-7	-11	21	105		
$(0),(0,1,1)$	0	0	0	0	0	-32		
$(0),(1,0,0,1)$	0	0	0	0	0	0		
$(0),(1,2)$	0	0	0	0	0	0	0	
$(0),(2,0,1)$	8208	3156	1056	252	-16	-28	192	
$(0),(3,1)$	0	0	0	0	0	0	0	
$(0),(5)$	18264	9096	4272	1872	744	248	64	64
$(0,0,0,0,1),(0)$	189	33	-7	-11	21			
$(0,0,0,1),(1)$	-5184	-1600	-352	32	128	0		
$(0,0,1),(0,1)$	0	0	0	0	0	-32		
$(0,0,1),(2)$	4320	1764	640	188	32	20		
$(0,1),(0,0,1)$	-1080	-352	-72	16	-24	-192		
$(0,1),(1,1)$	0	0	0	0	0	0		
$(0,1),(3)$	-18192	-8544	-3744	-1488	-496	-128	-128	
$(0,1,1),(0)$	-864	-264	-48	8	-64			
$(0,2),(1)$	9792	3904	1376	352	0	128		
$(1),(0,0,0,1)$	0	0	0	0	0	0		
$(1),(0,2)$	0	0	0	0	0	0		
$(1),(1,0,1)$	3888	1392	416	64	-48	-48		
$(1),(2,1)$	0	0	0	0	0	0	0	
$(1),(4)$	18264	9096	4272	1872	744	248	64	
$(1,0,0,1),(0)$	-1728	-416	-32	64	0			
$(1,0,1),(1)$	2736	1012	320	68	-16	-12		
$(1,1),(0,1)$	0	0	0	0	0	64		
$(1,1),(2)$	-16272	-7392	-3104	-1168	-368	-128		
$(1,2),(0)$	4032	1440	416	64	128			
$(2),(0,0,1)$	1152	380	96	-4	-32	-36		
$(2),(1,1)$	0	0	0	0	0	0		
$(2),(3)$	18264	9096	4272	1872	744	248	64	
$(2,0,1),(0)$	1044	336	84	0	-12			
$(2,1),(1)$	-11664	-5024	-1984	-688	-176	-64		
$(3),(0,1)$	0	0	0	0	0	-32		
$(3),(2)$	17304	8520	3952	1712	680	248		
$(3,1),(0)$	-5088	-2016	-720	-208	-64			
$(4),(1)$	13560	6472	2912	1232	488	216		
$(5),(0)$	6504	2928	1248	504	216			

(Theorem 2.29), connecting these numbers to a tropical count, is stated for any real toric unnodal Del Pezzo Surface. Gathmann, Markwig and Schroeter state Theorem 3.52 and Corollary 3.53 for any "toric Del Pezzo

Table 7: Welschinger invariants of the projective plane.

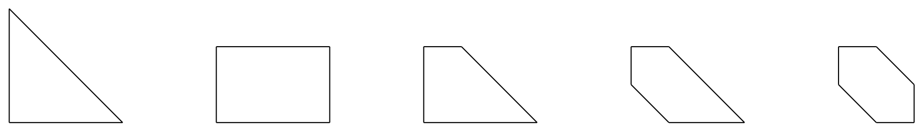
d	$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$	$s=7$	$s=8$
1	1	1							
2	1	1	1						
3	8	6	4	2	0				
4	240	144	80	40	16	0			
5	18264	9096	4272	1872	744	248	64	64	

degree".

In this section we discuss a possible generalisation of the formulæ of Theorem 4.11 and Theorem 4.13 to F_1 ($\mathbb{P}^2$ blown up at a point) and how this could be done for other real unnodal toric Del Pezzo surfaces.

We begin by making what we mean by a "toric Del Pezzo degree" precise.

Definition 5.1 (Toric Del Pezzo degrees)**.** A tropical degree Δ is said to be a *toric Del Pezzo degree* if it consists of the primitive normal directions of the edges of one of the polygons shown below, where each direction appears d times if d is the lattice length of the corresponding edge.



$\mathbb{P}^2$ $\qquad$ $\mathbb{P}^1 \times \mathbb{P}^1$ $\qquad$ F_1 $\qquad$ $\mathbb{P}^2_2$ $\qquad$ $\mathbb{P}^2_3$

Here, $\mathbb{P}^2_k$ denotes $\mathbb{P}^2$ blown up in k points.

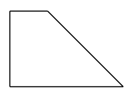
- In the first case, the polygon is a triangle with vertices $(0,0)$, $(0,d)$ and $(d,0)$ for some d. These degrees correspond to curves in $\mathbb{P}^2$.
- In the second case, the polygon is a rectangle with vertices $(0,0)$, $(d_1,0)$, $(0,d_2)$ and (d_1,d_2) for positive integers d_1 and d_2. These degrees correspond to curves in $\mathbb{P}^1 \times \mathbb{P}^1$.
- In the third case, the polygon is a trapezoid with vertices $(0,0)$, $(d,0)$, $(0,d-q)$ and $(q,d-q)$ for positive integers d and q such that $d > q$. These degrees correspond to curves in F_1 ($\mathbb{P}^2$ blown up at a point).
- In the fourth case, the polygon is a pentagon with vertices $(0,p)$, $(p,0)$, $(0,d-q)$, $(q,d-q)$ and $(d,0)$ for positive integers d,q and p such that $d > q+p$. These degrees correspond to curves in $\mathbb{P}^2_2$ ($\mathbb{P}^2$ blown up in two points).

- In the fifth case, the polygon is a hexagon with vertices $(0, d_2)$, $(d_2, 0)$, $(0, d - d_1)$, $(d_1, d - d_1)$, $(d - d_3, d_3)$ and $(d - d_3, 0)$ for positive integers d, d_1, d_2 and d_3 such that $d > d_1 + d_2 + d_3$. These degrees correspond to curves in $\mathbb{P}^2_3$ ($\mathbb{P}^2$ blown up in three points).

When we want to use our (r, s)-marked curves to compute Welschinger invariants of these surfaces, an important point to note is that every toric Del Pezzo degree consists only of directions of weight one, so we can apply Lemma 3.44. Hence, Remark 3.42 and Proposition 3.46 can be stated for any toric del Pezzo degree and its corresponding surface. Thus, the multiplicities of the curves are exactly what we need to compute the Welschinger invariants of these surfaces too.

5.1 Notation

We will look at curves in F_1. They are the curves dual to trapezoidal Newton polygons as the one below.



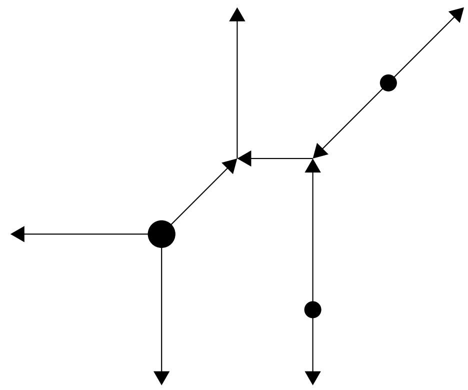
Just as in the previous section, our idea is to decompose a curve into a left and right part.

Definition 5.2 (Broccoli curves of type (q, α, β))**.** Let $d > q \geq 0$, and let α and β be sequences such that $I\alpha + I\beta = d - q$. Let $\Delta(q, \alpha, \beta)$ be a degree consisting of d times the vector $(0, -1)$, $d - q$ times the vector $(1, 1)$, q times the vector $(0, 1)$ and $(\alpha)_i + (\beta)_i$ times $(-i, 0)$ for all i (in any fixed order). Let $F(q, \alpha, \beta) \subseteq \{1, \ldots, |\Delta(q, \alpha, \beta)|\}$ be a fixed subset with $|\alpha|$ elements such that the entries of $\Delta(q, \alpha, \beta)$ with index in F are $(\alpha)_i$ times $(-i, 0)$ for all i.

A broccoli curve in $M^{\mathrm{B}}_{(r,s)}(\Delta(q, \alpha, \beta), F(q, \alpha, \beta))$ will be called a *curve of type* (q, α, β). Its unmarked ends with directions $(-i, 0)$ will be referred to as *left ends*. Unmarked ends with direction $(0, 1)$ will be referred to as *upper ends*.

A curve of type $(0, \alpha, \beta)$ is precisely a curve of type (α, β) in the sense of Definition 4.1.

Example 5.3 (A curve of type $(1, (0), (1))$)**.** The curve below is a curve of type $(1, (0), (1))$ with one complex and two real markings. It has one non-fixed left end of weight 1 and one non-fixed upper end of weight 1.



Again, the dimension condition $|\Delta| - 1 - |F| = r + 2s$ must be satisfied. For a curve of type (q, α, β), $|\Delta| = 2d + |\alpha| + |\beta|$, and $|F| = |\alpha|$. Hence, we must have

$$r = 2d + |\beta| - 2s - 1.$$

Definition 5.4 (Relative broccoli numbers)**.** Let $\Delta = \Delta(q, \alpha, \beta)$ and $F = F(q, \alpha, \beta)$ be as in Definition 5.2 and let $r, s \geq 0$ such that $|\Delta| - 1 - |F| = 2d + |\beta| - 1 = r + 2s$. We then use $N_q^d(\alpha, \beta, s)$ as a shorthand expression for $N^{\mathrm{B}}_{(r,s)}(\Delta(q, \alpha, \beta), F(q, \alpha, \beta))$.

In the proof of Lemma 4.5 by Gathmann and Markwig (see the proof of Theorem 4.3 in [GM07]), nothing would be changed when allowing direction vectors $(0, 1)$, $(-1, -1)$ and $(1, 0)$ in the degree Δ.

Lemma 5.5 (Decomposing a curve into a left and a right part)**.** *Let Δ be a tropical degree consisting only of the direction vectors $(-i, 0)$ for positive integers i, $(1, 0)$, $\pm(0, 1)$ and $\pm(1, 1)$ (each of these vectors may appear any finite number of times). Let $F \subseteq \{1, \ldots, |\Delta|\}$ be a subset such that the entries of Δ with index in F consist only of vectors of the form $(-i, 0)$.*

Let r, s be non-negative integers such that $\Delta - |F| - 1 = r + 2s$. Fix a small number ϵ and a large number $N > 0$. Choose $r + s$ points $P_1, \ldots, P_{r+s}$ and $|\alpha|$ y-coordinates for the fixed left ends such that these points are in general position and

- *the y-coordinates of all P_i and the fixed ends are in the open interval $(-\epsilon, \epsilon)$,*
- *the x-coordinates of $P_2, \ldots, P_{r+s}$ are in $(-\epsilon, \epsilon)$,*
- *the x-coordinate of P_1 is smaller than $-N$.*

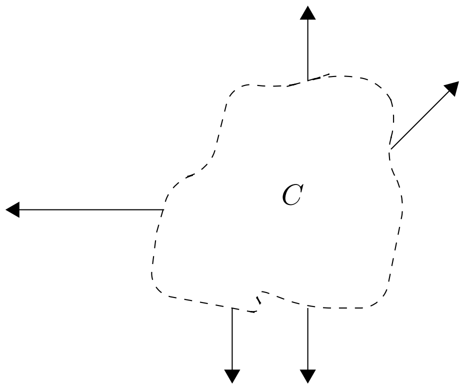
Let $C = (\Gamma, x_1, \ldots, x_{r+s}, y_1, \ldots, y_n, h) \in M^B_{(r,s)}(\Delta, F)$ be a broccoli curve satisfying these conditions. Then no vertex of C can have its y-coordinate *below $-\epsilon$ or above ϵ, and there exists a rectangle $R = [a, b] \times [-\epsilon, \epsilon]$ (with $a \geq -N$, $b \leq -\epsilon$ depending only on Δ) such that $R \cap h(\Gamma)$ contains only horizontal edges of C.*

Lemma 5.5 implies that when one of the points is moved to the far left, the broccoli curves of degree Δ and fixed ends F such that $\mathrm{ev}_F(C) = \mathscr{P}$ may be decomposed by a cutting procedure just as in Example 4.6. This means that we can count the curves by looking at possible shapes of the decomposition.

5.2 Examples

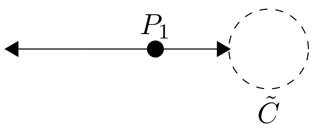
As in the last section, we start by looking at some concrete decompositions of curves before moving on to the general case.

Example 5.6. We will compute $N_1^2\left((0),(1),1\right)$, i.e. the number of broccoli curves (with multiplicity) C with 2 real and 1 complex markings fitting in the picture below.



i) Assume we have moved a real point to the left. Then this marked end must be adjacent to a vertex of type (1).

   C_0 could be contained in a horizontal line as in the curve below.



Then $\tilde{C}$ consists of one component; a curve of type $(1,(1),(0))$ through one real and one complex point. The multiplicity of the unique vertex of C_0 is 1, while the contribution from the left end of C_0 is outweighed by the contribution from the left end of $\tilde{C}$ which is not an end of C (in this case, they are both 1 anyway). The multiplicity of C is therefore equal to the multiplicity of $\tilde{C}$. Hence, the contribution from this kind of decomposition is given by

$$N_1^2\left((1),(0),1\right).$$

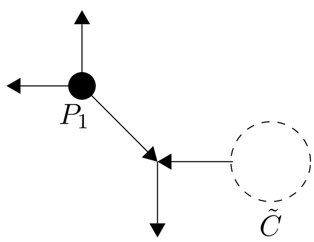
As in part i) of Example 4.7, Lemma 3.28 implies that this is the *only* possible decomposition. Thus,

$$N_1^2\left((0),(1),1\right) = N_1^2\left((1),(0),1\right).$$

ii) Assume we moved a complex point to the left instead. Then the complex marking must be adjacent to a vertex of type (5) or (6).

Firstly, C_0 cannot be contained in a horizontal line. C_0 has exactly one left end, the non-fixed end of weight 1. Hence, if C_0 consisted of a single vertex of type (5) or (6), the two right ends of C_0 would have weights summing to 1, which is impossible.

Secondly, the left end of C_0 may be adjacent to the complex marking.



In this case, $\tilde{C}$ consists of a single component, a curve of type $(0,(0),(1))$.

The multiplicity of the unique vertex of C_0 is 1, while the contribution from the left end of C_0 is outweighed by the contribution from the left end of $\tilde{C}$ which is not an end of C (in this case, they are both 1 anyway). The multiplicity of C is therefore equal to the multiplicity of $\tilde{C}$.

No other decomposition exists. To see why, note that Lemma 3.28 implies that the complex marking must be adjacent to the left end. Otherwise, the non-fixed left end would be in the same connected component of $C_0 \setminus \{x_1\}$ as one of the two other ends. They are both non-fixed, contradicting the lemma. The balancing condition ensures that the only possible decomposition is the one above.

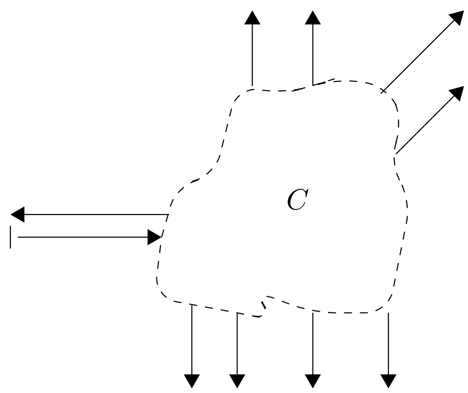
Hence,

$$N_1^2((0),(1),1) = N_0^1((0),(1),0).$$

In fact, we have already computed this last term in Section 4;

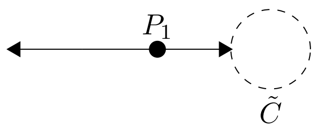
$$N_0^1((0),(1),0) = N^1((0),(1),0) = 1.$$

Example 5.7. We will compute $N_2^4((1),(1),1)$. Which curves C with 6 real and 1 complex markings fit in the picture below?



i) Assume we moved a real point to the left.

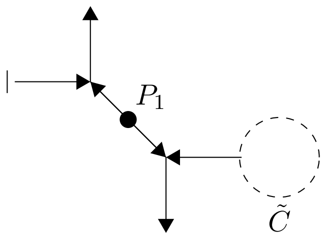
Firstly, C_0 could be contained in a horizontal line. Then P_1 must be adjacent to the non-fixed left end of C.



$\tilde{C}$ consists of one component; a curve of type $(2,(2),(0))$ through five real and one complex point. The multiplicity of the unique vertex of C_0 is 1, while the contribution from the left end of C_0 is outweighed by the contribution from the left end of $\tilde{C}$ which is not an end of C (in this case, they are both 1 anyway). The multiplicity of C is therefore equal to the multiplicity of $\tilde{C}$. Hence, the contribution from this kind of decomposition is given by

$$N_2^4((2),(0),1).$$

Secondly, C_0 could contain bounded edges. Then, by Lemma 3.28, only the fixed left end of C can be a left end of C_0. By the balancing condition, we cannot have any end of C_0 with direction $(1,1)$, so C_0 is as in the picture below.



Now, $\tilde{C}$ is a curve of type $(1,(0),(2))$. That is, it has one upper end, no fixed left ends and two non-fixed left ends of weight 1.

The multiplicity of the vertex adjacent to the marking is 1, while the contribution from the ends of C_0 which are ends of C are all 1. The left end of $\tilde{C}$ which is not an end of C is also of weight 1. The multiplicity of C is therefore equal to the multiplicity of $\tilde{C}$. Hence, the contribution from this kind of decomposition is given by

$$N_1^3\left((0),(2),1\right).$$

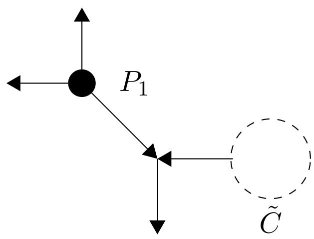
Summing over the possible contributions, we see that

$$N_2^4\left((1),(1),1\right) = N_2^4\left((2),(0),1\right) + N_1^3\left((0),(2),1\right).$$

ii) Now, assume we moved a complex point to the left instead.

Note that C_0 can *not* be contained in a horizontal line. If it were, the complex marking would have to be adjacent to three non-fixed ends, and only one of the left ends of C is non-fixed. Now, the balancing condition makes sure that two right ends and one left end of weight 1 from the same vertex is impossible. Therefore, we only have to consider "floor cases" in which C_0 has bounded edges.

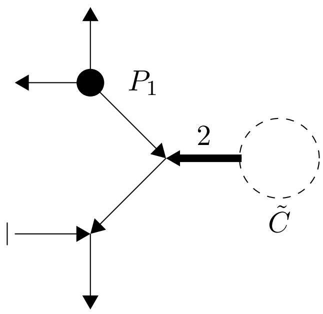
First, we take a look at the case when P_1 is adjacent to the non-fixed left end of C, and the fixed left end of C is not an end of C_0. As in the real case, the balancing condition ensures that there is only one such shape of C_0.



Here, $\tilde{C}$ is a curve of type $(1,(1),(1))$. That is, it has one upper end, one fixed left end and one non-fixed left end of weight 1. The multiplicity of C is equal to the multiplicity of $\tilde{C}$, so the contribution from this kind of decomposition is given by

$$N_1^3\left((1),(1),0\right).$$

Secondly, we look at the case when P_1 is adjacent to the non-fixed left end of C, C_0 has an upper end and the fixed left end of C is an end of C_0. Now $\tilde{C}$ could consist of one or two components. When $\tilde{C}$ consists of a single component, this is a curve of type $(1,(0),(0,1))$:



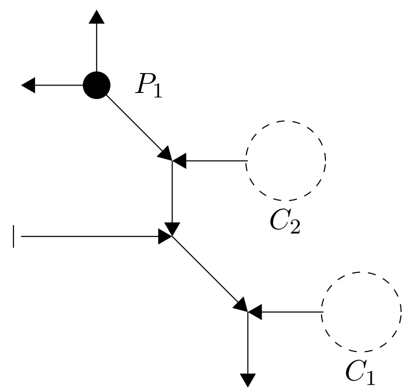
The vertex adjacent to P_1 has multiplicity 1, while the vertex adjacent to the connecting edge has multiplicity $2i$. The multiplicity of the vertex adjacent to the fixed left end of C_0 is 1. All ends of C_0 which are ends of C contribute a factor 1 to the multiplicity, while the left end of $\tilde{C}$ contributes a factor i to the multiplicity of $\tilde{C}$ which is not a factor of m_C. Hence, the multiplicity of C is given by

$$m_C = 2m_{\tilde{C}}$$

and the contribution from this kind of decomposition is

$$2\cdot N_1^3\left((0),(0,1),0\right).$$

When $\tilde{C}$ consists of two components, C_0 looks like the curve below.



The components C_1 and C_2 are curves of type $\left(q_1, \alpha^1, \beta^1\right)$ and $\left(q_2, \alpha^2, \beta^2\right)$, respectively.

Then we must have $q_1 + q_2 = 1$, $\alpha^1 = \alpha^2 = (0)$ and $\beta^1 = \beta^2 = 1$, so C_1 and C_2 are curves of types $(1, (0), (1))$ and $(0, (0), (1))$.

Letting C_1 be the curve with an upper end, it must have 4 real markings, while C_2 has 2 real markings.

The multiplicity of C is given by

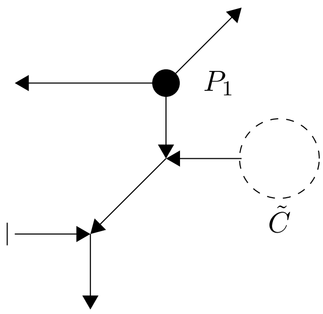
$$m_C = m_{C_1} \cdot m_{C_2},$$

and the number of ways to distribute the real markings among the C_i's is given by $\binom{6}{4,2} = 15$.

Hence, the contribution from this kind of decomposition is given by

$$15 \cdot N_1^2\left((0), (1), 0\right) \cdot N_0^1\left((0), (1), 0\right).$$

Thirdly, there exists a shape where P_1 is adjacent to the non-fixed left end of C, C_0 has no upper end and the fixed left end of C is an end of C_0.

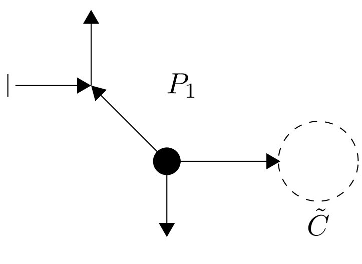


$\tilde{C}$ consists of a single component. It is a curve of type $(2,(0),(1))$.

The multiplicity of C is equal to the multiplicity of $\tilde{C}$ by arguments similar to those of the other cases. Hence, the contribution from this sort of decomposition is

$$N_2^3\left((0),(1),0\right).$$

Fourthly, P_1 could be adjacent to a right end of C_0. Then the non-fixed left end of C can not be an end of C_0, otherwise we would violate Lemma 3.28. Hence, there is only one such possibility:



In this case, $\tilde{C}$ consists of a single component, it is a curve of type $(1,(1),(1))$. The multiplicity of C is equal to the multiplicity of $\tilde{C}$, so the contribution from this sort of decomposition is

$$N_2^3\left((1),(1),0\right).$$

Summing up over all possible decompositions, we see that

$$\begin{aligned} N_2^4\left((1),(1),1\right) =& N_1^3\left((1),(1),0\right) + 2\cdot N_1^3\left((0),(0,1),0\right) \\ &+ 15\cdot N_1^2\left((0),(1),0\right)\cdot N_0^1\left((0),(1),0\right) \\ &+ N_2^3\left((0),(1),0\right) + N_2^3\left((1),(1),0\right). \end{aligned}$$

Now we could continue by computing the different terms in the sums of cases i) and ii) to find the number we seek. Without doing this, we may already note the identity

$$\begin{aligned} N_2^4((2),(0),1) + N_1^3((0),(2),1) =& N_1^3((1),(1),0) \\ &+ 2 \cdot N_1^3((0),(0,1),0) \\ &+ 15 \cdot N_1^2((0),(1),0) \cdot N_0^1((0),(1),0) \\ &+ N_2^3((0),(1),0) + N_2^3((1),(1),0). \end{aligned}$$

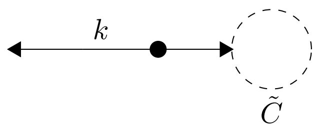
This is one of the features of our formulæ. Just as in the previous section, the formulæ may yield quite distinct expressions for the invariant.

5.3 The general case

The possible shapes of C_0 when decomposing a curve are very similar to those in Section 4. Before we start, we note that C_0 can not have both an upper end and an end of direction $(1,1)$, for in this case C_0 must have two ends of direction $(0,-1)$ by the balancing condition, violating Lemma 3.28.

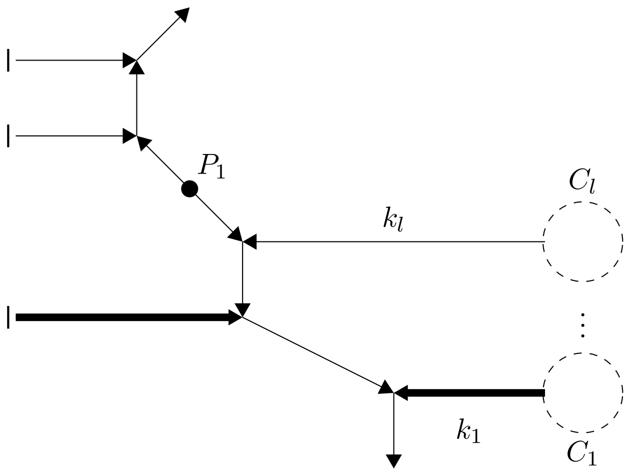
Proposition 5.8 (Possible shapes of C_0 when moving a real point to the left)**.** *Assume we have decomposed a curve of type (q,α,β) after moving a real point to the left. Below is a list of all possible shapes of its left part C_0.*

(A): If C_0 has no bounded edges, it looks like the picture below.



In this case, the curve $\tilde{C}$ is a curve of type $(q,\alpha+e_k,\beta-e_k)$, where k is the weight of the left non-fixed end in the picture. The multiplicity of C equals the multiplicity of $\tilde{C}$.

(B1): If C_0 has bounded edges and no upper end, it is similar to the curve below. C_0 has a number of left ends, all of which are fixed, while $\tilde{C}$ consists of zero or more connected components. The edge connecting C_0 to C_j may have any weight k_j, and is a fixed right end of C_0 and a non-fixed left end of C_j. All the bounded edges of C_0 must be odd.

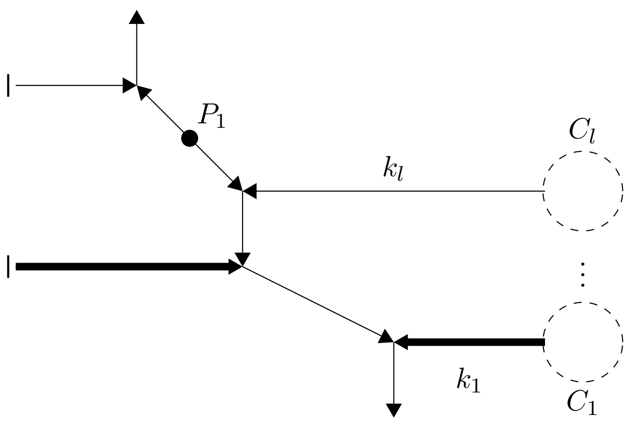


Each of the connected components of $\tilde{C}$ is a curve of type (q_j, α^j, β^j). We must have $\sum_{j=1}^{l} q_j = q$, $\sum_{j=1}^{l} d_j = d - 1$ and $\sum_{j=1}^{l} s_j = s$. The left ends of C_0 are fixed ends of C without being left ends of $C_1, \ldots, C_l$, while C_0 has at least one fixed left end, hence $\sum \alpha^l < \alpha$. The left ends of $\tilde{C}$ which are not connections to C_0 correspond exactly to the left ends of C, so $\sum(\beta^j - e_{k_j}) = \beta$.

The multiplicity of C is computed just as in case (B) of Proposition 4.9;

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m} \cdot \prod_{k_j \text{ even}} k_j.$$

(B2): If C_0 has bounded edges and an upper end, it is similar to the curve below. C_0 has a number of left ends, all of which are fixed, while $\tilde{C}$ consists of zero or more connected components. The edge connecting C_0 to C_j may have any weight k_j, and is a fixed right end of C_0 and a non-fixed left end of C_j. All the bounded edges of C_0 must be odd.



The components $C_1, \dots, C_l$ are irreducible curves of types (q_j, α^j, β^j). We must have $\sum_{j=1}^{l} q_j = q - 1$, $\sum_{j=1}^{l} d_j = d - 1$ and $\sum_{j=1}^{l} s_j = s$. The left ends of C_0 are fixed ends of C without being left ends of $C_1, \dots, C_l$, while C_0 has at least one fixed left end, hence $\sum \alpha^l < \alpha$. The left ends of $\tilde{C}$ which are not connections to C_0 correspond exactly to the left ends of C, so $\sum(\beta^j - e_{k_j}) = \beta$.

The multiplicity of C is computed just as in case (B) of Proposition 4.9;

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m} \cdot \prod_{k_j \text{ even}} k_j.$$

Theorem 5.10 summarises these results to give a recursive formula for relative broccoli invariants of F_1 in the case when r is non-zero.

Convention 5.9. Given α, β, s, we define r by

$$r := 2d + |\beta| - 2s - 1.$$

Similarly, the number of real markings of the component C_i is

$$r_i := 2d_i + |\beta^i| - 2s_i - 1.$$

$N_q^d(\alpha, \beta, s)$ will be interpreted as 0 if

- $r < 0$,
- $s < 0$,
- $q < 0$,
- $q \geq d$,
- $(\alpha)_i < 0$ for some i,
- $(\beta)_i < 0$ for some i, or
- $I\alpha + I\beta \neq d - q$.

The sequence of numbers of fixed left ends of C of weight i which are not left ends of $\tilde{C}$ is given by

$$(\alpha')_i := (\alpha)_i - \sum_{j=1}^{l} (\alpha^j)_i.$$

Theorem 5.10 (Computing $N_q^d(\alpha,\beta,s)$ when r is non-zero)**.** *Let $r>0$, let $d>q\geq 0$ such that $I\alpha+I\beta=d-q$. To find $N_q^d(\alpha,\beta,s)$, all we have to do is sum over all possible decompositions after a real point is moved to the far left.*

$$
\begin{aligned}
N_q^d(\alpha,\beta,s) &= \sum_{k\ odd} N_q^d(\alpha+e_k,\beta-e_k,s)\\
&+\sum_{I_1}\frac{1}{l!}\binom{s}{s_1,\dots,s_l}\binom{r-1}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l}\prod_{m\ even}(-m)^{(\alpha')_m}\\
&\cdot\prod_{\substack{j=1\\ k_j\ even}}^{l} k_j\cdot\prod_{j=1}^{l}\Big((\beta^j)_{k_j}N_{q_j}^{d_j}(\alpha^j,\beta^j,s_j)\Big)\\
&+\sum_{I_2}\frac{1}{l!}\binom{s}{s_1,\dots,s_l}\binom{r-1}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l}\prod_{m\ even}(-m)^{(\alpha')_m}\\
&\cdot\prod_{\substack{j=1\\ k_j\ even}}^{l} k_j\cdot\prod_{j=1}^{l}\Big((\beta^j)_{k_j}N_{q_j}^{d_j}(\alpha^j,\beta^j,s_j)\Big),
\end{aligned}
$$

where the index set I_1 runs over all $l\geq 0$ and all $\alpha^j,\beta^j,k_j\geq 1, s_j\geq 0, q_j\geq 0$ for $1\leq j\leq l$ such that

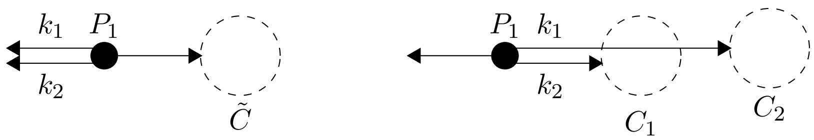
- $\sum_{j=1}^{l}\alpha^j<\alpha$,
- $\sum_{j=1}^{l}\left(\beta^j-e_{k_j}\right)=\beta$,
- $\sum_{j=1}^{l}d_j=d-1$,
- $\sum_{j=1}^{l}s_j=s$,
- $\sum_{j=1}^{l}q_j=q$,

and I_2 runs over all $l\geq 0$ and all $\alpha^j,\beta^j,k_j\geq 1, s_j\geq 0, q_j\geq 0$ for $1\leq j\leq l$ such that

- $\sum_{j=1}^{l}\alpha^j<\alpha$,
- $\sum_{j=1}^{l}\left(\beta^j-e_{k_j}\right)=\beta$,
- $\sum_{j=1}^{l}d_j=d-1$,
- $\sum_{j=1}^{l}s_j=s$,

- $\sum_{j=1}^{l} q_j = q - 1$,

Proposition 5.11 (Possible shapes of a curve of type (q, α, β) when moving a complex point to the left)**.** *Assume we have decomposed a curve of type (q, α, β) after moving a complex point to the left. Then there are two "elevator cases":*



In both cases, at least one of k_1 and k_2 have to be even by the balancing condition.

(C): In the first case, $\tilde{C}$ is a curve of type $(q, \alpha + e_{k_1+k_2}, \beta - e_{k_1} - e_{k_2})$. As in case (C) of Proposion 4.12, the multiplicity of C is given by

$$m_C = -m_{\tilde{C}}.$$

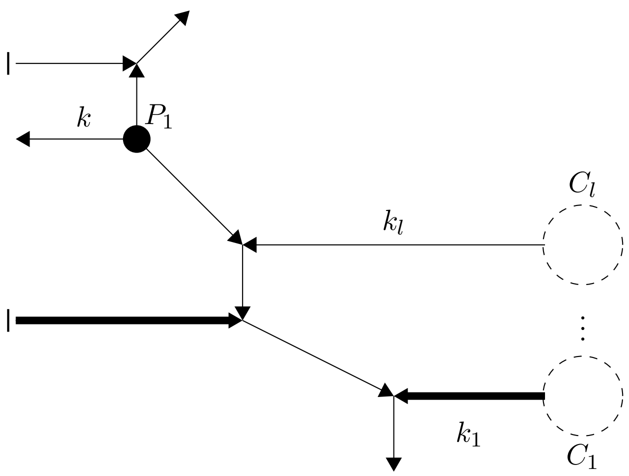
(D): In the second case, C_1 is a curve of type $(q_1, \alpha^1 + e_{k_1}, \beta^1)$ and C_2 is a curve of type $(q_2, \alpha^2 + e_{k_2}, \beta^2)$. Then $q_1 + q_2 = q$, $\alpha^1 + \alpha^2 = \alpha$ and $\beta^1 + \beta^2 = \beta$.

As in case (D) of Proposion 4.12, the multiplicity of C is given by

$$m_C = m_{C_1} \cdot m_{C_2}.$$

In addition there are four "floor cases":

(E1): The complex marking could be adjacent to a non-fixed left end of C. We must distinguish between two cases. First, we consider the case when C_0 has no upper end:

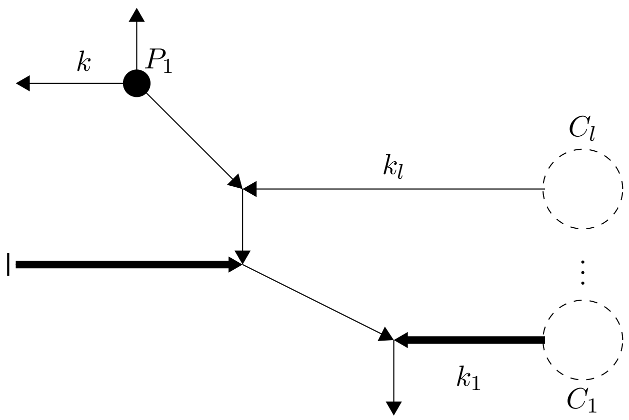


With the exception of the non-fixed left end of C_0, all left and right ends of C_0 must be fixed. Now $\tilde{C}$ consists of $l \geq 0$ connected components, each of type (q_j, α^j, β^j), where $\sum_{j=1}^{l} q_j = q$. The edge connecting C_j to C_0 must be a non-fixed left end of C_j, but all other non-fixed left ends of $\tilde{C}$ are non-fixed ends of C. The only non-fixed left end of C_0 is the one adjacent to P_1. Hence, the condition $\sum(\beta^j - e_{k_j}) = \beta - k$ must be satisfied, where k is the weight of the left end adjacent to P_1. Every fixed left end of $\tilde{C}$ is a fixed end of C, so we must have $\sum \alpha^j \leq \alpha$.

The computation of m_C is just like the one in case (E) of 4.9;

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m} \cdot \prod_{k_j \text{ even}} k_j \cdot M_k.$$

(E2): There exists a similar decomposition where C_0 has an upper end:

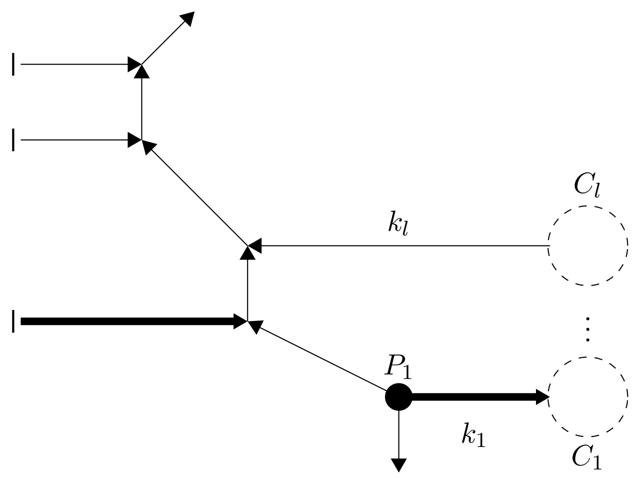


With the exception of the non-fixed left end of C_0, all left and right ends of C_0 must be fixed. Now $\tilde{C}$ consists of $l \geq 0$ connected components, each of type (q_j, α^j, β^j), where $\sum_{j=1}^{l} q_j = q - 1$. The edge connecting C_j to C_0 must be a non-fixed left end of C_j, but all other non-fixed left ends of $\tilde{C}$ are non-fixed ends of C. The only non-fixed left end of C_0 is the one adjacent to P_1. Hence, the condition $\sum(\beta^j - e_{k_j}) = \beta - k$ must be satisfied, where k is the weight of the left end adjacent to P_1. Every fixed left end of $\tilde{C}$ is a fixed end of C, so we must have $\sum \alpha^j \leq \alpha$.

The computation of m_C is just like the one in case (E) of 4.9;

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m} \cdot \prod_{k_j \text{ even}} k_j \cdot M_k.$$

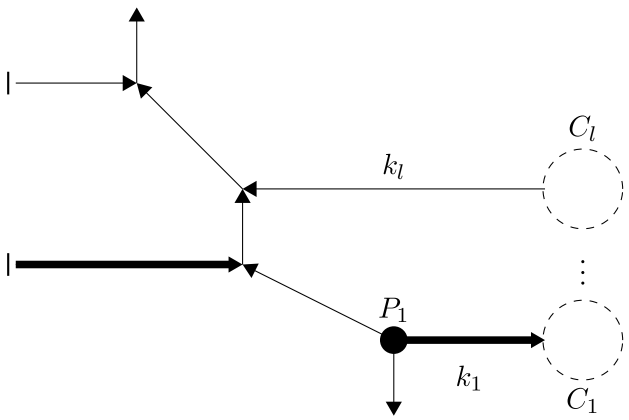
(F1): The complex marking could be adjacent to a non-fixed right end of C_0. Again, there are two such cases. First, we consider the case when C_0 has no upper ends:



With the exception of the non-fixed right end of C_0, all left and right ends of C_0 must be fixed. $\tilde{C}$ consists of a number of connected components $C_1, \dots, C_l$. One of them, C_1 is connected to C_0 through a non-fixed end of weight k_1. With the exception of the edge connecting C_0 to C_1, all right and left ends of C_0 are fixed. We let q_j, α^j and β^j be sequences such that C_j is a curve of type (q_j, α^j, β^j) for $j = 2, \dots, l$, and C_1 is a curve of type $\big(q_1, \alpha^1 + e_k, \beta\big)$. Then, the condition on the fixed left ends will be $\sum_{j=1}^{l} \alpha^j < \alpha$, while the q_j must sum to q. The computation of m_C is just like the one in case (F) of 4.9;

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \tilde{M}_{k_1} \cdot \prod_{\substack{j=2 \\ k_j \text{ even}}}^{l} k_j \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m}.$$

(F2): There is a similar decomposition with an upper end for C_0:



With the exception of the non-fixed right end of C_0, all left and right ends of C_0 must be fixed. $\tilde{C}$ consists of a number of connected components $C_1, \ldots, C_l$. One of them, C_1 is connected to C_0 through a non-fixed end of weight k_1. With the exception of the edge connecting C_0 to C_1, all right and left ends of C_0 are fixed. We let q_j, α^j and β^j be sequences such that C_j is a curve of type (q_j, α^j, β^j) for $j = 2, \ldots, l$, and C_1 is a curve of type $\left(q_1, \alpha^1 + e_k, \beta\right)$. Then, the condition on the fixed left ends will be $\sum_{j=1}^{l} \alpha^j < \alpha$, while the q_j must sum to $q - 1$. The computation of m_C is just like the one in case (F) of 4.9;

$$m_C = \prod_{j=1}^{l} m_{C_j} \cdot \tilde{M}_{k_1} \cdot \prod_{\substack{j=2 \\ k_j \text{ even}}}^{l} k_j \cdot \prod_{m \text{ even}} (-m)^{(\alpha')_m}.$$

Theorem 5.12 (Computing $N_q^d(\alpha, \beta, s)$ when s is non-zero)**.** *Let $s > 0$, let $d > q \geq 0$ such that $I\alpha + I\beta = d - q$. To find $N_q^d(\alpha, \beta, s)$, all we have to do is sum over all possible decompositions after a real point is moved to the far* *left.*

$$
\begin{aligned}
N_q^d(\alpha,\beta,s) =& \sum_{I_1} -\frac{1}{2} N_q^d(\alpha+e_{k_1+k_2},\beta-e_{k_1}-e_{k_2},s-1) \\
&+\sum_{I_2} \frac{1}{2}\binom{s-1}{s_1,s_2}\binom{r}{r_1,r_2}\binom{\alpha}{\alpha^1,\alpha^2}\prod_{j=1}^{2} N_{q_j}^{d_j}(\alpha^j+e_{k_j},\beta^j,s_j) \prod_{\substack{j=1\\ k_j \ even}}^{2} k_j \\
&+\sum_{I_3} \frac{1}{l!}\binom{s-1}{s_1,\dots s_l}\binom{r}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l} M_k \prod_{m \ even} (-m)^{(\alpha')_m} \\
&\cdot \prod_{\substack{j=1\\ k_j \ even}}^{l} k_j \cdot \prod_{j=1}^{l}\Big((\beta^j)_{k_j} N_{q_j}^{d_j}(\alpha^j,\beta^j,s_j)\Big) \\
&+\sum_{I_4} \frac{1}{l!}\binom{s-1}{s_1,\dots s_l}\binom{r}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l} M_k \prod_{m \ even} (-m)^{(\alpha')_m} \\
&\cdot \prod_{\substack{j=1\\ k_j \ even}}^{l} k_j \cdot \prod_{j=1}^{l}\Big((\beta^j)_{k_j} N_{q_j}^{d_j}(\alpha^j,\beta^j,s_j)\Big) \\
&+\sum_{I_5} \frac{1}{(l-1)!}\binom{s-1}{s_1,\dots s_l}\binom{r}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l} \tilde{M}_{k_1} \prod_{m \ even} (-m)^{(\alpha')_m} \\
&\cdot \prod_{\substack{j=2\\ k_j \ even}}^{l} k_j \cdot N_{q_1}^{d_1}(\alpha^1+e_k,\beta^1,s_1)\prod_{j=2}^{l}\Big((\beta^j)_{k_j} N_{q_j}^{d_j}(\alpha^j,\beta^j,s_j)\Big) \\
&+\sum_{I_6} \frac{1}{(l-1)!}\binom{s-1}{s_1,\dots s_l}\binom{r}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l} \tilde{M}_{k_1} \prod_{m \ even} (-m)^{(\alpha')_m} \\
&\cdot \prod_{\substack{j=2\\ k_j \ even}}^{l} k_j \cdot N_{q_1}^{d_1}(\alpha^1+e_k,\beta^1,s_1)\prod_{j=2}^{l}\Big((\beta^j)_{k_j} N_{q_j}^{d_j}(\alpha^j,\beta^j,s_j)\Big).
\end{aligned}
$$

I_1 consists of $k_1, k_2 \geq 1$ such that at least one of them is odd.

I_2 consists of all $\alpha^1, \alpha^2, \beta^1, \beta^2, k_1 \geq 1, k_2 \geq 1, s_1 \geq 0, s_2 \geq 0, q_1 \geq 0, q_2 \geq 0$ such that

- *at least one of k_1, k_2 is odd,*
- $\alpha^1 + \alpha^2 = \alpha$,

- $\beta^1 + \beta^2 = \beta - e_{k_1+k_2}$,
- $d_1 + d_2 = d$,
- $s_1 + s_2 = s - 1$,
- $q_1 + q_2 = q$.

I_3 consists of all $l \geq 0$ and all $\alpha^j, \beta^j, k \geq 1, k_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j \leq \alpha$,
- $\sum_{j=1}^{l} \left(\beta^j - e_{k_j}\right) = \beta - e_k$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$,
- $\sum_{j=1}^{l} q_j = q$.

I_4 consists of all $l \geq 0$ and all $\alpha^j, \beta^j, k \geq 1, k_j \geq 1, s_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j \leq \alpha$,
- $\sum_{j=1}^{l} \left(\beta^j - e_{k_j}\right) = \beta - e_k$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$,
- $\sum_{j=1}^{l} q_j = q - 1$.

I_5 consists of all $l > 0$ and all $\alpha^j, \beta^j, k_j \geq 1, s_j \geq 0, q_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j < \alpha$,
- $\beta^1 + \sum_{j=2}^{l} \left(\beta^j - e_{k_j}\right) = \beta$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$,
- $\sum_{j=1}^{l} q_j = q$.

I_6 consists of all $l > 0$ and all $\alpha^j, \beta^j, k_j \geq 1, s_j \geq 0, q_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j < \alpha$,
- $\beta^1 + \sum_{j=2}^{l} \left(\beta^j - e_{k_j}\right) = \beta$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$,
- $\sum_{j=1}^{l} q_j = q - 1$.

In the case of $N_0^1\left((0),(1),1\right)$, the index sets I_1, I_2, I_4, I_5 and I_6 are all empty, while I_3 consists of a single element; $l = 0, k = 1$. When computing $N_0^1\left((0),(1),0\right)$ by formula 5.10, the first and third index sets are empty, while I_2 consists of a single element, $l = 0$. Both $N_0^1\left((0),(1),0\right)$ and $N_0^1\left((0),(1),1\right)$ are 1 and these numbers are sufficient initial conditions to get our recursion starting:

Corollary 5.13. *The formulæ of Theorems 5.10 and 5.12 are sufficient to compute all relative broccoli invariants of F_1 (and therefore all Welschinger invariants of F_1).*

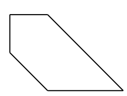
Note that when $q = 0$, the formulæ of Theorems 5.10 and 5.12 reduce to those of Theorems 4.11 and 4.13.

6 Calculations in $\mathbb{P}_2^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$

Extending the formulæ of the previous two sections to the other degrees described in Definition 5.1 is similar. In this section, we will look at the case of degrees corresponding to $\mathbb{P}_2^2$, the projective plane blown up in two points. The recursive formulæ presented to compute the Welschinger invariants of $\mathbb{P}_2^2$ also compute all Welschinger invariants of $\mathbb{P}^1 \times \mathbb{P}^1$.

Using Lemma 5.5, we just have to consider all possible shapes of the left part after we decompose the curve. In principle, the method is exactly the same as in the cases of $\mathbb{P}^2$ and F_1. The only difference is that the number of possible shapes for the left part after the decomposition is larger.

Let us move on to a concrete example of how this is done. Curves in $\mathbb{P}_2^2$ correspond to tropical curves with Newton polygons of the shape below.



6.1 Notation

Definition 6.1 (Broccoli curves of type (q,p,α,β))**.** Let d, p and q be non-negative integers such that $d \geq q+p$, $d > q$ and $d > p$. Let α and β be sequences such that $I\alpha + I\beta = d - q - p$. Let $\Delta(q,p,\alpha,\beta)$ be a degree consisting of $d-p$ times the vector $(0,-1)$, p times the vector $(-1,-1)$, $d-q$ times the vector $(1,1)$, q times the vector $(0,1)$ and $(\alpha)_i + (\beta)_i$ times $(-i,0)$ for all i (in any fixed order). Let $F(q,p,\alpha,\beta) \subseteq \{1,\ldots,|\Delta(\alpha,\beta)|\}$ be a fixed subset with $|\alpha|$ elements such that the entries of $\Delta(\alpha,\beta)$ with index in F are $(\alpha)_i$ times $(-i,0)$ for all i.

A broccoli curve in $M^{\mathrm{B}}_{(r,s)}(\Delta(q,p,\alpha,\beta), F(q,p,\alpha,\beta))$ will be called a *curve of type* (q,p,α,β). Its unmarked ends with directions $(-i,0)$ will be referred to as *left ends*. Unmarked ends with direction $(0,1)$ will be referred to as *upper ends*. Unmarked ends with direction $(-1,-1)$ will be referred to as *south-western ends*.

A curve of type $(q,0,\alpha,\beta)$ is precisely a curve of type (q,α,β) in the sense of Definition 5.2, so a curve of type $(0,0,\alpha,\beta)$ is a curve of type (α,β) (Definition 4.1).

Again, the dimension condition $|\Delta| - 1 - |F| = r + 2s$ must be satisfied. Since $|\Delta| = 2d + |\alpha| + |\beta|$, and $|F| = |\alpha|$, we must have

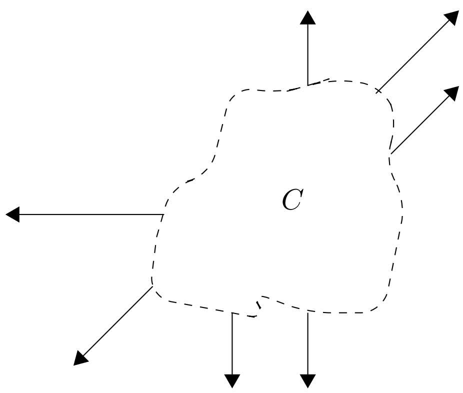
$$r = 2d + |\beta| - 2s - 1$$

for a curve of type (q,p,α,β).

Definition 6.2 (Relative broccoli numbers)**.** Let $\Delta = \Delta(q,p,\alpha,\beta)$ and $F = F(q,p,\alpha,\beta)$ be as in Definition 6.1 and let $r,s \geq 0$ such that $|\Delta| - 1 - |F| = 2d + |\beta| - 1 = r + 2s$. We then use $N^d_{q,p}(\alpha,\beta,s)$ as a shorthand expression for $N^{\mathrm{B}}_{(r,s)}(\Delta, F)$.

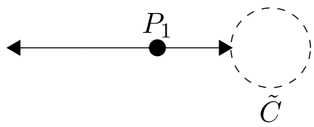
6.2 An example

Example 6.3. We will compute $N^3_{1,1}((0),(1),1)$. Which curves with 1 complex and 4 real markings fit in the figure below?



i) Assume we moved a real point to the left.

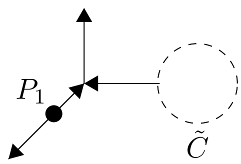
First, there is one "elevator case".



Here, $\tilde{C}$ consists of a single component, a curve of type $(1, 1, (1), (0))$. The multiplicity of C is equal to the multiplicity of $\tilde{C}$, so the contribution from this kind of decomposition is given by

$$N^3_{1,1}((1), (0), 1).$$

Secondly, the non-fixed south-western end of C could be an end of C_0:



In this case, the upper end of C must be an end of C_0 by the balancing condition. Hence, $\tilde{C}$ is a curve of type $(0, 0, (0), (2))$. The multiplicity of C is equal to the multiplicity of $\tilde{C}$, so the contribution from this kind of decomposition is given by

$$N^2_{0,0}((0), (2), 1)$$

which is 1 (look up $N^2((0), (2), 1)$ in Table 3).

No other decompositions are possible. To see why, note that either the left or the south-western end of C must be an end of C_0. By Lemma 3.28 and the balancing condition they can not both be ends of C_0.

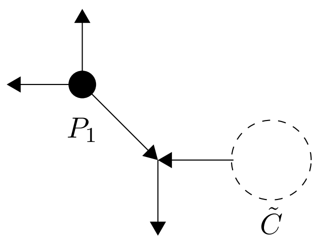
Summarising, we see that

$$N^3_{1,1}((0), (1), 1) = N^3_{1,1}((1), (0), 1) + N^2_{0,0}((0), (2), 1).$$

ii) Assume we moved a complex point to the left.

Note that C_0 cannot be contained in a horizontal line. If it were, the complex marking would have to be adjacent to three non-fixed ends, and C_0 can have at most one non-fixed left end. Now, the balancing condition makes sure that two right ends and one left end of weight 1 from the same vertex is impossible. Therefore, we only have to consider "floor cases" in which C_0 has bounded edges.

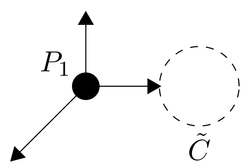
Firstly, the left end of C could be an end of C_0 without the south-western end of C being an end of C_0.



Here, $\tilde{C}$ is a curve of type $(0, 1, (0), (1))$. That is, it has no upper ends, one south-western end, no fixed left ends and one non-fixed left end of weight 1. The multiplicity of C is equal to the multiplicity of $\tilde{C}$, so the contribution from this sort of decomposition is

$$N^2_{0,1}\left((0), (1), 0\right).$$

Secondly, the south-western end of C could be an end of C_0 without the left end being an end of C_0.

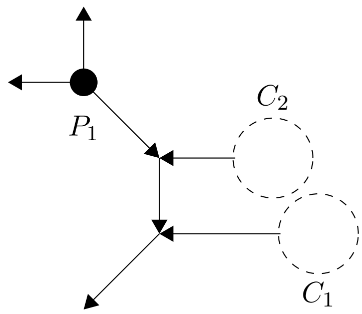


Here, $\tilde{C}$ consists of a single component, it is a curve of type $(0, 0, (1), (1))$. The multiplicity of C is equal to the multiplicity of $\tilde{C}$, so the contribution from this sort of decomposition is

$$N^2_{0,0}\left((1), (1), 0\right),$$

which is 1 (see Table 3).

Thirdly, we get three possibilities where both the left end and the south-western end of C are ends of C_0. In all of these shapes, the complex marking must be adjacent to the left or the south-western end. Below we see such a shape with an upper end.



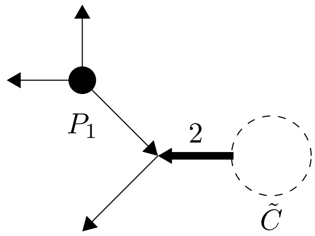
$\tilde{C}$ consists of two components. C_1 and C_2 are curves of type $\left(q_1, p_1, \alpha^1, \beta^1\right)$ and $\left(q_2, p_2, \alpha^2, \beta^2\right)$, where $q_1 + q_2 = 0$, $p_1 + p_2 = 0$, $\alpha^1 = \alpha^2 = (0)$, $\beta^1 = \beta^2 = (1)$. So the curves are both of type $(0, 0, (0), (1))$.

The multiplicity of C is equal to the product of the multiplicities of C_1 and C_2. There are 6 ways in which to distribute the 4 real points among C_1 and C_2 such that they have 2 each. Now we have overcounted by a factor 2 as the labelling of C_1 and C_2 is irrelevant. Hence, the contribution from this kind of decomposition is

$$3 \cdot N^1_{0,0}\left((0), (1), 0\right) \cdot N^1_{0,0}\left((0), (1), 0\right),$$

which is equal to 3 (look up $N^1\left((0), (1), 0\right)$ in Table 2).

We have a similar case in which $\tilde{C}$ consists of a single component:

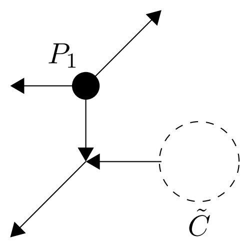


$\tilde{C}$ is a curve of type $(0, 0, (0), (0, 1))$. The vertex adjacent to the right end of C_0 has multiplicity $2i$, while the left end of $\tilde{C}$ contributes a factor i which is not a factor of m_C. All other vertices and ends of $\tilde{C}$ are vertices and ends of C, while the other ends and vertices of C_0 all contribute a factor 1 to m_C. Hence, the contribution from this kind of decomposition is given by

$$2N^2_{0,0}\left((0),(0,1),0\right),$$

which is equal to 0 (see Table 3).

Lastly, we have the case below:



Here, $\tilde{C}$ is a curve of type $(1,0,(0),(1))$. Its multiplicity is equal to the multiplicity of C, so the contribution from this kind of decomposition is

$$N^2_{1,0}\left((0),(1),0\right).$$

Noting that the complex marking can *not* be adjacent to a right end of C_0 (as in (F1) and (F2) of Theorem 5.11), since this would require a fixed left or south-western end of C_0, we have exhausted the possible decompositions.

Summing all up,

$$\begin{aligned} N^3_{1,1}\left((0),(1),1\right) =& N^2_{0,1}\left((0),(1),0\right) + N^2_{0,0}\left((1),(1),0\right) \\ &+ 6\cdot N^1_{0,0}\left((0),(1),0\right)\cdot N^1_{0,0}\left((0),(1),0\right) \\ &+ 2\cdot N^2_{0,0}\left((0),(0,1),0\right) \\ =& N^2_{0,1}\left((0),(1),0\right) + 1 + 6 + 0 \end{aligned}$$

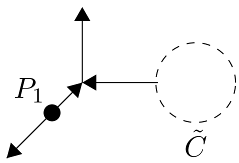
6.3 The general case

We move on to the general case.

Proposition 6.4 (Possible shapes of the left part of a curve of type (q,p,α,β) when moving a real point to the left)**.** *Assume we have decomposed a curve of type (q,p,α,β) after moving a real point to the left.*

We have an "elevator case" similar to case (A) of Proposition 4.9. There are four different versions of case (B) from Proposition 4.9; C_0 could have *one or zero upper ends and one or zero south-western ends. In all of these cases the multiplicity is as described in the two previous sections.*

We also have a new kind of shape:



$\tilde{C}$ *consists of a single component, it is a curve of type* $(q-1, p-1, \alpha, \beta + e_1)$.

Convention 6.5. Given α, β, s, we define r by

$$r := 2d + |\beta| - 2s - 1.$$

Similarly, the number of real markings of the component C_i is

$$r_i := 2d_i + |\beta^i| - 2s_i - 1.$$

$N^d_{q,p}(\alpha, \beta, s)$ will be interpreted as 0 if

- $r < 0$,
- $s < 0$,
- $q < 0$,
- $p < 0$,
- $q \geq d$,
- $p \geq d$,
- $(\alpha)_i < 0$ for some i,
- $(\beta)_i < 0$ for some i, or
- $I\alpha + I\beta \neq d - p - q$.

The sequence of numbers of fixed left ends of C of weight i which are not left ends of $\tilde{C}$ is given by

$$(\alpha')_i := (\alpha)_i - \sum_{j=1}^{l} (\alpha^j)_i.$$

Theorem 6.6 (Computing $N^d_{q,p}(\alpha,\beta,s)$ when r is non-zero)**.** *Let $r>0$, and let d, p and q be non-negative integers such that $d \geq q+p$, $d>q$ and $d>p$. Let α and β be sequences such that $I\alpha+I\beta=d-q-p$. To find $N^d_{q,p}(\alpha,\beta,s)$, all we have to do is sum over all possible decompositions after a real point is moved to the far left.*

$$\begin{aligned} N^d_{q,p}(\alpha,\beta,s) =& N^{d-1}_{q-1,p-1}(\alpha,\beta+e_1,s) \\ &+\sum_{k\ odd} N^d_{q,p}(\alpha+e_k,\beta-e_k,s) \\ &+\sum_I \frac{1}{l!}\binom{s}{s_1,\dots,s_l}\binom{r-1}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l}\prod_{m\ even}(-m)^{(\alpha')_m} \\ &\cdot \prod_{\substack{j=1\\ k_j\ even}}^{l} k_j \cdot \prod_{j=1}^{l}\Big((\beta^j)_{k_j} N^{d_j}_{q_j,p_j}(\alpha^j,\beta^j,s_j)\Big), \end{aligned}$$

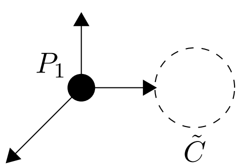
where I runs over all $l \geq 0$ and all $\alpha^j,\beta^j,k_j \geq 1, s_j \geq 0, q_j \geq 0, p_j \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^l \alpha^j < \alpha$,
- $\sum_{j=1}^l \left(\beta^j - e_{k_j}\right) = \beta$,
- $\sum_{j=1}^l d_j = d-1$,
- $\sum_{j=1}^l s_j = s$,
- $\sum_{j=1}^l q_j = q$ *or* $\sum_{j=1}^l q_j = q-1$,
- $\sum_{j=1}^l p_j = p$ *or* $\sum_{j=1}^l p_j = p-1$.

Proposition 6.7 (Possible shapes of the left part of a curve of type (q,p,α,β) when moving a complex point to the left)**.** *Assume we have decomposed a curve of type (q,p,α,β) after moving a real point to the left.*

We have elevator cases similar to cases (C) and (D) of Proposition 4.12. There are four different versions of cases (E) and (F) from Proposition 4.12; C_0 could have one or zero upper ends and one or zero south-western ends. In all of these cases the multiplicity is as described in the two previous sections.

We also have a new kind of shape:



Here, $\tilde{C}$ consists of a single component. It is a curve of type $(q-1, p-1, \alpha+e_1, \beta)$. Its multiplicity is equal to the multiplicity of C.

Summing up, we get the following theorem:

Theorem 6.8 (Computing $N^d_{q,p}(\alpha,\beta,s)$ when s is non-zero)**.** *Let $s>0$, and let d, p and q be non-negative integers such that $d \geq q+p$, $d>q$ and $d>p$. Let α and β be sequences such that $I\alpha + I\beta = d-q-p$. To find $N^d_{q,p}(\alpha,\beta,s)$, all we have to do is sum over all possible decompositions after a complex point is moved to the far left.*

$$
\begin{aligned}
N^d_{q,p}(\alpha,\beta,s) =& N^{d-1}_{q-1,p-1}(\alpha+e_1,\beta,s-1) \\
&+\sum_{I_1} -\frac{1}{2} N^d_{q,p}(\alpha+e_{k_1+k_2}, \beta-e_{k_1}-e_{k_2}, s-1) \\
&+\sum_{I_2} \frac{1}{2}\binom{s-1}{s_1,s_2}\binom{r}{r_1,r_2}\binom{\alpha}{\alpha^1,\alpha^2}\prod_{j=1}^{2} N^{d_j}_{q_j,p_j}(\alpha^j+e_{k_j},\beta^j,s_j) \prod_{\substack{j=1 \\ k_j \text{ even}}}^{2} k_j \\
&+\sum_{I_3} \frac{1}{l!}\binom{s-1}{s_1,\dots s_l}\binom{r}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l} M_k \prod_{m \text{ even}} (-m)^{(\alpha')_m} \\
&\cdot \prod_{\substack{j=1 \\ k_j \text{ even}}}^{l} k_j \cdot \prod_{j=1}^{l}\Big((\beta^j)_{k_j} N^{d_j}_{q_j,p_j}(\alpha^j,\beta^j,s_j)\Big) \\
&+\sum_{I_4} \frac{1}{(l-1)!}\binom{s-1}{s_1,\dots s_l}\binom{r}{r_1,\dots,r_l}\binom{\alpha}{\alpha^1,\dots,\alpha^l} \tilde{M}_{k_1} \prod_{m \text{ even}} (-m)^{(\alpha')_m} \\
&\cdot \prod_{\substack{j=2 \\ k_j \text{ even}}}^{l} k_j \cdot N^{d_1}_{q_1,p_1}(\alpha^1+e_k,\beta^1,s_1) \prod_{j=2}^{l}\Big((\beta^j)_{k_j} N^{d_j}_{q_j,p_j}(\alpha^j,\beta^j,s_j)\Big).
\end{aligned}
$$

I_1 consists of $k_1, k_2 \geq 1$ such that at least one of them is odd.

I_2 consists of all $\alpha^1, \alpha^2, \beta^1, \beta^2, k_1 \geq 1, k_2 \geq 1, s_1 \geq 0, s_2 \geq 0, q_1 \geq 0, q_2 \geq 0, p_1 \geq 0, p_2 \geq 0$ such that

- *at least one of k_1, k_2 is odd,*
- *$\alpha^1 + \alpha^2 = \alpha$,*
- *$\beta^1 + \beta^2 = \beta - e_{k_1+k_2}$,*

- $d_1 + d_2 = d$,
- $s_1 + s_2 = s - 1$,
- $q_1 + q_2 = q$,
- $p_1 + p_2 = p$,

I_3 consists of all $l \geq 0$ and all $\alpha^j, \beta^j, k \geq 1, k_j \geq 1, s_j \geq 0, q_{1j} \geq 0, q_{2j} \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j \leq \alpha$,
- $\sum_{j=1}^{l} \left(\beta^j - e_{k_j}\right) = \beta - e_k$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$,
- $\sum_{j=1}^{l} q_j = q$ *or* $\sum_{j=1}^{l} q_j = q - 1$,
- $\sum_{j=1}^{l} p_j = p$ *or* $\sum_{j=1}^{l} p_j = p - 1$.

I_4 consists of all $l > 0$ and all $\alpha^j, \beta^j, k_j \geq 1, s_j \geq 0, q_j \geq 0, q_{1j} \geq 0, q_{2j} \geq 0$ for $1 \leq j \leq l$ such that

- $\sum_{j=1}^{l} \alpha^j < \alpha$,
- $\beta^1 + \sum_{j=2}^{l} \left(\beta^j - e_{k_j}\right) = \beta$,
- $\sum_{j=1}^{l} d_j = d - 1$,
- $\sum_{j=1}^{l} s_j = s - 1$,
- $\sum_{j=1}^{l} q_j = q$,
- $\sum_{j=1}^{l} q_j = q$ *or* $\sum_{j=1}^{l} q_j = q - 1$,
- $\sum_{j=1}^{l} p_j = p$ *or* $\sum_{j=1}^{l} p_j = p - 1$.

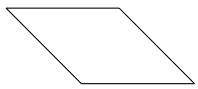
In particular, we note that when $p = 0$, the formulæ of Theorems 6.6 and 6.8 reduce to those of Theorem 5.10 and Theorem 5.12. When both p and q are zero, the formulæ are equivalent to the formulæ of Gathmann, Markwig and Schroeter (Theorem 4.11 and Theorem 4.13).

In the case of $N_{0,0}^{1}\left((0),(1),1\right)$, the index sets I_1, I_2, I_4, are all empty, while I_3 consists of a single element; $l = 0$, $k = 1$. When computing $N_{0,0}^{1}\left((0),(1),0\right)$ by formula 6.6, the first and third index sets are empty, while I_2 consists of a single element, $l = 0$. Both $N_0^1\left((0),(1),0\right)$ and $N_0^1\left((0),(1),1\right)$ are 1 and these numbers are sufficient initial conditions to get our recursion starting:

Corollary 6.9. *The formulæ of Theorems 6.6 and 6.8 are sufficient to compute all relative broccoli invariants of* $\mathbb{P}^2_2$ *(and therefore all Welschinger invariants of* $\mathbb{P}^2_2$*).*

Say we wanted to compute the Welschinger invariants of $\mathbb{P}^1 \times \mathbb{P}^1$. Let Δ be the degree consisting of p times the vectors $(0,1)$ and $(0,-1)$ and q times the vectors $(1,0)$ and $(0,1)$, i.e. curves corresponding to $\mathbb{P}^1 \times \mathbb{P}^1$.

We can identify the curves in $M^B_{(r,s)}(\Delta)$ with the curves of type $(q,p,(0),(0))$ with r real and s complex markings, i.e. the curves with degrees corresponding to Newton polygons shaped as the parallelogram below.

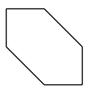


Hence, $N^W_{(r,s)}(\Delta) = N^B_{(r,s)}(\Delta) = N^d_{q,p}((0),(0),s)$, where $d = q + p$.

Corollary 6.10. *The formulæ of Theorems 6.6 and 6.8 are sufficient to compute all Welschinger invariants of* $\mathbb{P}^1 \times \mathbb{P}^1$*.*

6.4 Extensions to $\mathbb{P}^2_3$

To complete the picture, we could define curves of type $(d_1, d_2, d_3, \alpha, \beta)$ and look at possible decompositions of curves of degrees corresponding to the final hexagonal Newton polygon.



In principle, the computation of the Welschinger numbers of degrees dual to the hexagon could follow exactly the same path as in the cases we have looked at.

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