THE CONNES-MARCOLLI GL$_2$-SYSTEM
MASTER THESIS

Bjarte D. Berntsen

Fifteen years ago Bost and Connes constructed a C$^*-$dynamical system with the Calois group $G(\mathbb{Q}^{ab}/\mathbb{Q})$ as symmetry group and with phase transition related to properties of L-functions . Since then there have been numerous , and only partially succesful , attempts to generalize the system to arbitrary number fields . A few years ago , in order to extend that construction to imaginary quadratic feilds , Connes and Marcolli constructed a GL$_2-$system , an analogue of the BC-system with $\mathbb{Q}^*$ replaced by $GL_2(\mathbb{Q})$. They classified the $KMS_\beta-$states of the system for $\beta > 2$. Later Laca , Larsen and Neshveyev classified the $KMS_\beta-$states for all $\beta \neq 0, 1$.

1. Proper Gruppevirkning og Gruppoide C*-Algebraer

Let G be a group and X be a set. A Group action on a set X is a homomorphism ρ from the group G to the group Homeo(X) of all homeomorphisms from X to itself ($Aut(X)$). Thus to each $g \in G$ is associated a homeomorphism $\rho(g) : X \to X$, which for notational simplicity we write simply as $g : X \to X$.With this notation for the map :

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & gx \end{array}$$

with conditions :

$$\begin{gathered} (e, x) \longmapsto ex = x, \forall x \epsilon X \\ (g, (hx)) = (gh)x, \forall x \epsilon X, \forall g, h \epsilon G \end{gathered} \tag{1}$$

are equivalent to requiring ρ to be a homomorphism.

Under these conditions , we say that X is a left G-set and we have a left group action by G on X. Similarly , one can define a right action by letting the elements of the group act on the space from the right instead.

We consider cases where the group G is countable and X is a locally compact second countable topological space.

A continous map $f : X \to Y$ is called proper if for every compact $K \subset Y$, the space $f^{-1}(K)$ is also compact. Accordingly , an action of G on X is called proper if the map :

$$\begin{aligned} G \times X &\to X \times X \\ (g, x) &\to (gx, x) \end{aligned} \tag{2}$$

is proper. Then the space G/X , where points are identitied by the equivalence relation of laying on the same G- orbit {Gx} is Hausdorff. Assume that G is a discrete group. Consider $G \times X$. The space X , which is a G-space is called the unit space of $G \times X$. $G \times X$ has the product topology and the two maps , called the source map (s) and the range map (r) :

$$\begin{aligned} S, R \quad &: \quad G \times X \to X \\ s(g, x) &\longmapsto x \\ r(g, x) &\longmapsto gx \end{aligned}$$

define a law of composition : $((g, y), (h, x))\epsilon(G \times X)^2 \mapsto (g, y) \cdot (h, x)\epsilon G \times X$, where :

$$(G \times X)^2 := \{((g, y), (h, x))\epsilon(G \times X) \times (G \times X)|\, r(h, x) = s(g, y) = y\}$$

We see that the product on $G \times X$ are defined by the formula :

$$(g, hx)(h, x) = (gh, x)$$

In this way $G \times X$ becomes a groupoid (called the transformation groupoid) , since every element has an inverse :

$$(g, x)^{-1} = (g^{-1}, gx)$$

$G \times X$ has stabilizer subgroup $G_x = \{g\epsilon G|\, gx = x\}$ If G has stabilizer subgroup equal to $\{e\}$ for every x in X is equivalent to saying that the action of G on X is free i.e. an action whitout fixpoints for other elements of G than the identity.

The set $C_c(G \times X)$ of all continous functions on $G \times X$ with compact support has a structure of involutive algebra given by :

$$\begin{aligned}(f_1 \star f_2)(g,x) &= \sum_{h \epsilon G} f_1(gh^{-1}, hx) f_2(h,x) \\ f^*(g,x) &= (f((g,x)^{-1}\overline{)}) = (f(g^{-1}, gx)\bar{)}\end{aligned}$$

, where $(g,x)^{-1} = (g^{-1},gx)$ Let $C_0(X)$ be the algebra of continous functions on X that vanish at infinity. The product in $C_0(X)$ is the usual pointwise product.

If the restriction of the action to a subgroup Γ of G is free and proper , we can introduce a new groupoid : $\Gamma \backslash G \times_\Gamma X$ by taking the quotient of $G \times X$ by the action of $\Gamma \times \Gamma$ defined by :

$$(\gamma_1, \gamma_2)(g,x) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 x)$$

The unit space of $\Gamma \backslash G \times_\Gamma X$ is $\Gamma \backslash X$, and the product is induced from that on $G \times X$. If the action of Γ is proper but not free , the quotient space $\Gamma \backslash G \times_\Gamma X$ is no longer a groupoid , since the composition of classes using representatives will in general depend on the choice of representatives. Nevertheless , the same formulas for convolution and involution as in the groupoid case give us a well -defined algebra. To see this , consider the space $C_c(\Gamma \backslash G \times_\Gamma X)$ of continous compactly supported functions on $\Gamma \backslash G \times_\Gamma X$. The elements can be considered as $(\Gamma \times \Gamma)$ -INVARIANT functions on $G \times X$. The convolution of two such functions are defined accordingly :

1. (1.1)

$$(f_1 * f_2)(g,x) = \sum_{h \in \Gamma \backslash G} f_1(gh^{-1}, hx) f_2(h,x).$$

To see that the convolution is well-defined :

Assume the support of f_i is contained in $(\Gamma \times \Gamma)(\{g_i\} \times U_i)$, where $g_i \in G$ and U_i is a compact subset of X.(i=1,2). Let $\{\gamma_1, ..., \gamma_n\}$ be the set of elements $\gamma \in \Gamma$ such that $\gamma g_2 U_2 \cap U_1 \neq \emptyset$.This set is finite since the action of Γ is assumed to be proper.

If $f_2(h,x) \neq 0$, then there exist $\gamma \in \Gamma$ such that $h\gamma^{-1} \in \Gamma g_2$ and $\gamma x \in U_2$. Since the number of γ´s such that $\gamma x \in U_2$ is finite , the above sum must be finite. If furthermore $f_1(gh^{-1}, hx) \neq 0$, then $gh^{-1} = \gamma_a g_1 \gamma_b^{-1}$ for some γ_a, γ_b

, since (gh^{-1}, hx) is

contained in the support of f_1 . We can replace h by another representative of the right coset Γh . If we replace h by $\gamma_b h$, then $gh^{-1} = \gamma_a g_1 \in \Gamma g_1$, and also $hx \in U_1$. If now $h\gamma^{-1} = \tilde{\gamma} g_2$ with $\tilde{\gamma} \in \Gamma$, we get $hx = \tilde{\gamma} g_2 \gamma x \in \tilde{\gamma} g_2 U_2$.

Hence $\tilde{\gamma}$ must be equal to γ_i , for some i , and therefore $g \in \Gamma g_1 h = \Gamma g_1 \gamma_i g_2 \gamma$. Thus the support of $f_1 * f_2$ is contained in $\cup_i (\Gamma \times \Gamma)(\{g_1 \gamma_i g_2\} \times U_2)$. Thus the set of representatives γg_i giving a nonzero contribution to the above sum are finite and independent of the choice of $\gamma \in \Gamma$. The support of $f_1 * f_2$ is

contained in a compact set , so $f_1 * f_2 \in C_c(\Gamma \backslash G \times_\Gamma X)$, and the latter space becomes a well-defined algebra. The convolution is also associative :

$$\begin{aligned} (f_1 * (f_2 * f_3))(g,x) &= \sum_{t \in \Gamma \backslash G} f_1(gt^{-1}, tx)(f_2 * f_3)(t,x) = \sum_{t,h \in \Gamma \backslash G} f_1(gt^{-1}, tx) f_2(th^{-1}, hx) f_3(h,x) \\ ((f_1 * f_2) * f_3)(g,x) &= \sum_{h \in \Gamma \backslash G} (f_1 * f_2)(gh^{-1}, hx) f_3(h,x) = \sum_{t,h \in \Gamma \backslash G} f_1(gh^{-1}t^{-1}, thx) f_2(t, hx) f_3(h, x \end{aligned}$$

1. (1.2) Define also an *involution* on $C_c(\Gamma \backslash G \times_\Gamma X)$ by :

$$f^*(g,x) = f((g,x)^{-1}\bar{)} = f(g^{-1}, gx\bar{)}$$

If the support of f is contained in $(\Gamma \times \Gamma)(\{g_0\} \times U)$ for $g_0 \in G$ and compact $U \subset X$, then the support of f^* is contained in :

$$((\Gamma \times \Gamma)(\{g_0\} \times U))^{-1} = (\Gamma \times \Gamma)(\{g_0\} \times U)^{-1} = (\Gamma \times \Gamma)(\{g_0^{-1}\} \times g_0 U) \ ,$$

which is a compact set in $(\Gamma \backslash G \times_\Gamma X)$ and therefore $f^* \in C_c(\Gamma \backslash G \times_\Gamma X)$ for every $f \in C_c(\Gamma \backslash G \times_\Gamma X)$.

For each $x \in X$, define a representation :

1. (1.3)

$$\begin{aligned} \pi_x &: C_c(\Gamma \backslash G \times_\Gamma X) \longrightarrow B(l^2(\Gamma \backslash G)) \\ \pi_x(f)\delta_{\Gamma h} &= \sum_{g \in \Gamma \backslash G} f(gh^{-1}, hx)\delta_{\Gamma g} \end{aligned}$$

Here $\delta_{\Gamma g}$ denotes the characteristic function of the coset Γg . Consider $\delta_{\Gamma g}$ as a one of the unit basis vectors in the (standard) orthonormal basis $\{\delta_{\Gamma g}\}_{g\in\Gamma\backslash G}$ for $l^2(\Gamma\backslash G)$.

Lemma 1 1.1 *For each* $f \in C_c(\Gamma\backslash G \times_\Gamma X)$ *the operators* $\pi_x(f)$, $x \in X$, *are uniformly bounded.*

Proof. For $\xi_1, \xi_2 \in l^2(\Gamma\backslash G)$ we have :

$$
\begin{aligned}
|\langle \pi_x(f)\cdot\xi_1, \xi_2\rangle| &\le \sum_{g,h\in\Gamma\backslash G} \left|f(gh^{-1}, hx)\right| \cdot |\xi_1(h)| \cdot |\xi_2(g)| \\
&\le \left(\sum_{g,h\in\Gamma\backslash G} \left|f(gh^{-1}, hx)\right| \cdot |\xi_1(h)|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{g,h\in\Gamma\backslash G} \left|f(gh^{-1}, hx)\right| \cdot |\xi_2(g)|^2\right)^{\frac{1}{2}} .
\end{aligned}
$$

(Applying Hølders inequality.)

Thus if we denote by $\|f\|_I$ the quantity :

$$
\max\left\{\sup_{x\in X, h\in G} \sum_{g\in\Gamma\backslash G} \left|f(gh^{-1}, hx)\right|, \sup_{x\in X, g\in G} \sum_{h\in\Gamma\backslash G} \left|f(gh^{-1}, hx)\right|\right\},
$$

we get$\|\pi_x(f)\| \le \|f\|_I$ for any $x \in X$, so it suffices to show that $\|f_I\|$ is finite. Replacing x by $h^{-1}x$ and g by gh in the first supremum above , we see that this supremum equals :

$$
\|f\|_{I,s} := \sup_{x\in X} \sum_{g\in\Gamma\backslash G} |f(g,x)|
$$

As $f^*(hg^{-1}, gx) = (f((hg^{-1}, gx)^{-1}))^* = (f(gh^{-1}, hg^{-1}gx))^* = (f(gh^{-1}, hx)^*$, we see that $f(gh^{-1}, hx) = (f^*(hg^{-1}, gx))^*$. Then the second supremum must be equal to $\|f^*\|_{I,s}$. Therefore $\|f\|_I = \max\left\{\|f\|_{I,s}, \|f^*\|_{I,s}\right\}$. Now , the claim is that $\|f\|_{I,s}$ is finite for every $f \in C_c(\Gamma\backslash G \times_\Gamma X)$. If this claim is true , the Lemma is proved.

Proof of Claim :

We may assume without loss of generality that the support of f is contained in $(\Gamma \times \Gamma)(\{g_0\} \times U)$ for some $g_0 \in G$, and compact $U \subset X$. Since the action of Γ is proper , there exists $n \in \mathbb{N}$ such that the sets $\gamma_i U$, $i = 1, ..., n+1$ have trivial intersection for any different $\gamma_1, ..., \gamma_{n+1} \in \Gamma$. Now if $f(g,x) \neq 0$ for some g and x,there exists $\gamma \in \Gamma$ such that $g\gamma^{-1} \in \Gamma g_0$ and $\gamma x \in U$. Since the number of γ's such that $\gamma x \in U$ is at most n , we see that for each $x \in X$ the sum in the definition of $\|f\|_{I,s}$ is finite .

To see that π_x is a representation , one has to check :

$$\begin{aligned} \iota)\ \pi_x(f^*) &= (\pi_x(f))^* \\ \iota\iota)\ \pi_x(f_1 * f_2) &= \pi_x(f_1) \cdot \pi_x(f_2) \end{aligned}$$

$$\pi_x(f) \cdot \delta_{\Gamma h} = \sum_{g \in \Gamma \backslash G} f(gh^{-1}, hx) \cdot \delta_{\Gamma g}$$

Consider $f \in C_c(\Gamma \backslash G \times_\Gamma X)$. As $f = \sum_{s \in \Gamma \backslash G} f(s,x) \cdot \delta_{\Gamma s}$ observe that the vector $\xi = \delta_{\Gamma e}$ in $l^2(\Gamma \backslash G)$ is both cyclic and tracial for operators in $B(l^2(\Gamma \backslash G))$

$$\begin{aligned} \langle U_g \delta_{\Gamma e}, \delta_e \rangle &= 1 \text{ if } g = e \text{ , 0 else ,} \\ \text{therefore , for all } g_i &\in G \text{ we have :} \\ \langle U_{g_1} U_{g_2} \delta_{\Gamma e}, \delta_{\Gamma e} \rangle &= \langle U_{g_1} U_{g_2} \delta_{\Gamma e}, \delta_{\Gamma e} \rangle \end{aligned}$$

$\iota)$

$$(\pi_x(f)\delta_{\Gamma h}, \delta_{\Gamma t}) = \sum_{g \in \Gamma \backslash G} f(gh^{-1}, hx)(\delta_{\Gamma g}, \delta_{\Gamma t}) = f(th^{-1}, hx)$$

similarly ,

$$(\delta_{\Gamma h}, \pi_x(f^*)\delta_{\Gamma t}) = f^*(ht^{-1}, tx\bar{)} = f(th^{-1}, hx).$$

Hence

$$\pi_x(f)^* = \pi_x(f^*).$$

$\iota\iota)$ Can checked similarly to associativity of the convolution. ■

But let me see this from another perspective :

For each $x \in X$, f can be thought of as a vector in $l^2(\Gamma \backslash G)$) . Let U_g be the unitary operator on $l^2(\Gamma \backslash G)$ defined by $U_g \delta_{\Gamma h} = \delta_{\Gamma h g^{-1}}$. Expanding on the cyclic and tracial vector $\delta_{\Gamma e}$ gives :

$$f = (\sum_{g \in \Gamma \backslash G} f(g,x) U_g^*) \delta_{\Gamma e}.$$

For each $x \in X$, f can be thought of as a vector in $l^2(\Gamma\backslash G)$) , expanding its adjoint on the cyclic and tracial vector $\delta_{\Gamma e}$ gives :

$$f^* = (\sum_{g\in\Gamma\backslash G} U_g(f^*(g,x)))\delta_{\Gamma e} = \sum_{g\in\Gamma\backslash G} U_g(f^*(g,x)U_g^* U_g \delta_{\Gamma e} = \sum_{g\in\Gamma\backslash G} U_g(f^*(g,x))U_g^* \delta_{\Gamma g^{-1}}$$

Proof. Let f_1 and f_2 be two functions in $C_c(\Gamma\backslash G \times_\Gamma X)$. Then , for arbitrary $h \in \Gamma\backslash G$:

$$\begin{aligned}
(\pi_x(f_1 * f_2))\delta_{\Gamma h} &= \sum_{g\in\Gamma\backslash G} (f_1 * f_2)(gh^{-1}, hx) \cdot \delta_{\Gamma g} = \sum_{g\in\Gamma\backslash G} \sum_{t\in\Gamma\backslash G} f_1(gh^{-1}t^{-1}, thx) \cdot f_2(t, hx) \cdot \delta_{\Gamma g} \\
&= \sum_{g\in\Gamma\backslash G} \sum_{t\in\Gamma\backslash G} U_h f_1(gt^{-1}, thx) \cdot f_2(t, hx) \cdot \delta_{\Gamma g} = \sum_{g\in\Gamma\backslash G} \sum_{t\in\Gamma\backslash G} U_h f_1(gt^{-1}, thx) \cdot U_h^{-1} \\
&= \sum_{g\in\Gamma\backslash G} \sum_{t\in\Gamma\backslash G} U_h f_1(gt^{-1}, thx) \cdot U_h^* \cdot f_2(th^{-1}, hx) \cdot \delta_{\Gamma g} = \sum_{t\in\Gamma\backslash G} \sum_{g\in\Gamma\backslash G} U_h f_1(gt^{-1}, th \\
&= \sum_{t\in\Gamma\backslash G} U_h(\pi_{hx}(f_1))U_h^* \cdot \delta_{\Gamma t} \cdot f_2(th^{-1}, hx) = (\pi_x(f_1) \cdot \pi_x(f_2)) \cdot \delta_{\Gamma h}
\end{aligned}$$

so $\iota\iota$) is checked in this way of thinking . Hence π_x is a representation for every (fixed) $x \in X$. ■

Definition 2 *We denote by* $C_r^*(\Gamma\backslash G \times_\Gamma X)$ *the completion of* $C_c(\Gamma\backslash G \times_\Gamma X)$

in the norm defined by the representation :

$$(\oplus_{x\in X}\pi_x) : C_c(\Gamma\backslash G \times_\Gamma X) \to B(\oplus_{x\in X} l^2(\Gamma\backslash G)$$

, that is ,

$$\|f\| = \sup_{x\in X} \|\pi_x(f)\|$$

Remark 3 *As we observed above , for every* $s \in G$ *and its associated unitary* $U_s \in B(l^2(\Gamma\backslash G))$ *such that* $U_s\delta_{\Gamma h} = \delta_{\Gamma hs^{-1}}$ *,* $f \in C_c(\Gamma\backslash G \times_\Gamma X)$ *and* π_x *the representation defined above , we have*

$$U_s \pi_x(f) U_s^* = \pi_{sx}(f).$$

Proof. Observe first that , for every g , s and h in $\Gamma\backslash G$, we have :

$$\begin{aligned} U_s\pi_x(f)U_s^* \cdot \delta_{\Gamma h} &= \pi_x(f)\cdot\delta_{\Gamma hs} = U_s \cdot \sum_{g\in\Gamma\backslash G} f(gs^{-1}h^{-1}, hsx)\cdot\delta_{\Gamma g} = \sum_{g\in\Gamma\backslash G} U_s \cdot f(gs \\ \sum_{g\in\Gamma\backslash G} f(gs^{-1}h^{-1}, hsx)\cdot\delta_{\Gamma gs^{-1}} &= {}^{g\rightarrow l=gs^{-1}} \sum_{l\in\Gamma\backslash G} f(lh^{-1}, hsx)\cdot\delta_{\Gamma l} \\ &\Updownarrow \\ U_s\pi_x(f)U_s^* \cdot \delta_{\Gamma h} &= \sum_{g\in\Gamma\backslash G} f(gh^{-1}, hsx)\cdot\delta_{\Gamma h} = \pi_{sx}(f)\cdot\delta_{\Gamma h}\ . \end{aligned}$$

Hence

$$U_s\pi_x(f)U_s^* = \pi_{sx}(f)$$

Therefore

$$\|\pi_x(f)\| = \|\pi_{sx}(f)\|$$

, and so ■

Remark 4

$$\|f\| = \sup_{x\in G\backslash X} \|\pi_x(f)\|$$

Closely related is the notion of a C^*-*dynamical system* (A, G, α) , where A is a C^*-algebra, G a locally compact group and α is a homomorphism from G into $Aut(A)$. A covariant representation of (A, G , α) is a pair (π, U) , where π is a *-representation of A on a Hilbertspace H and

$$s \longmapsto U_s$$

is a unitary representation of G on the same H such that :

$$U_s\pi(A)U_s^* = \pi(\alpha_s(A))\ ,$$

for all $a \in A$, $s \in G$.

Denote by α_g the automorphism $\alpha(g)$ for g in G. The Cross Product , $A \rtimes_\alpha G$ of a C*-algebra A and a group G is the universal C*-algebra generated by A and unitaries v_g , $g\ \epsilon G$ such that :

$$\begin{aligned} 1)\ v_g a v_g^* &= \alpha_g(a) \\ 2)\ g &\longmapsto v_g \text{ is a homomorphism , } g\ \epsilon\ G \end{aligned}$$

If G is countable and discrete , the space $C_c(A,G)$ of continous compactly supported A-valued functions on G is the algebra of all finite sums :

$$f = \sum_{t \in G} A_t \cdot v_t$$

with coefficients in A.

One defines a C^*-norm by :

$$\|f\| = \sup_{\sigma} \|\sigma(f)\|$$

, as σ runs over all *-representations of $C_c(A,G)$.

The supremum is always bounded by :

$$\|f\|_1 = \sum_{t \in G} \|A_t\|$$

The supremum is always taken over a nonempty family of representations because certain representations can be explicitly constructed. Let π be any *-representation of A on a Hilbertspace H.Then one can always construct the representation :

$$\begin{aligned} \tilde{\pi} &: A \rtimes_{\alpha} G \rightarrow B(H \otimes l^2(G) = B(H)\bar{\otimes}B(l^2(G)) \\ \tilde{\pi}(a)(\xi \otimes \delta_g) &= \pi(\alpha_g^{-1}(a))(\xi \otimes \delta_g) \\ \tilde{\pi}(v_g)(\xi \otimes \delta_h) &= \xi \otimes \delta_{gh} \ , \end{aligned}$$

for $\xi \in H$ and $g, h \in G$.

Due to constuction , this representation is covariant :

$$\begin{aligned} \tilde{\pi}(v_g) \cdot \tilde{\pi}(a) \cdot (\tilde{\pi}(v_g))^*(\xi \otimes \delta_h) &= \tilde{\pi}(v_g) \cdot \tilde{\pi}(a)(\xi \otimes \delta_{g^{-1}h}) = \tilde{\pi}(v_g)(\alpha_{h^{-1}g}(a)\xi \otimes \delta_{g^{-1}h}) \\ &= \pi(\alpha_{h^{-1}g}(a))(\xi \otimes \delta_h) = \tilde{\pi}(\alpha_g(a))(\xi \otimes \delta_h) \end{aligned}$$

hence

$$\tilde{\pi}(v_g) \cdot \tilde{\pi}(a) \cdot \tilde{\pi}(v_g)^* = \tilde{\pi}(\alpha_g(A))$$

The Reduced Cross-Product , $A \rtimes_{\alpha r} G$ is defined to be : $= (A \rtimes_{\alpha} G))/Ker(\tilde{\pi})$, where π is any faithful representation of A.

The functions $f \in C_c(\Gamma \backslash G \times_{\Gamma} X)$ can be considered as $(\Gamma \times \Gamma)$-invariant functions on $G \times X$. Define an action of G on $C_c(\Gamma \backslash G \times_{\Gamma} X)$ by :

$$\alpha_g(f) = f(h, (g^{-1}x)).$$

Define for each $g \in G$ the following unitaries v_g on $C_c(\Gamma\backslash G \times_\Gamma X)$:

$$\begin{aligned} v_g f(s,x) &= f(sg, g^{-1}x) \\ f(s,x)v_g^* &= f(s,x)v_{g^{-1}} = f(sg^{-1}, x) \end{aligned}$$

For these

$$v_g f(s,x) v_g^* = f(s, g^{-1}x) \ ,$$

and as we have seen , $C_0(\Gamma\backslash X)$ can be considered as a subalgebra of $C_c(\Gamma\backslash G \times_\Gamma X)$, so we have a C^*-dynamical system $(C_0(\Gamma\backslash X), G, \alpha)$.

Now , for each $x \in X$, define a map :

$$\begin{aligned} \tilde{\pi}_x &: C_c(\Gamma\backslash G \times_\Gamma X) \rtimes_\alpha G \to B(l^2(\Gamma\backslash G) \otimes l^2(G)) \approxeq B(l^2(\Gamma\backslash G)) \otimes B(l^2(G)) \\ \tilde{\pi}_x(f)(\delta_{\Gamma h} \otimes \delta_g) &= \pi_x(\alpha_{g^{-1}}(f))\delta_{\Gamma h} \otimes \delta_g \\ \tilde{\pi}_x(v_g)(\delta_{\Gamma l} \otimes \delta_h) &= \delta_{\Gamma l} \otimes \delta_{gh} \end{aligned}$$

By the calculation above , this representation is covariant for any Hilbertspace H to which $C_0(\Gamma\backslash X) \subset C_c(\Gamma\backslash G \times_\Gamma X)$ can be represented on, so also for $l^2(\Gamma\backslash G)$. With U_s the unitary operator defined above , observe that :

$$(U_s \pi_x(f) U_s^* \otimes 1)(\xi \otimes \delta_s) = (\pi_{sx}(f) \otimes 1)(\xi \otimes \delta_s) = (\pi_x(\alpha_s^{-1}(f)) \otimes 1)(\xi \otimes \delta_s) = \tilde{\pi}_x(f)(\xi \otimes \delta_s)$$

, for $f \in C_0(\Gamma\backslash X) \subset C_c(\Gamma\backslash G \times_\Gamma X)$ and $\xi \in l^2(\Gamma\backslash G)$. Then we have :

$$\begin{aligned} \tilde{\pi}_x(f)(\xi \otimes \delta_g) &= \pi_x(\alpha_{g^{-1}}(f))(\xi \otimes \delta_g) = (U_g \pi_x(f) U_g^*)\xi \otimes \delta_g = (U_g \otimes 1)(\pi_x(f) \otimes 1)(U_g^* \otimes 1)(\xi \otimes \\ \tilde{\pi}_x(v_g)(\xi \otimes \delta_h) &= (1 \otimes v_g)(\xi \otimes \delta_h) = \xi \otimes \delta_{gh} \end{aligned}$$

and we get :

$$\begin{aligned} \tilde{\pi}_x(v_g) \cdot \tilde{\pi}_x(f) \cdot (\tilde{\pi}_x(v_g))^*(\delta_{\Gamma l} \otimes \delta_h) &= \tilde{\pi}_x(v_g) \cdot \tilde{\pi}_x(f) \cdot (\tilde{\pi}_x(v_g))^*(\delta_{\Gamma l} \otimes \delta_h) \\ &= (1 \otimes v_g) \cdot \tilde{\pi}_x(f) \cdot (1 \otimes v_g^*)(\delta_{\Gamma l} \otimes \delta_h) \\ &= (1 \otimes v_g) \cdot \tilde{\pi}_x(f)(\delta_{\Gamma l} \otimes \delta_{g^{-1}h}) = (1 \otimes v_g)(U_{g^{-1}h} \otimes 1)(\pi_x(f) \\ &= (1 \otimes v_g) \cdot (U_{g^{-1}h} \pi_x(f) U_{h^{-1}g} \otimes 1)(\delta_{\Gamma l} \otimes \delta_{g^{-1}h}) \\ &= (\pi_x(\alpha_{h^{-1}g}(f)) \otimes 1)(\delta_{\Gamma l} \otimes \delta_h) = (\pi_x(\alpha_{h^{-1}}(\alpha_g(f))) \otimes 1)(\delta_{\Gamma l} \\ &= \tilde{\pi}_x(\alpha_g(f))(\delta_{\Gamma l} \otimes \delta_h) \end{aligned}$$

From this we conclude that $\tilde{\pi}_x$, for every $x \in X, \tilde{U}_g = (1 \otimes v_g)$ and hence also $(\oplus_{x \in X} \tilde{\pi}_x) := \tilde{\pi}$, $\tilde{U}$ is a covariant representation of $(C_0(\Gamma\backslash X), G, \alpha)$.$(C_0(\Gamma\backslash X)$ can be considered as a subalgebra of $C_c(\Gamma\backslash G \times_\Gamma X)$. The embedding : $X \hookrightarrow G \times X$, $x \mapsto (e, x)$. In this way $\Gamma\backslash X$ is an open subset of $\Gamma\backslash G \times_\Gamma X$, and then the algebra $C_0(\Gamma\backslash X)$ is a subalgebra of $C_r^*(\Gamma\backslash G\times_\Gamma X)$.) Then , by the Universal Property of the Crossed Product , there exists a representation σ of $C_0(\Gamma\backslash X)\rtimes_\alpha G$ into $C^*(\tilde{\pi}(C_0(\Gamma\backslash X)),\tilde{U}_g,\ g\in G)$ obtained by setting $\sigma(f)=\tilde{\pi}(f(s,x))\cdot\tilde{U}_s$, for $f=f(s,x)\cdot\delta_{\Gamma s}=f(s,x)\cdot U_s^*\cdot\delta_{\Gamma e}$.

Observe that $\|f\| := \sup_{x\in X}\|\pi_x(f)\| = \sup_{x\in X}\|\pi_x(\alpha_g(f)\| = \sup_{x\in X}\|\pi_x(f(\cdot,g^{-1}x)\|$ by replacing x by gx , since for every $x\in X$: $\|\pi_{gx}(f)\| = \left\|U_g\pi_x(f)U_g^*\right\|$, where U_g is the unitary operator on $l^2(\Gamma\backslash G)$: $U_g\delta_{\Gamma h}=\delta_{\Gamma hg^{-1}}$. Therefore , and since $\|\pi_x(f)\|=\|\tilde{\pi}_x(f)\|$, for every $x\in X$, i conclude that the kernel of the representation $\tilde{\pi}_x$ is isomorphic to G , since $s\mapsto\alpha_s$ is a homomorphism and $\ker\tilde{\pi}=\bigcap\limits_{x\in X}\ker\tilde{\pi}_x=\bigcap\limits_{x\in X}G=G$.

By the universal property of $C^*(\tilde{\pi}(C_0(\Gamma\backslash X)),\tilde{U}_g,\ g\in G):=\mathrm{A}$, there is a Homomorphism H from this algebra onto $C_0(\Gamma\backslash X)\rtimes_\alpha G$ taking $\tilde{\pi}(f)\in B(\oplus_{x\in X}(\mathbb{C}\delta_x\otimes l^2(\Gamma\backslash G)\otimes l^2(G)))$ to $f\in C_0(\Gamma\backslash X)$ and $\tilde{U}_g$ to U_g^*.

(The point is that . . . . the composed map $C_c(\Gamma\backslash G\times_\Gamma X)\to C_0(\Gamma\backslash X)\rtimes_{\alpha r}G$ extends to an isomorphism : $C_r^*(\Gamma\backslash G\times_\Gamma X)\to C_0(\Gamma\backslash X)\rtimes_{\alpha r}G$. I will import a diagram above here to clarify this .)

$C_r^*(\Gamma\backslash G\times_\Gamma X)$ is the completion of $C_c(\Gamma\backslash G\times_\Gamma X)$ with respect to the norm defined by the representation $\pi=(\oplus_{x\in X}\pi_x)$, $\|f\|=\sup_{x\in X}\|\pi_x(f)\|_{l^2}$. Then by the first iso thm , $C_r^*(\Gamma\backslash G\times_\Gamma X)\approxeq C_0(\Gamma\backslash G)\rtimes_{\alpha r}G$.

For the special case when $\Gamma=\{e\}$, we have the following :

Claim 5 *$C_r^*(G\times X)$ is isomorphic to $C_0(X)\rtimes_{\alpha r}G$.*

Proof. For each $x\in X$, define a map :

$$\begin{array}{rcl} \Pi_x & : & C_c(G\times X)\to B(l^2(G)\otimes l^2(G))\approxeq B(l^2(G))\otimes B(l^2(G)) \\ \Pi_x(f)(\delta_h\otimes\delta_g) & = & \pi_x(\alpha_{g^{-1}}(f))\delta_h\otimes\delta_g \\ \Pi_x(v_g)(\delta_l\otimes\delta_h) & = & \delta_l\otimes\delta_{gh} \end{array}$$

where $\alpha_g(f)(x)=f(g^{-1}x)$, for $f\in C_0(X)$.

Lemma 6 *1.2*

There exists a conditional expectation

$$E : C_r^*(\Gamma\backslash G \times_\Gamma X) \to C_0(\Gamma\backslash X)$$

such that :

$$E(f)(x) = f(e,x) \ ,$$

for $f \in C_c(\Gamma\backslash G \times_\Gamma X)$ *.*

Proof. *If* $B \subset A$ *are* C^*-*Algebraes , a map* $E : A \to B$ *is called a Conditional Expectation if :* ι) E *is a projection onto* B*.i.e.* $(E(x) = x \ , \forall x \in B)$ $\iota\iota$) E *is* $B-$*bilinear :* $E(xy) = E(x)y$ *and* $E(yx) = yE(x)$ *, for all* $x \in A$ *,* $y \in B$ *and* $\iota\iota\iota$) E *is Positive.*

For each $x \in X$ *define a state* ω_x *on* $C_r^*(\Gamma\backslash G \times_\Gamma X)$ *by :*

$$\omega_x(a) = (\pi_x(a) \cdot \delta_\Gamma, \delta_\Gamma).$$

Then the function $E(a)$ *on* X *defined by :*

$$E(a)(x) = \omega_x(a)$$

is bounded by $\|a\|$*. As* $E(f)(x) = f(e,x)$ *, for* $f \in C_c(\Gamma\backslash G \times_\Gamma X)$ *(since* $\omega_x(f) = (\pi_x(f)\delta_\Gamma, \delta_\Gamma) = (\sum_{s\in\Gamma\backslash G} f(s,x) \cdot \delta_{\Gamma s}, \delta_{\Gamma e}) = f(e,x)$ *) , we conclude that* $E(a) \in C_0(\Gamma\backslash X)$ *for every* $a \in C_r^*(\Gamma\backslash G \times_\Gamma X)$*.Thus* E *is such a conditional expectation.* ■

■

The Boxproduct ⊠

Let $Y \subset X$ be a Γ-invariant clopen subset ($\Gamma Y \subset Y$).Then the characteristic function $1_{\Gamma\backslash Y}$ of the set $\Gamma\backslash Y$ is an element of the multiplier Algebra of $C_r^*(\Gamma\backslash G \times_\Gamma X)$. See this by using the embedding $X \hookrightarrow G \times X$, $x \mapsto (e,x)$, to consider $\Gamma\backslash X$ as an open subset of $\Gamma\backslash G \times_\Gamma X$, and then the algebra $C_0(\Gamma\backslash X)$ as a subalgebra of $C_r^*(\Gamma\backslash G \times_\Gamma X)$.

Denote by $\Gamma\backslash G \boxtimes_\Gamma Y$ the quotient of the space :

$$\{(g,x) \ , \ g \in G \ , \ x \in Y \ , \ gx \in Y\}$$

- by the action of $\Gamma \times \Gamma$:

$$(\gamma_1, \gamma_2)(g,x) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 x)$$

Then

$$1_{\Gamma\backslash Y} C_c(\Gamma\backslash G \times_\Gamma X) 1_{\Gamma\backslash Y} = C_c(\Gamma\backslash G \boxtimes_\Gamma Y).$$

Therefore the algebra $1_{\Gamma\backslash Y} C_r^*(\Gamma\backslash G \times_\Gamma X) 1_{\Gamma\backslash Y}$, which we denote $C_r^*(\Gamma\backslash G \boxtimes_\Gamma Y)$ is a completion of the algebra of compactly supported functions on $\Gamma\backslash G \boxtimes_\Gamma Y$ with convolution product given by :

$$(f_1 * f_2)(g, y) = \sum_{h \in \Gamma\backslash G : hy \in Y} f_1(gh^{-1}, hy) \cdot f_2(h, y)$$

,

and involution :

$$f^*(g, y) = f(g^{-1}, gy\bar{)}$$

Observe that $\pi_x(1_{\Gamma\backslash Y})$ is the projection onto the subspace $l^2(\Gamma\backslash G_x)$, where the subset G_x of G is defined by :

$$G_x = \{g \in G |\ gx \in Y\}$$

Then , for $f \in C_c(\Gamma\backslash G \boxtimes_\Gamma Y)$ and $h \in G_x$ we have :

$$\pi_x(f)\delta_{\Gamma h} = \sum_{g \in \Gamma\backslash G_x} f(gh^{-1}, hx)\delta_{\Gamma g}$$

So if $x \notin GY$, $\pi_x(f) = 0$ in particular. We saw above that the representations π_x and π_{gx} are unitarily equivalent for any $g \in G$.Therefore we can conclude that $C_r^*(\Gamma\backslash G \boxtimes_\Gamma Y)$ is the completion of $C_c(\Gamma\backslash G \boxtimes_\Gamma Y)$ in the norm

$$\|f\| = \sup_{y \in Y} \|\pi_y(f)\| .$$

Hecke Pairs

Consider the algebra $C_r^*(\Gamma\backslash G \times_\Gamma X)$. Our next goal is to show that under an extra assumtion on the pair (G, Γ) , the multiplier algebra contains other interessting elements in addition to the Γ-invariant functions on X.

The pair (G, Γ) is called a Hecke pair if Γ and $g\Gamma g^{-1}$ are commensurable for any $g \in G$. That $(\Gamma, g\Gamma g^{-1})$ are commensurable means that $\Gamma \bigcap g\Gamma g^{-1}$

$\subset \Gamma$ is a subgroup of finite index. Equivalently , every double coset of Γ contains finitely many right (and left) cosets of Γ , i.e. :

$$R_\Gamma(g) := |\Gamma \backslash \Gamma g \Gamma| < \infty \ ,$$

for any $g \in G$.

If (G, Γ) is a Hecke pair , the space $H(G, \Gamma)$ of finitely supported functions on $\Gamma \backslash G / \Gamma$ is a $*$-algebra with product :

$$(f_1 * f_2)(g) = \sum_{h \in \Gamma \backslash G} f_1(gh^{-1}) f_2(h),$$

and involution :

$$f^*(g) = \bar{f(g^{-1})}.$$

We can consider the functions $f \in H(G, \Gamma)$ as bounded operators on the Hilbertspace $l^2(\Gamma \backslash G)$ represented as :

$$f \cdot \delta_{\Gamma h} = \sum_{g \in \Gamma \backslash G} f(gh^{-1}) \cdot \delta_{\Gamma g}$$

The corresponding completion is called the reduced Hecke C^*-algebra of (G, Γ) and denoted by $C_r^*(G, \Gamma)$. Denote by $[g]$ the characteristic function of the double coset $\Gamma g \Gamma$, considered as an element of the Hecke algebra.

The elements of $H(G, \Gamma)$ may be considered as continous functions on $\Gamma \backslash G \times_\Gamma X$. Although these functions are not compactly supported in general , the formulas defining the $*-$algebra structure and the regular representation of $H(G, \Gamma)$ coincide with (1.2)-(1.4).

Moreover , the convolution of an element of $H(G, \Gamma)$ with a compactly supported function on $\Gamma \backslash G \times_\Gamma X$ gives a compactly supported function : If $f_1 = [g_1]$, and the support of $f_2 \in C_c(\Gamma \backslash G \times_\Gamma X)$ is contained in $(\Gamma \times \Gamma)(\{g_2\} \times U)$ for a compact $U \subset X$, then the support of $f_1 * f_2$ is contained in $(\Gamma \times \Gamma)(g_1 \Gamma g_2 \times U)$. Since $\Gamma \backslash \Gamma g_1 \Gamma g_2$ is finite , we see that $f_1 * f_2$ is compactly supported on $\Gamma \backslash G \times_\Gamma X$. Therefore , we have :

Lemma 7 *1.3*

If (G, Γ) is a Hecke pair , then the reduced Hecke C^*-algebra $C_r^*(G, \Gamma)$ is contained in the multiplier algebra of the C^*-algebra $C_r^*(\Gamma \backslash G \times_\Gamma X)$.

2. Dynamics and KMS-states

Assume as above that we have an action of G on X such that the action of $\Gamma \subset G$ is proper , and $Y \subset X$ is a Γ-invariant ($\Gamma Y \subset Y$) clopen set. Assume now that we are given a homomorphism :

$$N : G \rightarrow \mathbb{R}_+^* = (0, +\infty)$$

such that Γ is contained in the kernel of N. We define a one-parameter group of automorphisms of $C_r^*(\Gamma \backslash G \times_\Gamma X)$ by :

$$\sigma_t(f)(g,x) = N(g)^{it} \cdot f(g,x)$$

, for $f \in C_c(\Gamma \backslash G \times_\Gamma X)$. More precisely : We denote by $\bar{N}$ the selfadjoint operator on $l^2(\Gamma \backslash G)$ defined by :

$$\bar{N} \cdot \delta_{\Gamma g} = N(g) \cdot \delta_{\Gamma g}$$

Since $\bar{N}$ is selfadjoint (easy to check) , then by applying functional calculus for bounded operators on Hilbertspace with $f_t(z) = z^{it}$, the operator $\bar{N}^{it}$ $\in B(l^2(\Gamma \backslash G))$ is unitary , implementing the dynamics σ_t spatially by its associated unitary operator $(\oplus_{x \in X} \bar{N}^{it})$ on $(\oplus_{x \in X} l^2(\Gamma \backslash G))$.

In other words ,

$$\pi_x(\sigma_t(a)) = \bar{N}^{it} \pi_x(a) \bar{N}^{-it}$$

for all $x \in X$. See this by considering the operatoraction as represented on $l^2(\Gamma \backslash G)$:

$$\begin{aligned}
\pi_x(\sigma_t(f)) \cdot \delta_{\Gamma h} &= \sum_{g \in \Gamma \backslash G} \sigma_t(f)(gh^{-1}, hx) \cdot \delta_{\Gamma g} \\
&= \sum_{g \in \Gamma \backslash G} N(gh^{-1})^{it} \cdot f(gh^{-1}, hx) \cdot \delta_{\Gamma g} = \sum_{g \in \Gamma \backslash G} N(g)^{it} N(h^{-1})^{it} f(gh^{-1}, hx) \cdot \delta_{\Gamma g} \\
&= \sum_{g \in \Gamma \backslash G} N(g)^{it} N(h)^{-it} f(gh^{-1}, hx) \cdot \delta_{\Gamma g} = \sum_{g \in \Gamma \backslash G} N(g)^{it} f(gh^{-1}, hx) N(h)^{-it} \cdot \delta_{\Gamma g} \\
&= \sum_{g \in \Gamma \backslash G} \bar{N}^{it} f(gh^{-1}, hx) N(h)^{-it} \cdot \delta_{\Gamma g} = \bar{N}^{it} \pi_x(f) \bar{N}^{-it} \cdot \delta_{\Gamma h}.
\end{aligned}$$

A semifinite $\sigma-$invariant weight φ is called a $\sigma - KMS_\beta-$weight if , or equivalently , it satisfies the $\sigma - KMS$ condition at inverse temperatures $\beta \in \mathbb{R}$ if :

$$\varphi(aa^*) = \varphi(\sigma_{i\beta/2}(a)^* \sigma_{i\beta/2}(a)),$$

for any $\sigma-$analytic element a.(An element is called $\sigma-$analytic if the map $\mathbb{R} \to C_r^*(\Gamma\backslash G \times_\Gamma X)$, $t \mapsto \sigma_t(a)$ extends to an analytic map $\mathbb{C} \to C_r^*(\Gamma\backslash G \times_\Gamma X)$. A map $f : \mathbb{C} \to C_r^*(\Gamma\backslash G \times_\Gamma X)$ is called analytic if $\varphi \circ f$ is an analytic function for any $\varphi \in (C_r^*(\Gamma\backslash G \times_\Gamma X))^*$.)

If φ is finite , then the $KMS-$condition is equivalent to

$$\varphi(xy) = \varphi(y\sigma_{i\beta}(x)) ,$$

for any $\sigma-$analytic x, y. This follows from

$$\varphi(y\sigma_{i\beta}(x)) \ = \varphi(\sigma_{-i\beta/2}(y)\sigma_{i\beta/2}(x)) \ = \varphi(\sigma_{i\beta/2}(y^*)^*\sigma_{i\beta/2}(x))$$

and the identity :

$$xy = \frac{1}{4}((x+y^*)(x+y^*)^* - (x-y^*)(x-y^*)^* + i(x+iy^*)(x+iy^*)^* - i(x-iy^*)(x-iy^*)^*).$$

The following result will be the basis of our analysis of $KMS-$weights.

Proposition 8 2.1 Assume the action of G on X is an action without fixpoints (free action) , so that in particular $\Gamma\backslash G \boxtimes_\Gamma Y$ is a genuine groupoid. Then for any $\beta \in \mathbb{R}$, there exists a one-to-one correspondence between $\sigma - KMS_\beta$ weights φ *on* $C_r^*(\Gamma\backslash G \boxtimes_\Gamma Y)$ with domain of definition containing $C_c(\Gamma\backslash Y)$ Radon measures μ on Y such that

$$\mu(gZ) = N(g)^{-\beta}\mu(Z)$$

,

for every $g \in G$ and every compact subset $Z \subset Y$ such that $gZ \subset Y$. Namely , such a measure μ is $\Gamma-$invariant , so it determines a measure ν *on* $\Gamma\backslash Y$ such that :

$$\int_Y f(y)d\mu(y) = \int_{\Gamma\backslash Y} \left(\sum_{y \in p^{-1}(\{t\})} f(y) \right) d\nu(t)$$

for $f \in C_c(Y)$, where $p : Y \to \Gamma\backslash Y$ is the quotient map , and the associated weight φ is given by

$$\varphi(a) = \int_{\Gamma\backslash Y} E(a)(x)d\nu(x) ,$$

where E is the conditional expectation from Lemma 1.2.

Proof. For $\Gamma = \{e\}$ the result is well-known , see e.g. [19, Proposition II.5.4] . For arbitrary Γ , a way to argue is as follows :

Since the action of Γ on Y is free , the quotient space $\Gamma\backslash G \boxtimes_\Gamma Y$ is an etale groupoid. In fact it is an etale equivalence relation on $\Gamma\backslash Y$, or an r-discrete principial groupoid in the terminology of [19] .To veryfy this , we have to check that the isotropy group of every point in $\Gamma\backslash Y$ is trivial , *that is* , if $g \in G$ is such that $gy \in Y$ and $p(gy) = p(y)$, for some $y \in Y$, then (g, y) belongs to the $(\Gamma \times \Gamma)$ - orbit of (e, y). But on the other hand , if $p(gy) = p(y)$, there exist $\gamma \in \Gamma$ such that $\gamma gy = y$. Then $\gamma g = e$, since the action of G is free , and therefore $(g, y) = (\gamma^{-1}, e)(e, y)$. Then by [19, Proposition 11.5.4] ,

$\sigma - KMS_\beta-$weights with domain of definition containing $C_c(\Gamma\backslash Y)$ on the C^*-algebra $C_r^*(\Gamma\backslash G \boxtimes_\Gamma Y)$ of the etale equivalence relation are in *ono-to-one* correspondence with Radon measures ν on $\Gamma\backslash Y$ with Radon-Nikodym cocycle $(p(y), p(gy)) \longmapsto N(g)^\beta$.

This means that :

If we assume Y_0 is an open subset of Y such that the map $p : Y \to \Gamma\backslash Y$ is injective on Y_0 , and $g \in G$ is such that $gY_0 \subset Y$. Define an injective map

$$\begin{aligned} \tilde{g} &: p(Y_0) \to p(gY_0) \\ \text{by } \tilde{g}(p(y)) &\mapsto p(gy) \end{aligned}$$

for $y \in Y_0$, and let $\tilde{g}_*\nu$ be the push-forward of the measure ν under the map $\tilde{g}$, which again means that : $\tilde{g}_*\nu(Z) = \nu(\tilde{g}^{-1}(Z))$, for $Z \subset p(gY_0)$. Then :

$$\frac{d\tilde{g}_*\nu}{d\nu} = N(g)^\beta \text{ on } p(gY_0).$$

Therefore ; if we denote by μ the $\Gamma-$invariant measure on Y corresponding to ν via (2.2 below) , then to say that the Radon-Nikodym cocycle of ν is $(p(y), p(gy)) \mapsto N(g)^\beta$ is the same as saying that μ satisfies : $\mu(gZ) = N(g)^{-\beta}\mu(Z)$, *for every* $g \in G$ *and every compact subset* $Z \subset Y$ *such that* $gZ \subset Y$.(the scaling condition).

Recall that a Radon measure on Y is a Borel measure which is finite on compact sets , outer regular (*) on all Borel sets , and inner regular(**) on all open sets. Then , by The Riesz Representation Theorem , for each positive linear functional , and hence also for each $\sigma - KMS_\beta-$weight with domain of definition containing $C_c(\Gamma\backslash Y)$ on the C^*-algebra $C_r^*(\Gamma\backslash G \boxtimes_\Gamma Y)$, there exist an unique Radon measure ν on $\Gamma\backslash Y$ such that $\varphi(f) = \int f \, d\nu$, for $f \in C_c(\Gamma\backslash Y)$, φ a $\sigma - KMS_\beta-$weight. This establishes the *one-to-one* correspondence above. ■

The next lemma is about extension of the Radon measure μ from Y to GY :

Lemma 9 *2.2. If μ is a measure on Y as in Proposition 2.1 , then it extends uniquely to a Radon measure on $GY \subset X$ satisfying (2.1) for $Z \subset GY$ and $g \in G$.*

Proof. We can choose Borel subsets $Y_i \subset Y$ and elements $g_i \in G$ such that $GY = \sqcup_i g_i^{-1} Y_i$, where $\sqcup$ denotes disjoint union. There is only one choice for a measure extending μ and satisfying (2.1) on GY , namely , for a Borel subset $Z \subset GY$ let

$$\mu(Z) = \sum_i N(g_i)^\beta \mu(g_i Z \cap Y_i).$$

To show that $\mu(Z)$ is independent of any choices and that the extension satisfies (2.1) , assume $GY = \sqcup_j h_j Z_j$ for some $h_j \in G$ and Borel $Z_j \subset Y$. Let $g \in G$. Then :

$$\begin{aligned}
\sum_i N(g_i)^\beta \mu(g_i g Z \cap Y_i) &= \sum_i N(g_i)^\beta \cdot \sum_j \mu(g_i g Z \cap Y_i \cap g_i g h_j^{-1} Z_j) \\
&= \sum_i N(g_i)^\beta \cdot \sum_j N(g_i g h_j^{-1})^{-\beta} \mu(h_j Z) \cap h_j g^{-1} g_i^{-1} Y_i \cap Z_j) \\
&= N(g)^{-\beta} \sum_j N(h_j)^\beta \sum_i \mu(h_j Z \cap h_j g^{-1} g_i^{-1} Y_i \cap Z_j) \\
&= N(g)^{-\beta} \sum_j N(h_j)^\beta \mu(h_j Z \cap Z_j).
\end{aligned}$$

Taking $g = e$ we see that the extension of μ to GY is well-defined. But then for arbitrary g the above identity reads as :

$$\mu(gZ) = N(g)^{-\beta} \mu(Z).$$

■

Lemma 10 *2.4. Let Y_0 be a $\Gamma-$invariant Borel subset of Y such that :*

(i) if $gY_0 \cap Y_0 \neq \emptyset$ for some $g \in G$, then $g \in \Gamma$;

(ii) for any $y \in Y$, there exists $g \in G$ such that $gy \in Y_0$.

Then any $\Gamma-$invariant Borel measure on Y_0 extends uniquely to a Borel measure on Y satisfying the scaling condition from Proposition 2.1.

Proof. Let μ_0 be a $\Gamma-$invariant measure on Y_0. Since the assumptions imply

that Y is a disjoint union of translates of Y_0 by representatives of the right cosets of Γ , that is , $Y = \sqcup_{h:\Gamma\backslash G}(h^{-1}Y_0 \cap Y)$, there is only one choice for a measure μ extending μ_0 and satisfying Proposition 2.1 , namely ,

$$\mu(Z) = \sum_{h:\Gamma\backslash G} N(h)^{\beta}\mu_0(hZ \cap Y_0).$$

Since μ_0 is $\Gamma-$invariant , $\mu(Z)$ is independent of the choice of representatives , so all we need to check is that Proposition 2.1 holds : Let $g \in G$. Then

$$\mu(gZ) = \sum_{h:\Gamma\backslash G} N(h)^{\beta}\mu_0(hgZ\cap Y_0) = N(g)^{-\beta}\sum_{h:\Gamma\backslash G} N(hg)^{\beta}\mu_0(hgZ\cap Y_0) = N(g)^{-\beta}\mu(Z) \ ,$$

which proves the Lemma. ■

Although the condition for a measure ν on $\Gamma\backslash Y$ to define a KMS-weight is easier to formulate in terms of the corresponding $\Gamma-$ invariant measure on Y , it will also be important to work directly with ν. For this we introduce the following operators on functions on $\Gamma\backslash X$. We shall often consider functions on $\Gamma\backslash X$ as $\Gamma-$invariant functions on X.

Definition 11 2.5. *Let G act on a set X and suppose (G, Γ) is a Hecke pair.* The Hecke operator *associated to $g \in G$ is the operator T_g on $\Gamma-$invariant functions on X defined by :*

$$(T_g f)(x) = \frac{1}{R_\Gamma(g)} \sum_{l \,\in\, \Gamma\backslash\Gamma g\Gamma \text{ (finite)}} f(lx).$$

Clearly $T_g f$ is again $\Gamma-$invariant. Recall that $[g^{-1}]$ denotes the characteristic function of the double coset $\Gamma g^{-1}\Gamma$ considered as an element of the Hecke algebra. The map :

$$\left[g^{-1}\right] \mapsto R_\Gamma(g)T_g$$

is a representation of the Hecke algebra $H(G, \Gamma)$ on the space of $\Gamma-$invariant functions.

Notice that for $X = G$, this is exactly the way we defined the regular representation of $H(G,\Gamma)$ on $l^2(\Gamma\backslash G)$: For $f \in H(G,\Gamma)$, considered as operator on $l^2(\Gamma\backslash G)$, we defined its action by :

$$f \cdot \delta_{\Gamma h} = \sum_{l\ \in\ \Gamma\backslash G} f(lh^{-1}) \cdot \delta_{\Gamma l}$$

Indeed , for $f = [g^{-1}]$, using the regular representation (on $l^2(\Gamma\backslash G)$) we get :

$$\left[g^{-1}\right] \cdot \delta_{\Gamma h} = \sum_{l\in\Gamma\backslash G} \left[g^{-1}\right](lh^{-1}) \cdot \delta_{\Gamma l} = \sum_{l\in\Gamma\backslash G} \delta_{\Gamma g^{-1}\Gamma}(lh^{-1}) \cdot \delta_{\Gamma l}$$

so $([g^{-1}] \cdot \delta_{\Gamma h})(s) = 1$, if $sh^{-1} \in \Gamma g^{-1}\Gamma \iff s \in \Gamma g^{-1}\Gamma h$ and $= 0$ otherwise.

On the other hand ,

using the representation $\sigma : C_r^*(G,\Gamma) \to B(l^2(\Gamma\backslash G))$ defined as above by $[g^{-1}] \mapsto R_\Gamma(g)T_g$, we get :

$$\sigma(\left[g^{-1}\right]) \cdot \delta_{\Gamma h}(s) = R_\Gamma(g)T_g(\delta_{\Gamma h})(s) = \sum_{l\ \in\ \Gamma\backslash\Gamma g\Gamma} \delta_{\Gamma h}(ls)\ ,$$

so $(\sigma([g^{-1}]) \cdot \delta_{\Gamma h})(s) = \{1$, if $h \in \Gamma g\Gamma s \iff s \in \Gamma g^{-1}\Gamma h$,and $= 0$ otherwise.

By decomposing an arbitrary X into $G-$orbits one can obtain that $[g^{-1}] \mapsto R_\Gamma(g)T_g$ is a representation without any computations.

The following three lemmas will be our main computational tools :

Lemma 12 2.6. *Suppose μ is as in Proposition 2.1 and that ν is the measure on $\Gamma\backslash Y$ determined by (2.2). Assume further that $Y = X$, the action of G on X is free and that (G,Γ) is a Hecke pair with modular function $\Delta_\Gamma(g) := \frac{R_\Gamma(g^{-1})}{R_\Gamma(g)}$. Then for any positive measurable function f on $\Gamma\backslash X$ and $g \in G$, we have :*

$$\int_{\Gamma\backslash X} T_g f\ d\nu = \Delta_\Gamma(g) \cdot N(g)^\beta \cdot \int_{\Gamma\backslash X} f\ d\nu.$$

Proof. Let us first prove the following claim : ■

Claim 13 *There exist a neigbourhood U of x such that the sets hU are disjoint for different h in $\Gamma g^{-1}\Gamma$. Fix a point $x \in X$. Choose representatives $h_1, h_2, ..., h_n$ of the right $\Gamma-$cosets contained in $\Gamma g^{-1}\Gamma$. Since the action of Γ is Proper , there exist a neighbourhood U of x such that if $h_iU \cap \gamma h_jU \neq \varnothing$ for some i, j and $\gamma \in \Gamma$ then $h_ix = \gamma h_jx$. But since the action of G is free , the latter equality is possible only when $h_i = \gamma h_j$, so that $i = j$ and $\gamma = e$. Thus $h_iU \cap \gamma h_jU = \varnothing$ if $i \neq j$ or $\gamma \neq e$. Since $\Gamma g^{-1}\Gamma = \cup_{k=1}^{n} \Gamma h_k$, this proves the claim.*

Proof. We conclude from the claim that the set $\Gamma g^{-1}\Gamma U$ is a disjoint union of the sets hU , $h \in \Gamma g^{-1}\Gamma$. So we can write :

$$\sum_{h:\Gamma\backslash\Gamma g\Gamma} 1_{h^{-1}\Gamma U} = 1_{\Gamma g^{-1}\Gamma U} = \sum_{h:\Gamma\backslash\Gamma g^{-1}\Gamma} 1_{\Gamma hU} \ ,$$

Denoting by $p : X \to \Gamma\backslash X$ the quotient map , we can rewrite the above in terms of functions on $\Gamma\backslash X$ as

$$R_\Gamma(g)T_g(1_{p(U)}) = 1_{p(\Gamma g^{-1}\Gamma U)} = \sum_{h\,:\,\Gamma\backslash\Gamma g^{-1}\Gamma} 1_{p(hU)}.$$

It follows that

$$R_\Gamma(g)\int_{\Gamma\backslash X} T_g(1_{p(U)})\,d\nu = \sum_{h\,:\,\Gamma\backslash\Gamma g^{-1}\Gamma} \nu(p(hU)) = \sum_{h\,:\,\Gamma\backslash\Gamma g^{-1}\Gamma} \mu(hU) = R_\Gamma(g^{-1})N(g)^\beta\nu(p(U)).$$

In other words , the identity in the lemma holds for $f = 1_{p(U)}$. Since this is true for any x and sufficiently small neigbourhood U of x , we get the result. ■

Notice that by applying the above lemma to the characteristic function of X , we get the following :

If a group G acts freely on a space X with a $G-$invariant measure μ , and Γ is an *almost normal subgroup of* G (that is , (G, Γ) is a Hecke pair) such that the action of Γ on X is Proper and $0 < \mu(\Gamma\backslash X) < \infty$, then $\Delta_\Gamma(g) = 1$ for any $g \in G$. The same is true if we assume that the action of G on (X, μ) is only essentially free.

Lemma 14 *2.7. Suppose μ is as in Proposition 2.1 and ν is the measure on $\Gamma\backslash Y$ determined by (2.2) . Assume that the action of G on X is free and that (G, Γ) is a Hecke pair. Assume further that Y_0 is a $\Gamma-$invariant measurable* *subset of Y such that if $gY_0 \cap Y_0 \neq \varnothing$ for some $g \in G$, then $g \in \Gamma$. Then for any $g \in G$ such that $gY_0 \subset Y$, measurable $Z \subset \Gamma\backslash Y_0$ and positive measurable function f on $\Gamma\backslash Y$, we have :*

$$\int_{\Gamma g Z} f \; d\nu = N(g)^{-\beta} R_\Gamma(g) \int_Z T_g f \; d\nu \ ,$$

where $\Gamma g Z = p(\Gamma g p^{-1}(Z))$ and $p : X \to \Gamma\backslash X$ is the quotient map. In particular , $\nu(\Gamma g Z) = N(g)^{-\beta} \cdot R_\Gamma(g) \cdot \nu(Z)$.

Proof. Suppose $Z \subset \Gamma\backslash Y_0$ is measurable , and choose $U \subset Y_0$ measurable such that $Z = p(U)$ and p is injective on U. For $g \in G$ let $h_1, ..., h_n$ be representatives of the right $\Gamma-$cosets contained in $\Gamma g \Gamma$. Then we claim : ■

Claim 15 *The quotient map of Γ , p , is injective on $h_1 U, ..., h_n U$, and the images under p of these sets are disjoint.*

Proof. Assume $p(h_i x) = p(h_j y)$ for some i, j and $x, y \in U$, so that $\gamma h_i x = h_j y$ for some $\gamma \in \Gamma$. Since $U \subset Y_0$, our assumption on Y_0 implies that $h_j^{-1} \gamma h_i \in \Gamma$. But then , since p is injective on U , we get $x = y$,and since the action of Γ is free , we conclude that $h_j^{-1} \gamma h_i = e$. It follows that $i = j$ and $h_i x = h_j y$ which proves the claim. ■

Proof. Furthermore , the union of the disjoint sets $p(h_1 U)$, ..., $p(h_n U)$ is the set $\Gamma g Z = p(\Gamma g \; p^{-1}(Z))$.Hence , since $\Gamma \subset \ker N$, $N(h_i) = N(g)$ for $i = 1, ..., n$,

$$\int_{\Gamma g Z} f \; d\nu = \sum_{i=1}^{n} \int_{h_i U} f \circ p \; d\mu = N(g)^{-\beta} \sum_{i=1}^{n} \int_U f(p(h_i)) \; d\mu = N(g)^{-\beta} R_\Gamma(g) \int_Z T_g f \; d\nu.$$

The last assertion of the lemma , that $\nu(\Gamma g Z) = N(g)^{-\beta} \cdot R_\Gamma(g) \cdot \nu(Z)$ follows by taking $f = 1_{\Gamma g Z}$ and observing that in this case $(T_g f)(z) = 1$, for $z \in Z$. ■

For the next lemma , we introduce the following notation.

Definition 16 **2**.**8**. *If $\beta \in \mathbb{R}$ and S is a subsemigroup of G containing Γ , then we define*

$$\zeta_{S,\Gamma}(\beta) := \sum_{s : \Gamma\backslash S} N(s)^{-\beta} = \sum_{s \, : \, \Gamma\backslash S/\Gamma} N(s)^{-\beta} R_\Gamma(s).$$

Lemma 17 ***2.9.*** *Suppose μ is as in Proposition 2.1 and ν is the measure on $\Gamma\backslash Y$ determined by (2.2) . Assume that the action of G on X is free and that (G,Γ) is a Hecke pair. Assume further that Y_0 is a measurable $\Gamma-$invariant subset of Y , and S a subsemigroup of G containing Γ such that :*

(ι) if $gY_0 \cap Y_0 \neq \varnothing$ for some $g \in G$ then $g \in \Gamma$;

($\iota\iota$) $\cup_{s\in S} sY_0$ is a subset of Y of full measure ;

($\iota\iota\iota$) $\zeta_{S,\Gamma}(\beta) < \infty$.

Let H_S be the subspace of $S-$invariant functions in $L^2(\Gamma\backslash Y,\nu)$, that is , functions f such that $f(y) = f(sy)$ for all $s \in S$ and a.a. $y \in Y$. Then :

(1) if $f \in H_S$ then $\|f\|_2^2 = \zeta_{S,\Gamma}(\beta)\int_{\Gamma\backslash Y_0} |f(t)|^2\, d\nu(t)$;

(2) the orthogonal projection $P : L^2(\Gamma\backslash Y, d\nu) \longrightarrow H_S$ is given by

$$Pf\mid_{Sy} = \zeta_{S,\Gamma}(\beta)^{-1} \sum_{s\,:\,\Gamma\backslash S/\Gamma} N(s)^{-\beta} R_\Gamma(s)(T_s f)(y)\ ,$$

(2.3)

for $y \in Y_0$.

Proof. By condition (ι) the sets $\Gamma s Y_0$ are disjoint for s in different double cosets of Γ. Since the union of such sets is the whole space Y (modulo a set of measure zero) , by Lemma 2.7 applied to $Z = \Gamma\backslash Y_0$ for any $f \in L^2(\Gamma\backslash Y, d\nu)$ we get :

$$\|f\|_2^2 = \sum_{s\,:\,\Gamma\backslash S/\Gamma} \int_{\Gamma s Z} |f|^2\, d\nu = \sum_{s\in\Gamma\backslash S/\Gamma} N(s)^{-\beta} R_\Gamma(s) \int_{\Gamma\backslash Y_0} T_s(|f|^2) d\nu.$$

(2.4)

Since $T_s(|f|^2) = |f|^2$ for $f \in H_S$, this gives (1) .

To prove (2) , denote by T the operator on $L^2(\Gamma\backslash Y, d\nu)$ defined by the asserted formula for P. To see that it is well-defined , notice first that the summation in the right hand side of (2.3) is finite for f in the subspace of L^2-functions supported on a finite collection of sets of the form $p(sY_0)$, $s \in S$, which is a *dense* subspace of $L^2(\Gamma\backslash Y, d\nu)$. Thus the function Tf is well-defined for f in this subspace and , putting $\alpha_s = \zeta_{S,\Gamma}(\beta)^{-1} N(s)^{-\beta} R_\Gamma(s)$ and using (2.4) twice , we get :

$$\|Tf\|_2^2 = \zeta_{S,\Gamma}(\beta) \int_{\Gamma\backslash Y_0} |Tf|^2\ d\nu \le \zeta_{S,\Gamma}(\beta) \int_{\Gamma\backslash Y_0} \left(\sum_{s\in\Gamma\backslash S/\Gamma} \alpha_s T_s(|f|^2) \right) d\nu = \|f\|_2^2\ .$$

It follows that T extends to a well-defined contraction. Since $Tf = f$ for $f \in H_S$, we conclude that $T = P$. ■

3. THE CONNES-MARCOLLI SYSTEM

Consider the group $G = GL_2^+(\mathbb{Q})$ of invertible 2 by 2 matrices with rational coefficients and positive determinant , and its subgroup $\Gamma = SL_2(\mathbb{Z})$. For a prime number p consider the field $\mathbb{Q}_p$ of $p-$adic numbers and its compact subring $\mathbb{Z}_p$ of $p-$adic integers. We denote by A_f the space of finite adeles of $\mathbb{Q}$, that is , the restricted product of the fields of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$; $\mathbf{A}_f := \{(a_p)_{p\in\mathbf{P}} | a_p \in \mathbb{Q}_p \forall p, a_p \in \mathbb{Z}_p \text{ for all sufficiently large } p\}$ and by $\hat{\mathbb{Z}} = \prod\limits_p \mathbb{Z}_p = \{(a_p)_{p\in\mathbf{P}} \mid a_p \in \mathbb{Z}_p\}$ its maximal compact subring . The field $\mathbb{Q}$ is a subfield of $\mathbb{Q}_p$ since $\mathbb{Q}_p$ is a closure of $\mathbb{Q}$ in the p-norm (if $q = p^n \frac{a}{b}$, $(p \nmid a$, $p \nmid b)$, then $\|q\|_p = p^{-n}$) . Therefore $GL_2^+(\mathbb{Q})$ can be considered as a subgroup of $GL_2(\mathbb{Q}_p)$. In particular , we have an action of $GL_2^+(\mathbb{Q})$ on $Mat_2(\mathbb{Q}_p)$ by matrix-multiplication on the left.

Moreover , we have the following diagonal embedding of $\mathbb{Q}$ into $\mathbf{A}_f$:

$$\begin{array}{ccl} \mathbb{Q} & \subset & \mathbb{Q}_p \\ a & \mapsto & (a_p)_{p\in\mathbf{P}} \in \mathbf{A}_f \end{array}$$

for every $a \in \mathbb{Q}$,

$$a = \frac{n}{m} = \frac{n}{\left(p_1^{k_1} \cdot \ , \ , \ , \ , \ \cdot \ p_i^{k_i}\right)} \ ,$$

where we assume n and $m \in \mathbb{Z}$ with $\gcd(n,m) = 1$ (n and m are relatively prime) and $k_j \geq 1$. Then $a \notin \mathbb{Z}_p$ if $p = p_i$ for some i , so $a \notin \hat{\mathbb{Z}} = \prod\limits_{p\in\mathbf{P}} \mathbb{Z}_p$. Contrary $a \in \mathbb{Z}_p$ if $p \neq p_i$ for any i .From this we see that for any $a \in \mathbb{Q}$, eventually , for $p \in \mathbf{P}$ large enough $a \in \mathbb{Z}_p$. Hence the map : $a \longmapsto (a_p)_{p\in\mathbf{P}}$ embeds $\mathbb{Q}$ diagonally into $\mathbf{A}_f$. Extending this on the matrix entries , we get an embedding of $GL_2^+(\mathbb{Q})$ into $GL_2(\mathbf{A}_f)$, and thus an action of $GL_2^+(\mathbb{Q})$ on $Mat_2(\mathbf{A}_f)$.

In addition $GL_2^+(\mathbb{Q})$ acts by Møbius transformations on the upper half-plane $\mathbf{H}$. Therefore we have an action of $GL_2^+(\mathbb{Q})$ on $\mathbf{H} \times Mat_2(\mathbf{A}_f)$ such that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau \in \mathbf{H}$ and $m = (m_p)_p \in Mat_2(\mathbf{A}_f)$,

$$g(\tau, (m_p)_p) = (\frac{a\tau + b}{c\tau + d}, (gm_p)_p).$$

Note that the action of $SL_2(\mathbb{Z})$ is proper , since already the action of $SL_2(\mathbb{Z})$ on $\mathbf{H}$ is proper.

The GL_2-system of Connes and Marcolli is now defined as follows :

Definition 18 3.1. *The Connes-Marcolli algebra is the* C^*-*algebra* $A = C_r^*(\Gamma\backslash G \boxtimes_\Gamma Y)$, *where* $G = GL_2^+(\mathbb{Q})$, $\Gamma = SL_2(\mathbb{Z})$, G *acts diagonally on* $X = \mathbf{H} \times Mat_2(\mathbf{A}_f)$, *and* $Y = \mathbf{H} \times Mat_2(\mathbb{Z})$. *The dynamics* σ *on* A *is defined by the homomorphism* $N : GL_2^+(\mathbb{Q}) \to \mathbb{R}_+^*$, $N(g) = \det(g)$.

Notice that since $\Gamma\backslash\mathbf{H}$ is not compact , the algebra A is nonunital . By [5, Lemma 1.28] , the action of $GL_2^+(\mathbb{Q})$ on $X\backslash(\mathbf{H}\times\{0\})$ is free. *Recall briefly the reason :* If for $g \in GL_2^+(\mathbb{Q})$ $gm = m$ for some prime number $p \in \mathbf{P}$ ($\mathbf{P}$ denotes the set of all Prime numbers) and nonzero $m \in Mat_2(\mathbb{Q}_p)$, then the spectrum of the matrix g contains 1 , and hence $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, ($a, b, c, d \in \mathbb{Q}$ and $ad - bc > 0$) is conjugate in $GL_2^+(\mathbb{Q})$ to an upper-triangular matrix (by Linear Algebra) : $\tilde{g}‘ = \begin{pmatrix} \tilde{a} & \tilde{b} \\ 0 & \tilde{d} \end{pmatrix}$. But then g has no fixed points in $\mathbf{H}$, since the corresponding Møbius transformation for any upper triangular matrix only has fixpoints in $\bar{\mathbb{R}}$, but not in the upper halfplane $\mathbf{H}$. Note that this actually implies that the action of $GL_2^+(\mathbb{Q})$ on $\mathbf{H}\times Mat_2(\mathbb{Q}_p)^\times$, where $Mat_2(\mathbb{Q}_p)^\times = Mat_2(\mathbb{Q}_p) \setminus \{0\}$, is free for any prime number p. Although the action of $GL_2^+(\mathbb{Q})$ on $\mathbf{H}\times\{0\}$ is not free , this set can be ignored in the analysis of $KMS_\beta-$states for $\beta \neq 0$. This is proved in [5 , Proposition 1.30] .

Again , *recall briefly the reason :*

Consider the action of G on $\tilde{X} = X\backslash(\mathbf{H}\times\{0\})$, put $\tilde{Y} = Y\backslash(\mathbf{H}\times\{0\}) \subset \tilde{X}$, and then define $I = C_r^*(\Gamma\backslash G\boxtimes_\Gamma\tilde{Y})$. Then I can be considered as an ideal in A , and the quotient algebra A/I is isomorphic to $C_r^*(\Gamma\backslash G \times_\Gamma \mathbf{H})$. Now , if φ is a KMS_β state on A , the restriction $\varphi|_I := \varphi_I$ canonically extends to a $KMS-$functional on the multiplier algebra of I in the following sense : Consider the GNS-representation of $I \subset A$ given by the triple $(H_{\varphi_I}, \pi_{\varphi_I}, \xi_{\varphi_I})$. Then , if we let I_* denote the multiplier algebra of $I \subset A$, the GNS representation : $\pi_{\varphi_I} : I \to B(H_{\varphi_I})$ canonically extends to $\pi : I_* \to B(H_{\varphi_I})$, for if $x \in I$, $b \in I_*$, then

$$\pi(b)\pi_{\varphi_I}(x)\xi_{\varphi_I} = \pi(bx)\xi_{\varphi_I} = \pi_{\varphi_I}(bx)\xi_{\varphi_I}$$

Now , if we check that the extension π is bounded on I_* as extension of π_{φ_I} from I to I_* , it is welldefined by the above equation. For this , let

$\{e_i\}$ be an approximate unit in I with $\pi_{\varphi_I}(e_i) \nearrow 1$ in the Strong Operator Topology . Then we have :

$$\pi(b)\pi_{\varphi_I}(x)\xi_{\varphi_I} = \pi(bx)\xi_{\varphi_I} = \lim_i \pi_{\varphi_I}(be_ix)\xi_{\varphi_I} = \lim_i \pi_{\varphi_I}(be_i)\pi_{\varphi_I}(x)\xi_{\varphi_I}$$

from which we conclude that :

$$\|\pi(b)\| \leq \lim_i \left\|\pi_{\varphi_I}(be_i)\right\| \leq \lim_i \|be_i\| \leq \|b\|$$

Then what is called the canonical extension of φ_I to $\tilde{\varphi}$ on I_* is defined accordingly ; again if $0 \leq b \in I_*$, and $0 \leq x \in I$:

$$\tilde{\varphi}(b) = (\pi(b)\xi_{\varphi_I}\ ,\ \xi_{\varphi_I}) = \lim_i \varphi_I(be_i) = \lim_i(\pi(b)\pi_{\varphi_I}(e_i)\xi_{\varphi_I}, \pi_{\varphi_I}(e_i)\xi_{\varphi_I}) = \lim_i \varphi_I(e_ibe_i)$$

As $I \subset A \subset I_*$, $\tilde{\varphi}$ is a (positive) $KMS-$functional on A.But then $\tilde{\varphi} \leq \varphi$: For if $a \in A$, $0 \leq a$, then evaluating

$$\begin{aligned}\tilde{\varphi}(a) &= (\pi(a)\xi_{\varphi_I}, \xi_{\varphi_I}) = (\pi(a)^{\frac{1}{2}}\xi_{\varphi_I}, \pi(a)^{\frac{1}{2}}\xi_{\varphi_I}) = \lim_i(\pi_{\varphi_I}(e_i)\pi(a^{\frac{1}{2}})\xi_{\varphi_I}, \pi(a^{\frac{1}{2}})\xi_{\varphi_I}) \\ &= \lim_i\ (\pi_{\varphi_I}(a^{\frac{1}{2}}e_ia^{\frac{1}{2}})\xi_{\varphi_I}, \xi_{\varphi_I})\end{aligned}$$

The last equality since $e_i \nearrow 1$ in the strong operator topology. Then further

$$\tilde{\varphi}(a) = \lim_i\ (\pi_{\varphi_I}(a^{\frac{1}{2}}e_ia^{\frac{1}{2}})\xi_{\varphi_I}, \xi_{\varphi_I}) = \lim_i \varphi_I(a^{\frac{1}{2}}e_ia^{\frac{1}{2}}).$$

Now , since $0 \leq e_i \leq 1$, $\forall\ i$ we have $a^{\frac{1}{2}}e_ia^{\frac{1}{2}} \leq a^{\frac{1}{2}}a^{\frac{1}{2}} = a$, and thus

$$\varphi_I(a^{\frac{1}{2}}e_ia^{\frac{1}{2}}) = \varphi(a^{\frac{1}{2}}e_ia^{\frac{1}{2}}) \leq \varphi(a).$$

Therefore

$$\tilde{\varphi}(a) = \lim_i \varphi_I(a^{\frac{1}{2}}e_ia^{\frac{1}{2}}) \leq \varphi(a).$$

Thus we get a $KMS-$functional $\tilde{\varphi} \leq \varphi$ on A. If $\tilde{\varphi} \neq \varphi$ then $(\varphi - \tilde{\varphi})$ is a positive nonzero $KMS-$functional on A which vanishes on I . It follows that it factors through the canonical quotient map $q : A \to A/I$ since it is constant on equivalence classes . Hence we get a $KMS-$state on $A/I \approxeq C_r^*(\Gamma\backslash G \times_\Gamma \mathbf{H})$. By Lemma 1.3 the multiplier algebra of $C_r^*(\Gamma\backslash G \times_\Gamma \mathbf{H})$ contains the reduced Hecke C^*-algebra $C_r^*(G, \Gamma)$. The latter algebra contains in turn the C^*-algebra $Z(G)/(Z(G) \cap \Gamma)$, where $Z(G)$ is the center of $GL_2^+(\mathbb{Q})$, that is , the group of scalar matrices. But since the dynamics scales nontrivially some unitaries in this algebra , the algebra can not have any $KMS_\beta-$states for $\beta \neq 0$. This contradiction shows that $\varphi = \tilde{\varphi}$, so that φ is completely determined by φ_I.

Since the action of G on $\tilde{X} = \mathbf{H} \times Mat_2(\mathbf{A}_f)^{\times}$, where $Mat_2(\mathbf{A}_f)^{\times} = Mat_2(\mathbf{A}_f) \backslash \{0\}$, is free , we can apply Proposition 2.1 and conclude that there is a *one-to-one correspondence* between $KMS_\beta-$weights on I with domain of definition containing $C_c(\Gamma \backslash \tilde{Y})$ and measures μ on $\tilde{Y} = \mathbf{H} \times Mat_2(\hat{\mathbb{Z}})^{\times}$ such that :

$$\mu(gZ) = \det(g)^{-\beta} \mu(Z)$$

if both Z and gZ are subsets of $\tilde{Y}$. Then by Lemma 2.2 , we can uniquely extend any such measure to a measure on $\tilde{X} = G\tilde{Y} = \mathbf{H} \times Mat_2(\mathbf{A}_f)^{\times}$ such that :

$$\mu(gZ) = \det(g)^{-\beta} \mu(Z)$$

, but now for all $Z \subset \tilde{X}$.

To get a state on $I = C_r^*(\Gamma \backslash G \boxtimes_\Gamma \tilde{Y})$ we need the normalization condition $\mu(\Gamma \backslash \tilde{Y}) = 1$ (that is , the $\Gamma-$invariant measure μ on $\tilde{Y}$ defines a probability measure on $\Gamma \backslash \tilde{Y}$) . Note also that if $\beta \neq 0$ and we have a measure on $X = \mathbf{H} \times Mat_2(\mathbf{A}_f)$ with the same properties as above , then $\mathbf{H} \times Mat_2(\mathbf{A}_f)^{\times}$ is a subset of full measure , since scalar matrices act trivially on $\mathbf{H}$ and so $\mathbf{H}$ cannot support a measure scaled nontrivially by them.

Summarizing the above discussion we get the following :

Proposition 19 3.2. *For $\beta \neq 0$ there is a* one-to-one correspondence *between $\sigma - KMS_\beta-$states on the Connes-Marcholli system and $\Gamma-$invariant measures μ on $\mathbf{H} \times Mat_2(\mathbf{A}_f)$ such that :*

$$\begin{aligned} \mu(\Gamma \backslash \mathbf{H} \times (Mat_2(\hat{\mathbb{Z}}))) &= 1 \text{ and } \mu(gZ) = \det(g)^{-\beta} \mu(Z) \\ \text{for any } g &\in GL_2^+(\mathbb{Q}) \text{ and compact } Z \subset \mathbf{H} \times Mat_2(\mathbf{A}_f). \end{aligned}$$

Denote by $Mat_2^i(\mathbf{A}_f)$ the set of matrices $m = (m_p)_p \in Mat_2(\mathbf{A}_f)$ such that $\det(m_p) \neq 0$ for every prime p. Notice that $Mat_2^i(\mathbf{A}_f)$ is the set of *non-zero divisors* in $Mat_2(\mathbf{A}_f)$. Our next goal is to show that if $\beta \neq 0, 1$ then $\mathbf{H} \times$ $Mat_2^i(\mathbf{A}_f)$ is a subset of full measure for any measure μ as in Proposition 3.2. First let us recall the following simple properties of the Hecke pair $(G, \Gamma) = (GL_2^+(\mathbb{Q}), SL_2(\mathbb{Z}))$.

Put $Mat_2^+(\mathbb{Z}) = GL_2^+(\mathbb{Q}) \cap Mat_2(\mathbb{Z})$.

Lemma 20 3.3. *Every double coset of* Γ *in* $Mat_2^+(\mathbb{Z})$ *has an unique representative of the form* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *with* $a\ , d \in \mathbb{N}$ *and* $a \mid d$ *. Furthermore ,*

$$R_\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \frac{d}{a} \prod_{p\ prime\ :pa|d} (1 + p^{-1}) \ \ ,$$

and as representatives of the right cosets of Γ *contained in* $\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma$ *we can take the matrices* :

$$\begin{pmatrix} ak & am \\ 0 & al \end{pmatrix}$$

with $k,\ l \in \mathbb{N}$ *and* $m \in \mathbb{Z}$ *such that* $kl = d/a\ ,\ 0 \leq m \leq l$ *and* $\gcd(k, l, m) = 1$.

In particular , $R_\Gamma(g) = R_\Gamma(g^{-1})$ *, for every* $g \in GL_2^+(\mathbb{Q})$.

Before the proof of the above Lemma , let us recall the following facts from matrix factorization and elementary number theory taken from A. Krieg :

Fact 1 (Lemma)

Given $0 \neq \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{Z}^2$, there exist $\mathbf{U} \in \Gamma$ satisfying :

$$\mathbf{U} \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \delta \\ 0 \end{pmatrix} \ \ , \ \delta = \gcd(a, c).$$

Proof :

We may replace $\begin{pmatrix} a \\ c \end{pmatrix}$ by $\frac{1}{\delta} \cdot \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{Z}^2$ and therefore assume $\gcd(a, c) = 1$ without restriction. Hence there exist $b, d \in \mathbb{Z}$ such that $ad - bc = 1$. Now choose

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \Gamma.$$

Fact 2 (Proposition)

Given $A \in Mat_2(\mathbb{Z})$, the right coset ΓA contains an unique representative of the form :

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ , \ a, d \ \in \ \mathbb{N} \ , \ 0 \leq b < d \ .$$

This immediately leads to the

Fact 3 (Corollary)

Given $l \in \mathbb{N}$ the set $M(l) = \{A \in Mat_2(\mathbb{Z}) \mid \det A = l\}$ decomposes into :

$$\sigma_1(l) := \sum_{d \in \mathbb{N} \ , \ d|l} d$$

right cosets relative to Γ . A set of representatives is given by :

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ , \ \text{where } d \in \mathbb{N} \ , \ d \mid l \ \ , \ 0 \leq b < d \ \text{ and } \ a = \frac{l}{d} \ .$$

And : In particular ($SL_2(\mathbb{Z})$ *,* $GL_2^+(\mathbb{Q})$ *) is a Hecke pair .*

Proof : The first part follows by applying the above Proposition . For the second part ; Given $A \in GL_2^+(\mathbb{Q})$, choose $\alpha \in \mathbb{N}$ such that $\alpha A \in Mat_2^+(\mathbb{Z})$ .The assertion follows from $\sharp \ (\Gamma \backslash \Gamma A \Gamma) = \sharp \ (\Gamma \backslash \Gamma \alpha A \Gamma)$.

Fact 4 (Proposition 2)

Given $A \in Mat_2^+(\mathbb{Z})$ the right coset ΓA contains an unique representative of the form :

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \ , \ a, d \in \mathbb{N} \ , \ 0 \leq c < a \ .$$

Proof. (Omitted)

Fact 5 (Observation)

Now , let $\delta(A)$:= gcd of the entries of A , whenever A is a non-zero integral matrix . Then : $\delta(A)\delta(B) \mid \delta(AB)$, holds for all $A, B \in Mat_2^+(\mathbb{Z})$. Another well-known number theoretical assertion we need is :

Fact 6

Let $a, c, d \in \mathbb{Z}$ such that $a \neq 0$ and $\gcd(a, c, d) = 1$. Then there exist an integer $x \in \mathbb{Z}$ satisfying

$$\gcd(a, c + xd) = 1.$$

Proof. *a)* The *uniqueness* of the entries a , d in Lemma 3.3. follows from the latter Observation . For the *existence* , we may assume $\delta(A) = 1$, since A can otherwise be replaced by $\frac{1}{\delta(A)} \cdot A$. In view of *Fact 4 (Proposition 2)* , we may already suppose that A has the form :

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \ , \ a > 0 \ , \ d > 0 \ , \ \gcd(a, c, d) = 1 \ .$$

Next apply Fact 6 and determine $x \in \mathbb{Z}$ with $\gcd(a, c + xd) = 1$. The entries of the first column of :

$$\bar{A} = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c + xd & d \end{pmatrix}$$

are relatively prime . Due to the Lemma (Fact 1) , there exist $\mathbf{U} \in \Gamma$ such that :

$$\mathbf{U}\bar{A} = \begin{pmatrix} 1 & \bar{b} \\ 0 & ad \end{pmatrix}.$$

Now choose $\mathbf{V} = \begin{pmatrix} 1 & -\bar{b} \\ 0 & 1 \end{pmatrix} \in \Gamma$ to get :

$$\mathbf{U}\bar{A}\mathbf{V} = \begin{pmatrix} 1 & \bar{b} \\ 0 & ad \end{pmatrix} \begin{pmatrix} 1 & -\bar{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \Gamma A \Gamma.$$

b) By the first part , it suffices to consider $\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Since $a \mid d$, $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in Mat_2^+(\mathbb{Z})$ and from Fact 2 , $\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ posesses an unique representative of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $a, d \in \mathbb{N}$ and $0 \leq b < d$.

Inwoke the Corollary (Fact) , second part above to get that a set of representatives of the right cosets of Γ contained in $\Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma$ is given by $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, where $d \in \mathbb{N}$, $d \mid l$, $0 \leq b < d$ and $a = l/d$.

This is equivalent to the statement which is to be proved here if : As $a \mid d$, let $\frac{d}{a} = p_1^{k_1} .... p_n^{k_n}$ . . . . , $(1 \leq k_i)$. , we see that if p is a prime such that $pa \mid d$, then $p \in \{p_1, p_2, ..., p_n\}$, so by counting the number of representatives of the form : $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, $a, d \in \mathbb{N}$ and $0 \leq b < d$ and $ad = l$ such that $d \mid l$, we get that it equals : $\frac{d}{a} \prod_{p \text{ prime } : \ pa|d} (1 + p^{-1})$. Therefore this set of representatives could be explicitly given as : $\begin{pmatrix} ak & am \\ 0 & al \end{pmatrix}$, with $k, l \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $kl = d/a$ and $\gcd(k, l, m) = 1$.

The last statement , that $R_\Gamma(g) = R_\Gamma(g^{-1})$ for every $g \in GL_2^+(\mathbb{Q})$ follows from the fact that for every $g \in GL_2^+(\mathbb{Q})$, there exist $\alpha \in \mathbb{N}$ such that $\alpha g \in Mat_2^+(\mathbb{Z})$. Hence , since

$$\sharp\ (\Gamma \backslash \Gamma g \Gamma) = \sharp\ (\Gamma \backslash \Gamma \alpha g \Gamma)$$

and $\begin{pmatrix} ak & am \\ 0 & al \end{pmatrix}^{-1} = \begin{pmatrix} ak & 0 \\ am & al \end{pmatrix} \cdot \frac{1}{a^2 kl}$, so in view of Fact 4 above we see that : $\sharp\ (\Gamma \backslash \Gamma g \Gamma) = \sharp\ (\Gamma \backslash \Gamma g^{-1} \Gamma)$.

■

For a prime p put $G_p = GL_2^+(\mathbb{Z}[p^{-1}]) \subset GL_2^+(\mathbb{Q})$. Observe that if $g \in G_p$ then $\det(g)$ is a power of p , and if we multiply g by a sufficiently large power of $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$, we get an element in $Mat_2^+(\mathbb{Z})$ with determinant a power of p. But by Lemma 3.3 the double coset of Γ containing such an element has a (unique) representative of the form : $\begin{pmatrix} p^k & 0 \\ 0 & p^l \end{pmatrix}$, $0 \leq k \leq l$. We may therefore conclude that G_p is the subgroup of $GL_2^+(\mathbb{Q})$ generated by Γ and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. This since : $\Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma$ and if we set $g = \begin{pmatrix} p^{-l} & 0 \\ 0 & p^{-l} \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ we see that $\Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \sqcup_{i=0}^{p-1} \Gamma \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}$ and hence $\Gamma g \Gamma = \sqcup_{i=0}^{p-1} \Gamma (\begin{pmatrix} p^{-l} & 0 \\ 0 & p^{-l} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix})$ .As matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$,where $a \mid d$ constitutes a basis for the double coset decomposition of $G_p = GL_2^+(\mathbb{Z}[p^{-1}]) \subset GL_2^+(\mathbb{Q})$, we get that Γ and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ generates G_p . Furthermore , using the fact that a positive rational number is a power of p *if and only if* it belongs to the group of units $\mathbb{Z}_q^*$ of the ring $\mathbb{Z}_q$ for all primes $q \neq p$, we may also conclude that $g \in GL_2^+(\mathbb{Q})$ belongs to G_p *if and only if* it belongs to $GL_2(\mathbb{Z}_q)$ for all $q \neq p$.

Lemma 21 3.4. *We have $GL_2(\mathbb{Q}_p) = G_p GL_2(\mathbb{Z}_p)$.*

Proof. Let $r \in GL_2(\mathbb{Q}_p)$. Then $r\mathbb{Z}_p^2$ is a $\mathbb{Z}_p-$lattice in $\mathbb{Q}_p^2$, that is , an open compact $\mathbb{Z}_p-$submodule. By [22, Theorem V.2] there exist a subgroup $L \approxeq \mathbb{Z}^2$ of $\mathbb{Q}^2$, such that the closure of L in $\mathbb{Q}_p^2$ coincides with $r\mathbb{Z}_p^2$, and the closure of L in $\mathbb{Q}_q^2$ is $\mathbb{Z}_q^2$ for $q \neq p$.

Choose $g \in GL_2^+(\mathbb{Q})$ such that $g\mathbb{Z}^2 = L$. Since $g\mathbb{Z}_p^2 = r\mathbb{Z}_p^2$, we have $g^{-1}r \in GL_2(\mathbb{Z}_p)$. Since $g\mathbb{Z}_q^2 = \mathbb{Z}_q^2$ for $q \neq p$, we also have $g \in GL_2(\mathbb{Z}_q)$. Hence $g \in G_p$. ■

Lemma 22 3.5. *Let p be a prime and μ_p a $\Gamma-$invariant measure on $\mathbf{H}\times Mat_2(\mathbb{Q}_p)$ such that*

$$\mu_p(\mathbf{H}\times\{0\}) = 0\ ,\ \mu_p(\Gamma\backslash(\mathbf{H}\times Mat_2(\mathbb{Z}_p))) < \infty \ \ \textit{and}\ \mu_p(gZ) = \det(g)^{-\beta}\mu_p(Z)$$

for $g \in G_p$ and $Z \subset \mathbf{H}\times Mat_2(\mathbb{Q}_p)$. If $\beta \neq 1$, then the set $(\mathbf{H}\times GL_2(\mathbb{Q}_p))$ is a subset of full measure in $\mathbf{H}\times Mat_2(\mathbb{Q}_p)$.

Proof. Denote by $\tilde{\nu}$ the measure on $\Gamma\backslash(\mathbf{H}\times Mat_2(\mathbb{Q}_p))$ defined by the $\Gamma-$invariant measure μ_p . For a $\Gamma-$invariant subset $Z \subset Mat_2(\mathbb{Q}_p)$, the set $\mathbf{H}\times Z$ is $\Gamma-$invariant . We can thus define a measure ν on the $\sigma-$algebra of $\Gamma-$invariant Borel subsets of $Mat_2(\mathbb{Q}_p)$ by $\nu(Z) = \tilde{\nu}(\Gamma\backslash(\mathbf{H}\times Z))$. Note that since the action of Γ on $Mat_2(\mathbb{Q}_p)$ is not proper and , accordingly , the quotient space $\Gamma\backslash Mat_2(\mathbb{Q}_p)$ is quite bad , we do not want to consider $\Gamma-$invariant subsets of $Mat_2(\mathbb{Q}_p)$ as subsets of this quotient space and do not try to define a measure on all Borel subsets of $Mat_2(\mathbb{Q}_p)$ out of ν .

If $g \in G_p$ and f is a positive Borel $\Gamma-$invariant function on $Mat_2(\mathbb{Q}_p)$ then by Lemma 2.6 applied to the function $F : (\tau, m) \longmapsto f(m)$ on $\Gamma\backslash(\mathbf{H}\times Mat_2(\mathbb{Q}_p))$ we conclude that

$$\int_{Mat_2(\mathbb{Q}_p)} T_g f\ d\nu = \int_{\Gamma\backslash(\mathbf{H}\times Mat_2(\mathbb{Q}_p))} T_g F\ d\tilde{\nu} = \det(g)^{\beta}\int_{\Gamma\backslash(\mathbf{H}\times Mat_2(\mathbb{Q}_p))} F\ d\tilde{\nu} = \det(g)^{\beta}\int_{Mat_2(\mathbb{Q}_p)} f\ d\nu$$

By assumption we also have $\nu(Mat_2(\mathbb{Z}_p)) < \infty$. *We have to show* that the measure of the set of nonzero singular matrices is zero.

We claim that the set of nonzero singular matrices with coefficients in $\mathbb{Q}_p$ is the disjoint union of the sets :

$$Z_k = SL_2(\mathbb{Z}_p)\begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix} GL_2(\mathbb{Z}_p)\ ,\ k \in \mathbb{Z}.$$

This is proved in a standard way : given a nonzero singular matrix we use multiplication by elements of $GL_2(\mathbb{Z}_p)$ on the right to get a matrix with zero first column , and then multiplication by elements of $SL_2(\mathbb{Z}_p)$ on the left to get the required form . To show that the sets do not intersect , observe that the maximum of the p-adic valuations of the coefficient of a matrix does not change under multiplication by elements g of $GL_2(\mathbb{Z}_p)$ on either side , since if the maximum of the p-adic valuations should change , then such a g must lie in $GL_2^+(\mathbb{Z}\left[p^{-1}\right])$. We saw above that this is equivalent to $g \in GL_2(\mathbb{Z}_q)$ for all $q \neq p$. But then the coefficients of $g \notin \mathbb{Z}_p$, which is a contradiction.

Consider the functions $f_k = 1_{Z_k}$, $k \in \mathbb{Z}$. For $g = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}$ we claim that

$$T_g f_0 = \frac{1}{p+1} f_0 + \frac{p}{p+1} f_1.$$

Indeed , since the action of G_p commutes with the right action of $GL_2(\mathbb{Z}_p)$, the function $T_g f_0$ is $GL_2(\mathbb{Z}_p)-$invariant . $f_0 = 1_{SL_2(\mathbb{Z}_p)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_p)}$. As $Z_0 = \sqcup_{A \in GL_2(\mathbb{Z}_p)} SL_2(\mathbb{Z}_p) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A$ is the sum of right coset of $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} A$ with respect to $SL_2(\mathbb{Z}_p)$. We have $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \Gamma = \sqcup_{i=1}^n \Gamma h_i$, so $(T_g f_0)(x) = \frac{1}{R_\Gamma(g)} \sum_{i=1}^n f_0(h_i x)$ does not depend on the choice of representatives $h_i \in \Gamma g \Gamma$.

On the other hand , the sets Z_k are clopen subsets of the set of singular matrices (see * below) , so that the function f_0 is continous on this set . But then $T_g f_0$ is also continous . Since f_0 is right $GL_2(\mathbb{Z}_p)-$invariant , $T_g f_0$ is right $GL_2(\mathbb{Z}_p)-$invariant . Furthermore f_0 is left $GL_2(\mathbb{Z}_p)-$invariant and hence also $\Gamma-$invariant as $\Gamma \subset GL_2(\mathbb{Z}_p)$.Therefore $T_g f_0$ is left $\Gamma-$invariant . As Γ is dense in $SL_2(\mathbb{Z}_p)$, and $T_g f_0$ is continous , we conclude that $T_g f_0$ is left $SL_2(\mathbb{Z}_p)-$invariant since if $\gamma_n \in \Gamma$ and

$$\gamma_n \to \gamma \in SL_2(\mathbb{Z}_p)$$

then

$$(T_g f)(\gamma_n x) \to_n (T_g f)(\gamma x).$$

Hence $T_g f_0$ is constant on the sets Z_k . So to prove the claim that $T_g f_0 = \frac{1}{p+1} f_0 + \frac{p}{p+1} f_1$, it suffices to check it on the matrices : $\begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix}$, $k \in \mathbb{Z}$. Since $g = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} = \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, by Lemma 3.3 we can take the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} , \begin{pmatrix} p^{-1} & np^{-1} \\ 0 & 1 \end{pmatrix} , \ 0 \leq n \leq p-1 ,$$

as representatives of the right cosets of Γ contained in $\Gamma g \Gamma$.Then

$$(T_g f_0)\begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix} = \frac{1}{p+1} f_0(\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix}) + \frac{1}{p+1}\sum_{n=0}^{p-1} f_0(\begin{pmatrix} p^{-1} & np^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix})$$

Since the matrices $\begin{pmatrix} 0 & 0 \\ 0 & p^{k-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & np^{k-1} \\ 0 & p^k \end{pmatrix}$, $1 \leq n \leq p-1$,belong to Z_{k-1} , we see that

$$T_g f_0|_{Z_1} = \frac{p}{p+1} , \ T_g f_0|_{Z_0} = \frac{1}{p+1} \text{ and } T_g f_0|_{Z_k} = 0 \text{ for } k \neq 0, 1.$$

This is exactly what was claimed .
It follows from (3.1) that

$$p^{-\beta}\nu(Z_0) = \frac{1}{p+1}\nu(Z_0) + \frac{p}{p+1}\nu(Z_1).$$

On the other hand , for $g = \begin{pmatrix} p^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix}$ we get $T_g f_k = f_{k+1}$, so that

$$p^{-2\beta}\nu(Z_k) = \nu(Z_{k+1}).$$

If $\nu(Z_0) \neq 0$ this implies that $p^{-\beta}$ is a solution of the quadratic equation

$$(p+1)x = 1 + px^2 ,$$

Thus either $p^{-\beta} = p^{-1}$ or $p^{-\beta} = 1$. Since $\beta \neq 1$ we get $\beta = 0$. But then $\nu(Z_k) = \nu(Z_0)$ for any k , and this contradicts $\nu(Mat_2(\mathbb{Z}_p) < \infty$. The contradiction shows that $\nu(Z_0) = 0$ for any k , and *we conclude that the measure of the set of singular matrices is zero.*

(*) To see that the sets Z_k are clopen , define a function :

$$\begin{aligned} h &: \{\text{nonzero singular matrices}\} \to \mathbb{R}\left\{p^k\right\}_{k\in\mathbb{Z}} \\ h(A) &= \max_{i,j} \|a_{i,j}\|_p \ , \text{ for } A = (a_{i,j}) \text{ a nonzero singular matrix.} \end{aligned}$$

As h is a continous function and

$$Z_k = SL_2(\mathbb{Z}_p)\begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix} GL_2(\mathbb{Z}_p) = h^{-1}(\{p^{-k}\}) = h^{-1}((p^{-k-1}, p^{-k+1}))$$

, we see that the sets Z_k are open .

On the other hand , for every $k \in \mathbb{Z}$ the sets $Z_k = SL_2(\mathbb{Z}_p) \begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix} GL_2(\mathbb{Z}_p)$ is the image of the compact space $SL_2(\mathbb{Z}_p) \times GL_2(\mathbb{Z}_p)$ under the map : $(A, B) \to A \begin{pmatrix} 0 & 0 \\ 0 & p^k \end{pmatrix} B$, and hence can be considered as closed sets . ■

We are now ready to show that for $\beta \neq 0, 1$ the set $Mat_2(\mathbf{A}_f) \backslash Mat_2^i(\mathbf{A}_f)$ of zero-divisors has measure zero .

Corollary 23 3.6. *Assume* $\beta \neq 0, 1$ *and* μ *is a measure with properties as in Proposition* 3.2. *Then* $\mathbf{H} \times Mat_2^i(\mathbf{A}_f)$ *is a subset of full measure in* $\mathbf{H} \times Mat_2(\mathbf{A}_f)$.

Proof. Fix a prime p. First of all note that the set

$$\{(\tau, m) \in \mathbf{H} \times Mat_2(\mathbf{A}_f) \mid m_p = 0\}$$

has measure zero. Indeed , as we already remarked before Proposition 3.2 , the set $\mathbf{H} \times \{0\}$ has measure zero . So if our claim is not true , the set

$$\left\{(\tau, m) \in \mathbf{H} \times Mat_2(\hat{\mathbb{Z}})^{\times} \mid m_p = 0\right\}$$

has positive measure . Since the action of Γ on this set is free , there is a subset U of positive measure such that $\gamma U \cap U = \varnothing$ for $\gamma \in \Gamma$, $\gamma \neq e$. Then for $g = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ the set $U_k = g^k U$, $k \in \mathbb{Z}$ still has the property that $\gamma U_k \cap U_k = \varnothing$ for $\gamma \in \Gamma$, $\gamma \neq e$, since g commutes with Γ. As U_k is contained in $\mathbf{H} \times Mat_2(\hat{\mathbb{Z}})$, it follows that $\mu(U_k) \leq 1$. On the other hand , $\mu(U_k) = p^{-2\beta k} \mu(U)$. Letting $k \to -\infty$ if $\beta > 0$ and $k \to +\infty$ if $\beta < 0$, we get a contradiction.

Consider now the restriction of μ to the set

$$\mathbf{H} \times Mat_2(\mathbb{Q}_p) \times \prod_{q \neq p} Mat_2(\mathbb{Z}_q) ,$$

and use the projection onto the first two factors to get a measure μ_p on $\mathbf{H} \times Mat_2(\mathbb{Q}_p)$. By the first part of the proof the set $\mathbf{H} \times \{0\}$ has μ_p -measure zero. Since the image of G_p in $GL_2(\mathbb{Q}_q)$ lies in $GL_2(\mathbb{Z}_q)$ for $q \neq p$, the scaling property of μ implies that

$$\mu_p(gZ) = \det(g)^{-1} \mu_p(Z) \text{ for } Z \subset \mathbf{H} \times Mat_2(\mathbb{Q}_p) \ , \ g \in G_p.$$

Since the action of Γ on $\mathbf{H} \times Mat_2(\mathbb{Q}_p)^{\times}$ is free , the normalization condition on μ implies that $\mu_p(\Gamma\backslash(\mathbf{H} \times Mat_2(\mathbb{Z}_p)) = 1$. Thus μ_p satisfies the assumptions of Lemma 3.5. Hence $\mathbf{H} \times GL_2(\mathbb{Q}_p)$ is a set of full μ_p-measure. This means that the set of points $(\tau, m) \in \mathbf{H} \times Mat_2(\hat{\mathbb{Z}})$ with $\det(m_p) = 0$ has $\mu-$measure zero. By taking the union of such sets for all primes p and multiplying it by elements of $GL_2^+(\mathbb{Q})$ we get a set of measure zero , which is the complement of the set $\mathbf{H}\times Mat_2^i(\mathbf{A}_f)$. $\blacksquare$

To get further properties of a measure μ as above , let us recall the following well-known computation . Denote by S_p the semigroup $G_p \cap Mat_2^+(\mathbb{Z})$. Alternatively , S_p is the of elements $m \in Mat_2^+(\mathbb{Z})$ with determinant a nonnegative power of p. Then from Lemma 3.3 we know that as representatives of the right cosets of Γ in S_p we can take the matrices $\begin{pmatrix} p^k & m \\ 0 & p^l \end{pmatrix}$, $0 \leq k, l$, $0 \leq m < p^l$. Therefore

$$\begin{aligned} \zeta_{s_p,\Gamma}(\beta) \quad &= \sum_{s \in \Gamma\backslash S_p} \det(s)^{-\beta} = \sum_{k,l=0}^{\infty} p^{-\beta(k+l)} p^l = \\ +\infty \ , \ \text{if } \beta \quad &\leq \quad 1 \ , \ \text{and } (1-p^{-\beta})^{-1}(1-p^{-\beta+1})^{-1} \ , \ \text{if } \beta > 1. \quad (3.2) \end{aligned}$$

Since $\Gamma = G_p \cap GL_2(\mathbb{Z}_p)$, we can apply Lemma 2.7 to the group G_p acting on $\mathbf{H}\times Mat_2(\mathbf{A}_f)^{\times}$ and the set

$$Y_0 = \mathbf{H} \times GL_2(\mathbb{Z}_p) \times \prod_{q \neq p} Mat_2(\mathbb{Z}_q).$$

Then for any $s \in S_p$ we get

$$\mu(\Gamma\backslash\Gamma s Y_0) = \det(s)^{-\beta} R_{\Gamma}(s) \mu(\Gamma\backslash Y_0).$$

The sets $\Gamma s Y_0$ are disjoint for s in different double cosets of Γ , and their union is the set

$$\mathbf{H} \times Mat_2^i(\mathbb{Z}_p) \times \prod_{q \neq p} Mat_2(\mathbb{Z}_q) \ ,$$

where $Mat_2^i(\mathbb{Z}_p) = Mat_2(\mathbb{Z}_p) \cap GL_2(\mathbb{Q}_p)$. By Corollary 3.6 the above set is a subset of $\mathbf{H} \times Mat_2(\hat{\mathbb{Z}})$ of full measure for $\beta \neq 0, 1$. Therefore we obtain :

$$1 = \sum_{s \in \Gamma\backslash S_p/\Gamma} \mu(\Gamma\backslash\Gamma s Y_0) = \sum_{s \in \Gamma\backslash S_p/\Gamma} \det(s)^{-\beta} R_{\Gamma}(s) \mu(\Gamma\backslash Y_0) = \zeta_{s_p,\Gamma}(\beta) \mu(\Gamma\backslash Y_0). \quad (3.3)$$

This gives a contradiction if $\beta < 1$. Thus for $\beta < 1$, $\beta \neq 0$, there are no $KMS_{\beta}-$states. On the other hand , for $\beta > 1$ we get :

$$\mu(\Gamma\backslash Y_0) = \varsigma_{S_p,\Gamma}(\beta)^{-1} = (1-p^{-\beta})(1-p^{-\beta+1}).$$

Assuming now that $\beta > 1$ we can perform a similar computation for any finite set of primes instead of just one prime . Given a finite set F of primes consider the group G_F generated by G_p for all $p \in F$. Put also $S_F = Mat_2^+(\mathbb{Z}) \cap G_F$. Then S_F is the set of matrices $m \in Mat_2^+(\mathbb{Z})$ such that all prime divisors of $\det(m)$ belong to F. Let

$$Y_F = \mathbf{H} \times \left(\prod_{p \in F} GL_2(\mathbb{Z}_p) \right) \times \left(\prod_{q \notin F} Mat_2(\mathbb{Z}_q) \right) .$$

Then a computation similar to (3.2) and (3.3) yields :

$$\zeta_{S_F,\Gamma}(\beta) = \prod_{p \in F} (1-p^{-\beta})^{-1}(1-p^{-\beta+1})^{-1} \text{ and } \mu(\Gamma \backslash Y_F) = \prod_{p \in F} (1-p^{-\beta})(1-p^{-\beta+1}). \quad (3.4)$$

The intersection of the sets Y_F over all finite subsets F of prime numbers is the set $\mathbf{H} \times GL_2(\hat{\mathbb{Z}})$. So for $\beta > 2$ we get :

$$\mu(\Gamma \backslash (\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))) = \prod_p (1 - p^{-\beta})(1 - p^{-\beta+1}) = \zeta(\beta)^{-1}\zeta(\beta - 1)^{-1} ,$$

where ζ is the Riemann $\zeta-$function . On the other hand , for $\beta \in (1, 2]$ we get $\mu(\Gamma \backslash (\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))) = 0$.

Assume now that $\beta > 2$. In this case similarly to (3.2) we have

$$\zeta_{Mat_2^+(\mathbb{Z}),\Gamma}(\beta) = \zeta(\beta)\zeta(\beta - 1).$$

So analogously to (3.3) we get

$$\mu(\Gamma \backslash Mat_2^+(\mathbb{Z})(\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))) = \zeta_{Mat_2^+(\mathbb{Z}),\Gamma}(\beta)\mu(\Gamma \backslash (\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))) = 1.$$

We thus see that $Mat_2^+(\mathbb{Z})(\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))$ is a subset of $\mathbf{H} \times Mat_2(\hat{\mathbb{Z}})$ of full measure . Hence $GL_2^+(\mathbb{Q})(\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))$ is a subset of $\mathbf{H} \times Mat_2(\mathbf{A}_f)$ of full measure . By Lemma 3.4 the set $GL_2^+(\mathbb{Q})(\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))$ is nothing but $\mathbf{H} \times GL_2(\mathbf{A}_f)$.

To summarize , we have shown that for $\beta > 2$ the problem of finding all measures μ on $\mathbf{H} \times Mat_2(\mathbf{A}_f)$ satisfying the conditions in Proposition 3.2 reduces to finding all measures on $\mathbf{H} \times GL_2(\mathbf{A}_f)$ such that

$$\mu(gZ) = \det(g)^{-\beta}\mu(Z) \text{ and } \mu(\Gamma \backslash (\mathbf{H} \times GL_2(\hat{\mathbb{Z}}))) = \zeta(\beta)^{-1}\zeta(\beta - 1)^{-1}.$$

By Lemma 2.4 any $\Gamma-$invariant measure on $\mathbf{H} \times GL_2(\hat{\mathbb{Z}})$ extends uniquely to a measure on $\mathbf{H} \times GL_2(\mathbf{A}_f)$ satisfying the scaling condition . Thus we get a *one-to-one correspondence* between measures μ with properties as in Proposition 3.2 and measures on $\Gamma\backslash(\mathbf{H}\times GL_2(\hat{\mathbb{Z}})$ of total mass $\zeta(\beta)^{-1}\zeta(\beta-1)^{-1}$. Clearly , extremal measures μ correspond to point masses .

We have thus recovered the following result of Connes and Marcolli [5 , Theorem 1.26 and Cor
.

Theorem 24 3.7. *For the Connes-Marcolli* GL_2-*system we have :*
(ι) *for* $\beta\in(-\infty,0)\cup(0,1)$ *there are no* $KMS_\beta-$*states ;*
$(\iota\iota)$ *for* $\beta>2$ *there is a* one-to-one *affine correspondence between* $KMS_\beta-$*states and probability measures on* $\Gamma\backslash(\mathbf{H}\times GL_2(\hat{\mathbb{Z}}))$; in particular , *extremal* $KMS_\beta-$*states are in* bijection *with* $\Gamma-$*orbits in* $\mathbf{H}\times GL_2(\hat{\mathbb{Z}})$.