Backward Stochastic Partial Differential Equations and their Applications in Financial Mathematics and Life Insurance

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Blindern, November 2010 Mark Rubtsov

Contents

Preface

This thesis presents the results of 3 years of research work under a Ph.D. grant at the Centre of Mathematics for Applications, University of Oslo. It consists of five articles linked by the same general topic: Backward stochastic partial differential equations (BSPDEs) and their applications in financial mathematics and life insurance. The articles are presented in separate chapters and appear in chronological order.

The first paper - written together with Ta Thi Kieu An and Frank Proske - aims at establishing a necessary and sufficient maximum principle for partial information control of general stochastic differential games, where the controlled process is described by a stochastic reaction-diffusion equation with jumps. BSPDEs feature prominently in the formulation of the maximum principle. Their use enables us to extend the existing results to a more general setting, in which modelling objects are functions of both time and space parameters. That setting is particularly suitable for dealing with constant maturity products in finance. We apply the established results to study a zero-sum stochastic differential game on a fixed income market. In particular, we investigate the problem of finding an optimal strategy for portfolios of constant maturity interest rate derivatives, managed by a trader who plays against various "market scenarios". Moreover, the trader is assumed to have restricted access to market information. We consider several utility based examples and derive some closed-form solutions.

The second article - written again in cooperation with the above co-authors - studies the problem of risk indifference pricing of interest rate claims in the presence of partial information. The latter are considered functionals of the entire bond yield surface, which results in market incompleteness and renders traditional pricing techniques inappropriate. Our approach to pricing and hedging of functional claims of the yield surface relies on risk indifference pricing with respect to generalized bond portfolios and involves the use of BSPDEs. Like in the previous paper, we employ a maximum principle for partial information control of stochastic differential games based on generalized bond portfolios. The latter method enables us to establish a representation formula for the risk indifference price of such claims.

In the third article - written in cooperation with Paul C. Kettler and Frank Proske - we aim at generalizing the existing concept of bond duration to a more realistic stochastic setting. This effort leads to the introduction of the concept of stochastic duration, whose formulation is based on a Malliavin derivative in the direction of a forward curve process, which is modelled by an SPDE. This is a formulation, examplified by the Musiela equation, which naturally calls for the use of BSPDE techniques. As an application of our results, we use the concept of stochastic duration to propose a mathematical framework for the construction of immunization strategies of portfolios of interest-rate-sensitive securities with respect to the fluctuations of the whole yield surface.

The fourth project is application oriented. It investigates a problem arising in asset-liability management in life insurance. As shown by other authors, an insurance company can guarantee its solvency by purchasing a Margrabe option enabling it to exchange its assets for a certain portfolio replicating its insurance liabilities in terms of available financial instruments. The objective of the paper is to investigate numerically a valuation technique for such an option in a situation when the insurance company is a "large" investor, implying that its trading decisions can affect asset prices. This setting contradicts the assumptions underlying traditional financial models and requires alternative pricing techniques. One existing approach to dealing with such problems relies on the use of forward-backward stochastic differential equations (FBSDEs). We use this framework to formulate a pricing equation and solve the latter numerically to obtain the price of the option. Our findings, similarly to those of other authors, show that the replication strategy for the large investor is more expensive than that for a Black-Scholes trader. This makes it particularly compelling for a large insurance company to purchase a Margrabe option at the Black-Scholes price.

In the final paper we derive an explicit representation formula for strong solutions of forward stochastic differential equations with reflections (FSDERs). Our approach relies on techniques from white noise analysis. Adopting ideas in (Meyer-Brandis and Proske 2010), we mention that the results obtained in this paper are relevant for the construction of solutions of FSDER's with discontinuous coefficients.

Chapter I

An SPDE Maximum Principle for Stochastic Differential Games under Partial Information with Application to Optimal Portfolios on Fixed Income Markets

with Ta Thi Kieu An and Frank Proske

(published in Stochastics 82, No. 1-3, pp. 3-23 (2010).)

1 Introduction

The field of game theory initiated by the path breaking works of von Neumann and Morgenstern (von Neumann and Morgenstern 1944) has been an indispensable tool in economics to analyze complex strategic interactions between agents. Game theory as a branch of mathematics has also received much attention in other areas of applied sciences. For example, it has been proven useful in social sciences as an approach to model decision making of interacting individuals in certain social situations. Other applications of this theory pertain e.g. to the description of evolutionary processes in biology, modelling of interactive computation or the design of fair division in political science.

In this paper we study a zero-sum stochastic differential game under partial information: the total benefit of the players who follow a strategy based on partial information, always adds up to zero. In other words, we consider the antagonistic interaction of two players A and B: there is a payoff function depending on the partial information strategies of players A and B, which stands for the reward for player A but the cost for player B. More specifically, the player A in our game is represented by a trader who tries to optimize his portfolio of constant maturity interest rate derivatives against various "market scenarios" symbolized by player B. The trader aims at maximizing his payoff, that is he attempts to maximize the expected terminal (cumulative) utility of his portfolio under the constraint of limited market information. On the other hand, the market endeavours to create "reasonable" market prices by minimizing the payoff function. The portfolio managed by the trader is composed of fixed income instruments with constant time-to-maturity. Thus the portfolio value evolves in time and space (i.e. time-to-maturity) and necessitates the use of an infinite dimensional modelling approach. Here in this paper we use stochastic partial differential equations (SPDE's) to describe the portfolio dynamics. In order to solve the min-max problem we want to employ the stochastic maximum principle for SPDE's.

We remark that there is a rich literature on the stochastic maximum principle. See e.g. (Bensoussan 1983; Baghery and Øksendal 2007; Framstad, Øksendal, and Sulem 2004; Tang 1998; Zhou 1993) and references therein. The authors in (An and Øksendal 2008) derive a stochastic maximum principle for stochastic differential games, where the controlled process is given by a stochastic differential equation (SDE) and the control processes are assumed to be adapted to a sub-filtration of a filtration generated by a Lévy process. Our paper is an extension of the latter to the setting of SPDE's. Finally, we would like to mention (Mataramvura and Øksendal 2008), where the authors invoke stochastic dynamic programming to study stochastic differential games.

The plan of the paper is the following. In Section 2 we prove a sufficient (and necessary) maximum principle for zero-sum games (Theorems 2.1 and 2.2). Then, in Section 3, we apply the results of the previous section to construct an optimal strategy for the above mentioned stochastic differential game on fixed income markets.

2 The stochastic maximum principle for zero-sum games

In this section we intend to study the stochastic maximum principle for stochastic differential games in the framework of SPDE control.

2.1 A sufficient maximum principle

Let $\Gamma(t,x)$ be our *controlled process* described by stochastic reaction-diffusion equation:

$$\begin{aligned}\Gamma(t,x) &= \xi(x) + \int_0^t [L\Gamma(s,x) + b(s,x,\Gamma(s,x),u_0(s,x))]\,ds \\ &+ \int_0^t \sigma(s,x,\Gamma(s,x),u_0(s,x))dB_s \\ &+ \int_0^t \int_{\mathbb{R}} \psi(s,x,\Gamma(s,x),u_1(s,x,z))\widetilde{N}(ds,dz),\ (t,x)\in[0,T]\times G,\end{aligned} \tag{2.1}$$

with boundary conditions

$$\begin{aligned}\Gamma(0,x) &= \xi(x),\ x\in\overline{G},\\ \Gamma(t,x) &= \eta(t,x),\ (t,x)\in(0,T)\times\partial G,\end{aligned}$$

where $\{B_s\}_{0\le s\le T}$ is a 1-dimensional Brownian motion and $\widetilde{N}(ds,dz) = N(ds,dz) - ds\,\nu(dz)$ a compensated Poisson random measure associated with a Lévy process defined on the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\le t\le T},P)$. Here L is a partial differential operator of order m acting on the space variable $x\in\mathbb{R}^d$ and $G\subset\mathbb{R}^d$ is an open set. Further $U\subset\mathbb{R}^n$ is a closed set and the functions

$$\begin{aligned} b &: [0,T]\times G\times\mathbb{R}\times U \longrightarrow \mathbb{R},\\ \sigma &: [0,T]\times G\times\mathbb{R}\times U \longrightarrow \mathbb{R},\\ \psi &: [0,T]\times G\times\mathbb{R}\times U\times\mathbb{R}_0 \longrightarrow \mathbb{R},\\ \xi &: \overline{G}\longrightarrow\mathbb{R},\\ \eta &: (0,T)\times\partial G\longrightarrow\mathbb{R}\end{aligned}$$

are Borel measurable. The processes

$$u_0 : [0,T]\times G\times\Omega \longrightarrow U \text{ and } u_1 : [0,T]\times G\times\mathbb{R}_0\times\Omega\longrightarrow U$$

are the *control processes* which are required to be càdlàg and adapted to a given subfiltration

$$\mathcal{E}_t \subseteq \mathcal{F}_t,\ t\ge 0.$$

We shall define a *performance criterion* by

$$J(u) = \mathbb{E}\left[\int_0^T \int_G f(t,x,\Gamma(t,x),u_0(t,x))dxdt + \int_G g(x,\Gamma(T,x))dx\right], \tag{2.2}$$

provided that, for $u = (u_0, u_1)$,

$$\Gamma = \Gamma^{(u)} \text{ admits a unique strong solution of (2.1)} \tag{2.3}$$

and that

$$\mathbb{E}\left[\int_0^T \int_G |f(t,x,X(t,x),u_0(t,x))|\,dxdt + \int_G |g(x,X(T,x))|\,dx\right] < \infty, \tag{2.4}$$

for some given continuous functions

$$\begin{aligned} f &: [0,T] \times G \times \mathbb{R} \times U \longrightarrow \mathbb{R}, \\ g &: G \times \mathbb{R} \longrightarrow \mathbb{R}. \end{aligned}$$

We call $u = (u_0, u_1)$ an *admissible control* if conditions (2.3) and (2.4) are satisfied. As for general conditions which guarantee the existence and uniqueness of strong solutions of SPDE's of the type (2.1) the reader is referred to (Da Prato and Zabczyk 1992). From now on we assume that our controls $u = (u_0, u_1)$ have components of the form

$$u_0(t,x) = (\theta_0(t,x), \pi_0(t,x)),\ (t,x) \in [0,T] \times G, \tag{2.5}$$

$$u_1(t,x,z) = (\theta_1(t,x,z), \pi_1(t,x,z)),\ (t,x,z) \in [0,T] \times G \times \mathbb{R}_0. \tag{2.6}$$

Further we shall denote by Θ (resp. Π) the class of $\theta = (\theta_0, \theta_1)$ (resp. $\pi = (\pi_0, \pi_1)$) such that controls u of the form (2.5) and (2.6) are admissible.

The *partial information control problem for zero-sum stochastic differential games* amounts to determining a $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$\Phi_{\mathcal{E}} = J(\theta^*, \pi^*) = \sup_{\pi \in \Pi}(\inf_{\theta \in \Theta} J(\theta, \pi)). \tag{2.7}$$

A control $(\theta^*, \pi^*) \in \Theta \times \Pi$ solving the min-max problem (2.7) is called *optimal control.* The min-max problem (2.7) is inspired by game theory and arises e.g. from antagonistic actions of two players, I and II, where player I pursues to minimize and player II to maximize the cost functional $J(\theta, \pi)$.

In the following denote by $\mathcal{R}$ the collection of functions

$$r : [0,T] \times G \times \mathbb{R}_0 \longrightarrow \mathbb{R}.$$

In order to solve problem (2.7) we shall proceed as in (An and Øksendal 2008) and apply a SPDE maximum principle for stochastic differential games. In our setting the *Hamiltonian function* $H : [0,T] \times G \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \longrightarrow \mathbb{R}$ gets the following form:

$$\begin{aligned} H(t,x,\gamma,u,p,q,r(t,x,\cdot)) &= f(t,x,\gamma,u) + b(t,x,\gamma,u)p \\ &+ \sigma(t,x,\gamma,u)q + \int_{\mathbb{R}} \psi(t,x,\gamma,u,z)r(t,x,z)\nu(dz), \end{aligned} \tag{2.8}$$

and the *adjoint equation* which fits into our framework is given by the following backward stochastic partial differential equation (BSPDE) in the unknown predictable processes $p = p(t,x), q = q(t,x)$ and $r = r(t,x,z)$:

$$\begin{aligned} dp(t,x) = & -\Big[\frac{\partial H}{\partial \gamma}(t,x,\Gamma^{(u)}(t,x),u(t,x),p(t,x),q(t,x),r(t,x,\cdot)) \\ & + L^* p(t,x)\Big]dt + q(t,x)dB_t + \int_{\mathbb{R}_0} r(t,x,z)\widetilde{N}(dt,dz), \end{aligned} \tag{2.9}$$

for all $(t,x) \in [0,T) \times G$, and

$$\begin{aligned} p(T,x) &= \frac{\partial g}{\partial \gamma}(x,\Gamma^{(u)}(T,x)),\ x \in \overline{G}; \\ p(t,x) &= 0,\ (t,x) \in (0,T) \times \partial G. \end{aligned}$$

Here L^* is the adjoint of the operator L, that is

$$(L^* f, g)_{L^2(G)} = (f, Lg)_{L^2(G)},$$

for all $f, g \in C_0^\infty(G)$. Let us mention that BSPDE's of the form (2.9) have been studied e.g. in (Øksendal, Proske, and Zhang 2005).

We are now coming to a verification theorem for the optimization problem (2.7):

Theorem 2.1. *Let* $(\hat{\theta},\hat{\pi}) \in \Theta \times \Pi$ *and denote by* $\widehat{\Gamma}(t,x) = \Gamma^{(\hat{\theta},\hat{\pi})}(t,x)$ *the corresponding solution of* (2.1)*. Set* $\Gamma^{\theta}(t,x) = \Gamma^{(\theta,\hat{\pi})}(t,x)$ *and* $\Gamma^{\pi}(t,x) = \Gamma^{(\hat{\theta},\pi)}(t,x)$*. Suppose that* $\hat{p}(t,x), \hat{q}(t,x)$ *and* $\hat{r}(t,x,z)$ *solve the adjoint equation* (2.9) *in the strong sense and assume that the following conditions are fulfilled, for all* $u \in \mathcal{A}$,

$$E\Big[\int_G \int_0^T \big(\Gamma^{\theta}(t,x) - \widehat{\Gamma}(t,x)\big)^2 \Big\{\hat{q}^2(t,x) + \int_{\mathbb{R}_0} \hat{r}^2(t,x,z)\nu(dz)\Big\}dtdx\Big] < \infty, \tag{2.10}$$

$$E\Big[\int_G \int_0^T \big(\Gamma^{\pi}(t,x) - \widehat{\Gamma}(t,x)\big)^2 \Big\{\hat{q}^2(t,x) + \int_{\mathbb{R}_0} \hat{r}^2(t,x,z)\nu(dz)\Big\}dtdx\Big] < \infty, \tag{2.11}$$

and

$$\begin{aligned} & E\Big[\int_G \int_0^T \hat{p}^2(t,x)\Big\{\sigma^2(t,x,\Gamma^{\theta}(t,x),\theta_0(t,x),\hat{\pi}_0(t,x)) \\ & + \int_{\mathbb{R}_0} \psi^2(t,x,\Gamma^{\theta}(t,x),\theta_1(t,x,z),\hat{\pi}_1(t,x,z),z)\Big\}\nu(dz)dtdx\Big] < \infty, \end{aligned} \tag{2.12}$$

$$\begin{aligned} & E\Big[\int_G \int_0^T \hat{p}(t,x)^2\Big\{\sigma^2(t,x,\Gamma^{\pi}(t,x),\hat{\theta}_0(t,x),\pi_0(t,x)) \\ & + \int_{\mathbb{R}_0} \psi^2(t,x,\Gamma^{\pi}(t,x),\hat{\theta}_1(t,x,z),\pi_1(t,x,z),z)\Big\}\nu(dz)dtdx\Big] < \infty. \end{aligned} \tag{2.13}$$

Furthermore, assume that for all $(t,x) \in [0,T] \times G$ *the following* partial information maximum principle *holds:*

$$\begin{aligned}
&\inf_{\theta\in\Theta} E[H(t,x,\Gamma^\theta(t,x),\theta(t,x),\hat\pi(t,x),\hat p(t,x),\hat q(t,x),\hat r(t,x,\cdot))\,|\mathcal{E}_t] \\
&= E[H(t,x,\hat\Gamma(t,x),\hat\theta(t,x),\hat\pi(t,x),\hat p(t,x),\hat q(t,x),\hat r(t,x,\cdot))\,|\mathcal{E}_t] \qquad (2.14)\\
&= \sup_{\pi\in\Pi} E[H(t,x,\Gamma^\pi(t,x),\hat\theta(t,x),\pi(t,x),\hat p(t,x),\hat q(t,x),\hat r(t,x,\cdot))\,|\mathcal{E}_t]\,.
\end{aligned}$$

(i) Suppose that, for all $\gamma \in \mathbb{R}$ *and* $(t,x) \in [0,T] \times G$*, the functions*

$$\gamma \mapsto g(x,\gamma), \tag{2.15}$$

and

$$(\gamma,\pi) \mapsto H(t,x,\gamma,\hat\theta(t,x),\pi,\hat p(t,x),\hat q(t,x),\hat r(t,x,\cdot)) \tag{2.16}$$

are concave. Then,

$$J(\hat\theta,\hat\pi) \geq J(\hat\theta,\pi) \text{ for all } \pi \in \Pi,$$

and

$$J(\hat\theta,\hat\pi) = \sup_{\pi\in\Pi} J(\hat\theta,\pi).$$

(ii) Suppose that, for all $\gamma \in \mathbb{R}$ *and* $(t,x) \in [0,T] \times G$*, the functions*

$$\gamma \mapsto g(x,\gamma) \tag{2.17}$$

and

$$(\gamma,\theta) \mapsto H(t,x,\gamma,\theta,\hat\pi(t,x),\hat p(t,x),\hat q(t,x),\hat r(t,x,\cdot)) \tag{2.18}$$

are convex. Then,

$$J(\hat\theta,\hat\pi) \leq J(\theta,\hat\pi) \text{ for all } \theta \in \Theta,$$

and

$$J(\hat\theta,\hat\pi) = \inf_{\theta\in\Theta} J(\theta,\hat\pi).$$

(iii) Suppose the conditions in (i) and (ii) hold, then $(\theta^*,\pi^*) := (\hat\theta,\hat\pi)$ *is an optimal control and*

$$\Phi_{\mathcal{E}} = \sup_{\pi\in\Pi}\left(\inf_{\theta\in\Theta} J(\theta,\pi)\right) = \inf_{\theta\in\Theta}\left(\sup_{\pi\in\Pi} J(\theta,\pi)\right). \tag{2.19}$$

Proof. i) Fix $\hat{\theta} \in \Theta$. Let $\pi \in \Pi$ be an arbitrary admissible control with corresponding solution $\Gamma^{\pi}(t,x) = \Gamma^{(\hat{\theta},\pi)}(t,x)$. Then we have

$$\begin{aligned} J(\hat{\theta},\hat{\pi}) - J(\hat{\theta},\pi) = E\Big[\int_0^T \int_G \{ & f(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)) \\ & - f(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x))\} dx dt \\ & + \int_G \{g(x,\widehat{\Gamma}(T,x)) - g(x,\Gamma^{\pi}(T,x))\} dx\Big]. \end{aligned} \tag{2.20}$$

Putting

$$I_1 = E\Big[\int_0^T \int_G \{\hat{f} - f^{\pi}\} dx dt\Big], \tag{2.21}$$

and

$$I_2 = E\Big[\int_G \{\hat{g} - g^{\pi}\} dx\Big], \tag{2.22}$$

where

$$\begin{aligned} \hat{f} &= f(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)), \\ f^{\pi} &= f(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x)), \\ \hat{g} &= g(x,\widehat{\Gamma}(T,x)) \text{ and } g^{\pi} = g(x,\Gamma^{\pi}(T,x)). \end{aligned}$$

Similarly, we put

$$\begin{aligned} \hat{b} &= b(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)), \\ b^{\pi} &= b(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x)), \\ \hat{\sigma} &= \sigma(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x)), \\ \sigma^{\pi} &= \sigma(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x)), \\ \hat{\psi} &= \psi(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),z), \\ \psi^{\pi} &= \psi(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x),z). \end{aligned}$$

Moreover, we set

$$\begin{aligned} \widehat{H} &= H(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)), \\ H^{\pi} &= H(t,x,\Gamma^{\pi}(t,x),\hat{\theta}(t,x),\pi(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)). \end{aligned}$$

Since $g(x,\gamma)$ is concave in γ, we have

$$\hat{g} - g^{\pi} \geq \frac{\partial g}{\partial \gamma}(x,\widehat{\Gamma}(T,x)) \cdot (\widehat{\Gamma}(T,x) - \Gamma^{\pi}(T,x)). \tag{2.23}$$

Putting $\widetilde{\Gamma}(t,x)=\widehat{\Gamma}(t,x)-\Gamma^{\pi}(t,x)$ and using integration by parts, we get

$$
\begin{aligned}
I_2 &\geq E\Big[\int_G \frac{\partial g}{\partial \gamma}(x,\widehat{\Gamma}(T,x)).\widetilde{\Gamma}(T,x)dx\Big] = E\Big[\int_G \hat{p}(T,x).\widetilde{\Gamma}(T,x)dx\Big] \\
&= E\Big[\int_G \Big(\hat{p}(0,x).\widetilde{\Gamma}(0,x) \\
&\qquad + \int_0^T \{\widetilde{\Gamma}(t,x)d\hat{p}(t,x)+\hat{p}(t,x)d\widetilde{\Gamma}(t,x)+(\hat{\sigma}-\sigma^{\pi})\hat{q}(t,x)\}dt \\
&\qquad + \int_0^T \int_{\mathbb{R}} (\widehat{\psi}-\psi^{\pi})\hat{r}(t,x,z)\nu(dz)dt\Big)dx\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= E\Big[\int_G \Big(\int_0^T \widetilde{\Gamma}(t,x)\Big\{-\Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge} - L^*\hat{p}(t,x)\Big\}dt \\
&\qquad + \int_0^T \Big\{\hat{p}(t,x)[L\widetilde{\Gamma}(t,x)+(\hat{b}-b^{\pi})]+(\hat{\sigma}-\sigma)\hat{q}(t,x)\Big\}dt \\
&\qquad + \int_0^T \int_{\mathbb{R}} (\hat{\psi}-\psi^{\pi})\hat{r}(t,x,z)\nu(dz)dt\Big)dx\Big],
\end{aligned} \tag{2.24}
$$

where

$$
\Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge} = \frac{\partial H}{\partial \gamma}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)). \tag{2.25}
$$

By definition of H we have

$$
\begin{aligned}
I_1 = E\Big[\int_0^T \int_G \Big\{&\hat{H}-H^{\pi}-(\hat{b}-b^{\pi})\hat{p}(t,x)-(\hat{\sigma}-\sigma)\hat{q}(t,x) \\
&- \int_{\mathbb{R}} (\hat{\psi}-\psi)\hat{r}(t,x,z)\nu(dz)\Big\}dxdt\Big].
\end{aligned} \tag{2.26}
$$

On the other hand, we have for all $(t,x)\in(0,T)\times\partial G$

$$
\widetilde{\Gamma}(t,x)=\hat{p}(t,x)=0,
$$

and

$$
\int_G \{\widetilde{\Gamma}(t,x)L^*\hat{p}(t,x)-\hat{p}(t,x)L\widetilde{\Gamma}(t,x)\}dx = 0 \text{ for all } t\in(0,T). \tag{2.27}
$$

Combining these with (2.24) and (2.26), we obtain

$$
J(\hat{\theta},\hat{\pi})-J(\hat{\theta},\pi) \geq E\Big[\int_G \Big(\int_0^T \Big\{\hat{H}-H^{\pi}-\Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge}\cdot\widetilde{\Gamma}(t,x)\Big\}dt\Big)dx\Big]. \tag{2.28}
$$

Since H is concave in γ and π, we have

$$\hat{H} - H^{\pi} \geq \Big(\frac{\partial H}{\partial \gamma}\Big)^{\wedge} \cdot \widetilde{\Gamma}(t,x) + \Big(\frac{\partial H}{\partial \pi}\Big)^{\wedge} \cdot (\hat{\pi} - \pi), \tag{2.29}$$

where

$$\Big(\frac{\partial H}{\partial \pi}\Big)^{\wedge} = \frac{\partial H}{\partial \pi}(t, x, \hat{\Gamma}(t,x), \hat{\theta}(t,x), \pi(t,x), \hat{p}(t,x), \hat{q}(t,x), \hat{r}(t,x,.)).$$

On the other hand, since $\pi \to E[H^{\pi}(t, x, \Gamma^{\pi}(t,x), \hat{\theta}(t,x), \pi(t,x), \hat{p}(t,x), \hat{q}(t,x), \hat{r}(t,x,.))|\mathcal{E}_t]$ attains a maximum at $\pi(t,x) = \hat{\pi}(t,x)$ and $\pi(t,x), \hat{\pi}(t,x)$ are $\mathcal{E}_t$-measurable, we get

$$E\Big[\Big(\frac{\partial H}{\partial \pi}\Big)^{\wedge} (\hat{\pi} - \pi)\Big|\mathcal{E}_t\Big] = (\hat{\pi} - \pi)\Big(\frac{\partial}{\partial \pi}\Big)^{\wedge} E[H|\mathcal{E}_t]_{\pi = \hat{\pi}} \geq 0. \tag{2.30}$$

Combining (2.28), (2.29) and (2.30), we get

$$J(\hat{\theta}, \hat{\pi}) - J(\hat{\theta}, \pi) \geq 0. \tag{2.31}$$

Since $\pi \in \Pi$ is arbitrary, this proves (i).

ii) Fix $\hat{\pi} \in \Pi$. Let $\theta \in \Theta$ be an arbitrary admissible control. Just as in (i) we can show that

$$J(\hat{\theta}, \hat{\pi}) - J(\theta, \hat{\pi}) \leq 0. \tag{2.32}$$

iii) If both (i) and (ii) hold, then

$$J(\hat{\theta}, \pi) \leq J(\hat{\theta}, \hat{\pi}) \leq J(\theta, \hat{\pi}),$$

for any $(\theta, \pi) \in \Theta \times \Pi$. Thereby,

$$J(\hat{\theta}, \hat{\pi}) \leq \inf_{\theta \in \Theta} J(\theta, \hat{\pi}) \leq \sup_{\pi \in \Pi} \big(\inf_{\theta \in \Theta} J(\theta, \pi)\big).$$

On the other hand,

$$J(\hat{\theta}, \hat{\pi}) \geq \sup_{\pi \in \Pi} J(\hat{\theta}, \pi) \geq \inf_{\theta \in \Theta} \big(\sup_{\pi \in \Pi} J(\theta, \pi)\big).$$

Now due to the inequality

$$\inf_{\theta \in \Theta} \big(\sup_{\pi \in \Pi} J(\theta, \pi)\big) \geq \sup_{\pi \in \Pi} \big(\inf_{\theta \in \Theta} J(\theta, \pi)\big)$$

we have

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi \in \Pi} \big(\inf_{\theta \in \Theta} J(\theta, \pi)\big) = \inf_{\theta \in \Theta} \big(\sup_{\pi \in \Pi} J(\theta, \pi)\big).$$

∎

2.2 A necessary maximum principle for zero-sum games

In (An and Øksendal 2008) the authors gave a necessary stochastic maximum principle for zero-sum games based on SDE's. In this Section, we aim at presenting this result in the setting of SPDE's. The proof of this extension closely follows the arguments in (An and Øksendal 2008). Therefore, we omit the proof and refer the reader to the latter article.

In addition to the conditions in Section 2.1, we shall now assume the following:

(A1) For all $t \in (0,T)$ and all $\mathcal{E}_t$-measurable random variables α, ρ, the controls

$$\beta_\alpha(s,x) := \alpha(\omega)\chi_{[t,T]}(s)\chi_G(x),$$

and

$$\eta_\rho(s,x) := \rho(\omega)\chi_{[t,T]}(s)\chi_G(x)$$

belong to Θ and Π, respectively.

(A2) For given $\theta, \beta \in \Theta$ and $\pi, \eta \in \Pi$ with β, η bounded, there exists a $\delta > 0$ such that

$$\theta + y\beta \in \Theta \text{ and } \pi + v\eta \in \Pi,$$

for all $y, v \in (-\delta, \delta)$.

Set $\Gamma^{\theta+y\beta}(t,x) = \Gamma^{(\theta+y\beta,\pi)}(t,x)$ and $\Gamma^{\pi+v\eta}(t,x) = \Gamma^{(\theta,\pi+v\eta)}(t,x)$. For given $\theta, \beta \in \Theta$ and $\pi, \eta \in \Pi$ with β, η bounded, we define the processes $Y^\theta(t,x)$ and $Y^\pi(t,x)$ (if existing) by,

$$Y^\theta(t,x) = \frac{d}{dy}\Gamma^{\theta+y\beta}(t,x)\Big|_{y=0}, \tag{2.33}$$

$$Y^\pi(t,x) = \frac{d}{dv}\Gamma^{\pi+v\eta}(t,x)\Big|_{v=0}. \tag{2.34}$$

Further, let us assume that $Y^\theta(t,x)$ and $Y^\pi(t,x)$ satisfy the equations:

$$\begin{aligned} dY^\theta(t,x) &= (LY^\theta(t,x) + \lambda^\theta(t,x))dt \\ &\quad + \xi^\theta(t,x)dB(t) + \int_{\mathbb{R}} \zeta^\theta(t,x,z)\widetilde{N}(dt,dz), \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} dY^\pi(t,x) &= (LY^\pi(t,x) + \lambda^\pi(t,x))dt \\ &\quad + \xi^\pi(t,x)dB(t) + \int_{\mathbb{R}} \zeta^\pi(t,x,z)\widetilde{N}(dt,dz), \end{aligned} \tag{2.36}$$

where

$$\begin{cases} \lambda^\theta(t,x) = & \frac{\partial b}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^\theta(t,x) \\ & +\frac{\partial b}{\partial \theta}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \\ \xi^\theta(t,x) = & \frac{\partial \sigma}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^\theta(t,x) \\ & +\frac{\partial \sigma}{\partial \theta}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \\ \zeta^\theta(t,x) = & \frac{\partial \psi}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^\theta(t,x) \\ & +\frac{\partial \psi}{\partial \theta}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x), \end{cases} \tag{2.37}$$

and

$$
\left\{\begin{array}{rl}
\lambda^{\pi}(t,x)= & \frac{\partial b}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^{\pi}(t,x)\\
& +\frac{\partial b}{\partial \pi}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x),\\
\xi^{\pi}(t,x)= & \frac{\partial \sigma}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^{\pi}(t,x)\\
& +\frac{\partial \sigma}{\partial \pi}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x),\\
\zeta^{\pi}(t,x)= & \frac{\partial \psi}{\partial \gamma}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))Y^{\pi}(t,x)\\
& +\frac{\partial \psi}{\partial \pi}(t,x,\Gamma(t,x),\theta(t,x),\pi(t,x))\beta(t,x).
\end{array}\right. \tag{2.38}
$$

Theorem 2.2. *Suppose $\hat{\theta}\in\Theta$ and $\hat{\pi}\in\Pi$ are respectively a local minimum and a maximum for $J(\theta,\pi)$, in the sense that, for all bounded $\beta\in\Theta$ and $\eta\in\Pi$, there exists $\delta>0$ such that for all $y,v\in(-\delta,\delta)$, $\hat{\theta}+y\beta\in\Theta$, $\hat{\pi}+v\eta\in\Pi$ and*

$$
h(y,v):=J(\hat{\theta}+y\beta,\hat{\pi}+v\eta),\qquad y,v\in(-\delta,\delta),
$$

attains a minimum at $y=0$ and a maximum at $v=0$.

Suppose there exists a solution $\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.)$ of the associated adjoint equation

$$
\left\{\begin{array}{rcl}
d\hat{p}(t,x) & = & -\Big(\frac{\partial H}{\partial \gamma}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.))\\
& & +L^{*}\hat{p}(t,x)\Big)\,dt+\hat{q}(t,x)\,dB(t)+\int_{\mathbb{R}^n}\hat{r}(t^-,x,z)\widetilde{N}(\,dt,\,dz);\\
\hat{p}(T,x) & = & \frac{\partial g}{\partial \gamma}(x,\widehat{\Gamma}(T,x)),\ x\in\bar{G};\\
p(t,x) & = & 0,\ (t,x)\in(0,T)\times\partial G.
\end{array}\right. \tag{2.39}
$$

Moreover, adopting the notation in (2.35)-(2.38), *assume that*

$$
E\Big[\int_G\int_0^T Y^{\hat{\theta}}(t,x)^2\Big\{\hat{q}^2(t,x)+\int_{\mathbb{R}}\hat{r}^2(t,x,z)\nu(dz)\Big\}dxdt\Big]<\infty, \tag{2.40}
$$

$$
E\Big[\int_G\int_0^T Y^{\hat{\pi}}(t,x)^2\Big\{\hat{q}^2(t,x)+\int_{\mathbb{R}}\hat{r}^2(t,x,z)\nu(dz)\Big\}dxdt\Big]<\infty, \tag{2.41}
$$

and

$$
\begin{aligned}
E\Big[\int_G\int_0^T \hat{p}^2(t,x)\Big\{&\xi^{\hat{\theta}}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x))\\
&+\int_{\mathbb{R}}\psi^2(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x))\nu(dz)\Big\}dxdt\Big]<\infty,
\end{aligned} \tag{2.42}
$$

$$
\begin{aligned}
E\Big[\int_G\int_0^T \hat{p}^2(t,x)\Big\{&\xi^{\hat{\pi}}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x))\\
&+\int_{\mathbb{R}}\psi^2(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x))\nu(dz)\Big\}dxdt\Big]<\infty.
\end{aligned} \tag{2.43}
$$

Then, for a.a. $t \in [0,T]$, *we have*

$$E\Big[\frac{\partial H}{\partial \theta}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.))\Big|\mathcal{E}_t\Big]$$
$$= E\Big[\frac{\partial H}{\partial \pi}(t,x,\widehat{\Gamma}(t,x),\hat{\theta}(t,x),\hat{\pi}(t,x),\hat{p}(t,x),\hat{q}(t,x),\hat{r}(t,x,.))\Big|\mathcal{E}_t\Big] = 0. \tag{2.44}$$

Proof. See (An and Øksendal 2008). ∎

3 Application to portfolios of constant maturity interest rate derivatives

In the following denote by $F(t,T)$ the (market) price of an interest rate derivative at time $t \geq 0$ which expires at maturity $T < \infty$. In this Section we want to study optimal portfolio strategies for constant maturity interest rate derivatives, that is we aim at constructing optimal hedging strategies with respect to fixed income market contracts with constant time-to-maturity $x := T - t$. In our framework the price of such contracts at time t is assumed to be $F(t,t+x)$. Examples of such financial instruments are bonds on 6 month LIBOR rates or more general contracts on forward rates with constant time-to-maturity. In a wider sense such instruments also include constant maturity swaps. See e.g. (Hull 2000). We shall mention that these derivatives steadily gain importance in asset liability management and are e.g. used by life insurance companies to match their liabilities. Suppose that for each $x \geq 0$ our portfolio S^x is a portfolio made up of a risk-free asset and a derivative contract with constant time-to-maturity x . We are interested in finding an optimal portfolio strategy for the entirety of portfolios $\{S^x\}_{x\in J}$ (J is a subset of $[0,\infty)$) managed by a trader who only has limited access to market information. In the sequel let us consider a market model consisting of a risk-free asset and an interest rate derivative with maturity T specified by

$$\text{(risk-free asset)} \qquad dP_0(t) = \rho(t)P_0(t)dt,\ P_0(0) = 1. \tag{3.1}$$

$$\text{(interest rate derivative)} \qquad dF(t,T) = F(t^-,T)\Big[\alpha(t,T)dt + \sigma(t,T)dW_t + \int_{\mathbb{R}_0}\gamma(t,T,z)\widetilde{N}(dt,dz)\Big], \tag{3.2}$$
$$F(0,T) > 0, \qquad \text{for all } T > 0,$$

where $(\rho(t))_{t\geq 0}$, $(\alpha(t,T))_{0\leq t\leq T<\infty}, (\sigma(t,T))_{0\leq t\leq T<\infty}$ and $(\gamma(t,T,z))_{0\leq t\leq T}$ are $\mathcal{F}_t-$ predictable processes such that, for all $T \geq 0$,

$$E\Big[\int_0^\infty \Big\{ \mid \rho(s) \mid + |\alpha(s,T)| + \frac{1}{2}\sigma^2(s,T) + \int_{\mathbb{R}_0} |\log(1+\gamma(s,T,z)) - \gamma(s,T,z)|\nu(dz)\Big\}ds\Big] < \infty, \tag{3.3}$$

and

$$\gamma(t,T,z) > -1, \qquad \text{for } (\omega,t,z) \in \Omega \times [0,T] \times \mathbb{R}_0 \text{ a.e., } T \geq 0.$$

We assume that the dynamics of the short rate $\rho(t)$ is stochastic and governed by

$$\left\{ \begin{array}{rcl} d\rho(t) & = & a(t)dt + b(t)dW_t + \int_{\mathbb{R}_0} c(t,z)\widetilde{N}(dt,dz), \\ \rho(0) & = & 0, \end{array} \right. \tag{3.4}$$

where $a(t), b(t)$ and $c(t,z)$ are predictable processes such that (3.4) is well-defined.

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$ be a given sub-filtration. Denote by $\phi(t,T)$, $t \geq 0$, the fraction of wealth invested in $F(t,T)$ based on the partial market information $\mathcal{E}_t \subseteq \mathcal{F}_t$ being available at time t. Thus we require that $\{\phi(t,T)\}_{t\geq 0, T\geq 0}$ must be $\mathcal{E}_t-$ predictable. Then for each T the total wealth $V^{(\phi)}(t,T)$ of the portfolio S^T is given by the SDE

$$\left\{ \begin{array}{rcl} dV^{(\phi)}(t,T) & = & V^{(\phi)}(t^-,T)\Big[\{\rho(t) + (\alpha(t,T) - \rho(t))\phi(t,T)\}dt \\ & + & \phi(t,T)\sigma(t,T)dW_t + \phi(t,T)\int_{\mathbb{R}_0}\gamma(t,T,z)\widetilde{N}(dt,dz)\Big], \\ V^{(\phi)}(0,T) & = & w(T). \end{array} \right. \tag{3.5}$$

Let us rewrite the dynamics of the total wealth as an integral evolution equation in infinite dimensions by viewing terms of (3.5) as functions of maturity T. So we see that

$$\begin{aligned} V^{(\phi)}(t,\cdot) &= w(\cdot) + \int_0^t V^{(\phi)}(s,\cdot)\{\rho(s) + (\alpha(s,\cdot) - \rho(s))\phi(s,\cdot)\}ds \\ &+ \int_0^t V^{(\phi)}(s,\cdot)\phi(s,\cdot)\sigma(s,\cdot)dW_s \\ &+ \int_0^t \int_{\mathbb{R}_0} V^{(\phi)}(s^-,\cdot)\phi(s,\cdot)\gamma(s,\cdot,z)\widetilde{N}(ds,dz). \end{aligned} \tag{3.6}$$

Define

$$\begin{aligned} V_t^{(\phi)}(x) &= V^{(\phi)}(t,t+x),\ \phi_t(x) = \phi(t,t+x), \alpha_t(x) = \alpha(t,t+x), \\ \sigma_t(x) &= \sigma(t,t+x),\ \gamma_t(x,z) = \gamma(t,t+x,z), t,x \geq 0, z \in \mathbb{R}_0. \end{aligned}$$

Set $T = t+x$ in (3.5). Then differentiation of both sides of (3.5) w.r.t. time t (formally) yields,

$$\begin{aligned} dV_t^{(\phi)}(x) &= \Big(AV_t^{(\phi)}(x) + V_{t^-}^{(\phi)}(x)\{\rho(t) + (\alpha_t(x) - \rho(t))\phi_t(x)\}\Big)dt \\ &+ V_{t^-}^{(\phi)}(x)\phi_t(x)\Big\{\sigma_t(x)dW_t + \int_{\mathbb{R}_0}\gamma_t(x,z)\widetilde{N}(dt,dz)\Big\}, \end{aligned} \tag{3.7}$$

where A is the densely defined operator given by

$$A = \frac{d}{dx}.$$

We may think of A as the generator of a strongly continuous left shift operator on an appropriate Hilbert space H. One could e.g. choose H to be the weighted Sobolev space $H_\gamma, \gamma > 0$, consisting of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\| f \|_\gamma^2 := \int_0^\infty f^2(x) e^{-\lambda x} dx + \int_0^\infty \Big(\frac{d}{dx} f(x)\Big)^2 e^{-\lambda x} dx < \infty,$$

where the derivative $\frac{d}{dx}$ is in the distributional sense (See e.g. (Filipović 2001)). Criteria ensuring the existence and uniqueness of (strong) solutions of first order (quasi-) linear SPDE's of the type (3.7) can be found in e.g. (Kunita 1987).

Let us also mention that the type of SPDE obtained in (3.7) is often referred to as "Musiela equation" in the theory of interest rate modelling (Carmona and Tehranchi 2006). Usually a no-arbitrage condition in terms of a volatility process and a risk premium is imposed on the Musiela equation to enforce a risk-free evolution of forward curves (see e.g. (Carmona and Tehranchi 2006)). In this paper we won't necessarily require such a condition on the dynamics of the portfolio value $V_t^{(\phi)}(x)$, since we are interested in a general portfolio optimization problem.

Definition 3.1. The set $\mathcal{A}$ of *admissible portfolios* consists of all processes $\phi = \phi(t,x), t \in [0,T]$, such that

(i) $0 \le \phi_t(x) \le 1$;

(ii) ϕ permits a strong solution of the SPDE (3.7);

(iii) $\int_0^\infty \{|\rho(s) + (\alpha_s(x) - \rho(s))\phi_s(x)| + \phi_s^2(x)(\sigma_s^2(x) + \int_{\mathbb{R}_0} \gamma_s^2(x,z)\nu(dz))\} ds < \infty$;

(iv) $\phi_t(x)\gamma_t(x,z) > -1 \qquad (\omega,t,z) -$ a.e..

We now introduce a family $\mathcal{Q}$ of measures Q_θ parametrized by a process $\theta = (\theta^0(t,x), \theta^1(t,x,z))$ such that

$$dQ(\omega) = Z^{(\theta)}(T,x) dP(\omega) \qquad \text{on } \mathcal{F}_t, \tag{3.8}$$

where

$$\begin{cases} dZ^{(\theta)}(t,x) &= Z^{(\theta)}(t^-,x)[-\theta^0(t,x)dW_t - \int_{\mathbb{R}} \theta^1(t,x,z)\widetilde{N}(dt,dz)], \\ Z^\theta(0,x) &= 1. \end{cases} \tag{3.9}$$

We assume that

$$\theta^1(t,x,z) \le 1, \qquad \text{for } (\omega,t,z) \text{ a.s.,} \tag{3.10}$$

and

$$\int_0^t \Big\{\theta^0(s,x)^2 + \int_{\mathbb{R}} \theta^1(s,x,z)^2\Big\} ds < \infty \qquad \text{a.s..} \tag{3.11}$$

Setting

$$Z_t^{(\theta)}(x) = Z^{(\theta)}(t,x);\ \theta_t^0(x) = \theta^0(t,x);\ \theta_t^1(x,z) = \theta^1(t,x,z), \tag{3.12}$$

we get

$$dZ_t^{(\theta)}(x) = -Z_t^{(\theta)}(x)\theta_t^0(x)dW_t - \int_{\mathbb{R}} Z_t^{(\theta)}(x)\theta_t^1(x,z)\widetilde{N}(dt,dz) \tag{3.13}$$

The set of all $\theta = (\theta^0, \theta^1)$ such that (3.10)-(3.11) hold is denoted by Θ. These are the *admissible controls of the market.*

In the sequel we let G be an interval. Fix a utility function $U : G \times [0,\infty) \to [-\infty,\infty)$, assumed to be increasing, concave and twice continuously differentiable on $(0,\infty)$.

The problem is to find $\theta^* \in \Theta$ and $\phi^* \in \mathcal{A}$ such that

$$\Phi(y_1,y_2) = \inf_{\theta\in\Theta}\Big(\sup_{\phi\in\mathcal{A}} E_{Q_\theta}\Big[\int_G U(x, V_T^{(\phi)}(x))dx\Big]\Big). \tag{3.14}$$

This is a problem of the type described in the previous section. Here, player I is the trader and player II is the market. The trader wants to find an optimal strategy for portfolios that maximizes the (expected) cumulative utility of the terminal wealth of portfolios $V_T^{(\phi)}(x)$ with respect to time-to-maturity x in G. On the other hand, the market "wants" to choose a scenario (represented by a probability measure) which minimizes this maximal cumulative (or average) utility. Thus, to solve (3.14) by stochastic control methods, we have to look at the following three-dimensional state process $Y(t,x)$:

$$dY(t,x) = \begin{bmatrix} dY_1(t,x) \\ dY_2(t,x) \\ dY_3(t,x) \end{bmatrix} = \begin{bmatrix} d\rho(t) \\ dZ_t^{\theta}(x) \\ dV_t^{(\phi)}(x) \end{bmatrix}$$

$$= \begin{bmatrix} a(t) \\ 0 \\ AV_t^{(\phi)}(x) + V_{t^-}^{(\phi)}\{\rho(t) + (\alpha_t(x) - \rho(t))\phi_t(x)\} \end{bmatrix} dt$$

$$+ \begin{bmatrix} b(t) \\ -Z_{t^-}^{\theta}(x)\theta_t^0(x) \\ V_{t^-}^{(\phi)}(x)\sigma_t(x)\phi_t(x) \end{bmatrix} dW_t + \int_{\mathbb{R}} \begin{bmatrix} c(t,z) \\ -Z_{t^-}^{(\theta)}(x)\theta_t^1(x,z) \\ V_{t^-}^{(\phi)}(x)\phi_t(x)\gamma_t(x,z) \end{bmatrix} \widetilde{N}(dt,dz). \tag{3.15}$$

The Hamiltonian is defined as

$$\begin{aligned} &H(t,x,y_1,y_2,y_3,\theta,\phi,p,q,r(t,x,\cdot)) \\ &= a(t)p_1(t,x) + y_3\{y_1 + (\alpha_t(x) - y_1)\phi_t(x)\}p_3 \\ &+ b(t)q_1(t,x) - y_2\theta_t^0(x)q_2 + y_3\sigma_t(x)\phi_t(x)q_3 \\ &+ \int_{\mathbb{R}}\{c(t)r_1(t,x,z) - y_2\theta_t^1(x,z)r_2(t,x,z) \\ &\qquad + y_3\phi_t(x)\gamma_t(x,z)r_3(t,x,z)\}\nu(dz). \end{aligned} \tag{3.16}$$

And the adjoint equations are defined by

$$
\left\{
\begin{array}{lcl}
dp_1(t,x) & = & -y_3(1-\phi_t(x))p_3(t,x)dt + q_1(t,x)dW_t \\
 & + & \int_{\mathbb{R}} r_1(t,x,z)\widetilde{N}(dt,dz); \\
p_1(T,x) & = & U_{y_1}(x,y_3), x \in \bar{G}; \\
p_1(t,x) & = & 0, (t,x) \in (0,T) \times \partial G,
\end{array}
\right.
\tag{3.17}
$$

$$
\left\{
\begin{array}{lcl}
dp_2(t,x) & = & \Big[\theta_t^0(x)q_2(t,x) + \int_{\mathbb{R}} \theta_t^1(x,z)r_2(t,x,z)\nu(dz)\Big]dt \\
 & & +q_2(t,x)dW_t + \int_{\mathbb{R}} r_2(t,x,z)\widetilde{N}(dt,dz); \\
p_2(T,x) & = & U_{y_2}(x,y_3), x \in \bar{G}; \\
p_2(t,x) & = & 0, (t,x) \in (0,T) \times \partial G,
\end{array}
\right.
\tag{3.18}
$$

and

$$
\left\{
\begin{array}{lcl}
dp_3(t,x) & = & \Big[-\{y_1 + (\alpha_t(x) - y_1)\phi_t(x)\}p_3(t,x) - \sigma_t(x)\phi_t(x)q_3(t,x) \\
 & - & \int_{\mathbb{R}} \phi_t(x)\gamma_t(x,z)r_3(t,x,z)\nu(dz) - A^*p_3(t,x)\Big]dt \\
 & + & q_3(t,x)dW_t + \int_{\mathbb{R}} r_3(t,x,z)\widetilde{N}(dt,dz); \\
p_3(T,x) & = & U_{y_3}(x,y_3), x \in \bar{G}; \\
p_3(t,x) & = & 0, (t,x) \in (0,T) \times \partial G.
\end{array}
\right.
\tag{3.19}
$$

Suppose $(\hat{\theta},\hat{\phi})$ is an optimal control and $\widehat{Y}(t) = (\widehat{Y}_1(t,x), \widehat{Y}_2(t,x), \widehat{Y}_3(t,x))$ is the corresponding optimal process associated with the solution $\hat{p}(t,x) = (\hat{p}_1(t,x), \hat{p}_2(t,x))$, $\hat{q}(t,x) = (\hat{q}_1(t,x), \hat{q}_2(t,x)), \hat{r}(t,x,\cdot) = (\hat{r}_1(t,x,\cdot), \hat{r}_2(t,x,\cdot))$ of the adjoint equations. Maximizing the Hamiltonian $E[H(t,x,y_1,y_2,\theta,\phi,p,q,r) \mid \mathcal{E}_t]$ over all $\phi \in \mathcal{A}$ leads to the following first order condition for the maximum point $\hat{\phi}$:

$$
\begin{aligned}
&E[(\alpha_t(x) - y_1)\hat{p}_3(t,x) \mid \mathcal{E}_t] + E[\sigma_t(x)\hat{q}_3(t,x) \mid \mathcal{E}_t] \\
&\quad + \int_{\mathbb{R}} E[\gamma_t(x,z)\hat{r}_3(t,z) \mid \mathcal{E}_t]\nu(dz) = 0.
\end{aligned}
\tag{3.20}
$$

We then minimize $E[H(t,x,y_1,y_2,\theta,\phi,p,q,r) \mid \mathcal{E}_t]$ over all $\theta = (\theta^0, \theta^1)$ and get the following first order conditions for a minimum point $\hat{\theta} = (\hat{\theta}^0, \hat{\theta}^1)$:

$$
E[-\widehat{Y}_2(t,x)\hat{q}_2(t,x) \mid \mathcal{E}_t] = 0,
\tag{3.21}
$$

and

$$
\int_{\mathbb{R}} E[-\widehat{Y}_2(t,x)\widehat{r}_2(t,x,z) \mid \mathcal{E}_t]\nu(dz) = 0.
\tag{3.22}
$$

We try a process $\hat{p}_2(t,x)$ of the form

$$
\hat{p}_2(t,x) = f(t, \widehat{Y}_1(t,x))U(x, \widehat{Y}_3(t,x)),
\tag{3.23}
$$

with $f(T, y_1) = 1$, for all y_1. In the following, we will write $U^{'}$ instead of U_{y_3}. Differentiating (3.23), we get

$$\begin{aligned} d\hat{p}_2(t,x) = \Big\{ & f_t + \widetilde{A}(t,x)f + \widetilde{B}(t,x)f_{y_1} + \frac{1}{2}b^2(t)f_{y_1y_1} \\ & + \int_{\mathbb{R}} \{f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z)f_{y_1}\}\nu(dz)\Big\}dt \\ & + \Big(b(t)f_{y_1} + \widehat{Y}_3\sigma_t\phi_t\frac{U^{'}}{U}f\Big)dW_t \\ & + \int_{\mathbb{R}} \Big\{\frac{f}{U}[U(\widehat{Y}_3(1+\gamma_t\phi_t)) - U(\widehat{Y}_3)] \\ & + [f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)]\Big\}\widetilde{N}(dt,dz), \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} \widetilde{A}(t,x) &= \Big(\widehat{Y}_3\big(\widehat{Y}_1 + (\alpha_t - \widehat{Y}_1)\phi_t\big)\Big)\frac{U^{'}}{U} + \frac{1}{2}\widehat{Y}_3^2\sigma_t^2\phi_t^2\frac{U^{''}}{U} \\ &+ \frac{1}{U}\int_{\mathbb{R}}\{U(\widehat{Y}_3(1+\gamma_t\phi_t)) - U(\widehat{Y}_3) - \widehat{Y}_3\gamma_t\phi_tU^{'}\}\nu(dz); \end{aligned} \tag{3.25}$$

$$\widetilde{B}(t,x) = a(t) + \widehat{Y}_3b(t)\sigma_t\phi_t\frac{U^{'}}{U}. \tag{3.26}$$

Comparing this with equation (3.18) by equating the dt, dW_t and $\widetilde{N}(dt,dz)$ coefficients respectively, we get

$$\hat{q}_2(t,x) = b(t)f_{y_1} + \widehat{Y}_3\sigma_t\phi_t\frac{U^{'}}{U}f, \tag{3.27}$$

$$\hat{r}_2(t,x) = \frac{f}{U}[U(\widehat{Y}_3(1+\gamma_t\phi_t)) - U(\widehat{Y}_3)] + [f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)]; \tag{3.28}$$

and a second-order PDE for f of the form

$$\begin{aligned} 0 =& f_t + \widetilde{A}(t,x)f + \widetilde{B}(t,x)f_{y_1} + \frac{1}{2}b^2(t)f_{y_1y_1} \\ & + \int_{\mathbb{R}}\{f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z)f_{y_1}\}\nu(dz). \end{aligned} \tag{3.29}$$

Combining (3.27) and (3.21), we get

$$\phi_t(x) = -E\Big[\frac{b(t)}{\sigma_t(x)}\frac{U}{\widehat{Y}_3U'}\frac{f_{y_1}}{f}\Big|\mathcal{E}_t\Big]. \tag{3.30}$$

Try the process $\hat{p}_3(t,x)$ of the form

$$\hat{p}_3(t,x) = f(t,\widehat{Y}_1(t,x))\widehat{Y}_2(t,x)U^{'}(x,\widehat{Y}_3(t,x)), \tag{3.31}$$

with $f(T,y_1)=1$, for all y_1. Differentiating both sides of equation (3.31), we have

$$\begin{aligned} d\hat{p}_3(t,x) = \Big\{ & U^{'} f_t + A\hat{p}_3(t,x) + \widetilde{C}(t,x) f + \widetilde{D}(t,x) f_{y_1} + \frac{1}{2} b^2(t) f_{y_1 y_1} \\ & + \int_{\mathbb{R}} \{ f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z) f_{y_1} \} \nu(dz) \Big\} dt \\ & + \Big(\widehat{Y}_3 \sigma_t \phi_t U^{''} f - \theta_t^0 U^{'} f + b(t) U^{'} f_{y_1} \Big) dW_t \\ & + \int_{\mathbb{R}} \Big\{ f[U^{'}(\widehat{Y}_3(1+\gamma_t \phi_t)) - U^{'}(\widehat{Y}_3)] \\ & \quad + U^{'}[f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)] - \theta_t^1 U^{'} f \Big\} \widetilde{N}(dt,dz), \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} \widetilde{C}(t,x) = {} & \widehat{Y}_3(\widehat{Y}_1 + (\alpha_t - \widehat{Y}_1)\phi_t) U^{''} + \frac{1}{2} \widehat{Y}_3^2 \sigma_t^2 \phi_t^2 U^{'''} + \widehat{Y}_3 \sigma_t \phi_t \theta_t^1 U^{''} \\ & + \int_{\mathbb{R}} \{ U^{'}(\widehat{Y}_3(1+\gamma_t\phi_t)) - U^{'}(\widehat{Y}_3) - \widehat{Y}_3 \gamma_t \phi_t U^{''} \} \nu(dz); \end{aligned} \tag{3.33}$$

and

$$\widetilde{D}(t,x) \;=\; a(t) U^{'} + \widehat{Y}_3 b(t) \sigma_t \phi_t U^{''} - b(t) \theta_t^0 U^{'}. \tag{3.34}$$

Comparing this with equation (3.19) by equating the dt, dW_t and $\widetilde{N}(dt,dz)$ coefficients respectively, we get

$$\hat{q}_3(t,x) = \widehat{Y}_3 \sigma_t \phi_t U^{''} f - \theta_t^0 U^{'} f + b(t) U^{'} f_{y_1}; \tag{3.35}$$

$$\begin{aligned} \hat{r}_3(t,x) = {} & f[U^{'}(\widehat{Y}_3(1+\gamma_t\phi_t)) - U^{'}(\widehat{Y}_3)] \\ & + U^{'}[f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1)] - \theta_t^1 U^{'} f; \end{aligned} \tag{3.36}$$

and

$$\begin{aligned} & U^{'} f_t + A\hat{p}_3(t,x) + \widetilde{C}(t,x) f + \widetilde{D}(t,x) f_{y_1} + \frac{1}{2} b^2(t) f_{y_1 y_1} \\ & + \int_{\mathbb{R}} \{ f(\widehat{Y}_1 + c(t,z)) - f(\widehat{Y}_1) - c(t,z) f_{y_1} \} \nu(dz) \\ = {} & -\{ \widehat{Y}_1 + (\alpha_t - \widehat{Y}_1)\phi_t \} \hat{p}_3(t,x) - \sigma_t \phi_t \hat{q}_3(t,x) \\ & - \int_{\mathbb{R}} \phi_t \gamma_t \hat{r}_3(t,x,z) \nu(dz) - A^* \hat{p}_3(t,x). \end{aligned} \tag{3.37}$$

Substituting $\hat{p}_3(t,x), \hat{q}_3(t,x)$ and $\hat{r}_3(t,x,z)$ into equation (3.20), we obtain

$$\begin{aligned}
&\theta_t^0(x)E[\sigma_t(x) \mid \mathcal{E}_t] - \int_{\mathbb{R}} \theta_t^1(x,z)E[\gamma_t(x,z) \mid \mathcal{E}_t]\nu(dz) \\
&= E[(\alpha_t(x) - \rho(t))|\mathcal{E}_t] + E\Big[b(t)\sigma_t(x)\frac{f_{y_1}}{f}\Big|\mathcal{E}_t\Big] - E\Big[b(t)\sigma_t(x)\frac{UU''}{U'U'}\frac{f_{y_1}}{f}\Big|\mathcal{E}_t\Big] \\
&+ \int_{\mathbb{R}} E\Big[\gamma_t(x,z)\Big(\frac{1}{U'}[U'(\hat{Y}_3(1+\gamma_t(x)\phi_t(x))) - U'(\hat{Y}_3)] \\
&\qquad + \frac{1}{f}[f(\hat{Y}_1 + c(t,z)) - f(\hat{Y}_1)]\Big)\Big|\mathcal{E}_t\Big]\nu(dz). \qquad (3.38)
\end{aligned}$$

We have proved the following result:

Theorem 3.2. *A portfolio $\phi(t,x) \in \mathcal{A}$ is a maximum point for the problem* (3.14) *if it satisfies the equation* (3.30) *and if the optimal measure $Q_{\hat{\theta}}$ has an optimizer $\hat{\theta}(t,x) = (\hat{\theta}_t^0(x), \hat{\theta}_t^1(x))$ which fulfills the equation* (3.38).

Remark. When the short rate $\rho(t)$ is deterministic, we can easily see from (3.30) and (3.38) that

$$\phi(t,x) = 0,$$

and

$$\theta_t^0(x)E[\sigma_t(x)|\mathcal{E}_t] + \int_{\mathbb{R}} \theta_t^1((x,z)E[\gamma_t(x,z) \mid \mathcal{E}_t]\nu(dz) = E[(\alpha_t(x)|\mathcal{E}_t] - \rho(t).$$

This case is analogous to the result obtained in (An and Øksendal 2008), where the authors deal with SDE control.

Example 3.1. Let us consider an example in the continuous case, i.e. $c(t,z) = 0$, $\gamma_t(x) = 0$, $\theta_t^1(x) = 0$, and the *power utility*, i.e.,

$$U(x,u) = \frac{1}{\eta}u^\eta, \qquad u > 0, \qquad (3.39)$$

where $\eta \in (-\infty, 1)\backslash\{0\}$ is a constant. Using the separation

$$f(t,y_1) = g(t)e^{\beta(t)y_1} \qquad (3.40)$$

with terminal conditions $\beta(T) = 0$ and $g(t) = 1$ we get an optimal portfolio for

$$\phi_t(x) = -\frac{1}{\eta}\frac{E[b(t)\beta(t)|\mathcal{E}_t]}{E[\sigma_t(x)|\mathcal{E}_t]}, \qquad (3.41)$$

provided that

$$0 \leq -\frac{1}{\eta}\frac{E\,[b(t)\beta(t)\;|\mathcal{E}_t]}{E\,[\sigma_t(x)\;|\mathcal{E}_t]} \leq 1.$$

In this case the equation (3.29) becomes

$$
\begin{aligned}
0 &= g' + \Big(\beta' + \frac{b(t)}{\sigma_t(x)}\beta + \eta\Big) y_1 g \\
&+ \Big\{\frac{1}{2} b(t)\Big(\frac{\eta-1}{\eta} - b(t)\Big)\beta^2 + \Big(a(t) - \frac{\alpha_t(x) b(t)}{\sigma_t(x)}\Big)\beta\Big\} g. \qquad (3.42)
\end{aligned}
$$

The function f will be meaningful if we get an ODE for g which does not include the short rate y_1. Hence β should be calculated so that the term of y_1 in (3.42) becomes zero, i.e.,

$$
\beta' = -\frac{b(t)}{\sigma_t(x)}\beta - \eta \qquad \text{with } \beta(T) = 0. \qquad (3.43)
$$

This leads to

$$
\beta(t) = -\frac{\eta\sigma_t(x)}{b(t)}\Big(e^{-\frac{b(t)}{\sigma_t(x)}(T-t)} - 1\Big). \qquad (3.44)
$$

Then the optimal market strategy subject to the scenario $Q_{\hat{\theta}}$ satisfies the equation

$$
\begin{aligned}
\theta_t^0(x) E[\sigma_t(x) \mid \mathcal{E}_t] &= E[(\alpha_t(x) - \rho(t))|\mathcal{E}_t] + E[b(t)\sigma_t(x)\beta|\mathcal{E}_t] \\
&- \frac{\eta-1}{\eta} E[b(t)\sigma_t(x)\beta|\mathcal{E}_t]. \qquad (3.45)
\end{aligned}
$$

Example 3.2. Keep the utility function as in the previous example and consider the case when the dynamics of the short rate ρ are described by the Vasicek model:

$$
d\rho(t) = (\zeta - \mu\rho(t))dt + b dW_t \qquad (3.46)
$$

where ζ, μ, b are constants. The Vasicek model is an affine rate model and now $\beta(t) = \frac{1}{\mu}(1 - e^{-\mu(T-t)})$. In this case the optimal controls for the portfolio manager and for the market simplify to

$$
\phi_t(x) = -\frac{bE[(1 - e^{\mu(T-t)}) \mid \mathcal{E}_t]}{\mu\eta E[\sigma_t(x) \mid \mathcal{E}_t]}, \qquad (3.47)
$$

and

$$
\begin{aligned}
&\theta_t^0(x) E[\sigma_t(x)|\mathcal{E}_t] + \int_{\mathbb{R}} \theta_t^1(x,z) E[\gamma_t(x,z) \mid \mathcal{E}_t]\nu(dz) \\
&= E[(\alpha_t(x) - \rho(t))|\mathcal{E}_t] + \frac{b}{\mu\eta} E[\sigma_t(x)(1 - e^{-\mu(T-t)}) \mid \mathcal{E}_t] \qquad (3.48) \\
&+ \int_{\mathbb{R}} E[\gamma_t(x,z)\{(1 + \gamma_t(x,z)\phi_t(x))^{\eta-1} + (e^{\frac{c(t,z)}{\mu}(1-e^{-\mu(T-t)})} - 1)\} \mid \mathcal{E}_t]\nu(dz).
\end{aligned}
$$

Remark. a) Let us consider the case, when $Z_t^{(\theta)}(x) \equiv 1$ in (3.8). So our stochastic differential game reduces to an ordinary optimization problem for the SPDE (3.7) w.r.t. the portfolio strategy $\phi_t(x)$. In this case one can compare the optimal strategy $\phi_t(x)$ for constant maturity contracts with the corresponding strategy in the classical portfolio optimization problem of Merton in (Øksendal and Sulem 2007): As a result one finds that optimal hedging based on constant maturity instruments presumes knowledge of the *whole* "term structure of volatility" $x \mapsto \sigma_t(x)$, whereas derivatives expiring at a fixed maturity only require information of *single* points (i.e. $\sigma(t, T)$ for T fixed) on volatility curves.

b) Our optimization problem can be easily generalized to the case of an investor who is allowed to consume portfolio wealth.

c) In the framework of Malliavin calculus an SPDE optimization problem related to (3.7) is studied in (Menoukeu, Meyer-Brandis, Proske, and Salleh 2007).

Chapter II

Risk Indifference Pricing of Functional Claims of the Yield Surface in the Presence of Partial Information

with Ta Thi Kieu An and Frank Proske

1 Introduction

In this paper we aim at analyzing the pricing (and hedging) of functional claims of the yield surface in the presence of partial information. To be more precise, we want to consider interest rate derivatives which are functions of the yield surface

$$((t,x) \mapsto R(t,t+x)), \tag{1.1}$$

where $R(t,T)$ denotes the interest rate at time t with time-to-maturity $x = T - t$. Here we assume that pricing of such claims is based on limited access to market information.

Examples - out of a vast variety of claims traded on fixed income or over-the-counter markets worldwide - are bond options, swaptions, floors or caps (see e.g. (Hull 2000)). For example, a cap (or a caplet), which provides the holder with protection against rising interest rates, has the following payoff at time $T = t + x$:

$$Caplet_x(t) = N \cdot x \cdot \max(R(t,t+x) - K, 0), \tag{1.2}$$

where N is the notional amount and K the fixed cap rate.

Another type of a claim, which - in contrast to (1.2) - is a function of the whole yield surface (1.1) is the Asian option of a cap, with payoff given by

$$\frac{1}{(T_2 - T_1)(x_2 - x_1)} \int_{T_1}^{T_2} \int_{x_1}^{x_2} Caplet_x(t)\, dx\, dt. \tag{1.3}$$

We remark that due to its averaging property the latter claim exhibits the advantage of reducing the volatility risk inherent in the option.

Popular stochastic models for the dynamics of $R(t,T)$, $0 \le t \le T$ (T fixed), which can be found in the financial literature, are e.g. the Heath-Jarrow-Morton or the LIBOR model. See (Heath, Jarrow, and Morton 1992) or (Musiela and Rutkowski 1992) and the references therein. Assuming full access to market information in such models, it is well known that replicating strategies with respect to bonds of a given maturity can be used to determine the fair price of the cap in (1.2). On the other hand, taking into account the existence of maturity-specific risk of bonds with different maturities, pricing of functional claims of the yield surface - such as the Asian option (1.3) - is in general impossible within the above mentioned models. A model that takes into account maturity-specific risk is e.g. the Musiela equation. See e.g. (Carmona and Tehranchi 2006) or (Filipović 2001). This model, which is based on a stochastic partial differential equation, describes the fluctuations of the entire yield surface. This approach leads to an infinite dimensional model, which has the attractive feature that hedging strategies of claims for generalized bond portfolios (i.e. portfolios of bonds of arbitrary maturities) are unique.

A deficiency of a bond market model based on the Musiela equation is that it is in general incomplete, even if there exists a unique martingale measure (see e.g. (Carmona and Tehranchi 2006)). Thus, the determination of the arbitrage-free price of a claim based on exact replicating trading strategies is not always possible. Of course, if we in addition assume that the portfolio manager only has restricted access to market information, then pricing of both types of options (1.2), (1.3) converts into a pricing problem on incomplete markets.

One approach to option pricing on incomplete markets is e.g. utility indifference pricing. This method has been studied by many authors in literature from different points of view. See e.g. (Hodges and Neuberger 1989), where the authors consider a hedging problem under certain model constraints. Further, (Grasselli and Hurd 2005) apply similar techniques to a stochastic volatility model. The work of (Takino 2007) also deals with a financial application under incomplete information. See also (El Karoui and Rouge 2000), (Davis 1999), (Henderson 2002), (Monoyios 2004) and (Mandrekar and Zhang 1993).

The utility indifference price of a claim is defined at a level which makes the issuer of the claim utility indifferent between the investment strategies of either selling the claim and entering the market with the collected initial payment, or entering the market without selling the contract. In contrast to that approach, in this paper we want to employ *risk indifference pricing* to address the problem of pricing (and hedging) of functional claims of the yield surface under incomplete market information. The latter pricing principle is related to utility indifference pricing but it is based on a risk measure instead of the utility function. For more information on risk measures the reader may consult (Föllmer and Schied 2004) and the references therein. Regarding the topic of risk measure pricing we refer the reader to (Xu 2006), (Barrieu and Karoui 2004) and (Klöppel and Schweizer 2007).

The main result of our paper is a formula for the risk indifference price of an interest rate claim under partial information with respect to a certain class of risk measures. Our approach to deriving this formula rests on a stochastic maximum principle for differential games based on generalized bond portfolios, which are described by a stochastic evolution equation on a Hilbert space. This technique is inspired by (An, Øksendal, and Proske 2008), where the authors study a jump diffusion market modelled by an SDE. See also (An and Øksendal 2008). A paper related to the latter article is (Øksendal and Sulem 2009), which treats the case of Markovian controls in the framework of stochastic dynamic programming. Finally, we mention (Ekeland and Taflin 2005), where the authors analyze hedging of generalized bond portfolios in a Markovian setting by means of Hamilton-Jacobi-Bellman equations on Hilbert spaces.

Our paper is organized as follows: In Section 2 we introduce the mathematical tools we will use throughout the paper. Further, in Section 3 we give the precise statement of our pricing problem in the context of generalized bond portfolios. Sections 4 and 5 are devoted to establishing a stochastic maximum principle based on stochastic evolution equations, which is used in Section 6 to derive a formula for the risk indifference price of functional interest rate claims.

2 The general model

In this section we elaborate on some concepts essential for our further presentation. We begin by briefly recalling the classical Heath-Jarrow-Morton (HJM) framework for term structure modelling.

Let us denote by $P(t,T)$ the price at time t of a zero-coupon bond, that is a security that pays one unit of a given currency at maturity T. In the sequel the bond prices are modelled by non-negative adapted processes $\{P(t,T)\}_{0\leq t\leq T}$ for each $T>0$ on a filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P}), \tag{2.1}$$

where $\mathcal{F}_t$ is $\mathbb{P}$-completed and generated by independent one-dimensional Brownian motions $B_t^{(j)}$, $0\leq t\leq T$, $j=1,\ldots,d$.

In the HJM model the bond prices $P(t,T)$ are modelled as

$$P(t,T) = exp\Big(-\int_t^T f(t,s)\,ds\Big), \tag{2.2}$$

where $f(t,T)$, $0\leq t\leq T<\infty$ are instantaneous forward rates described by the SDE

$$df(t,T) = \alpha(t,T)\,dt + \sum_{j=1}^{d} \sigma^{(j)}(t,T)dB_t^{(j)}, \tag{2.3}$$

where $\alpha(t,T)$, $\sigma^{(j)}(t,T)$, $0\leq t\leq T$ are predictable processes. In order to rule out arbitrage opportunities in this setting one has to impose the following restriction on the drift coefficient $\alpha(t,T)$ in (2.3):

$$\alpha(t,T) = \sum_{j=1}^{d} \sigma^{(j)}(t,T)\left(\int_t^T \sigma^{(j)}(t,s)ds + \lambda(t)\right), \tag{2.4}$$

where $\lambda(t)$ is a risk premium process.

A shortcoming of the HJM model is that the implied hedging strategies are not unique. This is a consequense of the finite dimensional character of the model, i.e. it assumes that the noise is driven by finitely many Brownian motions. This assumption leads to the situation that e.g. in the HJM model driven by 3 Brownian motions, an option writen on a 5-year bond can be hedged with bonds of maturities e.g. 20, 25 and 30 years - a rather unrealistic implication from the point of view of a fixed income trader.

One way to extend the HJM model is to incorporate the notion of a *maturity specific risk*. This is done by explicitly recognizing the infinite dimensional character of the term structure. The latter leads to the Musiela formulation of the HJM model, which is given by the following stochastic partial differential equation (SPDE).

$$df_t(x) = \left(\frac{d}{dx}f_t(x) + \alpha_t(x)\right)\,dt + \sum_{j=1}^{\infty} \sigma_t^{(j)}(x)dB_t^{(j)}, \tag{2.5}$$

where $B_t^{(j)}$, $j \geq 1$ are independent one-dimensional Brownian motions. Here we use the notation $f_t(x) := f(t, t+x)$ and $x := T-t$ is the time-to-maturity of the forward rate;$\alpha_t(x) := \alpha(t, t+x)$, $\sigma_t^{(j)}(x) := \sigma^{(j)}(t, t+x)$ for predictable processes $\sigma_t^{(j)}(T)$, $j \geq 1$, $0 \leq t \leq T$.

One can now look at the forward curve $x \mapsto f_t(x)$ as a single element of an appropriate function space $\mathcal{H}$. It is natural to require that this space has the property that the evaluation functionals

$$\delta_x : \mathcal{H} \to \mathbb{R},\ f \to f(x) \tag{2.6}$$

are continuous for all x. In addition we shall assume that the generator $A := \frac{d}{dx}$ in (2.5) has a strongly continuous semigroup S_t on $\mathcal{H}$. The semigroup S_t is the left shift operator given by

$$\big(S_t f\big)(x) = f(t+x) \tag{2.7}$$

An example of a suitable function space on which one can properly describe the evolution of forward curves is the Hilbert space of Sobolev type:

$$\mathcal{H} := \left\{ f\ :\ [0,\infty) \to \mathbb{R}\ :\ f \text{ is absolutely continuous and } \int_0^\infty \left(\frac{d}{dx} f(x)\right)^2 w(x)\, dx < \infty \right\} \tag{2.8}$$

with the scalar product given by

$$\langle f, g\rangle_{\mathcal{H}} := f(0) \cdot g(0) + \int_0^\infty \frac{d}{dx} f(x) \cdot \frac{d}{dx} g(x)\, w(x)\, dx \tag{2.9}$$

The function $w : [0,\infty) \to (0,\infty)$ is required to be increasing and to satisfy the following condition

$$\int_0^\infty \frac{1}{w(x)}\, dx < \infty \tag{2.10}$$

See e.g. (Carmona and Tehranchi 2006) for details.

In what follows suppose that

$$\alpha_t(\cdot),\ \sigma_t^{(j)}(\cdot) \in \mathcal{H},\ \text{a.e.},\quad \forall t \geq 0$$

Now we want to rewrite Equation (2.5) as a stochastic evolution equation on the Hilbert space $\mathcal{H}$. For that purpose consider a Q-Wiener process W_t, where Q is a symmetric non-negative operator on a separable Hilbert space U with $\mathrm{Trace}(Q) < \infty$. Define the Hilbert space $U_0 = Q^{1/2}(U)$, with norm

$$\|h\|_0 := \|Q^{-1/2}(h)\|, \quad h \in U_0$$

Further, we shall denote by $L_2(U, \mathcal{H})$ the space of Hilbert-Schmidt operators from U to $\mathcal{H}$ with the norm $\|\cdot\|_{L_2}$. Let u_j, $j \geq 1$, be an orthonormal basis of U, and suppose that there exists a Borel-measurable function

$$\sigma\colon [0,T] \longrightarrow L(U_0, \mathcal{H})$$

such that

$$\sigma_t \left[Q^{1/2}(u_j) \right] = \sigma_t^{(j)}(\cdot)$$

and

$$\sigma_t \circ Q^{1/2} \in L_2(U, \mathcal{H})$$

for all t, j in Equation (2.5), where $\circ$ refers to the composition of mappings. Then $\left\{ B_t^{(k)} \right\}_{0 \leq t \leq T}$, $k \geq 1$, in Equation (2.5) can be regarded as a Wiener process B_t cylindrically defined on U, and Equation (2.5) can be recast as

$$\mathrm{d}f_t = \left(A f_t + \alpha_t \right) \mathrm{d}t + \sigma_t \, \mathrm{d}B_t \tag{2.11}$$

In the following we assume that there is a predictable unique mild solution

$$\left(t \longmapsto f_t(\cdot) \right) \in C([0, T]; \mathcal{H})$$

to the SPDE (2.11). As for sufficient criteria for the existence and uniqueness of mild, weak or even strong solutions of SPDE's we refer the reader to (Kai 2006).

In order to rule out arbitrage opportunities with respect to our forward curve model (2.11) we shall require that the forward curves f_t satisfy the generalized HJM no-arbitrage condition:

$$\alpha_t(x) = \sum_{j \geq 1} \sigma_t^{(j)}(x) \left(J_x(\sigma_t^{(j)}) + \lambda_t^{(j)} \right), \tag{2.12}$$

where J_x is a continuous linear functional on $\mathcal{H}$ defined by

$$J_x(f) := \int_0^x f(u) \, du$$

and where the processes $\lambda_t^{(j)}$, $j \geq 1$ are the components of the $\mathcal{H}$-valued process

$$\lambda_t = \sum_{j \geq 1} \lambda_t^{(j)} \, v_j \tag{2.13}$$

Here v_j, $j \geq 1$ is an orthonormal basis of $\mathcal{H}$. The processes $\lambda_t^{(j)}$, $j \geq 1$ can be financially interpreted as risk premiums with respect to different times-to-maturity, that is these premiums entice investors to bear the volatility risk of bonds of different maturities.

3 The risk indifference price of an interest rate claim as a solution of a stochastic differential game

This Section explains the concept of risk indifference pricing. In simple words, this pricing technique relies on minimization of a chosen risk measure. We need to resort to this pricing method because of incompleteness of the infinite dimensional bond market that we are studying. Our approach involves reformulating the risk indifference pricing problem into a stochastic differential game and then using available mathematical tools to obtain a simplified pricing formula. The particular choice of a benchmark risk measure is unimportant. Instead, in our derivations we use a general representation formula for a convex risk measure. In accordance with that representation formula, we choose a risk measure that will enable us to obtain closed-form results.

We begin by describing the market and the problem faced by the investor. Assume that the filtration $\{\mathcal{F}_t\}$ in (2.1) is generated by the Wiener process B_t in (2.11). Define $P_t(x) := P(t, t+x)$ to be the bond price at time t with constant time to maturity x. Further, let $m : [0,\infty) \times \mathcal{H} \to \mathbb{R}$ and $g : \mathcal{H} \to \mathbb{R}$ be Borel measurable functions, where $\mathcal{H} \subseteq C([0,\infty))$ is a Hilbert space as in Section 2. Our objective is to price an option of the following form:

$$G_\tau := \int_0^\tau m\left(t, P_t(\cdot)\right)\, dt \;+\; g(P_\tau(\cdot)) \tag{3.1}$$

where τ is the time at which the option expires. All prices are measured in the units of the bank account, so we consider discounted quantities. We assume that there are the following investment possibilities:

- Bank account: $B_t^0 = 1$, $\forall t \in [0, \tau]$
- Bonds with date of maturity $T < \infty$, $P(t, T)$.

In the sequel let us assume that the conditions

$$\mathbb{E}\Big[exp\Big\{\int_0^t \langle \lambda_s, dB_s\rangle_0 - \frac{1}{2}\int_0^t \|\lambda_s\|_0^2\, ds\Big\}\Big] = 1 \tag{3.2}$$

and

$$\int_0^t \Big(\int_0^s \|\delta_{s-u} \circ \sigma_s\|_{L_0^2}^2\, du\Big)^{\frac{1}{2}} ds < \infty \tag{3.3}$$

hold for all $t \geq 0$, where $\|L\|_{L_0^2} := \|L \circ Q^{\frac{1}{2}}\|_{L_2}$ for each $L \in L_2(U_0, \mathcal{H})$. Then in our HJM framework one can show by Itô's formula and Girsanov's theorem that

$$P(t,T) \;=\; P(0,T) - \int_0^t P(s,T)\, J_{T-s} \circ \sigma_s\, d\tilde{B}_s, \tag{3.4}$$

where $\tilde{B}_t = B_t - \int_0^t \lambda_s\, ds$ is a Wiener process under a local martingale measure $\tilde{\mathbb{P}}$. Further, let us require that $\tilde{\sigma}$ given by

$$\tilde{\sigma}_t(\omega, x) := P_t(x)\, J_x \circ \sigma_t \tag{3.5}$$

is a predictable $L_2(U_0, \mathcal{H})$-valued process, such that $\int_0^T \|\tilde{\sigma}_s\|^2_{L_0^2}\, ds < \infty$ a.e. Then the bond price curves P_t are $\mathcal{H}$-valued and fulfil

$$dP_t \;=\; AP_t dt - \tilde{\sigma}_t\, d\tilde{B}_t \tag{3.6}$$

or

$$dP_t \;=\; \Big(AP_t + \tilde{\sigma}_t(\lambda_t)\Big) dt - \tilde{\sigma}_t\, dB_t \tag{3.7}$$

in the mild sense, where as before $A = \frac{d}{dx}$.

Using our notation in Section 2, Equation (3.7) can be equivalently written as

$$\begin{aligned} dP_t(x) &= (AP_t(x) + P_t(x) \cdot b_t(x))\; dt \\ &- \sum_{j\geq 1} P_t(x)\, \delta_t^{(j)}(x)\, dB_t^{(j)}, \end{aligned} \tag{3.8}$$

where $\delta_t^{(j)}(x) := J_x(\sigma_t^{(j)})$ and $b_t(x) := \sum_{j\geq 1} J_x(\sigma_t^{(j)})\lambda_t^{(j)}$.

In the sequel we assume (the rather strong condition) that there exists a unique strong solution $P_t \in \mathcal{H}$ to Equation (3.6). See (Kai 2006) for sufficient criteria.

In this paper we aim at using *risk indifference pricing* to price options of the form (3.1) in the presence of partial information. We are now going to explain the idea behind this pricing concept, but first we introduce the concept of a convex risk measure. Let $\mathbb{F}$ be the space of all equivalence classes of real-valued random variables defined on Ω.

Definition 3.1. ((Föllmer and Schied 2002a), (Frittelli and Gianin 2002)) A *convex risk measure* $\rho : \mathbb{F} \to \mathbb{R} \cup \{\infty\}$ is a mapping satisfying the following properties, for $X, Y \in \mathbb{F}$,

(i) (convexity): $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \lambda \in (0,1)$;

(ii) (monotonicity): If $X \leq Y$, then $\rho(X) \geq \rho(Y)$;

(iii) (translation invariance): $\rho(X+m) = \rho(X) - m, \qquad m \in \mathbb{R}$.

As its name suggests, a risk measure serves to evaluate the risk exposure associated with a certain financial asset or a project. The defining properties of the risk measure have concrete economic interpretations. Thus, the latter property in the above definition means that adding an amount of cash m to the portfolio reduces the portfolio's risk by the same amount, while the second property implies that a financial project Y, which generates higher profits than another project X, must have a lower risk measure. The first property, which is a relaxation of a stronger sub-additivity property, i.e. $\rho(X+Y) \leq \rho(X) + \rho(Y)$, that characterizes coherent risk measures, demonstrates the virtue of diversification. It can be illustrated as follows. The risk measure associated with e.g. financial operations of a bank must not exceed the sum of risk measures associated with the work of its individual departments. Had it been otherwise, it would have made more sense to split the bank and operate its departments as separate entities.

A popular example of a convex risk measure is the *Expected Shortfall*, which has the following interpretation. The expected shortfall at a q % confidence level is the expected loss of the portfolio in the worst $(1-q)$ % of the cases. This risk measure is computed according to the folrmula

$$ES_q(X) := \mathbb{E}\left[x | x < \mu\right], \tag{3.9}$$

where μ is the $(1-q)$ %-quantile of the distribution of X. Another risk measure routinely used in practice is *Value at Risk*. However, there is a lot of criticism against the use of this risk measure. In particular, it is not convex as it often violates the convexity requirement.

Coming back to our issue at hand, if an investor sells a liability to pay out the amount G_τ at the time moment τ and receives an initial payment p for such a contract, then the minimal risk involved for the seller is

$$\Phi_G(v+p) = \inf_{\varphi \in \mathcal{P}} \rho\left(V_\tau^{v+p}(\varphi) - G_\tau\right), \tag{3.10}$$

where $V_\tau^{v+p}(\varphi)$ denotes a replicating portfolio at the time moment τ under a self-financing strategy φ with initial wealth being equal to v, and $\mathcal{P}$ is the set of self-financing strategies such that $V_t^v(\varphi) \geq c$, for some finite constant c and for $0 \leq t \leq \tau$.

If the investor does not issue a claim (and hence no initial payment p is received), then the minimal risk for the investor is

$$\Phi_0(v) = \inf_{\varphi \in \mathcal{P}} \rho(V_\tau^v(\varphi)). \tag{3.11}$$

We formulate the risk indifference pricing principle in the form of the following definition.

Definition 3.2. The seller's *risk indifference price*, $p = p_{risk}^{seller}$, of the claim G is the solution p of the equation:

$$\Phi_G(v+p) = \Phi_0(v). \tag{3.12}$$

Thus p_{risk}^{seller} is the initial payment p that makes an investor risk indifferent between selling the contract with liability payoff G and not selling the contract.

We are now going to recast the risk indifference pricing problem in the context of stochastic differential games. For that purpose we are going to need the following representation formula for a convex risk measure, suggested by (Föllmer and Schied 2002b).

Theorem 3.3. (Representation Theorem (Föllmer and Schied 2002b), (Föllmer and Schied 2002a), (Frittelli and Gianin 2002)) *A map $\rho : \mathbb{F} \to \mathbb{R}$ is a convex risk measure if and only if there exists a family $\mathcal{L}$ of measures $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_\tau$ and a convex "penalty" function $\zeta : \mathcal{L} \to (-\infty, +\infty)$ with $\inf_{\mathbb{Q}\in\mathcal{L}} \zeta(\mathbb{Q}) = 0$ such that*

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{L}} \{\mathbb{E}_{\mathbb{Q}}[-X] - \zeta(\mathbb{Q})\}, \qquad X \in \mathbb{F}. \tag{3.13}$$

This representation shows that every convex risk measure ρ is defined by the corresponding family of measures $\mathcal{L}$, and the penalty function ζ. Equalities (3.10) and (3.11) now look as follows:

$$\Phi_G(v+p) = \inf_{\varphi\in\mathcal{P}} \Big(\sup_{\mathbb{Q}\in\mathcal{L}} \{\mathbb{E}_{\mathbb{Q}}[-V_\tau^{v+p}(\varphi) + G_\tau] - \zeta(\mathbb{Q})\} \Big), \tag{3.14}$$

and

$$\Phi_0(v) = \inf_{\varphi\in\mathcal{P}} \Big(\sup_{\mathbb{Q}\in\mathcal{L}} \{\mathbb{E}_{\mathbb{Q}}[-V_\tau^{v}(\varphi)] - \zeta(\mathbb{Q})\} \Big), \tag{3.15}$$

for a given penalty function ζ and the family of measures $\mathcal{L}$.

In the case of (local) martingale measures $\mathbb{Q} \in \mathcal{L}$, these equalities can be seen as two stochastic differential games, in which Player 1 - the trader - wants to minimize his risk exposure by choosing an appropriate trading strategy φ; while Player 2 - the market - seeks to maximize the corresponding expectation defining the risk measure ρ, by choosing the optimal measure $\mathbb{Q}$. As we will show in the following sections, one can use the tools, such as the stochastic maximum principle, available in the field of stochastic differential games to simplify these problems in a way that will enable us to give a simplified formula for the risk indifference price of an interest rate claim.

4 Modelling framework

We consider the situation in which the investor is able to construct a replicating portfolio only by holding traditional bonds, i.e. bonds with fixed *dates of maturity*, $T \in (0, \infty)$. In such a situation, to replicate the payoff of an option written on bonds with constant *time to maturity* will in general require an infinite dimensional portfolio, i.e. the one containing infinitely many bonds with different dates of maturity. In order to better explain the construction of such an infinite dimensional portfolio we begin with a simple case. Suppose there are just 2 bonds with dates of maturity T_1 and T_2. Then the portfolio value will be given by:

$$V_t(\pi) := \pi_t^0 \cdot 1 + \pi_t^1 \cdot P(t, T_1) + \pi_t^2 \cdot P(t, T_2), \tag{4.1}$$

where π_t^0 is the number of units of the bank account held in the portfolio; and π_t^i, $i = 1, 2$ are the number of units of bonds with dates of maturity T_1 and T_2 correspondingly.

The dynamics of the portfolio value will look as follows:

$$
\begin{aligned}
(4.2)\, dV_t(\pi) &:= \pi_t^1 \cdot dP(t,T_1) + \pi_t^2 \cdot dP(t,T_2) \\
&= \pi_t^1 \cdot [P(t,T_1)\, b_t(T_1 - t)]\; dt - \\
&\qquad -\pi_t^1 \cdot \sum_{j\geq 1} P(t,T_1)\, \delta_t^{(j)}(T_1 - t)\, dB_t^{(j)} \\
&\quad +\pi_t^2 \cdot [P(t,T_2)\, b_t(T_2 - t)]\; dt - \\
&\qquad -\pi_t^2 \cdot \sum_{j\geq 1} P(t,T_2)\, \delta_t^{(j)}(T_2 - t)\, dB_t^{(j)} = \\
&= \Big[\pi_t^1 \cdot P(t,T_1) \cdot b_t(T_1 - t) + \pi_t^2 \cdot P(t,T_2) \cdot b_t(T_2 - t)\Big]\, dt \\
&\quad - \sum_{j\geq 1} \Big[\pi_t^1 \cdot P(t,T_1)\, \delta_t^{(j)}(T_1 - t) + \pi_t^2 \cdot P(t,T_2)\, \delta_t^{(j)}(T_2 - t)\Big]\; dB_t^{(j)}
\end{aligned}
$$

Consider an $\mathcal{H}^*$-valued process φ_t given by

$$\varphi_t := \beta_1 \cdot \delta_{T_1 - t} + \beta_2 \cdot \delta_{T_2 - t}, \tag{4.3}$$

where δ_x is the evaluation functional and $\beta_i(t) := \frac{\pi_t^i \cdot P(t,T_i)}{V_t(\pi)}$, if $V_t(\pi) \neq 0$, is a fraction of wealth invested in the bond with date of maturity T_i, $i = 1, 2$. Then equation (4.2) becomes

$$
\begin{aligned}
dV_t(\varphi) &= V_t(\varphi) \cdot \varphi_t(b_t(\cdot))\, dt \\
&\qquad -V_t(\varphi) \cdot \sum_{j\geq 1} \varphi_t(\delta_t^{(j)}(\cdot))\, dB_t^{(j)}
\end{aligned}
\tag{4.4}
$$

We can view the process φ_t in (4.4) as representing a generalized portfolio strategy, which can now be infinite dimensional.

In the sequel we say that an $\mathcal{H}^*$-valued process φ_t is a *self-financing strategy* if the risk-neutral evolution of the discounted portfolio value is given by

$$dV_t(\varphi) \;=\; -V_t(\varphi) \cdot \sum_{j\geq 1} \varphi_t(\delta^{(j)}(\cdot))\, d\tilde{B}_t^{(j)}, \tag{4.5}$$

where $\tilde{B}_t^{(j)} = B_t^{(j)} - \int_0^t \lambda_s^{(j)}\, ds$, $j \geq 1$ are Brownian motions under a martingale measure and $\lambda_t^{(j)}$, $j \geq 1$ are the risk premium processes.

Let $\mathcal{P}$ be the class of such self-financing strategies. In what follows we want to consider hedging strategies $\varphi \in \mathcal{P}$ of traders with limited access to market information, i.e. we assume that $\varphi \in \mathcal{P}$ is $\mathcal{E}_t$-predictable, where $\mathcal{E}_t \subseteq \mathcal{F}_t$. We shall also call a strategy $\varphi \in \mathcal{P}$ *admissible* if φ is $\mathcal{E}_t$-predictable, solves (4.5) in the strong sense and satisfies

$$\int_0^{\tau} \Big\{ |V_t(\varphi) \cdot \varphi_t(b_t(\cdot))| + \sum_{j\geq 1} V_t(\varphi)^2 \cdot \varphi_t(\delta_t^{(j)}(\cdot))^2 \Big\} dt < \infty.$$

The collection of such strategies is denoted by Π.

Let us consider the case of unrestricted access to market information. Then a market with respect to our model is referred to as *complete* if each contingent claim can be replicated. This means that for all square-integrable (non-negative) $\mathcal{F}_\tau$-measurable random variables h there exists an admissible strategy φ such that

$$V_\tau(\varphi) = h$$

An advantage of our generalized bond model (3.8) is that replicating strategies are unique (under certain conditions on $\tilde{\sigma}_t$ in (3.5)). See (Carmona and Tehranchi 2006). Furthermore, this model satisfies the intuitive requirement that bond maturities used in the hedging strategies do correspond to those of the underlying of the claim. These natural properties, however, cannot be captured by finite-rank models, such as (2.3). In such models replicating hedging strategies are not unique in general and call options written on a 5-year bond can be hedged by e.g. a 30-year bond. This is a shortcoming that contradicts market practice.

On the other hand, a deficiency of our infinite-dimensional HJM framework is that the existence of the unique martingale measure does not in general imply the completeness of our bond market model. This is actually a property not exhibited by finite rank models. However, one can show that if the kernel of $\tilde{\sigma}_t$ in (3.5) is zero (t, ω)-a.e. then our bond market is *approximately* complete, that is for all contingent claims h and all $\epsilon > 0$ there is an admissible strategy ϕ^ϵ such that

$$\mathbb{E}_{\tilde{\mathbb{P}}}\Big[\Big(\mathbb{E}_{\tilde{\mathbb{P}}}(h) + \int_0^\tau \phi_t^\epsilon \circ \tilde{\sigma}_t \, d\tilde{B}_s - h\Big)^2\Big] < \epsilon$$

See (Carmona and Tehranchi 2006).

Now we define the measures $\mathbb{Q}_q$ parametrized by given $\mathcal{E}_t$-predictable processes $q_t := \Big\{q_t^{(j)}\Big\}_{j\geq 1}$ such that

$$d\mathbb{Q}_q(\omega) := K_\tau \cdot d\mathbb{P}(\omega) \qquad \text{on } \mathcal{F}_\tau, \tag{4.6}$$

where $\mathbb{P}$ is the objective probability measure and $\mathbb{Q}_q$ is a measure absolutely continuous with respect to $\mathbb{P}$. The Radon-Nikodym derivative K_τ is defined as follows:

$$dK_t := \sum_{j\geq 1} K_t \, q_t^{(j)} \, dB_t^{(j)}, \qquad K_0 = k \tag{4.7}$$

We say that the control q is *admissible*, and write $q \in \Theta$, if $q_t^{(j)}$ is adapted to the sub-filtration $\mathcal{E}_t$ for all j, such that

$$\int_0^\tau \sum_{j\geq 1} \Big(q_t^{(j)}\Big)^2 \, dt < \infty$$

and

$$\mathbb{E}[K_\tau] = k > 0. \tag{4.8}$$

Further, we define $\mathcal{L}$ in Theorem 3.3 to be the class of measures given by

$$\mathcal{L} := \{\mathbb{Q}_q \,:\, q \in \Theta\} \tag{4.9}$$

Thus, the control process - denoted by u_t - in our stochastic control problems (3.14) and (3.15) consists of the processes $\left\{q_t^{(j)}\right\}_{j\geq 1}$ determining the risk measure, chosen by the market, and the portfolio strategy φ_t chosen by the investor:

$$u_t = \begin{bmatrix} \left\{q_t^{(j)}\right\}_{j\geq 1} \\ \varphi_t \end{bmatrix} \tag{4.10}$$

Our state process is given by

$$Y_t = \begin{bmatrix} K_t \\ P_t(\cdot) \\ V_t(\varphi) \end{bmatrix} := \begin{bmatrix} \tilde{Y}_t \\ V_t(\varphi) \end{bmatrix}, \qquad y := Y_0 = \begin{bmatrix} k \\ P_0(\cdot) \\ V_0(\varphi) \end{bmatrix} \tag{4.11}$$

Its dynamics is described by the following SPDE:

$$\begin{aligned} dY_t = & \begin{bmatrix} 0 \\ AP_t(\cdot) + P_t(\cdot)\cdot b_t(\cdot) \\ V_t(\varphi)\cdot\varphi_t(b_t(\cdot)) \end{bmatrix} dt + \\ & + \begin{bmatrix} K_t\, q_t^{(1)} & K_t\, q_t^{(2)} & \dots \\ -P_t(\cdot)\,\delta_t^{(1)}(\cdot) & -P_t(\cdot)\,\delta_t^{(2)}(\cdot) & \dots \\ -V_t(\varphi)\cdot\varphi_t(\delta_t^{(1)}(\cdot)) & -V_t(\varphi)\cdot\varphi_t(\delta_t^{(2)}(\cdot)) & \dots \end{bmatrix} \cdot \begin{bmatrix} dB_t^{(1)} \\ dB_t^{(2)} \\ \vdots \end{bmatrix} \end{aligned} \tag{4.12}$$

We now define another set $\mathcal{M}$ of measures as follows:

$$\mathcal{M} := \{\mathbb{Q}_q;\, q \in \mathbb{M}\}, \tag{4.13}$$

where

$$\mathbb{M} := \{q \in \Theta \,:\, \mathbb{E}[b_t(x) - \sum_{j\geq 1} \delta_t^{(j)}(x) q_t^{(j)} | \mathcal{E}_t] = 0,\, \forall t, x\}. \tag{4.14}$$

Thus, if $k = 1$ in (4.8) then the measures $\mathbb{Q}_q$ in $\mathcal{M}$ become equivalent martingale measures with respect to bond prices given by

$$\begin{aligned} d\overline{P}_t(x) \;\; = & \;\; \Big(A\overline{P}_t(x) + \overline{P}_t(x)\mathbb{E}[b_t(x)|\mathcal{E}_t]\Big)\, dt \\ & + \overline{P}_t(x) \sum_{j\geq 1} \mathbb{E}[\delta_t^{(j)}(x)|\mathcal{E}_t]\, dB_t^{(j)} \end{aligned} \tag{4.15}$$

To complete the definition of our benchmark risk measure, as given in (3.13), we require that the penalty function ζ takes the form

$$\zeta(\mathbb{Q}_q) := \mathbb{E}_{\mathbb{P}}\left[\int_0^{\tau} \Lambda\left(t, q_t, \tilde{Y}_t\right)\, dt \,+\, h(\tilde{Y}_\tau)\right] \tag{4.16}$$

for some convex functions $\Lambda : [0,\infty) \times \mathcal{H} \times \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ and $h : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$, such that

$$\mathbb{E}\Big[\int_0^{\tau} |\Lambda\left(t, q_t, \tilde{Y}_t\right)| dt + |h(\tilde{Y}_\tau)|\Big] < \infty,$$

for all $q = (q_j)_{j\geq 1} \in \Theta$. Thus, the risk measure ρ, which we are going to use, is given in Equation (3.13) with $\mathcal{L}$ defined in (4.9) and $\zeta(\mathbb{Q})$ as given above, in Equation (4.16).

Now we formulate our stochastic differential game problem corresponding to equation (3.14), incorporating the form of the option payoff (3.1) and the representation formula (3.13) for our benchmark risk measure ρ.

Problem A: Determine $\Phi_G^{A,\mathcal{E}}(t,y)$ and $(q^*, \varphi^*) \in \Theta \times \Pi$, such that

$$\Phi_G^{A,\mathcal{E}}(t,y) = \inf_{\varphi\in\Pi}\Big(\sup_{q\in\Theta} J_A^{q,\varphi}(t,y)\Big) = J_A^{q^*,\varphi^*}(t,y), \tag{4.17}$$

where

$$\begin{aligned} J_A^{q,\varphi}(t,y) &:= \mathbb{E}_{\mathbb{P}}^{y}\left[\int_0^{\tau} -\Lambda\left(s, q_s, \tilde{Y}_s\right)\, ds \,-\, h(\tilde{Y}_\tau) \,+ \int_0^{\tau} K_s \cdot m(s, P_s(\cdot))\, ds + \right. \\ &+ \left. K_\tau \cdot g(P_\tau(\cdot)) - K_\tau \cdot V_\tau(\varphi)\right] \\ &= \mathbb{E}_{\mathbb{P}}^{y}\left[\int_t^{\tau} -\tilde{\Lambda}\left(s, q_s, \tilde{Y}_s\right)\, ds \,+ \Psi(Y_\tau)\right], \end{aligned} \tag{4.18}$$

where the functions $\tilde{\Lambda} : [0,\infty) \times \mathcal{H} \times \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ and $\Psi : \mathbb{R} \times \mathcal{H} \times \mathbb{R} \to \mathbb{R}$ are defined as

$$\Psi(K_t, P_t(\cdot), V_t(\varphi)) := -h(K_t, P_t(\cdot)) \,+\, K_t \cdot g(P_t(\cdot)) - K_t \cdot V_t(\varphi) \tag{4.19}$$

and

$$\tilde{\Lambda}\left(t, q_t, K_t, P_t(\cdot)\right) := \Lambda\left(t, q_t, K_t, P_t(\cdot)\right) - K_t \cdot m(t, P_t(\cdot)). \tag{4.20}$$

Here we assume that $\tilde{\Lambda} \in C_b^{1,1}\big([0,\infty) \times \mathcal{H} \times \tilde{\mathcal{H}}\big)$ for $\tilde{\mathcal{H}} := \mathbb{R} \times \mathcal{H}$, i.e. $\tilde{\Lambda}$ is continuously Fréchet differentiable w.r.t. $(t, q_t) \in (0,\infty) \times \mathcal{H}$ and $(K_t, P_t(\cdot)) \in \tilde{\mathcal{H}}$, $\forall t$, with bounded partial derivatives, which have continuous extensions to $[0,\infty) \times \mathcal{H} \times \tilde{\mathcal{H}}$. Further, suppose that $\Psi \in C_b^1(X)$, where $X := \mathbb{R} \times \mathcal{H} \times \mathbb{R}$.

Later in this paper we want to exploit a certain connection between Problem A and the following stochastic control problem:

$$\Phi_G^{B,\mathcal{E}} = \sup_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}_{\mathbb{Q}}[G_\tau] - \zeta(\mathbb{Q})\} \tag{4.21}$$

The latter will enable us to simplify the problem setting by removing one of the controls, namely the trading strategy φ. Using our notation for $\tilde{Y}_t$, this new problem can be stated as follows:

Problem B: Search for $\Phi_G^{B,\mathcal{E}}(t,\tilde{y})$ and $\check{q} \in \mathbb{M}$, such that

$$\Phi_G^{B,\mathcal{E}}(t,\tilde{y}) = \sup_{q\in\mathbb{M}} J_B^q(t,\tilde{y}) = J_B^{\check{q}}(t,\tilde{y}), \tag{4.22}$$

where

$$\tilde{y}_t = \begin{bmatrix} k \\ P_0(\cdot) \end{bmatrix} \tag{4.23}$$

and

$$\begin{aligned} J_B^q(t,\tilde{y}) &:= \mathbb{E}_{\mathbb{P}}^{\tilde{y}}\left[\int_t^\tau -\Lambda\left(s,q_s,\tilde{Y}_s\right)\,ds \,-\, h(\tilde{Y}_\tau) \,+ \int_t^\tau K_s\cdot m(s,P_s(\cdot))\,ds+\right.\\ &+ \left. K_\tau\cdot g(P_\tau(\cdot))\right]\\ &= \mathbb{E}_{\mathbb{P}}^{\tilde{y}}\left[\int_t^\tau -\tilde{\Lambda}\left(s,q_s,\tilde{Y}_s\right)\,ds \,+\Phi(\tilde{Y}_\tau)\right], \end{aligned} \tag{4.24}$$

where the function $\Phi : \mathbb{R}\times\mathcal{H}\to\mathbb{R}$ is given by

$$\Phi(K_t,P_t(\cdot)) := -h(K_t,P_t(\cdot)) \,+\, K_t\cdot g(P_t(\cdot)). \tag{4.25}$$

We require here that $\Phi \in C_b^1(V)$, for $V := \mathbb{R}\times\mathcal{H}$.

As for Problem A, we aim at introducing the following *Hamiltonian* $H^A : [0,\infty)\times\mathbb{R}\times\mathcal{H}\times\mathbb{R}\times\mathcal{H}\times\mathcal{H}^*\times(\mathbb{R}\times\mathcal{H}\times\mathbb{R})\times(\mathcal{H}\times L_2(U,\mathcal{H})\times\mathcal{H})\to\mathbb{R}$ given by

$$\begin{aligned} H^A(t,K_t,P_t(\cdot),V_t(\varphi),q_t,\varphi_t,\mathbf{p}^A,\mathbf{q}^A) &:= -\tilde{\Lambda}\left(t,q_t,K_t,P_t(\cdot)\right) + \langle (P_t\,b_t)(\cdot),p_2^A\rangle_K\\ &+ V_t(\varphi)\cdot\varphi(b_t(\cdot))\cdot p_3^A\\ &+ K_t\cdot\langle q_t,q_1^A\rangle_K - \sum_{j\ge1}\langle (P_t\cdot\delta_t^{(j)}(\cdot),q_2^{A,(j)}\rangle_K\\ &- \sum_{j\ge1} V_t(\varphi)\cdot\varphi(\delta_t^{(j)}(\cdot))\cdot q_3^{A,(j)}, \end{aligned} \tag{4.26}$$

where

$$\mathbf{p}^A = \begin{bmatrix} p_1^A \\ p_2^A \\ p_3^A \end{bmatrix}, \qquad \text{and} \qquad \mathbf{q}^A = \begin{bmatrix} q_1^A \\ q_2^A \\ q_3^A \end{bmatrix}, \tag{4.27}$$

with $q_i^A = \sum_{j\ge1} q_i^{A,(j)} u_j$, $i=1,3$, $q_2^A = \{q_2^{A,(j)}\}_{j\ge1}$ for an orthonormal basis u_j, $j\ge1$ of $\mathcal{H}$.

On the other hand, we can define the Hamiltonian for Problem B as a map $H^B : [0,\infty) \times \mathbb{R} \times \mathcal{H} \times \mathcal{H} \times (\mathbb{R} \times \mathcal{H}) \times (\mathcal{H} \times L_2(U,\mathcal{H})) \to \mathbb{R}$ given by

$$\begin{aligned} H^B(t, K_t, P_t(\cdot), q_t, \mathbf{p}^B, \mathbf{q}^B) \;&:=\; -\tilde{\Lambda}\left(t, q_t, K_t, P_t(\cdot)\right) + \langle (P_t(\cdot)\, b_t)(\cdot), p_2^B \rangle_K \\ &+\; K_t \cdot \langle q_t, q_1^B \rangle_K \;-\; \sum_{j\geq 1} \langle (P_t \cdot \delta_t^{(j)}(\cdot), q_2^{B,(j)} \rangle_K, \end{aligned} \tag{4.28}$$

where

$$\mathbf{p}^B = \begin{bmatrix} p_1^B \\ p_2^B \end{bmatrix}, \qquad \text{and} \qquad \mathbf{q}^B = \begin{bmatrix} q_1^B \\ \{q_2^{B,(j)}\}_{j\geq 1} \end{bmatrix}, \tag{4.29}$$

Let us require that H^A and H^B are Fréchet differentiable with respect to $(K_t, P_t(\cdot), V_t(\varphi)) \in \mathbb{R}\times\mathcal{H}\times\mathbb{R}$ and $(K_t, P_t(\cdot)) \in \mathbb{R}\times\mathcal{H}$, respectively. In the sequel we denote by ∇g the gradient of a function $g : Z \to Z$ on a Hilbert space Z. We recall that $\nabla g : Z \to Z$ is a function characterized by the equation

$$\langle (\nabla g)(x),\, h \rangle_Z = (Dg)(x)(h), \tag{4.30}$$

for all $x, h \in Z$, where $(Dg)(x)(h)$ is the directional derivative at point x in the direction of h.

The adjoint equations with respect to H^A are given by the following backward stochastic (partial) differential equations:

$$\begin{cases} dp_1^A(t) &= \left[\nabla_{K_t}\tilde{\Lambda}\left(t, q_t, \tilde{Y}_t\right) \;-\; \langle q_t, q_1^A(t)\rangle_K\right] dt + \sum_{j\geq 1} q_1^{A,(j)}(t)\, dB_t^{(j)} \\ p_1^A(\tau) &= -\nabla_{K_t} h(\tilde{Y}_\tau) - g(P_\tau(\cdot)) - V_\tau(\varphi) \end{cases} \tag{4.31}$$

$$\begin{cases} dp_2^A(t,x) &= \left[-\nabla_{P_t(\cdot)} F(t, q_t, \tilde{Y}_t, p_t^*, q_t^*) - A^* p_2^A(t,x)\right] dt \\ &\quad + \sum_{j\geq 1} q_2^{A,(j)}(t,x)\, dB_t^{(j)} \\ p_2^A(\tau, x) &= -\nabla_{P_t(\cdot)} h(\tilde{Y}_\tau) \;+\; K_\tau \cdot \nabla_{P_t(\cdot)} g(P_\tau(\cdot)), \end{cases} \tag{4.32}$$

where A^* is the adjoint operator for the differential operator A in (3.6) and F is a function given by

$$F(t, q_t, K_t, P_t(\cdot), \mathbf{p}^A, \mathbf{q}^A) := -\tilde{\Lambda}\left(t, q_t, K_t, P_t(\cdot)\right) + \langle (P_t \cdot b_t)(\cdot), p_2^A\rangle_K - \sum_{j\geq 1} \langle (P_t \cdot \delta_t^{(j)})(\cdot), q_2^{A,(j)} \rangle_K \tag{4.33}$$

$$\begin{cases} dp_3^A(t) &= \left[-\varphi_t(b_t(\cdot)) \cdot p_3^A(t) + \sum_{j\geq 1} \varphi_t(\delta_t^{(j)}(\cdot)) \cdot q_3^{A,(j)}(t)\right] dt \\ &\quad + \sum_{j\geq 1} q_3^{A,(j)}(t)\, dB_t^{(j)} \\ p_3^A(\tau) &= -K_\tau \end{cases} \tag{4.34}$$

On the other hand, the adjoint equations with respect to the Hamiltonian H^B take the form

$$(4.35)\begin{cases} dp_1^B(t) &= \left[\nabla_{K_t}\tilde{\Lambda}\left(t, q_t, \tilde{Y}_t\right) - \langle q_t, q_1^B(t)\rangle_K\right] dt + \sum_{j\geq 1} q_1^{B,(j)}(t)\cdot dB_t^{(j)} \\ p_1^B(\tau) &= -\nabla_{K_t} h(K_\tau, P_\tau(\cdot)) + g(P_\tau(\cdot)) \end{cases}$$

$$(4.36)\qquad \begin{cases} dp_2^B(t,x) &= \left[-\nabla_{P(\cdot)}\tilde{F}(t, q_t, \tilde{Y}_t, \mathbf{p}_t^B, \mathbf{q}_t^B) - A^* p_2^B(t,x)\right] dt \\ & \quad + \sum_{j\geq 1} q_2^{B,(j)}(t,x)\cdot dB_t^{(j)} \\ p_2^B(\tau, x) &= -\nabla_{P(\cdot)} h(\tilde{Y}_\tau) + K_\tau \cdot \nabla_{P(\cdot)} g(P_\tau(\cdot)), \end{cases}$$

where $\tilde{F}$ is a function defined by
(4.37)

$$\tilde{F}(t, q_t, K_t, P_t(\cdot), \mathbf{p}^B, \mathbf{q}^B) := -\tilde{\Lambda}\left(t, q_t, K_t, P_t(\cdot)\right) + \langle (P_t\cdot b_t)(\cdot), p_2^B\rangle_K - \sum_{j\geq 1}\langle (P_t\cdot \delta_t^{(j)})(\cdot), q_2^{B,(j)}\rangle_K$$

Regarding the conditions ensuring the existence and uniqueness of (strong) solutions of such B(S)PDEs the reader may consult e.g. (Hu and Peng 1996), (Øksendal, Proske, and Zhang 2005) and the references therein.

The next auxiliary result gives a link between the solutions of the adjoint equations (4.31), (4.32) and (4.34) for Problem A and (4.35) and (4.36) for Problem B, as well as the relation between Hamiltonians H^A and H^B in Problems A and B, respectively.

Lemma 4.1. *Choose $\forall q \in \Theta$ and $\forall \varphi \in \Pi$. If the chosen $q \in \mathbb{M}$, then the solutions of the adjoint equations for Problem A and Problem B are connected as follows:*

$$(4.38)\qquad p_1^A(t) := p_1^B(t) - V_t(\varphi)$$

$$(4.39)\qquad p_2^A(t,x) = p_2^B(t,x)$$

$$(4.40)\qquad p_3^A(t) = -K_t$$

where $\mathbf{p}^B(t) = (p_1^B(t), p_2^B(t))$ is a (strong) solution of the corresponding adjoint equations (4.35) *and* (4.36) *for Problem B, and $\mathbf{p}^A(t) = (p_1^A(t), p_2^A(t), p_3^A(t))$ is a (strong) solution of the adjoint equations* (4.31), (4.32) *and* (4.34) *for Problem A. Moreover, the Hamiltonians in Problem A and Problem B are related to each other as follows:*

$$(4.41)\qquad \begin{aligned} H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) &= H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) \\ + \quad K_t\cdot V_t(\varphi) & \left[\varphi_t\left(2\sum_{j\geq 1} q_t^{(j)}\cdot \delta_t^{(j)}(\cdot) - b_t(\cdot)\right)\right] \end{aligned}$$

Proof. Our proof closely follows the arguments in (An, Øksendal, and Proske 2008), Lemma 3.1, where the finite dimensional case was treated. Using the dynamics of $p_1^A(t)$, $p_1^B(t)$ and $V_t(\varphi)$ we find that

$$
\begin{aligned}
(4.42)\qquad dp_1^A(t) &= dp_1^B(t) - dV_t(\varphi) \\
&= \left[\nabla_{K_t}\tilde{\Lambda}\left(t, q_t, \tilde{Y}_t\right) - \sum_{j\geq 1} q_t^{(j)} \cdot q_1^{B,(j)}(t)\right] dt + \sum_{j\geq 1} q_1^{B,(j)}(t)\, dB_t^{(j)} \\
&\quad - V_t(\varphi)\cdot\varphi_t(b_t(\cdot))\, dt + V_t(\varphi)\cdot\sum_{j\geq 1}\varphi_t(\delta_t^{(j)}(\cdot))\, dB_t^{(j)} \\
&= \left[\nabla_{K_t}\tilde{\Lambda}\left(t, q_t, \tilde{Y}_t\right) - \sum_{j\geq 1} q_t^{(j)} \cdot q_1^{B,(j)}(t) - V_t(\varphi)\cdot\varphi_t(b_t(\cdot))\right] dt \\
&\quad + \sum_{j\geq 1}\left[q_1^{B,(j)}(t) + V_t(\varphi)\cdot\varphi_t(\delta_t^{(j)}(\cdot))\right] dB_t^{(j)}
\end{aligned}
$$

So, it follows from (4.31) that

$$
(4.43)\qquad -\sum_{j\geq 1} q_t^{(j)} \cdot q_1^{A,(j)}(t) = -\sum_{j\geq 1} q_t^{(j)} \cdot q_1^{B,(j)}(t) - V_t(\varphi)\cdot\varphi_t(b_t(\cdot)),
$$

and

$$
(4.44)\qquad q_1^{A,(j)}(t) = q_1^{B,(j)}(t) + V_t(\varphi)\cdot\varphi_t(\delta_t^{(j)}(\cdot))
$$

One can see that (4.43) holds, provided that $\varphi_t(\sum_{j\geq 1}\delta_t^{(j)}(\cdot)\, q_t^{(j)}) = \varphi_t(b_t(\cdot))$. Since the latter equality must be satisfied for every admissible strategy φ_t one concludes that $\sum_{j\geq 1}\delta_t^{(j)}(x)\, q_t^{(j)} = b_t(x)$, for all x, which also implies that $q \in \mathbb{M}$, as claimed.

Doing the same thing for equation (4.40) we observe that

$$
(4.45)\qquad -\varphi_t(b_t(\cdot))\cdot p_3^A(t) + \sum_{j\geq 1}\varphi_t(\delta_t^{(j)}(\cdot))\cdot q_3^{A,(j)}(t) = 0
$$

and

$$
(4.46)\qquad q_3^{A,(j)}(t) = -K_t\, q_t^{(j)}.
$$

Substituting (4.46) into (4.45) we see that the latter is satisfied provided that $p_3^A(t) = -K_t$ and $\sum_{j\geq 1}\delta_t^{(j)}(x)\, q_t^{(j)} = b_t(x)$, for all x, as claimed.

Now, the Hamiltonian in Problem A and the one in Problem B are related to each other as follows:

$$
\begin{aligned}
H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) &= H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^A, \mathbf{q}^A) + \\
&+ V_t(\varphi)\cdot\varphi_t(b_t(\cdot))\cdot p_3^A(t) - \\
(4.47)\qquad &- \sum_{j\geq 1} V_t(\varphi)\cdot\varphi_t(\delta_t^{(j)}(\cdot))\cdot q_3^{A,(j)}(t)
\end{aligned}
$$

Using (4.38), (4.39), (4.40) and (4.44), as well as assuming that $q_t \in \mathbb{M},\ \forall t \in [0,\tau)$, we obtain

$$
\begin{aligned}
H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) &= H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) \\
&\quad + \sum_{j\geq 1} K_t\, q_t^{(j)} \cdot V_t(\varphi) \cdot \varphi_t(\delta_t^{(j)}(\cdot)) - V_t(\varphi) \cdot \varphi_t(b_t(\cdot)) \cdot K_t \\
&\quad + \sum_{j\geq 1} V_t(\varphi) \cdot \varphi_t(\delta_t^{(j)}(\cdot)) \cdot K_t\, q_t^{(j)} \qquad (4.48) \\
&= H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) + K_t \cdot V_t(\varphi) \left[\varphi_t \left(2 \sum_{j\geq 1} q_t^{(j)} \cdot \delta_t^{(j)}(\cdot) - b_t(\cdot)\right)\right]
\end{aligned}
$$

∎

Thus, Lemma 4.1 claims that the Hamiltonians, as well as the solutions to adjoint equations for Problems A and B are connected in the above stated way, provided that $q \in \mathbb{M}$. The following Lemma states the connection between Problems A and B working in the opposite direction. Namely, if Equations (4.38), (4.39) and (4.40) hold and certain optimum conditions are satisfied, then indeed $q \in \mathbb{M}$.

Lemma 4.2. *Suppose that $p_1^A(t)$, $p_2^A(t)$ and $p_3^A(t)$ are given by Equations* (4.38), (4.39) *and* (4.40), *with $\mathbf{p}^B(t) = (p_1^B(t), p_2^B(t))$ being a (strong) solution of the adjoint equations* (4.35) *and* (4.36) *for Problem B, as in Lemma 4.1. Also, let the function*

$$q = \{q^{(j)}\}_{j\geq 1} \mapsto \mathbb{E}[H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t], \qquad q \in \Theta,$$

have a maximum point at $\hat{q} = \{\hat{q}^{(j)}\}_{j\geq 1} = \{\hat{q}^{(j)}(\varphi)\}_{j\geq 1}$, for all $\varphi \in \Pi$, and the function

$$\varphi \mapsto \mathbb{E}[H^A(t, Y_t, \hat{q}_t(\varphi), \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t], \qquad \varphi \in \Pi,$$

attain a minimum point at $\hat{\varphi} \in \Pi$. Then,

$$\hat{q}(\hat{\varphi}) \in \mathbb{M}. \tag{4.49}$$

Proof. In what follows we want to use the following notation: $q = \{q_j\}_{j\geq 1}$ and $\varphi = \{\varphi_i\}_{i\geq 1}$ if $q = \sum_{j\geq 1} q_j\, u_j$ and $\varphi = \sum_{j\geq 1} \varphi_i\, v_i$ for an orthonormal basis u_j and v_i in $\mathcal{H}$ and $\mathcal{H}^*$, respectively.

The assumption that the function $\mathbb{E}[H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t]$ has a maximum at $\hat{q}^{(j)} = \hat{q}^{(j)}(\varphi)$ implies that

$$\mathbb{E}[\nabla_{q^{(j)}}(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)_{q^{(j)}=\hat{q}^{(j)}(\varphi)}|\mathcal{E}_t] = 0, \qquad j \geq 1,\ \forall \varphi \in \Pi \tag{4.50}$$

Similarly, the necessary condition for the function $\mathbb{E}[H^A(t, Y_t, \hat{q}_t(\varphi), \varphi_t, \mathbf{p}^A, \mathbf{q}^A)|\mathcal{E}_t]$ to attain a minimum at $\hat{\varphi}$ is

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\sum_{j\geq 1} \nabla_{q^{(j)}}(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A) \cdot \nabla_{\varphi_i}\Big(\hat{q}^{(j)}(\varphi)\Big) \qquad (4.51) \\
&\quad + \nabla_{\varphi_i}\Big(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)\Big)\Big)_{\substack{\varphi_i=\hat{\varphi}_i \\ q^{(j)}=\hat{q}^{(j)}(\hat{\varphi})}} \Big|\mathcal{E}_t\Big] = 0, \qquad i \geq 1,
\end{aligned}
$$

Choose $\varphi = \hat{\varphi}$. Then, by (4.50) and (4.51), we obtain

$$\mathbb{E}\Big[\nabla_{\varphi_i}\big(H^A(t, Y_t, q_t, \varphi_t, \mathbf{p}^A, \mathbf{q}^A)\big)_{\substack{\varphi=\hat{\varphi}\\ q=\hat{q}(\hat{\varphi})}}\Big|\mathcal{E}_t\Big] = 0, \qquad i \geq 1 \tag{4.52}$$

Thus, after differentiating the Hamiltonian we obtain

$$\mathbb{E}\Bigg[\Bigg(\Big(V_t(\varphi)\cdot\nabla_{\varphi_i}\varphi(b_t(\cdot)\Big)\cdot p_3^A(t) - V_t(\varphi)\cdot\nabla_{\varphi_i}\Big(\sum_{j\geq 1}\varphi(\delta_t^{(j)}(\cdot)\Big)\cdot q_3^{A,(j)}(t)\Big)\Bigg)_{\substack{\varphi=\hat{\varphi}\\ q=\hat{q}(\hat{\varphi})}}\Bigg|\mathcal{E}_t\Bigg] = 0. \tag{4.53}$$

Combining this result with Lemma 4.1 yields

$$V_t(\varphi)\cdot K_t\cdot v_i\Bigg(\mathbb{E}\Bigg[\Big(b_t(\cdot) - \sum_{j\geq 1}\delta_t^{(j)}(\cdot)\cdot q_t^{(j)}\Big)\Big|\mathcal{E}_t\Bigg]\Bigg)_{\substack{\varphi=\hat{\varphi}\\ q=\hat{q}(\hat{\varphi})}} = 0, \qquad i \geq 1, \tag{4.54}$$

where we have used the fact that $v_i \in \mathcal{H}^*$. The condition that equality (4.54) holds for all $i \geq 1$ implies that for $\varphi = \hat{\varphi}$ and $q = \hat{q}(\hat{\varphi})$ $\mathbb{E}\Bigg[\Big(b_t(x) - \sum_{j\geq 1}\delta_t^{(j)}(x)\cdot q_t^{(j)}\Big)\Big|\mathcal{E}_t\Bigg] = 0$, for all $j \geq 1$ and any $x \in [0, \infty)$, i.e. if $\hat{q}(\hat{\varphi}) \in \mathbb{M}$, as claimed. ∎

5 Maximum principle for stochastic differential games on a generalized bond market

Analogues of Problem A were studied by a number of authors. See e.g. (An and Øksendal 2008), (An, Øksendal, and Proske 2008) and (An, Proske, and Rubtsov 2010). Adapting their results to the present setting, we formulate the following result, which is an extension of Theorem 2.1 in (An and Øksendal 2008).

Theorem 5.1. (Maximum principle for stochastic differential games (An and Øksendal 2008; Ferris and Mangasarian 1992)) *For controls* $(\hat{q}, \hat{\varphi}) \in \Theta \times \Pi$*, suppose that the following partial information maximum principle holds*

$$\begin{aligned}&\sup_{q\in\Theta}\mathbb{E}[H^A(t, Y_t, q_t, \hat{\varphi}_t, \hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A)) \mid \mathcal{E}_t]\\ &= \mathbb{E}[H^A(t, Y_t, \hat{q}_t, \hat{\varphi}_t, \hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A) \mid \mathcal{E}_t]\\ &= \inf_{\varphi\in\Pi}\mathbb{E}[H^A(t, Y_t, \hat{q}_t, \varphi_t, \hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A) \mid \mathcal{E}_t].\end{aligned} \tag{5.1}$$

for all $t \in [0, \tau]$*, with* $(\hat{\mathbf{p}}^A, \hat{\mathbf{q}}^A))$ *being the strong solutions of the adjoint equations* (4.31)*,* (4.32) *and* (4.34) *in Problem A. Moreover, require that the function* $q \mapsto J_A^{q,\varphi}(t, y)$ *defined in* (4.18) *is concave, while* $\varphi \mapsto J_A^{q,\varphi}(t, y)$ *is convex. Then* $(q^*, \varphi^*) := (\hat{q}, \hat{\varphi})$ *is the optimal* *control and*

$$\begin{aligned}\Phi_G^{A,\mathcal{E}}(t,y) &= \inf_{\varphi\in\Pi}\Big(\sup_{q\in\Theta} J_A^{q,\varphi}(t,y)\Big) = \sup_{q\in\Theta}\Big(\inf_{\varphi\in\Pi} J_A^{q,\varphi}(t,y)\Big)\\ &= \sup_{q\in\Theta} J_A^{q,\hat{\varphi}}(t,y) = \inf_{\varphi\in\Pi} J_A^{\hat{q},\varphi}(t,y) = J_A^{\hat{q},\hat{\varphi}}(t,y)\end{aligned} \tag{5.2}$$

We have come to the main theorem of the article. It provides the key result, which is used in the following Section to derive a formula for the risk indifference price of an interest rate claim. Its proof relies on the maximum principle stated above.

Theorem 5.2. *Let $p_1^B(t)$, $p_2^B(t,x)$ be strong solutions of the adjoint equations* (4.35) *and* (4.36) *of Problem B and $p_1^A(t)$, $p_2^A(t,x)$, $p_3^A(t)$ be defined by Equations* (4.38), (4.39) *and* (4.40) *as in Lemma 4.1. Then, if the map $q \mapsto H^B(t,\tilde{Y}_t,q_t,\mathbf{p}^B,\mathbf{q}^B)$ of Problem B is concave, then the optimal control $\check{q}$ for Problem B is the same as the corresponding optimal control $\hat{q}(\hat{\varphi})$ for Problem A, i.e.*

$$\check{q} = \hat{q}(\hat{\varphi}) \tag{5.3}$$

Proof. The proof is similar to that of Theorem 4.2 in (An, Øksendal, and Proske 2008). Applying Theorem 5.1 to Problem B, one finds that $\check{q}$ is the optimal control, provided that

$$\begin{aligned}&\sup_{q\in\mathbb{M}} \mathbb{E}[H^B(t,\tilde{Y}_t,q_t,\mathbf{p}^B,\mathbf{q}^B)|\mathcal{E}_t]\\ &= \mathbb{E}[H^B(t,\tilde{Y}_t,\check{q},\mathbf{p}^B,\mathbf{q}^B)|\mathcal{E}_t],\end{aligned} \tag{5.4}$$

The corresponding first order conditions for the constrained maximization problem (5.4) imply that

$$\mathbb{E}\left[\bigtriangledown_{q^{(j)}}\left(H^B(t,\tilde{Y}_t,q_t,\mathbf{p}^B,\mathbf{q}^B) + C_t\cdot\left(\sum_{j\geq 1}\delta_t^{(j)}(x)\,q^{(j)} - b_t(x)\right)\right)_{q=\check{q}}|\mathcal{E}_t\right] = 0, \tag{5.5}$$

for $\forall j \geq 1$ and $x \in [0,\infty)$, with C_t being the corresponding Lagrange multiplier. Moreover,

$$\mathbb{E}\left[\left(\sum_{j\geq 1}\delta_t^{(j)}(x)\,q^{(j)} - b_t(x)\right)_{q=\check{q}}|\mathcal{E}_t\right] = 0,\ \forall x\in[0,\infty) \tag{5.6}$$

On the other hand, let $\hat{\varphi}$, $\hat{q}(\hat{\varphi})$ be the optimal controls for Problem A. Then,

$$\mathbb{E}[\bigtriangledown_{q^{(j)}}(H^A(t,Y_t,q_t,\hat{\varphi}_t,\mathbf{p}^A,\mathbf{q}^A)_{q=\hat{q}(\hat{\varphi}(t))}|\mathcal{E}_t] = 0,\ j\geq 1 \tag{5.7}$$

and by Lemma 4.2, $\hat{q}(\hat{\varphi}) \in \mathbb{M}$. Hence, using equality (4.41) in Lemma 4.1 yields

$$\begin{aligned}&\mathbb{E}\Big[\bigtriangledown_{q^{(j)}}\Big\{H^B(t,\tilde{Y}_t,q_t,\mathbf{p}^B,\mathbf{q}^B) + \\ &+\ K_t\cdot V_t(\varphi)\left(\varphi_t\left(2\sum_{j\geq 1} q_t^{(j)}\cdot\delta_t^{(j)}(\cdot) - b_t(\cdot)\right)\right)\Big\}_{q=\hat{q}(\hat{\varphi}(t))}\Big|\mathcal{E}_t\Big] = 0,\end{aligned} \tag{5.8}$$

for all $j \geq 1$ and all $\varphi \in \Pi$. Then, for any fixed $x \in [0, \infty)$ we can rewrite (5.8) as follows

$$\begin{aligned}(5.9)\qquad & \mathbb{E}\Big[\nabla_{q^{(j)}} \Big\{ H^B(t, \tilde{Y}_t, q_t, \mathbf{p}^B, \mathbf{q}^B) + \\ & + \; K_t \cdot V_t(\varphi) \cdot \beta_t(x) \cdot \Big(2 \sum_{j \geq 1} q_t^{(j)} \cdot \delta_t^{(j)}(x) - b_t(x) \Big) \Big\}_{q = \hat{q}(\hat{\varphi}(t))} \Big| \mathcal{E}_t \Big] = 0,\end{aligned}$$

where $\beta_t(x)$ is a fraction of wealth invested in $P_t(x)$ at the time moment t.

Since neither $b_t(\cdot)$ nor any of the terms outside of the brackets depend on $q^{(j)}$, we see that equation (5.9) is the same as equation (5.5), with $C_t(\cdot) = 2\,K_t \cdot V_t(\varphi)\,\beta_t(x)$. Moreover, by Lemma 4.2 the optimal market control in Problem A corresponds to a martingale measure, i.e. $\hat{q}(\hat{\varphi}) \in \mathbb{M}$, which implies that

$$(5.10)\qquad \mathbb{E}\Big[\Big(\sum_{j \geq 1} \delta_t^{(j)}(x)\, q^{(j)} - b_t(x) \Big)_{q = \hat{q}(\hat{\varphi})} |\mathcal{E}_t \Big] = 0, \; \forall x \in [0, \infty)$$

We immediately observe that the optimal control $\hat{q}(\hat{\varphi})$ for Problem A also satisfies the first order conditions (5.5) and (5.6) corresponding to Problem B. Hence, by the uniqueness of the solution, we conclude that $\check{q} = \hat{q}(\hat{\varphi})$, as claimed. ∎

6 Risk indifference pricing of claims of the yield curve

In this section we aim at establishing a relation between the value function in Problem A and that in Problem B. Theorem 5.2 provides the key result needed for this purpose. Let $(q^*, \varphi^*) = (\check{q}, \hat{\varphi})$ be the optimal controls for Problem A with $\check{q}$ being optimal for Problem B, as in Theorem 5.2. Also, denote by $\tilde{Y}^* = \tilde{Y}^{q^*}$ the state process corresponding to the optimal control $\check{q}$. The value function $\Phi_G^{A,\mathcal{E}}$ of Problem A then becomes

$$\begin{aligned}(6.1)\qquad \Phi_G^{A,\mathcal{E}}(t, y) &= \inf_{\varphi \in \Pi} \Big(\sup_{q \in \Theta} J^{q,\varphi}(t, y) \Big) \\ &= \inf_{\varphi \in \Pi} \Big(\sup_{q \in \Theta} \mathbb{E}_{\mathbb{P}}^{y} \Big[\int_t^\tau -\tilde{\Lambda}\left(s, q_s, \tilde{Y}_s\right) ds \; - \; h(\tilde{Y}_\tau) + \\ &\qquad + K_\tau \cdot g(P_\tau(\cdot)) - K_\tau \cdot V_\tau(\varphi) \Big] \Big) \\ &= \inf_{\varphi \in \Pi} \Big(\mathbb{E}_{\mathbb{P}}^{y} \Big[\int_t^\tau -\tilde{\Lambda}\left(s, q_s^*, \varphi_s, \tilde{Y}_s^*\right) ds \; - \; h(\tilde{Y}_\tau^*) + \\ &\qquad + K_\tau^* \cdot g(P_\tau(\cdot)) - K_\tau^* \cdot V_\tau(\varphi) \Big] \Big)\end{aligned}$$

Since the first part of equation (6.1) does not depend on the parameter φ, it can be rewritten as follows

$$\begin{aligned}\Phi_G^{A,\mathcal{E}}(t,y) = \mathbb{E}_{\mathbb{P}}^{y}\Bigg[\int_t^{\tau} -\tilde{\Lambda}\left(s, q_s^*, \varphi_s, \tilde{Y}_s^*\right) ds \;-\; h(\tilde{Y}_\tau^*) + \\ + K_\tau^* \cdot g(P_\tau(\cdot))\Bigg] - \inf_{\varphi\in\Pi}\left(\mathbb{E}_{\mathbb{P}}^{y}\left[K_\tau^* \cdot V_\tau(\varphi)\right]\right)\end{aligned} \tag{6.2}$$

Also, by the original assumption, φ^* is optimal for Problem A and by Theorem 5.2, $\check{q} = q^*$ is optimal for Problem B. Hence, by the formulation of Problem B $\check{q} \in \mathbb{M}$ and $\mathbb{Q}_{q^*}$ defined by the Radon-Nikodym derivative K_τ^* is a martingale measure. Therefore, $\mathbb{E}_{\mathbb{P}}^{y}[K_\tau^* \cdot V_\tau(\varphi)] = k \cdot V_0$, for all $\varphi \in \Pi$, and the previous expression becomes

$$\begin{aligned}\Phi_G^{A,\mathcal{E}}(t,y) &= \mathbb{E}_{\mathbb{P}}^{y}\Bigg[\int_t^{\tau} -\tilde{\Lambda}\left(s, q_s^*, \varphi_s, \tilde{Y}_s^*\right) ds \;-\; h(\tilde{Y}_\tau^*) + \\ &\quad + K_\tau^* \cdot g(P_\tau(\cdot))\Bigg] - k \cdot V_0 \\ &= \sup_{q\in\mathbb{M}} J_B^q(t,\tilde{y}) - k \cdot V_0 \\ &= \tilde{\Phi}_G^{B,\mathcal{E}}(t,\tilde{y}) - k \cdot V_0,\end{aligned} \tag{6.3}$$

where we once again used the claim of Theorem 5.2. This result is analogous to the one stated in (An, Proske, and Rubtsov 2010).

Coming back to our original problem, we want to find the risk indifference price $p = p_{risk}^{seller}$ of an interest rate claim, which is determined by the Equation (3.12):

$$\Phi_G^{A\mathcal{E}}(V_0 + p) = \Phi_0^{A\mathcal{E}}(V_0). \tag{6.4}$$

By the result in Equation (6.3), one can immediately see that the equality (3.12) becomes

$$\Phi_G^{B\mathcal{E}}(t,\tilde{y}) - k \cdot (V_0 + p) = \Phi_0^{B\mathcal{E}}(t,\tilde{y}) - k \cdot V_0, \tag{6.5}$$

which implies that the risk indifference price is given by

$$p = p_{risk}^{seller} = k^{-1} \cdot \left(\Phi_G^{B\mathcal{E}}(t,\tilde{y}) - \Phi_0^{B\mathcal{E}}(t,\tilde{y})\right) \tag{6.6}$$

The latter expression provides the main result of this paper. For $k = 1$, we obtain the following representation for the risk indifference price of functional claims of the yield curve under partial information, which is similar to the one derived in (An, Øksendal, and Proske 2008). We formulate it in the form of a theorem.

Theorem 6.1. (Risk indifference price of functional claims of the yield curve under partial information) *Given that the conditions of Theorem 5.2 hold, the risk indifference price $p_{risk}^{seller}(G_\tau, \mathcal{E})$ for the seller of an interest rate claim G_τ is given by*

$$p_{risk}^{seller}(G_\tau, \mathcal{E}) = \sup_{\mathbb{Q}\in\mathcal{M}}\{\mathbb{E}_{\mathbb{Q}}^{\tilde{y}}[G_\tau] - \zeta(\mathbb{Q})\} - \sup_{\mathbb{Q}\in\mathcal{M}}\{-\zeta(\mathbb{Q})\}. \tag{6.7}$$

Chapter III

Sensitivity with respect to the yield curve: Duration in a stochastic setting

with Paul Kettler and Frank Proske[1]

[1]The authors wish to thank Vidyadhar S. Mandrekar for his valuable comments on this work.

1 Introduction

The concept of bond duration dates to a foundational book defining the idea (Macaulay 1938). Through the years there have been many presentations on the idea. One of note is (Jarrow 1978). Other tracts obtain, most frequently addressing the bond with periodic coupons and a terminal payment of principal. Such discussions tend to concentrate on the idea of an annuity as the sum of a geometric series, presented in a variety of flavors. We eschew these notions as being of scant academic interest, and focus on the continuously compounded zero coupon bond as a building block, leaving the construction of instruments with component payments to others.

The bond market worldwide has about \$45 trillion outstanding, with about \$1 trillion trading on a typical day. Other than price and yield, the most widely quoted parameter in the market, without question, is duration. It appears on quotation screens, on traders' lips, and in all manner of literature on the market. Yet the concept, which addresses the sensitivity of a bond's price with respect to changes in yield, assumes a uniform rate of interest through the life of a bond, an unrealistic hypothesis.

In basic bond analysis one considers a zero coupon bond with present value (or price) v given as a function of a level interest rate r, maturing to future value \$1 at time T. The relationship of variables is this:

$$v = \mathrm{e}^{-rT} \tag{1.1}$$

The quantity

$$d := \frac{1}{v}\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\log v = -T$$

is known as the *duration*, and the quantity

$$c := \frac{1}{2v}\frac{\partial^2 v}{\partial r^2} = \frac{1}{2}T^2$$

is known as the *convexity.* Note that d and c are the coefficients, respectively, of r and r^2 in the Taylor series expansion of v.

$$v = 1 - Tr + \frac{1}{2}T^2 r^2 - \dots \tag{1.2}$$

Bond traders routinely employ duration and convexity in market analysis to estimate the effects of rate changes.

An important fact about duration, which makes it useful for portfolio analysis, is that the duration of a portfolio is the average of the component durations weighted by present values. A two security case is sufficient to illustrate. Let

$$v = \alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 \exp(-rT_1) + \alpha_2 \exp(-rT_2)$$

Then

$$d = -\frac{\alpha_1 v_1}{\alpha_1 v_1 + \alpha_2 v_2} T_1 - \frac{\alpha_2 v_2}{\alpha_1 v_1 + \alpha_2 v_2} T_2$$

One may generalize this concept of bond to incorporate a piecewise constant interest rate $r(s)$, where

$$r(s) = \begin{cases} r_1 & \text{, if} \quad 0 =: s_0 \le s < s_1 \\ r_2 & \text{, if} \quad s_1 \le s < s_2 \\ \cdots & \\ r_n & \text{, if} \quad s_{n-1} \le s \le s_n := T \end{cases}$$

Then Equation (1.1) becomes

$$v = \exp\left[-\sum_{i=1}^{n} r_i (s_i - s_{i-1}) \right] \tag{1.3}$$

From this expression we obtain the i^{th} *partial duration*

$$d_i := \frac{\partial}{\partial r_i} \log v = -(s_i - s_{i-1}) \qquad , \ 1 \le i \le n$$

and the i^{th} *partial convexity*

$$c_i := \frac{1}{2}(s_i - s_{i-1})^2 \qquad , \ 1 \le i \le n$$

Observe that the partial durations add to the total duration, whereas the partial convexities (and higher order related partial terms) do not.

One may elaborate further on the themes of Equations (1.1) and (1.3) by putting r and the $\{r_i\}$ on stochastic paths. To start, denote by $P(t,T)$ the price at time t of a zero coupon bond, which pays \$1 at maturity T. Then one can define instantaneous forward rates as

$$f(t;T) = -\frac{\partial \log\big(P(t,T)\big)}{\partial T}, \quad 0 \le t \le T \tag{1.4}$$

for each maturity T. See (Heath, Jarrow, and Morton 1992). So we can recast Equation (1.1) as

$$v = P(t,T) = \exp\left(-\int_t^T f(t,s)\,\mathrm{d}s \right) \tag{1.5}$$

Since the outcome of future interest rates is not known in advance it is reasonable to model instantaneous forward rates $\{f(t,s)\}_{0\leq t\leq s}$ as stochastic processes. In this context we may interpret $f(t,s)$ as the overnight interest rate at (future) time s as seen from the current time t. The case $f(t,t) =: r(t)$ is simply the overnight rate, or short rate.

The literature is replete with examples on interest rates. A small sample of papers, not otherwise cited in the text, is this (Vašíček 1977; Rendleman and Bartter 1980; Cox, Ingersoll Jr., and Ross 1985; Lee and Ho 1986; Black, Derman, and Toy 1990; Ritchken and Sankarasubramanian 1995; Musiela 1995; Chen 1996a; Chen 1996b; Björk, Christensen, and Gombani 1998; Björk and Gombani 1999; Vargiolu 1999; Filipović and Zabczyk 2002; Aihara and Bagchi 2005; Filipović and Tappe 2008). All address stochastic interest rates in financial modelling. Of interest within are these references including co-author Marek Musiela: (Brace and Musiela 1994; Brace, Gątarek, and Musiela 1997; Musiela and Rutkowski 1997; Goldys, Musiela, and Sondermann 2000).

As mentioned above the classical duration is based on the assumption that interest rates are flat or piecewise flat. This assumption is quite unrealistic and only applies to sensitivity measurements with respect to parallel shifts of interest rates. The latter is especially unsatisfying for a trader who manages a complex portfolio of interest-rate-sensitive securities (as, *e.g.*, caps, swaps, bond options, *etc.*) In this case it would be desirable to measure the interest rate risk of the portfolio with respect to the *stochastic fluctuations* of the *whole* term structure or even the *yield surface*, that is

$$(t,x) \longmapsto Y(t,t+x), \tag{1.6}$$

where $Y(t,T)$ is the yield given by

$$Y(t,T) = -\frac{1}{T-t}\log P(t,T)$$

Here x in Equation (1.6) stands for the time-to-maturity.

Using the relation of Equation (1.5) we can represent the yield surface $Y_t(x) := Y(t,t+x)$ as

$$Y_t(x) = \frac{1}{x}\int_0^x f_t(s)\,\mathrm{d}s, \tag{1.7}$$

where $f_t(s) := f(t,t+s)$. Because of the linear correspondence of Equation (1.7) between the yield curves $Y_t(\cdot)$ and the forward curves $f_t(\cdot)$ we can and will refer to

$$(t,x) \longmapsto f_t(x) \tag{1.8}$$

as the yield surface in this paper.

Assuming, *e.g.*, the Heath–Jarrow–Morton [HJM] model for the dynamics of instantaneous interest rates, one shows under certain conditions that the yield surface of Mapping (1.8) is described by a stochastic partial differential equation [SPDE], called the

Musiela equation. See (Heath, Jarrow, and Morton 1992; Goldys, Musiela, and Sondermann 2000).

In this paper we wish to develop an analogous concept to the classical duration of Macaulay in the HJM setting. More precisely, we want to measure the sensitivity of interest rate claims with respect to the Musiela dynamics of the yield surface, as in Equation (1.8).

An apparently analogous way to the classical case would be to define the duration of an interest-rate security by means of the Fréchet derivative for each interest rate scenario. However, interest rate securities, or even dynamically hedged portfolios composed of them, are in general complicated functionals of the yield surface, and are usually not even continuous.

In order to overcome this problem one may think of weaker forms of derivatives than the Fréchet derivative to measure interest rate sensitivities. A possible candidate could be the Malliavin derivative. This derivative, which is treated in Section 2, can be considered as a stochastic Gateaux derivative.

In this paper we want to base the stochastic duration concept on this stochastic Gateaux derivative. This concept is analogous to the classical one in the sense that it relies on the derivative of an infinite-dimensional version of the Taylor expansion as in Equation (1.2). Using this concept we also define *stochastic convexity* as a measure for the "curvature" of yield surface movements.

The paper is organized as follows: In Section 2 we define the concept of stochastic duration by using Malliavin calculus for general Gaussian random fields. In Section 3 we propose a mathematical framework for the construction of immunization strategies of portfolios, which are composed of interest rate instruments.

2 An expanded concept of duration via Malliavin calculus

In this section we want to elaborate a duration concept for stochastic yield curves. This definition extends the classical duration of Macaulay to a stochastic setting.

Denote by $P(t,T)$ the price at time t of a zero coupon bond, which pays \$ 1 at maturity T. Suppose that the bond prices are modelled by non-negative adapted processes $\{P(t,T)\}_{0\leq t\leq T}$ for each $T>0$ on a filtered probability space

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) \tag{2.1}$$

In the following we assume that the bond prices $P(t,T)$ are described by the HJM model (Heath, Jarrow, and Morton 1992); that is, the bond prices take the form

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\,\mathrm{d}s\right), \tag{2.2}$$

where $f(t,T)$, $0 \le t \le T < \infty$, are instantaneous forward rates modelled by the stochastic differential equation [SDE]

$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t + \sigma(t,T)\,\mathrm{d}B_t, \quad 0 \le t \le T < \infty \tag{2.3}$$

Here we require that $\sigma(\cdot,T)$ be a deterministic Borel-measurable function and $\alpha(\cdot,T)$ a predictable process for all T *wrt* the P-completed filtration $\mathcal{F}_t$ generated by a one-dimensional Brownian motion B_t, $t \ge 0$.

Now, let us reparametrize the forward rates by the *time-to-maturity* $x = T - t$; that is, let us consider the forward curves

$$f_t(x) := f(t, t+x)$$

An application of the generalized Itô formula shows that under certain conditions on $\alpha(\cdot,T)$, $\sigma(\cdot,T)$ the forward curves $f_t(x)$ satisfy the first order SPDE

$$\mathrm{d}f_t(x) = \Big(\frac{\mathrm{d}}{\mathrm{d}x} f_t(x) + \alpha_t(x)\Big)\,\mathrm{d}t + \sigma_t(x)\,\mathrm{d}B_t, \tag{2.4}$$

as in (Kunita 1997, Theorem 3.3.1). Here we use the notation $\alpha_t(x) := \alpha(t,t+x)$, $\sigma_t(x) := \sigma(t,t+x)$. Note that Equation (2.4) is referred to as the *Musiela equation* in the literature. See, *e.g.*, (Carmona and Tehranchi 2006). See also (Da Prato and Zabczyk 1992) and the references therein for more information about SPDE's.

A deficiency of the model of Equation (2.4) is that it does not capture the feature of *maturity-specific risk*. A model with such a property would enable hedging of bond options with unique portfolio strategies. On the other hand, it would meet the intuitive requirement that maturities of the bonds underlying the bond option are used in the hedging portfolio.

A more realistic model than that of Equation (2.4), which takes into account maturity-specific risk, would consequently have the form

$$\mathrm{d}f_t(x) = \Big(\frac{\mathrm{d}}{\mathrm{d}x} f_t(x) + \alpha_t(x)\Big)\,\mathrm{d}t + \sigma_t(x)\,\mathrm{d}B_t(x), \tag{2.5}$$

where each noise $B_t(x)$ stands for the risk arising from the time-to-maturity x. Here we may think of $B_t(x)$ as a Brownian sheet in t and x. So Equation (2.5) can be recast as

$$\mathrm{d}f_t(x) = \Big(\frac{\mathrm{d}}{\mathrm{d}x} f_t(x) + \alpha_t(x)\Big)\,\mathrm{d}t + \sum_{k\ge 1} \sigma_t^{(k)}(x)\,\mathrm{d}B_t^{(k)}, \tag{2.6}$$

where $\sigma_\cdot^{(k)}(\cdot)$, $k \ge 1$, are deterministic measurable functions and $B_t^{(k)}$, $k \ge 1$, are independent one-dimensional Brownian motions.

In what follows we want to assume that the forward curves are modelled by functions on a Hilbert space H. This space should exhibit the natural feature that evaluation functionals on it are continuous; that is,

$$\begin{aligned} \delta_x \colon H &\longmapsto \mathbb{R} \\ f &\longmapsto f(x) \end{aligned} \tag{2.7}$$

is continuous on H for all x. Further, it is desirable that the generator $A\colon = \frac{\mathrm{d}}{\mathrm{d}x}$ in Equation (2.6) admits a strongly continuous semigroup S_t on H. The semigroup S_t is the left shift operator given by

$$(S_t f)(x) = f(t+x) \tag{2.8}$$

The following family $\{H_w\}$ of Hilbert spaces of Sobolev type introduced by (Filipović 2001) fulfills the above-mentioned conditions: Let $w\colon [0,\infty) \to (0,\infty)$ be a non-decreasing function such that

$$\int_0^\infty \frac{1}{w(x)}\,\mathrm{d}x < \infty$$

Then H_w is defined as

$$H_w = \left\{ f\colon [0,\infty) \longrightarrow \mathbb{R} \;\middle|\; f \text{ is absolutely continuous and } \int_0^\infty \big(f'(x)\big)^2 w(x)\,\mathrm{d}x < \infty \right\},$$

and is equipped with the scalar product

$$\langle f, g\rangle_{H_w} = f(0)g(0) + \int_0^\infty f'(x)g'(x)w(x)\,\mathrm{d}x$$

In the sequel we require that

$$\alpha_t(\cdot),\ \sigma_t^{(k)}(\cdot) \in H, \text{ a.e.}, \quad \forall t \geq 0$$

Consider the special case that $\alpha_t(x) = \alpha_t^*(x) f_t(x)$ for a deterministic function $\alpha_t^*(x)$. Then, using integrating factors we observe that the mild solution of the SDE of Equation (2.6) is explicitly given by the Gaussian random field

$$\begin{aligned} f_t(x) = {} & \exp\Big(\int_0^t \alpha^*(s,t+x)\,\mathrm{d}s\Big) f(0,t+x) \\ & + \sum_{k\geq 1} \int_0^t \exp\Big(\int_s^t \alpha^*(u,t+x)\,\mathrm{d}u\Big)\sigma^{(k)}(s,t+x)\,\mathrm{d}B_t^{(k)} \end{aligned} \tag{2.9}$$

Now, let W_t be a Q-Wiener process, where Q is a symmetric non-negative operator on a separable Hilbert space U with $\mathrm{Trace}(Q) < \infty$. Set $U_0 = Q^{1/2}(U)$, which is a Hilbert space with norm

$$\|h\|_0 := \|Q^{-1/2}(h)\|, \quad h \in U_0$$

Denote by $L_2(U,H)$ the space of Hilbert–Schmidt operators from U to H with the operator norm $\|\cdot\|_{L_2}$. Further, let u_k, $k \geq 1$, be an orthonormal basis of U, and suppose that there exists a Borel-measurable map

$$\sigma\colon [0,T] \longrightarrow L(U_0, H)$$

such that

$$\sigma_t \left[Q^{1/2}(u_k)\right] = \sigma_t^{(k)}(\cdot)$$

and

$$\sigma_t \circ Q^{1/2} \in L_2(U, H)$$

for all (t, k) in Equation (2.6), where $\circ$ represents the composition of operators. Then we can view $\{B_t^{(k)}\}_{0 \le t \le T}$, $k \ge 1$, in Equation (2.6) as a Wiener process W_t cylindrically defined on U, and rewrite Equation (2.6) as

$$\mathrm{d}f_t = \Big(Af_t + \alpha_t\Big)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t \tag{2.10}$$

In the sequel we assume that there exists a predictable unique strong solution

$$\big(t \longmapsto f_t(\cdot)\big) \in C([0, T]; H)$$

to Equation (2.10).

Remark. Suppose that $\alpha_t = b(t, f_t)$ in Equation (2.10), where $b\colon [0, T] \times H \to H$ is a Borel-measurable map. Then the following set of conditions provide sufficient criteria for the existence of a unique strong solution of Equation (2.10).

(i) f_t is a unique mild solution of Equation (2.10).

(ii) $f_0 \in \mathrm{Dom}(A)$; $S_{t-s}b(s, x) \in \mathrm{Dom}(A)$; $S_{t-s}\sigma_s u \in \mathrm{Dom}(A)$, $\forall u \in U_0$, $t \ge s$.

(iii)
$$\|AS_{t-s}b(s, x)\|_H \le q(t - s)\|x\|_H, \text{ for some } q \in L^1([0, T]; \mathbb{R}_+).$$

(iv)
$$\|AS_{t-s}\sigma_s\|_H = g(t - s), \text{ for some } g \in L^2([0, T]; \mathbb{R}_+).$$

See, *e.g.*, (Kai 2006).

Assume that σ_t is invertible for all $0 \le t \le T$ a.e. and that the integrability condition

$$\sup_{t \in [0,T]} \mathrm{E}\left[\exp\left(\delta \|\sigma_t^{-1}[Af_t + \alpha_t]\|_0^2\right)\right] < \infty \tag{2.11}$$

holds for some $\delta > 0$. Then Girsanov's Theorem [see, *e.g.* (Bensoussan 1971)] applied to Equation (2.10) entails that

$$\mathrm{d}f_t = \sigma_t\,\mathrm{d}\widehat{W}_t, \tag{2.12}$$

where

$$\widehat{W}_t = W_t - \int_0^t \psi(s)\,\mathrm{d}s$$

is a Q-Wiener process under the change of measure $\widehat{\mathbb{P}}$ given by

$$\widehat{P}(A) = \mathrm{E}\left[\mathbf{1}_A \exp\left(\int_0^T \langle \psi(s), \mathrm{d}W_s\rangle_0 - \tfrac{1}{2}\int_0^T \|\psi(s)\|_0^2\,\mathrm{d}s\right)\right],$$

with

$$\psi(t) := \sigma_t^{-1}[Af_t + \alpha_t] \tag{2.13}$$

Consequently f_t is a Gaussian $\mathcal{F}_t$-martingale with respect to $\widehat{\mathbb{P}}$. Define

$$\widehat{f}_t = f_t - f_0 = \int_0^t \sigma_s\,\mathrm{d}\widehat{W}_s \tag{2.14}$$

Thus $\widehat{f}_t(x)$ is a centered Gaussian random field with respect to time and time-to-maturity under $\widehat{\mathbb{P}}$. We wish to use these forward curves to define an expanded concept of duration which serves as a tool to measure interest rate sensitivities of bond options or bond portfolios with respect to the *whole* yield surface

$$(t, x) \longmapsto f_t(x)$$

In view of the relation between Malliavin derivatives and Gateaux derivatives it is reasonable to define the duration of an interest rate instrument as the Malliavin derivative of a square integrable functional of $\widehat{f}_t(x)$. To this end we have to introduce a Malliavin calculus with respect to $\widehat{f}_t(x)$, which is the centered forward curve in the risk neutral world. For this purpose let $(\Omega, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ be our reference probability space, where $\widehat{\mathcal{F}}$ is generated by $\widehat{f}_t(x)$. In the following, denote by I the index set with respect to the tuples (t, x), and set $\widehat{f}(u) = \widehat{f}_t(x)$ if $u = (t, x) \in I$. Let

$$C(u, r) = \mathrm{E}\left[\widehat{f}(u)\widehat{f}(r)\right]$$

be the covariance function of $\widehat{f}$. Further consider the reproducing kernel Hilbert space [RKHS] K of C with norm $\|\cdot\|_K$. See, *e.g.*, (Chatterji and Mandrekar 1978). Then K is isometrically isomorphic to the closure of the linear span of $\widehat{f}(u)$, $u \in I$, in $L^2(\Omega, \widehat{\mathcal{F}}, \widehat{P})$. Using in addition the continuity of evaluation functionals on H and the theorem of Banach–Steinhaus we find that K is isometrically isomorphic to the space

$$H(\widehat{f}) := \left\{\lambda\colon [0, T] \longrightarrow H^* \text{ Borel measurable} \;\Big|\; \int_0^T \|\lambda_s \circ \sigma_s\|_{L_2^0}^2\,\mathrm{d}s < \infty\right\}, \tag{2.15}$$

where $\|B\|_{L_2^0} := \left\|B \circ Q^{1/2}\right\|_{L^2} < \infty$ for $B \in L(H,H)$. Here H^* stands for the topological dual of H.

By (Chatterji and Mandrekar 1978) we obtain the following chaos decomposition.

$$L^2(\Omega, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}) = \sum_{p\geq 0} \oplus I_p(K^{\widehat{\otimes} p}),$$

where $K^{\widehat{\otimes} p}$ is the p-fold symmetric tensor product of K, and where $I_p\colon K^{\otimes p} \to L^2(\Omega, \widehat{\mathcal{F}}, \widehat{P})$ are linear operators such that the following properties hold.

$$\mathrm{E}[I_p(f)] = 0$$

$$\mathrm{E}[I_p(f)I_q(g)] = \begin{cases} 0 & , \ p \neq q \\ p!\langle \widetilde{f}, \widetilde{g}\rangle_K & , \ p = q \end{cases}$$

for $f \in K^{\otimes p}, \ g \in K^{\otimes q}$, where $\widetilde{f}$ is the symmetrization of f. Here I_p is recursively defined by

$$I_{p+1}(gh) = I_p(g)I_1(h) - \sum_{k=1}^{p} I_{p-1}(g \underset{k}{\otimes} h)$$

for $g \in K^{\otimes p}$, $h \in K$, where

$$I_1(h) := \int_0^T h_s \,\mathrm{d}(\sigma_s \widehat{W}_s) = \int_0^T h_s \circ \sigma_s \,\mathrm{d}\widehat{W}_s.$$

for $h \in H(\widehat{f})$. See (Mandrekar and Zhang 1993).

Now let $u \in L^2(\Omega; K)$ and let u_t have the chaos representation

$$u_t = \sum_{p\geq 0} I_p\left(f_p^t\right)$$

for unique $f_p^t \in K^{\widehat{\otimes} p}$ and each $t \in I$. Denote by $\widetilde{f}_p$ the symmetrization of an appropriate version of $f_p^t(t_1, \ldots, t_p)$ *wrt* $t_1, \ldots, t_p$, and t. Then the Skorohod integral of the process u_t is defined as

$$\delta(u_\cdot) = \sum_{p\geq 1} I_{p+1}\big(\widetilde{f}_p\big) \tag{2.16}$$

if

$$\sum_{p\geq 1} (p+1)! \big\|\widetilde{f}_p\big\|^2_{K^{\widehat{\otimes} p+1}} < \infty$$

is fulfilled.

The *Malliavin derivative* $D_u F \in L^2(\Omega; K)$ of a square integrable functional F of the forward curve $\widehat{f}$ can be defined as the adjoint operator of δ in Equation (2.16). In the sequel we shall denote by $\mathbb{D}^{1,2} \subset L^2(\Omega, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ the domain of the Malliavin derivative D.

In view of the financial applications we have in mind it is important to note that the Malliavin derivative can be regarded as a sensitivity measure with respect to the fluctuations of the yield surface $(t, x) \longmapsto f_t(x)$. The latter can be justified by the following relationship between the Malliavin derivative and the stochastic Gateaux K-derivative: Let X be the support of the image measure μ of $\widehat{f}$ under $\widehat{\mathbb{P}}$ in $C([0,T]; H)$. Then by (Borel 1976) we find that X is the closure of K in $C([0,T]; H)$. Further, in (Gawarecki and Mandrekar 1993, Proposition 4.1) we have that if for $F \in L^2(\mu)$,

$$\frac{F(x + \epsilon k) - F(x)}{\epsilon} \tag{2.17}$$

converges in $L^2(\mu)$ as $\epsilon \to 0$ for $k \in K$, then $D.F \in L^2(\mu; K)$ exists and the above limit equals $(D.F, k)_K$.

Since the measure P in Equation (2.1) is equivalent to $\widehat{\mathbb{P}}$ we see that the convergence of Expression (2.17) to $\langle D.F, k \rangle_K$ also holds in probability with respect to the image measure of the forward curves under the original measure $\mathbb{P}$. Therefore, if $F = \xi_T$ is the terminal value of a bond portfolio, we may interpret the Malliavin derivative $D.F$ as a sensitivity measure of ξ_T to the fluctuations of the whole yield surface in this portfolio. The latter observation gives rise to the introduction of an expanded concept of duration as follows.

Definition 2.1 (Stochastic duration)**.** Let F be a square integrable functional of the forward curve $\widehat{f}$ *wrt* $\widehat{\mathbb{P}}$. Assume that F is Malliavin differentiable *wrt* $\widehat{f}$. Then the *stochastic duration of* F is a stochastic process

$$D.F \in L^2(\Omega, \widehat{\mathcal{F}}, \widehat{P}; K)$$

Remark. We shall mention that we also could have introduced our concept of stochastic duration *wrt* mild solutions f_t of Equation (2.10). In this case one can replace Condition (2.11) by assuming that

$$\sup_{t \in [0,T]} \mathrm{E}\left[\exp\left(\delta \left\|\sigma_t^{-1}[\alpha_t]\right\|_0^2\right)\right] < \infty$$

for some $\delta > 0$. Compared to mild solutions, strong solutions are rare. However, from the viewpoint of applications we have in mind it is technically more convenient to deal with strong solutions. See Section 3.

We want to illustrate this concept by calculating the stochastic duration of certain interest rate claims. For this purpose we need the following auxiliary results.

The first Lemma gives a chain rule for the Malliavin derivative D.

Lemma 2.2 (Chain Rule)**.** *Let F be Malliavin differentiable with respect to $\widehat{f}$,* i.e., *$F \in \mathbb{D}^{1,2}$. Further, suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable with bounded derivative. Then $g(F) \in \mathbb{D}^{1,2}$ and*

$$D_u g(F) = g'(F) D_u F$$

for each $u \in K$. Here g' stands for the derivative of g.

Proof. The proof follows from arguments in the Brownian motion case. See (Di Nunno, Øksendal, and Proske 2008, Theorem 3.5) or (Nualart 1995, Proposition 1.2.2). ∎

The next Lemma pertains to the closability of the Malliavin derivative.

Lemma 2.3 (Closability)**.** *Let $F \in L^2(\widehat{\mathbb{P}})$ and $(F_k)_{k\geq 1} \subset \mathbb{D}^{1,2}$ such that*

$$F_k \underset{k\to\infty}{\longrightarrow} F \text{ in } L^2(\widehat{\mathbb{P}})$$

and

$$D.F_k \text{ converges in } L^2(\widehat{\mathbb{P}}; K)$$

Then $F \in \mathbb{D}^{1,2}$ and

$$D.F_k \underset{k\to\infty}{\longrightarrow} D.F \text{ in } L^2(\widehat{\mathbb{P}}; K)$$

Proof. See the arguments in (Di Nunno, Øksendal, and Proske 2008, Theorem 3.3). ∎

Example 2.1 (Zero Coupon Bond)**.** *As before, let $P(t,T)$ be the price at time t of a zero coupon bond, which pays \$1 at maturity T. Then using the instantaneous forward rates $f(t,s)$, $0 \leq t \leq s$, we have that*

$$\begin{aligned} P(t,T) &= \exp\Big(- \int_t^T f(t,s)\,\mathrm{d}s \Big) \\ &= \exp\Big(- \int_0^{T-t} f_t(x)\,\mathrm{d}x \Big) \end{aligned}$$

We find that

$$\begin{aligned} D_{r,y}\Big(\int_0^{T-t} f_t(x)\,\mathrm{d}x \Big) &= \int_0^{T-t} D_{r,y}\big(f_t(x)\big)\,\mathrm{d}x \\ &= \int_0^{T-t} \mathbf{1}_{[0,t]}(r)\,\mathrm{d}x \\ &= (T-t)\mathbf{1}_{[0,t]}(r), \end{aligned}$$

where $\mathbf{1}_{[0,t]}$ *is the indicator function of* $[0,t]$*. Then the chain rule of Lemma 2.2 (in connection with Lemma 2.3) shows that the stochastic duration* $D.P(t,T)$ *of* $P(t,T)$ *in the HJM model is given by*

$$D_{r,y}P(t,T) = \begin{cases} -(T-t)P(t,T) & \text{, if } 0 \leq r \leq t \\ 0 & \text{, otherwise} \end{cases}$$

So $D_{r,y}P(t,T)/P(t,T)$*,* $0 \leq r \leq t$*, has the form of the classical duration in Section 1. The latter expression seems to suggest that we should rather use* $D.F/F$ *as a generalized duration than* $D.F$*. However, a general interest rate claim* F *may be zero for a positive probability. Therefore it is reasonable to introduce* $D.F$ *as an expanded concept of duration.*

Note that our definition does not generalize Macaulay's duration in the sense that $D.F$ *gives the classical duration if the interest rate claim* F *is deterministic, that is, a functional of a deterministic (piecewise flat) yield surface. The explanation for this is that the duration concepts are based on different interest rate models.*

The classical duration presumes yield surfaces which are flat or piecewise flat. Such a model is fundamentally different from a stochastic interest rate model. For example, under our conditions yield surfaces in our [risk-neutral] HJM model only assume a certain constant value with probability zero. In view of this, we may therefore consider the stochastic duration as a concept which is analogous to the classical one in the HJM setting.

Example 2.2 (Interest Rate Cap)**.** *Consider a cap of the form*

$$F = \big(R(t,T) - K\big)^{+},$$

where K *is the cap rate and* $R(t,T)$ *the average interest rate given by*

$$R(t,T) = \frac{1}{T-t}\int_t^T r(s)\,\mathrm{d}s$$

Here $r(t) = f(t,t)$ *is the overnight interest rate, also known as the short rate. We observe that*

$$\begin{aligned} D_{r,y}\left(\frac{1}{T-t}\int_t^T r(s)\,\mathrm{d}s\right) &= \frac{1}{T-t}\int_t^T D_{r,y}\big(r(s)\big)\,\mathrm{d}s \\ &= \frac{1}{T-t}\int_t^T D_{r,y}\big(f_s(0)\big)\,\mathrm{d}s \\ &= \mathbf{1}_{[0,t]}(r) \end{aligned}$$

Now let us approximate the $\varphi(x) := (x-K)^{+}$ *by functions* $\{\varphi_n\}$ *with*

$$\varphi_n(x) = \varphi(x) \text{ for } |x-K| \geq \frac{1}{n}$$

and

$$0 \leq \varphi_n'(x) \leq 1 \text{ for all } x$$

Then it follows from Lemma 2.2 and Lemma 2.3 that

$$D_{r,y}F = \mathbf{1}_{[K,\infty)}\big(R(t,T)\big) \cdot \mathbf{1}_{[0,t]}(r)$$

Example 2.3 (Asian Option). *Let us also have a look at the following Asian type of option defined as*

$$F = \frac{1}{(\overline{x}_2 - \overline{x}_1)(T_2 - T_1)} \int_{\overline{x}_1}^{\overline{x}_2} \int_{T_1}^{T_2} f_t(x)\, \mathrm{d}t\, \mathrm{d}x$$

Then

$$\begin{aligned} D_{r,y}F &= \frac{1}{(\overline{x}_2 - \overline{x}_1)(T_2 - T_1)} \int_{\overline{x}_1}^{\overline{x}_2} \int_{T_1}^{T_2} \mathbf{1}_{[0,t]}(r)\, \mathrm{d}t\, \mathrm{d}x \\ &= \mathbf{1}_{[0,t]}(r) \end{aligned}$$

3 Estimation of Stochastic Duration and the Construction of Immunization Strategies

In the previous section we introduced the concept of stochastic duration $D_{t,y}F$ and gave examples of interest rate derivatives F whose stochastic duration can be computed explicitly. In general, the stochastic duration of an interest rate claim or a complex bond portfolio cannot be determined explicitly. The latter is also due to the fact that, *e.g.*, a dynamically hedged bond portfolio is a stochastically weighted sum of interest rate claims. The weights of the portfolio or hedging strategy at any time point are usually complicated functionals of the stochastic forward curve. In order to overcome this deficiency we aim at resorting to an estimate of $D_{t,y}F$. A reasonable estimate of $D_{t,y}F$ could be the expected stochastic duration of F given the observed forward curves $\widehat{f}_s$, $0 \leq s \leq t$. This estimate naturally appears in the Clark–Ocone formula or as a solution of a backward stochastic differential equation [BSDE].

Using the fact that the set

$$\left\{\exp\left\{I_1(h) - \tfrac{1}{2}\|h\|_K^2\right\} \;\middle|\; h \in K\right\}$$

is total in $L^2(\Omega, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ one finds in connection with Relation (2.15) that the Clark–Ocone formula *wrt* the forward curves $\widehat{f}_t$ takes the following form. See also (Di Nunno, Øksendal, and Proske 2008).

$$F = \mathrm{E}_{\widehat{\mathbb{P}}}[F] + \int_0^T \mathrm{E}_{\widehat{\mathbb{P}}}[D_s^*(F) \mid \widehat{\mathcal{F}}_s] \, \mathrm{d}\widehat{f}_s,$$

where the $\mathcal{B}([0,T]) \otimes \widehat{\mathcal{F}}$, $\mathcal{B}(H^*)$-measurable map $D_\cdot^*(F)\colon [0,T] \times \Omega \to H^*$ can be linearly isometrically identified with the Malliavin derivative, *i.e.*, stochastic duration, $D.F$ in Definition 2.1. Further, $F \in L^2(\Omega, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ is in the domain of D^* and $\widehat{\mathcal{F}}_t$ is the $\widehat{\mathbb{P}}$-completed filtration generated by $\widehat{f}_s$, $0 \le s \le t$.

The H^*-valued conditional expectation

$$\mathrm{E}[D_t^*(F) \mid \widehat{\mathcal{F}}_t], \quad 0 \le t \le T$$

can be regarded as an estimation of $D.F$. Now let us have a look at the BSDE

$$Y_t = Y_T - \int_t^T Z_s \, \mathrm{d}\widehat{f}_s, \tag{3.1}$$

where $Y_T = F$. Then we observe that

$$Z_t = \mathrm{E}[D_t^*(F) \mid \widehat{\mathcal{F}}_t] \quad \widehat{P} \text{ a.e.}$$

for $0 \le t \le T$, a.e.

We wish to recast the dynamics of the solution (Y_t, Z_t) in Equation (3.1) *wrt* the original measure $\mathbb{P}$. Since σ_t is invertible t-a.e. we see that the natural filtration of $\widehat{W}_t$ coincides with the filtration $\widehat{\mathcal{F}}_t$. Assume that there exists a unique strong solution f_t^* of the SPDE

$$f_t^* = \int_0^t \sigma_s^{-1}[Af_s^* + \alpha_s(s,\cdot)] \, \mathrm{d}s + W_t, \quad 0 \le t \le T, \tag{3.2}$$

where W_t is the Q-cylindrical Wiener process in Equation (2.12). See, *e.g.*, (Prévôt and Röckner 2007) for criteria about the existence and uniqueness of solutions of non-linear SPDE's.

Remark. Let $\alpha_t = b(t, f_t)$ in Equation (3.2) for a Borel measurable map $b\colon [0,T] \times H \to H$. Impose on A the rather strong condition to be a bounded operator on H. Further assume that the drift coefficient $F(t,x) := \sigma_t^{-1}[Ax + b(t,x)]$ satisfies a linear growth and Lipschitz condition *wrt* x, uniformly in t. Then the Picard iteration gives a unique strong solution of Equation (3.2).

The Assumption of Equation (3.2) entails that the natural filtration of W_t is given by $\widehat{\mathcal{F}}_t$. Then it follows from Equation (2.12) that the solution (Y_t, Z_t) in Equation (3.1) has the following BSDE dynamics under P.

$$\begin{aligned} Y_t &= Y_T + \int_t^T Z_s[Af_s + \alpha_s(s,\cdot)]\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s^* \\ Y_T &= F, \end{aligned}$$

where W^* is the square integrable H-valued martingale given by

$$W_t^* = \int_0^t \sigma_s\,\mathrm{d}W_s$$

So we see that the estimate Z_t of the stochastic duration of F satisfies the forward-backward stochastic partial differential equation [FBSPDE]

$$\begin{aligned} \mathrm{d}f_t &= Af_t + \alpha_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t \\ Y_t &= Y_T + \int_t^T Z_s[Af_s + \alpha_s(s,\cdot)]\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s^* \\ Y_T &= F, \end{aligned} \tag{3.3}$$

where F is a measurable functional of the solution of the forward SPDE, *i.e.*, of the forward curves f_t. For more information about linear forward-backward S(P)DE's the reader may consult (Ma and Yong 1999). See also (Øksendal, Proske, and Zhang 2005).

Remark. In view of financial applications it would be desirable to develop a numerical approximation scheme for solutions (Y_t, Z_t) of FBSPDE's of the type of Equation (3.3). In general, this is a challenging task. A possible *ansatz* to this problem (in some special cases) would be to employ the results in (Zhang 2004) or in (Nakayama 2002) in connection with Galerkin approximation. Another approach could be based on finite element or finite difference schemes in a backward stochastic partial differential equation [BSPDE] setting. In the framework of the linear Gaussian model, as in Equation (2.9), for the forward curves one can simplify further the numerical analysis by using dimension reduction techniques as, *e.g.*, principal component analysis of interest rate data. See (Carmona and Tehranchi 2006).

Remark. Using stochastic distribution theory the concept of stochastic duration for interest rate claims $F \in \mathbb{D}^{1,2}$ can be extended to the case of claims contained in a space of generalized random variables which comprises the space of square integrable functionals of the forward curves *wrt* $\widehat{\mathbb{P}}$. See, *e.g.*, (Üstünel 1995) or (Da Prato and Zabczyk 1992). As a consequence we may still interpret Z_t in Equation (3.3) as an estimate of the stochastic duration of a claim F, when $F \in L^2(\mathbb{P}) \cap L^2(\widehat{\mathbb{P}})$.

Finally, we want to discuss an extension of the concept of delta hedge of interest rate sensitive securities developed by (Hull and White 1994) to a stochastic setting, which involves the fluctuations of the whole yield surface. The purpose of delta hedge is to immunize portfolios of interest-rate-sensitive securities under Ho's interest rate scenario (Ho 1992). In other words, the idea devised by (Hull and White 1994) is to neutralize given financial positions in interest-rate derivatives against parallel shifts of i-years spot rates (or *key* rates).

We want to propose a mathematical framework which facilitates the construction of immunization strategies of interest-rate-sensitive portfolios in the sense of (Hull and White 1994) *wrt* stochastic fluctuations of the yield surface. In fact, we aim at minimizing the exposure of given financial positions to interest rate risk by going short in bonds of a generalized bond portfolio, that is, of self-financing portfolios composed of infinitely many bonds of any maturity.

To this end we need some notions and conditions. Suppose that the generalized HJM-model [see Equation (2.10)] for the forward curves f_t fulfills the HJM no-arbitrage condition

$$\alpha_t(x) = \sum_{k \geq 1} \sigma_t^{(k)}(x) \Big(I_x\big(\sigma_t^{(k)}\big) + \lambda_t^{(k)} \Big),$$

where the processes $\lambda_t^{(k)}$, $k \geq 1$, are the Fourier coefficients of a predictable H-valued process

$$\lambda_t = \sum_{k \geq 1} \lambda_t^{(k)} e_k$$

Here $\{e_k\}$ is an orthonormal basis of H. Further $\sigma_t^{(k)}$, $k \geq 1$, is given as in Equation (2.6) and I_x is a linear functional in H^* defined by

$$I_x(f_t) = \int_0^x f_t(u)\,\mathrm{d}u$$

We remark that the processes $\lambda_t^{(k)}$, $k \geq 1$, admit the interpretation of market prices of risk *wrt* different bond maturities.

Now let us consider the discounted bond price curve $\widetilde{P}_t(\cdot)$ given by

$$\widetilde{P}_t(x) = \exp\Big(- \int_0^t f_s(0)\,\mathrm{d}s - \int_0^x f_s(x)\,\mathrm{d}s \Big)$$

We require that the conditions

$$\mathrm{E}\left[\exp\Big(\int_0^t \langle \lambda_s, \mathrm{d}W_s \rangle_0 - \tfrac{1}{2} \int_0^t \|\lambda_s\|_0^2\,\mathrm{d}s \Big) \right] = 1$$

and

$$\int_0^t \left(\int_0^s \|\delta_{s-u} \circ \sigma_s\|^2_{L_2^0} \, \mathrm{d}u \right)^{1/2} \mathrm{d}s < \infty$$

hold for all $t \geq 0$.

Then, using Itô's Formula and Girsanov's Theorem one finds that

$$\widetilde{P}(t,T) = P(0,T) - \int_0^t P(s,T) I_{T-s} \circ \sigma_s \, \mathrm{d}\widetilde{W}_s, \tag{3.4}$$

where

$$\widetilde{W}_t = W_t + \int_0^t \lambda_s \, \mathrm{d}s$$

is a Q-Wiener process under a local martingale measure $\widetilde{\mathbb{P}}$.

Define

$$\widetilde{\sigma}_t(\omega, x) = P_t(x) I_x \circ \sigma_t \tag{3.5}$$

Let G be a separable Hilbert space in $C\big([0,\infty)\big)$ such that evaluation functionals δ_x on G are continuous and the semigroup S_t of left shift operators is strongly continuous on G. See Equations (2.7) and (2.8). From now forward we assume that $\widetilde{\sigma}_t$ in Equation (3.5) is a predictable $L(U_0, G)$-valued process such that $\int_0^T \|\widetilde{\sigma}_s\|^2_{L_2^0} \, \mathrm{d}s < \infty$ a.e. The latter implies that the bond price curves $\widetilde{P}_t$ are G-valued and satisfy

$$\mathrm{d}\widetilde{P}_t = A\widetilde{P}_t \, \mathrm{d}t - \widetilde{\sigma}_t \, \mathrm{d}\widetilde{W}_t$$

or

$$\mathrm{d}\widetilde{P}_t = (A\widetilde{P}_t - \widetilde{\sigma}_t[\lambda_t]) \, \mathrm{d}t - \widetilde{\sigma}_t \, \mathrm{d}W_t$$

in the mild sense.

Now let us consider generalized bond portfolios. See (Björk, Masi, Kabanov, and Runggaldier 1997). That is, the wealth process V_t of such portfolios is given by

$$V_t = V_t(\phi) := \phi_t[P_t(\cdot)], \quad t \geq 0,$$

where ϕ_t is a predictable G^*-valued process. The process ϕ_t can be regarded as the trading strategy of an investor who manages a portfolio with infinitely many bonds of any maturity.

For example, the strategy $\phi_t = \delta_{T-t}$ stands for buying and holding a zero-coupon bond with maturity T, since $\phi_t[P_t(\cdot)] = P(t,T)$.

Assume that

$$\mathrm{E}_{\widetilde{\mathbb{P}}}\left[\int_0^t \|\phi_s \circ \widetilde{\sigma}_s\|^2_{L_2^0}\,\mathrm{d}s\right] < \infty$$

for all $t \geq 0$. Then we shall say that a trading strategy ϕ_t, $t \geq 0$, is *self-financing* if there is a $V_0 \in \mathbb{R}$ such that

$$\widetilde{V}_t(\phi) - \int_0^t \phi_s \circ \widetilde{\sigma}_s\,\mathrm{d}\widetilde{W}_s = V_0 \tag{3.6}$$

for all $t \geq 0$ a.e. where $\widetilde{V}_t(\phi)$ is the discounted wealth process given by

$$\widetilde{V}_t(\phi) = \phi_t[\widetilde{P}_t(\cdot)]$$

See, *e.g.*, (Björk, Masi, Kabanov, and Runggaldier 1997). We denote the set of all self-financing strategies by $\mathcal{A}$.

Remark. In the infinite-dimensional HJM-framework the existence of a unique martingale measure does not imply in general that the bond market given by Equation (3.4) is complete. The latter is a deficiency not shared by finite-rank models. However, since the kernels of $\widetilde{\sigma}_t$, as in Equation (3.5), are zero t-a.e. our bond market is approximately complete in the following sense. For all $\epsilon > 0$ there exists a strategy $\phi^\epsilon \in \mathcal{A}$, such that

$$\mathrm{E}_{\widetilde{\mathbb{P}}}\left[\left(\mathrm{E}_{\widetilde{\mathbb{P}}}[\widetilde{h}] + \int_0^T \phi^\epsilon_s \circ \widetilde{\sigma}_s\,\mathrm{d}\widetilde{W}_s - \widetilde{h}\right)^2\right] < \epsilon,$$

where $\widetilde{h}$ a discounted contingent claim. See, *e.g.*, (Björk, Masi, Kabanov, and Runggaldier 1997).

Suppose that a trader is long in interest rate securities at time $t \geq 0$ whose price process is L_t. In order to neutralize the risk coming from the fluctuations of the yield surface the trader wishes to go short in the generalized bond portfolio, as in Equation (3.6), for a self-financing strategy $\phi^* \in \mathcal{A}$, such that ϕ^* minimizes at any time point the worst-scenario interest rate sensitivity of the resulting portfolio. More precisely, the trader tries to find a $\phi^* \in \mathcal{A}$ such that

$$\inf_{\phi^* \in \mathcal{A}} \mathrm{E}\left[\int_0^T \|D.\big(L_t - V_t(\phi)\big)\|^2_K\,\mathrm{d}t\right] = \mathrm{E}\left[\int_0^T \|D.\big(L_t - V_t(\phi^*)\big)\|^2_K\,\mathrm{d}t\right] < \infty, \tag{3.7}$$

where K is the RKHS of the forward curves. Note that

$$\sup_{\|k\|_K = 1} \langle D.F, k\rangle_K = \|D.F\|_K$$

for an interest claim $F \in \mathbb{D}^{1,2}$. So, by Equation (2.17), $\|D.F\|_K$ admits the interpretation that it is the worst-scenario sensitivity with respect to all directional interest rate changes $k \in K$.

Using the estimate $Z. = Z.(F)$ for the stochastic duration $D.(F)$ in the FBSPDE of Equation (3.3) for $F = L_t - V_t(\phi)$ [see Remark 3 and Relation 2.15] the optimization problem of Equation (3.7) then takes the form

$$\begin{aligned}\inf_{\phi^* \in \mathcal{A}} \mathrm{E}\left[\int_0^T \int_0^T \|Z_u\big(L_t - V_t(\phi)\big) \circ \sigma_u\|^2_{L_2^0} \,\mathrm{d}u\,\mathrm{d}t\right] \\ = \mathrm{E}\left[\int_0^T \int_0^T \|Z_u\big(L_t - V_t(\phi^*)\big) \circ \sigma_u\|^2_{L_2^0} \,\mathrm{d}u\,\mathrm{d}t\right] < \infty\end{aligned}$$

for $\phi^* \in \mathcal{A}$.

We see that the construction of an immunized bond portfolio reduces to an optimal control problem of the FBSPDE of Equation (3.3) or the FBSPDE

$$\begin{aligned}\widetilde{V}_t(\phi) &= \widetilde{V}_0(\phi) - \int_0^t \phi_s \circ \widetilde{\sigma}_s \,\mathrm{d}\widetilde{W}_s \\ Y_t &= Y_T + \int_t^T Z_s[Af_s + \alpha_s(s,\cdot)]\,\mathrm{d}s - \int_t^T Z_s \,\mathrm{d}W_s^* \\ Y_T &= F,\end{aligned}$$

where $F = L_t - V_t(\phi)$ for each t, if L_t is a measurable functional of $\widetilde{V}.(\phi)$.

An approach to tackle this problem could be based on a stochastic maximum principle for FBSPDE's. See (Haadem and Mandrekar 2010). From a practical point of view it would be important to find numerical approximation schemes for a delta hedge $\phi^* \in \mathcal{A}$.

Remark.

1. It is conceivable that the concept of g-expectation by (Peng 1997) for BSDE's can be generalized to FBSPDE's of the type of Equation (3.3). The latter would enable the construction of risk measures of functionals of forward curves. Such a construction would reveal the role of the stochastic duration as a building block for general interest rate risk measures.

2. We point out that our framework also allows for the definition of *stochastic convexity*, that is, a measure of "curvature" *wrt* the fluctuations of the yield surface. It makes sense to define the stochastic convexity of a twice Malliavin differentiable interest rate claim F as
$$D.D.(F) \in L^2(\Omega, \widehat{\mathcal{F}}, \widehat{P}; K \otimes K)$$

Chapter IV

Pricing of Margrabe Options for Large Investors with Application to Asset-Liability Management in Life Insurance

1 Introduction

Usual continuous-time financial models assume that individual investor's behaviour does not affect stock prices. This idea is known under the name "small investor hypothesis", implying that each investor is assumed to be unable to move market prices by his trading decisions and acts as a price taker. In contrast to that hypothesis, many authors investigated a setting in which market prices depended on the size of the transaction.

This research direction can be classified into two broad categories. The first one deals with liquidity risk and investigates its impact on asset prices and hedging strategies. In particular, (Cetin, Jarrow, and Protter 2004) and (Cetin, Soner, and Touzi 2010) assume that market prices result from interaction of small price taking investors with an exogenously given asset supply curve. The latter determines the price for a given transaction size. This research category also includes (Cetin and Rogers 2007), (Rogers and Singh 2007), as well as (Soner and Gökay 2009), among others.

The second type of literature in this research direction focuses on feedback effects of hedging strategies on underlying asset prices. It is assumed that a large investor can - directly or indirectly - affect market prices. In particular, (Platen and Schweizer 1998) show that large investor's hedging decisions can lead to increased volatility of the underlying asset. The authors in (Frey and Stremme 1997) also present a model in which the existence of a large trader results in higher market volatility. These authors, as well as (Jarrow 1994), (Frey 1998), and (Papanicolaou and Sircar 1998) model stock prices as being directly dependent on large investor's stock holdings through a certain reaction function. The authors in (Bank and Baum 2004) use a similar reaction function setting to the one employed in (Frey and Stremme 1997) to specify the dynamics of the asset price as a semi-martingale parametrized by the large investor's position in the stock. In contrast to their approach, (Cvitanić and Ma 1996), as well as (Cuoco and Cvitanić 1998), assume that parameters in the diffusion driving the evolution of the stock price, rather than the price itself, depend on large investor's trading decisions. (De Marzo and Urosevic 2006) develop a general equilibrium model that justifies the models used by Cvitanić and Ma, and Cuoco and Cvitanić.

The present paper belongs to the second group. We adopt the framework used in (Cvitanić and Ma 1996) to study the problem faced by an insurance company that wants to achieve solvency. This can be done by means of a Margrabe option that gives the company the right to exchange its asset portfolio for a valuation portfolio. The latter is a replicating portfolio for the insurance company's liabilities. One can obtain the price of a Margrabe option within the Black-Scholes setting by representing it as a standard European Put option written on a certain artificial asset. However, if the company is a large investor whose hedging strategies can have a feedback effect on the price of the underlying asset, one needs to use a different pricing methodology. In this paper it is assumed that market volatility depends on the value of the replicating portfolio held by the large investor and the amount of money he invests in the asset. Our objective is to study the hedging problem faced by the large investor. Namely, we intend to find the price of and the hedging strategy for the Margrabe option in a situation where the insurance company is no longer a price taker. In mathematical terms, the problem translates into a forward-backward stochastic differential equation. The solution of its backward part gives us the quantities we are looking for.

The paper is organised as follows. Section two presents the model, explains the derivation of the associated FBSDE and PDE and ends with an explanation of a numerical scheme that will be used to obtain the solution. Section three discusses the application at hand. It gives a brief overview of the relevant material in (Wüthrich, Bühlmann, and Furrer 2008) and puts it in the context of the large investor problem. Section four gives details of numerical simulations and presents the findings.

2 Model

We study the hedging problem for a large investor on a finite time horizon $[0, T]$, given the initial stock price $S(0)$ and the terminal wealth $g(S(T))$ referring to the pay-off of the option to be hedged. The objective of the hedger is to find a portfolio process and the minimal initial wealth $x = X(0)$, such that at the option's expiration date $X(T) = g(S(T))$.

We assume that there are two investment possibilities on the market:

- Bond with the following dynamics:

$$\begin{aligned} dB(t) &= B(t) \cdot r \cdot dt \\ B(0) &= 1, \end{aligned} \tag{2.1}$$

  where r is a riskless interest rate.

- Stock that follows the dynamics:

$$\begin{aligned} dS(t) &= S(t) \cdot b \cdot dt + S(t) \cdot \gamma(t, S(t), X(t), \pi(t))\, dW \\ &= \mu \cdot dt + \sigma\, dW \\ S(0) &= s, \end{aligned} \tag{2.2}$$

  where b is a constant drift, γ is an appropriate volatility function, $X(t)$ is the the value of the replicating portfolio held by large investor and $\pi(t)$ is the amount of money he invests in the stock. The main thing to be noticed here is that stock's volatility is assumed to depend on large investor's strategies. $W(t)$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq \bar{T}}, \mathbb{P})$ with filtration generated by $W(t)$.

The investor begins with a certain endowment $x > 0$ and allocates his wealth in the stock and the bond according to a certain strategy. The portfolio process $\pi = \pi(t), t \in [0, T]$ is assumed to be a real-valued, progressively measurable, square-integrable stochastic process.

At each point of time the value of the replicating portfolio equals:

$$X(t) = \frac{\pi(t)}{S(t)} \cdot S(t) + \frac{(X(t) - \pi(t))}{B(t)} \cdot B(t) \tag{2.3}$$

and has the following dynamics:

$$\begin{aligned} dX(t) &= \frac{\pi(t)}{S(t)} \cdot dS(t) + \frac{(X(t) - \pi(t))}{B(t)} \cdot dB(t) = \\ &= \big(\pi(t)b + (X(t) - \pi(t)) \cdot r\big)dt + \\ &\qquad \pi(t) \cdot \gamma(t, S(t), X(t), \pi(t))\, dW(t) \end{aligned} \tag{2.4}$$

Following (Cvitanić and Ma 1996), we introduce the following assumptions.

- The drift and volatility functions in the stock price dynamics, μ, and σ, are twice continuously differentiable. The functions b and γ together with their first order partial derivatives are bounded uniformly in (t, s, x, π). Further, it is assumed that partial derivatives of b and γ in s satisfy

$$sup_{(t,s,x,\pi)} \left(\left| s \cdot \frac{\partial b}{\partial s} \right|, \left| s \cdot \frac{\partial \gamma}{\partial s} \right| \right) < \infty \tag{2.5}$$

- There exists a positive constant k, such that $\gamma^2 > k$ for all (t, s, x, π).

According to Lemma 2.3 in (Cvitanić and Ma 1996) these assumptions guarantee that the stock price remains almost surely positive.

- Also, we assume that option's pay-off function g is non-negative and $lim_{s \to \infty} g(s) = \infty$. Moreover, g has bounded, continuous partial derivatives up to third order and there exist constants $K > 0$ and $M > 0$, such that

$$\left| s \cdot \frac{dg}{ds} \right| \le K \cdot \big(1 + g(s)\big) \tag{2.6}$$

and

$$sup_{s>0} \left| s^2 \cdot \frac{d^2 g}{ds^2} \right| = M < \infty \tag{2.7}$$

In order to satisfy these assumptions we use the following smoothed pay-off function (suggested in (Frey 1998)):

$$g(x) = \frac{1}{2} \cdot \left(K - x + \sqrt{(K - x)^2 + \alpha} \right), \tag{2.8}$$

where α is a smoothing parameter.

- Finally, we assume that partial derivatives of γ in x and π satisfy the following condition.

$$sup_{(t,s,x,\pi)}\left(\left|x\cdot\frac{d\gamma}{dx}\right|+\left|\pi\cdot\frac{d\gamma}{d\pi}\right|\right)<\infty \tag{2.9}$$

These assumptions are satisfied by the constant drift coefficient b in SDE (2.2) and the following volatility function (suggested in (Cvitanić and Ma 1996))

$$\gamma\big(t,S(t),X(t),\pi(t)\big)=\sigma+\frac{1}{4}\cdot arctan\big(X(t)^2+\pi(t)^2\big) \tag{2.10}$$

Definition 2.1. ((Cvitanić and Ma 1996)) For any given initial wealth $x>0$ the portfolio process π is called admissible (with respect to x) if for any $s>0$ the corresponding price process $S(\cdot)$ and wealth process $X(\cdot)$ satisfy $S(t)>0$ and $X(t)\geq 0,\ \forall t\in[0,T]$. For each initial wealth x, we denote the set of admissible portfolio strategies by $\mathcal{A}$.

We are now going to formulate the hedging problem in terms of a forward-backward stochastic differential equation. Consider a FBSDE given by the stock price SDE - the forward part - and the BSDE describing the evolution of the portfolio wealth - the backward part.

$$\begin{cases} dS(t) &= S(t)\cdot b\cdot dt+S(t)\cdot\gamma(t,S(t),X(t),\pi(t))\,dW \\ dX(t) &= \big(\pi(t)b+(X(t)-\pi(t))\cdot r\big)dt+\pi(t)\cdot\gamma(t,S(t),X(t),\pi(t))\,dW(t) \\ S(0) &= s \\ X(T) &= g(S(T)) \end{cases} \tag{2.11}$$

Definition 2.2. ((Cvitanić and Ma 1996)) A triple (S,X,π) is called an adapted solution of FBSDE (2.11) if S, X and π are $\{\mathcal{F}_t\}$-adapted, square-integrable stochastic processes satisfying (2.11) a.s.

The assumptions in the previous section guarantee that the FBSDE (2.11) has a unique adapted solution, as shown in (Cvitanić and Ma 1996).

We now intend to obtain a partial differential equation characterising the FBSDE (2.11). Denote $S_t^{\tau,y}:=\big(S(t)|S(\tau)=y\big)$ and $X_t^{\tau,y}:=V(t,S_t^{\tau,y})$. Then by the Itô's formula the price of the option written on the stock has the following dynamics:

$$\begin{aligned} dV(t,S_t^{\tau,y}) &= \frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial S}dS+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \left(\frac{\partial V}{\partial t}+\frac{\partial V}{\partial S}Sb+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2\Big[\sigma+arctan\big(X(t)^2+\pi(t)^2\big)\Big]^2\right)dt \\ &\quad +\frac{\partial V}{\partial S}S\cdot\Big[\sigma+arctan\big(X(t)^2+\pi(t)^2\big)\Big]dW(t), \end{aligned} \tag{2.12}$$

where $V(t, S_t^{\tau,y})$ denotes the option price. At the same time, as can be seen from SDE (2.4):

$$dX(t) = \big(\pi(t)b + (X(t) - \pi(t)) \cdot r\big)dt + \pi(t) \cdot \Big[\sigma + arctan\big(X(t)^2 + \pi(t)^2\big)\Big]\, dW(t) \tag{2.13}$$

Matching the dt and dW terms we find the following:

$$\pi(t) = \frac{\partial V}{\partial S} S \tag{2.14}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2 \Big[\sigma + arctan\big(X(t)^2 + \pi(t)^2\big)\Big]^2 = \Big(V - \frac{\partial V}{\partial S} S\Big) \cdot r \tag{2.15}$$

Condition (2.14) gives us the hedging strategy right away. Combining it with equation (2.15) produces the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} S^2 \Big[\sigma + arctan\big(X(t)^2 + \pi(t)^2\big)\Big]^2 - \Big(V - \frac{\partial V}{\partial S} S\Big) \cdot r = 0 \tag{2.16}$$

This is a quasi-liner parabolic partial differential equation that needs to be solved in order to obtain function $V(t, S(t))$, which represents the option price, as well as its first derivative $\frac{\partial V}{\partial S}$, which is needed to compute the hedging strategy.

Obtaining an analytical solution for PDE (2.16) is rather formidable. Instead, it can be solved by means of the Finite Differences method. In particular, we are going to employ the Predictor-Corrector method, which is successfully used to solve non-linear PDEs and described in (Duffy 2006).

The idea of the method is the following. First, we split the time interval $[0, T]$ and the space interval $(0, S^{max})$ into N and M subintervals, correspondingly, and denote by $V_{t,n} := V(t \cdot \Delta t, n \cdot \Delta S)$, where $\Delta t := \frac{T}{N}$ and $\Delta S := \frac{S^{max}}{M}$. Then, we write down an explicit finite difference scheme for PDE (2.16). We use the backward finite difference in time and the central finite difference in space in order to approximate the first order partial derivatives of function $V(t, S(t))$ with respect to t and S respectively.

$$\begin{aligned} V_{t-1,n} = V_{t,n} + \Delta t \cdot \Bigg[&\frac{1}{2} \cdot \frac{V_{t,n+1} - 2V_{t,n} + V_{t,n-1}}{(\Delta S)^2} S_n^2 \cdot \\ &\cdot \Big[\sigma + arctan\Big(V_{t,n}^2 + \Big(\frac{V_{t,n+1} - V_{t,n-1}}{2\Delta S} S_n\Big)^2\Big)\Big]^2 - \\ &- \Big(V_{t,n} - \frac{V_{t,n+1} - V_{t,n-1}}{2\Delta S} S_n\Big) \cdot r \Bigg] \end{aligned} \tag{2.17}$$

The value $V_{t-1,\cdot}$ obtained this way is a preliminary estimate, which is to be "corrected" through the use of the implicit scheme, which looks as follows:

$$\begin{aligned} V_{t-1,n} &= V_{t,n} + \Delta t \cdot \Bigg[\frac{1}{2} \cdot \frac{V_{t-1,n+1} - 2V_{t-1,n} + V_{t-1,n-1}}{(\Delta S)^2} S_n^2 \cdot \\ &\quad \cdot \Big[\sigma + arctan\Big(V_{t-1,n}^2 + \Big(\frac{V_{t-1,n+1} - V_{t-1,n-1}}{2\Delta S} S_n \Big)^2 \Big) \Big]^2 - \\ &\quad - \Big(V_{t-1,n} - \frac{V_{t-1,n+1} - V_{t-1,n-1}}{2\Delta S} S_n \Big) \cdot r \Bigg] \end{aligned} \tag{2.18}$$

In the traditional case, the use of the implicit scheme (2.18) would involve a solution of a non-liner system of equations, which would require numerical methods for itself. Instead, the predictor-corrector method suggests using the preliminary estimate of $V_{t-1,\cdot}$ obtained in (2.17) to substitute for the corresponding values on the right hand side of (2.18). Thus, we obtain a cyclical algorithm for consecutive computation of $V_{t-1,\cdot}$:

$$\begin{aligned} V_{t-1,n}^{k+1} &= V_{t,n} + \Delta t \cdot \Bigg[\frac{1}{2} \cdot \frac{V_{t-1,n+1}^k - 2V_{t-1,n}^k + V_{t-1,n-1}^k}{(\Delta S)^2} S_n^2 \cdot \\ &\quad \cdot \Big[\sigma + arctan\Big((V_{t-1,n}^k)^2 + \Big(\frac{V_{t-1,n+1}^k - V_{t-1,n-1}^k}{2\Delta S} S_n \Big)^2 \Big) \Big]^2 - \\ &\quad - \Big(V_{t-1,n}^k - \frac{V_{t-1,n+1}^k - V_{t-1,n-1}^k}{2\Delta S} S_n \Big) \cdot r \Bigg], \end{aligned} \tag{2.19}$$

where the initial $V_{t,n}^0$ needed to initiate the cycle is obtained from the explicit scheme (2.17). The algorithm is exited after a predefined number of iterations, or when a certain measure of distance between the consecutive solutions falls below a pre-specified value.

The initial condition in this numerical scheme comes from the option's pay-off function, while boundary conditions are obtained by using the put-call parity.

3 Application: Asset-Liability management in life insurance

This section relies on material in (Wüthrich, Bühlmann, and Furrer 2008). The central problem in insurance business is solvency. In simple words, a company is called solvent if it is able to meet all its financial obligations. Formally,

Definition 3.1. (Wüthrich, Bühlmann, and Furrer 2008) A company is solvent at time t if

$$\mathcal{E}_t[S] \geq \mathcal{E}_t[VaPo], \qquad \forall t \geq 0, \tag{3.1}$$

where $VaPo$ is a valuation portfolio, which can be viewed as a replicating portfolio for the insurance liabilities in terms of financial instruments; S is the existing asset portfolio of the insurance company; $\mathcal{E}$ is the accounting principle used to value assets and liabilities. A usual choice is the economic accounting principle, which corresponds to the valuation at current market prices.

The goal of asset-liability management is to maximize returns on company's assets under the solvency constraint (3.1). An obvious way to guarantee solvency is to invest in the valuation portfolio, i.e. company's assets consist of $VaPo$ and a certain excess capital F, which must always be non-negative. However, a mismatch between the actual company's assets and $VaPo$ is often preferred due to company's desire to maximize its returns. In this case solvency is guaranteed through the use of a Margrabe option. The latter is the right to exchange one asset for another at a pre-specified time moment. We consider a time period $[t, t+1]$. At t we decompose company's assets as follows:

$$S = \tilde{S} + M + F, \tag{3.2}$$

where $\tilde{S}$ is an asset portfolio satisfying the solvency condition (3.1); F denotes free reserves or excess capital; and M denotes the value of the Margrabe option, giving the holder the right to switch from $\tilde{S}$ to $VaPo$ at time $t+1$, if needed.

The objective now is to price such an option in case company's trading actions can affect market prices. This situation is not unrealistic, as insurance companies normally manage large asset portfolios and their market trades can also be quite substantial. To formalize the problem we consider two stochastic processes

$$Y_t = \mathcal{E}_t[\tilde{S}], \tag{3.3}$$

and

$$V_t = \mathcal{E}_t[VaPo], \tag{3.4}$$

Then by the risk-neutral valuation the price of the Margrabe option is given by

$$P(t, t+1) \cdot \mathbb{E}^*[V_{t+1} - Y_{t+1} | \mathcal{F}_t], \tag{3.5}$$

where $P(t, t+1)$ is a zero coupon bond maturing at $t+1$ and $\mathbb{E}^*$ is an expectation under the corresponding risk-neutral measure. Changing the numeraire from $P(t, t+1)$ to V_t and introducing a new process $\tilde{Y}_t := \frac{Y_t}{V_t}$, (3.5) can be rewritten as follows

$$V_t \cdot \mathbb{E}^{**}[1 - \tilde{Y}_{t+1} | \mathcal{F}_t], \tag{3.6}$$

where $\mathbb{E}^{**}$ is an expectation with respect to the risk-neutral measure under a new numeraire. Thus, the price of the Margrabe option is equivalent to the price of a European put written on the artificial asset $\tilde{Y}$, having strike $K = 1$ and maturing at $t+1$.

4 Numerical simulations

We assume that $\tilde{Y}$ has the following dynamics

$$\left\{ \begin{array}{rcl} d\tilde{Y}(t) & = & \tilde{Y}(t) \cdot \gamma\big(X(t), \pi(t)\big)\, dW \\ \tilde{Y}(0) & = & 100, \end{array} \right. \tag{4.1}$$

where the volatility function $\gamma\big(X(t), \pi(t)\big)$ is the same as in (2.10). For the sake of computational stability we assume the initial value of $\tilde{Y}$ equal 100 instead of 1. The replicating portfolio has the following dynamics

$$\left\{ \begin{array}{rcl} dX(t) & = & \big(X(t) - \pi(t)\big) r\, dt + \pi(t) \cdot \gamma(X(t), \pi(t))\, dW(t) \\ X(T) & = & g(\tilde{Y}(T)), \end{array} \right. \tag{4.2}$$

where the terminal condition is given by the smoothed European Put pay-off function with strike equal to 100, as in (2.8)

$$g(x) = \frac{1}{2} \cdot \left(100 - x + \sqrt{\left(x - 100\right)^2 + \alpha}\right), \tag{4.3}$$

where we take $\alpha = 10$. The PDE to solve is the same as (2.16), with boundary conditions coming from the well known Put-Call parity condition:

$$\left\{ \begin{array}{l} V(t, \tilde{Y}^{max}) \approx 0 \\ V(t, 0) \approx e^{-r \cdot (T-t)}, \end{array} \right. \tag{4.4}$$

where T is the time of maturity of the option, assumed equal 1. The space interval, represented by possible values of $\tilde{Y}$ is assumed to be given by $(0, \tilde{Y}^{max}) = (0, 200)$. The time interval spans from 0 to 1. To apply the Finite Difference approach we split the time and space intervals into $N = 2000$ and $M = 20$ subintervals, correspondingly. Also, we use 3 iteration cycles in the predictor-corrector scheme. The results of the numerical simulations are shown below.

Figure IV.1 shows the dependence of the price of the European Put option for a large investor on time and the price of the underlying asset. The same graph for a small investor is shown in Figure IV.2. A comparison graph in Figure IV.3 shows the difference between option prices for the large and small investors. It also contains a graph of a theoretical option price computed according to the Black-Scholes formula.

As could have been expected, the price for the large investor is higher than that for a small investor, highlighting the fact that large investor's attempts to hedge the option would result in higher volatility of the underlying. It is also seen at the graphs that the derivative $\frac{\partial V}{\partial S}$ used in computing the hedging strategy, is always negative, implying that the investor must always go short in the underlying asset to hedge the option.

The qualitative situation does not change if we alter the underlying Black-Scholes volatility. Figure IV.4 shows comparison results for increased Black-Scholes volatility. We still observe that the large investor's price is considerably higher than the Black-Scholes

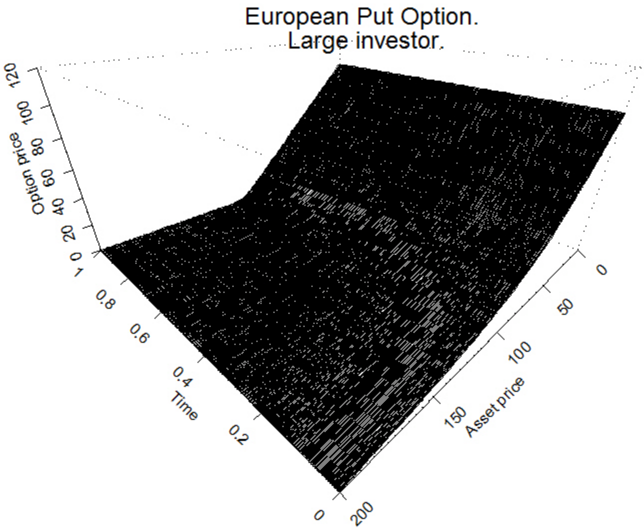

Figure IV.1: The value of the European Put option for the "large" investor.

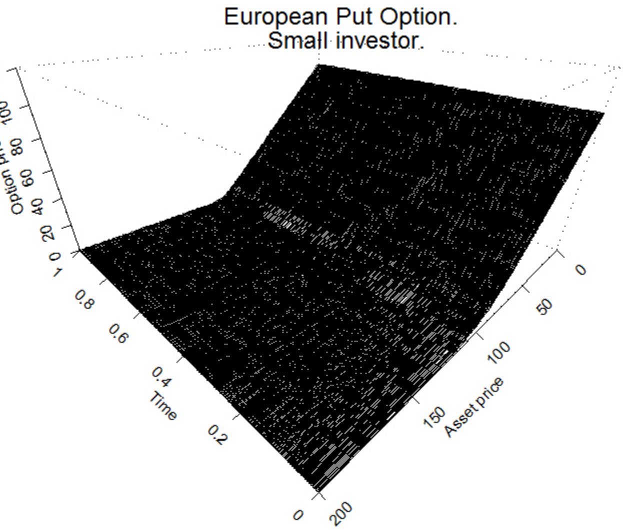

Figure IV.2: The value of the European Put option for a "small" investor.

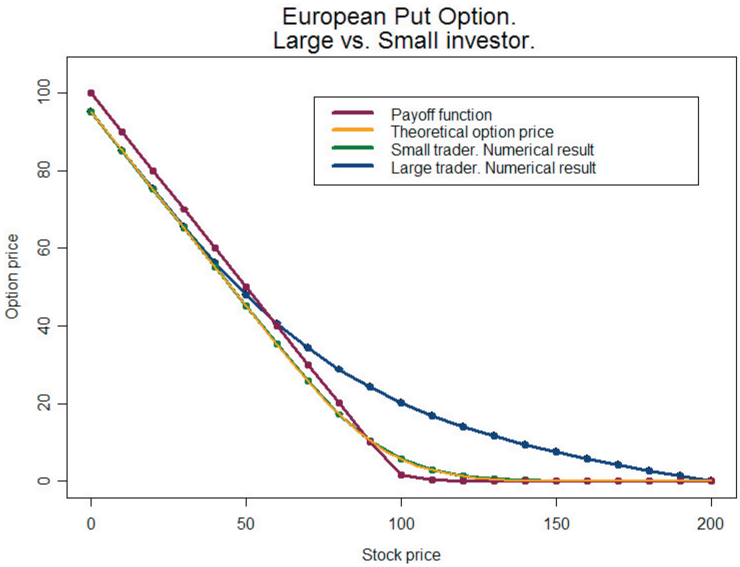

Figure IV.3: Price of a European Put option for the "large" and "small" investors.

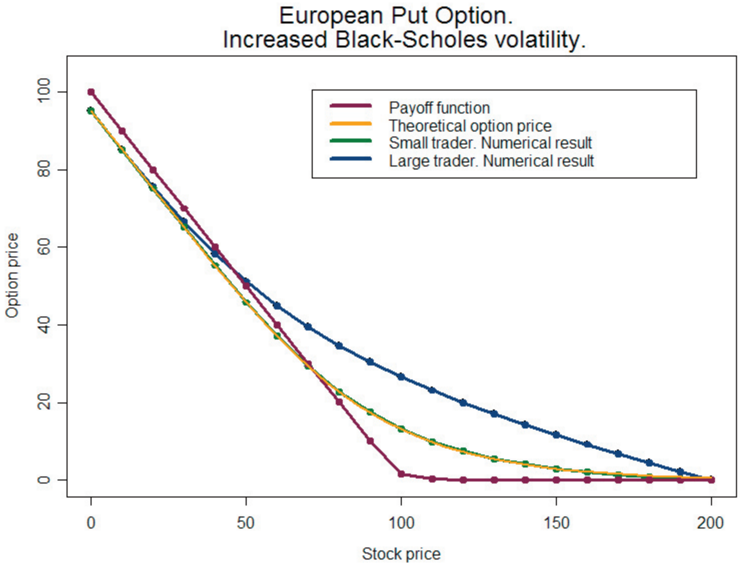

Figure IV.4: Price of a European Put option for the "large" and "small" investors. Black-Scholes volatility has been increased from 0.2 to 0.4.

one. Our computations confirm that a large investor suffers from adverse conditions when it comes to pricing and hedging options. The necessity to take into account increased volatility of the underlying makes large investor's hedging strategy more expensive, thus making the option itself more valuable. As for the practical situation under our investigation, one can conclude that buying a Margrabe option from an investor who uses the Black-Scholes formula is a good deal for the large investor, since the replicating alternative would cost him more. Large investor's direct market involvement in an attempt to construct a replicating portfolio for the Margrabe option would mean a short position in the underlying, thus resulting in higher asset volatility. This situation, in its turn, would compel the large investor to expand his market position even further, leading to higher replicating costs. Our main recommendation to an insurance company in this situation would be to avoid option replication and instead try to buy the Margrabe option from the market.

Chapter V

An Explicit Representation of Solutions of Forward SDE's with Reflections via White Noise Analysis

1 Introduction

This paper derives an explicit representation formula for strong solutions of forward stochastic differential equations with reflections (FSDER). Our approach relies on techniques from white noise analysis. Adopting ideas in (Meyer-Brandis and Proske 2010) we mention that the results obtained in this paper are relevant for the construction of solutions of FSDER's with discontinuous coefficients.

The plan of the paper is the following. The first Section is based on the material presented in (Hida, Kuo, Potthoff, and Streit 1993), (Obata 1994), (Potthoff and Streit 1991), (Kuo 1996) and (Holden, Øksendal, Ubøe, and Zhang 1996). Its objective is to introduce briefly some main concepts of the Gaussian white noise theory. In Section 2 some relevant results from the FSDER theory are reviewed. Finally, Section 3 provides a derivation of our main result.

2 Framework

We begin with giving a construction of Hida distributions on $\mathbb{R}^d$. Let A be a positive self-adjoint operator on $L^2([0,T])$ with $Spec(A) > 1$ and a fixed time horizon $0 < T < \infty$. Assume that A^{-r} is a Hilbert-Schmidt operator for some $r > 0$. Then the set of eigenfunctions of A forms a complete orthonormal basis $\{e_j\}_{j\geq 0}$ of $L^2([0,T])$ in $Dom(A)$ with eigenvalues $\lambda_j > 0$, $j \geq 0$ such that

$$Ae_j = \lambda_j e_j \tag{2.1}$$

for all $j \geq 0$. We assume that $1 < \lambda_0 \leq \lambda_1 \leq ... \longrightarrow \infty$, each e_j is continuous on $[0,T]$ and that there exists an open covering $[0,T] = \cup_\lambda O_\lambda$ and $\alpha(\lambda) \geq 0$ such that

$$\sup_{j\geq 0} \lambda_j^{-\alpha(\lambda)} \sup_{t\in O_\lambda} |e_j(t)| < \infty. \tag{2.2}$$

Following the notation in (Obata 1994), we denote by $\mathcal{S}([0,T])$ the standard countably Hilbertian space constructed from $(L^2([0,T]), A)$. $\mathcal{S}([0,T])$ is a nuclear space which is contained in $L^2([0,T])$, with a topological dual denoted by $\mathcal{S}'([0,T])$ being a conuclear space. Denote by $\mathcal{B}(\mathcal{S}'([0,T]))$ the Borel $\sigma-$algebra of $\mathcal{S}'([0,T])$. One can now apply the Bochner-Minlos theorem to find a unique probability measure π on $\mathcal{B}(\mathcal{S}'([0,T]))$ such that

$$\int_{\mathcal{S}'([0,T])} e^{i\langle\omega,\phi\rangle}\pi(d\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2([0,T])}} \tag{2.3}$$

for $\phi \in \mathcal{S}([0,T])$, where $\langle\omega,\phi\rangle$ denotes the action of $\omega \in \mathcal{S}'([0,T])$ on $\phi \in \mathcal{S}([0,T])$.

We now define the *d-dimensional white noise probability measure* μ given by the product measure

$$\mu = \otimes_{i=1}^d \pi. \tag{2.4}$$

on the measurable space

$$(\mathcal{S}', \mathcal{B}) := \left(\prod_{i=1}^{d} \mathcal{S}'([0,T]), \otimes_{i=1}^{d} \mathcal{B}(\mathcal{S}'([0,T])) \right) \tag{2.5}$$

For $\omega = (\omega_1, ..., \omega_d) \in \mathcal{S}'$ and $\phi = (\phi^{(1)}, ..., \phi^{(d)}) \in (\mathcal{S}([0,T]))^d$ we introduce the exponential functional

$$\widetilde{e}(\phi, \omega) = \exp\left(\langle \omega, \phi \rangle - \frac{1}{2} \|\phi\|^2_{L^2([0,T];\mathbb{R}^d)} \right), \tag{2.6}$$

where $\langle \omega, \phi \rangle := \sum_{i=1}^{d} \langle \omega_i, \phi_i \rangle$. Let us denote by $\left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} n}$ the $n-$th completed symmetric tensor product of $(\mathcal{S}([0,T]))^d$ with itself. One can observe that $\widetilde{e}(\phi, \omega)$ is holomorphic in ϕ around zero, which implies that there exist generalized Hermite polynomials $H_n(\omega) \in \left(\left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} n}\right)'$ such that

$$\widetilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \left\langle H_n(\omega), \phi^{\otimes n} \right\rangle \tag{2.7}$$

for ϕ in a certain neighbourhood of zero in $(\mathcal{S}([0,T]))^d$. One can also observe that

$$\left\{ \left\langle H_n(\omega), \phi^{(n)} \right\rangle : \phi^{(n)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} n},\ n \in \mathbb{N}_0 \right\} \tag{2.8}$$

is a total set of $L^2(\mu)$. Moreover, one can show that the following orthogonality relation is valid for all n, m, $\phi^{(n)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} m}$:

$$\int_{\mathcal{S}'} \left\langle H_n(\omega), \phi^{(n)} \right\rangle \left\langle H_m(\omega), \psi^{(m)} \right\rangle \mu(d\omega) = \delta_{n,m} n! \left(\phi^{(n)}, \psi^{(n)} \right)_{L^2([0,T]^n;(\mathbb{R}^d)^{\otimes n})}, \tag{2.9}$$

where $\delta_{n,m} = 1$ if $n = m$ and 0 otherwise. The latter expression implies that $\phi^{(n)} \longmapsto \left\langle H_n(\omega), \phi^{(n)} \right\rangle$ has a unique extension to $L^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})$ for ω a.e.

The functional $\left\langle H_n(\omega), \phi^{(n)} \right\rangle$ can be regarded as a $n-$fold iterated stochastic integral of functions $\phi^{(n)} \in L^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})$ with respect to a $d-$dimensional Wiener process $B_t = \left(B_t^1, ..., B_t^d\right)$ defined on the white noise space

$$(\Omega, \mathcal{F}, \mu) = \left(\mathcal{S}', \mathcal{B}, \mu\right) \tag{2.10}$$

Let us now denote by $\widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})$ the space of square integrable symmetric functions $f(x_1, ..., x_n) \in (\mathbb{R}^d)^{\otimes n}$. An important consequence of (2.7), (2.8) and (2.9) is that square integrable functionals of B_t admit a Wiener-Itô chaos representation. The latter can be considered an infinite-dimensional Taylor expansion, i.e. for all $F \in L^2(\mu)$ there exist unique kernels $\phi^{(n)} \in \widehat{L}^2([0,T]^n;(\mathbb{R}^d)^{\otimes n})$ such that

$$F(\omega) = \sum_{n\geq 0} \left\langle H_n(\omega), \phi^{(n)} \right\rangle \tag{2.11}$$

for ω a.e. The Wiener-Itô chaos expansion (2.11) can now be used to define the Hida stochastic test function and distribution space.

We construct the *Hida stochastic test function space* $(\mathcal{S})$ through a second quantization argument, that is we define $(\mathcal{S})$ to be the space of all $f = \sum_{n\geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \in L^2(\mu)$ such that

$$\|f\|_{0,p}^2 := \sum_{n\geq 0} n! \left\| \left((A^d)^{\otimes n} \right)^p \phi^{(n)} \right\|^2_{L^2([0,T]^n;(\mathbb{R}^d)^{\otimes n})} < \infty \tag{2.12}$$

for all $p \geq 0$, where $A^d = (A, ..., A)$. The space $(\mathcal{S})$ is a nuclear Fréchet algebra with respect to multiplication of functions and its topology is given by the seminorms $\|\cdot\|_{0,p}$, $p \geq 0$. In particular, one can use (2.7) to show that

$$\widetilde{e}(\phi, \omega) \in (\mathcal{S}) \tag{2.13}$$

for all $\phi \in (\mathcal{S}([0,T]))^d$.

We define the *Hida stochastic distribution space* $(\mathcal{S})^*$ as the topological dual of $(\mathcal{S})$ and get the Gelfand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*. \tag{2.14}$$

An important property of the Hida distribution space $(\mathcal{S})^*$ is that it contains the *white noise* of the coordinates of the $d-$dimensional Wiener process B_t. That is the time derivatives of the Wiener process

$$W_t^i := \frac{d}{dt} B_t^i, \ i = 1, ..., d \tag{2.15}$$

belong to $(\mathcal{S})^*$.

We now introduce another key concept of the white noise theory, namely the S-transform. The $S-$transform of $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$, is defined through the dual pairing

$$S(\Phi)(\phi) = \langle \Phi, \widetilde{e}(\phi, \omega) \rangle \tag{2.16}$$

for $\phi \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$, where $(\mathcal{S}_{\mathbb{C}}([0,T])$ is the complexification of $\mathcal{S}([0,T]))$. The $S-$transform is a monomorphism from $(\mathcal{S})^*$ to $\mathbb{R}$. In particular, if

$$S(\Phi) = S(\Psi) \text{ for } \Phi, \Psi \in (\mathcal{S})^*$$

then

$$\Phi = \Psi.$$

One can also show that

$$S(W_t^i)(\phi) = \phi^i(t),\ \ i = 1, ..., d \tag{2.17}$$

for $\phi = (\phi^{(1)}, ..., \phi^{(d)}) \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$.

Finally, we introduce the concept of the *Wick product*, which can be considered a tensor algebra multiplication on the Fock space. The Wick product of two distributions $\Phi, \Psi \in (\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi)S(\Psi)(\phi) \tag{2.18}$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$. As an example of the use of the Wick product one can verify that

$$\left\langle H_n(\omega), \phi^{(n)} \right\rangle \diamond \left\langle H_m(\omega), \psi^{(m)} \right\rangle = \left\langle H_{n+m}(\omega), \phi^{(n)} \widehat{\otimes} \psi^{(m)} \right\rangle \tag{2.19}$$

for $\phi^{(n)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} m}$. The latter result, as well as (2.7) imply that

$$\widetilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \tag{2.20}$$

for $\phi \in (\mathcal{S}([0,T]))^d$. The Wick exponential $\exp^{\diamond}(X)$ of a $X \in (\mathcal{S})^*$ is defined as

$$\exp^{\diamond}(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}, \tag{2.21}$$

where $X^{\diamond n} = X \diamond ... \diamond X$, provided the sum on the right hand side converges in $(\mathcal{S})^*$.

3 Forward SDEs with reflections

This section passes in review conditions for the existence and uniqueness of (global strong) solutions of a forward stochastic differential equation with reflections (FSDER). For more information on FSDER's and their applications the reader may consult the excellent book of (Ma and Yong 1999).

A general form of a forward SDE with reflections is the following:

$$\begin{aligned} X(t) &= x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s) + \eta(t) \\ x &\in \mathcal{O} \end{aligned} \tag{3.1}$$

Here $\mathcal{O}$ is a closed convex domain in $\mathbb{R}^n$; b and σ are functions of $(t, x, \omega) \in [0,T] \times \mathbb{R}^n \times \Omega$; and $\eta \in BV_{\mathcal{F}}([0,T], \mathbb{R}^m)$, the set of all $\mathbb{R}^m$-valued $\{\mathcal{F}_t\}_{t \geq 0}$-adapted processes η with paths of bounded variation. In the sequel we shall also denote by $L^2_{\mathcal{F}}([0,T], \mathbb{R}^m)$ the space of all $\{\mathcal{F}_t\}_{t \geq 0}$-progressively measurable processes $X(t)$ in $\mathbb{R}^n$, such that $\int_0^T \mathbb{E}[\|X(t)\|^2 \, dt] < \infty$.

Definition 3.1. (Ma and Yong 1999) A pair of continuous, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $(X, \eta) \in \mathcal{L}^2_{\mathcal{F}}([0,T], \mathbb{R}^m) \times BV_{\mathcal{F}}([0,T], \mathbb{R}^m)$ is called a solution to the FSDER (3.1) if

1) $X(t) \in \mathcal{O}$, $\forall t \in [0,T]$, a.s.;
2) $\eta(t) = \int_0^t 1_{\{X(s)\in\partial\mathcal{O}\}}\gamma(s)d|\eta|(s)$, where $\gamma(s) \in \mathcal{N}_{X(s)}$, $0 \geq s \geq t \geq T$, $d|\eta|$-a.e. and $|\eta|(T)$ denotes the total variation of η on $[0,T]$;
3) equation (3.1) is satisfied a.s.

Here $\mathcal{N}_x$ is the set of inward normals to $\mathcal{O}$ at x defined as follows:

$$\mathcal{N}_x = \{\gamma : |\gamma| = 1, \langle \gamma, x - y\rangle \leq 0, \forall y \in \mathcal{O}\}, \tag{3.2}$$

for $x \in \partial\mathcal{O}$.

The issue of existence and uniqueness of a solution of an FSDE with reflections can be related to the so-called Skorohod problem. The latter is defined as follows:

Let the domain $\mathcal{O}$ and a function $\psi \in C([0,T], \mathbb{R}^n)$ with $\psi(0) \in \mathcal{O}$ be given. Find a pair $(\phi, \eta) \in C([0,T], \mathbb{R}^n) \times BV([0,T], \mathbb{R}^n)$, such that

1) $\phi(t) = \psi(t) + \eta(t)$, $\forall t \in [0,T]$, and $\phi(0) = \psi(0)$;
2) $\phi(t) \in \mathcal{O}$, $\forall t \in [0,T]$;
3) $|\eta(t)| = \int_0^t 1_{\{\phi(s)\in\partial\mathcal{O}\}}d|\eta|(s)$;
4) there exists a measurable function $\gamma : [0,T] \to \mathbb{R}^n$, such that $\gamma(t) \in \mathcal{N}_{\phi(t)}$ $d|\eta|$ a.s. and $\eta(t) = \int_0^t \gamma(s)d|\eta|(s)$.

A pair (ϕ, η) satisfying the above conditions is called a solution of the Skorohod problem. We define a mapping $\Gamma : C([0,T], \mathbb{R}^n) \to C([0,T], \mathbb{R}^n)$, such that $\Gamma(\psi)(t) = \phi(t)$, $t \in [0,T]$, where (ϕ, η) is the unique solution of the corresponding Skorohod problem. We call Γ the solution mapping of the Skorohod problem, and we call a convex domain $\mathcal{O} \in \mathbb{R}^n$ *regular* if the solution mapping of the corresponding Skrohod problem satisfies the Lipschitz condition:

$$|\Gamma(\psi_1)(\cdot) - \Gamma(\psi_2)(\cdot)|^*_T \leq K|\psi_1(\cdot) - \psi_2(\cdot)|^*_T, \tag{3.3}$$

where $|\epsilon|^*_T$ denotes the sup-norm on $[0,T]$ for a function $\epsilon \in C([0,T], \mathbb{R}^n)$, and $K > 0$.

As shown in (Ma and Yong 1999), for a regular convex domain $\mathcal{O}$ the FSDER (3.1) has a unique strong solution provided that the following conditions are satisfied:

1) for every fixed $x \in \mathbb{R}^n$, $b(\cdot, x, \cdot)$ and $\sigma(\cdot, x, \cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-progressively measurable;
2) there exists a constant $K > 0$, such that for all $(t, \omega) \in [0,T] \times \Omega$ and $x, y \in \mathbb{R}^n$ it holds that

$$\begin{aligned} |b(t,x,\omega) - b(t,y,\omega)| &\leq K|x-y| \\ |\sigma(t,x,\omega) - \sigma(t,y,\omega)| &\leq K|x-y| \end{aligned} \tag{3.4}$$

4 White noise representation for FSDEs with reflections

In this section we derive a representation formula for solutions of FSDEs with reflections. Our results stem from the ideas presented in (Lanconelli and Proske 2004) and (Meyer-Brandis and Proske 2006)

Theorem 4.1. *Suppose that $\mathcal{O} \in \mathbb{R}^n$ is a regular, convex domain and assume that the drift term b and the diffusion matrix σ in equation* (3.1) *are deterministic. Further, require that σ be independent of the space variable, continuous in time, invertible t a.e., and that $\sigma^{-1}(t)$ be continuously differentiable on* $(0, T)$ *(with continuous extensions to* $[0, T]$*). For a fixed initial value $x \in \mathbb{R}^n$ in* (3.1) *define the Borel measurable functions $\tilde{b} : [0, T] \times C([0, T], \mathbb{R}^n) \to \mathbb{R}^n$, and $\Psi : C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)$ as follows:*

$$\tilde{b}(t, \phi) = \Big(\frac{\partial}{\partial t}\sigma^{-1}(t)\Big)\sigma(t)\Gamma(\phi)(t) + \sigma^{-1}(t)b\Big(t, \sigma(t)\big(\Gamma(\phi)(t) + \sigma^{-1}(0)x\big)\Big) \tag{4.1}$$

and

$$\Psi(\phi)(t) = \Gamma(\sigma(\cdot)(\phi + \sigma^{-1}(0)x))(t), \tag{4.2}$$

where the mapping Γ is the solution mapping of the corresponding Skorohod problem. In addition suppose that the following integrability conditions are satisfied.

$$E\Big[\int_0^T \|\Psi(B.)(t)\|^2 dt\Big] < \infty, \tag{4.3}$$

for $\forall t \in [0, T]$, and

$$E\Big[exp\Big(36\int_0^T \|\tilde{b}(s, B.)\|^2 ds\Big)\Big] < \infty \tag{4.4}$$

Denote by $\Psi^i(\phi)(t)$ and $\tilde{b^i}(\phi)$ the i-th component of $\Psi(\phi)(t)$ and $\tilde{b}(\phi)$ for all $\phi \in C([0, T], \mathbb{R}^n)$ and t respectively. Then the unique solution $X_t = (X_t^i)_{i=1,\dots,d}$ of the FSDER (3.1) *is explicitly represented by*

$$X_t^i = E_{\hat{\mu}}\Big[\Psi^i(\hat{B}.(\hat{\omega}))(t)\mathcal{E}_T^{\diamond}(\tilde{b})\Big],\ i = 1, ..., d \tag{4.5}$$

where the random element $\mathcal{E}_T^{\diamond}(\tilde{b}) : \tilde{\Omega} \to (S)^*$ is defined as

$$\mathcal{E}_T^{\diamond}(\tilde{b})(\omega, \hat{\omega}) = exp^{\diamond}\left(\sum_{j=1}^{d}\int_0^T [\tilde{b^j}(s, \hat{B}.(\hat{\omega})) + W_s^j]d\hat{B}_s^j(\hat{\omega}) - \frac{1}{2}\int_0^T [\tilde{b^j}(s, \hat{B}.(\hat{\omega})) + W_s^j(\omega)]^{\diamond 2} ds\right) \tag{4.6}$$

The 4-tuple $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mu}\right),\left(\widehat{B}_t\right)_{t\geq 0}$ *is a copy of* $(\Omega,\mathcal{F},\mu)\,,(B_t)_{t\geq 0}$ *in* (2.10). *The* $E_{\widehat{\mu}}$ *stands for the Pettis integral of random elements* $\Phi:\widehat{\Omega}\longrightarrow(\mathcal{S})^*$ *with respect to the measure* $\widehat{\mu}$. *Here* W_t^i *in the Wick exponential of* (4.6) *also denotes the white noise in the Hida space* $(\mathcal{S})^*$. *The Wick product* $\diamond$ *in* (4.6) *refers to the measure* μ. *The* ds*-integral in* (4.6) *is defined in the Pettis sense and the other integrals* $\int_0^T\phi(t,\omega)d\widehat{B}_s^j(\widehat{\omega})$ *in* (4.6) *are stochastic integrals of predictable* $(\mathcal{S})^*$*-valued integrands* $\phi(t,\omega)$. *(see e.g. (Kallianpur and Xiong 1995))*

The proof of Theorem 4.1 relies on the following auxiliary result (see e.g. (Kuo 1996, Theorem 13.4).

Lemma 4.2. *Let* $(M,\mathcal{B},\rho)$ *be a measure space. Let a function* $\Phi:M\longrightarrow(\mathcal{S})^*$ *be such that* $S(\Phi(\cdot))(\phi)$ *is measurable for all* $\phi\in(\mathcal{S}_{\mathbb{C}}([0,T]))^d$. *Denote by* $(|\cdot|_p)_{p\geq 0}$ *the family of increasing compatible seminorms of* $(\mathcal{S}_{\mathbb{C}}([0,T]))^d$. *Further, require that there exist* K, a, $p\geq 0$ *such that*

$$\int_M |S(\Phi(u))(\phi)|\,\rho(du)\leq K\exp(a\,|\phi|_p^2)$$

for all $\phi\in(\mathcal{S}_{\mathbb{C}}([0,T]))^d$. *Then* Φ *is Pettis integrable and for any* $\mathcal{A}\in\mathcal{B}$ *and all* $\phi\in(\mathcal{S}_{\mathbb{C}}([0,T]))^d$ *we have that*

$$S\left(\int_{\mathcal{A}}\Phi(u)\rho(du)\right)(\phi)=\int_{\mathcal{A}}S(\Phi(u))(\phi)\rho(du)$$

Proof of Theorem 4.1. The proof is based on ideas developed in (Lanconelli and Proske 2004) and (Meyer-Brandis and Proske 2006). Without loss of generality let us confine ourselves to the case when $d=1$. To simplify notation we set $\Psi:=\Psi^1$. Using the properties of the mapping Γ and the regularity of the domain $\mathcal{O}$ one finds that $b^*(t,\phi)=b(t,\Gamma(\phi)(t))$, $\phi\in\mathbb{C}([0,T])$ is a progressively measurable Lipschitz continuous functional (see e.g. (Ma and Yong 1999)). Thus, there exists a unique strong solution $\tilde{X}_t$ to the SDE

$$d\tilde{X}_t=x+\int_0^t b^*(s,\tilde{X}_\cdot)ds+\int_0^t\sigma(s)dB_s \tag{4.7}$$

On the other hand, invoking the definition of the Skorohod problem it follows that $X_t=\Gamma(\tilde{X})(t)$, $t\in[0,T]$ solves the FSDER (4.7) uniquely. Set $Y_t=\sigma^{-1}(t)\tilde{X}_t$. Then Ito's Lemma implies that

$$Y_t=\sigma^{-1}(0)x+\int_0^t\tilde{b}(s,Y_\cdot)ds+B_t,\ \ t\in[0,T] \tag{4.8}$$

So it follows from our assumptions that $Y_t\in L^2(\mu)$ for all t.

Suppose that $\pi:[0,T]\times C([0,T])\to\mathbb{R}$ is a bounded Borel measurable function. We aim at deriving a representation of the S-transform of $\pi(t,Y_\cdot)$. Using the definition of the S-transform we see that

$$S(\pi(t,Y_\cdot))=E_\mu\left[\pi(t,Y_\cdot(\omega+\phi))\right] \tag{4.9}$$

for all $\phi \in \mathcal{S}([0,T])$. Girsanov's theorem shows that $Y_t^*(\omega) = Y_t(\omega+\phi)$ satisfies the following SDE

$$dY_t^* = \tilde{b}(t, Y_\cdot^*) + \phi(t)dt + dB_t,\ Y_0^* = \sigma^{-1}(0)x, 0 \le t \le T.$$

Applying Girsanov's theorem to (4.9) once again yields that

$$S(Y_t)(\phi) = E_{\widehat{\mu}}\left[\pi\left(t, \widehat{B}_\cdot\right)\mathcal{E}(M_t^\phi)\right] \tag{4.10}$$

for all $\phi \in \mathcal{S}([0,T])$, where the 4-tuple $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu}\right), \left(\widehat{B}_t\right)_{t\ge 0}$ is a copy of $(\Omega, \mathcal{F}, \mu), (B_t)_{t\ge 0}$ and where $\mathcal{E}(M_t^\phi)$ is the usual notation for the Doleans-Dade exponential for the martingale

$$M_s^\phi(\widehat{\omega}) = \int_0^s \left(\tilde{b}(t, \widehat{B}_\cdot(\widehat{\omega})) + \phi(t)\right) d\widehat{B}_t(\widehat{\omega}).$$

So

$$\begin{aligned} & \mathcal{E}(M_t^\phi) \\ = \ & \exp\left(\int_0^T \left(\tilde{b}(t, \widehat{B}_\cdot) + \phi(t)\right) d\widehat{B}_t - \frac{1}{2}\int_0^T \left(\tilde{b}(t, \widehat{B}_\cdot) + \phi(t)\right)^2 dt\right) \end{aligned}$$

One can show that relation (4.10) also holds for $\phi \in \mathcal{S}_{\mathbb{C}}([0,T])$.

On the other hand, we know by (2.17) suggests that

$$S(W_t)(\phi) = \phi(t)$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. Further we find that

$$\mathcal{E}(M_t^\phi) = \mathcal{E}(M_t^0)\exp\left(\int_0^T \phi(t)d\widehat{B}_t - \frac{1}{2}\int_0^T (\phi(t))^2\, dt\right)\exp\left(\int_0^T \tilde{b}(t, \widehat{B}_\cdot)\phi(t)dt\right) \tag{4.11}$$

for all $\phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$. The second factor of the right hand side of (4.11) is the S-transform of the Kubo-Yokoi delta function (see (Kuo 1996, Theorem 13.4)) and it can be written as follows:

$$\mathcal{E}_T^\diamond(0) = \exp^\diamond\left(\int_0^T W_s(\omega)d\widehat{B}_s^j(\widehat{\omega}) - \frac{1}{2}\int_0^T (W_s(\omega))^{\diamond 2}\, ds\right).$$

The last factor in (4.11) is $G-$entire and its $S-$transform is bounded from above by

$$K^*\exp(a^*\,|\phi|_0^2),$$

with some constants $K^*, a^*$and $|\phi|_0 = \|\phi\|_{L^2_{\mathbb{C}}([0,T])}$. Using the characterization theorem of Hida distributions (see (Potthoff and Streit 1991)) and employing the properties of the S-transform we find that

$$S(\Phi(\widehat{\omega}, \cdot))(\phi) = \pi\left(t, \widehat{B}_\cdot(\widehat{\omega})\right)\mathcal{E}(M_t^\phi)(\widehat{\omega}),$$

where the map $\Phi : \Omega \times \widehat{\Omega} \longrightarrow (\mathcal{S})^*$ is given by

$$\Phi(\widehat{\omega}, \omega) = \pi\left(t, \widehat{B}_\cdot(\widehat{\omega})\right) \mathcal{E}_T^{\diamond}(\tilde{b})(\omega, \widehat{\omega})$$

with $\mathcal{E}_T^{\diamond}(\tilde{b})$ as in (4.6). Note that $S(\Phi(\widehat{\omega}, \cdot))(\phi)$ is $\widehat{\omega}$-measurable for all ϕ.

Now the Hölder inequality and the supermartingale property of Doleans-Dade exponentials imply

$$\begin{aligned}
& E_{\widehat{\mu}}\left[|S(\Phi(\widehat{\omega}, \cdot))(\phi)|\right] \\
= \;& E_{\widehat{\mu}}\left[\left|\pi\left(t, \widehat{B}_\cdot\right) \mathcal{E}(M_t^{\phi})\right|\right] \\
\leq \;& K \cdot E_{\widehat{\mu}}^{\frac{1}{2}}\left[\mathcal{E}\left(\int_0^T 2\left(\tilde{b}(t, \widehat{B}_\cdot) + \operatorname{Re}\phi(t)\right) d\widehat{B}_t\right)\right] \exp(a \int_0^T |\phi(t)|^2\, dt) \\
\leq \;& K \exp(a\, |\phi|_0^2),
\end{aligned}$$

with some constants a, $K \geq 0$. Then Lemma 4.2 above shows that

$$S(\pi(t, Y_\cdot))(\phi) = S(E_{\widehat{\mu}}[\Phi])(\phi)$$

The injectivity of the $S-$transform yields

$$\pi(t, Y_\cdot) = E_{\widehat{\mu}}[\Phi]\,. \tag{4.12}$$

Finally, by Itô's Lemma we get that $X_t = \Psi(Y_\cdot)(t)$. So choosing $\pi(t, \phi) = \Psi(\phi)(t)$ in (4.12) yields the result. ∎

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