From Hilbert's Axioms to Circle-squaring in the Hyperbolic Plane

by

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THESIS

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Abstract

This thesis is based on M. J. Greenberg's article "Old and New Results in the Foundation of Elementary Plane Euclidean and Non-Euclidean Geometries" (American Mathematical Monthly , Vol 117, No 3 pp. 198-219). The aim of this thesis is to give a more complete description of some of the interesting topics in this article. We will start with Hilbert's axioms and Euclid's propositions, and then focus on hyperbolic geometry. We will proceed to give a complete proof of the uniformity theorem by using Saccheri's quadrilateral. Further, implication relations between the axioms and statements which can eliminate the obtuse angle hypothesis. Lastly, we shall discuss the famous mathematic problem of "squaring the circle" in a hyperbolic plane to Fermat primes, based on W. Jagy's discovery.

Preface

This thesis represents the completion of my master degree in LAP at the Department of Mathematics, University of Oslo. It was written during the autumn of 2011.

I would like to express gratitude to my supervisor Professor Hans Brodersen for suggesting the topic of this thesis, and otherwise providing valuable guidance through the entire writing process.

Oslo, December 2011

Jin Hasvoldseter

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Chapter 1

Introduction

Euclid (c. 300 B.C.) is considered the founder of Euclidean geometry. By Euclidean geometry, we shall mean geometry as it is developed in the books of Euclid. The Euclidean geometry that we are familiar with depends on the hypothesis that, given a line and a point not on that line, there exists one and only one line through the point parallel to the line. However, some of his proofs his books are less than perfect.

Much work was done during the nineteenth century to attain the perfection that Euclid sought. In 1899, David Hilbert proposed a set of axioms in his book *The Foundations of Geometry* ([DH]). Hilbert's axiom system is generally recognized as a flawless version of what Euclid had in mind to begin with.

The purpose of this chapter is to create a common base and language with which to begin our more formal study of geometry in the following chapters. We will start with a short presentation of Hilbert's axioms. For more details, we refer to the rich literature in this field - e.g. the books "Euclidean and non-Euclidean geometries" by M. J. Greenberg, "Euclid and beyond" by R. Hartshorne and "The Thirteen Books of The Elements" by Euclid. We will refer to Euclid's 48 propositions as Euclid I.1 - I.48, and you can find them in attached appendix.

1. Axioms

1.1 Hilbert's axioms

Hilbert divided his axioms into five groups: Incidence, betweenness, congruence, continuity and parallelism. We are going to briefly describe the axioms. Let us begin with incidence axioms.

1.1.1 Incidence axioms:

I-1: For any two points A, B, with $A \neq B$, there exists a unique line l containing both A and B.

I-2: For every line l there exists at least two distinct points that are contained in l.

I-3: There exist three distinct points with the property that no line contains all three of them.

As we can see, the axioms of incidence deal with points, lines and their intersections. By *incidence*, we mean points lying on a line. We will not define *points* and *lines* here. Indeed, most will have a general notion of what points and lines are from their basic knowledge of geometry, but frequently, precise definitions will not have been given. We leave these definitions as undefined notions that obey certain axioms. Later on, when we discuss non-Euclidean geometry, redefining the terms *lines* and *points* might be necessary.

We might also use expressions like "A lies on l" or "l passes through A" etc. We denote this by $A \in l$. If a point A lies on both l and m, we might say that "the lines l and m intersect at the point A."
We will give a few definitions before listing the betweenness axioms.

Definition 1.1.1: Let A, B be distinct points, then $\overleftrightarrow{AB}$ is defined to be the line that passes through points A and B. This definition requires the existence and uniqueness assertion from axiom I-1 for its validity.

Note that A and B cannot be the same point, otherwise, it will not be a line because axiom I-2 requires that lines have at least two points.

Proposition 1.1.2: Let A, B be distinct points on a line l, then line l is equal to $\overleftrightarrow{AB}$. The notation $\overleftrightarrow{AB}$ is equal to $\overleftrightarrow{BA}$.

Definition 1.1.3 (Collinear): Let $A_1, A_2, \ldots, A_n$ be points. If there is a line l such that $A_i \in l$ for all i, then we say that points $A_1, A_2, \ldots, A_n$ are collinear.

Then by axiom I-3 we know that there exist three distinct non-collinear points. Thus, axiom I-3 can be replaced with the following axiom:

> **I-3a**: There exists a line l and a point A not on l.

Definition 1.1.4: Given two lines *l* and *m*. Suppose that these two lines do not meet no matter how far they are extended in either direction. Then lines *l* and *m* are parallel, we denote this by *l // m*.

Now we move on to the betweenness axioms. Simply said, when one point is in between two others.

Notation: $A * B * C$ means that "the point *B* is between the points *A* and *C*."

1.1.2 Betweenness axioms:

B-1: If $A * B * C$, then *A*, *B*, and *C* are three distinct points all lying on the same line, and also $C * B * A$.

Axiom B-1 ensures that C is not equal to either A or B.

B-2: For any two distinct points *B* and *D*, there exists points *A*, *C*, and *E* lying on $\overleftrightarrow{BD}$ such that $A * B * D$, $B * C * D$, and $B * D * E$.

Figure 1

Axiom B-2 ensures that there are points between *B* and *D* and that the line $\overleftrightarrow{BD}$ does not end at either *B* or *D*.

B-3: Given three distinct points on a line, then one and only one of them is between the other two.

Axiom B-3 ensures that a line is not circular; if these three distinct points were on a simple closed curve like a circle, depending on how you look at it, you would then have to say that each is between the other two or that none is between the other two (Figure 2).

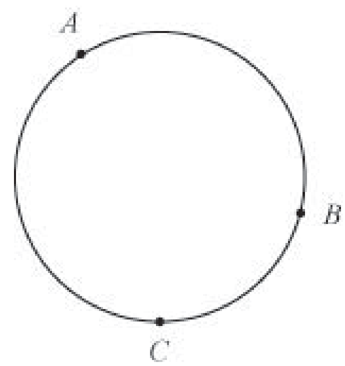


Figure 2

B-4 (Plane Separation Axiom): For any line *l*, and for any three points *A*, *B*, and *C* not lying on *l*, we have the following:

1. If *A* and *B* are on the same side of *l*, and *B* and *C* are on the same side of *l*, then *A* and *C* are on the same side of *l*.
2. If *A* and *B* are on opposite sides of *l*, and *B* and *C* are on opposite sides of *l*, then *A* and *C* are on the same side of *l*.

If *A* and *B* are on opposite sides of *l*, and *B* and *C* are on the same side of *l*, then *A* and *C* are on opposite sides of *l*.

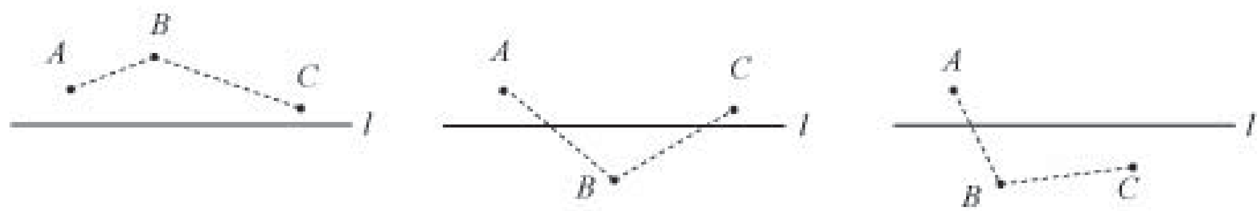


Figure 3

As a consequence of the four betweenness axioms, every line must contain infinitely many points. These axioms make it possible to define line segment.

Definition 1.1.5: A segment $\overline{AB}$ is defined as the set of all points between its end points *A* and *B* (including *A*, *B*). $\overline{AB} = \overline{BA}$.

Definition 1.1.6: A ray is a part of a line which is finite in one direction, but infinite in the other. The ray $\overrightarrow{AB}$ can be defined as the set of all points on the segment $\overline{AB}$ together with all points C such that *A* * *B* * *C*. The point *A* is the vertex of the ray $\overrightarrow{AB}$. And the point *A* is the unique point on the ray that is not between two other points on the ray.

Proposition 1.1.7: Let *A* and *B* be points. Then $\overline{AB} \subseteq \overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.

Axiom B-1 ensures that C is not equal to either A or B. so the ray $\overrightarrow{AB}$ is larger than the segment $\overline{AB}$. B-1 also ensures that all points on the ray $\overrightarrow{AB}$ lie on the line $\overleftrightarrow{AB}$.

Definition 1.1.8 (Angle): An angle is the union of two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ originating at the same point, its vertex *A*, and not lying on the same line.

Definition 1.1.9 (Triangle): Let *A*, *B*, *C* be non-collinear points. The triangle Δ*ABC* is defined to be the union of the three segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$. The points *A*, *B*, *C* are the vertices of Δ*ABC*. The segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called the sides of Δ*ABC*. The angles ∢*ABC*, ∢*BCA* and ∢C*AB* are angles of Δ*ABC* ([H] p.74).

Definition 1.1.10: Let *B* * *A* * *C*, and let *D* be a point not on $\overleftrightarrow{BC}$. Then ∢*BAD* and ∢*DAC* are called supplementary angles ([H] p.92).

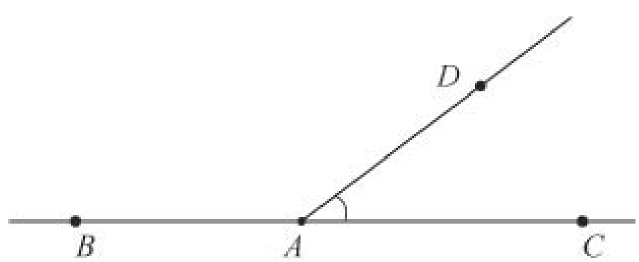


Figure 4

Definition 1.1.11 (Vertical Angles**):** Suppose lines *l* and *m* intersecting at a point *A*. Suppose *B* and *D* are on *l*, and *C* and *E* are on *m* such that *B* * *A* * *D* and *C* * *A* * *E*. Then ∢*BAE* and ∢*DAC* are called vertical angles (Figure 5).

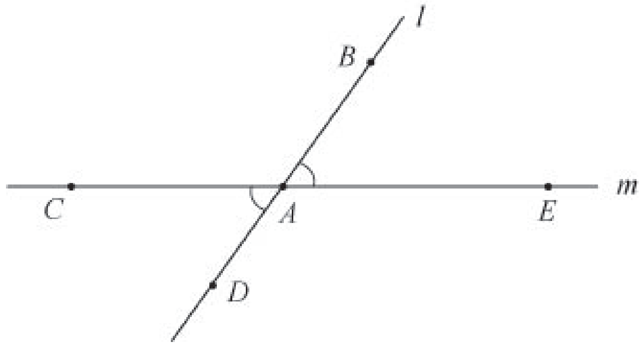


Figure 5

Proposition 1.1.12 (Euclid I.15): If two straight lines cut one another, then they make the vertical angles equal to one another.

The angles ∢*BAE* and ∢*DAC* mentioned above are both supplementary to the same angle, namely ∢*BAC*.

B-4, we say that two points *A* and *B* are on the same side of line *l*, if segment $\overline{AB}$ does not intersect *l*. If the segment $\overline{AB}$ do intersect *l*, then we say that *A* and *B* are on the opposite side of *l*. As for angles, we say that a point *D* is interior of the angle ∢*BAC* if *B* and *D* are on the same side of line $\overleftrightarrow{AC}$, and *C*, *D* are on the same side of $\overleftrightarrow{AB}$. We can use this to determine whether or not a point is inside a triangle (Figure 6).

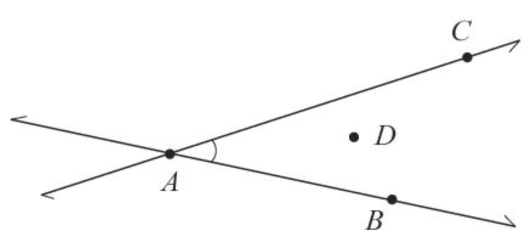


Figure 6

If points *C* and *D* are on the same(/opposite) side of the ray $\overrightarrow{AB}$, then angles ∢*BAC* and ∢*BAD* are on the same(/opposite) side of line $\overleftrightarrow{AB}$ (Figure 7).

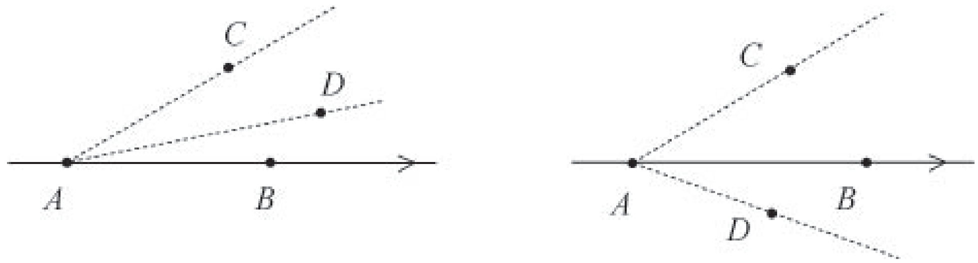


Figure 7

If we think of *A*, *B* and *C* as the vertices of a triangle, then Axiom B-4 can be replaced by the Pasch's axiom.

Theorem 1.1.13 (Pasch's Theorem**)**: If *A*, *B*, *C* are distinct non-collinear points, then we can construct a triangle Δ*ABC* with *A*, *B*, and *C* as vertices (Figure 8). Let *l* be any line intersecting *AB* in a point *D* between *A* and *B*; then *l* also intersect either *AC* or *BC*. If *C* does not lie on *l*, then *l* does not intersect both *AC* and *BC*. ([EM] p.74 & [G1] p.114)
Simply said, if a line "goes into" a triangle through one side, it must "come out" though another side.

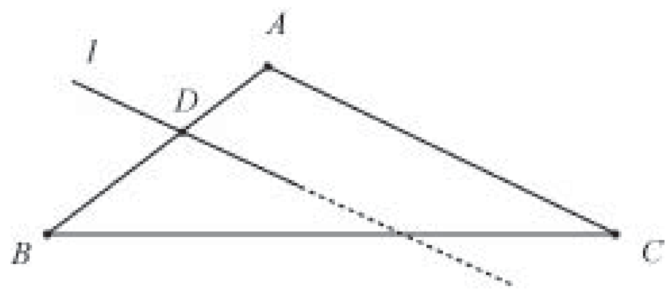


Figure 8

B-4 and Pasch's theorem leads us to another useful theorem, called the Crossbar theorem.

Theorem 1.1.14 (The Crossbar Theorem): ([G1] p.116 & [H] p.77 & [M] p.146) Let ∢*BAC* be an angle that is composed by rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. If the point *D* is in the interior of the angle, then the ray $\overrightarrow{AD}$ intersect the segment$\overline{BC}$ (Figure 9).

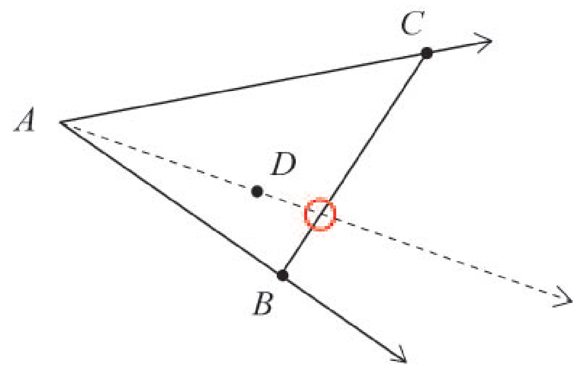


Figure 9

The axioms we are going to talk about now are congruence axioms. There are two primitive notions of congruence - congruence of segments and congruence of angles. The congruence relation between two segments or two angles will be denoted by the familiar symbol "≅".

1.1.3 Congruence axioms:

C-1: Given a line segment $\overline{AB}$, and given a ray *r* emanating from a arbitrary point *C*, then there is a unique point *D* on *r* such that *D* ≠ *C* and $\overline{AB} \cong \overline{CD}$.

Figure 10

C-2: If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Moreover, every segment is congruent to itself.

C-3 (Addition): If *A* * *B* * *C*, *D* * *E* * *F*, $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$. Then $\overline{AC} \cong \overline{DF}$.

C-4: Given any angle ∢*BAC*, and given any ray $\overrightarrow{DF}$ emanating from a arbitrary point *D*, there is a *unique* ray $\overrightarrow{DE}$ on a given side of the line $\overleftrightarrow{DF}$ such that ∢*BAC* ≅ ∢*EDF* (Figure 11).

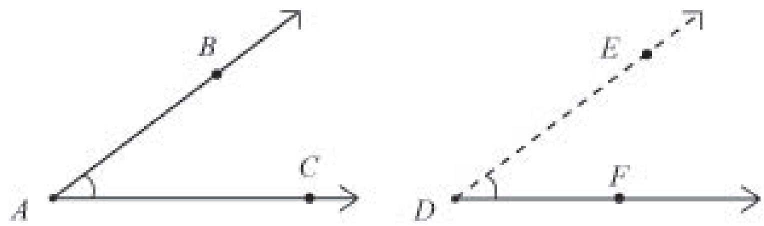


Figure 11

C-5: If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

A triangle is called *isosceles* if two of its sides are congruent, and in that case it follows by Euclid I.5 that the two base angles of such a triangle are congruent.

The final congruence axiom concerns copying of triangles. Before we give this axiom, we need a definition concerning triangles.

Definition 1.1.15: Two triangles ΔABC and ΔDEF are said to be congruent iff *i)*$\overline{AB} \cong \overline{DE}$, *ii)* $\overline{BC} \cong \overline{EF}$, *iii)* $\overline{AC} \cong \overline{DF}$, *iv)*$\sphericalangle A \cong \sphericalangle D$, *v)* $\sphericalangle B \cong \sphericalangle E$, *vi)* $\sphericalangle C \cong \sphericalangle F$.

C-6 (SAS): Given two triangles ΔABC and ΔDEF, such that $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\sphericalangle BAC \cong \sphericalangle EDF$, then $\Delta ABC \cong \Delta DEF$. In general, if two sides and an included angle are congruent, respectively, to two sides and the included angle of another triangle, then the two triangles are congruent.

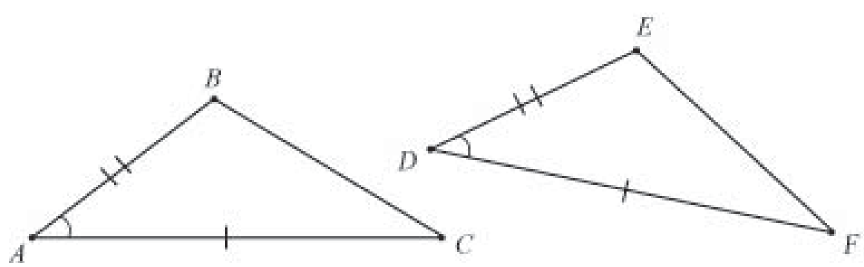


Figure 12

Besides side-angle-side (*SAS*), there are other familiar triangle congruence criteria, such as *ASA*, *AAS* (Euclid I.26) and *SSS*, and they are provable; we refer to Moise's book for proof details ([EM] p.106). Hilbert gave a model to show that *SAS* cannot be proven from the first twelve axioms ([DH] p.20). If there is a correspondence between the vertices of two triangles such that corresponding angles are congruent (*AAA*), those triangles are by definition similar. Later we will show that there exists planes in which *AAA* can be the criterion for congruence of triangles.

Definition 1.1.16 (Right Angle)**:** Let θ be an angle. Suppose that θ has a supplementary angle congruent to itself; then θ is said to be a right angle.

Assume that we have two lines *l* and *m* which intersect at a point *A*. Let *B* be a point on *l* and *C* be a point on *m*, such that neither *B* nor *C* are the point *A*. We say that the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular if ∢*BAC* is a right angle; we then write $\overrightarrow{AB} \perp \overrightarrow{AC}$. It follows from this that the lines *l* and *m* are perpendicular.

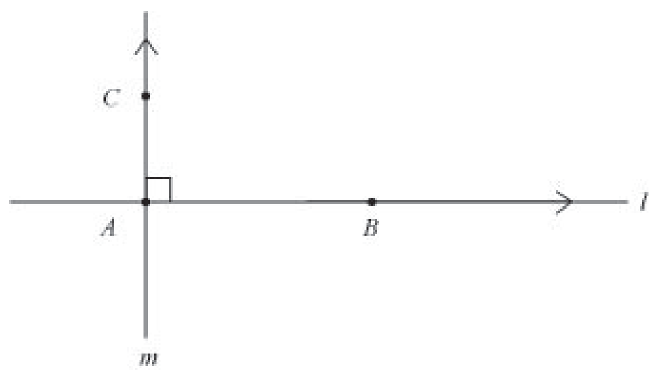


Figure 13

By the axioms I-2, C-2 and C-5, we know that:

Proposition 1.1.17: for a line *l* and a point *A*, there is a line *m* through *A* that is perpendicular to *l* ([G1] p.125).

By the congruence axioms, we can compare two segments/angles:

Proposition 1.1.18: Given two segments $\overline{AB}$ and $\overline{CD}$, we say that the segment $\overline{AB}$ is shorter than the segment $\overline{CD}$, denoted by $\overline{AB} < \overline{CD}$. There exists a point *E* with *C* * *E* * *D* such that $\overline{AB} \cong \overline{CE}$ (Axiom *C-1*).

Proposition 1.1.19: Given two angles ∢*BAC* and ∢*EDF* (Figure 14), we say that ∢*BAC* < ∢*EDF* if there exists a ray $\overrightarrow{DG}$ in the interior of the angle ∢*EDF* such that ∢*BAC* ≅ ∢*GDF* (Axiom C-4).

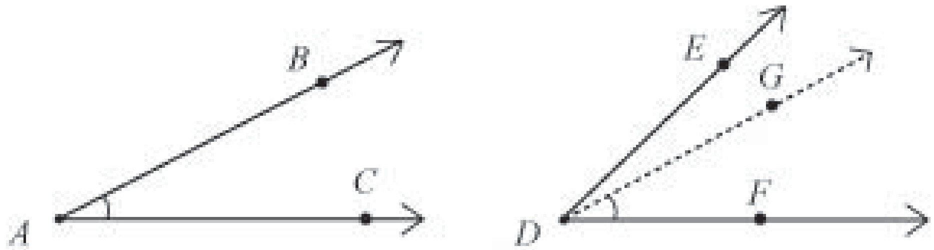


Figure 14

Definition 1.1.20: If an angle is less than a right angle, then it is called an *acute* angle; if an angle is greater than an right angle, then it is called an *obtuse* angle. Two angles are called *complementary* if the sum of them is equal to a right angle.

Proposition 1.1.21 (Trichotomy)**:** For every pair of angles, exactly one of the following three conditions holds: $\measuredangle A < \measuredangle B$; $\measuredangle A \cong \measuredangle B$; $\measuredangle A < \measuredangle B$. For every pair of segments, exactly one of the following three conditions holds: $\overline{AB} < \overline{CD}$; $\overline{AB} \cong \overline{CD}$; $\overline{AB} < \overline{CD}$.

Definition 1.1.22: Given a distinct point *O* and a segment $\overline{AB}$, we call the set of all points *C* such that $\overline{OC} \cong \overline{AB}$ the circle with *O* as *center* and $\overline{OC}$ as *radius.*

Note that this set of all points *C* is nonempty. By Axiom *C-1* we know that any line through *O* intersects the circle in two points.

1.2 Hilbert plane

Definition 1.2.1: A Hilbert plane is a plane where these thirteen axioms holds - the incidence axioms (I1 - I3), the betweenness axioms (B1 - B4), and the congruence axioms (C1 - C6).

Proposition 1.2.2 (Euclid I.31)**:** Through any point *A* not on the line *l*, there exists at least one line *m* which is parallel to *l*. ([G1] p.163)

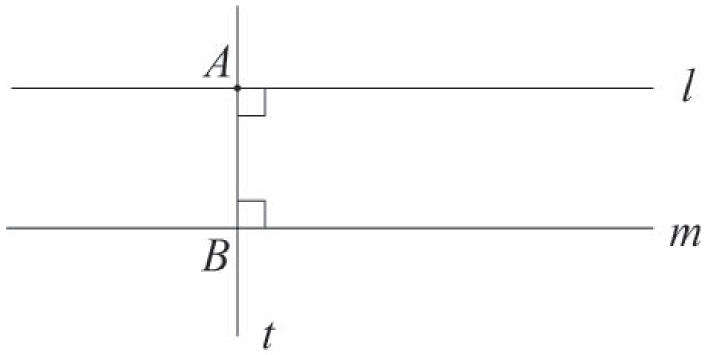


Figure 15 The standard construction

By proposition 1.1.17 and proposition 1.2.2 we get a construction as shown in the figure above. We refer this as *the standard construction* (Figure 15). We can find a proof for this Proposition in Greenberg's book ([G1] p.164).

Definition 1.2.3 (Midpoints)**:** If $A * M * B$ and $AM \cong MB$, then M is the midpoint of segment $\overline{AB}$.

Proposition 1.2.4 (Euclid I.10): Every segment has a unique midpoint.

This means that, every segment $\overline{AB}$ has a midpoint M ([M] p.89), and every segment has exactly one midpoint ([EM] p.71).

A proof that $\overline{AB}$ has a midpoint details, we refer to Greenberg's book ([G1] p.167).

Proposition 1.2.5 (Bisectors)**:** a) Every angle has exactly one bisector. b) Every segment has exactly one perpendicular bisector. ([G1] p.168)

When two lines are crossed by another line (which is called the *transversal*), the pairs of angles on the opposite sides of the transversal, but inside the two lines, are called *alternate interior angles*.

Theorem 1.2.6 (Alternate Interior Angle Theorem) (Euclid I.27). In a Hilbert plane, if two lines cut by a transversal have a pair of congruent alternate interior angles with respect to that transversal, then the two lines are parallel.([G1] p.162 & [H] p.38) (Figure 16)

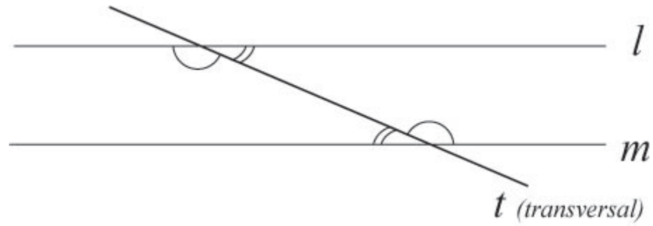


Figure 16

Theorem 1.2.7 (Exterior Angle Theorem) (Euclid I.16): In any Hilbert plane, an exterior angle of a triangle is greater than either of the opposite interior angles ([G1] p.164 & [H] p.101)

Proof: Let ΔABC be a given triangle. Extend the side BC. Let D be a point on the ray $\overrightarrow{BC}$ such that $B * C * D$ (Figure 17). We want to prove that the exterior angle $\sphericalangle ACD$ is greater than the opposite interior angles at $A(\sphericalangle BAC)$ and B ($\sphericalangle ABC$). We will just show that $\sphericalangle ACD$ is greater than the opposite interior angle at A ($\sphericalangle BAC$), since both proofs utilize the same method.

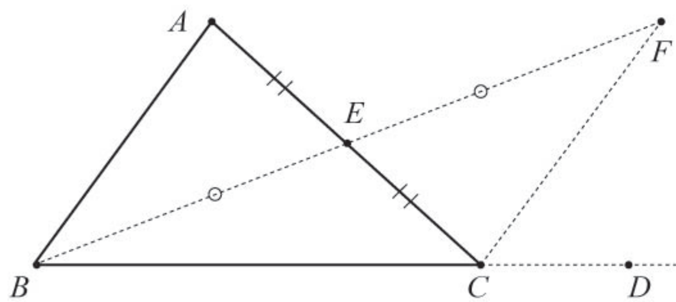


Figure 17

Let E be the midpoint of $\overline{AC}$ ([G1] p.167), then extend BE to F so that $\overline{BE} \cong \overline{EF}$ (Axiom C-1). Draw the line CF. Since $\overline{BE} \cong \overline{EF}$, $\overline{AE} \cong \overline{EC}$ and the vertical angles at E are equal (Proposition 1.1.12), so by *SAS* (Axiom C-6), the triangles ΔABE and ΔCFE are congruent. Hence $\sphericalangle BAC \cong \sphericalangle ACF$.

Because F is in the interior of $\sphericalangle ACD$, we have $\sphericalangle ACF < \sphericalangle ACD$ (Proposition 1.1.19). Thus, the opposite interior angle at $A(\sphericalangle BAC)$ satisfies $\sphericalangle BAC \cong \sphericalangle ACF < \sphericalangle ACD$, as required.◄

1.2.1 Eliminating spherical and elliptic geometry

The study of Hilbert planes is often referred to as elementary plane geometry. The axioms for a Hilbert plane did eliminate the possibility that there are no parallels at all - they eliminate spherical and elliptic geometry.

In spherical and elliptic geometry, where "lines" are great circles on the sphere, two points on the sphere are called *antipodal* if they lie on a diameter of the sphere. We can get infinitely many "lines" through any pair of antipodal points. Hence the uniqueness of axiom I-1 does not hold.

Let *A* and *B* be the antipodal points to the line *l* with the property that every line through *A* and *B* are perpendicular to *l*. If we think of *l* as the equator, then the points *A* and *B* are the north and south pole. Let two "lines" through the antipodal points *A* and *B* intersect *l* on *C* and *D*, respectively. Then as the figure below shows (Figure 18), the triangle Δ*ACD* have both an exterior angle and its opposite interior angle equal to right angles; obviously, the exterior angle theorem does not hold.

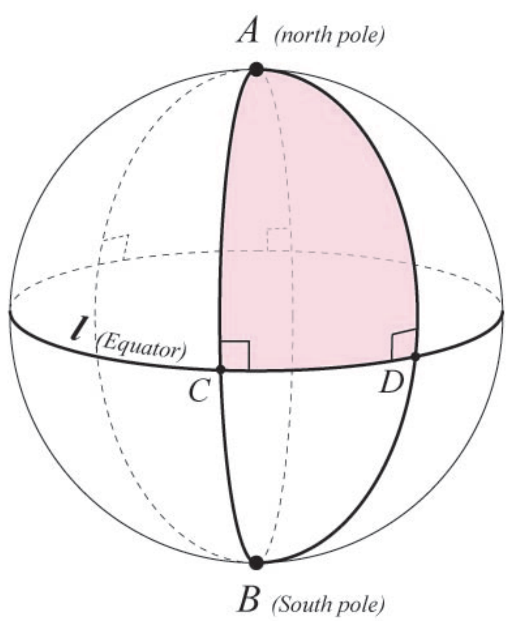


Figure 18

We see that in a Hilbert plane, we cannot expect to be able to associate real numbers to segments and angles, and segments and angles to real numbers. By introducing the continuity axioms, we are allowed to measure segments and angles by real numbers. We are going to introduce three continuity axioms - Dedekind's axiom, Archimedes' axiom and Aristotle's axiom. A relation between these three axioms will be given later.

1.3 Continuity Axioms

Dedekind's Axiom: Suppose the points of a line *l* are divided into two disjoint non-empty subsets $\sum_1 \cup \sum_2$ such that no point of either subset is between two points of the other. Then there exists a unique point O on *l* such that one of the subsets is equal to a ray of *l* with vertex O, and the other subset is equal to the complement. Namely, for any two points *A*, *B* on *l*, where $A \in \sum_1$, and $B \in \sum_2$, then one of the following must hold: 1) $A = O$; 2) $B = O$; 3) $A * O * B$ ([G1] p.134).

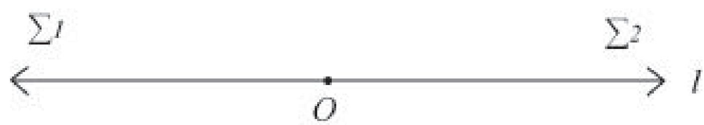


Figure 19

We should notice that the intersection of $\sum_1$ and $\sum_2$ is the empty set, but the union is all of *l*.

An informal way to describe the purpose of the Dedekind's axiom is that the axiom ensures that a line *l* has no "holes" in it. In order to understand Dedekind's axiom, we need to define the notion of a Dedekind cut.

Definition 1.3.1 (Dedekind Cut): A pair of non-empty subsets $\sum_1$ and $\sum_2$ with the properties described in Dedekind's axiom is called a Dedekind cut of the line *l* ([G1] p.135).

Any separation of points on *l* into left and right is produced by a unique point *O;* this unique cut point is in some sense between $\sum_1$ and $\sum_2$. If a Dedekind cut did not have a cut point, then there would be a "hole" between $\sum_1$ and $\sum_2$.

Archimedes' Axiom: Given a segment $\overline{CD}$, and two points *A*, *B* on a ray *r*, where $A \neq B$, there we can find a natural number *n* such that when we lay off *n* copies of *CD* on *r* starting from A, a point E on r is reached such that $n\cdot \overline{CD} \cong \overline{AE}$. Then either $B = E$, or $A * B * E$. ([G1] p.132)

As shown in the figure below, we can find points $a_0, ..., a_n$ on r, such that $a_i a_{i+1} \cong \overline{CD}$ (Axiom C-1) for every $i \leq n$. This means that if we arbitrary choose one segment $\overline{CD}$ as a unit of length, then every other segment has finite length with respect to this unit.

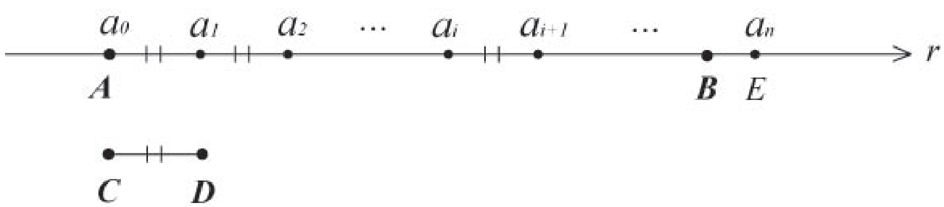


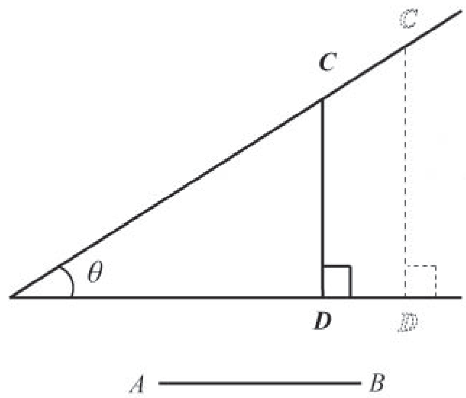
Figure 20

Another way to look at it is to choose $\overline{AB}$ as the unit of length. The axiom says that no other segment can be infinitesimally small with respect to this unit (the length of $\overline{CD}$ with respect to $\overline{AB}$ as unit is at least $1/n$ units).

Archimedes' axiom is needed to measure with real numbers; that is why the Archimedes' axiom is also called "the axiom of measurement". We can find more about the angle/segment measuring properties in Greenberg's book ([G1] p.170).

Aristotle assumed that the distance between two rays of an acute angle tends to infinity as you go out along a ray. This holds both in Euclidean and Hyperbolic plane, the definition of a hyperbolic plane will be given in the next topic. We will show proofs and discussions later.

Aristotle's Axiom: Given any segment $\overline{AB}$, and an acute angle θ (Figure 21). There exists a point C on the given side of the angle such that if D is the foot of the perpendicular from C to the other side of the angle, $\overline{CD} > \overline{AB}$. (In other words, the perpendicular segments from one side of an acute angle to the other are unbounded) ([G1] p.133)



Aristotle's Axiom

Figure 21

C **statement:** Give any segment $\overline{AB}$, and a line *l* through *B* perpendicular to $\overline{AB}$. Let *r* be a ray of *l* with vertex *B*. If θ is any acute angle, then there exists a point *C* on *r* such that $\sphericalangle ACB < \theta$ ([G] p.207).

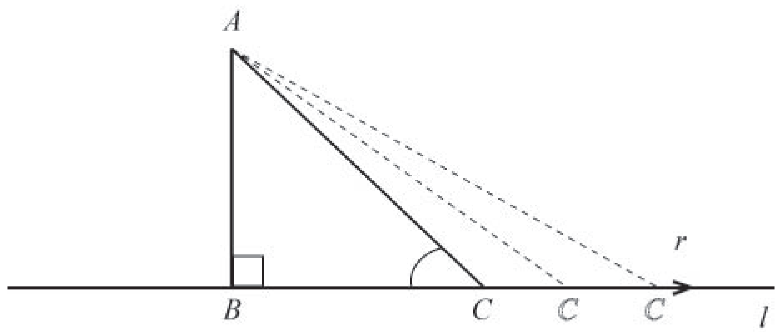


Figure 22

1.3.1 Aristotle's axiom ⇒ C statement

Proposition 1.3.2: In a Hilbert plane with Aristotle's axiom, the C statement holds.

Proof: Let $\overline{AB}$, *l, r,* and θ be given as in the C statement. Now we apply Aristotle's axiom, letting $\overline{AB}$ be the given segment and letting θ be the acute angle we use, to produce a perpendicular segment $\overline{EF}$ that is greater than the given segment $\overline{AB}$. Where *E* is on one side of the angle θ and *F* is the foot of the perpendicular from *E* to the other side of θ (Figure 23).

Let us call the vertex of angle θ, O. Lay off segment $\overline{EF}$ on ray $\overrightarrow{AB}$ starting at B and ending at some point D. and lay off segment $\overline{FO}$ on ray r of l starting at B and end at some point C. Then we have $\overline{EF} \cong \overline{BD}$, and $\overline{FO} \cong \overline{BC}$. Both $\sphericalangle EFO$ and $\sphericalangle DBC$ are right angles, then by the *SAS* axiom, $\Delta EFO \cong \Delta DBC$. Hence, $\sphericalangle BCD \cong \theta$. Since $EF > AB$, A is between B and D, so ray CA is between rays $\overrightarrow{CD}$ and $\overrightarrow{CB}$. Hence $\sphericalangle BCA < \sphericalangle BCD \cong \theta$. Then we achieved the conclusion of statement C.◄

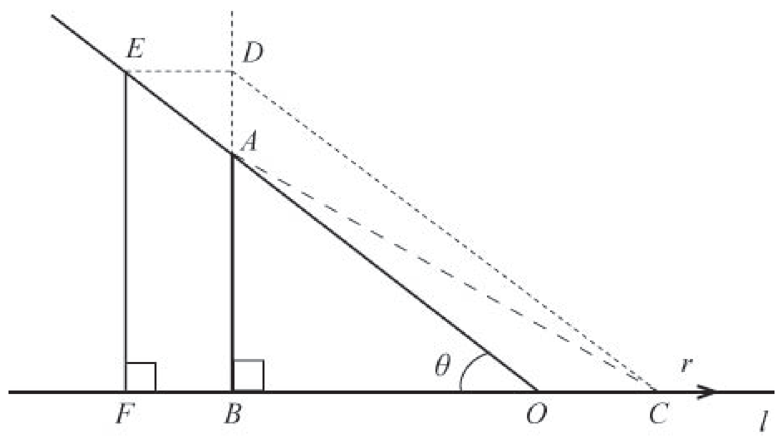


Figure 23

Definition 1.3.3: Given a circle γ with center *O* and radius *OA*. We say that a point *B* is inside γ if $OB < OA$ (outside if $OB > OA$).

Circle-circle continuity principle: Given two circles γ and γ'. If γ contains at least one point inside and at least one point outside γ', then these two circles intersect at two points. ([G1] p.130)

Line-circle continuity principle: If a line *l* contains a point inside a circle γ , then the line *l* intersects γ at two points. ([G1] p.131)

The circle-circle continuity principle implies the Line-circle continuity principle ([G1] p.201).

1.3.2 About Dedekind's axiom

Dedekind's axiom allows us to measure with real numbers, enabling us to calculate basic formulas for length, area, and trigonometry. A Hilbert plane satisfying Dedekind's axiom is either *real Euclidean* or *real hyperbolic* ([G1] p.262). We will discuss more about Dedekind's axiom in hyperbolic plane later.

Dedekind's axiom implies Archimedes' axiom; we can find a complete proof of this in Greenberg's book ([G1] p.136). It also implies the circle-circle continuity principle; we can find the proof for this in Euclid's book with nice illustrations ([E] p.238).

Furthermore, Archimedes axiom implies Aristotle's axiom, we will show that proof later on.

The figure below shows an overview of the implication relations we have discussed.

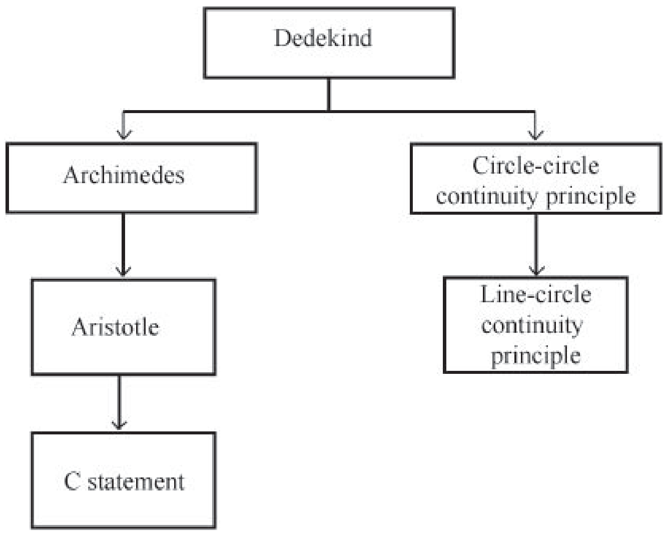


Figure 24

1.4 Parallelism Axioms

Finally, there is much in geometry that depends on parallelism axioms. This is the last group of Hilbert's axioms. But we will just mention shortly about parallelism axioms here. More about parallelism will discussed in later topics.

Hilbert's Euclidean Axiom of parallelism (Playfair's axiom): For every line *l* and every point *A* not lying on *l* there is at most one line *m* through *A* such that *m* is parallel to *l.* ([G1] p.138)

Negation of Hilbert's Euclidean Axiom of parallelism: For every line *l* and every point *A* not lying on *l* there is at least one line *m* through *A* such that *m* is parallel to *l.* ([G1] p.250)

Definition 1.4.1: A Hilbert plane satisfying the circle-circle principle and Hilbert's Euclidean Axiom of parallelism is called *Euclidean plane*.

Definition 1.4.2: A Hilbert plane satisfying the Dedekind's axiom and Hilbert's Euclidean Axiom of parallelism is called *real Euclidean plane* ([G1] p.139)

Chapter 2

2. Neutral geometry

2.1 The fifth postulate

We are all quite familiar with Euclidean geometry, especially students who took advanced mathematics courses in high school. It is the study of geometry that is based on the assumptions of the mathematician Euclid (300 B.C.), whose "Elements" became the most widely read geometry textbook in the world. The geometry taught in high school today is essentially a part of the Elements, with a few unimportant changes. The list below contains the first five of Euclid's postulates.

Postulate 1: You can draw a straight line through any two points.

Postulate 2: You can extend any segment indefinitely.

Postulate 3: You can draw a circle with any given point as center and any given radius.

Postulate 4: All right angles are equal.

Postulate 5: Given two lines and a transversal, if the two interior angles on one side add to less than a right angle, then the two lines intersect on that side of the transversal ([E] p.154).

The fifth postulate, known as the *Parallel Postulate*.

However, if we consider the axioms of geometry as abstractions from experience, we can see a difference between the fifth postulate and the other four. The first postulate implies that any two points determine a unique line; this is an abstraction from our experience drawing with straightedge. The second postulate implies that any line is of infinite length. The third postulate implies that we can uniquely determine a circle by a given point and a given segment; this derives from our experience drawing with compass. The fourth postulate is less obvious as an abstraction. One could argue that it derives from our experience measuring angles with a protractor; if we think of congruence for angles in terms of having the same number of degrees when measured by a protractor, then indeed all right angles are congruent ([G1] p.21). We can draw two segments and extend them to see if the lines containing them meet, but we cannot go on extending them forever; that is one of the reasons why the fifth postulate is so hard to verify. Euclid did come up with a suggestion to test whether two lines are parallel without using the definition, by draw a transversal and compare the alternate interior angles (Proposition 1.2.6).

2.1.1 Equivalence of Euclid's fifth postulate

A Hilbert plane will be said to posses *neutral geometry* if we neither affirm nor deny the parallel axiom ([H] p.305). *Euclidean geometry* is obtained by adding a parallel axiom to Neutral geometry. What parallel axiom do you add? It turns out there are several axioms that work equally well. For example, there is Euclid's fifth postulate, and any statement that is equivalent to Euclid's fifth postulate.

We will only discuss the most straightforward equivalences. For more details, we will refer to the other geometry literatures ([G1] p.173 & [M] p.160).

First, we will reformulate the definition of the fifth postulate:

Definition 2.1.1 (Euclid's Fifth Postulate**)**: Suppose that $l = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{CD}$ are distinct lines such that $A \neq C$. B and D are on the same side of line $n = \overleftrightarrow{AC}$, and angle sum of $\sphericalangle BAC$ and $\sphericalangle DCA$ is less than two right angles. Then l and m intersect in a points E, and points B, D and E are all on the same side of line n.

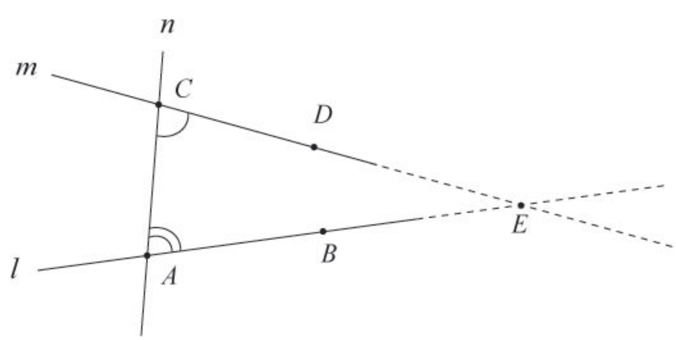


Figure 25

Proposition 2.1.2: Euclid's fifth postulate is equivalent to the Hilbert's Euclidean parallel postulate in the context of neutral geometry.

Recall the Alternating Interior angle theorem (Proposition 1.2.6). We can now consider the converse, and the converse to the Alternating interior angle theorem is the following definition:

Definition 2.1.3: For any two distinct parallel lines *l* and *m*, and any transversal *t* to these lines, the alternating interior angles are congruent.

In other words, if $l = \overleftrightarrow{AB}$ and $m = \overleftrightarrow{CD}$ are parallel, and if *B* and *D* are on the opposite side of line $n = \overleftrightarrow{AC}$, then $\measuredangle BAC \cong \measuredangle DCA$.

Proposition 2.1.4: The Hilbert's Euclidean parallel postulate equivalent to the converse to the Alternating interior angle theorem in the context of neutral geometry.

Because of the parallel postulate, some might be confused by the definitions of parallel lines with lines that are "equidistant". After all, we are most familiar with the elementary geometry, and we learn from high school geometry that parallels lines are lines that are equidistant ([R] p.241). For some, the first definition of parallel line might be "lines in the same plane that never meet." ([S] p.48). The second one is a right way to define parallel lines, but as we can see that, it is quite abstract. The first one is more understandable for our common knowledge, but the image of parallel lines as equidistant led to several confused attempts to prove Euclid's parallel postulate. For centuries, many mathematicians did not believe the fifth postulate was a postulate at all and tried to show that it could be proved using the other postulates. Attempting to use indirect proof, mathematicians began by assuming that the fifth postulate was false and then tried to reach a logical contradiction. If the parallel postulate is false, then one of these assumptions must be true.

Assumption 1: Through a given point not on a given line, you can draw more than one line parallel to the given line.

Assumption 2: Through a given point not on a given line, you can draw no line parallel to the given line.

Interestingly, neither of these assumptions contradict any of Euclid's other postulates. When one of the assumptions is true, we can call it non-Euclidean geometry. We have mentioned the definition of a Hilbert plane earlier. So a more formal way to describe a *non-Euclidean geometry* would be a Hilbert plane in which the parallel axiom does not hold.

Before we go any further in neutral geometry, we should mention that although we can understand what Euclid meant geometrically without using numbers, we are still going to use number measurement as a language for situations where it simplifies the statements. This means that we will assume the Archimedes' axiom in the later topics. For simplicity we will drop the bars over *AB* in the notation for a line segment, so long as no confusion can result.

2.2 The Uniformity Theorem

Quadrilaterals played a important part in the history of the parallel postulate. One of the most remarkable attempts to prove the parallel postulate was that of Girolamo Saccheri (1667 - 1733), who based his work on a special quadrilateral, which we will refer to as the Saccheri quadrilateral.

2.2.1 Saccheri quadrilateral

Definition 2.2.1 (Saccheri Quadrilateral): In a general Hilbert plane, suppose that two congruent perpendiculars *AD* and *BC* stand at the end of the interval *AB*, and we join *CD* (Figure 26). This quadrilateral □*ABCD* is called a *Saccheri quadrilateral*.

Side *AB* joining the right angles will be called the *base*; its opposite side *CD* will be called the *summit*.∢*C* and ∢*D* will be called the *summit angles*.

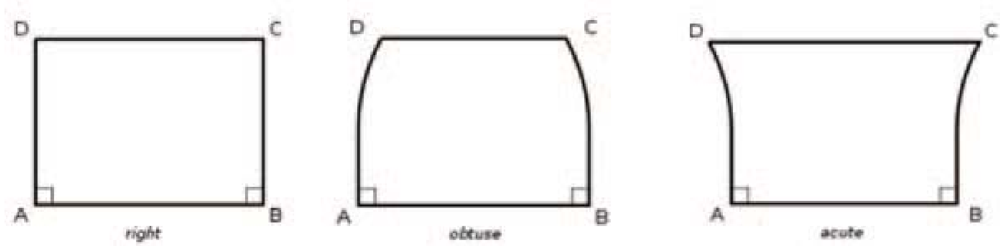


Figure 26

Proposition 2.2.2: The summit angles (angles at *C* and *D*) of a Saccheri quadrilateral □*ABCD* are congruent to each other, and furthermore, the line joining the midpoints of *AB* and *CD*, the midline, is perpendicular to both ([G1 p.177] & [H] p.306).

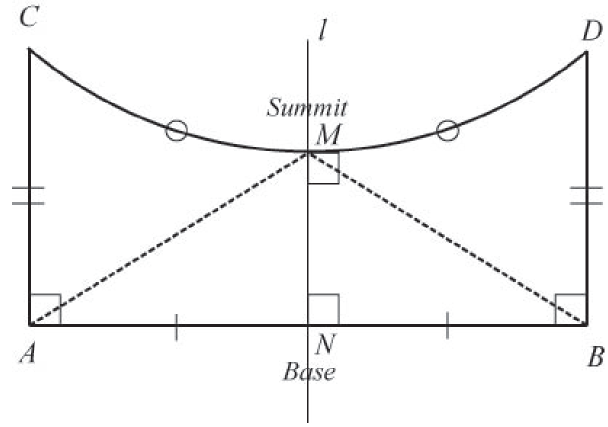


Figure 27

Proof: Given □*ABCD* as above, let *N* be the midpoint of *AB* and let *l* be the perpendicular to *AB* at *N*. Since *l* is perpendicular bisector of *AB*, the points *A*, *C* lie on the one side of *l* while *BD* lie on the other side. Hence *l* meets the segment *CD* in a point *M*. $\Delta ANM \cong \Delta BNM$ by *SAS*. Hence $\measuredangle MAN \cong \measuredangle MBN$, and $AM \cong BM$.

By subtraction from the right angles at *A* and *B*, $\measuredangle MAC \cong \measuredangle MBD$. So by *SAS* again, $\Delta AMC \cong \Delta BMD$. This shows that angles at *C* and *D* are congruent, and that *M* is the midpoint of *CD*.

The two pairs of congruent triangles also imply that $\measuredangle CMN \cong \measuredangle DMN$. And they are both right angles. ◄

The base $\overleftrightarrow{AB}$ and the summit $\overleftrightarrow{CD}$ in a Saccheri quadrilateral □*ABCD* will not intersect, they are parallel.

Definition 2.2.3 (Hyperparallel): When two parallel lines *l* and *m* have a common perpendicular, we call it *hyperparallel***.**

In this case, line $\overleftrightarrow{AB}$ is hyperparallel to line $\overleftrightarrow{CD}$, since they have the common perpendicular $\overleftrightarrow{MN}$.

There are two types of parallel lines in the Hyperbolic plane. The definition of a hyperbolic plane will be given later. One(we just mentioned) is hyperparallel, and the other is horoparallel (some may call it critically parallel), or as we may refer to as limiting parallel. We will discuss more about limiting parallel later on. ([M] p.339/341)

2.2.2 Lambert quadrilateral

Fifty years after Saccheri's attempt, Lambert (1728 - 1777) followed the same general program, using a quadrilateral with at least three right angles. We will refer to as the *Lambert quadrilateral*. The remaining angle, about which we are not assuming anything for now, is referred to as *the fourth angle* with respect to the three given right angles. The Lambert quadrilateral can be regarded as one half of the Saccheri's quadrilateral. If we observe the figure 27, we will see that Saccheri quadrilateral □*ABDC* with midline *MN* can be "divided" into two equal Lambert quadrilaterals □*ANMC* and □*NBDM*, with the fourth angle equal to the summit angle. Furthermore, the type of the fourth angle in a Lambert quadrilateral is the same as the type of the summit angles of a Saccheri quadrilateral.

2.2.3 Proof of the Uniformity theorem

Uniformity Theorem: A Hilbert plane must be one of three distinct types:

i) **Semi-Euclidean:** The angle sum of every triangle is equal to two right angles (equivalently, every Lambert or Saccheri quadrilateral is a rectangle).

ii) **Semi-hyperbolic:** The angle sum of every triangle is less than two right angles (equivalently, the fourth angle of every Lambert quadrilateral and the summit angles of every Saccheri quadrilateral are acute).

***iii*)** **Semi-elliptic:** The angle sum of every triangle is greater than two right angles (equivalently, the fourth angle of every Lambert quadrilateral and the summit angles of every Saccheri quadrilateral are obtuse).

Moreover, all three types of the Hilbert plane exists ([G] p.208). In case *ii*) we say that it satisfies the acute angle hypothesis. In case *iii*) we say that it satisfies the obtuse angle hypothesis. We can call case *ii*) and *iii*) non-Euclidean.

The following lemmas are proofs of uniformity. We will denote two right angles by 2*RA* for simplicity.

Lemma 2.2.4: Let □*ABCD* be a bi-right quadrilateral with right angles at *A* and *B*, and sides *AC* and *BD* (Figure 28).

a) $AC < BD$ iff $\sphericalangle\beta > \sphericalangle\gamma$.

b) $AC \cong BD$ iff $\sphericalangle\beta \cong \sphericalangle\gamma$.

c) $AC > BD$ iff $\sphericalangle\beta < \sphericalangle\gamma$.

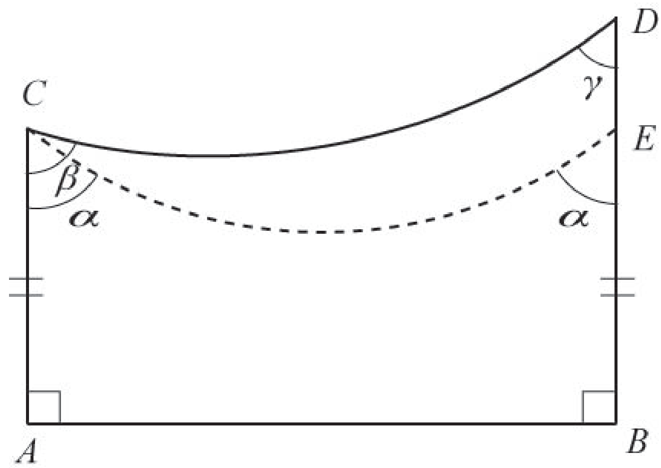


Figure 28

Proof: Assume first $AC < BD$. We choose *E* on *BD* such that $AC \cong BE$. Then □*ABEC* is a Saccheri quadrilateral and the summit angles $\sphericalangle ACE \cong \sphericalangle BEC \cong \alpha$. In the triangle Δ*DEC*, the exterior angle theorem (Theorem 1.2.7) yields $\gamma < \alpha$. Since points *A* and *D* lie on different sides of line *CE*, angle comparison at vertex *D* implies $\alpha < \beta$. Hence $\gamma < \alpha < \beta$, and transitivity of angle comparison yields $\gamma < \beta$, as to be shown.

By a similar argument, we prove that $AC > BD$ implies $\gamma > \beta$. Finally $AC \cong BD$ implies $\gamma \cong \beta$, since the summit angles of a Saccheri quadrilateral are congruent. The other direction of these three cases follows from trichotomy.◄

Lemma 2.2.5. Given a Saccheri quadrilateral □*ABDC* and a point *P* between *C* and *D*. Let *Q* be the foot of the perpendicular from *P* to the base *AB*. Then

a) $PQ < BD$ iff the summit angles of □*ABDC* are acute.

b) $PQ \cong BD$ iff the summit angles of □$ABDC$ are right angles.

c) $PQ > BD$ iff the summit angles of □$ABDC$ are obtuse.

Proof: Since □$ABDC$ is a Saccheri quadrilateral, we know that the summit angles $\sphericalangle C$ and $\sphericalangle D$ are congruent, and $AC \cong BD$ (Proposition 2.2.1). Also observe that □$AQPC$ and □$QBDP$ are bi-right quadrilaterals; thus we can apply Lemma 2.2.4.

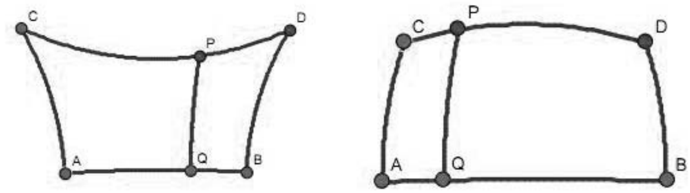


Figure 29

At vertex P there occur the supplementary angles $\alpha \cong \sphericalangle CPQ$, $\beta \cong \sphericalangle DPQ$, $\gamma \cong \sphericalangle PDB$ and $\delta \cong \sphericalangle ACP$. We begin by assuming $PQ < BD$, and look for a result about the summit angle γ. Using the previous Lemma 2.2.4 for the quadrilateral □$AQPC$, we conclude $\delta < \alpha$. Using the Lemma 2.2.4 once more for the quadrilateral □$QBDP$, we conclude $\gamma < \beta$. Hence angle addition yields $2\gamma = \gamma + \delta < \beta + \alpha = 2RA$ and hence $\gamma < RA$. Hence the summit angles are acute.

By a similar argument, the assumption $PQ > BD$ implies $\gamma > RA$, and finally, indeed, the assumption $PQ \cong BD$ implies $\gamma = RA$

The reverse direction follows from trichotomy. ◄

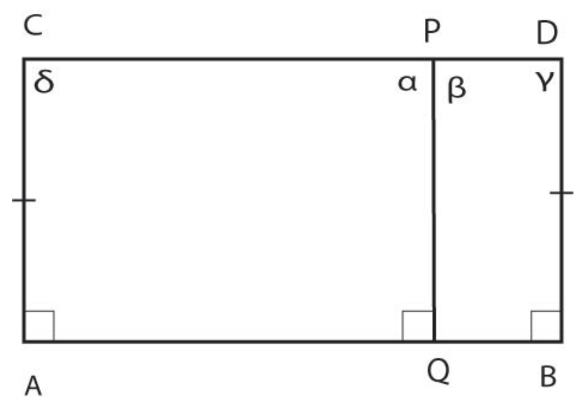


Figure 30

Lemma 2.2.6. Given a Saccheri quadrilateral □$ABDC$ and a point P such that $C * D * P$. Let Q be the foot of the perpendicular from P to $\overleftrightarrow{AB}$. Then

a) $PQ > BD$ iff the summit angles of □$ABDC$ are acute.

b) $PQ \cong BD$ iff the summit angles of □$ABDC$ are right angles.

c) $PQ < BD$ iff the summit angles of □$ABDC$ are obtuse.

Proof: Since □*ABDC* is a Saccheri quadrilateral, we know that the summit angles ∢*C* and ∢*D* are congruent, and $AC \cong BD$ (Proposition 2.2.1). Let the summit angles be congruent to α.

Choose *E* in $\overleftrightarrow{PQ}$ such that $BD \cong QE$. Draw *CE* and *DE*, then □*AQEC* is a Saccheri quadrilateral. Let the summit angel be congruent to γ. We get the third Saccheri quadrilateral □*BQED*, denote its summit angel by β.

First, let us establish the forward direction of all three cases:

Suppose $PQ > BD$, and check that $\gamma < RA$ .The three points *A*, *B* and *Q* on the base line lie on the same side of the summit line *CE*, since baseline and summit line of the Saccheri quadrilaterals □*AQEC* have the middle line as their common perpendicular, and hence are parallel. Since $QP > BD \cong QE$, points *Q* and *P* lie on the different sides of line *CE*. Hence points *P* and *D* lie on the (upper) side of *CE*, where *A*, *B* and *Q* lie on the lower side.

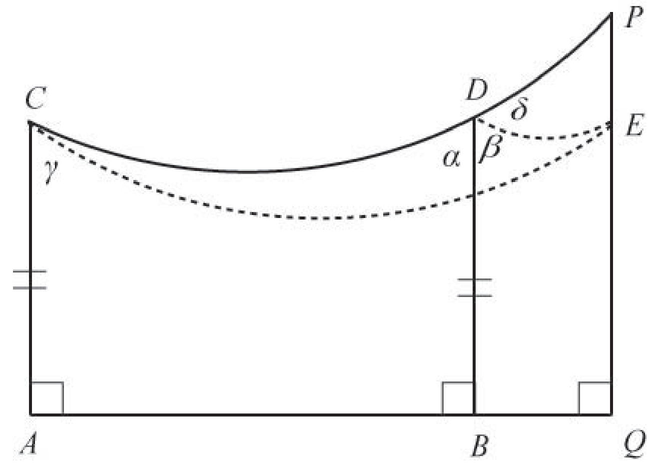


Figure 31

Let $\delta =$ ∢*PDE*. Then by the angle addition at vertex *D* yields $\alpha + \beta + \delta = 2RA$. In the triangle Δ*CED*, the exterior angel $\delta > \alpha - \gamma$. Finally, comparison of angles at vertex E shows that $\beta > \gamma$. Put together, we get $2RA = \alpha + \beta + \delta > \alpha + \beta + \alpha - \gamma > 2\alpha$. Hence α is acute, as required.

Under the assumption $PQ < BD$. this proof is quite similar to the last one. Indeed, because of $PQ < BD \cong QE$, points *Q* and *P* lie on the same side of line *DE*. Hence

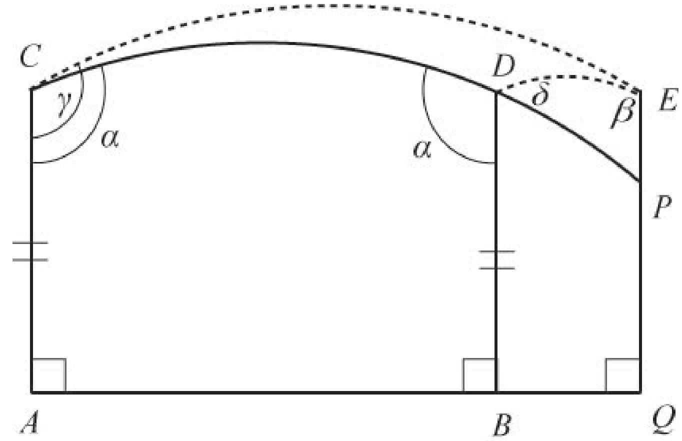


Figure 32

points A, B and Q lie on the same (lower) side of summit line CE.

Let $\delta = \sphericalangle PDE$. Angle addition at vertex D yields $\alpha + \beta - \delta = 2RA$. In the triangle ΔCED, the exterior angle $\delta > \gamma - \alpha$. Finally, comparison of angles at vertex E shows that $\beta < \gamma$. Put together, we get $2RA = \alpha + \beta - \delta < \alpha + \beta + \alpha - \gamma < 2\alpha$. Hence α is obtuse, as required.

Finally, if $PQ \cong BD$, then □$AQPC$ is Saccheri, so by part b) of Lemma 2.2.4, α must be right.

The reverse direction follows from trichotomy. ◄

Lemma 2.2.7 Given Saccheri quadrilaterals □$ABCD$ and □$A'B'C'D'$, let □$ABCD$ have midline MN, then □$A'B'C'D'$ must have midline that is congruent to EF.

Proof: Given two Saccheri quadrilaterals □$ABCD$ and □$A'B'C'D'$ with a common middle segment MN. Let α and α' be the summit angles of these two quadrilaterals, respectively (Figure 33). Assume $C * C' * M * D' * D$ as order of the vertices on the base line. Using the Lemma 2.2.5 for the Saccheri quadrilateral □$ABDC$, we get the three equivalences

$\alpha < RA$ iff $A'C' < BD$

$\alpha \cong RA$ iff $A'C' \cong BD$

$\alpha > RA$ iff $A'C' > BD$

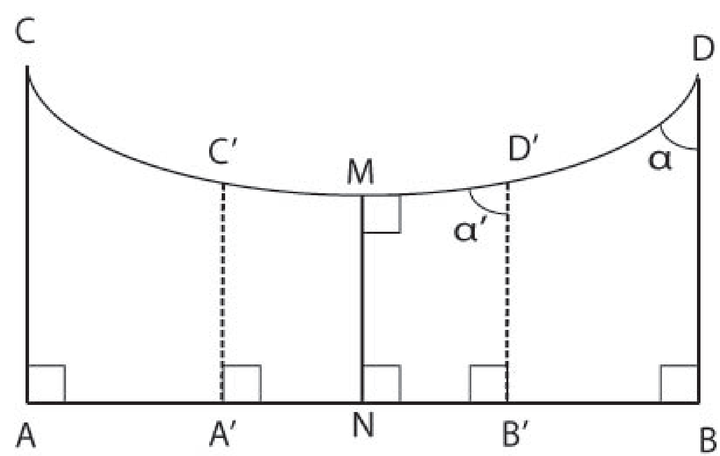


Figure 33

Using Lemma 2.2.6 for the Saccheri quadrilateral □$A'B'D'C'$, we get the three equivalences

$\alpha' < RA$ iff $A'C' < BD$

$\alpha' \cong RA$ iff $A'C' \cong BD$

$\alpha' > RA$ iff $A'C' > BD$

Put together, we see that angles α and α' are either both acute, both right, or both obtuse. As required. ◄

The proof of the Uniformity theorem is indicated in these four lemmas. That is, for any Hilbert plane, if one Saccheri quadrilateral has acute(respectively, right, obtuse) summit angles, then so do all Saccheri quadrilaterals.

Next we show that for any other segment, there exists a Saccheri quadrilateral with midline equal to that segment and same summit angel type(namely, acute, right or obtuse) as the other Saccheri quadrilaterals in the Hilbert plane.

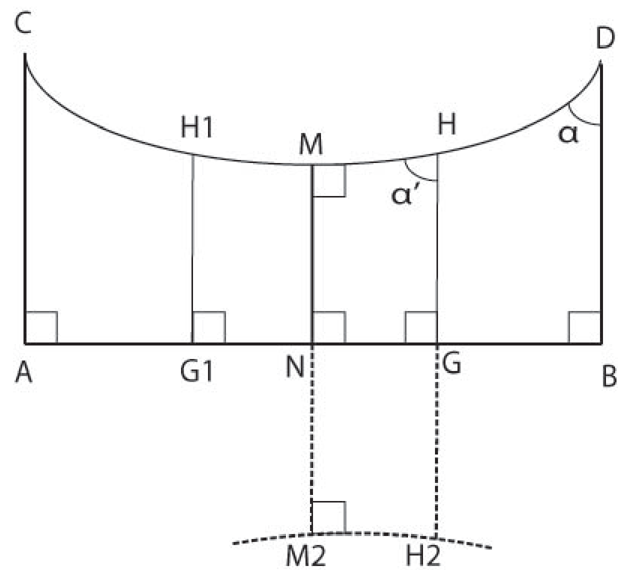


Figure 34

Given a Saccheri quadrilateral □*ABDC* with midline *MN*, and a segment *NG* on the ray $\overleftrightarrow{NB}$. Let the perpendicular to *AB* at *G* meet *CD* at *H*. Reflect *M* and *H* in *MN* to get H_1 and G_1. Reflect *M* and *H* in *AB* to get M_2 and H_2. Now □G_1GHH_1 is a Saccheri quadrilateral with midline *MN*. By the previous argument, the summit angles α and α' of the Saccheri quadrilaterals are either both acute, right or obtuse. □M_2H_2HM is another Saccheri quadrilateral with the same summit angel α' and midline *NG*. Now by the earlier argument, every other Saccheri quadrilaterals with midline equal to *NG* has the same summit angel type as □*ABDC*. But *NG* was arbitrary, so it applies for all the other Saccheri quadrilaterals in the Hilbert plane. ◄

The uniformity theorem applies also to Lambert quadrilaterals. The proof will be similar to the one we just did, since a Lambert quadrilateral can be regarded as one half of some Saccheri quadrilateral, and its fourth angle is the same type as the summit angles of some Saccheri quadrilateral. Now we are going to show, by using the Saccheri quadrilateral, that the uniformity theorem also applies to the angle sum of triangles in a Hilbert plane.

Proposition 2.2.8: Given a triangle ΔABC, there is a Saccheri quadrilateral for which the sum of its two summit angles is equal to the sum of the three angles of the triangle.

Proof: Let ΔABC be a given triangle. Let *D* and *E* be the midpoints of *AB* and *AC*. Connect the midpoints *D* and *E* by line *l*. Then drops the perpendiculars from all three vertices *A, B, C* onto *l*. Let *G, F, H* be the foot points of the perpendiculars, respectively (Figure 35).

By *AAS* , $\Delta ADG \cong \Delta BDF$, because of the right angles at vertices *F* and *G*, the congruent vertical angles at vertex *D*, and because segments $AD \cong DB$ by construction. By similar reasoning we get $\Delta AEG \cong \Delta CEH$. From these triangle congruences we obtain $\overline{BF} \cong \overline{AG} \cong \overline{CH}$. The quadrilateral $\square FHBC$ has right angles at *F* and *H*, so it is a Saccheri quadrilateral (upside down). Its summit angles at vertices *B* and *C* are congruent (Proposition 2.2.2).

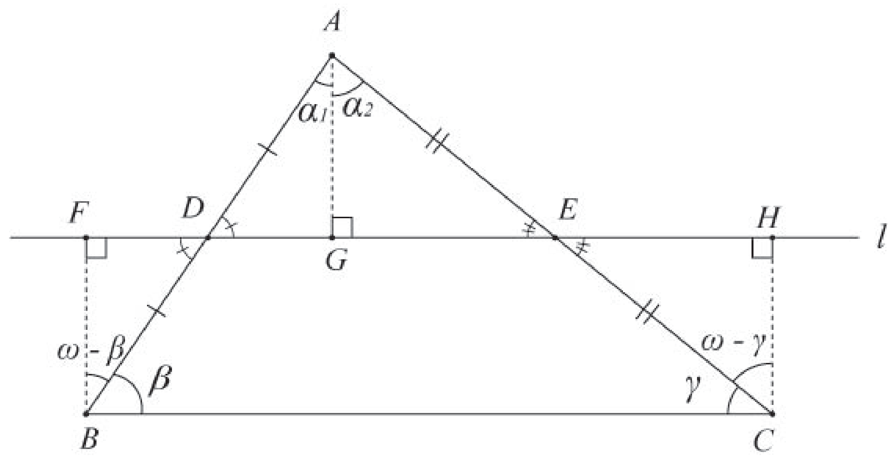


Figure 35

Let us denote the summit angles in the Saccheri quadrilateral $\square FHBC$ by ω, and denote the angles in the triangle ΔABC by α, β, γ, where $\alpha = \alpha_1 + \alpha_2$, as shown in the figure. Then, the sum of the angles of ΔABC is

$$\alpha + \beta + \gamma = \alpha_1 + \alpha_2 + \beta + \gamma = (\omega - \beta) + (\omega - \gamma) + \beta + \gamma = 2\omega$$

as required.

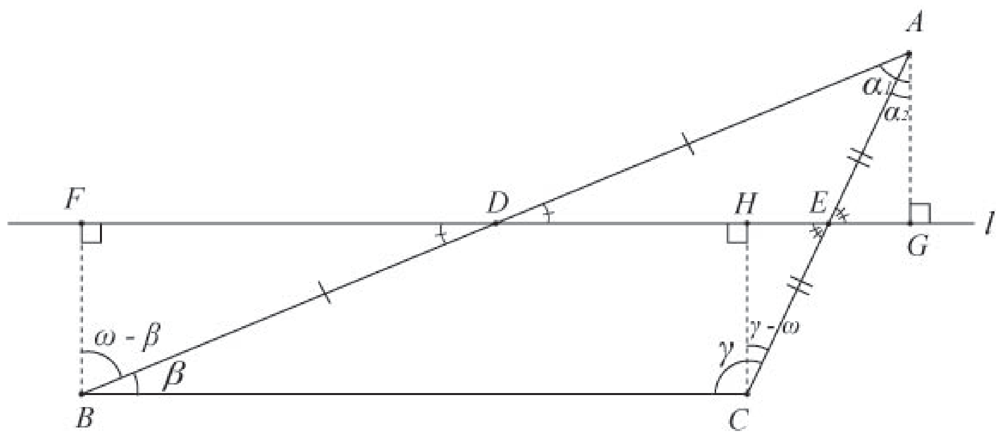


Figure 36

If G happens to fall outside the interval FH (Figure 36), the same argument works; $\alpha = \alpha_1 - \alpha_2$. Then the sum of the angles of ΔABC is

$$\alpha + \beta + \gamma = \alpha_1 - \alpha_2 + \beta + \gamma = (\omega - \beta) - (\gamma - \omega) + \beta + \gamma = 2\omega$$

as required. ◄

As we mentioned earlier, by adding Hilbert's Euclidean parallel postulate to neutral geometry, we can obtain Euclidean geometry. By case *i)* of the uniformity theorem, we know that the angle sum of every triangle is equal to two right angles (equivalently, every Lambert or Saccheri quadrilateral is a rectangle). What we will discuss next, is triangles in non-Euclidean geometry (case *ii)* and *iii)* of the uniformity theorem).

Before we determine whether or not there is similar triangles in non-Euclidean plane, we will clarify the definition of defect. Since we assume the Archimedes' axiom in the neutral geometry, we will denote right angle by angle measure "*180*" (instead of "2*RA*").

2.3 Is there "similar noncongruent triangles" in non-Euclidean geometry?

The defect of a triangle ΔABC is noted by $\delta(\Delta ABC)$, and is given by $\delta(\Delta ABC) = 180$ - (angle sum of ΔABC). Thus $\delta = 0$ for a Euclidean triangle, δ is a positive angle for a triangle in a semi-hyperbolic plane, and δ is the negative of an angle for a triangle in a semi-elliptic plane (or we can call it "excess" instead of "defect" in a semi-elliptic plane).

Proposition 2.3.1: Given a triangle ΔABC, and let D be any point such that $A * D * C$. Then $\delta(ABC) = \delta(ABD) + \delta(BCD)$.

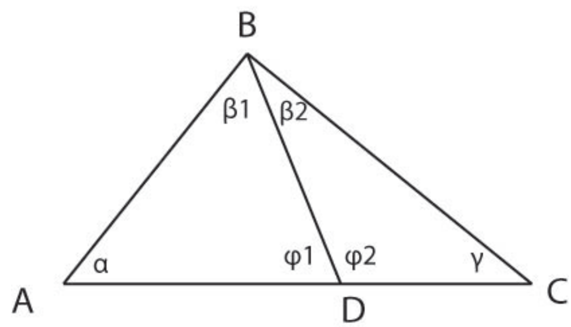


Figure 37

Proof: Let $\sphericalangle A = \alpha$, $\sphericalangle B = \beta_1 + \beta_2$, $\sphericalangle C = \gamma$, $\varphi_1 = \sphericalangle ADB$, $\varphi_2 = \sphericalangle CDB$, since $\sphericalangle ADB$ and $\sphericalangle CDB$ are supplementary, $\varphi_1 + \varphi_2 = 180$.

$$\delta(ABD) + \delta(BCD) = 180 - (\alpha + \beta_1 + \varphi_1) + 180 - (\gamma + \beta_2 + \varphi_2)$$

$$= 360 - (\alpha + \beta_1 + \beta_2 + \gamma) - (\varphi_1 + \varphi_2)$$

$$= 360 - (\alpha + \beta_1 + \beta_2 + \gamma) - 180$$

$$= 180 - (\alpha + \beta_1 + \beta_2 + \gamma)$$

$= \delta(ABC)$, as required.◄

Defect of a quadrilateral □*ABCD* is noted $\delta(\square ABCD)$, and $\delta(\square ABCD) = 360$ - (angle sum of □*ABCD*). It works pretty much the same way as defects for triangles. Thus $\delta = 0$ for a Euclidean quadrilateral, δ is a positive angle for a quadrilateral in a semi-hyperbolic plane, and δ is the negative of an angle for a quadrilateral in a semi-elliptic plane.

Proposition 2.3.2: In a Hilbert plane, if a triangle has defect $\delta = 0$ (respectively positive, negative), then so do all quadrilaterals.

Proof: It works much the same as the last proof. We divide quadrilateral □*ABCD* into two triangles Δ*ABD* and Δ*BCD* (Figure 38), since any the quadrilaterals can be divided into two triangles. Let $\sphericalangle A = \alpha$, $\sphericalangle B = \beta_1 + \beta_2$, $\sphericalangle C = \gamma$, $\sphericalangle D = \varphi_1 + \varphi_2$.

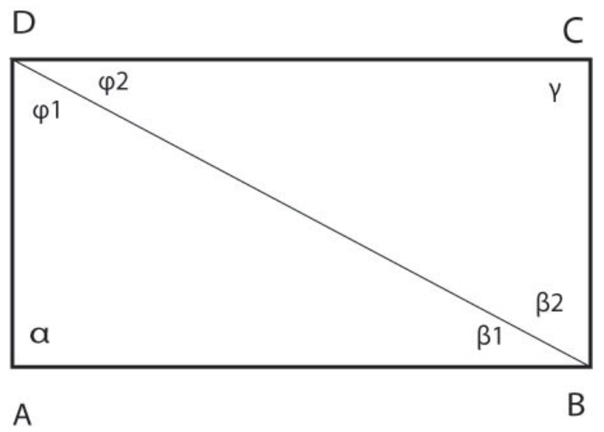


Figure 38

Then

angle sum of $\square ABCD = \measuredangle A + \measuredangle B + \measuredangle C + \measuredangle D$

$= \alpha + \beta_1 + \beta_2 + \gamma + \varphi_1 + \varphi_2$

$= (\alpha + \beta_1 + \varphi_1) + (\beta_2 + \gamma + \varphi_2)$

= angle sum of ΔABD + angle sum of ΔBCD

and

$\delta(\square ABCD) = 360 - (\measuredangle A + \measuredangle B + \measuredangle C + \measuredangle D)$

$= 360 - (\alpha + \beta_1 + \beta_2 + \gamma + \varphi_1 + \varphi_2)$

$= (180 - \alpha - \beta_1 - \varphi_1) + (180 - \gamma - \beta_2 - \varphi_2)$

$= \delta(\Delta ABD) + \delta(\Delta BCD)$ ◄

As we all know from high school elementary geometry, *ASA*, *SAS*, *AAS* and *SSS* implies two angles are congruent, while *AAA* implies that two triangles are similar, but not necessarily congruent. Let us see how *AAA* works in non-Euclidean plane.

Proposition 2.3.3 (No similarity): In a plane satisfying the acute or obtuse angle hypothesis, if two triangles are similar, then they are congruent. Thus, *AAA* implies congruence of triangles.

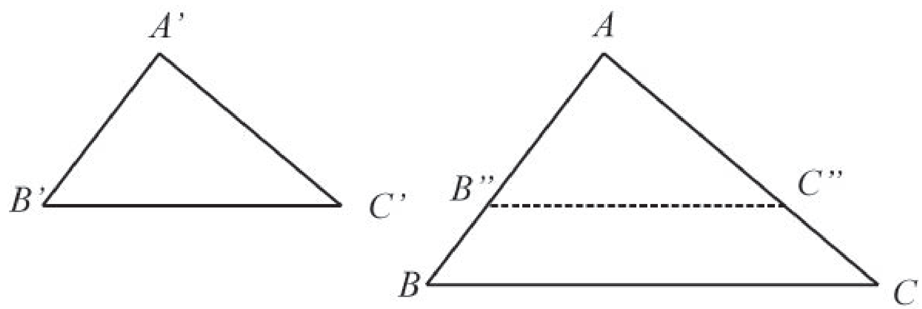


Figure 39

Proof: In order to proof this, we can assume on the contrary that there exists triangles ΔABC and $\Delta A'B'C'$ which are similar, but not congruent. Thus, corresponding angles are congruent, but corresponding sides are not. In fact, no corresponding pair of sides may be congruent, or by ASA, the triangles would be congruent. Consider the triples (*AB, AC, BC*) and (*A'B', A'C', B'C'*) of sides of these triangles. One of these triples must contain at least two segments that are larger than the two corresponding segments of the other triple. Suppose that $AB > A'B'$ and $AC > A'C'$. This means that we can find points *B''* and *C''* on sides *AB* and *AC* respectively such that $AB'' \cong A'B'$ and $AC'' \cong A'C'$. By *SAS*, $\Delta AB''C'' \cong \Delta A'B'C'$ and corresponding angles are congruent, in particular, angle $\measuredangle AB''C'' \cong \measuredangle A'B'C' \cong \measuredangle ABC$ and $\measuredangle AC''B'' \cong \measuredangle A'C'B' \cong \measuredangle ACB$.

We have $\delta(\Delta ABC) = \delta(\square BCC''B'') + \delta(\Delta B''C''A) = \delta(\square BCC''B'') + \delta(\Delta A'B'C') = \delta(\square BCC''B'') + \delta(\Delta ABC)$. Thus $\delta(\square BCC''B'') = 0$. This contradicts the fact that the defect of a triangle cannot be zero in a non-Euclidean plane. Thus $\Delta ABC \cong \Delta A'B'C'$. ◄

This implies that these are no similar triangles that are not congruent in a non-Euclidean plane. Furthermore, the length of the sides of any triangle are determined by its angles.

We will see more about angle determining length when we come to the topics of limiting parallel and angle of parallelism.

2.4 Eliminating the obtuse angle type *iii)*

2.4.1 Eliminating the obtuse angle by Archimedes' axiom

Theorem 2.4.1 (Saccheri - Legendre Theorem): In a Hilbert plane in which Archimedes' axiom holds, the angle sum of every triangle is less than or equal to two right angles. ([H] p.320 & [G1] p.186)

Proof: We begin the proof with a contradiction. Let ΔABC be a triangle with angle sum that is greater than two right angles (Figure 40). Let ε be a non-zero angle, such that the angle-sum of ΔABC is equal to two right angles plus ε. Let ΔDEF be another triangle with the same angle sum as ΔABC, which also has one angle $\alpha < \varepsilon$. Then sum of the remaining two angles in ΔDEF will be more than two right angles, which contradicts Euclid I.17.

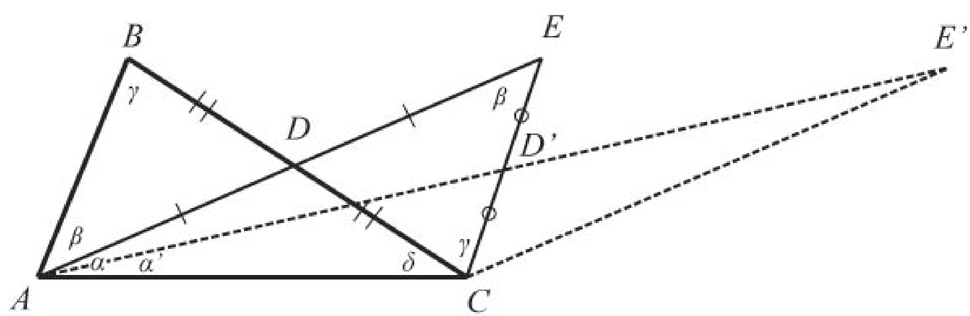


Figure 40

Now we are going to show how we can construct a triangle with one angle that is less than ε by using Archimedes' axiom. Given triangle ΔABC, let D be the midpoint of BC. Let E be the point on the ray opposite to $\overrightarrow{DA}$ such that $DE \cong DA$. Now the vertical angles at D are equal (Proposition 1.1.12). The triangles ΔABD and ΔECD are congruent by *SAS*. This means $\sphericalangle BAD = \sphericalangle DEC = \beta$, and $\sphericalangle ABD = \sphericalangle DCE = \gamma$. Let $\sphericalangle DAC = \alpha$, and $\sphericalangle DCA = \delta$; then the angle sum of both triangles is equal to $\alpha + \beta + \gamma + \delta$. The angle at A is $\alpha + \beta$, so one of them satisfies α *(or* β*)* $\leq 1/2 \sphericalangle ABC$. Then we can repeat the process by construct a new triangle $\Delta AE'C$, where D' is the midpoint of EC, and E' a point on the ray opposite to $\overrightarrow{D'A}$ such that $D'E' \cong D'A$. $\Delta AE'C$ will have the same angle sum as ΔAEC. This means that the angle sum equals ΔABC, so $\alpha' \leq \alpha \leq 1/2 \sphericalangle ABC$. By repeating this construction enough times, let us say n times, by Archimedes' axiom, we will have a triangle with one angle less than ε. ◄

2.4.2 Eliminating the obtuse angle by Aristotle's axiom

Theorem 2.4.2 (Non-obtuse-angle theorem): A Hilbert plane satisfying Aristotle's axiom is either semi-Euclidean or satisfies the acute angle hypothesis (so that, by Uniformity theorem, the angle sum of every triangle is less than or equal to two right angles).

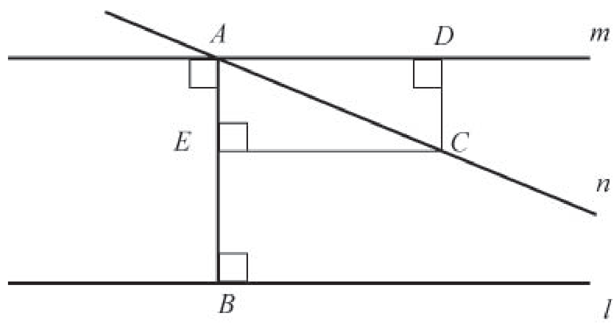


Figure 41

Proof: We begin the proof by assuming the contrary; that the angle sum of every triangle is greater than two right angles (using the uniformity theorem), hence the fourth angle of every Lambert quadrilateral is obtuse. When there exists no Lambert quadrilateral which is also a rectangle, the Hilbert's Euclidean parallel postulate fails in this plane. This means that there exists a line *l* and a point *A* not on *l* such that at least two distinct lines parallel to *l* pass through *A*. So drop a perpendicular from *A* to *l* with foot *B*. Let *m* be a line through *P* and parallel to *l*, and let *n* be another parallel. Let *C* be a point on *n* that is in the interior of ∢*mAB*. Drop a perpendicular from *C* to *m* with foot *D*. Let E be the foot of the perpendicular from *C* to *AB*, then *A* * *E* * *B*. In the Lambert quadrilateral □*AECD*, ∢C is obtuse. By Lemma 2.2.4, *CD* < *AE* < *AB*. Then the perpendicular segment *CD* for ∢*CAD* is bounded above by the fixed segment AB, contradicting Aristotle's axiom. ◄

2.4.3 Eliminating the obtuse angle by Ҁ stament

Ҁ **statement:** Give any segment *AB*, and a line *l* through *B* perpendicular to *AB*, we let *r* be a ray of *l* with vertex *B*. If θ is any acute angle, then there exists a point *C* on *r* such that ∢*ACB* < θ.

Theorem 2.4.3: A Hilbert plane which is non-Euclidean satisfying C statement satisfies the acute angle hypothesis (so that, by Uniformity theorem, the angle sum of every triangle is less than two right angles).

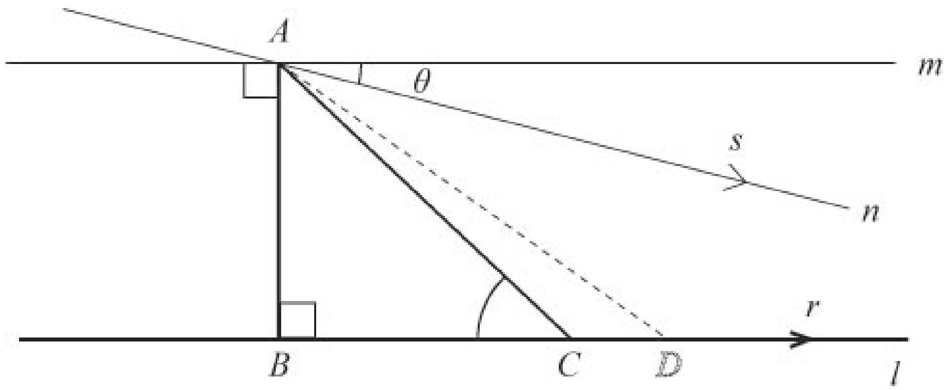


Figure 42

Proof: Given a line *l* and a point *A* not on *l*, we let a point *B* on *l* be the foot of the perpendicular from *A* to *l* (Proposition 1.1.17). Let *m* through *A* be a parallel to *l* (Proposition 1.2.2), and let *n* be a second parallel to *l* through *A*, making an acute angle θ with *m*. Let *s* be the ray of *n* with vertex *A* on the same side of *m* as *l*. If we let *D* be a point that is further away from *B* on *l*, such that $B * C * D$, then we will have an angle $\measuredangle ADB < \measuredangle ACB < \theta$, by Theorem 1.2.7. So the C statement says that as *D* moves further away from *B* on the ray *r* of *l* with vertex *B*, the angle $\measuredangle ADB$ goes to zero.

Since $l \,//\, n$, the ray *s* on *n* does not intersect the ray *r* on *l*. The ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and *s*; otherwise it would violate Theorem 1.1.14 (The Crossbar theorem). Then we have:

$$\measuredangle ADB + \measuredangle DAB < \theta + \measuredangle DAB < 90^o$$

Then the right triangle ΔADB has angle sum less than 180^o, which means that the obtuse angle hypothesis does not hold. ◄

2.4.4 Archimedes' axiom ⇒ Aristotles' axiom

Proposition 2.4.4: In a Hilbert plane with Archimedes' axiom, Aristotle's axiom holds, namely, given any acute angle, the perpendicular from a point on one arm of the angle to the other arm can be made to exceed any given segment. ([H] p.324)

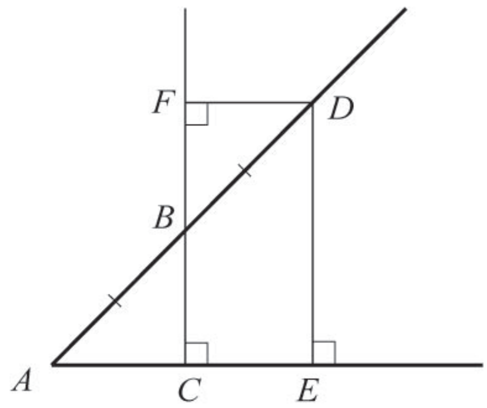


Figure 43

Proof: Given a segment *BC*, and an angle ∢*CAB* such that *BC* is perpendicular to the ray $\overrightarrow{AC}$. We want to show that there exists another such perpendicular with is greater or equal to 2BC. Let *D* be a point on the ray $\overrightarrow{AB}$ such that *AB* ≅ *BD*, and drop a perpendicular from *D* to $\overrightarrow{AC}$ at *E*. Then drop a perpendicular from *D* to $\overrightarrow{CB}$ on *F*, such that *C* * *B* * *F*. Now we have the congruent triangles Δ*ABC* and Δ*DBF* by *AAS*, and it follow that *CF* ≅ *2BC*. Since we have assumed Archimedes' axiom, the angle *D* at the quadrilateral □*FCDE* cannot be obtuse (Saccheri - Legendre theorem). Then by Lemma 2.2.4 *DE* ≥ *FC* ≅ *2BC*.
◄

From now on we will be focused on the hyperbolic geometries, where there will be more than one line though a given point parallel to a given line not containing that point. To be more precise, a hyperbolic plane is a Hilbert plane where Hilbert's hyperbolic axiom of parallelism holds. For such geometries case *ii)* of the uniformity theorem holds. More about hyperbolic geometry will be explained later. We will start with limiting parallel; it is an important part of hyperbolic geometry.

2.5 Limiting parallel

There are two types of parallel lines. One we can call hyperparallel: If the lines *l* and *m* have a common perpendicular, then *l* and *m* are hyperparallel. The other is horoparallel (some may call it critically parallel), but we are going to refer to it as limiting parallel. ([M] p.339/341). As we mentioned earlier, the base and the summit lines in a Saccheri quadrilateral are hyperparallel lines. Now we are going to focus on limiting parallel lines.

Definition 2.5.1: Given a line *l* and a point *A* not on *l*, we let *B* be the foot of the perpendicular from *A* to *l*. Let $\overrightarrow{Bb}$ be a ray on *l* with vertex *B*. *A* limiting parallel ray to $\overrightarrow{Bb}$ emanating from *A* is a ray $\overrightarrow{Aa}$ that does not intersect *l* and such that, every ray *n* in the interior of the angle ∢*BAa*, intersects the line *l*. In symbols we write $\overrightarrow{Aa}///\overrightarrow{Bb}$. ([H] p.312 & [G1] p.259)

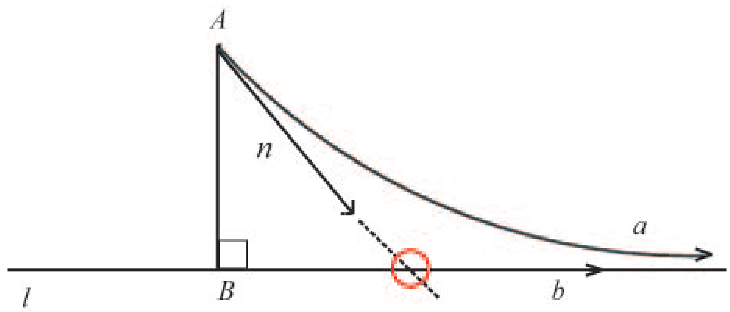


Figure 44

Ray $\overrightarrow{Aa}$ that is shown in this figure above is a right limiting parallel ray to *l* emanating from *A*. Similarly, there would be a unique left limiting parallel ray to *l* emanating from *A*. These two rays, situated symmetrically about *AB*, are the only limiting parallel rays to *l* through *A*.

Hyperbolic axiom (Hilbert's Hyperbolic Axiom Of Parallelisms): For each line *l* and each point *A* not on *l* , there are two rays $\overrightarrow{Aa}$ and $\overrightarrow{Aa'}$ from *A*, not lying on the same line, and not meeting *l* , such that any ray $\overrightarrow{An}$ in the interior of the angle ∢*aAa'* meets *l*. ([H] p.374) (Figure 45)

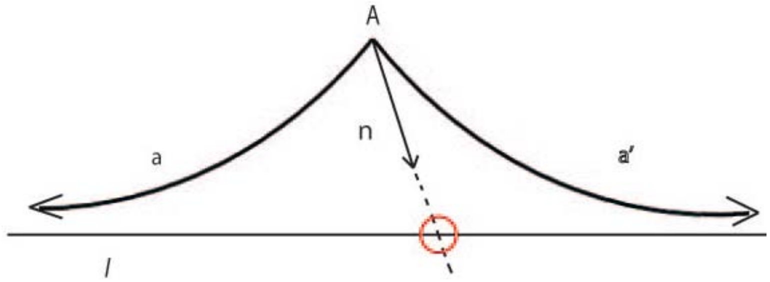


Figure 45

Euclid's fifth postulate, known as the parallel postulate obviously does not hold in this case. This implies that the geometry is non-Euclidean. Then assumption 1 of these two assumptions we talked about at the beginning of this topic seems to be fitting in here. Because the two rays $\overrightarrow{Aa}$ and $\overrightarrow{Aa'}$ lie on the distinct line through *A* that will both be parallel to *l* ([H] p.374).

Proposition 2.5.2: If a ray $\overrightarrow{Aa}$ is limiting parallel to another ray $\overrightarrow{Bb}$, then also $\overrightarrow{Bb}$ is limiting parallel to $\overrightarrow{Aa}$. ([H] p.314)

Theorem 2.5.3 : Given three rays $\overrightarrow{Aa}, \overrightarrow{Bb}, \overrightarrow{Cc}$, if $\overrightarrow{Aa}/// \overrightarrow{Bb}$ and $\overrightarrow{Bb}///\overrightarrow{Cc}$, then $\overrightarrow{Aa}///\overrightarrow{Cc}$ ([H] p.314 & [M] p.339 & [G1] p.277).

Definition 2.5.4 (Angle Of Parallelism): For any segment *AB*, let *l* be a line perpendicular to *AB* at *B*, and let $\overrightarrow{Aa}$ be the limiting parallel ray to ray $\overrightarrow{Bb}$ on *l*, which exists by the hyperbolic axiom. Then we call the acute angle ∢*BAa* the *angle of parallelism* of the segment *AB*, and we denote by *Π(AB)* ([H] p.374). Let *Π(AB)* = *θ*, then we will call segment $\overline{AB}$ the *segment of parallelism* of the angle *θ*.

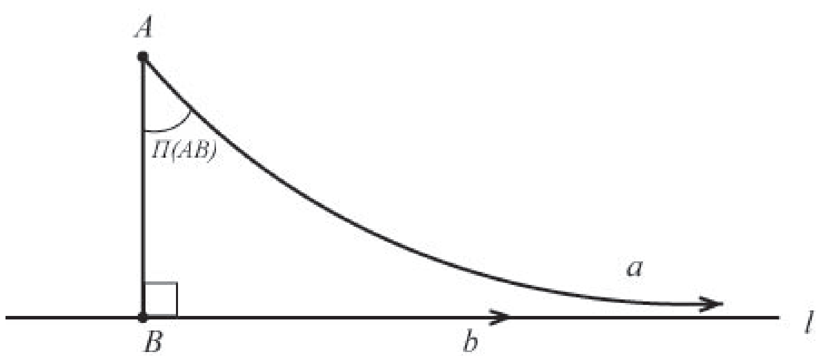


Figure 46

Definition 2.5.5: We call the figure consisting of the segment *AB* and the two limiting parallel rays $\overrightarrow{Aa}$ and $\overrightarrow{Bb}$ a *limit triangle aABb.* If in a limit triangle *aABb* we have $\measuredangle A \cong \measuredangle B$, then we say that the limit triangle is *isosceles.*

Proposition 2.5.6 (Limit Triangle Congruence): Let *aAA'a'* and *bBB'b*' be limit triangles (Figure 47). If $\measuredangle aAA' \cong \measuredangle bBB'$ and $\overline{AA'} \cong \overline{BB'}$, then the limit triangles *aAA'a'* and *bBB'b*' are congruent. Furthermore, we have that $\measuredangle a'A'A \cong \measuredangle b'B'B$ ([M] p.295).

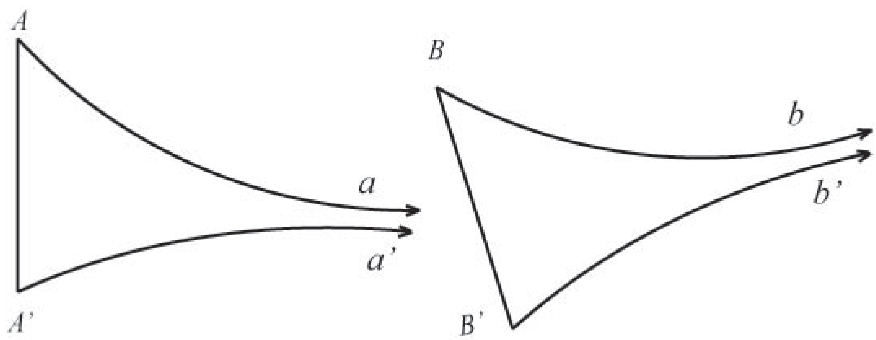


Figure 47

Proposition 2.5.7: There is a ray $\overrightarrow{Aa'}$ emanating from *A*, with *a'* on the opposite side of *AB* from *a,* such that $\overrightarrow{Aa'}$ is other limiting parallel ray to *l.* And $\measuredangle BAa' \cong \measuredangle BAa$.Rays $\overrightarrow{Aa'}$ and $\overrightarrow{Aa}$ are the only two limiting parallels rays to *l* through *A* (Figure 48). Note that also that the angle of parallelism *Π(AB)* is necessarily acute, because the two limiting parallels from *A* to *l* do not lie on the same line by Hyperbolic axiom. ([G1] p.260)

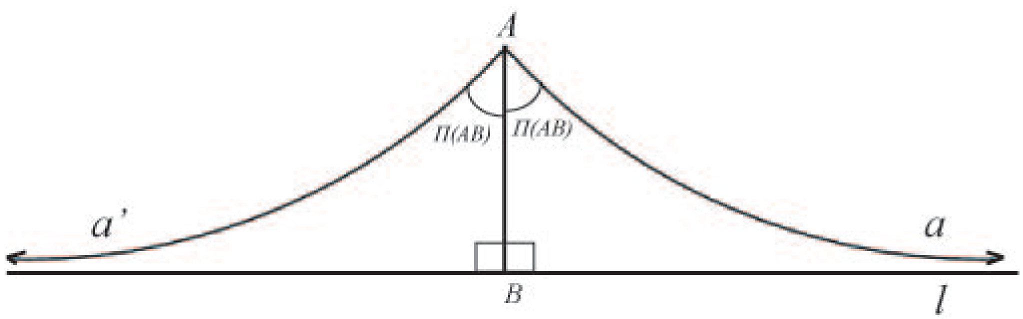


Figure 48

Proposition 2.5.8: The angle of parallelism varies inversely with the segment:

a) $AB < A'B' \Leftrightarrow \Pi(AB) > \Pi(A'B')$

b) $AB \cong A'B' \Leftrightarrow \Pi(AB) = \Pi(A'B')$

Proof:

a) Suppose $AB < A'B'$. Find a point C on ray $\overrightarrow{AB}$, such that $A * B * C$, and $AB < AC \cong A'B'$. Draw a perpendicular line l to AC at C, and let $\overrightarrow{Aa}$ be the ray emanating from A and is limiting parallel to ray $\overrightarrow{Cc}$ on l. Then we have the angle of parallelism $\Pi(AC) = \Pi(A'B') \cong \sphericalangle CAa$ (Figure 49).

Let the ray $\overrightarrow{Bb}$ be a ray that is perpendicular to AB at B, and which is on the same side of AB as ray $\overrightarrow{Aa}$.

- Suppose ray $\overrightarrow{Bb}$ does not intersect ray $\overrightarrow{Aa}$: Since $A * B * C$, ray $\overrightarrow{Bb}$ would be between $\overrightarrow{Aa}$ and $\overrightarrow{Cc}$, and it would not intersect either of these two rays. Then by Theorem 2.5.3, $\overrightarrow{Bb}$ is limiting parallel to both $\overrightarrow{Aa}$ and $\overrightarrow{Cc}$. Since $\overrightarrow{Bb}$ /// $\overrightarrow{Cc}$, the angle of parallelism to the segment BC will be a right angle, which contradict the fact that the angle of parallelism is always acute.

- Hence, ray $\overrightarrow{Bb}$ intersects ray$\overrightarrow{Aa}$: $\overrightarrow{Aa}$ is not limiting parallel to $\overrightarrow{Bb}$. Let ray $\overrightarrow{Aa'}$ be the ray emanating from A and is limiting parallel to $\overrightarrow{Bb}$. Then $\sphericalangle BAa' > \sphericalangle BAa \cong \sphericalangle CAa$. This means that the limiting parallel $\overrightarrow{Aa'}$ for the segment AB makes an angle $\Pi(AB)$, which is greater than $\Pi(AC)$.

Reversing the roles of AB and $A'B'$ we find that if $AB > A'B'$, then $\Pi(AB) < \Pi(A'B')$.

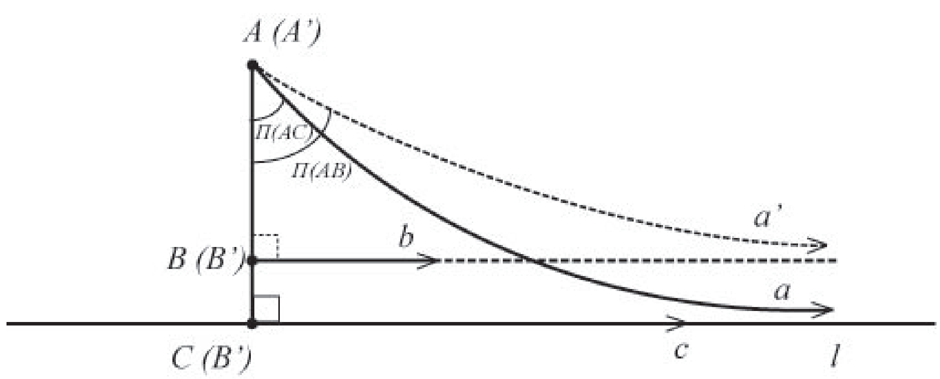


Figure 49

b) Assume $AB \cong A'B'$ and on the contrary $\Pi(AB) > \Pi(A'B')$. Draw a perpendicular line l to AB at B, and let $\overrightarrow{Aa}$ be the ray emanating from A which is limiting parallel to l. Then we have that angle of parallelism $\sphericalangle BAa \cong \Pi(AB)$. Let $A'B'$ also be perpendicular to l at B', and let the ray $\overrightarrow{A'a'}$ emanating from A' to l. Then $\sphericalangle BAa' \cong \Pi(A'B')$. As we know that the angle of parallelism is the smallest angle we can have before ray $\overrightarrow{Aa}$ from A (i.e. $\overrightarrow{Aa'}$) intersecting l. So $\overrightarrow{Aa}$ is limiting parallel to l, and $\sphericalangle BAa > \sphericalangle BAa'$. This means ray $\overrightarrow{Aa}$ must intersect l. This contradicts the definition of the limiting parallel. The other contradictions can be shown in a similar way. An easier way to prove that $AB \cong A'B'$ is by using Theorem 2.5.6; it then follows $\Pi(AB) = \Pi(A'B')$.◄

Proposition 2.5.9: If AB is a segment, with limiting parallel rays emanating from A and B, then the exterior angle β at B is greater than the interior angle α at A ([H] p.376).

Proposition 2.5.10: Let the limit triangle $aABb$ be isosceles. Let C and D be points on $\overleftrightarrow{Aa}$ and $\overleftrightarrow{Bb}$, respectively, where C and D are on the same side of $\overleftrightarrow{AB}$ with $\overline{AC} \cong \overline{BD}$. Then the limit triangle $aCDb$ is also isosceles (Figure 50).

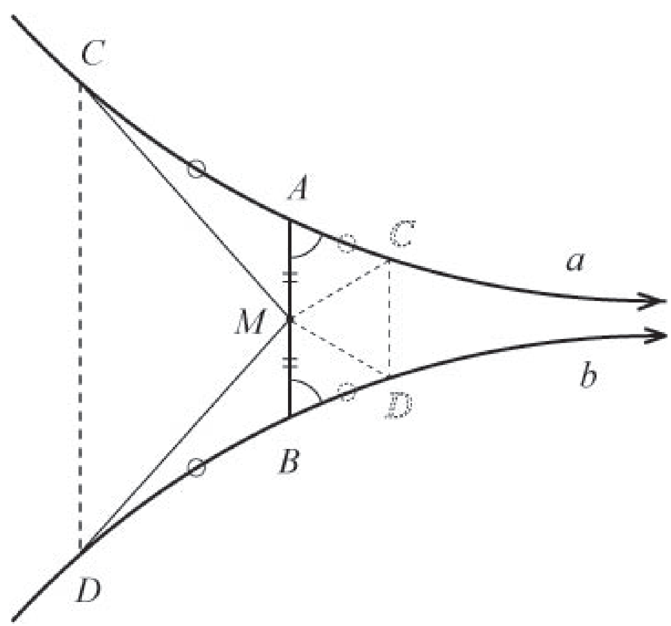


Figure 50

Proof: Let M be the midpoint of $\overline{AB}$; then by *SAS*, $\Delta AMC \cong \Delta BMD$. So $\measuredangle ACM \cong \measuredangle BDM$ and $\overline{CM} \cong \overline{DM}$. Then ΔCDM is isosceles, and $\measuredangle MCD \cong \measuredangle MDC$. Thus, $\measuredangle ACD \cong \measuredangle ACM + \measuredangle MCD \cong \measuredangle BDM + \measuredangle MDC \cong \measuredangle BDC$, and the limit triangle *aCDb* is also isosceles. ◄

Proposition 2.5.11: Suppose we have a line *l*, as well as rays $\overrightarrow{Aa}$ and $\overrightarrow{Bb}$, as in Definition 2.5.1. We can then find a point *B'* on *l* such that $\measuredangle AB'b \cong \measuredangle B'Aa$. Hence, the limit triangle a*AB'b* is isosceles (Figure 51).

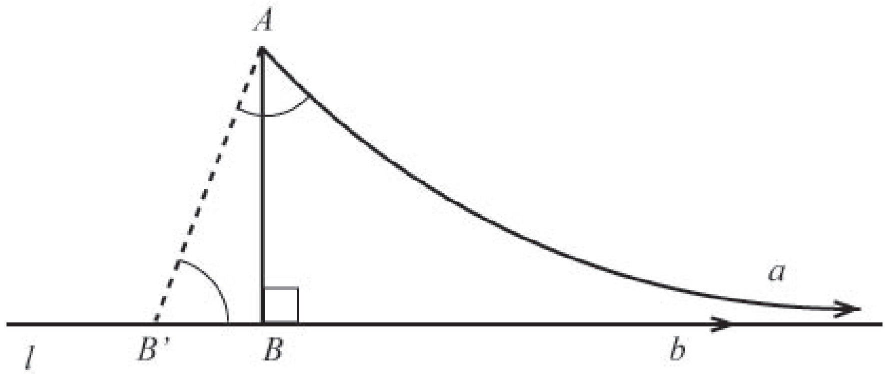


Figure 51

Proof: Since $\overrightarrow{Aa} /// \overrightarrow{Bb}$, the angle bisector of $\measuredangle BAa$ intersect $\overrightarrow{Bb}$ at some point *C*. Then the angle bisector of $\measuredangle ABb$ must intersect $\overline{AC}$ at some point *O* (Theorem 1.1.14). Let *X, Y, Z* be the foot of the perpendicular from *O* to $\overleftrightarrow{AB}$, $\overleftrightarrow{Aa}$ and $\overleftrightarrow{Bb}$, respectively (Figure 52).

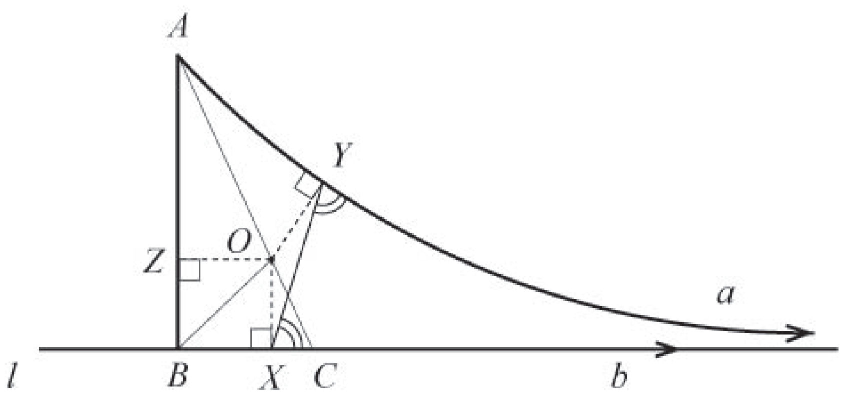


Figure 52

By *SAA*, $\Delta AZO \cong \Delta AYO$, then $\overline{OZ} \cong \overline{OY}$. Again by *SAA*, $\Delta BZO \cong \Delta BXO$; we then get $\overline{OZ} \cong \overline{OX}$. Hence $\overline{OY} \cong \overline{OX}$. The triangle ΔXYO is then isosceles, so $\measuredangle OXY \cong \measuredangle OYX$.

Then, $\sphericalangle YXb \cong$ *right angle* - $\sphericalangle OXY \cong$ *right angle* - $\sphericalangle OYX \cong \sphericalangle XYa$. Hence the limit triangle *aYXb* is isosceles. Then by Proposition 2.5.10, we can find a point *B′* on *l* such that the limit triangle a*AB'b* is isosceles. ◄

When the angle bisectors of $\sphericalangle BAa$ and $\sphericalangle ABb$ are perpendicular to $\overrightarrow{Bb}$ and $\overrightarrow{Aa}$, respectively, then by *SAA*, $\Delta AZO \cong \Delta BDO$, hence $\overline{AZ} \cong \overline{BZ}$. Then $\overleftrightarrow{ZO}$ is the perpendicular bisector of $\overline{AB}$ (Figure 53).

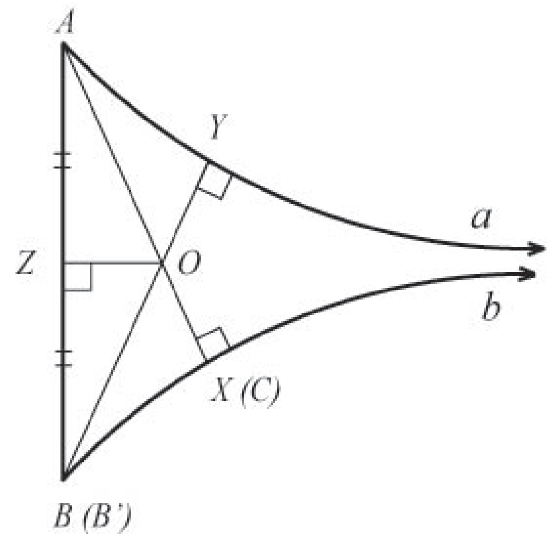


Figure 53

Proposition 2.5.12: Let the limit triangle *aAA'a* be isosceles with $\sphericalangle aAA' \cong \sphericalangle A'Aa$, then the perpendicular bisector of $\overline{AA'}$ is limiting parallel to both $\overrightarrow{Aa}$ and $\overrightarrow{A'a'}$ (Figure 54).

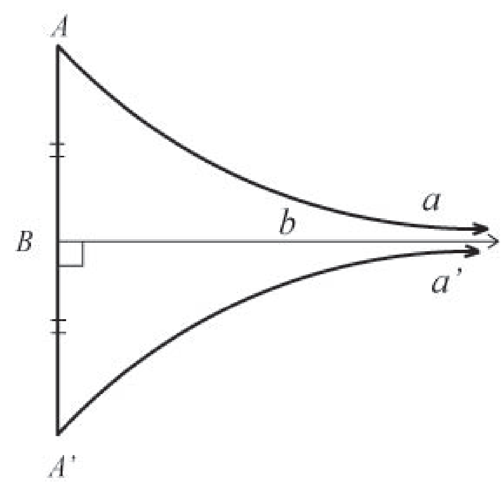


Figure 54

Proof: Let *B* be the be the midpoint of $\overline{AA'}$. Let $\overrightarrow{Bb}$ be a ray such that $\overrightarrow{Bb}$ ///$\overrightarrow{Aa}$. Then by Theorem 2.5.3 $\overrightarrow{Bb}$ ///$\overrightarrow{A'a'}$, since $\overrightarrow{Aa}$ ///$\overrightarrow{A'a'}$. The limit triangles *aABb* and *a'A'Bb* are congruent by Proposition 2.5.6. So ∢*ABb* ≅ ∢*A'Bb*, and they must both be right angles (Definition 1.1.16). Thus $\overleftrightarrow{Bb}$ is the perpendicular bisector of $\overline{AA'}$. ◄

Based on Proposition 2.5.12 and Definition 2.5.4, we know that $\overline{AB} \cong \overline{BA'} \cong 1/2\ \overline{AA'}$, and ∢*BAa* ≅ ∢*BA'a'* = *Π(1/2* $\overline{AA'}$).

Proposition 2.5.13 : For any acute angle *θ*, there exists a line that is limiting parallel to one arm of the angle and orthogonal to the other arm of the angle. In particular, there is a segment whose angle of parallelism is equal to *θ* ([H] p.380).

Combining with proposition 2.5.8, we see that there is a one-to-one correspondence between the set of congruence classes of line segments and the set of congruence equivalence classes of acute angles, given by associating a segment *AB* to its angle of parallelism *θ* = *Π(AB)*.

Recall that the angle of parallelism in Euclidean geometry is a constant right angle, regardless of the length of a segment *AB*. In hyperbolic plane, as the length of the segment *AB* gets longer, the angle of parallelism will get closer and closer to zero, but as the length of the segment *AB* gets shorter, the angle of parallelism will get closer and closer to a right angle. It is quite interesting that the closer the point *A* gets to line *l*, the more it behaves like it would in a Euclidean plane.

We have already shown that for every line *l* and any point *A* not on *l,* there exists a unique limiting parallel line from *A* to *l* with its corresponding angle of parallelism. Furthermore, for every acute angle, there exists a corresponding segment of limiting parallel. But, can we construct it? By construct we mean draw it by only using straightedge and compass. We will discuss constructability in more details later on.

Now we are going to look into the limiting parallel construction Bolyai provided. This construction gives the angle of parallelism corresponding to a given segment.

Bolyai's construction of the limiting parallel ray.

In the hyperbolic plane, let *l, m* be hyperparallel lines with a common perpendicular *AB*. Where *A* is on *m*, and *B* is on *l*. Let *C* be a point on *l* that is different from *B*. Let *D* on *m* be the foot of the perpendicular from *C* to *m*. Then the circle of radius *BC* and center *A* will intersect the segment *CD* at a point *E*. Ray $\overrightarrow{AE}$ is the limiting parallel to *l* through *A*. Therefore angle ∢*BAE* is the angle of limiting parallel to *AB*. We can find a proof for this construction in Hartshorne's book ([H] p.397).

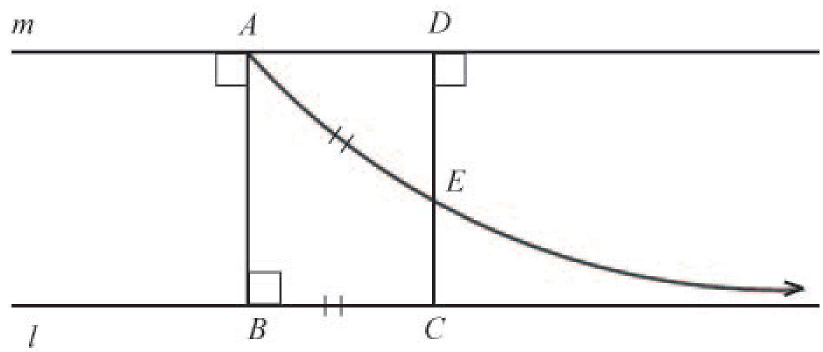


Figure 55

Now, the reverse of the previously construction. To construct a segment by a given acute angle.

Before we introduce the next construction, we will assume Dedekind's axiom.

The following construction gives the segment of parallelism corresponding to a given acute angle.

Theorem 2.5.14 (George Martin's Theorem): In a hyperbolic plane, given an acute angel ∢*BAC*, suppose *C* is the foot of the perpendicular from B to $\overleftrightarrow{AC}$. Let point D be constructed on $\overleftrightarrow{BC}$ such that ∢*CAD* ≅ *Π(AB)*. Then ∢*BAC* is the angle of parallelism for segment *AD*. ([M] p.484) (Figure 56)

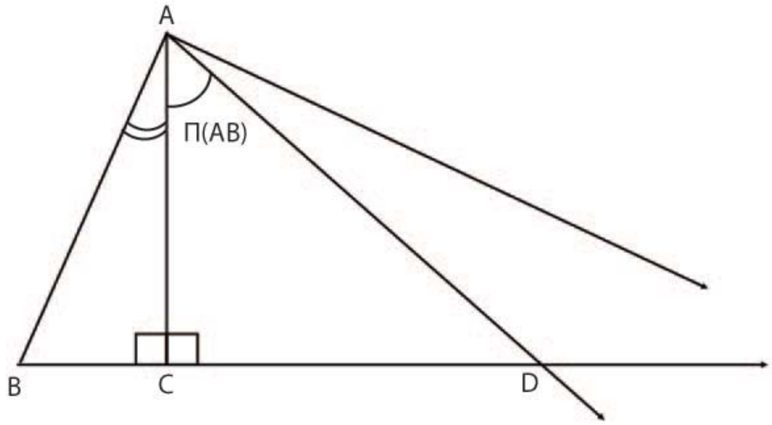


Figure 56

Here is the proof for this theorem: In the hyperbolic right triangle Δ*ABC*, *AB* > *AC* (hypotenuse greater than leg), then *Π(AB)* < *Π(AC)* (Proposition 2.5.8). Hence the point *D* exists. By the formulas below we get

$$\cos \sphericalangle BAC = \frac{\tanh AC}{\tanh AB} = \frac{\tanh AC}{\cos \Pi(AB)} = \tanh AD = \cos \Pi(AD), \text{ so } \sphericalangle BAC = \Pi(AD). \blacktriangleleft$$

Formulas of hyperbolic trigonometry that are used in this proof:

$\cos \Pi(x) = \tanh x$ ([G1] p.491)

In any right triangle Δ*ABC* in the hyperbolic plane, with ∢C right. Then $\cos A = \frac{\tanh b}{\tanh c}$ ([G1] p.492).

2.6 Hyperbolic plane

Definition 2.6.1: A *Hyperbolic plane* is a Hilbert plane in which Hilbert's hyperbolic axiom of parallelism holds.

This means that, given a line *l* and a point *A* not on *l*, there are at least two lines through *A* which do not intersect *l*. Obviously, a hyperbolic plane is non-Euclidean. Case *ii)* of the Uniformity theorem holds in hyperbolic plane. This means the angle sum of every triangle is less than two right angles (equivalently, the fourth angle of every Lambert quadrilateral and the summit angles of every Saccheri quadrilateral are acute). It is a famous result of hyperbolic geometry that the angle sum of a triangle is always less than two right angles.

Theorem 2.6.2: In a hyperbolic plane, the angle sum of a triangle is always less than two right angles.

Proof: As we proved earlier that, for any triangle there is a Saccheri quadrilateral for which the sum of its two summit angles is equal to the sum of the three angles of the triangle (Proposition 2.2.8). So we have only to prove that the summit angles for any Saccheri quadrilateral are acute.

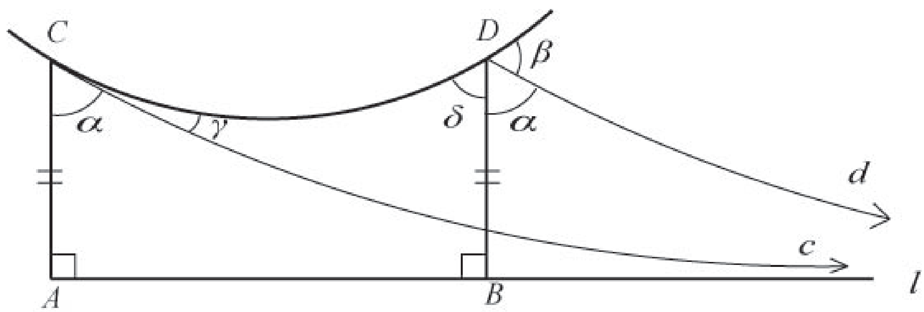


Figure 57

Let □*ABCD* be a Saccheri quadrilateral with base $\overleftrightarrow{AB} = l$. Let us denote the angles as shown in the figure 57. Draw limiting parallel rays $\overrightarrow{Cc}$, $\overrightarrow{Dd}$ to l, from *C* and *D*, respectively. Then by Proposition 2.5.8, the angles of parallelism α are equal.

By Proposition 2.5.9, $\beta > \gamma$. We know that the summit angles of a Saccheri quadrilateral are congruent to each other (Proposition 2.2.2), this means $\alpha + \gamma \cong \delta$.

We conclude that $\alpha + \beta > \alpha + \gamma = \delta$, then δ must be acute. ◄

Theorem 2.6.3 (Advanced Theorem)**:** A Hilbert plane satisfying the acute angle hypothesis (*ii*), Aristotle's axiom and the line-circle continuity principle is hyperbolic; and limiting parallel rays exist for every line l and every point P not on l. ([G] p.210 & [G1] p.258)

A slightly stronger version of the *Advanced Theorem***:** A non-Euclidean Hilbert plane is hyperbolic if and only if it satisfies the Line-Circle axiom and Aristotle's axiom ([G] p.210).

2.6.1 Aristotle's axiom holds in hyperbolic planes

Proposition 2.6.4: In a hyperbolic plane, Aristotle's holds, namely, given an acute angle α and a segment *AB* in hyperbolic plane, there exists a point *E* on one arm of the angle such that the perpendicular *EF* from *E* to the other arm of angle is greater than *AB* (In other words, the *perpendicular segments from one side of an acute angle to the other is unbounded*) ([H] p.380).

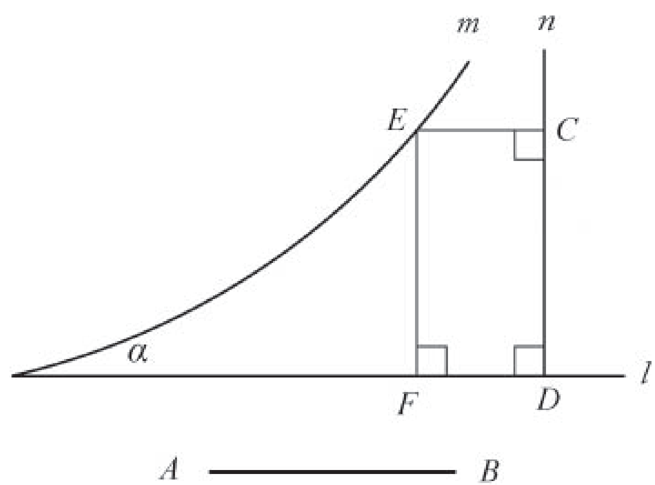


Figure 58

Proof: Given a segment *AB*, and an acute angel *α* with arms *l* and *m*. Let *n* be the limiting parallel to one arm *m*, and intersect the other arm perpendicularly at a point *D* (Proposition 2.5.13). Let *C* be a point on *n* such that *CD* ≅ *AB.* Draw a perpendicular to *m* at *C*, and let it intersect *m* at *E.* Drop a perpendicular from *E* to *l* with foot *F.* We now have a Lambert quadrilateral □*CDEF,* the angle at *E* (∢*CEF*) must be acute. therefore *EF* > CD ≅ *AB*, as required. ◄

2.6.2 More about Dedekind's axiom

We invoked Dedekind's axiom to prove the existence of limiting parallel rays as well as Aristotle's axiom and the acute angle hypothesis. Without Dedekind's axiom, there exist many non-Euclidean Hilbert planes which are not hyperbolic ([G] p.209).

Theorem 2.6.5: In a non- Euclidean plane satisfying Dedekind's axiom, Hilbert's hyperbolic axiom of parallels holds, as do Aristotle's axiom and acute angle hypothesis. ([G1] p.260)

As we mentioned earlier, a non-Euclidean plane is a Hilbert plane in which Hilbert's hyperbolic axiom of parallels holds.

Definition 2.6.6 (Real hyperbolic plane):A non-Euclidean plane satisfying Dedekind's axiom is called a *real hyperbolic plane* ([G1] p.262).

Chapter 3

3. Squaring circles in the hyperbolic plane

Before we start our discussion in this topic, we will assume Dedekind's axiom. As we mentioned in earlier topics - a Hilbert plane satisfying Dedekind's axiom is either *real Euclidean* or *real hyperbolic*. With respect to what Jagy wrote in his article, we are going to use number measurement as a language for situations where it simplifies the statements.

"Squaring the circle" is one of the most famous geometric problem in the world. There were many attempts to square the circle over centuries, and many approximate solutions. By "squaring the circle", we mean the act of drawing circles and squares of equal areas or perimeters by using only straightedge and compass. There is many interesting methods for achieving circles and squares of equal perimeters ([F]), but in this paper we will just focus on equal areas.

3.1 Circles and squares in Euclidean plane

In the Euclidean plane, a square is defined as a four sided regular polygon with all sides equal and all internal angles right angles. The area of a square is the product of the length of its sides. As for a circle, it is defined as the set of points in a plane that are equally distant from a given point in the plane, call the *center*. The distance between any of the points and the centre is called the *radius*. And the area of a circle is πr^2, where r is the radius.

A length, an angle or other geometric figures is constructible if it can be constructed by straightedge and compass. All constructions start with a given line segment (/angle) which is defined to be *one* unit in length. A real number a is constructible if one can construct a line segment of length $|a|$ in a finite numbers of steps from the given unit segment by using straightedge and compass.

All constructible numbers are algebraic. By algebraic numbers we mean the numbers that are a roots of some polynomial equation with integer coefficients. Also, if a and b are constructible numbers, then $a - b$, ab and a/b (with $b \neq 0$) are constructible.

A number that is not algebraic is said to be transcendental, and transcendental numbers are not constructible. Almost all the irrational numbers are transcendental, and all transcendental numbers are irrational.

Many irrational numbers (e.g., $\sqrt{2}$) are constructible. Let *a* be a positive real number. Figure 59 shows how to construct $\sqrt{a}$. ([B] p.2)

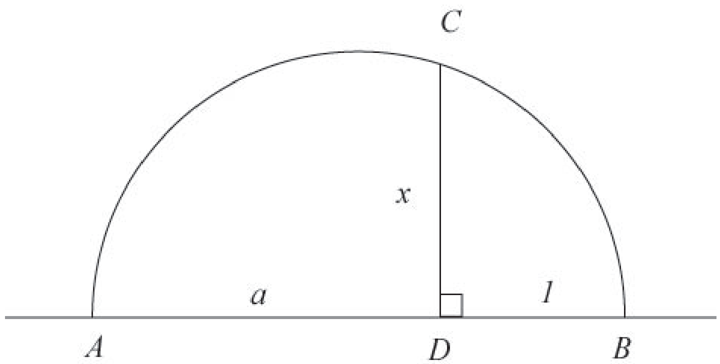


Figure 59

A semicircle is constructed on *AB* as a diameter. *CD* is the perpendicular to *AB* (*C* is the intersection of the perpendicular and the semicircle). Then $x = \sqrt{a}$. By repeat this construction, we can construct $\sqrt[4]{a}$, $\sqrt[8]{a}$ and so on.

To construct a square with area equal to a given circle by only using a straightedge and compass is impossible in the Euclidean plane. This problem is usually called "squaring the circle". If the circle has radius r, its area is πr^2, so we need a square of side a with $a^2 = \pi r^2$. Thus $a = \sqrt{\pi} \cdot r$. If a were constructible from r, then $\sqrt{\pi}$, and hence π would be constructible. But how do we determine a length *π?* To determine *π*, and whether or not *π* is constructible became a popular problem for many mathematicians. Many have tried and failed. Although many did find constructions with close approximations ([B]). The proof of impossibility wasn't obtained until 1882, when Lindemann proved that *π* is transcendental. That is, π is not the solution of any polynomial equation with integer coefficients. This implies that there is no straightedge and compass construction to square a circle.

3.2 Circles and squares in hyperbolic plane

From now on we will just focused on the real hyperbolic plane.

In hyperbolic plane, squares with right angles do not exist. Rather, squares in hyperbolic plane have angles of less than right angles. Indeed, calling them squares is not quite right, because squares don't exists in hyperbolic plane. Rather it should be called *a regular 4-gon* (a quadrilateral with all sides and angles congruent). However, we will still be using the word "square", seeing as how Jagy did that in his article; this is also more convenient for the readers. The area of such a square is equal to its defect. As we mentioned in the last topic, the defect of a quadrilateral $\square ABCD$ is noted $\delta(\square ABCD)$, and $\delta(\square ABCD) = 2\pi$ - (angle sum of $\square ABCD$). In this case, the area of a square is equal to $2\pi - 4\sigma$, where σ is the corner angle of a square. When σ increases, $2\pi - 4\sigma$ decreases. This is why larger squares have smaller angles. We know that both the corner angle σ and the area is non-negative; this means that areas of squares in the hyperbolic plane must be bounded above by 2π. Indeed, the area of a convex polygon with n sides is bounded by $(n - 2)\pi$ (the proof is similar to the proof for Proposition 2.3.2, we just divided it into hyperbolic triangles).

The area of a circle of radius r is $4\pi sinh^2(r/2) = 2\pi(cosh\ r - 1)$ ([G1] p.498). As in this figure for hyperbolic trigonometric functions, *sinh r* and *cosh r* increases when *r* increases; areas of circles are then unbounded.

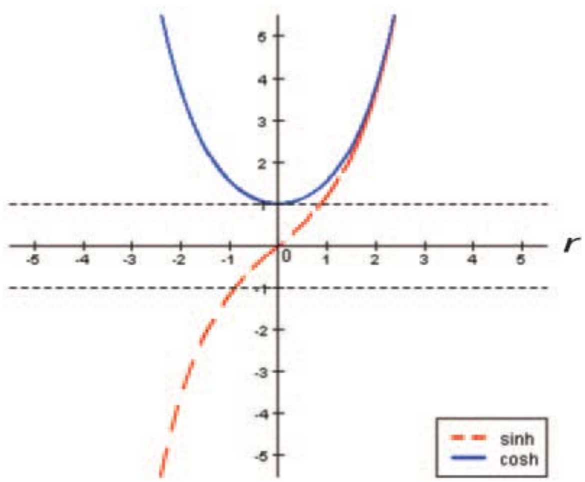


Figure 60

3.3 About constructability

An angle α can be constructed in hyperbolic plane if and only if it can be constructed in Euclidean plane ([G1] p.525). This follows from the relationships between the trigonometric functions and the hyperbolic trigonometric functions, as explained below.

By Mordukhai-Boltovski theorem in Greenberg's book, that in the real hyperbolic plane, a segment of length r is constructible if and only if $\sinh r$ (equivalently, $\cosh r$ or $\tanh r$ or e^r) is constructible. The equivalence follows from the following algebraic relations among e^r, $\sinh r$, $\cosh r$ and $\tanh r$. We will need them to relate hyperbolic distance to Euclidean distance. Recall that hyperbolic sine, hyperbolic cosine, and hyperbolic tangent are defined by following,

$$\sinh r = \frac{e^r - e^{-r}}{2}$$

$$\cosh r = \frac{e^r + e^{-r}}{2}$$

$$\tanh r = \frac{\sinh r}{\cosh r} = \frac{e^r - e^{-r}}{e^r + e^{-r}}$$

For example $r = 1$ is not constructible in the hyperbolic plane, because the number e is transcendental.

Furthermore, in Euclidean plane, by considering a right triangle with any side of length 1, a non-right angle θ is constructible if and only if $\tan\theta$ (equivalently, $\sin\theta$ or $\cos\theta$) is constructible. For an uniquely determined length r in hyperbolic plane, we have $\theta = \Pi(r)$ (refer to the limiting parallel we discussed earlier). θ is constructible in hyperbolic plane if and only if r is constructible in hyperbolic plane. We have already shown both constructions in the last topic (Bolyai's construction and George Martin's Theorem). In hyperbolic plane, consider a right triangle with one side equal to the *Schweikart's segment*, a segment whose angle of parallelism is $\pi/4$, has length *arcsinh 1*; we obtain the formula of Bolyai-Lobachevsky ([G1] p.480), which relates a length r and its angle of parallelism $\Pi(r)$:

$$\tan\frac{\Pi(r)}{2} = e^{-r} \quad \Rightarrow \quad \Pi(r) = 2\arctan e^{-r}$$

By straightforward calculation, we obtain the relationships between the trigonometric functions of θ and the hyperbolic trigonometric functions of r ([G1] p.491).

$$\sin\theta = \sin\Pi(r) = \operatorname{sech} r = \frac{1}{\cosh r} \qquad (1)$$

$$\cos\theta = \cos\Pi(r) = \tanh r \qquad (2)$$

$$\tan\theta = \tan\Pi(r) = \operatorname{csch} r = \frac{1}{\sinh r} \qquad (3)$$

θ is Euclidean constructible ⇔ $\sin\theta$ (equivalently, $\cos\theta$, $\tan\theta$) is Euclidean constructible $\overset{1)\ 2)\ 3)}{\Longleftrightarrow}$ $\cosh r$ (equivalently, $\tanh r$, $\sinh r$) is Euclidean constructible ⇔ r is hyperbolic constructible ⇔ θ is hyperbolic constructible.

This means that θ is constructible in hyperbolic plane iff θ is constructible in Euclidean plane.

To construct a square with area equal to a given circle in Euclidean plane has already been proven impossible. But how will it be in Hyperbolic plane? Is there possible to find a general method of construction that we can use to square the circle?

Bolyai did managed to construct a circle and a square with the same area. Here is an example where both have area equal to π. The area of a circle of radius r is $2\pi(\cosh r - 1)$, simplify the equation $2\pi(\cosh r - 1) = \pi$, we get $\cosh r = 3/2$, which is a constructible number. Hence the circle with radius r has area π. As for the square with corner angle σ , its area is equal to $2\pi - 4\sigma$. Thus, its equal is to π when $\sigma = \pi/4$.

The construction of this unique square with corner angle $\pi/4$ is obtained by first constructing a right angle triangle with acute angles $\pi/4$ and $\pi/8$ and then reflecting it seven times. Such right angle can be constructed by the method in the proof of the right triangle construction theorem ([G1] p.506).

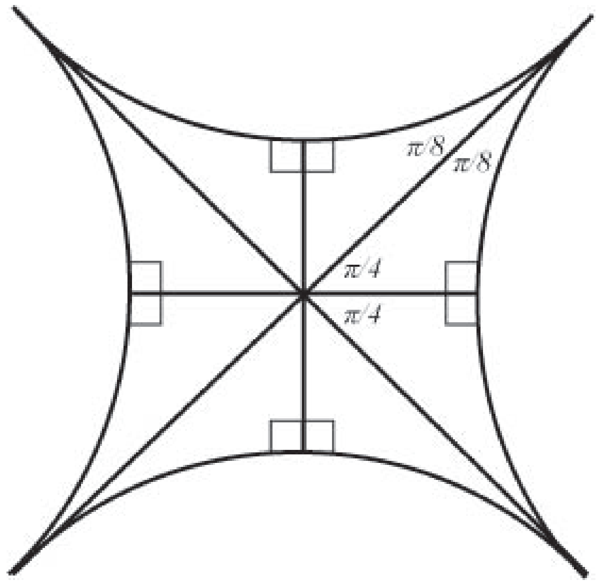


Figure 61

Bolyai did not provide a general method for squaring circles in the hyperbolic plane. And, as Jagy wrote in his article, such method do not exists - for one thing, because areas of circles are unbounded, whereas areas of squares are bounded by 2π. But even if the circle has area $< 2\pi$ and has a constructible radius, the square with the same area may not be constructible ([J]).

But Bolyai did find a remarkable theorem for angle radius construction. This theorem is called *Bolyai's circle-angle theorem* in Greenberg's book ([G1] p.521).

Theorem 3.3.1 (Bolyai's circle-angle theorem): Given an acute angle θ or a segment r, where r can be the radius of a circle, we can construct one from the other by satisfying $\tan \theta = 2\sinh(r/2)$.

As we mentioned earlier, the area of a circle of radius r is $4\pi\sinh^2(r/2)$. Then the area of the circle is also equal to $\pi\tan^2\theta$. There is a constructive correspondence between circles of radius r and acute angles θ such that the area of the circle is $\pi\tan^2\theta$. The constructions as following.

Two more formulas of hyperbolic trigonometry will be needed in the following proofs:

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \qquad (4)$$

In any right triangle ΔABC in the hyperbolic plane, with $\measuredangle$C right. Then

$$\sin A = \frac{\sinh b}{\sinh c} \qquad (5)$$

3.4 Relation between angle and radius

Given either an acute angle with measure θ or a segment of length r, we can construct the other such that $\tan \theta = 2\sinh(r/2)$. Let us begin with θ and construct r.

3.4.1 From angle to radius ($\theta \Rightarrow r$)

Given an acute angel $\measuredangle ABC$ of measure θ, let point D be constructed that $\measuredangle ABD$ is right, points D and C are on the same side of $\overleftrightarrow{AB}$, and $\measuredangle CBD$ is a angel of parallelism for BD, then $\measuredangle CBD = \pi/2 - \theta$. Point D exists by Theorem 2.5.14 (George Martin's Theorem). Let point E be constructed such that $\overrightarrow{BC} /// \overrightarrow{ED}$ and $\measuredangle CBE \cong \measuredangle DEB$, point E exists by Proposition 2.5.11. If BE has length r, then $\measuredangle BED = \Pi(r/2)$ (Proposition 2.5.12).

$$\tan \theta = \cot (\pi/2 - \theta) = \cot \Pi(BD) \overset{3)}{\Leftrightarrow} \sinh BD \overset{5)}{\Leftrightarrow} (\sin \Pi(r/2))(\sinh r) \overset{1)4)}{\Longleftrightarrow} (\operatorname{sech} (r/2))(2\sinh (r/2) \cosh (r/2) \overset{1)}{\Leftrightarrow} 2 \sinh (r/2)$$

$$\Rightarrow \tan \theta = 2\sinh(r/2).$$

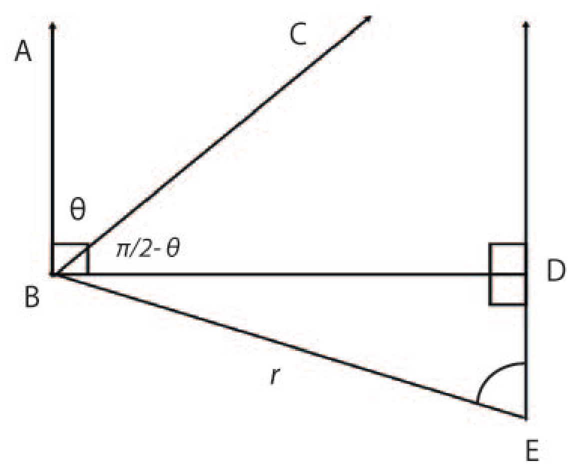


Figure 62

Now, the converse construction. Given a segment with length *r*, construct *θ,* such that *tan θ = 2sinh(r/2).*

3.4.2 From radius to angle ($r \Rightarrow \theta$)

Given a segment *AB* with length *r*. Construct a point *C* on the segment *AB* such that *AC* ≅ *BC = r/2* (Figure 63). Then we can construct a ray *l* through *C* that is perpendicular to *AB*. Construct a ray *m* through the point *A* that is limiting parallel to the ray *l* (Bolyai's construction), and a ray *n* through the point *B* which is also limiting parallel to *l*. Then by Proposition 2.5.12 limit triangle *mABn* is a isosceles. Construct a point *D* on *n* such that ∢*ADB* is right. Then we can construct a ray *k* through the point *A*, such that k is perpendicular to ray $\overleftrightarrow{AD}$. If the angle between the ray *m* and *k* has measure *θ*, then *2sinh(r/2) = tan θ.*

$$\sinh(r/2) \overset{4)}{\Leftrightarrow} \frac{\sinh r}{2\cosh(\frac{r}{2})} \overset{1)}{\Leftrightarrow} 1/2 \sin \Pi(r) \sinh r \overset{5)}{\Leftrightarrow} 1/2 \sinh BD = 1/2 \cot \Pi(BD) = 1/2 \tan \theta$$

$$\Rightarrow 2\sinh(r/2) = \tan \theta$$

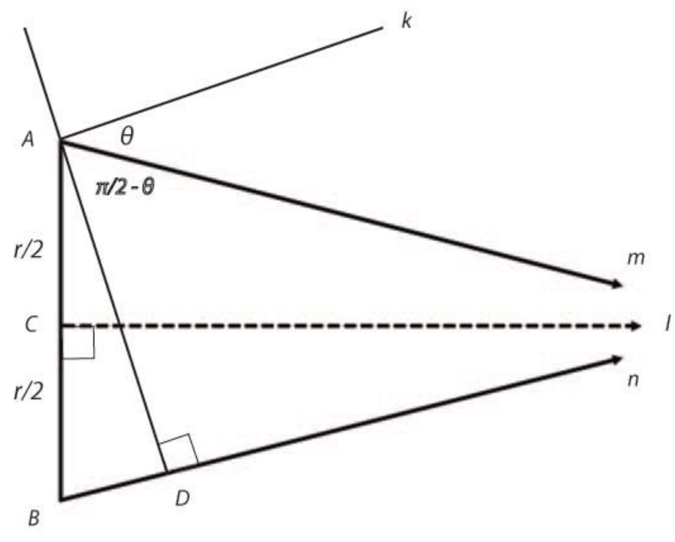


Figure 63

3.5 Constructible numbers

Now we are going to obtain a complete determination of the possible constructible regular *n-gons*.

Fermat numbers are of the form $p = 2^{2^k} + 1$. If we denote by F_k the kth Fermat number $2^{2^k} + 1$, then $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$, these five are known to be prime.

In Euclidean plane, a regular polygon of n sides is constructible by straightedge and compass if and only if n is a number on the form

$$n = 2^r p_1 ... p_s , \quad r, s \geq 0,$$

where the p_i are distinct Fermat numbers.([H] p.258)

Let's call the set of such number n Gauss numbers, because it was Gauss who determined those numbers. The prime 2 may occur with any exponent ≥ 0 in the factorization of a Gauss number, so there are infinitely many Gauss numbers.

This is the key to Jagy's theorem about how to determine exactly when a circle and a square having the same area are *both* constructible in hyperbolic plane.

3.6 Equal area

Again, we are back to our main question — How to construct these two equal area figures? The area of a circle of radius r is $4\pi sinh^2(r/2)$. Given a acute angle θ, we have already shown that we can construct one from the other by the formula $tan\ \theta = 2sinh(r/2)$. The area of a square with corner angle σ is equal to $2\pi - 4\sigma$, Now this problem of constructing comes down to constructing two angles — the auxiliary angle θ for which the circle has area $\pi\ tan^2\ \theta$ and the acute corner angle σ of the square. Let us denote the common area by ω, so that $\omega \leq 2\pi$. The equation for equal areas can now be written in terms of angles in the hyperbolic plane, that is:

$$\omega = \pi \tan^2\theta = 2\pi - 4\sigma$$

If θ and σ are both constructible, then $\omega = 2\pi - 4\sigma$ is an constructible angle. Let $x = \tan^2\theta$; then x is a constructible length in Euclidean plane.

So far, we have found several ways to express the area ω using various symbols; in summary, these are

$$\omega = \pi x = \pi \tan^2\theta = 2\pi - 4\sigma = 4\pi \sinh^2(r/2) = 2\pi(\cosh r - 1) .$$

We can get several equations from this equivalence:

$$2\pi - 4\sigma = 4\pi \sinh^2(r/2) \quad \Rightarrow \quad 2\pi = 2\sigma + \pi \cosh r$$

$$\pi x = 2\pi(\cosh r - 1) \quad \Rightarrow \quad x + 2 = 2 \cosh r$$

Let

$\mathbb{Q}$: rational numbers

$\mathbb{R}$**:** real numbers

$\mathbb{Z}$: integers

A: complex numbers that are algebraic over $\mathbb{Q}$**.**

E: the set of lengths in Euclidean plane that are constructible. **E**$\subset \mathbb{R}$.

E(*i*): a subset of **E**. $a + ib$, with $a, b \in$ E, $and\ b \neq 0$.

$\mathbb{C}$: complex numbers

then, **E** $\subset$ **E(*i*)** $\subset$ **A** $\subset$ $\mathbb{C}$**.**

Theorem 3.6.1 (Gelfond-Schneider Theorem): If α and β are algebraic numbers with $\alpha \neq 0$, $\alpha \neq 1$ and if $\beta \notin \mathbb{Q}$, then any value of $\alpha^\beta = exp(\beta \log \alpha)$ is a transcendental numbers.

For example i^{-2i} is a transcendental number, because $i \neq 1$, and $-2i$ is not a rational number.

We know that $e^{i\pi x} = \cos \pi x + i \sin \pi x$. Then $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i$. And $i^{-2i} = e^{i\frac{\pi}{2}(-2i)} = e^{\pi}$. This shows that e^{π} is transcendental.

We have another example to show that e^{π} is transcendental: $(-1)^{-i}$ is a transcendental number, because $i \neq 1$, and $-i$ is not a rational number. $(-1)^{-i} = e^{i\pi\,(-i)} = e^{\pi}$. This also shows that e^{π} is transcendental.

Proof that *x* is rational

We return to the equation $\omega = \pi x$. The angle ω is constructible because $\omega = 2\pi - 4\sigma$, and the length x is also constructible in because $x = \tan^2\theta$.

ω is constructible $\Rightarrow$ $\sin \omega = \sin \pi x$ and $\cos \omega = \cos \pi x$ are constructible, and are therefore both in **E**. Then $e^{i\pi x} = \cos \pi x + i \sin \pi x \in \mathbf{E}(\boldsymbol{i}) \subset \mathbf{A}$.

For example, since log(-1) = πi, this means that $(-1)^x = exp(x \log(-1)) = exp(i\pi x)$ is in **A.** On the other hand, $x \in \mathbf{E} \subset \mathbf{A}$, so that $(-1)^x \in \mathbf{A}$ implies $\boldsymbol{x} \in \mathbb{Q}$ by the Gelfond-Schneider Theorem**.** Furthermore, $\cosh r \in \mathbb{Q}$**,** because $x + 2 = 2 \cosh r$.

Proof that *2π/n* is an constructible angle

Now we have $\boldsymbol{x} \in \mathbb{Q}$**,** this means x can be expressed as m/n in "lowest terms", that is $m, n \in \mathbb{Z}$, $n \neq 0$, and the greatest common divisor gcd(m, n) = 1. Then $\omega = \pi x = \pi\, m/n$. There must exist $u, v \in \mathbb{Z}$ such that $um + vn = 1$. If we multiply both sides of this equality by π/n, we see that $u\pi m/n + v\pi = \pi/n$, or $u\omega + v\pi = \pi/n$. Then π/n must be constructible, because $u, v \in \mathbb{Z}$, and ω is an constructible angle. Then we have that $2\pi/n$ is an constructible angle, it means n is a Gauss number.

The corner angle σ

We know that $\omega = \pi x = \pi \tan^2\theta = 2\pi - 4\sigma$. Then

$\sigma = \frac{2\pi-\omega}{4} = \frac{2\pi-\pi tan^2\theta}{4} = \frac{2\pi-\pi x}{4} = \frac{2-x}{4}\pi$. We know that $x \in \mathbb{Q}$**,** then $\frac{2-x}{4} \in \mathbb{Q}$**.** So σ is a rational multiple of π.

We know that the area of a square is bounded above by 2π, and the corner angle varies between 0 and $\pi/2$. Now we are going to look into these boundaries. If $\sigma = \pi/2$, then the area would be zero; the "square" would be a single point. We shall then reject the value $\pi/2$ for σ. If $\sigma = 0$, the area would be 2π; thus, we shall allow this square with four infinite edges.

Theorem 3.6.2: Suppose a square with corner angle σ and a circle with radius r in the hyperbolic plane has the same area ω, with ω bounded by 2π. The square and circle are both constructible if and only if σ satisfies these conditions: $0 \leq \sigma < \pi/2$, and σ is an integer multiple of $2\pi/n$, n being a Gauss number. ([J] p.36)

However, what been said so far has not provide a general method to square any circle. This means we do not have a method which allows us to begin with either σ or r, and from that produce the other. Indeed, there is no such method ([J] p.36).

Here is one example Jagy gives in his article to show constructible r (whence θ) with the corresponding σ (whence ω) non-constructible. Let m/n be a rational number in lowest terms, such that n is not a power of 2, but has some odd prime factor d. Then $r = 2\operatorname{arcsinh}(m/2n)$ is a constructible length (formula $m/n = 2\sinh(r/2)$), and $\theta = \arctan(m/n)$ is an constructible angle (formula $m/n = \tan\theta$). But then $\omega = \pi \tan^2\theta = \pi m^2/n^2$ cannot be constructible, otherwise, it would imply constructability for the regular polygon of d^2 sides ([J] p.36). Here is the explanation: let us assume that $\omega = \pi \tan^2\theta = \pi m^2/n^2$ is constructible, and let $n = sd$, then $\omega = \pi m^2/n^2 = \pi m^2/(sd)^2 \Rightarrow s^2\omega = \pi m^2/d^2$, this implies $\pi m^2/d^2$ is constructible. The greatest common divisor gcd(m^2, d^2) = 1, since gcd(m, n) = 1. Then there must exist $u, v \in \mathbb{Z}$ such that $um^2 + vd^2 = 1$. If we multiply both sides of this equality by π/d^2, we see that $u\pi m^2/d^2 + v\pi = \pi/d^2$. Then π/d^2 must be constructible, because $u, v \in \mathbb{Z}$, $u\pi m^2/d^2$ and $v\pi$ are both constructible angles. This means π/d^2 is an constructible angle, which cannot be true since d^2 is not a Gauss number.

This then lead us to the conclusion that "squaring the circle" is solvable for some circles, and unsolvable for others.

What we called "square" here is actually a regular 4-gon in hyperbolic plane. For arbitrary $m \geq 4$, if we consider a regular m-gon having the same area as a circle, when will they both be constructible? The answer is quite the similar to that for a regular 4-gon.

That regular m-gon will be constructible if and only if σ *is an constructible angle and* π/m *is constructible*: by joining the center of the m-gon to the midpoint and an endpoint of one of its sides, a right triangle is formed with one acute angle $\sigma/2$ and the other π/m. Constructing the regular m-gon comes down to constructing that right triangle, which have constructible acute angles $\sigma/2$ and π/m. They are constructible by the right angle construction theorem ([G1] p. 506).

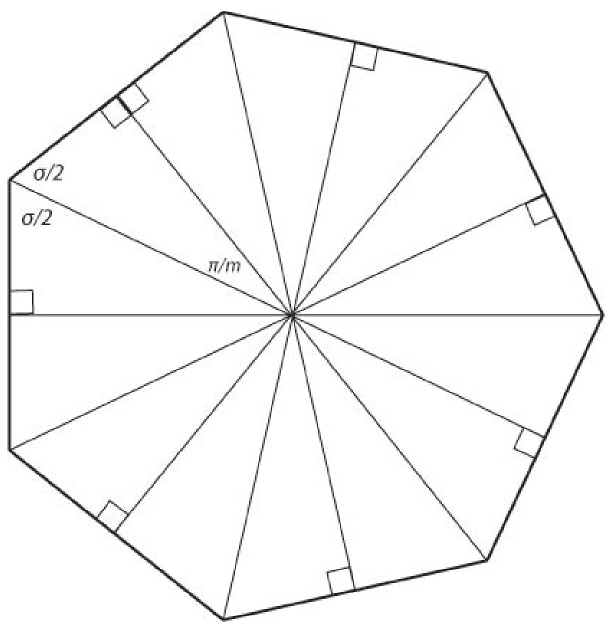


Figure 64

Suppose that a regular m-gon with acute corner angle σ such that $m\sigma < (m - 2)\pi$, and a circle in the hyperbolic plane of radius r, have the same area ω. Then both are constructible if and only if they satisfy the following conditions: $\omega < (m - 2)\pi$, *and* ω *is a rational multiple of* π, *and if that rational multiple is k/n in lowest terms, n is a Gauss number or* $n = 1$. *m is a Gauss number, and m and n have no odd prime factors in common.*([G] p.213)

The equation of areas to consider then becomes

$$\omega = \pi \tan^2\theta = (m - 2)\pi - m\sigma$$

$\Rightarrow \qquad \pi\ tan^2\theta = (m - 2)\pi - m\sigma$

$\Rightarrow \qquad m\sigma = (m - 2)\pi - \pi\ tan^2\theta$

$\Rightarrow \qquad \sigma = \frac{(m-2)}{m}\pi - \frac{tan^2\theta}{m}\pi$

$$\xRightarrow{tan^2\theta\, =\, \frac{k}{n}} \sigma = \frac{(m-2)}{m}\pi - \frac{k}{mn}\pi,$$

where k/n is a rational number in lowest terms and n is a Gauss number or $n = 1$. Then $\frac{k}{mn}\pi$ must be constructible.

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Appendix:

Euclid book I [E]

Refer to as Euclid I.1 - I.48 in our text

Proposition 1.

To construct an equilateral triangle on a given finite straight line.

Proposition 2.

To place a straight line equal to a given straight line with one end at a given point.

Proposition 3.

To cut off from the greater of two given unequal straight lines a straight line equal to the less.

Proposition 4.

If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.

Proposition 5.

In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition 6.

If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.

Proposition 7.

Given two straight lines constructed from the ends of a straight line and meeting in a point, there cannot be constructed from the ends of the same straight line, and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each equal to that from the same end.

Proposition 8.

If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines.

Proposition 9.

To bisect a given rectilinear angle.

Proposition 10.

To bisect a given finite straight line.

Proposition 11.

To draw a straight line at right angles to a given straight line from a given point on it.

Proposition 12.

To draw a straight line perpendicular to a given infinite straight line from a given point not on it.

Proposition 13.

If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

Proposition 14.

If with any straight line, and at a point on it, two straight lines not lying on the same side make the sum of the adjacent angles equal to two right angles, then the two straight lines are in a straight line with one another.

Proposition 15.

If two straight lines cut one another, then they make the vertical angles equal to one another.

Corollary. If two straight lines cut one another, then they will make the angles at the point of section equal to four right angles.

Proposition 16.

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Proposition 17.

In any triangle the sum of any two angles is less than two right angles.

Proposition 18.

In any triangle the angle opposite the greater side is greater.

Proposition 19.

In any triangle the side opposite the greater angle is greater.

Proposition 20.

In any triangle the sum of any two sides is greater than the remaining one.

Proposition 21.

If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides.

Proposition 22.

To construct a triangle out of three straight lines which equal three given straight lines: thus it is necessary that the sum of any two of the straight lines should be greater than the remaining one.

Proposition 23.

To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

Proposition 24.

If two triangles have two sides equal to two sides respectively, but have one of the angles contained by the equal straight lines greater than the other, then they also have the base greater than the base.

Proposition 25.

If two triangles have two sides equal to two sides respectively, but have the base greater than the base, then they also have the one of the angles contained by the equal straight lines greater than the other.

Proposition 26.

If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle.

Proposition 27.

If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.

Proposition 28.

If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

Proposition 29.

A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles.

Proposition 30.

Straight lines parallel to the same straight line are also parallel to one another.

Proposition 31.

To draw a straight line through a given point parallel to a given straight line.

Proposition 32.

In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles.

Proposition 33.

Straight lines which join the ends of equal and parallel straight lines in the same directions are themselves equal and parallel.

Proposition 34.

In parallelogrammic areas the opposite sides and angles equal one another, and the diameter bisects the areas.

Proposition 35.

Parallelograms which are on the same base and in the same parallels equal one another.

Proposition 36.

Parallelograms which are on equal bases and in the same parallels equal one another.

Proposition 37.

Triangles which are on the same base and in the same parallels equal one another.

Proposition 38.

Triangles which are on equal bases and in the same parallels equal one another.

Proposition 39.

Equal triangles which are on the same base and on the same side are also in the same parallels.

Proposition 40.

Equal triangles which are on equal bases and on the same side are also in the same parallels.

Proposition 41.

If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

Proposition 42.

To construct a parallelogram equal to a given triangle in a given rectilinear angle.

Proposition 43.

In any parallelogram the complements of the parallelograms about the diameter equal one another.

Proposition 44.

To a given straight line in a given rectilinear angle, to apply a parallelogram equal to a given triangle.

Proposition 45.

To construct a parallelogram equal to a given rectilinear figure in a given rectilinear angle.

Proposition 46.

To describe a square on a given straight line.

Proposition 47.

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Proposition 48.

If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.