AN IMPROVED ERROR ESTIMATE FOR THE FINITE DIFFERENCE APPROXIMATION TO DEGENERATE CONVECTION-DIFFUSION EQUATIONS

by

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Introduction

This thesis is based on the article [15] by K. H. Karlsen, U. Koley and N. H. Risebro. The subject of the article is to prove an L^1_{loc} error estimate for semidiscrete first-order finite difference schemes for nonlinear strongly degenerate convection-diffusion equations in one space dimension. More precisely they show that the L^1_{loc} difference between the approximate solution and the unique entropy solution converges at a rate $\mathcal{O}(\Delta x^{1/11})$ where Δx is the spatial mesh size. In addition they prove that it converges at a rate $\mathcal{O}(\sqrt{\Delta x})$ if the diffusion therm is linear. In this thesis I present a convergence rate of the order $\mathcal{O}(\Delta x^{1/7})$ in the nonlinear case, and prove a similar result for the fully discrete implicit scheme. The results are generalized in the sense that they apply to more general schemes. This is done by the introduction of the numerical entropy flux inspired by the article [17]. I also include the existence, uniqueness and some basic estimates on the semidiscrete scheme that is needed in the proof. Most of the ideas and techniques are taken from [15] and the results in this thesis should be seen as a continuation of the work therein.

Finite difference schemes for hyperbolic conservation laws is a well developed subject and there exists an extensive literature. See for instance [22] or [14]. For an approach using the kinetic formulation see [25]. These methods have long been known to converge at a rate $\mathcal{O}(\Delta x^{1/2})$ and this is indeed optimal for discontinuous solutions ([19, 14, 25]). In the case of strictly parabolic equations the solutions are smooth and more standard techniques have long been available. Concerning numerical methods for strongly degenerate problems, there has been a growing interest the last decade and there exists several articles considering different approaches. A list of articles is provided in [15]. As stated in [15], many of these articles show that the approximate solutions converge to the right entropy solution, but as far as we know, none of them provides a rate of convergence.

The improved convergence rate in this thesis relies on the use of the piecewise constant approximation (3.1) instead of the piecewise linear approximation applied in [15]. The use of a discontinuous approximation led to the use of difference quotients instead of derivatives. The advantage being that the difference quotients appear naturally in the numerical schemes. From a more technical point of view one should pay special attention to the equality (3.37) as the improved convergence rate relies on this observation.

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Contents

1 Degenerate convection-diffusion equations

Consider the nonlinear, possibly strongly degenerate convection-diffusion problem

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 A(u), & (x,t) \in \Pi_T, \\ u(x,0) = u^0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\Pi_T = \mathbb{R} \times (0,T)$ for some fixed $T > 0$, $u(x,t)$ is the scalar unknown function that is sought and u^0, f, A are given. The function f is called the flux function and A is called the diffusion function. We make the following assumptions:

- f is differentiable with $f(0) = 0$ and $f \in Lip_{loc}(\mathbb{R})$.
- A is differentiable with $A(0) = 0,\ A' \geq 0$ and $A \in Lip_{loc}(\mathbb{R})$.

The term *strongly degenerate* means that $A' = 0$ on some open interval. The class of equations under consideration therefore contains the heat equation, the porous medium equation, the two phase flow equation and conservation laws. We will be using the term *nondegenerate*. This means that (1.1) is uniformly parabolic, which is another way of saying that $A' \geq \eta$ for some $\eta > 0$. In the nondegenerate case of problem (1.1) it is well known that it admits a unique classical solution (See for instance [21]). However strongly degenerate equations must be considered in the weak sense and will in general possess discontinuous, "shock wave", solutions. Considering weak solutions it turns out that we need another condition to ensure that (1.1) is well-posed. The aim is to single out the physically most relevant solution. The conditions obtained are named *entropy conditions*, inspired by thermodynamical considerations (See for instance [22]). Under the assumption of strong degeneracy the notion of entropy solution goes back to Vol'pert and Hudjaev ([26]). These authors also proved general existence and stability results in the BV class. However the uniqueness result therein is incomplete (See [27]). The existence proof in [26] is obtained by certain viscous approximate solutions. These are of fundamental importance in the present thesis and provide a way to introduce the entropy conditions mentioned above.

1.1 Viscous approximations

Let $A^\eta(u) = A(u) + \eta u$ for some $\eta > 0$, u^0 be in $BV(\mathbb{R})$ and consider the nondegenerate problem

$$\begin{cases} u_t^\eta + f(u^\eta)_x = A^\eta(u^\eta)_{xx}, & (x,t) \in \Pi_T, \\ u^\eta(x,0) = u^0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

It is known that (1.2) admits a unique solution and that the solution operator has a strong smoothing effect. The problem (1.2) is an approximation of problem (1.1) and, inspired by fluid dynamics, u^η is referred to as a viscous approximation. Let us collect some properties in a lemma.

Lemma 1.1. *Let u^η be the solution of problem* (1.2). *Then* $\|u^\eta(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u^0\|_{L^1(\mathbb{R})}$, $\|u^\eta(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u^0\|_{L^\infty(\mathbb{R})}$ *and* $|u^\eta(\cdot,t)|_{BV(\mathbb{R})} \leq |u^0|_{BV(\mathbb{R})}$.

For a proof see [21]. Note that these results combined with the Lipschitz conditions on A and f imply that we have the same type of estimates on both $A(u^\eta)$ and $f(u^\eta)$.

Lemma 1.2. *Let u^η be a solution to* (1.2)*. Then*

$$\|f(u^\eta(\cdot,t)) - A(u^\eta(\cdot,t))_x\|_{L^\infty(\mathbb{R})} \leq \|f(u^0) - A(u^0)_x\|_{L^\infty(\mathbb{R})}, \tag{1.3}$$
$$|f(u^\eta(\cdot,t)) - A(u^\eta(\cdot,t))_x|_{BV} \leq |f(u^0) - A(u^0)_x|_{BV}. \tag{1.4}$$

Proof. This is outlined in [12]. Consider the equation

$$u_t^\eta + (f(u^\eta) - A(u^\eta)_x)_x = 0.$$

Integrating in space we obtain

$$\int_{-\infty}^{x} u_t^\eta(\xi,t)d\xi + f(u^\eta(x,t)) - (A(u^\eta(x,t)))_x = 0. \tag{1.5}$$

Now, let $v(x,t) = \int_{-\infty}^{x} u_t^\eta(\xi,t)d\xi$ and differentiate the equation with respect to t. Then v satisfies the following non degenerate linear parabolic equation

$$v_t(x,t) + f'(u^\eta(x,t))v_x - (A'(u^\eta(x,t))v_x)_x = 0.$$

It is then known that v satisfies the following estimates:

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|v^0\|_{L^\infty(\mathbb{R})} \quad \text{and} \quad |v(\cdot,t)|_{BV} \leq |v^0|_{BV}.$$

Hence solving (1.5) for v we obtain (1.3) and (1.4). □

Note that $\|A(u^\eta)_x\|_{L^\infty(\Pi_T)}$ and $\|A(u^\eta)_{xx}\|_{L^1(\Pi_T)}$ are bounded independently of η provided that $A(u^0)_x$ is in $BV(\mathbb{R})$. The next lemma is an easy but important consequence of the lemma above.

Lemma 1.3. *The solution u^η is L^1 Lipschitz continuous in the time variable. That is*

$$\|u^\eta(\cdot,t+h) - u^\eta(\cdot,t)\|_{L^1} \leq |f(u^0) - A(u^0)_x|_{BV} h$$

Proof. This follows from lemma 1.2.

$$\begin{aligned}\|u^\eta(\cdot,t+h) - u^\eta(\cdot,t)\|_{L^1} &\leq \int\int_t^{t+h} |u_t^\eta(x,s)|dsdx \\ &= \int_t^{t+h}\int |(f(u^\eta(x,s)) - A(u^\eta(x,s))_x)_x|dxds \leq |f(u^0) - A(u^0)_x|_{BV} h.\end{aligned}$$

□

The results above imply that the family $\{u^\eta\}_{\eta>0}$ is relatively compact in $C([0,T];L^1_{loc}(\mathbb{R}))$. To see this we apply Kolmogorov's compactness theorem (See [14]) to prove that $\{u^\eta(\cdot,t)\}_{\eta>0}$ is relatively compact in $L^1_{loc}(\mathbb{R})$ for each $t \in [0,T]$. Lemma 1.3 implies that the family $\{u^\eta\}_{\eta>0}$ is equicontinuous as functions from $[0,T]$ into $L^1(\mathbb{R})$. The relative compactness now follows by the Arzela-Ascoli theorem. The convergence rate in this thesis is going to rely on a much stronger result obtained by S. Evje and K.H. Karlsen in [10]. Let u be the limit obtained by considering some subsequence of $\{u^{\alpha(n)}\}_{n\in\mathbb{N}}$ where $\alpha : \mathbb{N} \to \mathbb{R}^+$ is some map such that $\alpha(n) \to 0$ as $n \to \infty$. That is, let u be the entropy solution defined below. Then there exists a constant C independent of η such that

$$\|u - u^\eta\|_{L^1(\Pi_T)} \leq C\sqrt{\eta}. \tag{1.6}$$

1.2 The entropy solution and its uniqueness

The intention of this section is to motivate the definition of entropy solution and introduce some common notions. The discussion leading to the introduction of an entropy solution is a mere generalization of the theory developed for conservation laws. In spite of this similarity, the question of its uniqueness was left unanswered until quite recently. Furthermore we include a brief overview of the historical development concerning this question.

Let u be the limit obtained by the viscous approximations. Under the assumption that u is somehow the most "reasonable" solution to (1.1) the obvious questions are; which properties does the limit u satisfies, and which properties should we demand of the solution in order to obtain uniqueness?

Let ψ be a convex C^2 function and introduce the corresponding functions q and ρ with the properties that

$$q'(u) = \psi'(u)f'(u) \quad \text{and} \quad \rho'(u) = \psi'(u)a(u)$$

where $a = A'$. We call ψ an *entropy function* and q, ρ *entropy fluxes.* The triple (ψ, q, ρ) is an *entropy, entropy flux triple.* Multiply the equation (1.2) by $\psi'(u^\eta)$ to obtain

$$\psi(u^\eta)_t + \psi'(u^\eta)f'(u^\eta)u^\eta_x = \psi'(u^\eta)A(u^\eta)_{xx} + \eta\psi'(u^\eta)u^\eta_{xx}.$$

Observe that

$$\begin{aligned} \rho(u^\eta)_{xx} &= \psi''(u^\eta)a(u^\eta)(u^\eta_x)^2 + \psi'(u^\eta)A(u^\eta)_{xx}, \\ \eta\psi(u^\eta)_{xx} &= \eta\psi''(u^\eta)(u^\eta_x)^2 + \eta\psi'(u^\eta)u^\eta_{xx} \end{aligned}$$

and so

$$\psi(u^\eta)_t + q(u^\eta)_x - (\rho(u^\eta) + \eta\psi(u^\eta))_{xx} = -\psi''(u^\eta)(a(u^\eta) + \eta)(u^\eta_x)^2.$$

Multiply this equation by a non-negative function $\varphi \in C_0^\infty([0,T) \times \mathbb{R})$. Then use integration by parts on the left hand side and note that the right hand side is negative. We obtain the inequality

$$\iint_{\Pi_T} \psi(u^\eta)\varphi_t + q(u^\eta)\varphi_x + (\rho(u^\eta) + \eta\psi(u^\eta))\varphi_{xx} dxdt + \int_{\mathbb{R}} \psi(u^0(x))\varphi(x,0)dx \geq 0.$$

It follows that the limit u satisfies the so called *entropy inequality*

$$\iint_{\Pi_T} \psi(u)\varphi_t + q(u)\varphi_x + \rho(u)\varphi_{xx} dxdt + \int_{\mathbb{R}} \psi(u^0(x))\varphi(x,0)dx \geq 0. \tag{1.7}$$

Demanding that this inequality is satisfied for all entropy, entropy flux triples turns out to be the right criteria. In fact this is also a sufficient condition for u to be a weak solution to (1.1). Let $\psi(u,c) = |u - c|$. By considering $\psi_\delta(u,c) = \sqrt{(u-c)^2 + \delta^2}$ one may show that (1.7) applies to the entropy, entropy flux triple $(\psi(\cdot,c), q(\cdot,c), \rho(\cdot,c))$ where

$$q(u,c) = \text{sign}(u-c)(f(u) - f(c)) \quad \text{and} \quad \rho(u,c) = |A(u) - A(c)|.$$

$\psi(u,c)$ is known as the *Kružkov entropy* function and it is sufficient to consider this class of entropies in (1.7). Note that using the Kružkov entropies it is seen that if u satisfies (1.7) then it is a weak solution to (1.1).

Definition 1.1. An *entropy solution* of (1.1) is a measurable function $u = u(x,t)$ satisfying:

(D.1) $u \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C((0,T); L^1(\mathbb{R}))$.

(D.2) For all non-negative test functions φ in $C_0^\infty(\Pi_T)$ and all entropy,entropy flux triples (ψ, q, ρ) u satisfies the entropy inequality

$$\iint_{\Pi_T} \psi(u)\varphi_t + q(u)\varphi_x + \rho(u)\varphi_{xx} dxdt \geq 0.$$

(D.3) The initial condition is satisfied in L^1 sense. That is

$$\lim_{t\to 0^+} \int_{\mathbb{R}} |u(t,x) - u^0(x)| dx = 0.$$

(D.4) $A(u)$ is continuous and $A(u)_x \in L^\infty(\mathbb{R})$.

The uniqueness of such a solution follows from [16]. Note that condition (D.4) applies to the limit u of the viscous approximations by lemma 1.2, and hence the solution is continuously differentiable in the regions where $A' > 0$.

The question of whether (1.1) is well posed or not is of fundamental importance. The discussion above describes how problem (1.1) can be reformulated in order to yield a well posed problem. For this to be successful we need the entropy solution to be unique. When discussing the uniqueness of entropy solutions for degenerate parabolic equations one should keep in mind the diversity of this class. The problem is not restricted to $x \in \mathbb{R}$, the functions f and A may depend on both space and time in different ways and there are different assumptions on the initial condition. Because of this there exist numerous articles with different assumptions. In the case of conservation laws, uniqueness in $L^1 \cap L^\infty$ is a result due to Kružkov presented in 1970 ([18]). This proof was not straight on adaptable to the more general case. The erroneous uniqueness proof in the space BV that Vol'pert and Hudjaev proposed in [26] from 1969 is based on the idea of developing a discontinuity condition. This type of discontinuity conditions goes back to Oleĭnik and is interesting not only as an entropy condition, but also because it gives a more geometric description of the solution ([22, 24]). The discontinuity condition was corrected by Wu and Yin in 1989 and uniqueness in the space BV was established ([27]). In the purely parabolic case (no convection term) the question of uniqueness in $L^1 \cap L^\infty$ was proved by Brezis and Crandall in 1979 ([2]). This combined with Kružkovs theory for conservation laws indicated that a $L^1 \cap L^\infty$ theory should be possible also for degenerate equations. In his uniqueness proof Kružkov applied the, by now, famous technique of doubling of variables. This technique was further developed by Carrillo who proved uniqueness in $L^1 \cap L^\infty$ in 1999 ([4]). This result was later generalized by Risebro and Karlsen in 2003 ([16]). Using the so called kinetic formulation Chen and Perthame proved well posedness in L^1 in 2001 ([6]).

1.3 An application to traffic flow

Degenerate convection diffusion equations are used to model phenomenons such as flow in porous media([9]), sedimentation-consolidation processes ([1]) and traffic flow ([3, 23]). Let us take a brief look at the last case. The intention is to sketch how problem (1.1) may appear in applications, and not to give an introduction to this field. Consider a one-lane highway. Let $\rho = \rho(x, t)$ denote the car density at a point x at time t. This should be understood the following way:

$$\text{“the number of cars between } x_1 \text{ and } x_2 \text{ at time } t\text{”} \approx \int_{x_1}^{x_2} \rho(x, t)dx.$$

Let $v = v(x, t)$ denote the speed of the car located at point x at time t and $f(x, t)$ the flux of cars. That is the approximate number of cars per time passing a point x at time t. Conservation of cars gives the following equation:

$$\frac{\partial}{\partial t} \int_{x_1}^{x_2} \rho(x, t)dx = f(x_1, t) - f(x_2, t). \tag{1.8}$$

The flux of cars is given by the product $v\rho$, so if we assume that the speed is a function of the car density only, we obtain from (1.8) the scalar conservation law

$$\partial_t \rho + \partial_x(\rho v(\rho)) = 0. \tag{1.9}$$

The model as it stands above assumes that the driver adjusts his speed instantaneously to the local car density. This is not a natural assumption. We follow the discussion given in [3]. The assumption is thus that there is a difference between the cars velocity v and the velocity the driver considers as reasonable $\hat{v}$. This is caused by both the drivers reaction time and his ability to observe. Let τ represent the drivers reaction time. Furthermore, let us assume that the driver tries to adjusts his velocity to the car density seen a distance L ahead, but are delayed by his reaction time. Hence we make the assumption that

$$v(x, t) = \hat{v}(\rho(x + L - \hat{v}\tau, t - \tau)).$$

Note that this expression is ambiguous since we have not specified where we should evaluate the $\hat{v}$ in the argument of ρ. Let us make some approximate calculations. For a proper derivation see either [3] or [23]. Let us expand ρ around the point (x, t). This gives

$$\rho(x + L - \hat{v}\tau, t - \tau) \approx \rho(x, t) + \partial_x\rho(x, t)(L - \hat{v}\tau) + \partial_t\rho(x, t)(-\tau).$$

By the expansion of $\hat{v}$ we obtain

$$v(x, t) \approx \hat{v}(\rho) + \hat{v}'(\rho)(\partial_x\rho(L - \hat{v}\tau) + \partial_t\rho(-\tau))$$

where $\rho = \rho(x, t)$. By the conservation law (1.9) we have $\partial_x\rho\hat{v} + \partial_t\rho \approx -\rho\partial_x\hat{v} = -\rho\hat{v}'(\rho)\partial_x\rho$. It follows that

$$v(x, t) \approx \hat{v}(\rho) + \hat{v}'(\rho)((L + \tau\rho\hat{v}'(\rho))\partial_x\rho).$$

We now make the assumption $L = L(\hat{v}(\rho)) = L(\rho)$. Let us define

$$\tilde{A}(\rho) = \int_0^{\rho} -s\hat{v}'(s)(L(s) + \tau s\hat{v}'(s))ds,$$

and note that

$$\hat{v}(\rho(x+L-\hat{v}\tau, t-\tau)) \approx \hat{v}(\rho) + \frac{1}{\rho}\partial_x \tilde{A}(\rho).$$

We may now use this expression for v in (1.9) to obtain

$$\partial_t \rho + \partial_x(\rho \hat{v}(\rho)) = \partial_x^2 \tilde{A}(\rho).$$

It turns out that it is natural to assume the existence of a critical density ρ_c up to which the effects discussed above are not present. Hence we exchange the function $\tilde{A}$ by the function

$$A(\rho) = \int_0^\rho a(s)ds, \quad \text{where} \quad a(s) = \begin{cases} 0 & \text{if } s \leq \rho_c, \\ -s\hat{v}'(s)(L(s) + \tau s \hat{v}'(s)) & \text{if } s > \rho_c. \end{cases}$$

Considering this expression one should note that it is natural that $\hat{v}$ should decrease with increasing car density. The question of whether $L(s) \geq -\tau \hat{v}'(s)s$ is closely related to the assumption that there are no collisions. It is therefore natural to pick $L, \hat{v}$ and τ in order to yield a problem described by (1.1).

2 Finite difference schemes

We are going to consider a particular class of numerical methods used to obtain approximate solutions to problem (1.2). However we are not interested in how these methods are obtained, but rather to ensure that our result applies to as many methods as possible. The schemes under consideration are obtained either by a finite difference method or by a finite volume method. Because these methods are so natural we will refer to them just as the discretization of the problem. We will consider two different classes of approximations. The first one is obtained by discretizing in space only and leaving the problem continuous in time. The idea is that it is simpler to work with the semidiscrete case than the fully discrete case and hence we may simplify things by dividing the discretization process into two stages. The second one is an implicit method obtained by discretizing both space and time.

One of the main concerns regarding the discretization is how to approximate the flux function f.

Definition 2.1. (Numerical flux) We call a function $F \in C^1(\mathbb{R}^2)$ a *numerical flux for f* given that $F(u,u) = f(u)$ for $u \in \mathbb{R}$. If

$$\frac{\partial}{\partial u}F(u,v) \geq 0 \quad \text{and} \quad \frac{\partial}{\partial v}F(u,v) \leq 0$$

holds for all $u, v \in \mathbb{R}$ we call it *monotone*.

Let F_u and F_v denote the partial derivatives of F with respect to the first and second variable respectively. We will also assume F to be Lipschitz continuous in each variable. That is, there exists a constant K such that for real numbers u, v and w

$$|F(u,w) - F(u,v)| \leq K|w - v| \text{ and } |F(u,v) - F(w,v)| \leq K|u - w|.$$

Let z also be a real number. Then

$$\begin{aligned} |F(u,w) - F(z,v)| &\leq |F(u,w) - F(z,w)| + |F(z,w) - F(z,v)| \\ &\leq K(|u - z| + |w - v|). \end{aligned} \tag{2.1}$$

2.1 The semidiscrete scheme

Let $\Delta x > 0$ and define $x_j = j\Delta x$. The discrete derivatives are defined by

$$D^{\pm}(\sigma_j) = \pm\frac{\sigma_{j\pm1} - \sigma_j}{\Delta x}.$$

for any sequence $\{\sigma_j\}$. Note that $D^{\pm}$ can be interpreted as operators. However it is in general just considered as a shorthand notation. We may now define the semidiscrete approximation of problem (1.1) as the solution to the scheme

$$\begin{cases} (u_j)_t + D^- F_j = D^- D^+ A(u_j), & j \in \mathbb{Z},\ t \in (0,T), \\ u_j(0) = \frac{1}{\Delta x}\int_{I_j} u^0(x)dx, & j \in \mathbb{Z}. \end{cases} \tag{2.2}$$

Here $F_j = F(u_j, u_{j+1})$ is a numerical flux function and $I_j = (x_{j-1/2}, x_{j+1/2}]$. Note that (2.2) is a system of ordinary differential equations. The method is therefore often refered to as the *method of lines*.

The problem above can be viewed as an abstract Cauchy problem in the Banach space $\ell^1(\mathbb{Z})$ (See for instance [20]). In order to get bounds independent of Δx we let

$$\|\sigma\|_1 = \Delta x \sum_j |\sigma_j| \quad \text{and} \quad |\sigma|_{BV} = \sum_j |\sigma_{j+1} - \sigma_j| = \|D^+\sigma\|_1.$$

If these are bounded we say that $\sigma = \{\sigma_j\}$ is in ℓ^1 and of bounded variation. Let $u(t) = \{u_j(t)\}, u^0 = \{u_j(0)\}$ and define the operator $\mathcal{A} : \ell^1 \to \ell^1$ by $(\mathcal{A}(u))_j := D^-(F(u_j, u_{j+1}) - D^+A(u_j))$. Then (2.2) takes the following form

$$\begin{cases} \frac{du}{dt} + \mathcal{A}(u) = 0, & t \in (0,T), \\ u(0) = u^0, \end{cases} \tag{2.3}$$

where the derivative is meant in the strong sense. That is

$$\frac{du}{dt}(t) = \lim_{h\to 0} \frac{u(t+h) - u(t)}{h}$$

where the limit is taken in the norm topology. The existence of a unique continuously differentiable solution to (2.3) may be established on $[0,T]$ the same way as in $\mathbb{R}^n$ provided that $\mathcal{A}$ is Lipschitz continuous. (See problem 1.3.1 [20]). To see that $\mathcal{A}$ is Lipschitz define the induced maps $\hat{A} : \ell^1 \to \ell^1$ and $\hat{F} : \ell^1 \to \ell^1$ by $(\hat{A}(u))_j = A(u_j)$ and $(\hat{F}(u))_j = F(u_j, u_{j+1})$. By the Lipschitz continuity of A and inequality (2.1) these maps are Lipschitz continuous. Considering $D^\pm$ as maps from ℓ^1 into ℓ^1 these are also Lipschitz continuous. Since

$$\mathcal{A}(u) = D^-(\hat{F}(u) - D^+\hat{A}(u)) \qquad u \in \ell^1$$

and the sum and composition of Lipschitz continuous maps are Lipschitz $\mathcal{A}$ is Lipschitz continuous. The solution of (2.3) then provides a solution to (2.2). For any $t \geq 0$ we let $\mathcal{S}(t) : \ell^1 \to \ell^1$ be the solution operator. That is, $\mathcal{S}(t)u^0 = u(t)$. Then $\mathcal{S}$ satisfies the following properties:

$$\mathcal{S}(t+\tau) = \mathcal{S}(t)\mathcal{S}(\tau), \qquad t,\tau \geq 0, \tag{2.4}$$

$$\lim_{t\to 0^+} \mathcal{S}(t)u = u, \qquad u \in \ell^1. \tag{2.5}$$

That is, the family $\{\mathcal{S}(t) : t \in \mathbb{R}^+\}$ is a *semigroup* on ℓ^1. In our case the map $t \mapsto \mathcal{S}(t)u$ is strongly continuous so we call it a strongly continuous semigroup and condition (2.5) may be replaced by $\mathcal{S}(0) = \mathcal{I}$. If $\mathcal{S}$ also satisfies

$$\|\mathcal{S}(t)u - \mathcal{S}(t)v\|_1 \leq \|u - v\|_1 \quad \text{for} \quad u, v \in \ell^1$$

we say that it is *nonexpansive.* The notions described above were given in the particular case of the Banach space ℓ^1 but are of course general. The next goal is to show that our semigroup is nonexpansive and thus obtain an ℓ^1 contraction property. This will follow from the theory of T.M. Liggett and M.G. Crandall presented in [7], but to describe the results provided there we need some notions regarding nonlinear operators on Banach spaces. The next definitions can be found in [13] and [11]. For a more elaborate introduction to this field see [8]. Suppose that X is a real Banach space and X^* its dual. A *duality mapping* J is a map $J : X \to X^*$ such that for all $x \in X$,

$$\|J(x)\|_{X^*} = \|x\|_X \quad \text{and} \quad \langle J(x), x\rangle = \|x\|_X^2.$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between X^* and X. A mapping $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ is called *accretive* if for all pairs $(u, \mathcal{A}(u))$ and $(v, \mathcal{A}(v))$ in the graph of $\mathcal{A}$, and for all duality mappings J we have

$$\langle J(u-v), \mathcal{A}(u) - \mathcal{A}(v) \rangle \geq 0.$$

If in addition $\mathcal{I} + \lambda\mathcal{A}$ is surjective for all $\lambda > 0$, then $\mathcal{A}$ is called *m-accretive*. To avoid the notion of multivalued operators we use the less general presentation of these theorems given in [8].

Theorem 2.1 (Crandall,Liggett). *Let X be a Banach space, $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ accretive and such that $R(\mathcal{I}+\lambda\mathcal{A}) \supseteq D$ for all small $\lambda > 0$. Then $\mathcal{A}$ generates a nonexpansive semigroup by means of "the exponential formula"*

$$\mathcal{U}(t)x = \lim_{n\to\infty} \left(\mathcal{I} + \frac{t}{n}\mathcal{A}\right)^{-n} x \qquad x \in D.$$

where the convergence is uniform on compact subintervals of $\mathbb{R}^+$, and $\mathcal{U}(\cdot)x$ is locally Lipschitz. If, moreover, $R(\mathcal{I} + \lambda\mathcal{A}) \supseteq \overline{D}$ for all small $\lambda > 0$, then the exponential formula holds on $\overline{D}$.

We are left with two questions to be answered: Does this theorem apply in our case, and do the two semigroups coincide? Observe that the domain of $\mathcal{A}$ is ℓ^1 and so we need to prove that $\mathcal{A}$ is m-accretive. The second theorem in [7] provides an answer to the last question.

Theorem 2.2 (Crandall,Liggett). *Let X and $\mathcal{A}$ be as in Theorem 2.1. Suppose also that $\mathcal{A}$ is closed, i.e. the graph($\mathcal{A}$) is closed, and let $\{\mathcal{U}(t) : t \geq 0\}$ be the semigroup from Theorem 2.1. Then u is a (strong) solution of $u' + \mathcal{A}u = 0$, $u(0) = x$ if and only if $u(t) = \mathcal{U}(t)x$ and $\mathcal{U}(t)x$ is differentiable a.e. with respect to t.*

The *graph*($\mathcal{A}$) is the subset $\{(u, \mathcal{A}(u)) | u \in \ell^1\} \subset \ell^1 \times \ell^1$ and is closed by the continuity of $\mathcal{A}$. It follows by the Lipschitz continuity of $\mathcal{A}$ that it is m-accretive if it is accretive. Let us show that $\mathcal{A}$ is accretive. In [11] this is done for a slightly more general F, but in our case we assume F to be monotone so we might as well do a straightforward calculation. Suppose that $X = L^1(\Omega)$ where $(\Omega, d\mu)$ is some measure space. Then every duality mapping can be written as an integral

$$\langle J(u), v \rangle = \int_\Omega j(u)(x)v(x)d\mu \quad \text{where} \quad j(u)(x) = \begin{cases} \text{sign}(u(x)) & \text{if } u(x) \neq 0, \\ a(x) & \text{if } u(x) = 0 \end{cases}$$

where $a(x)$ is any measurable function with $|a(x)| \leq 1$ almost everywhere w.r.t. μ. Let $u = \{u_j\}$ and $v = \{v_j\}$ be in ℓ^1. Then for any duality map J we have

$$\langle J(u-v), \mathcal{A}(u) - \mathcal{A}(v) \rangle = \Delta x \sum_j \text{sign}(u_j - v_j)(\mathcal{A}(u) - \mathcal{A}(v))_j$$

Let $w_j = u_j - v_j$. By $\text{int}(a, b)$ we mean the interval between a and b. By the mean value theorem we have for each $j \in \mathbb{Z}$

$$\begin{aligned} &F(u_j, u_{j+1}) - F(v_j, v_{j+1}) \\ &\quad = (F(u_j, u_{j+1}) - F(v_j, u_{j+1})) + (F(v_j, u_{j+1}) - F(v_j, v_{j+1})) \\ &\quad\quad = F_u(\alpha_j, u_{j+1})w_j + F_v(v_j, \beta_{j+1})w_{j+1} \end{aligned}$$

for sequences $\{\alpha_j\}$ and $\{\beta_j\}$ with $\alpha_j, \beta_j \in \mathrm{int}(u_j, v_j)$. Let $a = A'$, then

$$A(u_j) - A(v_j) = a(\xi_j) w_j \quad \text{for some } \xi_j \in \mathrm{int}(u_j, v_j).$$

Hence

$$\begin{aligned}(\mathcal{A}(u) - \mathcal{A}(v))_j &= D^-(F(u_j, u_{j+1}) - F(v_j, v_{j+1})) - D^- D^+(A(u_j) - A(v_j)) \\ &= D^-(F_u(\alpha_j, u_{j+1}) w_j + F_v(v_j, \beta_{j+1}) w_{j+1}) - D^- D^+ a(\xi_j) w_j\end{aligned}$$

Using this expression we get

$$\begin{aligned}&\Delta x \sum_j \mathrm{sign}(u_j - v_j)(\mathcal{A}(u) - \mathcal{A}(v))_j = \\ &\qquad \sum_j (F_u(\alpha_j, u_{j+1})|w_j| - F_u(\alpha_{j-1}, u_j) w_{j-1} \mathrm{sign}(w_j)) \\ &\qquad + \sum_j (F_v(v_j, \beta_{j+1}) w_{j+1} \mathrm{sign}(w_j) - F_v(v_{j-1}, \beta_j)|w_j|) \\ &+ \frac{1}{\Delta x} \sum_j (-a(\xi_{j-1}) w_{j-1} \mathrm{sign}(w_j) + 2a(\xi_j)|w_j| - a(\xi_{j+1}) w_{j+1} \mathrm{sign}(w_j)) \geq 0.\end{aligned}$$

Thus $\mathcal{A}$ is accretive if F is monotone. Let us collect the above results and some more in a lemma.

Lemma 2.1. *Suppose that F is monotone. Then there exists a unique solution $u = \{u_j\}$ to* (2.2) *on $[0, T]$ with the properties:*

(a) $\|u(t)\|_1 \leq \|u^0\|_1$.

(b) *For every $j \in \mathbb{Z}$ and $t \in (0, T)$*

$$\inf_k \{u_k^0\} \leq u_j(t) \leq \sup_k \{u_k^0\}.$$

(c) $|u(t)|_{BV} \leq |u^0|_{BV}$.

(d) *If $v = \{v_j\}$ is a solution the same problem with initial data v^0 then*

$$\|u(t) - v(t)\|_1 \leq \|u^0 - v^0\|_1.$$

Proof. The existence of a solution follows from above. The same applies to property (d). Note that $\mathcal{A}(0) = 0$ so $\|u\|_1 \leq \|u^0\|_1$ follows from (d) by letting $v^0 = 0$. The BV estimate follows by letting $v_j^0 = u_{j+1}^0$. To prove (b) note that there exists an index j_0 such that $\sup_j(u_j(t)) = u_{j_0}(t)$ if $\sup_j(u_j(t)) > 0$ since $u_j(t)$ is in ℓ^1. Then $D^+ u_{j_0}(t) \leq 0$ and $D^- u_{j_0}(t) \geq 0$. We skip the argument t in the computations.

$$\begin{aligned}(u_{j_0})' &= D^-(D^+ A(u_{j_0}) - F(u_{j_0}, u_{j_0+1})) \\ &= \frac{1}{(\Delta x)^2}(A(u_{j_0+1}) - 2A(u_{j_0}) + A(u_{j_0-1})) \\ &\qquad - \frac{1}{\Delta x}(F(u_{j_0}, u_{j_0+1}) - F(u_{j_0-1}, u_{j_0})) := T_1 - T_2.\end{aligned}$$

Note that $T_1 \le 0$ since A is increasing. Consider T_2.

$$T_2 = \frac{1}{\Delta x}\left((F(u_{j_0}, u_{j_0+1}) - F(u_{j_0}, u_{j_0})) + (F(u_{j_0}, u_{j_0}) - F(u_{j_0-1}, u_{j_0}))\right) \\ = F_v(u_{j_0}, \alpha)D^+u_{j_0} + F_u(\beta, u_{j_0})D^-u_{j_0} \ge 0.$$

for $u_{j_0+1} < \alpha < u_{j_0}$ and $u_{j_0-1} < \beta < u_{j_0}$. Similarily there exists an index j_1 such that $\inf_j\{u_j(t)\} = u_{j_1}$ if $\inf_j\{u_j(t)\} < 0$. That $(u_{j_1})' \ge 0$ follows by the computations above. □

Lemma 2.2. *If F is monotone, then*

$$\|F(u_j, u_{j+1}) - D^+A(u_j)\|_{l^\infty} \le \|F(u^0_j, u^0_{j+1}) - D^+A(u^0_j)\|_{l^\infty}, \tag{2.6}$$

$$|F(u_j, u_{j+1}) - D^+A(u_j)|_{BV} \le |F(u^0_j, u^0_{j+1}) - D^+A(u^0_j)|_{BV}. \tag{2.7}$$

Proof. We use the same strategy as the one applied in the continuous case. Let $v_j = \Delta x \sum_{k\le j}(u_k)_t$ for $t \in (0,T)$. Then v_j satisfies

$$v_j = \Delta x \sum_{k\le j} D^-(D^+A(u_k) - F(u_k, u_{k+1})) = D^+A(u_j) - F(u_j, u_{j+1}) \tag{2.8}$$

and we may define v_j for all $t \in [0,T]$. Note that $\{v_j(t)\}$ is in ℓ^1 for all t by lemma 2.1. Differentiating (2.8) with respect to t we obtain

$$(v_j)_t = \frac{1}{\Delta x}(a(u_{j+1})(u_{j+1})_t - a(u_j)(u_j)_t) \\ - F_u(u_j, u_{j+1})(u_j)_t - F_v(u_j, u_{j+1})(u_{j+1})_t.$$

Note that $D^-v_j = (u_j)_t$ and $D^+v_j = (u_{j+1})_t$. Then v_j satisfy

$$(v_j)_t = \left(\frac{1}{\Delta x}a(u_{j+1}) - F_v(u_j, u_{j+1})\right)D^+v_j \\ - \left(\frac{1}{\Delta x}a(u_j) + F_u(u_j, u_{j+1})\right)D^-v_j. \tag{2.9}$$

Assume that $v_{j_0}(t)$ is a local maximum. Then $D^+v_{j_0} \le 0$ and $D^-v_{j_0} \ge 0$ so $(v_{j_0})_t \le 0$ since F is monotone. If v_{j_0} is a local minimum then $(v_{j_0})_t \ge 0$. Then inequality (2.6) follows by the fact that $\{v_j(t)\} \in \ell^1$. Consider (2.7). We want to show that $(|v_j(t)|_{BV})_t \le 0$. Now,

$$\frac{\partial}{\partial t}\left(\sum_j |v_{j+1} - v_j|\right) = \sum_j \operatorname{sign}(v_{j+1} - v_j)(v_{j+1} - v_j)_t$$

so we may apply equation (2.9). Thus

$$
\begin{aligned}
(|v_j|_{BV})_t = &\sum_j \left(\frac{1}{\Delta x}a(u_{j+2}) - F_v(u_{j+1},u_{j+2})\right)(D^+v_{j+1})\text{sign}(v_{j+1}-v_j) \\
&- \sum_j \left(\frac{1}{\Delta x}a(u_{j+1}) + F_u(u_{j+1},u_{j+2})\right)|D^+v_j| \\
&- \sum_j \left(\frac{1}{\Delta x}a(u_{j+1}) - F_v(u_j,u_{j+1})\right)|D^+v_j| \\
&+ \sum_j \left(\frac{1}{\Delta x}a(u_j) + F_u(u_j,u_{j+1})\right)((D^-v_j)\text{sign}(v_{j+1}-v_j)) \\
&:= S_1 + S_2 + S_3 + S_4.
\end{aligned}
$$

Since $S_1 + S_3 \leq 0$ and $S_2 + S_4 \leq 0$ the result follows. □

By lemma (2.2) $\{u_j\}$ is ℓ^1 Lipschitz continuous by the same argument as in the continuous case.

2.2 The implicit scheme (Fully discrete case)

Let $\Delta t > 0$ and $t^n = n\Delta t$. Let $D_t^{\pm}$ denote the discrete derivative in time with parameter Δt and $D_x^{\pm}$ the discrete derivative in space. The implicit scheme is then defined by

$$
\begin{cases} D_t^- u_j^n + D_x^- F(u_j^n, u_{j+1}^n) = D_x^- D_x^+ A(u_j^n), & (j,n) \in \mathbb{Z}\times\mathbb{N}, \\ u_j^0 = \int_{I_j} u^0(x)dx, & j \in \mathbb{Z}. \end{cases} \tag{2.10}
$$

This scheme is studied by S. Evje and K. H. Karlsen. Under the assumption that F is monotone, all the properties corresponding to the ones obtained for the semidiscrete scheme in the previous section and some more are proved in [11]. Since the solution is discrete in the time variable the ℓ^1 Lipschitz continuity takes a slightly different form. We state it here as an easy reference.

Lemma 2.3. *Let $\{u_j^n\}$ be the solution to* (2.10). *Let m, n be non-negative integers. Then*

$$
\|u_j^m - u_j^n\|_1 \leq |F(u_j^0, u_{j+1}^0) - D_x^+ A(u_j^0)|_{BV}\Delta t|m-n|
$$

where $\|\cdot\|_1$ and $|\cdot|_{BV}$ are defined in section 2.1.

Observe thet there are no CFL conditions involved. Therefore the error estimate in section 4 is based on the assumption that Δt and Δx are independent. Suppose that we want to construct an algorithm in order to solve (2.10). We can do this inductively in time. Assume that we have found $u^k = \{u_j^k\}_{j\in\mathbb{Z}} \in \ell^1$ for $0 \leq k < n$ and want to find u^n. Using the notation from section 2.1 we define the mapping $\Phi_n : \ell^1 \to \ell^1$ by

$$
\Phi_n(z) = u^{n-1} - \Delta t \mathcal{A}(z)
$$

and observe that if Φ_n has a fixed point, then this is the sought sequence u^n. To see how the Lipschitz constant of Φ_n depends on Δt and Δx, take $u, v \in \ell^1$. By inequality (2.1)

$$\begin{aligned}
&\|\Phi_n(u) - \Phi_n(v)\|_1 = \Delta t \|\mathcal{A}(v) - \mathcal{A}(u)\|_1 \\
&\quad = \Delta t \Delta x \sum_j \left(|D_x^- (F(v_j, v_{j+1}) - F(u_j, u_{j+1})) - D_x^- D_x^+ (A(v_j) - A(u_j))| \right) \\
&\quad \leq \Delta t \Delta x \sum_j \left(\frac{K}{\Delta x} + \frac{\|A\|_{Lip}}{(\Delta x)^2} \right) (|v_{j+1} - u_{j+1}| + 2|v_j - u_j| + |v_{j-1} - u_{j-1}|) \\
&\quad = \frac{\Delta t}{(\Delta x)^2} 4(K\Delta x + \|A\|_{Lip}) \|u - v\|_1.
\end{aligned}$$

By Banach's contraction mapping theorem Φ_n has a unique fixed point provided that

$$\frac{\Delta t}{(\Delta x)^2} 4(K\Delta x + \|A\|_{Lip}) < 1.$$

Considering the result obtained in section 4 this condition on Δt is unfortunate. It is clear that another way of finding a solution to (2.10) or a better estimate on $\|\Phi_n\|_{Lip}$ would be of interest.

2.3 The numerical entropy flux

It turns out that we need some more conditions on F than just demanding it to be monotone. Lemma 2.4 provides us with a sufficient condition.

Definition 2.2. Given an entropy, entropy flux pair (ψ, q) and a numerical flux F. Suppose that $Q \in C^1(\mathbb{R}^2)$ satisfies

$$\begin{aligned}
Q(u, u) &= q(u), \\
\frac{\partial}{\partial v} Q(v, w) &= \psi'(v) \frac{\partial}{\partial v} F(v, w), \\
\frac{\partial}{\partial w} Q(v, w) &= \psi'(w) \frac{\partial}{\partial w} F(v, w),
\end{aligned}$$

then we call Q a *numerical entropy flux.*

A natural question would now be for what type of numerical fluxes, if any, does such a function exists.

Lemma 2.4. *The numerical flux F has a numerical entropy flux Q, independent of the chosen entropy, entropy flux pair, if there exist $F_1, F_2 \in C^1(\mathbb{R})$ such that*

$$F(u, v) = F_1(u) + F_2(v), \tag{2.11}$$

$$F_1'(u) + F_2'(u) = f'(u) \tag{2.12}$$

for all $u, v \in \mathbb{R}$.

Proof. Let (ψ, q) be an entropy, entropy flux pair. Then q has the form

$$q(u) = \int_c^u \psi'(z) f'(z) dz + C$$

for some constant C. Define Q by

$$Q(u,v) = \int_c^u \psi'(z)F_1'(z)dz + \int_c^v \psi'(z)F_2'(z)dz + C. \tag{2.13}$$

It is easily verified that Q is a numerical entropy flux for F. □

Note that if Q is supposed to have symmetric partial derivatives, then (2.11) and (2.12) are necessary conditions. Fortunatly there exist some numerical flux functions which satisfies lemma 2.4.

Example 2.1. (The Engquist-Osher flux) Let

$$f'_+(s) = \max(f'(s), 0) \qquad \text{and} \qquad f'_-(s) = \min(f'(s), 0).$$

Then, in the terminology of lemma 2.4, let $F(u,v) = F_1(u) + F_1(v)$ where

$$F_1(u) = f(0) + \int_0^u f'_+(s)ds \qquad \text{and} \qquad F_2(v) = \int_0^v f'_-(s)ds.$$

It is easily seen to satisfy the criteria given in lemma 2.4 and it is also clearly monotone.

Example 2.2. Let $a, b \in \mathbb{R}$ and define

$$F_1(u) = af(u) + bu \quad \text{and} \quad F_2(v) = (1-a)f(v) - bv.$$

Note that $F(u,v) = F_1(u) + F_2(v)$ is monotone if

$$a \inf_x \{f'(x)\} \geq -b \quad \text{and} \quad (1-a)\sup_x \{f'(x)\} \leq b$$

This example includes both the upwind scheme and the Lax-Friedrichs scheme (See for instance [14]).

From a more general point of view we may consider any flux splitting. That is $f(u) = f^+(u) + f^-(u)$ where $(f^+(u))' \geq 0$ and $(f^-(u))' \leq 0$ for all $u \in \mathbb{R}$. Then the numerical flux F defined by

$$F(u,v) = f^+(u) + f^-(v)$$

satisfies the assumptions of lemma 2.4. Note also that any convex combinations of numerical flux functions which satisfies the hypothesis of lemma 2.4 itself satisfies the lemma.

If lemma 2.4 holds we have a representation of Q given by (2.13). It follows that

$$Q(u,v) = q(u) + \int_u^v \psi'(z)F_2'(z)dz.$$

Note that we may obtain another representation depending on F_1 by splitting up the first integral. The next result is taken from [17] by I. Kröker and C. Rohde.

Lemma 2.5. *Let Q be a numerical entropy flux associated with the entropy, entropy flux pair (ψ, q) and the monotone numerical flux F. Then*

$$\psi'(u)(F(u,w) - F(v,u)) \geq Q(u,w) - Q(v,u)$$

for all $u, v, w \in \mathbb{R}$.

Proof. Let u be fixed. Define $p(v,w) = p_1(w) + p_2(v)$ where

$$\begin{aligned} p_1(w) &= -\psi'(u)F(u,w) + Q(u,w) + \psi'(u)f(u) - q(u), \\ p_2(v) &= \psi'(u)F(v,u) - Q(v,u) - \psi'(u)f(u) + q(u). \end{aligned}$$

Then we have

$$p(v,w) = -\psi'(u)(F(u,w) - F(v,u)) + (Q(u,w) - Q(v,u))$$

and so the lemma is proved if we can show that $p(v,w) \le 0$ for all $v, w \in \mathbb{R}$. Let us differentiate p_1 and p_2.

$$\begin{aligned} p_1'(w) &= -\psi'(u)\frac{\partial}{\partial w}F(u,w) + \psi'(w)\frac{\partial}{\partial w}F(u,w) \\ &= \psi''(\xi_1)\frac{\partial}{\partial w}F(u,w)(w-u) \end{aligned}$$

for some $\xi_1 \in \operatorname{int}(u,w)$. Similarly

$$\begin{aligned} p_2'(v) &= \psi'(u)\frac{\partial}{\partial v}F(v,u) - \psi'(v)\frac{\partial}{\partial v}F(v,u) \\ &= \psi''(\xi_2)\frac{\partial}{\partial v}F(v,u)(u-v) \end{aligned}$$

for some $\xi_2 \in \operatorname{int}(u,v)$. Since F is monotone and ψ is convex we may infer that if $z \in \mathbb{R}$ then $p_i'(z)(z-u) \le 0$ for both $i=1$ and $i=2$. It remains to observe that $p_1(u) = p_2(u) = 0$ and so

$$p_i(z) = \int_u^z p_i'(\xi)d\xi \le 0, \quad \text{for } i = 1, 2.$$

Hence $p(v,w) \le 0$. $\square$

3 An error estimate for the semidiscrete approximation

Let u_j be a solution to (2.2) with $A = A^\eta$ and let u be the viscous approximation defined by (1.2). We need to consider u_j not as a sequence, but rather as a piecewise constant function. Note that equation (2.2) also holds for all $(x,t) \in \Pi_T$ given that

$$u_j(x,t) = \sum_j u_j(t)\chi_{I_j}(x) \tag{3.1}$$

and the discrete differentials denote difference quotients with parameter Δx. Observe that the norm $\|\cdot\|_1$ defined in section 2.1 has the property that $\|u_j(t)\|_1 = \|u_j(\cdot,t)\|_{L^1(\mathbb{R})}$. In order to obtain the estimate we need many of the uniform bounds obtained in section 1 and 2.1. These are based on the properties of u^0 and so we need to assume that u^0 is sufficiently well behaved. We make the following assumptions on u^0:

(i) u^0 is contained in the space BV. That is $u^0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $T.V.(u^0) < \infty$.

(ii) $A(u^0)$ is differentiable and $A(u^0)_x$ is in the space BV.

We may now state the theorem. The proof is presented in the rest of this chapter.

Theorem 3.1. *Suppose that u is the entropy solution to* (1.1) *and that u_j^η is the semidiscrete approximation with $\eta = (\Delta x)^{\frac{2}{7}}$. If u^0 satisfies* (i) *and* (ii)*, then there exists a constant C independent of Δx such that*

$$\|u(\cdot,t) - u_j^\eta(\cdot,t)\|_{L^1(L_l(t),L_r(t))} \le C\sqrt[7]{\Delta x}, \qquad t \in (0,T)$$

where $L_l(t) = -L + Mt, L_r(t) = L - Mt$, $M \ge \|f\|_{Lip}$ and $L \ge MT + \Delta x$.

Note, C might depend on L. Let us define some of the functions we are going to work with. First, let the approximation of the sign function be given by

$$\operatorname{sign}_\varepsilon(\sigma) = \begin{cases} \sin(\frac{\pi\sigma}{2\varepsilon}) & \text{if } |\sigma| \le \varepsilon, \\ \operatorname{sign}(\sigma) & \text{otherwise} \end{cases}$$

where $\varepsilon > 0$. Note that $\operatorname{sign}_\varepsilon$ is a C^1 function which is nondecreasing and odd. Since the derivative of an odd function is even, $\operatorname{sign}'_\varepsilon$ is even. Having defined an approximation of the sign function we get a natural approximation of the absolute value function $|\cdot|$ given by

$$|u|_\varepsilon = \int_0^u \operatorname{sign}_\varepsilon(z)dz.$$

By a simple substitution argument

$$|u-c|_\varepsilon = \int_c^u \operatorname{sign}_\varepsilon(z-c)dz \quad \text{and} \quad \frac{d}{du}(|u-c|_\varepsilon) = \operatorname{sign}_\varepsilon(u-c).$$

In the semi-discrete case we are only going to work with difference quotients in the spatial direction, so we might as well let $D^\pm$ denote $D^\pm_{\Delta x}$. The following lemma is a kind of substitution for the chain-rule when working with the piecewise constant approximation (3.1) and difference quotients.

Lemma 3.1. *Given a sequence $\{u_j\}$ there exist sequences $\{\tau_j\}$ and $\{\theta_j\}$ such that both τ_j and θ_j lies in $int(u_j, u_{j+1})$ and*

$$D^+\mathrm{sign}_\varepsilon(A(u_j) - A(c)) = \mathrm{sign}'_\varepsilon(A(\tau_j) - A(c))D^+A(u_j),$$
$$D^+(|A(u_j) - A(c)|_\varepsilon) = \mathrm{sign}_\varepsilon(A(\theta_j) - A(c))D^+A(u_j).$$

The proof is a simple application of the mean value theorem. Note that both τ_j and θ_j depends on both $\{u_j\}$ and c.

3.1 The doubling of variables

This part contains the construction of an equality on which the sought inequality is based. The manipulations involved are inspired by the entropy inequality. We let (x, t, y, s) denote a point in $\Pi_T \times \Pi_T = \Pi_T^2$ where x and y are the spatial variables and s and t are the time variables. Let u_j defined by (3.1) be a function of x and t, and the viscous approximation u be a function of y and s. To avoid writing four integral signs we will in general write one for each domain Π_T and let $dX = dxdtdyds$. For a function φ on Π_T^2 we let $\varphi^{\Delta x}$ denote the function translated by Δx in the spatial variable x. That is, $\varphi^{\Delta x}(x, t, y, s) = \varphi(x + \Delta x, t, y, s)$. We write u_{j+1} instead of $u_j^{\Delta x}$.

3.1.1 Rewriting the continuous equation (Local to global)

Define an entropy, entropy flux pair $(\psi_\varepsilon, q_\varepsilon)$ by

$$\psi_\varepsilon(u, c) = \int_c^u \mathrm{sign}_\varepsilon(A(z) - A(c))dz,$$
$$q_\varepsilon(u, c) = \int_c^u \psi'_\varepsilon(z, c)(f(z) - f(c))'dz.$$

Let φ be a non-negative test function in Π_T^2 and observe that for each fixed point (x, t), $\varphi(x, t, y, s)$ is a test function in Π_T. Multiply equation (1.2) by $\psi'_\varepsilon(u, c)\varphi$ and integrate in both space and time to get

$$\int_{\Pi_T} \psi_\varepsilon(u, c)_s\varphi + \psi'_\varepsilon(u, c)(f(u) - f(c))_y\varphi dyds = \int_{\Pi_T} (\psi'_\varepsilon(u, c)\varphi)A(u)_{yy}dyds.$$

Integration by parts gives

$$\begin{aligned}&\int_{\Pi_T} \psi_\varepsilon(u, c)\varphi_s + q_\varepsilon(u, c)\varphi_y dyds \\ &= \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u) - A(c))A(u)_y\varphi_y + \mathrm{sign}'_\varepsilon(A(u) - A(c))(A(u)_y)^2\varphi dyds.\end{aligned} \tag{3.2}$$

Using the chain rule and integration by parts we get

$$\int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u) - A(c))(A(u))_y\varphi_y dyds = -\int_{\Pi_T} |A(u) - A(c)|_\varepsilon\varphi_{yy}dyds. \tag{3.3}$$

Next we are going to write zero in a rather complicated manner. The idea will be clear at a later point. Let τ_j be as in lemma 3.1. By Leibniz rule and integration by parts

$$
\begin{aligned}
0 &= \iint_{\Pi_T^2} D^+ \left(\mathrm{sign}_\varepsilon(A(u) - A(u_j))A(u)_y\varphi\right) dX \\
&= \iint_{\Pi_T^2} D^+(\mathrm{sign}_\varepsilon(A(u) - A(u_j)))A(u)_y\varphi^{\Delta x} - |A(u) - A(u_j)|_\varepsilon D^+\varphi_y dX \\
&= \iint_{\Pi_T^2} -\mathrm{sign}'_\varepsilon(A(\tau_j) - A(u))D^+A(u_j)A(u)_y\varphi^{\Delta x} - |A(u) - A(u_j)|_\varepsilon D^+\varphi_y dX.
\end{aligned}
\tag{3.4}
$$

Let $c = u_j$ in (3.3). By (3.3) and (3.4) we turn equation (3.2) into the following equation:

$$
\begin{aligned}
&\iint_{\Pi_T^2} |u - u_j|\varphi_s - q_\varepsilon(u, u_j)_y\varphi dX \\
&= \iint_{\Pi_T^2} \mathrm{sign}'_\varepsilon(A(\tau_j) - A(u))((A(u)_y)^2 - D^+A(u_j)A(u)_y)\varphi^{\Delta x} dX \\
&- \iint_{\Pi_T^2} |A(u) - A(u_j)|_\varepsilon(\varphi_{yy} + D^+\varphi_y)dX \\
&+ \iint_{\Pi_T^2} (|u - u_j| - \psi_\varepsilon(u, u_j))\varphi_s dX \\
&+ \iint_{\Pi_T^2} \left(\mathrm{sign}'_\varepsilon(A(u_j) - A(u))\varphi - \mathrm{sign}'_\varepsilon(A(\tau_j) - A(u))\varphi^{\Delta x}\right)(A(u)_y)^2 dX.
\end{aligned}
\tag{3.5}
$$

3.1.2 Rewriting the semidiscrete equation (Local to global)

Let us try to rewrite the semidiscrete equation (2.2) in a similar way. Multiply by $\psi'_\varepsilon(u_j, c)\varphi$ and integrate in both time and space to obtain

$$
\begin{aligned}
&\int_{\Pi_T} \psi_\varepsilon(u_j, c)_t\varphi + \psi'_\varepsilon(u_j, c)D^-F(u_j, u_{j+1})\varphi dxdt \\
&\qquad = \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u_j) - A(c))D^-D^+A(u_j)\varphi dxdt.
\end{aligned}
\tag{3.6}
$$

Using integration by parts for difference quotients we get

$$
\begin{aligned}
&\int_{\Pi_T} \psi_\varepsilon(u_j, c)\varphi_t - \psi'_\varepsilon(u_j, c)D^-F(u_j, u_{j+1})\varphi dxdt \\
&= \int_{\Pi_T} D^+\mathrm{sign}_\varepsilon(A(u_j) - A(c))D^+A(u_j)\varphi^{\Delta x} dxdt \\
&+ \int_{\Pi_T} \mathrm{sign}_\varepsilon(A(u_j) - A(c))D^+A(u_j)D^+\varphi dxdt.
\end{aligned}
\tag{3.7}
$$

Concerning the second term on the right of (3.7) we would like to use a type of chain rule. By lemma 3.1

$$\begin{aligned}
&\int_{\Pi_T} \operatorname{sign}_\varepsilon(A(u_j) - A(c)) D^+ A(u_j) D^+ \varphi dx dt \\
&= -\int_{\Pi_T} |A(u_j) - A(c)|_\varepsilon D^- D^+ \varphi dx dt \\
&+ \int_{\Pi_T} \left[\operatorname{sign}_\varepsilon(A(u_j) - A(c)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(c))\right] D^+ A(u_j) D^+ \varphi dx dt.
\end{aligned}$$

As in the continuous case observe that

$$\begin{aligned}
0 &= \iint_{\Pi_T^2} (\operatorname{sign}_\varepsilon(A(u_j) - A(u)) D^+ A(u_j) \varphi^{\Delta x})_y dX \\
&= \iint_{\Pi_T^2} \left((\operatorname{sign}_\varepsilon(A(u_j) - A(u)))_y \varphi^{\Delta x} + \operatorname{sign}_\varepsilon(A(u_j) - A(u)) \varphi_y^{\Delta x}\right) D^+ A(u_j) dX.
\end{aligned} \tag{3.8}$$

The last term on the right may be rewritten using lemma 3.1 as above. Note that $D^-(\varphi^{\Delta x}) = D^+\varphi$. Then

$$\begin{aligned}
&\iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_j) - A(u)) D^+ A(u_j) \varphi_y^{\Delta x} dX \\
&= -\iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^+ \varphi_y dX \\
&+ \iint_{\Pi_T^2} \left[\operatorname{sign}_\varepsilon(A(u_j) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(u))\right] D^+ A(u_j) \varphi_y^{\Delta x} dX.
\end{aligned}$$

Let $c = u(y,s)$ in (3.7). Integrate (3.7) in both y and s and add equation (3.8) to obtain

$$\begin{aligned}
&\iint_{\Pi_T^2} |u_j - u| \varphi_t - \psi_\varepsilon'(u_j, u) D^- F(u_j, u_{j+1}) \varphi dX \\
&= \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon'(A(\tau_j) - A(u)) \left([D^+(A(u_j))]^2 - D^+ A(u_j) A(u)_y\right) \varphi^{\Delta x} dX \\
&- \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon (D^- D^+ \varphi + D^+ \varphi_y) dX \\
&+ \iint_{\Pi_T^2} \left[\operatorname{sign}_\varepsilon(A(u_j) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(u))\right] D^+ A(u_j) \varphi_y^{\Delta x} dX \\
&+ \iint_{\Pi_T^2} (|u_j - u| - \psi_\varepsilon(u_j, u)) \varphi_t dX \\
&+ \iint_{\Pi_T^2} (\operatorname{sign}_\varepsilon'(A(\tau_j) - A(u)) - \operatorname{sign}_\varepsilon'(A(u_j) - A(u))) A(u)_y D^+ A(u_j) \varphi^{\Delta x} dX \\
&+ \iint_{\Pi_T^2} \left[\operatorname{sign}_\varepsilon(A(u_j) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(u))\right] D^+ A(u_j) D^+ \varphi dX.
\end{aligned} \tag{3.9}$$

3.1.3 Adding up the equations

Adding equation (3.5) to equation (3.9) we get

$$\iint_{\Pi_T^2} |u_j - u|(\varphi_t + \varphi_s) - (\psi'_\varepsilon(u_j,c)D^- F(u_j,u_{j+1}) + q_\varepsilon(u,c)_y)\varphi dX \tag{3.10}$$

$$= \iint_{\Pi_T^2} \operatorname{sign}'_\varepsilon(A(\tau_j) - A(u)) \left(D^+(A(u_j) - A(u)_y\right)^2 \varphi^{\Delta x} dX \tag{3.11}$$

$$- \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon (D^- D^+ \varphi + 2D^+ \varphi_y + \varphi_{yy}) dX \tag{3.12}$$

$$+ \iint_{\Pi_T^2} (|u_j - u| - \psi_\varepsilon(u_j,u))\varphi_t dX \tag{3.13}$$

$$+ \iint_{\Pi_T^2} (|u - u_j| - \psi_\varepsilon(u,u_j))\varphi_s dX \tag{3.14}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon(A(u_j) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(u))\big] D^+ A(u_j) \varphi_y^{\Delta x} dX \tag{3.15}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}'_\varepsilon(A(\tau_j) - A(u)) - \operatorname{sign}'_\varepsilon(A(u_j) - A(u))\big] A(u)_y D^+ A(u_j) \varphi^{\Delta x} dX \tag{3.16}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon(A(u_j) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(u))\big] D^+ A(u_j) D^+ \varphi dX \tag{3.17}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}'_\varepsilon(A(u_j) - A(u))\varphi - \operatorname{sign}'_\varepsilon(A(\tau_j) - A(u))\varphi^{\Delta x}\big] (A(u)_y)^2 dX. \tag{3.18}$$

3.2 Obtaining the inequality

Following lemma 2.4 we may define the numerical entropy flux $Q_\varepsilon^c(u_j,u_{j+1})$ by

$$Q_\varepsilon^c(u_j,u_{j+1}) = q_\varepsilon(u_j,c) + \int_{u_j}^{u_{j+1}} \psi'_\varepsilon(z,c) F'_2(z) dz. \tag{3.19}$$

By lemma 2.5

$$\psi'_\varepsilon(u_j,c) D^- F(u_j,u_{j+1}) \geq D^- Q_\varepsilon^c(u_j,u_{j+1}).$$

The term (3.11) is positive and so

$$\iint_{\Pi_T^2} |u_j - u|(\varphi_t + \varphi_s) - (D^- Q_\varepsilon^u(u_j,u_{j+1}) + q_\varepsilon(u,u_j)_y)\varphi dX + \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon (D^- D^+ \varphi + 2D^+ \varphi_y + \varphi_{yy}) dX \geq \Re, \tag{3.20}$$

where $\Re$ is the sum

$$\Re := (3.13) + \cdots + (3.18).$$

Integration by parts and (3.19) yields

$$
\begin{aligned}
-\iint_{\Pi_T^2} & D^- Q_\varepsilon^u(u_j, u_{j+1})\varphi + q_\varepsilon(u, u_j)_y \varphi dX \\
&= \iint_{\Pi_T^2} Q_\varepsilon^u(u_j, u_{j+1}) D^+\varphi + q_\varepsilon(u, u_j)\varphi_y dX \\
&= \iint_{\Pi_T^2} q_\varepsilon(u_j, u) D^+\varphi + q_\varepsilon(u, u_j)\varphi_y dX \\
&\qquad + \iint_{\Pi_T^2} \int_{u_j}^{u_{j+1}} \psi_\varepsilon'(z, u) F_2'(z) dz D^+\varphi dX.
\end{aligned}
$$

Next we need an expression for $q_\varepsilon(u_j, u)D^+\varphi + q_\varepsilon(u, u_j)\varphi_y$. Let us rewrite $q_\varepsilon(u, c)$ using integration by parts.

$$
\begin{aligned}
q_\varepsilon(u, c) &= \int_c^u \operatorname{sign}_\varepsilon(A(z) - A(c))(f(z) - f(c))' dz \\
&= \operatorname{sign}_\varepsilon(A(u) - A(c))(f(u) - f(c)) \\
&\qquad - \int_c^u (\operatorname{sign}_\varepsilon(A(z) - A(c)))'(f(z) - f(c)) dz.
\end{aligned}
$$

Observe that $\operatorname{sign}_\varepsilon(A(u_j) - A(u)) = -\operatorname{sign}_\varepsilon(A(u) - A(u_j))$ and so

$$
\begin{aligned}
q_\varepsilon(u_j, u) & D^+\varphi + q_\varepsilon(u, u_j)\varphi_y \\
&= \operatorname{sign}_\varepsilon(A(u_j) - A(u))(f(u_j) - f(u))(D^+\varphi + \varphi_y) \\
&\quad - \int_u^{u_j} (\operatorname{sign}_\varepsilon(A(z) - A(u)))'(f(z) - f(u)) dz D^+\varphi \\
&\qquad - \int_{u_j}^u (\operatorname{sign}_\varepsilon(A(z) - A(u_j)))'(f(z) - f(u_j)) dz \varphi_y.
\end{aligned}
$$

Let

$$
\gamma_1 = \iint_{\Pi_T^2} \int_u^{u_j} (\operatorname{sign}_\varepsilon(A(z) - A(u)))'(f(z) - f(u)) dz D^+\varphi dX, \tag{3.21}
$$

$$
\gamma_2 = \iint_{\Pi_T^2} \int_{u_j}^u (\operatorname{sign}_\varepsilon(A(z) - A(u_j)))'(f(z) - f(u_j)) dz \varphi_y dX, \tag{3.22}
$$

$$
\gamma_3 = \iint_{\Pi_T^2} \int_{u_j}^{u_{j+1}} \psi_\varepsilon'(z, u) F_2'(z) dz D^+\varphi dX. \tag{3.23}
$$

We obtain from 3.20 the inequality

$$
\begin{aligned}
\iint_{\Pi_T^2} & |u_j - u|(\varphi_t + \varphi_s) dX \\
&+ \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_j) - A(u))(f(u_j) - f(u))(D^+\varphi + \varphi_y) dX \\
&+ \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon (D^- D^+\varphi + 2D^+\varphi_y + \varphi_{yy}) dX + \gamma_3 \\
&\qquad \geq \gamma_1 + \gamma_2 + \Re.
\end{aligned}
\tag{3.24}
$$

The next task is to chose φ in a clever way. Let $\omega \in C_0^\infty(\mathbb{R})$ be a function satisfying

$$\mathrm{supp}(\omega) \subset [-1,1], \quad \omega(\sigma) \geq 0, \quad \int_{\mathbb{R}} \omega(\sigma) d\sigma = 1$$

and let $\omega_r(\sigma) = \frac{1}{r}\omega(\sigma/r)$. Let $\nu < \tau$ be two numbers in $(0,T)$. For any α define

$$H_{\alpha_0}(t) = \int_{-\infty}^{t} \omega_{\alpha_0}(\xi) d\xi, \quad \text{and} \quad \chi_{(\nu,\tau)}^{\alpha_0}(t) = H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau).$$

Furthermore let

$$K_\alpha(x,t) = (L - Mt)\alpha + 1 - \alpha|x|$$

and $\chi_{(L_l,L_r)}^{\alpha}(x,t) = \max(0, \min(1, K_\alpha(x,t)))$. That is

$$\chi_{(L_l,L_r)}^{\alpha}(x,t) = \begin{cases} 1 & \text{if} \quad |x| \leq (L-Mt), \\ K_\alpha(x,t) & \text{if} \quad (L-Mt) \leq |x| \leq (L-Mt) + \frac{1}{\alpha}, \\ 0 & \text{else.} \end{cases}$$

Here L_l and L_r denote the lines defined in theorem 3.1. Let

$$\Psi(x,t) = \chi_{(\nu,\tau)}^{\alpha_0}(t)\chi_{(L_l,L_r)}^{\alpha}(x,t)$$

and

$$\varphi(x,t,y,s) = \Psi(x,t)\omega_r(x-y)\omega_{r_0}(t-s).$$

To make sure that $supp(\varphi) \subset \Pi_T^2$ let $0 < \alpha_0 < \min(\nu, T-\tau)$ and $0 < r_0 < \min(\nu, T-\tau)$. We let $\omega = \omega_r\omega_{r_0}$ and remark that this should not be confused with the ω defined above. Observe that φ has some very important properties:

$$\begin{aligned} \varphi_t + \varphi_s &= \Psi_t\omega, \\ \varphi_x + \varphi_y &= \Psi_x\omega, \\ \varphi_{xx} + 2\varphi_{xy} + \varphi_{yy} &= \Psi_{xx}\omega. \end{aligned}$$

Using difference quotients instead of derivatives these properties are not directly involved, but as long as the difference quotient parameter Δx tend relatively fast to zero compared with r and r_0 these properties still approximatly apply. This will be seen in the following computations. Before going further we include a list of elementary results. These results will be used without reference.

Lemma 3.2. *Let $L > MT + \Delta x$ and define*

$$\Lambda = \{(x,t) | (L-Mt) \leq |x| \leq (L-Mt) + \frac{1}{\alpha}\},$$
$$\Lambda_{\Delta x} = \{(x,t) | (L-Mt) - \Delta x \leq |x| \leq (L-Mt) + \frac{1}{\alpha} + \Delta x\}.$$

Let $\chi_\Lambda, \chi_{\Lambda_{\Delta x}}$ denote the characteristic functions of Λ and $\Lambda_{\Delta x}$ respectively.

Then the following computations and estimates apply for all $t \in [0,T]$:

$$
\begin{aligned}
(\chi^\alpha(x,t))_t &= -M\alpha\chi_\Lambda(x,t) \\
(\chi^\alpha(x,t))_x &= -(sign(x))\alpha\chi_\Lambda(x,t) \\
D^+K(x,t) &= -\alpha D^+(|x|) \\
|D^+\chi^\alpha| &\le \alpha\chi_{\Lambda_{\Delta x}} \\
\int_{\mathbb{R}} |D^-D^+(\chi^\alpha)(x,t)|dx &= 4\alpha \\
\int_{\mathbb{R}} |D^+\chi^\alpha(x,t) - (\chi^\alpha)_x|dx &\le 4\Delta x\alpha \\
\int_{\mathbb{R}} \chi_\Lambda(x,t)dx &= \frac{2}{\alpha} \\
\int_{\mathbb{R}} \chi_{\Lambda_{\Delta x}}(x,t)dx &= \frac{2}{\alpha} + 4\Delta x \\
\int_{\mathbb{R}} \chi^\alpha(x,t)dx &= 2(L-Mt) + \frac{1}{\alpha} \\
\int_0^T \int_{\mathbb{R}} \chi^\alpha dxdt &= (2L-MT)T + \frac{T}{\alpha}
\end{aligned}
$$

The proof of these statements are left to the reader. The next task is to find out what type of inequality we may obtain from (3.24) with this choice of test function φ. Consider the first term on the left in (3.24). Let us split it into positive and negative parts.

$$
\begin{aligned}
&\iint_{\Pi_T^2} |u_j - u|(\varphi_t + \varphi_s)dX \\
&\quad = \iint_{\Pi_T^2} |u_j - u|\omega_{\alpha_0}(t-\nu)\chi^\alpha\omega dX - \iint_{\Pi_T^2} |u_j - u|\omega_{\alpha_0}(t-\tau)\chi^\alpha\omega dX \\
&\qquad\qquad - \iint_{\Pi_T^2} M\alpha|u_j - u|\chi_\Lambda\chi^{\alpha_0}\omega dX := \delta_1 + \delta_2 + \delta_3.
\end{aligned}
$$

Consider the second term in (3.24). We add and subtract to obtain

$$
\begin{aligned}
&\iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(u_j) - A(u))(f(u_j) - f(u))(D^+\varphi + \varphi_y)dX = \\
&\qquad \iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(u_j) - A(u))(f(u_j) - f(u))(\varphi_x + \varphi_y)dX \\
&\qquad + \iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(u_j) - A(u))(f(u_j) - f(u))(D^+\varphi - \varphi_x)dX \\
&\qquad =: \beta_1 + \beta_2. \quad (3.25)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\delta_3 + \beta_1 = \\
&\iint_{\Pi_T^2} \alpha\left[\mathrm{sign}_\varepsilon(A(u_j) - A(u))(-\mathrm{sign}(x))(f(u_j) - f(u)) - M|u_j - u|\right]\chi_\Lambda\chi^{\alpha_0}\omega dX
\end{aligned}
$$

$$\leq \iint_{\Pi_T^2} \alpha \left(|f(u_j) - f(u)| - M|u_j - u|\right) \chi_\Lambda \chi^{\alpha_0} \omega dX.$$

Since $M \geq \|f\|_{Lip}$ we have $\delta_3 + \beta_1 \leq 0$. By (3.24)

$$\begin{aligned}
&\iint_{\Pi_T^2} |u_j - u| \omega_{\alpha_0}(t-\nu)\chi^\alpha \omega dX + \beta_2 \\
&\qquad + \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon (D^- D^+ \varphi + 2D^+ \varphi_y + \varphi_{yy}) dX + \gamma_3 \\
&\qquad\qquad \geq \iint_{\Pi_T^2} |u_j - u| \omega_{\alpha_0}(t-\tau)\chi^\alpha \omega dX \\
&\qquad\qquad\qquad - (\delta_3 + \beta_1) + \gamma_1 + \gamma_2 + \Re.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\iint_{\Pi_T^2} |u_j - u| \omega_{\alpha_0}(t-\nu)\chi^\alpha \omega dX + \beta_2 + |\gamma_1| + |\gamma_2| + \gamma_3 \\
&\qquad + \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon (D^- D^+ \varphi + 2D^+ \varphi_y + \varphi_{yy}) dX + |\Re| \\
&\qquad\qquad \geq \iint_{\Pi_T^2} |u_j - u| \omega_{\alpha_0}(t-\tau)\chi^\alpha \omega dX. \qquad (3.26)
\end{aligned}$$

3.3 Finding the rate of convergence

The subject of this section is to show how and at which speed the "unwanted" terms in (3.26) tend to zero as the small parameters $\Delta x, \alpha, \alpha_0, \varepsilon, r$ and r_0 vanish. Note that Δx and η are the parameters which define the approximations while the other parameters can be picked freely in order to optimize the rate of convergence.

In these computations we let C denote a generic constant. By constant it is meant that it does not depend on the small variables but it might depend on L, M and T. Similarly we let $\Gamma = \Gamma(\Delta x, \eta, \alpha, \alpha_0, r, r_0)$ denote a generic function with the property that it is locally bounded, positive and increasing in each variable. Note that given Γ_1 and Γ_2 we can always pick $\Gamma = \max\{\Gamma_1, \Gamma_2\}$. That is, taking the maximum of two increasing functions we obtain an increasing function. Thus we may work with this class of functions in a similar way as with constants. The following simple computation should be kept in mind while simplifying the expressions below. Given two positive functions f and g we have

$$\Gamma_1 f + \Gamma_2 g \leq \Gamma f + \Gamma g = \Gamma(g + f)$$

We are going to need some more elementary results.

Lemma 3.3. *The following estimates apply:*

$$|D^{\pm}(\omega_r(x-y))| \leq \|\omega'\|_{L^\infty(\mathbb{R})}\frac{1}{r^2}\chi_{\{|x-y|\leq r+\Delta x\}} \tag{3.27}$$

$$|(\omega_r(x-y))_x| \leq \|\omega'\|_{L^\infty(\mathbb{R})}\frac{1}{r^2}\chi_{\{|x-y|\leq r\}} \tag{3.28}$$

$$|(\omega_r(x-y))_{xx}| \leq \|\omega''\|_{L^\infty(\mathbb{R})}\frac{1}{r^3}\chi_{\{|x-y|\leq r\}} \tag{3.29}$$

$$|D^{\pm}\omega_r - (\omega_r)_x| \leq \|\omega''\|_{L^\infty(\mathbb{R})}\frac{\Delta x}{2r^3}\chi_{\{|x-y|\leq r+\Delta x\}} \tag{3.30}$$

Proof. The proof of these statements are elementary computations. To show (3.27) note that

$$supp(D^+\omega_r(x-y)) \subset \{(x,y) \mid |x-y| \leq r+\Delta x\}$$

Now, consider ω_r as a function of one variable. By the mean value inequality $\|D^+\omega_r\|_{L^\infty(\mathbb{R})} \leq \|(\omega_r)'\|_{L^\infty(\mathbb{R})}$. Differentiating gives

$$|\omega_r'(\sigma)| = |\frac{1}{r^2}\omega'(\frac{\sigma}{r})| \leq \frac{1}{r^2}\|\omega'\|_{L^\infty(\mathbb{R})}.$$

The proof of (3.28) and (3.29) are similar. The proof of (3.30) follows by the Taylor expansion of ω_r. □

We start by considering the term β_2 defined in (3.25).

Estimate 3.1.

$$|\beta_2| \leq \Gamma(\Delta x, \alpha, r)\left(\frac{\Delta x}{r^2\alpha}\left(1+\frac{\Delta x}{r}\right)\right).$$

Proof. Since both $\|u_j\|_{L^\infty(\mathbb{R})} \leq \|u^0\|_{L^\infty(\mathbb{R})}$ and $\|u\|_{L^\infty(\mathbb{R})} \leq \|u^0\|_{L^\infty(\mathbb{R})}$ it follows by the Lipschitz continuity of f that $|f(u_j)-f(u)|$ is bounded independently of η and Δx. We need to estimate $\|D^+\varphi - \varphi_x\|_{L^1(\Pi_T^2)}$. Differentiating and comparing terms we obtain

$$D^+\varphi - \varphi_x = (D^+\Psi - \Psi_x)\omega_r\omega_{r_0} + \Psi(D^+\omega_r - (\omega_r)_x)\omega_{r_0} + (\Psi^{\Delta x} - \Psi)D^+\omega_r\omega_{r_0}.$$

Therefore

$$\begin{aligned}\|D^+\varphi - \varphi_x\|_{L^1(\Pi_T^2)} \leq \int_{\Pi_T} |D^+\Psi - \Psi_x| dxdt + \iint_{\Pi_T^2} \Psi|D^+\omega_r - (\omega_r)_x|\omega_{r_0} dX \\ + \iint_{\Pi_T^2} |\Psi^{\Delta x} - \Psi||D^+\omega_r|\omega_{r_0} dX.\end{aligned}$$

We can now consider each of these terms.

$$\int_{\Pi_T} |D^+\Psi - \Psi_x| dxdt \le \int_{\Pi_T} \chi^{\alpha_0} |D^+\chi^\alpha - (\chi^\alpha)_x| dxdt \le C\Delta x\alpha.$$

$$\iint_{\Pi_T^2} \Psi |D^+\omega_r - (\omega_r)_x| \omega_{r_0} dX \le C\frac{\Delta x}{r^3} \iint_{\Pi_T^2} \Psi \chi_{\{|x-y|\le r+\Delta x\}} \omega_{r_0} dX$$
$$\le C\frac{\Delta x}{r^3}(r+\Delta x) \int_{\Pi_T} \Psi dxdt \le C\frac{\Delta x}{r^3}(r+\Delta x)\left(1+\frac{1}{\alpha}\right).$$

$$\iint_{\Pi_T^2} |\Psi^{\Delta x} - \Psi||D^+\omega_r|\omega_{r_0} dX| \le \Delta x \iint_{\Pi_T^2} |D^+\Psi||D^+\omega_r|\omega_{r_0} dX|$$
$$\le C\frac{\alpha\Delta x}{r^2} \iint_{\Pi_T^2} \chi^{\alpha_0}\chi_{\Lambda_{\Delta x}}\chi_{\{|x-y|\le r+\Delta x\}}\omega_{r_0} dX$$
$$\le C\frac{\alpha\Delta x}{r^2}(r+\Delta x)\left(\frac{1}{\alpha}+\Delta x\right).$$

Collecting all the terms we obtain

$$\|D^+\varphi - \varphi_x\|_{L^1(\Pi_T^2)}$$
$$\le C\left(\Delta x\alpha + \frac{\Delta x}{r^2\alpha}\left(1+\frac{\Delta x}{r}\right)(1+\alpha) + \frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right)(1+\alpha\Delta x)\right).$$

The result follows by this inequality. □

Let us consider the double derivative term (3.12). Before doing the computations below, observe that $D^+D^- = D^-D^+$.

Estimate 3.2.

$$|(3.12)| \le C\alpha + \Gamma(\Delta x,\alpha)\left(\frac{\Delta x}{r^2}\left(1+\frac{\Delta x}{r}\right)\right).$$

Proof. Observe that

$$\begin{aligned}
D^-D^+(\varphi) &= D^-(\Psi^{\Delta x}D^+\omega + D^+\Psi\omega)\\
&= D^+\Psi(D^+\omega)^{-\Delta x} + \Psi^{\Delta x}D^-D^+\omega\\
&\quad + D^-D^+\Psi\omega^{-\Delta x} + D^+\Psi D^-\omega,\\
2D^+(\varphi)_y &= -D^+\Psi(\omega)_x^{\Delta x} + \Psi D^+(\omega)_y - D^+\Psi\omega_x - \Psi^{\Delta x}D^+\omega_x,\\
\varphi_{yy} &= -\Psi(\omega_x)_y.
\end{aligned}$$

By Leibniz rule

$$\begin{aligned}
D^-D^+\varphi + 2D^+\varphi_y + \varphi_{yy} = D^-D^+\Psi\omega^{-\Delta x} + D^+(\Psi(D^-\omega - \omega_x))\\
+ \Psi(D^+\omega - \omega_x)_y + D^+\Psi(D^-\omega - \omega_x^{\Delta x}). \quad (3.31)
\end{aligned}$$

Let us estimate each term separately. First note that $|A(u_j)-A(u)|_\varepsilon$ is bounded independently of η and Δx. Hence

$$|\iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^-D^+\Psi\omega_r^{-\Delta x}\omega_{r_0} dX| \le C\alpha.$$

The next two terms can be estimated using integration by parts.

$$\iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^+(\Psi(D^-\omega_r - (\omega_r)_x))\omega_{r_0} dX$$
$$= -\iint_{\Pi_T^2} D^-|A(u_j) - A(u)|_\varepsilon \Psi(D^-\omega_r - (\omega_r)_x)\omega_{r_0} dX$$
$$= -\iint_{\Pi_T^2} \mathrm{sign}_\varepsilon(A(\xi_j) - A(u))(D^-A(u_j))\Psi(D^-\omega_r - (\omega_r)_x)\omega_{r_0} dX$$

for some $\xi_j \in \mathrm{int}(u_{j-1}, u_j)$. Recall that $\|D^-A(u_j)\|_{L^1(\Pi_T)}$ and $\|A(u)_y\|_{L^1(\Pi_T)}$ is bounded independently of η and Δx. Applying lemma 3.3 it follows that

$$\iint_{\Pi_T^2} |D^-A(u_j)|\Psi|D^-\omega_r - (\omega_r)_x|\omega_{r_0} dX$$
$$\leq C\frac{\Delta x}{r^3}(r + \Delta x)\int_{\Pi_T} \Psi|D^-A(u_j)|dxdt \leq C\frac{\Delta x}{r^3}(r + \Delta x).$$

Similarly

$$|\iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon \Psi(D^+\omega_r - (\omega_r)_x)_y \omega_{r_0} dX| \leq C\frac{\Delta x}{r^3}(r + \Delta x).$$

To estimate the term associated with the last term on the right of (3.31) we split it the following way:

$$\iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^+\Psi(D^-\omega_r - (\omega_r)_x^{\Delta x})\omega_{r_0} dX$$
$$= \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^+\Psi(D^-\omega_r - (\omega_r)_x)\omega_{r_0} dX$$
$$- \Delta x \iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^+\Psi(D^+\omega_r)_x \omega_{r_0} dX.$$

Now,

$$|\iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^+\Psi(D^-\omega_r - (\omega_r)_x)\omega_{r_0} dX|$$
$$\leq C\Delta x \frac{\alpha}{r^3}(r + \Delta x)(\frac{1}{\alpha} + \Delta x)$$

and

$$\Delta x|\iint_{\Pi_T^2} |A(u_j) - A(u)|_\varepsilon D^+\Psi D^+(\omega_r)_x \omega_{r_0} dx|$$
$$\leq C\Delta x \frac{\alpha}{r^3}(r + \Delta x)(\frac{1}{\alpha} + \Delta x).$$

The result follows by collecting all the terms. □

Estimate 3.3.

$$|\gamma_1| + |\gamma_2| \leq \Gamma(\alpha, r)\left(\frac{\varepsilon}{\alpha\eta r}\left(1 + \frac{\Delta x}{r}\right)\right).$$

Proof. We start with the term γ_1. Consider the integral

$$I = \int_u^{u_j} \operatorname{sign}'_\varepsilon(A(z) - A(u))(f(z) - f(u))A'(z)dz.$$

By the mean value theorem $|A(z) - A(u)| \geq \eta|z - u|$. Let $K = \|f\|_{Lip}$, then

$$|f(z) - f(u)| \leq K|z - u| \leq \frac{K}{\eta}|(A(z) - A(u)|.$$

Observe that $\operatorname{sign}(u_j - u)(A(z) - A(u)) = |A(z) - A(u)|$ for all $z \in \operatorname{int}(u_j, u)$. Let $\sigma(z) = |A(z) - A(u)|$. Then $\sigma'(z) = \operatorname{sign}(u_j - u)A'(z)$. Recalling that $\operatorname{sign}'_\varepsilon$ is symmetric we can make a change of variables.

$$\begin{aligned} |I| \leq \frac{K}{\eta}| \int_u^{u_j} \operatorname{sign}'_\varepsilon(A(z) - A(u))|A(z) - A(u)|A'(z)dz| \\ \leq \frac{K}{\eta}| \int_0^{\sigma(u_j)} \operatorname{sign}'_\varepsilon(\sigma)(A(z) - A(u))d\sigma| \\ \leq \frac{K}{\eta} \int_0^{\varepsilon} \operatorname{sign}'_\varepsilon(\sigma)\sigma d\sigma = K\frac{\varepsilon}{\eta}\left(1 - \frac{2}{\pi}\right). \end{aligned}$$

To finish the estimate we need a bound on $\|D^+\varphi\|_{L^1(\Pi_T^2)}$. Note that

$$|D^+\varphi| \leq \alpha\chi_{\Lambda_{\Delta x}}\chi^{\alpha_0}\omega_r\omega_{r_0} + \Psi^{\Delta x}\frac{1}{r^2}\chi_{|x-y|\leq r+\Delta x}\omega_{r_0} \tag{3.32}$$

and so

$$\begin{aligned} \iint_{\Pi_T^2} |D^+\varphi|dX \leq \alpha \int_{\Pi_T} \chi_{\Lambda_{\Delta x}}\chi^{\alpha_0}dxdt + \frac{r + \Delta x}{r^2}\int_{\Pi_T} \Psi^{\Delta x}dxdt \\ \leq C\left((1 + \Delta x\alpha) + \frac{1}{\alpha r}(1 + \frac{\Delta x}{r})(1 + \alpha)\right). \quad (3.33) \end{aligned}$$

Therefore

$$|\gamma_1| \leq C\left(\frac{\varepsilon}{\alpha\eta r}(1 + \alpha + \alpha r) + \frac{\varepsilon\Delta x}{\eta\alpha r^2}(1 + r + \alpha^2 r^2)\right).$$

$|\gamma_2|$ is estimated in the same way. Just note that $|\varphi_y| \leq C\frac{1}{r^2}\Psi\chi_{|x-y|\leq r}\omega_{r_0}$ so

$$\iint_{\Pi_T^2} |\varphi_y|dX \leq \frac{C}{r}\int_{\Pi_T} \Psi dxdt \leq \frac{C}{r\alpha}(1 + \alpha).$$

Hence

$$|\gamma_2| \leq C\frac{\varepsilon}{\eta\alpha r}(1 + \alpha).$$

□

Estimate 3.4.

$$|\gamma_3| \leq \Gamma(\alpha, r)\left(\frac{\Delta x}{r}\left(1 + \frac{\Delta x}{r}\right)\right).$$

Proof. Since F is Lipschitz in each variable $F_2'(z)$ is bounded. Hence

$$|\int_{u_j}^{u_{j+1}} \mathrm{sign}_\varepsilon(A(z)-A(u))F_2'(z)dz| \leq C\Delta x|D^+u_j|.$$

Now, $\|D^+u_j(\cdot,t)\|_{L^1(\mathbb{R})} \leq C$ which is independent of Δx since $|u_j(\cdot,t)|_{BV} \leq |u_j^0|_{BV} \leq |u^0|_{BV}$ by lemma(5.1). We use (3.32) to obtain the following inequality:

$$\begin{aligned}\iint_{\Pi_T^2} &|D^+u_j||D^+\varphi|dX \\ &\leq \alpha\int_{\Pi_T}|D^+u_j|\chi_{\Lambda_{\Delta x}}dxdt + C\frac{r+\Delta x}{r^2}\int_{\Pi_T}|D^+u_j|\Psi^{\Delta x}dxdt.\end{aligned}$$

Hence

$$|\gamma_3| \leq C\Delta x\left(\alpha+\frac{1}{r}+\frac{\Delta x}{r^2}\right) \leq C\frac{\Delta x}{r}\left((1+\alpha r)+\frac{\Delta x}{r}(1+\alpha r)\right).$$

□

We are now left with the terms contained in $\Re$.

Estimate 3.5.

$$|\Re| \leq \Gamma(\alpha,r,r_0)\left(\frac{\varepsilon}{r_0\eta\alpha}+\left(\frac{\Delta x}{\varepsilon r}+\frac{\Delta x}{\alpha r}\right)\left(1+\frac{\Delta x}{r}\right)\right)$$

Proof. We start by the first term and continue until we have reached the end.

(3.13) Note that $|A(z)-A(u)| \geq \eta|z-u|$ and

$$\mathrm{sign}_\varepsilon(A(z)-A(u)) = \mathrm{sign}(u_j-u)\mathrm{sign}_\varepsilon(|A(z)-A(u)|)$$

for all $z \in \mathrm{int}(u,u_j)$. Let $\zeta(z) = \frac{\pi\eta}{2\varepsilon}|z-u|$. Then we have

$$\begin{aligned}||u_j-u|-\psi_\varepsilon(u_j,u)| &= \int_u^{u_j}\mathrm{sign}(u_j-u)-\mathrm{sign}_\varepsilon(A(z)-A(u))dz \\ &= \mathrm{sign}(u_j-u)\int_u^{u_j}1-\mathrm{sign}_\varepsilon(|A(z)-A(u)|)dz \\ &\leq \mathrm{sign}(u_j-u)\int_u^{u_j}1-\mathrm{sign}_\varepsilon(\eta|z-u|)dz \\ &= \frac{2\varepsilon}{\pi\eta}\int_0^{\zeta(u_j)}(1-\sin(\zeta))\chi_{\zeta\leq\pi/2}d\zeta \leq \frac{\varepsilon}{\eta}\end{aligned}$$

Consider $\|\varphi_t\|_{L^1(\Pi_T^2)}$. First differentiate to obtain

$$\iint_{\Pi_T^2}|\varphi_t|dX \leq \int_{\Pi_T}|\Psi_t|dxdt + \iint_{\Pi_T^2}\Psi\omega_r|(\omega_{r_0})_t|dX.$$

We may now estimate each of these terms.

$$\iint_{\Pi_T^2}\Psi\omega_r|(\omega_{r_0})_t|dX \leq C\frac{1}{r_0}\int_{\Pi_T}\Psi dxdt \leq \frac{C}{r_0}\left(1+\frac{1}{\alpha}\right),$$

$$\int_{\Pi_T} |\Psi_t| dxdt \leq \int_{\Pi_T} |(\chi^{\alpha_0})_t| \chi^\alpha dxdt + \int_{\Pi_T} \chi^{\alpha_0} |(\chi^\alpha)_t| dxdt$$

$$\chi^{\alpha_0}_{(\nu,\tau)}(t) = H_{\alpha_0}(t-\nu) - H_{\alpha_0}(t-\tau) \leq \int_0^T \int_{-(L+1/\alpha)}^{(L+1/\alpha)} (\omega_{\alpha_0}(t-\nu) + \omega_{\alpha_0}(t-\tau)) dxdt$$
$$+ M\alpha \int_{\Pi_T} \chi^{\alpha_0} \chi_\Lambda dxdt \leq 4\left(L + \frac{1}{\alpha}\right) + 2MT.$$

It follows that

$$\|\varphi_t\|_{L^1(\Pi_T^2)} \leq \frac{C}{r_0 \alpha}(1+r_0)(1+\alpha).$$

Hence

$$|(3.13)| \leq \Gamma(\alpha, r_0) \frac{\varepsilon}{r_0 \eta \alpha}$$

(3.14) This term may be estimated as (3.13). Note that $|\varphi_s| = \Psi \omega_r |(\omega_{r_0})_t|$ and thus by the above computations

$$|(3.14)| \leq \Gamma(\alpha) \frac{\varepsilon}{r_0 \eta \alpha}.$$

(3.15) This term cancels with the term (3.16). To see this we rewrite (3.16) according to the following equalities:

$$\begin{aligned}
\operatorname{sign}'_\varepsilon(A(u_j) - A(u)) A(u)_y D^+ A(u_j) &= -\left(\operatorname{sign}_\varepsilon(A(u_j) - A(u)) D^+ A(u_j)\right)_y \\
&= -(D^+ |A(u_j) - A(u)|_\varepsilon)_y \\
&\quad + (\operatorname{sign}_\varepsilon(A(\theta_j) - A(u)) - \operatorname{sign}_\varepsilon(A(u_j) - A(u)))_y D^+ A(u_j).
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{sign}'_\varepsilon(A(\tau_j) - A(u)) D^+ A(u_j) A(u)_y &= D^+ \operatorname{sign}_\varepsilon(A(u_j) - A(u)) A(u)_y \\
&= -(D^+ |A(u_j) - A(u)|_\varepsilon)_y.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\operatorname{sign}'_\varepsilon(A(\tau_j) - A(u)) - \operatorname{sign}'_\varepsilon(A(u_j) - A(u))\right) A(u)_y D^+ A(u_j) \\
&\quad = \left(\operatorname{sign}_\varepsilon(A(u_j) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(u))\right)_y D^+ A(u_j). \qquad (3.34)
\end{aligned}$$

By (3.34) and integration by parts the two terms cancels.

(3.17) To estimate this term note that $|\operatorname{sign}'_\varepsilon(z)| \leq \frac{C}{\varepsilon}$. So by the mean value inequality we obtain

$$|\operatorname{sign}_\varepsilon(A(u_j) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j) - A(u))| \leq C \frac{\Delta x}{\varepsilon} |D^+ A(u_j)| \qquad (3.35)$$

since $\theta_j \in \operatorname{int}(u_j, u_{j+1})$. We are left with a similar computation as in estimate 3.4. Hence

$$|(3.17)| \leq C \frac{\Delta x}{\varepsilon}\left(\alpha + \frac{1}{r} + \frac{\Delta x}{r^2}\right) \leq \Gamma(\alpha, r) \frac{\Delta x}{\varepsilon r}\left(1 + \frac{\Delta x}{r}\right).$$

(3.18) Let us first split (3.18) according to the following equality:

$$\begin{aligned}\operatorname{sign}'_\varepsilon(A(u_j)-A(u))\varphi-\operatorname{sign}'_\varepsilon(A(\tau_j)-A(u))\varphi^{\Delta x}\\ =\operatorname{sign}'_\varepsilon(A(u_j)-A(u))(\varphi-\varphi^{\Delta x})\\ +\left(\operatorname{sign}'_\varepsilon(A(u_j)-A(u))-\operatorname{sign}'_\varepsilon(A(\tau_j)-A(u))\right)\varphi^{\Delta x}.\end{aligned}\tag{3.36}$$

The first term on the right gives rise to the term

$$\iint_{\Pi_T^2}\operatorname{sign}'_\varepsilon(A(u_j)-A(u))(\varphi-\varphi^{\Delta x})(A(u)_y)^2dX$$
$$=\Delta x\iint_{\Pi_T^2}\operatorname{sign}_\varepsilon(A(u_j)-A(u))A(u)_yD^+\varphi dX.$$

Recall that $A(u)_y$ is bounded independently of Δx and η, and hence we may use (3.33) to obtain

$$|\iint_{\Pi_T^2}\operatorname{sign}'_\varepsilon(A(u_j)-A(u))(\varphi-\varphi^{\Delta x})(A(u)_y)^2dX|$$
$$\leq\Gamma(\alpha,r)\left(\frac{\Delta x}{\alpha r}\left(1+\frac{\Delta x}{r}\right)\right).$$

We now consider the second term on the right of (3.36). We would like to use equation (3.34), but the factor $D^+A(u_j)$ is missing. The key is to observe that whenever $D^+A(u_j)=0$ we have $u_j=u_{j+1}$. It follows since both τ_j and θ_j belongs to $\operatorname{int}(u_j,u_{j+1})$ that

$$\begin{aligned}\left(\operatorname{sign}'_\varepsilon(A(\tau_j)-A(u))-\operatorname{sign}'_\varepsilon(A(u_j)-A(u))\right)A(u)_y\\ =\left(\operatorname{sign}_\varepsilon(A(u_j)-A(u))-\operatorname{sign}_\varepsilon(A(\theta_j)-A(u))\right)_y.\end{aligned}\tag{3.37}$$

Using this equation and partial integration we obtain

$$\iint_{\Pi_T^2}\left(\operatorname{sign}'_\varepsilon(A(u_j)-A(u))-\operatorname{sign}'_\varepsilon(A(\tau_j)-A(u))\right)(A(u)_y)^2\varphi^{\Delta x}dX=$$
$$\iint_{\Pi_T^2}\left(\operatorname{sign}_\varepsilon(A(\theta_j)-A(u))-\operatorname{sign}_\varepsilon(A(u_j)-A(u))\right)(A(u)_y\varphi^{\Delta x})_ydX.$$

We may now apply (3.35). Since $D^+A(u_j)$ is bounded and both $A(u)_{yy}$ and $A(u)_y$ is in $L^1(\Pi_T)$ it follows that

$$\iint_{\Pi_T^2}|D^+A(u_j)||A(u)_{yy}\varphi^{\Delta x}+A(u)_y\varphi_y^{\Delta x}|dX\leq C\left(\frac{1}{r}+1\right).$$

Hence

$$|(3.18)|\leq\Gamma(\alpha,r,r_0)\left(\frac{\Delta x}{\alpha r}\left(1+\frac{\Delta x}{r}\right)\right)+\Gamma(r)\frac{\Delta x}{\varepsilon r}.$$

The result now follows by

$$|\Re|\leq|(3.13)|+|(3.14)|+|(3.17)|+|(3.18)|.$$

□

Let us turn back to inequality (3.26). Define κ by

$$\kappa_\mu(f) := \iint_{\Pi_T^2} f(x,t,y,s)\omega_{\alpha_0}(t-\mu)\chi^\alpha(x,t)\omega_r(x-y)\omega_{r_0}(t-s)dX.$$

By the triangle inequality we have

$$\begin{aligned}|u_j(x,t)-u(y,s)| &\le |u_j(x,t)-u(x,t)|+|u(x,t)-u(y,t)|+|u(y,t)-u(y,s)|,\\ |u_j(x,t)-u(x,t)| &\le |u_j(x,t)-u(y,s)|+|u(y,s)-u(y,t)|+|u(y,t)-u(x,t)|.\end{aligned}$$

Recall that $|u(\cdot,s)|_{BV}$ is bounded independently of η, s and Δx. Using lemma 5.2 and the L^1 Lipschitz continuity of u in the time variable we get the following bounds:

$$\kappa_\mu(|u(x,t)-u(y,t)|) \le Cr \quad \text{and} \quad \kappa_\mu(|u(y,s)-u(y,t)|) \le Cr_0.$$

It follows that

$$\begin{aligned}\kappa_\nu(|u_j(x,t)-u(y,s)|) &\le \kappa_\nu(|u_j(x,t)-u(x,t)|)+C(r+r_0),\\ \kappa_\tau(|u_j(x,t)-u(x,t)|) &\le \kappa_\tau(|u_j(x,t)-u(y,s)|)+C(r+r_0).\end{aligned} \tag{3.38}$$

We add $C(r+r_0)$ to both sides of (3.26). Using the inequalities (3.38) we obtain

$$\begin{aligned}\kappa_\nu(|u_j(x,t)-u(x,t)|)+2C(r+r_0)+\beta_2+|\gamma_1|+|\gamma_2|+\gamma_3\\ +\iint_{\Pi_T^2}|A(u_j)-A(u)|_\varepsilon(D^-D^+\varphi+2D^+\varphi_y+\varphi_{yy})dX+|\Re|\\ \ge \kappa_\tau(|u_j(x,t)-u(x,t)|).\end{aligned}$$

Combining all the estimates we get the following inequality:

$$\begin{aligned}\kappa_\tau(|u_j(x,t)-u(x,t)|) &\le \kappa_\nu(|u_j(x,t)-u(x,t)|)+C(r+r_0+\alpha)\\ &\quad+\Gamma(\alpha,\Delta x,r,r_0)\left(\frac{\varepsilon}{\alpha\eta r}+\frac{\varepsilon}{\alpha\eta r_0}+\frac{\Delta x}{r^2\alpha}+\frac{\Delta x}{\varepsilon r}\right)\left(1+\frac{\Delta x}{r}\right).\end{aligned}$$

Observe that we might let α_0 go to zero. Since

$$\kappa_\mu(|u_j(x,t)-u(x,t)|) \to \int |u_j(x,\mu)-u(x,\mu)|\chi^\alpha(x,\mu)dx$$

as $\alpha_0 \to 0$, we obtain

$$\begin{aligned}\int_{L_l(\tau)}^{L_r(\tau)}|u_j(x,\tau)-u(x,\tau)|dx &\le \int_{\mathbb{R}}|u_j(x,\nu)-u(x,\nu)|dx\\ +C(r+r_0+\alpha)+\Gamma(\alpha,\Delta x,r,r_0)&\left(\frac{\varepsilon}{\alpha\eta r}+\frac{\varepsilon}{\alpha\eta r_0}+\frac{\Delta x}{r^2\alpha}+\frac{\Delta x}{\varepsilon r}\right)\left(1+\frac{\Delta x}{r}\right).\end{aligned}$$

Let us pick a relation between the parameters. These are of course picked this way to optimize the convergence rate in the final result. Let $\alpha = r = r_0 = \sqrt{\eta}$, $\varepsilon = \alpha^5$ and $\Delta x = \alpha^7$. Why we pick η this way will become clear below. As Γ is increasing, and Δx is assumed to be smaller than some constant there is a constant C such that

$$\|u_j(\cdot,\tau)-u(\cdot,\tau)\|_{L^1(L_l(\tau),L_r(\tau))} \le \|u_j(\cdot,\nu)-u(\cdot,\nu)\|_{L^1(\mathbb{R})}+C\alpha.$$

Recall that $r_0 < \nu$. Let us pick $\nu = 2r_0$. Then the L^1 Lipschitz continuity of both u_j and u combined with lemma 5.1 implies that

$$\begin{aligned}
&\|u_j(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbb{R})} \leq \\
&\|u_j(\cdot,\nu) - u_j(\cdot,0)\|_{L^1(\mathbb{R})} + \|u_j(\cdot,0) - u(\cdot,0)\|_{L^1(\mathbb{R})} + \|u(\cdot,0) - u(\cdot,\nu)\|_{L^1(\mathbb{R})} \\
&\qquad\qquad \leq C(r_0 + \Delta x).
\end{aligned}$$

Therefore

$$\|u_j(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(L_l(\tau),L_r(\tau))} \leq C\alpha.$$

Let $u = u^\eta$ and $u_j = u_j^\eta$ and u denote the solution to (1.1). Then recall that

$$\|u(\cdot,\tau) - u^\eta(\cdot,\tau)\|_{L^1(\mathbb{R})} \leq C\sqrt{\eta}.$$

Using the triangle inequality we get

$$\|u(\cdot,\tau) - u_j^\eta(\cdot,\tau)\|_{L^1(L_l(\tau),L_r(\tau))} \leq C(\sqrt{\eta} + \alpha) = C\sqrt[7]{\Delta x}.$$

This finishes the proof.

4 An error estimate for the fully discrete approximation

The subject of this chapter is to obtain a similar result for the fully discrete approximation defined in section 2.2. The approach is exactly the same as for the semidiscrete approximation, and so the task is to track down the differences and the dependency on the discretization parameter Δt. Let $\{u_j^n\}$ satisfy the implicit monotone scheme (2.10) with $A = A^\eta$. That is

$$\begin{cases} D_t^- u_j^n + D_x^- F(u_j^n, u_{j+1}^n) = D_x^- D_x^+ A^\eta(u_j^n), & (j,n) \in \mathbb{Z} \times \mathbb{N}, \\ u_j^0 = \int_{I_j} u^0(x)dx, & j \in \mathbb{Z}. \end{cases}$$

Let $x_j = j\Delta x$ and $t^n = n\Delta t$. Then define the squares

$$C_j^n = \begin{cases} (x_{j-1/2}, x_{j+1/2}] \times (t^{n-1/2}, t^{n+1/2}] & \text{if } (j,n) \in \mathbb{Z} \times \mathbb{N}, \\ (x_{j-1/2}, x_{j+1/2}] \times [0, \Delta t/2] & \text{if } j \in \mathbb{Z} \text{ and } n = 0. \end{cases}$$

The discrete approximation is the stepfunction also denoted by u_j^n. That is

$$u_j^n(x,t) = \sum_j \sum_{n=0}^{\infty} u_j^n \chi_{C_j^n}(x,t).$$

Let us first state the result.

Theorem 4.1. *Suppose that u is the entropy solution to* (1.1) *and that u_j^n is the discrete approximation defined by* (2.10) *with $A = A^\eta$ where $\eta = (\Delta x)^{\frac{2}{7}}$. If u^0 satisfies* (i) *and* (ii) *from section 3, then there exists a constant C independent of Δx and Δt such that*

$$\|u(\cdot,t) - u_j^n(\cdot,t)\|_{L^1(L_l(t),L_r(t))} \le C \max\{\sqrt[7]{\Delta x}, \sqrt[3]{\Delta t}\}. \qquad t \in (0,T)$$

where $L_l(t) = -L + Mt, L_r(t) = L - Mt$, $M \ge \|f\|_{Lip}$ and $L \ge MT + \Delta x$.

The proof of this statement is provided in the rest of this chapter. It is very similar to the proof of theorem 3.1. In order to avoid repeating the steps already done in chapter 3, the approach is to give corresponding terms the same name and take the results obtained by merely interchanging u_j with u_j^n for granted. Define τ_j^n and θ_j^n analogous to the sequences defined in lemma 3.1. Let $u = u(y,s)$ be the viscous approximation defined by (1.2). We may rewrite the continuous equation as in section 3.1.1. Next, we want to reformulate (2.10) in a similar way as done in section 3.1.2 with the semidiscrete approximation. Using the Taylor series of $\psi_\varepsilon(\cdot,c)$ we obtain

$$D_t^- \psi_\varepsilon(u_j^n, c) = \psi_\varepsilon'(u_j^n, c) D_t^- u_j^n - \frac{\Delta t}{2} \psi_\varepsilon''(\zeta_j^n, c)(D_t^- u_j^n)^2 \tag{4.1}$$

for some $\zeta_j^n \in \operatorname{int}(u_j^n, u_j^{n-1})$. Multiply (2.10) with $\psi_\varepsilon'(u_j^n, c)\varphi$ and integrate in time and space. Using (4.1) we get

$$\begin{aligned} \int_{\Pi_T} D_t^- \psi_\varepsilon(u_j^n, c)\varphi + \psi_\varepsilon'(u_j^n, c) D_x^- F(u_j^n, u_{j+1}^n)\varphi dxdt \\ = \int_{\Pi_T} \psi_\varepsilon'(u_j^n, c) D_x^- D_x^+ A(u_j^n)\varphi dxdt \\ - \int_{\Pi_T} \frac{\Delta t}{2} \psi_\varepsilon''(\zeta_j^n, c)(D_t^- u_j^n)^2 \varphi dxdt. \end{aligned}$$

Except for the last term on the right, this equation is very similar to (3.6). By the same manipulations as in section 3.1.2 we obtain a similar equation. Adding up the two equations we get

$$\iint_{\Pi_T^2} |u_j^n - u|(D_t^+\varphi + \varphi_s) - (\psi_\varepsilon'(u_j^n, c)D_x^- F(u_j^n, u_{j+1}^n) + q(u,c)_y)\varphi dX \tag{4.2}$$

$$= \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon'(A(\tau_j^n) - A(u)) \left(D_x^+(A(u_j^n) - A(u)_y\right)^2 \varphi^{\Delta x} dX \tag{4.3}$$

$$- \iint_{\Pi_T^2} |A(u_j^n) - A(u)|_\varepsilon (D_x^- D_x^+ \varphi + 2D_x^+\varphi_y + \varphi_{yy}) dX \tag{4.4}$$

$$+ \iint_{\Pi_T^2} (|u_j^n - u| - \psi_\varepsilon(u_j^n, u)) D_t^+\varphi dX \tag{4.5}$$

$$+ \iint_{\Pi_T^2} (|u - u_j^n| - \psi_\varepsilon(u, u_j^n))\varphi_s dX \tag{4.6}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon(A(u_j^n) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j^n) - A(u))\big] D_x^+ A(u_j^n)\varphi_y^{\Delta x} dX \tag{4.7}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon'(A(\tau_j^n) - A(u)) - \operatorname{sign}_\varepsilon'(A(u_j^n) - A(u))\big] A(u)_y D_x^+ A(u_j^n)\varphi^{\Delta x} dX \tag{4.8}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon(A(u_j^n) - A(u)) - \operatorname{sign}_\varepsilon(A(\theta_j^n) - A(u))\big] D_x^+ A(u_j^n) D_x^+\varphi dX \tag{4.9}$$

$$+ \iint_{\Pi_T^2} \big[\operatorname{sign}_\varepsilon'(A(u_j^n) - A(u))\varphi - \operatorname{sign}_\varepsilon'(A(\tau_j^n) - A(u))\varphi^{\Delta x}\big] (A(u)_y)^2 dX \tag{4.10}$$

$$+ \iint_{\Pi_T^2} \frac{\Delta t}{2} \psi_\varepsilon''(\zeta_j^n, u)(D_t^- u_j^n)^2 \varphi dX. \tag{4.11}$$

4.1 Obtaining the inequality

Let us chose φ in the same way as in section 3.2 with one minor exception. Since we are using difference quotients instead of the ordinary derivative in the variable t, we do not need φ to be smooth in this variable. Hence we use $\chi_{[\nu,\tau]}(t)$ instead of $\chi^{\alpha_0}(t)$. Note that

$$D_t^+ \chi_{[\nu,\tau]}(t) = \delta_{\Delta t}(t - \nu) - \delta_{\Delta t}(t - \tau) \quad \text{where} \quad \delta_{\Delta t}(t) := \frac{1}{\Delta t}\chi_{[-\Delta t, 0]}(t).$$

Introduce the numerical entropy flux

$$Q_\varepsilon^c(u_j^n, u_{j+1}^n) = q_\varepsilon(u_j^n, c) + \int_{u_j^n}^{u_{j+1}^n} \psi_\varepsilon'(z, c) F_2'(z) dz,$$

and let

$$\Re := (4.5) + \cdots + (4.10).$$

Removing the positive terms on the right hand side and applying lemma 2.5 we obtain the inequality

$$\iint_{\Pi_T^2} |u_j^n - u|(D_t^+\varphi + \varphi_s) - (D_x^- Q_\varepsilon^u(u_j^n, u_{j+1}^n) + q_\varepsilon(u, u_j^n)_y)\varphi dX + \iint_{\Pi_T^2} |A(u_j^n) - A(u)|_\varepsilon (D_x^- D_x^+ \varphi + 2D_x^+\varphi_y + \varphi_{yy})dX \geq \Re. \quad (4.12)$$

Consider the term including the numerical entropy flux. We do the same computations as in the semidiscrete case and defines γ_1, γ_2 and γ_3 as in (3.21),(3.22) and (3.23). The inequality corresponding to (3.24) then takes the following form:

$$\begin{aligned}\iint_{\Pi_T^2} |u_j^n - u|(D_t^+\varphi + \varphi_s)dX &+ \iint_{\Pi_T^2} \operatorname{sign}_\varepsilon(A(u_j^n) - A(u))(f(u_j^n) - f(u))(D_x^+\varphi + \varphi_y)dX \\ &+ \iint_{\Pi_T^2} |A(u_j^n) - A(u)|_\varepsilon (D_x^- D_x^+\varphi + 2D_x^+\varphi_y + \varphi_{yy})dX + \gamma_3 \\ &\geq \gamma_1 + \gamma_2 + \Re. \quad (4.13)\end{aligned}$$

Consider the first term on the left in (4.13).

$$\begin{aligned}\iint_{\Pi_T^2} |u_j^n - u|(D_t^+\varphi + \varphi_s)dX &= \iint_{\Pi_T^2} |u_j^n - u|(D_t^+\Psi\omega^{\Delta t} + \Psi(D_t^+\omega - \omega_t))dX \\ &= \iint_{\Pi_T^2} |u_j^n - u| D_t^+\Psi\omega dX + \Delta t \iint_{\Pi_T^2} |u_j^n - u| D_t^+\Psi D_t^+\omega dX \\ &\quad + \iint_{\Pi_T^2} |u_j^n - u|\Psi(D_t^+\omega - \omega_t)dX =: T_1 + \zeta_1 + \zeta_2.\end{aligned}$$

Using Leibniz rule we obtain:

$$\begin{aligned}T_1 &= \iint_{\Pi_T^2} |u_j^n - u|\chi_t^\alpha \chi_{[\nu,\tau]}\omega dX + \iint_{\Pi_T^2} |u_j^n - u|(D_t^+\chi^\alpha - \chi_t^\alpha)\chi_{[\nu,\tau]}\omega dX \\ &\quad + \iint_{\Pi_T^2} |u_j^n - u|\chi^\alpha D_t^+\chi_{[\nu,\tau]}\omega dX + \Delta t \iint_{\Pi_T^2} |u_j - u| D_t^+\chi^\alpha D_t^+\chi_{[\nu,\tau]} dX \\ &=: \delta_3 + \zeta_3 + T_2 + \zeta_4.\end{aligned}$$

Furthermore

$$T_2 = \iint_{\Pi_T^2} |u_j^n - u|\chi^\alpha \delta_{\Delta t}(t-\nu)\omega dX - \iint_{\Pi_T^2} |u_j^n - u|\chi^\alpha \delta_{\Delta t}(t-\tau)\omega dX.$$

The terms β_1 and β_2 are defined as in (3.25). Collecting these computations we obtain the inequality

$$\begin{aligned}\iint_{\Pi_T^2} |u_j^n - u|\chi^\alpha \delta_{\Delta t}(t-\nu)\omega dX + \beta_2 + \sum_{k=1}^4 \zeta_k \\ + \iint_{\Pi_T^2} |A(u_j^n) - A(u)|_\varepsilon (D^- D^+ \varphi + 2D^+ \varphi_y + \varphi_{yy}) dX + \gamma_3 \\ \geq \iint_{\Pi_T^2} |u_j^n - u|\chi^\alpha \delta_{\Delta t}(t-\tau)\omega dX \\ - (\delta_3 + \beta_1) + \gamma_1 + \gamma_2 + \Re.\end{aligned}$$

As $M \geq \|f\|_{Lip}$ it follows that $\delta_3 + \beta_1 \leq 0$. Hence

$$\begin{aligned}\iint_{\Pi_T^2} |u_j^n - u|\chi^\alpha \delta_{\Delta t}(t-\nu)\omega dX + \beta_2 + |\gamma_1| + |\gamma_2| + \gamma_3 + \sum_{k=1}^4 \zeta_k \\ + \iint_{\Pi_T^2} |A(u_j^n) - A(u)|_\varepsilon (D^- D^+ \varphi + 2D^+ \varphi_y + \varphi_{yy}) dX + |\Re| \\ \geq \iint_{\Pi_T^2} |u_j^n - u|\chi^\alpha \delta_{\Delta t}(t-\tau)\omega dX. \quad (4.14)\end{aligned}$$

4.2 Finding the rate of convergence

Comparing (4.14) with (3.26) it is clear that there are only some minor changes. We consider these and take the results obtained by merely interchanging u_j with u_j^n for granted. Note that interchanging χ^{α_0} with $\chi_{[\nu,\tau]}$ amounts to taking the limit $\alpha_0 \to 0$. One may therefore apply the Lebesgue dominated convergence theorem to ensure that this interchange does not cause any problems. First consider some useful elementary estimates.

Lemma 4.1. *Let*

$$\Lambda_{\Delta t} = \{(x,t) | (L - Mt) - \Delta t/M \leq |x| \leq (L - Mt) + 1/\alpha + \Delta t/M\}.$$

Then the following estimates apply:

$$|D_t^+ \chi^\alpha| \leq M\alpha \chi_{\Lambda_{\Delta t}}, \quad (4.15)$$

$$|D_t^+ \omega_{r_0}| \leq \|\omega'\|_{L^\infty(\mathbb{R})} \frac{1}{r_0^2} \chi_{|s-t| \leq r_0 + \Delta t}, \quad (4.16)$$

$$|D_t^+ \omega_{r_0} - (\omega_{r_0})_t| \leq \|\omega''\|_{L^\infty(\mathbb{R})} \frac{\Delta t}{2r_0^2} \chi_{|s-t| \leq r_0 + \Delta t}, \quad (4.17)$$

$$\int |D_t^+ \chi^\alpha - \chi_t^\alpha| dt \leq 2\Delta t M\alpha \chi_{|x| \leq L + 1/\alpha}. \quad (4.18)$$

Let us start by considering the terms $\{\zeta_k\}_{k=1}^4$.

Estimate 4.1.

$$\sum_{k=1}^4 |\zeta_k| \leq \Gamma(\alpha, r_0, \Delta t) \frac{\Delta t}{\alpha r_0} \left(1 + \frac{\Delta t}{r_0}\right).$$

Proof. (ζ_1) Recall that $|u_j^n - u| \in L^\infty(\Pi_T^2)$ and so

$$|\zeta_1| \leq C\Delta t \frac{1}{r_0^2}(r_0 + \Delta t)\|D_t^+ \Psi\|_{L^1(\Pi_T)}.$$

Consider $\|D_t^+ \Psi\|_{L^1(\Pi_T)}$. Using Leibniz rule we obtain two integrals that may be estimated separately.

$$\iint_{\Pi_T} |D_t^+ \chi^\alpha (\chi_{[\nu,\tau]})^{\Delta t}| dxdt \leq T(M + 2\alpha\Delta t),$$
$$\iint_{\Pi_T} \chi^\alpha |D_t^+ \chi_{[\nu,\tau]}| dxdt \leq \int_{-L-1/\alpha}^{L+1/\alpha} \int_0^T |D_t^+ \chi_{[\nu,\tau]}| dtdx \leq 4\left(L + \frac{1}{\alpha}\right).$$

It follows that $\|D_t^+ \Psi\|_{L^1(\Pi_T)} \leq \Gamma(\alpha, \Delta t)\frac{1}{\alpha}$. Hence

$$|\zeta_1| \leq \Gamma(\alpha, \Delta t)\frac{\Delta t}{r_0 \alpha}\left(1 + \frac{\Delta t}{r_0}\right).$$

(ζ_2) By the above lemma

$$\iint_{\Pi_T^2} |u_j^n - u|\Psi(D_t^+ \omega - \omega_t) dX \leq C\frac{\Delta t}{r_0^2}(r_0 + \Delta t)\iint_{\Pi_T} \Psi dxdt.$$

Hence

$$|\zeta_2| \leq \Gamma(\alpha)\frac{\Delta t}{r_0 \alpha}\left(1 + \frac{\Delta t}{r_0}\right).$$

(ζ_3) This also follows by the above lemma.

$$|\zeta_3| \leq \iint_{\Pi_T^2} |u_j^n - u||D_t^+ \chi^\alpha - \chi_t^\alpha|\chi_{[\nu,\tau]}\omega dX$$
$$\leq C\iint_{\Pi_T} |D_t^+ \chi^\alpha - \chi_t^\alpha| dtdx \leq \Gamma(\alpha)\Delta t.$$

(ζ_4) Consider the L^1 norm of $D_t^+ \chi^\alpha D_t^+ \chi_{[\nu,\tau]}$.

$$\|D_t^+ \chi^\alpha D_t^+ \chi_{[\nu,\tau]}\|_{L^1(\Pi_T)}$$
$$\leq M\alpha \iint_{\Pi_T} \chi_{\Lambda_{\Delta t}}(\delta_{\Delta t}(t - \nu) + \delta_{\Delta t}(t - \tau)) dxdt$$
$$\leq 4(M + \alpha\Delta t).$$

Hence

$$|\zeta_4| \leq \Gamma(\alpha, \Delta t)\Delta t.$$

The proof follows by collecting all the terms. □

Let us look at the terms in $\Re$. Considering estimate 3.5 we observe that the computations apply in this case as well with one exception. In the term (4.5) we have exchanged φ_t with $D_t^+ \varphi$ so we need to find a bound on $\|D_t^+ \varphi\|_{L^1(\Pi_T^2)}$.

$$\|D_t^+ \varphi\|_{L^1(\Pi_T^2)} \leq \|D_t^+ \Psi\|_{L^1(\Pi_T)} + \frac{C}{r_0^2}(r_0 + \Delta t)\int_{\Pi_T} \Psi dxdt.$$

By the above computations

$$\|D_t^+\varphi\|_{L^1(\Pi_T^2)} \leq \Gamma(\alpha, r_0)\frac{1}{r_0\alpha}\left(1+\frac{\Delta t}{r_0}\right).$$

The above discussion implies the following result:

Estimate 4.2.

$$|\Re| \leq \Gamma(\alpha, r, r_0)\left(\frac{\varepsilon}{r_0\eta\alpha}\left(1+\frac{\Delta t}{r_0}\right)+\left(\frac{\Delta x}{\varepsilon r}+\frac{\Delta x}{\alpha r}\right)\left(1+\frac{\Delta x}{r}\right)\right).$$

Considering $\gamma_1, \gamma_2, \gamma_3$, the double derivative term (4.4) and β_2 we may apply the same estimates as in the semidiscrete case with u_j interchanged with u_j^n. Let

$$\kappa_\mu(f) := \iint_{\Pi_T^2} f(x,t,y,s)\delta_{\Delta t}(t-\mu)\chi^\alpha(x,t)\omega_r(x-y)\omega_{r_0}(t-s)dX.$$

By the same argument as in the semidiscrete case

$$\begin{aligned}\kappa_\nu(|u_j^n(x,t)-u(x,t)|) + 2C(r+r_0) + \beta_2 + |\gamma_1| + |\gamma_2| + \gamma_3 + \sum_{k=1}^{4}\zeta_k \\ + \iint_{\Pi_T^2} |A(u_j^n)-A(u)|_\varepsilon(D_x^- D_x^+\varphi + 2D_x^+\varphi_y + \varphi_{yy})dX + |\Re| \\ \geq \kappa_\tau(|u_j^n(x,t)-u(x,t)|). \quad (4.19)\end{aligned}$$

Combining the estimates we get

$$\begin{aligned}\kappa_\tau(|u_j^n(x,t)-u(x,t)|) \leq \kappa_\nu(|u_j^n(x,t)-u(x,t)|) + C(r+r_0+\alpha) \\ + \Gamma(\alpha, \Delta x, r, r_0)\left(\frac{\varepsilon}{\alpha\eta r}+\frac{\varepsilon}{\alpha\eta r_0}\left(1+\frac{\Delta t}{r_0}\right)+\frac{\Delta x}{r^2\alpha}+\frac{\Delta x}{\varepsilon r}\right)\left(1+\frac{\Delta x}{r}\right) \\ + \Gamma(\alpha, r_0, \Delta t)\frac{\Delta t}{\alpha r_0}\left(1+\frac{\Delta t}{r_0}\right).\end{aligned}$$

Let $\alpha = r = r_0 = \sqrt{\eta}$, $\varepsilon = \alpha^5$. Let $\alpha = \max(\sqrt[7]{\Delta x}, \sqrt[3]{\Delta t})$. Then $\alpha^7 \geq \Delta x$ and $\alpha^3 \geq \Delta t$. Recall that Δx is less than some constant. We may assume that the same applies to Δt. Thus there exists a constant C such that

$$\kappa_\tau(|u_j^n(x,t)-u(x,t)|) \leq \kappa_\nu(|u_j^n(x,t)-u(x,t)|) + C\alpha. \quad (4.20)$$

In the semidiscrete case we used the Lipschitz continuity of both u_j and u to finish the estimate. Since u_j^n is piecewise constant in the time variable we need a slightly different approach. Assume that $\mu = t^n + \Delta t/2$ for some $n \in \mathbb{N}$. Observe that $u_j^n(x,t)$ does not depend on t for $t \in (\mu - \Delta t, \mu]$, so by the L^1 Lipschitz continuity of u and the reversed triangle inequality

$$\begin{aligned}|\kappa_\mu(|u_j^n(x,t)-u(x,t)|\chi_{(L_l(\mu),L_r(\mu))}(x)) - \|u_j^n(\cdot,\mu)-u(\cdot,\mu)\|_{L^1(L_l(\mu),L_r(\mu))}| \\ \leq \frac{1}{\Delta t}\int_{\mu-\Delta t}^{\mu} |\|u_j^n(\cdot,t)-u(\cdot,t)\|_{L^1(L_l(\mu),L_r(\mu))} - \|u_j^n(\cdot,\mu)-u(\cdot,\mu)\|_{L^1(L_l(\mu),L_r(\mu))}|dt \\ \leq \frac{1}{\Delta t}\int_{\mu-\Delta t}^{\mu} \|u(\cdot,t)-u(\cdot,\mu)\|_{L^1(L_l(\mu),L_r(\mu))}dt \leq C\Delta t.\end{aligned}$$

Furthermore

$$|\kappa_\mu(|u_j^n(x,t) - u(x,t)|\chi_{(L_l(\mu),L_r(\mu))}(x)) - \kappa_\mu(|u_j^n(x,t) - u(x,t)|)| \leq C\Delta t.$$

Assume that $\tau = t^n + \Delta t/2$ and $\nu = t^m + \Delta t/2$ where $n, m \in \mathbb{N}$. Then by the above estimates and (4.20) we obtain:

$$\|u_j^n(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(L_l(\tau),L_r(\tau))} \leq \|u_j^n(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbb{R})} + C\alpha.$$

The next observation provides a useful substitution for the lack of L^1 Lipschitz continuity.

$$\|u_j^n(\cdot,t) - u_j^n(\cdot,s)\|_{L^1(\mathbb{R})} \leq C(|t-s| + \Delta t) \qquad t, s \geq 0. \tag{4.21}$$

This follows by observing that there exist integers $p, q \geq 0$ such that $|t^p - t| \leq \Delta t/2, |t^q - s| \leq \Delta t/2, u_j^n(\cdot,t) = u_j^n(\cdot,t^p)$ and $u_j^n(\cdot,s) = u_j^n(\cdot,t^q)$. By lemma 2.3

$$\|u_j^n(\cdot,t) - u_j^n(\cdot,s)\|_{L^1(\mathbb{R})} = \|u_j^p - u_j^q\|_1 \leq C(\Delta t|p-q|) \leq C(|s-t| + \Delta t).$$

Recall that we assumed $r_0 < \nu$. Let ν be such that $r_0 < \nu \leq r_0 + \Delta t$. By the triangle inequality, (4.21), lemma 5.1 and the L^1 Lipschitz continuity of u

$$\|u_j^n(\cdot,\nu) - u(\cdot,\nu)\|_{L^1(\mathbb{R})} \leq C(r_0 + \Delta t + \Delta x). \tag{4.22}$$

Hence

$$\|u_j^n(\cdot,\tau) - u(\cdot,\tau)\|_{L^1(L_l(\tau),L_r(\tau))} \leq C\alpha.$$

By the same reasoning as we used to obtain (4.22) we may show that this inequality applies to any $\tau \in (0,T)$. The theorem now follows by (1.6) as in the semidiscrete case.

5 Appendix

Here I have collected some results which I decided not to include in the main text. This is either because they are elementary results written down to convince the author, or because they serve rather as background than as a part of the proof.

Lemma 5.1. *Let* $x_j = j\Delta x$ *and*

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u^0(x)dx, \qquad I_j = (x_{j-1/2}, x_{j+1/2}).$$

Then

1. $|u_j^0|_{BV} \leq |u^0|_{BV}$
2. $\|u_j^0\|_{L^\infty} \leq \|u^0\|_{L^\infty}$
3. $\|u_j^0 - u^0\|_{L^1(\mathbb{R})} \leq \frac{1}{4}|u^0|_{BV}\Delta x$

Proof. Let

$$g(\xi) = \begin{cases} \int_{\mathbb{R}} \frac{|u^0(x+\xi)-u^0(x)|}{|\xi|}dx & \xi \neq 0 \\ 0 & \xi = 0 \end{cases}$$

and note that $|u^0|_{BV} = \|g\|_\infty$.

1.

$$\begin{aligned} \sum_j |u_{j+1}^0 - u_j^0| &= \sum_j \frac{1}{\Delta x}|\int_{I_{j+1}} u^0 dx - \int_{I_j} u^0 dx| \\ &= \sum_j \frac{1}{\Delta x}|\int_{I_j} (u^0)^{\Delta x} - u^0 dx| \leq \sum_j \int_{I_j} \frac{|(u^0)^{\Delta x} - u^0|}{\Delta x}dx \\ &= g(\Delta x) \leq |u^0|_{BV} \end{aligned}$$

2. This is obvious.

3.

$$\begin{aligned} \int_{\mathbb{R}} |u_j^0 - u^0|dx &= \int_{\mathbb{R}} \frac{1}{\Delta x}|\int_{-\Delta x/2}^{\Delta x/2} u^0(x+\xi) - u^0(x)d\xi|dx \\ &\leq \int_{\mathbb{R}} \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} |u^0(x+\xi) - u^0(x)|d\xi dx \\ &= \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} |\xi|g(\xi)d\xi \leq \frac{\Delta x}{4}|u^0|_{BV} \end{aligned}$$

□

Lemma 5.2. *Let* $u \in BV(\mathbb{R})$ *and* ω *be a mollifier. Then*

$$\iint |u(x) - u(y)|\omega_r(x-y)dxdy \leq (|u|_{BV})r.$$

Proof. Let

$$g(\xi) = \begin{cases} \int_{\mathbb{R}} \frac{|u(x+\xi)-u(x)|}{|\xi|} dx & \xi \neq 0 \\ 0 & \xi = 0 \end{cases}$$

and note that $|u|_{BV} = \|g\|_{\infty}$. Let $x = y + \xi$ and observe that

$$\iint |u(x) - u(y)|\omega_r(x-y)dxdy = \iint |u(y+\xi) - u(y)|\omega_r(\xi)dyd\xi$$
$$= \int |\xi| g(\xi)\omega_r(\xi)d\xi \leq (|u|_{BV})r$$

□

Lemma 5.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with lipschitz constant K. Then*

$$|\int_{\mathbb{R}} f(t)\delta_{\Delta t}(t-\mu)dt - f(\mu)| \leq \frac{K}{2}\Delta t$$

for any $\mu \in \mathbb{R}$ and $\delta_{\Delta t}(t) = \frac{1}{\Delta t}\chi_{[-\Delta t, 0]}(t)$.

Proof.

$$|\int_{\mathbb{R}} f(t)\delta_{\Delta t}(t-\mu)dt - f(\mu)| = \frac{1}{\Delta t}|\int_{\mu-\Delta t}^{\mu} f(t) - f(\mu)dt|$$
$$\leq \frac{K}{\Delta t}\int_0^{\Delta t} tdt \leq \frac{K}{2}\Delta t.$$

□

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