Optimization by controlled outphasing given a pasture – livestock system

by

KARI T. HYLLAND

THESIS

for the degree of

MASTER IN MATHEMATICS

(Master of Science)

Faculty of Mathematics and Natural Sciences
University of Oslo

May 2010

Preface

This project was written from January 2010 to May 2010. I would like to thank my supervisor, Professor Tom Lindstrøm, for his optimistic support, good advice and all the help he has given me during the writing process. Also, I am very grateful to Nikolay for his helpful suggestions and for taking such good care of me these four months. I would also like to thank the other people on the 6th floor for their good spirits and positivity, and my family for their patience and support during my time at the university.

Contents

1 Introduction

Contents

This project is based on a model for a pasture – livestock system and an optimization problem described in [2], where the owner of the livestock wants to maximize his utility from consumption given an infinite time horizon. Our approach is to study the same model, but we change the optimization problem to a problem where we optimize by controlled outphasing of the livestock. In this chapter, we will first give a brief overview the model and results in [2]. We then outline the problem we discuss further in this project.

1.1 Optimization with infinite time horizon

In [2], Brekke et al. consider a common pasture owned by N herders. Let us denote the total plant biomass available to grazing at time t by $x(t)$ and the animal biomass owned by herder number i by $y_i(t)$. The livestock of herder number i follows the dynamics

$$y_i'(t) = h(t, x(t))y_i(t) - c_i(t) \tag{1.1}$$

for $i = 1, \ldots, N$. The function h is bounded and increasing in x, and describes the growth rate of the livestock given its response to the food available and the natural mortality rate. (This h will later be referred to as the *natural growth rate*.) In addition, the herder is free to harvest a suitable amount of his livestock at every time t, described by the consumption rate c_i. $c_i : [0, \infty) \to [0, \infty)$ is a positive function such that $y_c(t) = y(t) \geq 0$, $\forall t$. We let the total amount of animal biomass be denoted by

$$Y(t) = \sum_{i=1}^{N} y_i(t).$$

The plant biomass follows the dynamics

$$x'(t) = g(t, x(t), Y(t))x(t) - kh(t, x(t))Y(t). \tag{1.2}$$

The function g is bounded, decreasing in x and increasing in Y, and describes the growth rate of animal biomass in the absence of grazing. h is defined as above and $k > 0$ is a constant. As pointed out in [5], p. 8, the effect of grazing is two-folded: on one hand it reduces the plant biomass by the term $kh(t, x(t))$, but it also has a positive effect on the plant growth rate, for example by stimulating regrowth. The dynamics of the complete pasture – livestock system is described by

$$\begin{aligned} x'(t) &= g(t, x(t), Y(t))x(t) - kh(t, x(t))Y(t) && (1.3)\\ Y'(t) &= h(t, x(t))Y(t) - \hat{c}(t), && (1.4) \end{aligned}$$

where $\hat{c}(t) = \sum_{i=1}^{N} c_i(t)$.

Every herder wishes to maximize his utility, given by

$$J_i(c_i) = E\left(\int_0^\infty u_i(c_i(t))e^{-\delta t}dt\right) \tag{1.5}$$

where $\delta > 0$ is a discount rate common to all the herders, and $u_i : [0, \infty) \to \mathbb{R}$ are given utility functions[1]. The herders are assumed to have an infinite planning horizon. Brekke et al. concentrate on the special case where $u_i(c_i) = \ln(c_i)$, and discuss both individual and cooperative optimization. In the following, we will only consider the case of individual optimization, i.e., a scenario where every herder is only interested in maximizing his own profit and is unable and/or unwilling to cooperate with the other herders. For a discussion on the cooperative case, see [2] and [5].

We can sum up the individual optimization problem of Brekke et al. the following way:

Problem 1.1 (The individual optimization problem of Brekke et al.) *We want to maximize*

$$J_i(c_i) = E\left(\int_0^\infty \ln(c_i(t))e^{-\delta t}dt\right), \tag{1.6}$$

subject to the differential equations

$$\begin{aligned} x'(t) &= g(t, x(t), Y(t))x(t) - kh(t, x(t))Y(t) && (1.7)\\ y_i'(t) &= h(t, x(t))y_i(t) - c_i(t) && (1.8) \end{aligned}$$

[1] A precise definition is given in Chapter 2.2.

and the boundary conditions

$$x(0) = x_0 > 0 \tag{1.9}$$
$$y_i(0) = y_0^i > 0. \tag{1.10}$$

In the case of individual optimization, the amount of animal biomass owned by herder number i is low relative to the total biomass owned by all the herders and we can assume that the effect of the individual herder on the ecosystem is neglectable. Further, we assume that the dynamics of x are beyond the herder's control. Hence, h will also be beyond the herder's control. In [1], Brekke et al. show that given a very general, possibly stochastic, h, it is optimal to harvest a constant fraction of the livestock, given by

$$\lambda^* = \frac{c_i^*(t)}{y_i(t)} = \delta.$$

Hence, the optimal consumption for the individual herder is

$$c_i^*(t) = \delta y_i(t). \tag{1.11}$$

Brekke et al. conclude that systems of this kind are likely to experience limit cycles when all N herders follow the optimization strategy described above. For more details, see [1], [2] and [5].

1.2 Optimization with controlled outphasing

In this project, we will consider an alternative scenario for individual optimization. We use the same model as in Chapter 1.1, but we assume that the herder wants to phase out the production at a finite time τ. This τ is not fixed, but has to be chosen to maximize

$$J_i(c_i, \tau) = E\left(\int_0^\tau u_i(c_i(t))e^{-\delta t}dt\right), \tag{1.12}$$

where

$$\tau = \inf\{t \geq 0 : y_i(t) \leq 0\}.$$

In this scenario of *controlled outphasing*, our task is to describe the maximizing τ and c_i.

In other words, we want to solve the following optimization problem:

Problem 1.2 (The general optimization problem) *Maximize*

$$J_i(c_i, \tau) = E\left(\int_0^\tau u_i(c_i(t))e^{-\delta t}dt\right), \tag{1.13}$$

where

$$\tau = \inf\{t \geq 0 : y_i(t) \leq 0\},$$

subject to the differential equations

$$x'(t) = g(t, x(t), Y(t))x(t) - kh(t, x(t))Y(t) \quad (1.14)$$
$$y_i'(t) = h(t, x(t))y_i(t) - c_i(t) \quad (1.15)$$

and the boundary conditions

$$x(0) = x_0 > 0 \quad (1.16)$$
$$x(\tau) \geq 0 \quad (1.17)$$
$$y_i(0) = y_0^i > 0 \quad (1.18)$$
$$y_i(\tau) = 0. \quad (1.19)$$

To be able to solve Problem 1.2 analytically, we make some simplifications and further assumptions in the model described in Chapter 1.1. First of all, we restrict our analysis to the deterministic case, i.e., we let h, g, c_i be deterministic functions. Further, we assume that

$$c_1 = c_2 = \ldots = c_N$$

and

$$u_1 = u_2 = \ldots = u_N,$$

i.e., that all N herders have the same consumption rate and utility function. For notational simplicity, we set

$$c := c_i$$

and

$$u := u_i.$$

Let $y_i(0) = y_0^i > 0$ be given, and assume that

$$y_0^1 = y_0^2 = \ldots = y_0^N$$

i.e., that we have the same initial condition on y_i for all i. We set

$$y_0 := y_0^i.$$

Then, from the theory of ordinary deterministic differential equations, we have

$$y_1 = y_2 = \ldots = y_N.$$

We set

$$y := y_i$$

for notational simplicity. With these assumptions on c, y, we see that the total amount of animal biomass can be expressed in the following way:

$$Y(t) = Ny(t).$$

We can then rewrite the dynamics of x:

$$x'(t) = g(t, x(t), y(t))x(t) - kNh(t, x(t))y(t). \quad (1.20)$$

Before we formulate a simplified version of the optimization Problem 1.2, we make some remarks:

Remark 1.1 We have that u is a utility function (a precise definition of u will be given in Chapter 2.2). In addition, we assume that $u \in C^2$. $\triangle$

Remark 1.2 To be able to use standard methods to solve the differential equations, we let h, g depend on t only. Let h be a continuous function such that $h(t) < \delta - \epsilon$, $\forall t \geq 0, \forall \epsilon \in (0, \delta)$. Further assume that $H(t) = \int_0^t h(s)ds$ is monotonous, i.e., strictly increasing or strictly decreasing, for $t \geq 0$. Let g be a bounded and continuous function. We recall that $k > 0$. $\triangle$

Remark 1.3 We define $\mathbf{x}(t) = (x(t), y(t))$ and assume that there exists a constant $d \in (0, \infty)$ such that

$$\| \mathbf{x}(t) \| \leq d, \forall t.$$

This is a reasonable restriction to make on $\mathbf{x}$: it is reasonable to assume that the pasture, and hence the amount of plant biomass, x, is limited. If we had an infinitely large livestock, this would not make any sense; first of all, it is an unlikely scenario given the limited food supply. Furthermore, we remember that we are aiming to maximize the herder's profit, not the herder's amount of livestock. It would certainly be profitabe to harvest the livestock before the size of the herd reach infinity. $\triangle$

Remark 1.4 **a)**We let τ vary in a closed interval $[0, T]$, where $0 < T < \infty$, but still sufficiently large. **b)**We let $c_i : [0, T] \to [0, K]$, where $0 < K < \infty$, but still sufficiently large. We now give a short argument for this "closing" of the intervals: **a)**We show that the utility

$$\int_T^{T_1} u(c(t))e^{-\delta t}dt$$

will be very small when T is large enough. We have $T_1 > T$, and typically, $T_1 = \infty$. From the definition of u in Chapter 2.2, we know that $u(c) \leq \mu c$, for some constant μ. We observe that we have

$$c(t) = -y'(t) + h(t)y(t)$$

from equation (1.15). Further, solving (1.15) with the initial condition (1.18), we obtain

$$y(t) = e^{H(t)} \left(y_0 - \int_0^t e^{-H(s)}c(s)ds \right),$$

which implies that, for all t,

$$y(t)e^{-H(t)} \leq y_0.$$

We recall that

$$h(t) < \delta - \epsilon, \forall t,$$

and hence

$$\delta t - H(t) \geq \epsilon t, \forall t.$$

We are now ready to estimate the utility:

$$\begin{aligned}
\int_T^{T_1} u(c(t))e^{-\delta t}dt &\leq \mu \int_T^{T_1} c(t)e^{-\delta t}dt \\
&= \mu \int_T^{T_1} (-y'(t) + h(t)y(t))e^{-\delta t}dt \\
&= \mu \int_T^{T_1} (-y'(t)e^{-H(t)} + h(t)e^{-H(t)}y(t))e^{-\delta t + H(t)}dt \\
&= \mu \int_T^{T_1} -(y(t)e^{-H(t)})'e^{-\delta t + H(t)}dt \\
&\leq \mu e^{-\epsilon T} \int_T^{T_1} -(y(t)e^{-H(t)})'dt \\
&= \mu e^{-\epsilon T}[(y(T)e^{-H(T)}) - (y(T_1)e^{-H(T_1)})] \\
&\leq \mu e^{-\epsilon T}(y(T)e^{-H(T)}) \\
&\leq \mu e^{-\epsilon T} y_0.
\end{aligned}$$

From the estimate above, we see that the utility in the interval $[T, T_1]$ is neglectable when T is large. Hence, we can restrict the interval that τ varies in to $[0, T]$ without losing any utility of significance when T is sufficiently large.

b)We proceed by showing that when K is big enough, it does not pay off to allow controls c that take on values larger than K. We let

$$\epsilon_K = \sup \left\{ \frac{u(c)}{c} : c > K \right\},$$

and observe that $\lim_{C\to\infty} \epsilon_K = 0$. Given a control c, let

$$c_K(t) = \begin{cases} c(t) & \text{if } c(t) \leq K \\ K & \text{otherwise} \end{cases}$$

We will show that, by chosing K large enough, we can get the difference

$$\int_0^T u(c(t))e^{-\delta t}dt - \int_0^T u(c_K(t))e^{-\delta t}dt$$

as small as we want:

$$\begin{aligned}
\int_0^T u(c(t))e^{-\delta t}dt - \int_0^T u(c_K(t))e^{-\delta t}dt &\leq \int_{\{c(t)>K\}} u(c(t))e^{-\delta t}dt \\
&\leq \epsilon_K \int_{\{c(t)>K\}} c(t)e^{-\delta t}dt \\
&\leq \epsilon_K \int_0^T c(t)e^{-\delta t}dt \\
&= \epsilon_K \int_0^T (-y'(t) + h(t)y(t))e^{-\delta t}dt \\
&= \epsilon_K \int_0^T (-y'(t)e^{-H(t)} \\
&+ h(t)e^{-H(t)}y(t))e^{-\delta t + H(t)}dt \\
&\leq \epsilon_K \int_0^T -(y(t)e^{-H(t)})'dt \\
&= \epsilon_K[y_0 - y(T)e^{-H(T)}] \\
&\leq \epsilon_K y_0.
\end{aligned}$$

We have $\lim_{K\to\infty} \epsilon_K y_0 = 0$. Hence, we can let c vary in $[0, K]$ without losing any utility of significance. $\triangle$

Now, we are finally ready to state our version of Problem 1.2:

Problem 1.3 (A simplified version of the optimization problem) *We want to maximize*

$$J(c, \tau) = \int_0^\tau u(c(t))e^{-\delta t}dt \tag{1.21}$$

$$\tau = \inf\{t \geq 0 \mid y(t) \leq 0\}, \tag{1.22}$$

subject to the differential equations

$$\begin{aligned}
x'(t) &= g(t)x(t) - kNh(t)y(t) \qquad (1.23)\\
y'(t) &= h(t)y(t) - c(t) \qquad (1.24)
\end{aligned}$$

and the boundary conditions

$$\begin{aligned}
x(0) &= x_0 > 0 \qquad (1.25)\\
x(\tau) &\geq 0 \qquad (1.26)\\
y(0) &= y_0 > 0 \qquad (1.27)\\
y(\tau) &= 0. \qquad (1.28)
\end{aligned}$$

1.3 Solving Problem 1.3

In the following chapters, we will show that it is possible to solve Problem 1.3. We have chosen the following approach:

Before we actually begin solving the problem, we will present the essential theoretical background in Chapter 2. Then, in Chapter 3, we simplify Problem 1.3 further by omitting equation (1.23). This way, we assume that there is always food available for the livestock. We show that we can find a unique solution for this simplified problem and we describe this solution. In Chapter 4, we go back to Problem 1.3. As it turns out, solving this problem generally is quite difficult. However, we use the results from Chapter 3 to find a solution to Problem 1.3 when $x(\tau^*) > 0$, $\forall \tau^*$. We also consider what we need to know about the initial condition x_0 to make sure that we always have $x(\tau^*) > 0$. The last chapter gives a brief discussion on the possible advantages of choosing our optimization strategy over the optimization strategy of Brekke et al.

2

Theoretical background

Contents

In this chapter, we will first place Problem 1.3 in a general control theoretical context. This will be done by presenting the main concepts from control theory and by outlining a general control problem. For a more detailed and general introduction to the subject, see [3] and [6]. In the second part of the chapter, we will give a definition of utility functions, based on a definition in [4], but with some more conditions added. This definition will be used throughout the project.

2.1 Control theory – a very brief introduction

Definition 2.1 (Control functions and control space) *Let $t \in [t_0, t_1]$, where t_0 is fixed and t_1 is allowed to vary in a closed interval $[t_0, T]$, $t_0 < T < \infty$. Let $c_1, \ldots, c_r$ be piecewise continuous functions and let $[c_1(t), \ldots, c_r(t)] \in U \subseteq \mathbf{R}^r$, where U is closed and bounded. The functions $c_1, \ldots, c_r$ are called* control functions. *The set U is called the* control space.

Remark 2.1 In the following, we will let $r = 1$ and $U = [0, K]$, where $0 < K < \infty$. △

Definition 2.2 (The state of the system) *Let $\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_n(t))$ be a vector function in $\mathbf{R}^n$ and let the dynamics of $\mathbf{x}(t)$ be described by*

$$\begin{aligned} x_1'(t) &= f_1(\mathbf{x}(t), c(t), t) \\ x_2'(t) &= f_2(\mathbf{x}(t), c(t), t) \\ &\vdots \\ x_n'(t) &= f_n(\mathbf{x}(t), c(t), t), \end{aligned} \tag{2.1}$$

where $f_i(\mathbf{x}(t), c(t), t)$ *and* $\frac{\partial f_i(\mathbf{x}(t),c(t),t)}{\partial x_j}$ *are continuous with respect to all the* $n+2$ *variables for* $i, j = 1, \ldots, n$. $\mathbf{x}(t)$ *is called the* state *of the system (2.1) at time* t.

Let us assume that the state of the system is known at the initial time t_0, i.e., that we are given

$$\begin{aligned} x_1(t_0) &= x_0^1 \\ x_2(t_0) &= x_0^2 \\ &\vdots \\ x_n(t_0) &= x_0^n. \end{aligned} \tag{2.2}$$

As the initial value of $\mathbf{x}$ is given, we will have a unique solution to the system (2.1). Further assume that we want the state of the system to hit a certain surface R at the terminal time t_1. More precisely, we want $\mathbf{x}(t_1)$ to satisfy the following transversality conditions:

$$\begin{aligned} R_j(\mathbf{x}(t_1), t_1) &\geq 0, j = 1, \ldots, k', \\ R_j(\mathbf{x}(t_1), t_1) &= 0, j = k'+1, \ldots, k. \end{aligned} \tag{2.3}$$

We assume that the R_j's are C^1-functions.

Definition 2.3 (Admissible triple) *If* $t_1 \in [t_0, T]$, $c(t) \in U$ *is any piecewise continuous function and* $\mathbf{x}(t)$ *is a continuously differentiable function such that (2.1), (2.2) and (2.3) are satisfied, we call* $(t_1, c(t), \mathbf{x}(t))$ *an* admissible triple. *A pair* $(c(t), \mathbf{x}(t))$ *that is such that* $(t_1, c(t), \mathbf{x}(t))$ *is an admissible triple, is called an* admissible pair.

The *optimal control problem* can be formulated in the following way:

Problem 2.1 (The optimal control problem) *Let the function* f_0 *be such that* $f_0(\mathbf{x}(t), c(t), t)$ *and* $\frac{\partial f_0(\mathbf{x}(t),c(t),t)}{\partial x_j}$ *are continuous with respect to all the* $n+2$ *variables. We want to find an admissible triple* $(t_1, c(t), \mathbf{x}(t))$ *such that the* criterion functional J, *defined as*

$$J(c, t_1) = \int_{t_0}^{t_1} f_0(\mathbf{x}(t), c(t), t)dt, \tag{2.4}$$

is maximized subject to the differential equations in (2.1), the initial conditions in (2.2) and the transversality conditions in (2.3). We denote this optimal triple *by* $(t_1^*, c^*(t), \mathbf{x}^*(t))$, *where* $c^*(t), \mathbf{x}^*(t)$ *are defined on* $[t_0, t_1^*]$.

Remark 2.2 The optimal triple will later be referred to as the *optimal solution* of the problem.

Definition 2.4 (The optimal value function) *The* optimal value function V *is defined to be*

$$V(t_0, \mathbf{x}_0, \mathbf{x}(t_1)) = \sup \left\{ \int_{t_0}^{t_1} f_0(\mathbf{x}(t), c(t), t)dt : (t_1, c(t), \mathbf{x}(t)) \text{ admissible} \right\}.$$

V is defined only for those triples $(t_0, \mathbf{x}_0, \mathbf{x}(t_1))$ for which admissible triples exist. If an optimal triple exists for a given triple $(t_0, \mathbf{x}_0, \mathbf{x}(t_1))$, then

$$V(t_0, \mathbf{x}_0, \mathbf{x}(t_1)) < \infty$$

and

$$V(t_0, \mathbf{x}_0, \mathbf{x}(t_1)) = J^*,$$

where

$$J^* = J(c^*, t_1^*) = \int_{t_0}^{t_1^*} f_0(\mathbf{x}^*(t), c^*(t), t)dt.$$

We observe that Problem 1.3 is just a special case of Problem 2.1. In order to find a solution for Problem 2.1, one possible strategy is to first use Pontryagin's principle, a result that provides necessary condition for optimality, to find a candidate for the optimal solution, and then prove that the solution actually exists using the Filippov-Cesari existence theorem. We present the theorems below:

Theorem 2.1 (Pontryagin's principle, [6], p. 180) *Let $(\mathbf{x}^*(t), c^*(t), t_1^*)$ be an optimal solution for Problem 2.1. Then, with $t_1 = t_1^*$, we have the following:*
There exist constants $p_0, \gamma_1, \ldots, \gamma_k$ and a continuous and piecewise continuously differentiable vector function $\mathbf{p}(t) = (p_1(t), \ldots, p_n(t))$ such that for all $t \in [t_0, T]$:

(1) $(p_0, \gamma_1, \ldots, \gamma_k) \neq (0, \ldots, 0)$;

(2) $c^*(t)$ *maximizes the Hamiltonian,* $H(\mathbf{x}^*(t), c(t), \mathbf{p}(t), t)$, *for* $c \in U$, *i.e.,*

$$H(\mathbf{x}^*(t), c^*(t), \mathbf{p}(t), t) \geq H(\mathbf{x}^*(t), c(t), \mathbf{p}(t), t), \forall c \in U. \tag{2.5}$$

The Hamiltonian is defined as

$$H(\mathbf{x}, c, \mathbf{p}, t) = p_0 f_0(\mathbf{x}, c, t) + \sum_{j=1}^{n} p_j f_j(\mathbf{x}, c, t); \tag{2.6}$$

(3) Except at the points of discontinuity of $c^*(t)$, *we have, for* $j = 1, \ldots, n$:

$$p_j'(t) = \frac{-\partial H^*}{\partial x_j} \tag{2.7}$$

where

$$\frac{\partial H^*}{\partial x_j} = \frac{\partial H(\mathbf{x}^*(t), c^*(t), \mathbf{p}(t), t)}{\partial x_j};$$

(4) $p_0 = 1$ *or* $p_0 = 0$;

(5) The following transversality conditions are satisfied for $j = 1, \ldots, n$:

$$p_j(t_1^*) = \sum_{l=1}^{k} \gamma_l \frac{\partial R_l(\mathbf{x}^*(t_1^*), t_1^*)}{\partial x_j} \tag{2.8}$$

where we have

$$\gamma_l \geq 0 \ (= 0 \text{ if } R_l(\mathbf{x}^*(t_1^*), t_1^*) > 0)$$

for $l = 1, \ldots, k'$, *and* γ_l *is a constant (also possibly negative) for* $l = k'+1, \ldots, k$;

(6)

$$H(\mathbf{x}^*(t_1^*), c^*(t_1^*), \mathbf{p}(t_1^*), t_1^*) + \sum_{l=1}^{k} \gamma_l \frac{\partial R_l(\mathbf{x}^*(t_1^*), t_1^*)}{\partial t} = 0 \text{ if } t_1^* \in (t_0, T) \quad (2.9)$$

$$\geq 0 \text{ if } t_1^* = T \quad (2.10)$$

If $t_1^* = t_0$, we have

$$\sup_{c \in U} H(\mathbf{x}_0, c, \mathbf{p}(t_1^*), t_1^*) + \sum_{l=1}^{k} \gamma_l \frac{\partial R_l(\mathbf{x}_0, t_0)}{\partial t} \leq 0. \quad (2.11)$$

Before we present our existence result, we define the set $N(\mathbf{x}, U, t) \in \mathbf{R}^{n+1}$:

$$\boxed{N(\mathbf{x}, U, t) := \{(f_0(\mathbf{x}, c, t) + \beta, f_1(\mathbf{x}, c, t), f_2(\mathbf{x}, c, t), \ldots, f_n(\mathbf{x}, c, t)) : \beta \leq 0, c \in U\}}$$

Theorem 2.2 (Filippov-Cesari existence, [6], p. 145) *We consider Problem 2.1. Assume that $[t_0, T]$ is a bounded interval and that t_1 varies in $[t_0, T]$. Further, assume that:*

(a) There exists an admissible pair $(\mathbf{x}(t), c(t))$.

(b) $N(\mathbf{x}, U, t)$ is convex for each $(\mathbf{x}, t)$.

(c) U is closed and bounded.

(d) There exists a number b such that $\| \mathbf{x} \| \leq b$, $\forall t \in [t_0, T]$ and all admissible pairs $(\mathbf{x}(t), c(t))$.

Then there exists an optimal, measurable control $c^*(t)$.

2.2 Utility functions

Definition 2.5 *A* utility function *is a concave, non-decreasing, continuous and differentiable function $u : \mathbb{R} \to [-\infty, \infty)$ satisfying:*

- *The half-line*

$$\text{dom(u)} := \{x \in \mathbb{R} : u(x) > -\infty\}$$

  is a non-empty subset of $[0, \infty)$.

- *u' is continuous, positive and strictly decreasing on the interior of* dom(u), *and*

$$\lim_{x \to \infty} u'(x) = 0$$

We set

$$\overline{x} := \inf\{x \in \mathbb{R} : u(x) > -\infty\}$$

so that $\overline{x} \in [0, \infty)$ *and either* $\mathrm{dom(u)} = [\overline{x}, \infty)$ *or* $\mathrm{dom(u)} = (\overline{x}, \infty)$. *We define*

$$u'(\overline{x}+) := \lim_{x \to \overline{x}^+} u'(x)$$

so that $u'(\overline{x}+) \in (0, \infty]$.

- *We assume* $u(\overline{x}) < 0$ *and that there exists an* x_0 *such that* $u(x) > 0, \forall x > x_0$.

With u a utility function and $\overline{x}$ as above, one can show that the strictly decreasing, continuous and surjective function

$$u' : (\overline{x}, \infty) \to (0, u'(\overline{x}+))$$

has a strictly decreasing, continuous and surjective inverse

$$(u')^{-1} : (0, u'(\overline{x}+)) \to (\overline{x}, \infty).$$

In order to have $(u')^{-1}$ defined, finite and continuous on $(0, \infty]$, we set $(u')^{-1}(y) = \overline{x}$ for $y \in [u'(\overline{x}+), \infty]$, and we set

$$u'((u')^{-1})(y) = \begin{cases} y & \text{if } y \in (0, u'(\overline{x}+)) \\ u'(\overline{x}+) & \text{if } y \in [u'(\overline{x}+), \infty] \end{cases} \tag{2.12}$$

$$(u')^{-1}(u')(x) = x, \text{for } x \in (\overline{x}, \infty). \tag{2.13}$$

Remark 2.3 We will assume $\overline{x} = 0$ throughout this project. △

Example 2.1 Examples of common utility functions satisfying the above conditions are:
• Logarithmic utility:

$$u(x) = \begin{cases} \ln x & \text{if } x \in (0, \infty) \\ -\infty & \text{if } x \in [-\infty, 0] \end{cases}$$

We have $\overline{x} = 0$ and dom (u) $= (\overline{x}, \infty)$.
• Power utility:

$$u(x) = \begin{cases} \frac{x^p}{p} - 1 & \text{if } x \in [0, \infty) \\ -\infty & \text{if } x \in [-\infty, 0) \end{cases}$$

where $p \in (0, 1)$. We have $\overline{x} = 0$ and dom (u) $= [\overline{x}, \infty)$. ♡

3 Finding a solution, part I: Simplifying Problem 1.3

Contents

In this chapter, we will simplify Problem 1.3 by omitting the differential equation (1.23). Given this simplification, we will first show that we can find a unique candidate for an optimal solution of the problem, and compute this explicitly in two examples. Then we will show the existence of the optimal solution.

3.1 The simplified version of Problem 1.3

Problem 3.1 (The simplified version of Problem 1.3) *We want to maximize*

$$J(c,\tau) = \int_0^\tau u(c(t))e^{-\delta t}dt \tag{3.1}$$

$$\tau = \inf\{t \geq 0 \mid y(t) \leq 0\}, \tag{3.2}$$

subject to the differential equation

$$y'(t) = h(t)y(t) - c(t) \tag{3.3}$$

with boundary conditions

$$y(0) = y_0 > 0 \tag{3.4}$$
$$y(\tau) = 0. \tag{3.5}$$

The conditions on τ, h and c are the same as in Chapter 1.2. We assume that $y(t) \leq \tilde{d}$, $\forall t$, where $\tilde{d} < \infty$ is a positive constant.

3.2 Finding candidates for an optimal solution

Let us assume that we have found an optimal solution (y^*, c^*, τ^*) for Problem 3.1. We now use Theorem 2.1 to determine the optimal solution. We observe that we have $R(y(\tau), \tau) = y(\tau)$.

Let us now restate the conditions (1)-(6) from Theorem 2.1 for Problem 3.1:

(1) $(p_0, \gamma) \neq (0, 0)$;

(2) $c^*(t)$ maximizes the Hamiltonian, $H(y^*(t), c(t), p(t), t)$, for $c \in [0, K]$, i.e.,

$$H(y^*(t), c^*(t), p(t), t) \geq H(y^*(t), c(t), p(t), t), \forall c \in [0, K]. \tag{3.6}$$

The Hamiltonian in this case is

$$H(y(t), c(t), p(t), t) = p_0 u(c(t)) e^{-\delta t} + p(t)[h(t)y(t) - c(t)]; \tag{3.7}$$

(3) Except at the points of discontinuity of $c^*(t)$, we have

$$p'(t) = \frac{-\partial H^*}{\partial y} \tag{3.8}$$

where

$$\frac{\partial H^*}{\partial y} = \frac{\partial H(y^*(t), c^*(t), p(t), t)}{\partial y};$$

(4) $p_0 = 1$ or $p_0 = 0$;

(5) The following transversality condition is satisfied:

$$p(\tau^*) = \gamma \frac{\partial R(y^*(\tau^*), \tau^*)}{\partial y} = \gamma \tag{3.9}$$

where γ is a constant (also possibly negative);

(6)

$$\begin{aligned} H(y^*(\tau^*), c^*(t_1^*), p(\tau^*), \tau^*) &= 0 \text{ if } \tau^* \in (0, T) && (3.10) \\ &\geq 0 \text{ if } \tau^* = T && (3.11) \end{aligned}$$

If $\tau^* = 0$, we have

$$\sup_{c \in U} H(y_0, c, p(\tau^*), \tau^*) \leq 0. \tag{3.12}$$

In the remaining part of the chapter, we will use the conditions above to find $p_0, p(t), c^*, \tau^*, \gamma$.

We observe that solving the differential equation (3.3) with the initial condition (3.4), we find, using standard methods:

$$y^*(t) = e^{H(t)} \left(y_0 - \int_0^t e^{-H(s)} c^*(s) ds \right) \tag{3.13}$$

where $H(t) = \int_0^t h(s)ds$.

To find $p(t)$, we begin by solving the differential equation

$$p'(t) = -\frac{\partial H^*}{\partial y} = -p(t)h(t).$$

Using standard methods, we get that

$$p(t) = Ce^{-H(t)}$$

where C is a constant. To determine C, we use the transversality condition from (5),

$$p(\tau^*) = \gamma.$$

This implies that $C = \gamma e^{H(\tau^*)}$, and hence

$$p(t) = \gamma e^{H(\tau^*)-H(t)}. \tag{3.14}$$

We observe that the sign of $p(t)$ only depends on the sign of γ. We will consider an interpretation of $p(\tau^*)$ in order to determine the sign of γ. As we have already seen, in our case condition (5) is

$$p(\tau^*) = \frac{\gamma \partial R(y(\tau^*), \tau^*)}{\partial y} = \gamma.$$

We have that τ is a free terminal time, $\tau \in [t_0, T]$, and that the initial time t_0, the initial state $y(t_0) = y_0$ and the terminal state $y(\tau) = y_\tau$ are fixed. As we saw in Chapter 2.1, we define the optimal value function V associated with the triple (t_0, y_0, y_τ) as

$$V(t_0, y_0, y_\tau) = \sup \left\{ \int_{t_0}^{\tau} u(c(t))e^{-\delta t} dt : (y(t), c(t), \tau) \text{ admissible} \right\}.$$

According to [6], p. 215, Theorem 3.11, the value function $V(t_0, y_0, y_\tau)$ is defined and differentiable with respect to y_τ, and we have

$$-p(\tau^*) = \frac{\partial V(t_0, y_0, y_{\tau^*})}{\partial y_{\tau^*}}.$$

In other words, we can say that $-p(\tau^*)$ measures the change in the value function given a certain change in the desired terminal value of y. In our case, V will be a decreasing function with respect to y_τ. This is because u is non-decreasing and we must assume that consumption is non-decreasing as the amount of lifestock to consume increases. Hence, we have

$$\frac{\partial V(t_0, y_0, y_{\tau^*})}{\partial y_{\tau^*}} \leq 0 \tag{3.15}$$

From the discussion above, we see that we can rewrite condition (5) as

$$\frac{\partial V(t_0, y_0, y_{\tau^*})}{\partial y_{\tau^*}} = -\gamma. \tag{3.16}$$

From (3.15) and (3.16), we conclude that $\gamma \geq 0$.

To determine $c^*(t)$, we use condition (2) from Theorem 2.1. We know that $c^*(t)$ maximizes

$$p_0 u(c(t))e^{-\delta t} + p(t)[h(t)y^*(t) - c(t)].$$

In other words, we need c^* such that

$$f(c) = p_0 u(c)e^{-\delta t} - p(t)c$$

is maximized.

First assume $p_0 = 1$. We observe that $p_0 u(c)e^{-\delta t}$ is concave in c and that the second term is linear (and hence concave) in c. Therefore, f is concave in c as well, and thus we can use standard methods to find the maximizing c. We differentiate $f(c)$ and set $f'(c) = 0$:

$$\begin{aligned} u'(c(t))e^{-\delta t} - p(t) &= 0 \\ u'(c(t)) &= e^{-\delta t}p(t) \\ c^*(t) &= (u')^{-1}(\gamma e^{H(\tau^*)+\delta t - H(t)}). \end{aligned} \tag{3.17}$$

With $p_0 = 1$, we observe that

$$\boxed{c^*(\tau^*) = (u')^{-1}(\gamma e^{\delta \tau^*}).} \tag{3.18}$$

Further, we observe that, given the definition of $(u')^{-1}$ in section 2.2, we must have $\gamma > 0$ if $p_0 = 1$.

Now assume $p_0 = 0$. Then $c^*(t)$ will maximize

$$d(c(t)) = -p(t)c(t).$$

If $\gamma = 0$, and hence $p(t) = 0$, the equation above will provide no information concerning c^*. So, we assume $\gamma > 0$, and we get

$$c^* = 0. \tag{3.19}$$

Let us now determine the value of p_0. From the beginning of this chapter, we had

$$y^*(t) = e^{H(t)} \left(y_0 - \int_0^t e^{-H(s)} c^*(s) ds \right),$$

given the initial condition $y^*(0) = y_0$. However, if we use the terminal condition

$$y^*(\tau^*) = 0$$

instead, we get

$$y^*(t) = e^{H(t)} \left(\int_0^{\tau^*} e^{-H(s)} c^*(s) ds - \int_0^t e^{-H(s)} c^*(s) ds \right).$$

Hence,

$$\boxed{y_0 = \int_0^{\tau^*} e^{-H(s)} c^*(s) ds.} \tag{3.20}$$

Now assume $p_0 = 0$. From (3.19), we have that $c^*(t) = 0$. But from (3.20), we will then get

$$y_0 = 0$$

which is a contradiction as we assumed $y_0 > 0$. Hence, we must have $p_0 = 1$.

Before considering condition (6), we make a remark:

Remark 3.1 We assume that $\tau^* \in (0, T)$: If $\tau^* = 0$, the production is terminated before it is started, and $J(c, \tau^*) = 0$. We assume that the herder is not this unlucky. As we can choose T arbitrarily large, we can also assume that we always have $\tau^* < T$. $\triangle$

Thus, from condition (6),

$$u(c^*(\tau^*))e^{-\delta\tau^*} + p(\tau^*)[h(\tau^*)y^*(\tau^*) - c^*(\tau^*)] = 0. \tag{3.21}$$

Using that $p(\tau^*) = \gamma$, $y^*(\tau^*) = 0$, we simplify (3.21) to

$$\boxed{u(c^*(\tau^*))e^{-\delta\tau^*} - \gamma c^*(\tau^*) = 0.} \tag{3.22}$$

So far, we have been able to find the following expressions for $c^*(t)$,

$$\boxed{c^*(t) = (u')^{-1}(\gamma e^{H(\tau^*)+\delta t - H(t)})}$$

and $y^*(t)$,

$$\boxed{y^*(t) = e^{H(t)}\left(y_0 - \int_0^t e^{-H(s)}c^*(s)ds\right).}$$

Observing the equations above, it is obvious that in order to find a candidate for the optimal solution we need to determine τ^* and γ. We will do this by showing that the system of equations generated from (3.22), (3.18) and (3.20), i.e.,

$$\boxed{\begin{array}{ccc} u(c^*(\tau^*))e^{-\delta\tau^*} - \gamma c^*(\tau^*) & = & 0 \\ c^*(\tau^*) & = & (u')^{-1}(\gamma e^{\delta\tau^*}) \\ y_0 & = & \int_0^{\tau^*} e^{-H(s)}c^*(s)ds \end{array}}$$

has a unique solution (τ^*, γ) for the functions u defined in Chapter 2.2. Before moving on in the general case, we consider a special case where we can find (τ^*, γ), and hence our candidate for an optimal solution, by direct computation.

3.2.1 Interlude: Finding an optimal solution in a special case

Let $u(c) = \ln c$. Inserting (3.18) in (3.22), we get

$$\begin{aligned} \ln\left(\frac{1}{\gamma e^{\delta\tau^*}}\right)e^{-\delta\tau^*} &= e^{-\delta\tau^*} \\ \ln(\gamma e^{\delta\tau^*}) &= -1 \end{aligned}$$

$\Leftrightarrow$

$$\boxed{\gamma = e^{-\delta\tau^* - 1}} \tag{3.23}$$

We remember that we have $\gamma > 0$. Solving (3.23) for τ^*, we obtain

$$\boxed{\tau^* = \frac{-(1 + \ln\gamma)}{\delta}.} \tag{3.24}$$

From (3.24), we see that we need $\gamma < e^{-1}$ in order to have $\tau^* > 0$. In other words, we have

$$\boxed{0 < \gamma < e^{-1}.}$$

Considering equation (3.20), we get the following:

$$\begin{aligned} y_0 &= \int_0^{\tau^*} \frac{e^{-H(s)}}{\gamma e^{H(\tau^*) - H(s) + \delta s}}ds \\ &= \frac{e^{-H(\tau^*)}}{\gamma}\int_0^{\tau^*} e^{-\delta s}ds \\ &= -\frac{e^{-H(\tau^*)}}{\gamma\delta}[e^{-\delta\tau^*} - 1] \end{aligned}$$

$\Leftrightarrow$

$$\gamma = \frac{e^{-H(\tau^*)}}{\delta y_0}[1 - e^{-\delta\tau^*}]. \tag{3.25}$$

If we insert (3.23) in (3.25), we get that

$$e^{-\delta\tau^*-1} = \frac{e^{-H(\tau^*)}}{\delta y_0}[1 - e^{-\delta\tau^*}]$$

or

$$\frac{e^{-H(\tau^*)+\delta\tau^*+1}}{\delta y_0}[1 - e^{-\delta\tau^*}] = 1. \tag{3.26}$$

For notational simplicity, let

$$f(t) = \frac{e^{-H(t)+\delta t+1}}{\delta y_0}[1 - e^{-\delta t}].$$

Hence, (3.26) can be rewritten as

$$f(\tau^*) = 1.$$

Lemma 3.1 *Equation (3.26) has a unique solution* $\tau^* \in (0, T)$.

Proof: We need to show that there exists a $\tau^* \in (0, T)$ such that

$$f(\tau^*) = 1$$

is satisfied. f is obviously continuous. We will use the intermediate value theorem to show that a solution τ^* exists. We have $f(0) = 0$. As T can be arbitrarily large, we consider $\lim_{T\to\infty} f(T)$. We want to show that

$$\lim_{T\to\infty} f(T) = \infty.$$

It is sufficient to show that $\delta t - H(t) \to \infty$, $\forall t$, i.e., that

$$\int_0^t (\delta - h(s))ds$$

diverges. As we assumed $h(t) < \delta - \epsilon$, $\forall t, \forall \epsilon \in (0, \delta)$, we see that the integral diverges. Hence $\lim_{T\to\infty} f(T) = \infty$. In other words, we have

$$f(0) < 1 < \lim_{T\to\infty} f(T),$$

and the intermediate value theorem tells us that the desired τ^* exists. To show that the τ^* in question is unique, we have to consider the monotonicity properties of f. We consider the two factors of f separately. First, we observe that $1 - e^{-\delta t}$ is strictly increasing as $e^{-\delta t}$ is strictly decreasing. Furthermore, we see that $\frac{e^{\delta t - H(t)+1}}{\delta y_0}$ is strictly increasing as we assumed $h(t) < \delta - \epsilon$ and H monotonous. Thus, f is strictly increasing and the τ^* that solves (3.26) is unique. □

Let us now consider a specific choice of h through some examples:

Example 3.1 Let us now consider the special case where $\underline{h(t) = \alpha, \forall t}$, where $\alpha < \delta$ is a constant. Equation (3.26) will then give us

$$e^{(\delta-\alpha)\tau^*}[1 - e^{-\delta\tau^*}] = \frac{\delta y_0}{e}$$

$\Leftrightarrow$

$$e^{(\delta-\alpha)\tau^*} - e^{-\alpha\tau^*} = \frac{\delta y_0}{e}. \tag{3.27}$$

For general α, we cannot solve (3.27) analytically. However, we observe that if we set $\underline{\alpha = -\delta}$, (3.27) becomes

$$e^{2\delta\tau^*} - e^{\delta\tau^*} = \frac{\delta y_0}{e}$$

$\Leftrightarrow$

$$(e^{\delta\tau^*})^2 - e^{\delta\tau^*} = \frac{\delta y_0}{e}$$

$\Leftrightarrow$

$$e^{\delta\tau^*} = \frac{1 + \sqrt{1 + \frac{4\delta y_0}{e}}}{2}$$

$\Leftrightarrow$

$$\boxed{\tau^* = \frac{\ln\left(\frac{1+\sqrt{1+\frac{4\delta y_0}{e}}}{2}\right)}{\delta}}. \tag{3.28}$$

Using (3.28) in (3.23), we get that

$$\boxed{\gamma = \frac{2}{e\left(1 + \sqrt{1 + \frac{4\delta y_0}{e}}\right)}.} \tag{3.29}$$

We see that $\gamma < e^{-1}$, as required. Using (3.28) and (3.29) in (3.17), we obtain the optimal control in this special case:

$$\boxed{c^*(t) = \left(\frac{1 + \sqrt{1 + \frac{4\delta y_0}{e}}}{2}\right)^2 e^{-2\delta t+1}.} \tag{3.30}$$

Further, inserting the optimal control from (3.30) in (3.13) and computing, we get the following expression for $y^*(t)$,

$$\boxed{y^*(t) = e^{-\delta t}\left(y_0 - \frac{e\left(\frac{1+\sqrt{1+\frac{4\delta y_0}{e}}}{2}\right)^2}{\delta}\right) + e^{-2\delta t}\frac{e\left(\frac{1+\sqrt{1+\frac{4\delta y_0}{e}}}{2}\right)^2}{\delta}.} \tag{3.31}$$

Using (3.28) in (3.31), we see that $y^*(\tau^*) = 0$, as was to be expected! ♡

We note that the assumption that $h(t) = -\delta$ in Example 3.1 means that the natural growth rate is negative, so we can interpret this either as the population having a very high natural mortality rate, or that the response to the food is very low. Clearly, this is an undesirable situation for the herder. Therefore, we will also consider a situation where $\underline{\alpha > 0}$:

Example 3.2 As we commented on earlier, we cannot solve equation (3.27) analytically when $\alpha \neq -\delta$. However, let us rearrange (3.27):

$$e^{(\delta-\alpha)\tau^*} - e^{-\alpha\tau^*} - \frac{\delta y_0}{e} = 0. \tag{3.32}$$

If we, for instance, let $y_0 = 1000, \delta = 0.05, \alpha = 0.02$ and plot

$$e^{(\delta-\alpha)\tau^*} - e^{-\alpha\tau^*} - \frac{\delta y_0}{e}$$

as a function of τ^*, see Figure 3.1, we can read from the plot that (3.32) is satisfied for approximately $\tau^* = 97.351$.

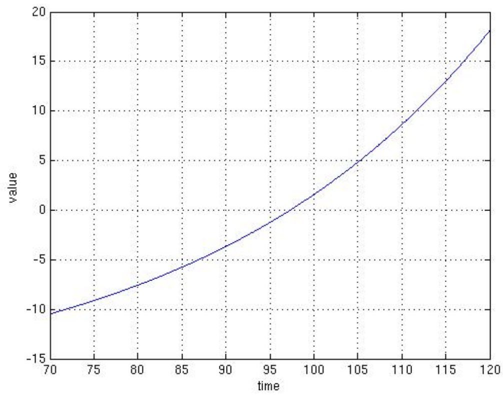

Figure 3.1: $e^{(\delta-\alpha)\tau^*} - e^{-\alpha\tau^*} - \frac{\delta y_0}{e}$ plotted as a function of τ^* with $y_0 = 1000, \delta = 0.05, \alpha = 0.02$.

Hence, we have

$$\boxed{\tau^* = 97.351} \tag{3.33}$$

$$\boxed{\gamma = 0.00283} \tag{3.34}$$

$$\boxed{c^*(t) = 50.42e^{-0.03t}} \tag{3.35}$$

and

$$\boxed{y^*(t) = 1008.54e^{-0.03t} - 8.54e^{0.02t}.} \tag{3.36}$$

In this case $y^*(\tau^*) = -5.4809 \neq 0$. However, compared to the value of y_0 the deviation from 0 is quite small, so we can conclude that this inaccuracy is probably due to the "approximate" nature of this example. ♡

3.2.2 The optimal solution in the general case

We now show that we can find a unique solution (τ^*, γ) of the system generated by (3.22), (3.18) and (3.20) for general utility functions u. As mentioned in Section 2.2, we will assume that $\overline{x} = 0$, as this is the case for the most common utility functions.

First we consider an auxiliary result:

Lemma 3.2 *The equation*

$$\frac{u(x)}{x} = u'(x) \tag{3.37}$$

has a unique solution $\hat{x} \in (0, \infty)$.

Proof: Set

$$g(x) = \frac{u(x)}{x} - u'(x).$$

We want to show that

$$g(x) = 0$$

has a unique solution $\hat{x} \in (0, \infty)$. Let us begin by showing that $\hat{x}$ is unique if it exists. First, recall that we assume $u \in C^2$. We then rewrite $g(x)$ slightly:

$$g(x) = \frac{u(x) - xu'(x)}{x}$$

If $g(x) = 0$, we need $v(x) := u(x) - xu'(x) = 0$. Thus, it suffices to show that $v'(x) > 0$, $\forall x$, as we then have at most one solution for the equation $v(x) = 0$. But this follows from

$$v'(x) = u'(x) - u'(x) - xu''(x) = -xu''(x) > 0$$

as u is concave.

We move on to showing the existence of a solution of (3.37). As g is obviously continuous, we will use the intermediate value theorem to prove this. Let us first consider $\lim_{x \to 0^+} g(x)$. As $u(0) < 0$ and $u'(x) > 0$, $\forall x$, we have

$$\lim_{x \to 0^+} g(x) < 0.$$

It is now sufficient to show that there exists $\tilde{x}$ such that $g(\tilde{x}) \geq 0$. Let us assume that for all x, we have

$$g(x) < 0. \tag{3.38}$$

From Chapter 2.2, we know that there exists x_0 such that $u(x) > 0$, $\forall x > x_0$. Then, by (3.38), we have $\forall x > x_0$,

$$\frac{u(x)}{x} < u'(x)$$

⇔
$$\frac{u'(x)}{u(x)} > \frac{1}{x}$$
⇔
$$(\ln u(x))' > (\ln x)'.$$
Letting a, b be such that $x_0 < a < b$ and integrating, we get
$$\int_a^b (\ln u(t))' dt > \int_a^b (\ln t)' dt$$
⇔
$$\ln\left(\frac{u(b)}{u(a)}\right) > \ln\left(\frac{b}{a}\right)$$
⇔
$$\frac{u(b)}{u(a)} > \frac{b}{a}$$
⇔
$$\frac{u(b)}{b} > \frac{u(a)}{a}.$$
As this holds for all a, b such that $x_0 < a < b$, we see that on (x_0, ∞) the statement (3.38) is equivalent to $L(x) := \frac{u(x)}{x}$ being strictly increasing. But if we consider (3.38), we see that $L(x)$ being strictly increasing on the entire interval (x_0, ∞) contradicts the fact that $u'(x)$ is strictly decreasing and that $\lim_{x\to\infty} u'(x) = 0$. Hence, we know that for some $\tilde{x} > x_0$, we will have $g(\tilde{x}) \geq 0$, and the intermediate value theorem guarantees the existence of a solution. □

Moving on to (3.22), (3.18) and (3.20), we see that by substituting (3.18) into (3.22), (3.17) into (3.20) and rewriting (3.22) slightly, we obtain the following system of equations:

$$u((u')^{-1}(\gamma e^{\delta\tau^*})) = \gamma e^{\delta\tau*}(u')^{-1}(\gamma e^{\delta\tau^*}) \tag{3.39}$$
$$y_0 = \int_0^{\tau^*} e^{-H(s)}(u')^{-1}(\gamma e^{H(\tau^*)+\delta s-H(s)})ds \tag{3.40}$$

Set
$$\boxed{x := (u')^{-1}(\gamma e^{\delta\tau^*}).}$$
Then we have
$$u'(x) = \gamma e^{\delta\tau^*}.$$
Equation (3.39) then becomes
$$u(x) = xu'(x)$$
or
$$\frac{u(x)}{x} = u'(x),$$
which we recognize as equation (3.37), From Lemma 3.2, we know that (3.37) has a unique solution $\hat{x} \in (x_0, \infty)$.

Lemma 3.3 *Given any $\tau^* > 0$, we can find a unique γ that solves (3.39), i.e., for any $\tau^* > 0$, there exists a unique γ such that*

$$\hat{x} = (u')^{-1}(\gamma e^{\delta\tau^*}),$$

or, equivalently,

$$\boxed{u'(\hat{x}) = \gamma e^{\delta\tau^*}} \tag{3.41}$$

is satisfied.

Proof: Remember that $u'(\hat{x})$ is just a fixed positive number. As $\delta > 0$ and $\tau^* > 0$ are given, the constant $A := e^{\delta\tau^*} > 1$. The larger we choose τ^*, the larger we will get A. Thus, if it exists, we must have $\boxed{\gamma \in (0, u'(\hat{x})).}$ Let us consider $f(\gamma) := A\gamma$. As $f(\gamma)$ is continuously and strictly increasing, we see that given the above restrictions, we can find a unique γ for any $\tau^* > 0$ such that

$$u'(\hat{x}) = \gamma e^{\delta\tau^*},$$

and we can write

$$\boxed{\gamma = u'(\hat{x})e^{-\delta\tau^*}.} \tag{3.42}$$

□

Let us now move on to equation (3.40). Substituting (3.42) into (3.40), we get the following equation

$$y_0 = \int_0^{\tau^*} e^{-H(s)}(u')^{-1}(u'(\hat{x})e^{H(\tau^*)-\delta\tau^*+\delta s-H(s)})ds. \tag{3.43}$$

Lemma 3.4 *We can find a unique solution τ^* of (3.43).*

Proof: Set

$$\boxed{\zeta(\tau^*) = \int_0^{\tau^*} e^{-H(s)}(u')^{-1}(u'(\hat{x})e^{H(\tau^*)-\delta\tau^*+\delta s-H(s)})ds.}$$

We begin by making some observations about the integrand of ζ. Let s be fixed and denote the integrand by $\kappa(\tau^*)$. First, observe that, for any choice of s, we have $\kappa(\tau^*) > 0, \forall\tau^*$. We further observe that, as we assume $h < \delta - \epsilon$, the term $e^{H(\tau^*)-\delta\tau^*}$ will decrease strictly with respect to τ^*. Moreover, we know that $(u')^{-1}$ is strictly decreasing with respect to $z(\tau^*, s) = u'(\hat{x})e^{H(\tau^*)-\delta\tau^*+\delta s-H(s)}$. This implies that $(u')^{-1}$ and, hence, $\kappa(\tau^*)$ are strictly increasing with respect to τ^*. Furthermore, we have

$$\lim_{\tau^*\to\infty} (u')^{-1}(z(\tau^*, s)) = \infty.$$

From the previous observations on κ and the definition of ζ, we see that ζ is strictly increasing with respect to τ^*. Moreover, we have

$$\zeta(0) = 0.$$

As we assume $y_0 > 0$, to show that (3.43) has a unique solution τ^*, it suffices to show that

$$\lim_{\tau^* \to \infty} \zeta(\tau^*) = \infty.$$

Let

$$I(\tau^*, s) := e^{-H(s)}(u')^{-1}(z(\tau^*, s)).$$

Choose $\tau^* \in \mathbb{N}$. $\{I(\tau^*, s)\}_{\tau^* \geq 0}$ will then be a strictly increasing sequence. Set

$$\hat{I} := \lim_{\tau^* \to \infty} I(\tau^*, s) = \infty.$$

Then, by the monotone convergence theorem,

$$\lim_{\tau^* \to \infty} \zeta(\tau^*) = \lim_{\tau^* \to \infty} \int_0^{\tau^*} I(\tau^*, s) ds = \int_0^{\tau^*} \hat{I} ds = \infty.$$

Hence, we have a unique solution τ^* of (3.43). □

To sum up: We have shown that given any τ^*, we can find a unique solution γ of (3.39) and this γ can be expressed as a function of τ^*. Using this expression for γ, we showed that (3.40) has a unique solution τ^*. Thus, we have shown that the system of equations (3.39) and (3.40) has a unique solution (γ, τ^*).

If we consider the expression for c^*, we see that if we can find (γ, τ^*), we can also find c^*, and ultimately y^*. Hence, we can find a candidate for the optimal solution of Problem 3.1 in the general case.

3.3 Existence of an optimal solution

We now show the existence of an optimal solution using Theorem 2.2. We have to make sure that the four conditions listed in the theorem are satisfied in our case.

Condition (a) is obviously satisfied, and so is (c) and (d), as we assume $U = [0, K]$, $0 < K < \infty$ and that $y(t) \leq d$, where $d < \infty$ is a positive constant.

Condition (b) is satisfied by the following proposition:

Proposition 3.1 *The set*

$$N(y, U, t) := \{(u(c(t))e^{-\delta t} + \beta, h(t)y(t) - c(t)) : \beta \leq 0, c \in U\}$$

is convex in (y, t).

Proof: Fix (y, t) and let $\lambda \in [0, 1], \beta_1, \beta_2 \leq 0$ and $c_1, c_2 \in U$. Let

$$\mathbf{v_1} = (u(c_1(t))e^{-\delta t} + \beta_1, h(t)y(t) - c_1(t))$$

and

$$\mathbf{v_2} = (u(c_2(t))e^{-\delta t} + \beta_2, h(t)y(t) - c_2(t)).$$

We clearly have $\mathbf{v_1}, \mathbf{v_2} \in N(y, U, t)$. We want to show that

$$\mathbf{v_3} = \lambda \mathbf{v_1} + (1-\lambda)\mathbf{v_2} \in N(y, U, t)$$

for $\lambda \in (0, 1)$. Let z_1, z_2 denote the components of the vector $\mathbf{v_3}$. We will show that z_1, z_2 have the same shape as the first and second components of $\mathbf{v_1}, \mathbf{v_2}$. We consider the first component,

$$z_1 = \lambda u(c_1(t))e^{-\delta t} + \lambda\beta_1 + (1-\lambda)u(c_2(t))e^{-\delta t} + (1-\lambda)\beta_2.$$

Being a utility function, u is concave in c. This implies that

$$\lambda u(c_1(t))e^{-\delta t} + (1-\lambda)u(c_2(t))e^{-\delta t} \leq e^{-\delta t}u(\lambda c_1(t) + (1-\lambda)c_2(t)).$$

Let $c_3 = \lambda c_1(t) + (1-\lambda)c_2(t)$. c_3 is clearly a control contained in U. The latter inequality implies

$$z_1 \leq e^{-\delta t}u(c_3) + \lambda\beta_1 + (1-\lambda)\beta_2.$$

Let $\beta_3 = z_1 - e^{-\delta t}u(c_3)$. Then we have $\beta_3 \leq \lambda\beta_1 + (1-\lambda)\beta_2 \leq 0$, as $\beta_1, \beta_2 \leq 0$.

Considering the second component, we observe that

$$z_2 = h(t)y(t) - (\lambda c_1(t) + (1-\lambda)c_2(t)) = h(t)y(t) - c_3$$

so z_2 is of the desired form. Hence we have found $c_3 \in U$ and $\beta_3 \leq 0$ such that $\mathbf{v_3} \in N(y, U, t)$, and we conclude that $N(y, U, t)$ is indeed a convex set. □

As all of the conditions in Theorem 2.2 are satisfied, there exists an optimal solution to Problem 3.1. This solution is the candidate we described in Chapter 3.2.

4 Finding a solution, part II: Problem 1.3

Contents

4.1 Recapitulation of Problem 1.3

We now return to our original problem, Problem 1.3, which we restate to refresh our memories: We want to maximize

$$J(c, \tau) = \int_0^\tau u(c(t))e^{-\delta t} dt \tag{4.1}$$

$$\tau = \inf\{t \geq 0 \mid y(t) \leq 0\}, \tag{4.2}$$

subject to the differential equations

$$x'(t) = g(t)x(t) - kNh(t)y(t) \tag{4.3}$$

$$y'(t) = h(t)y(t) - c(t), \tag{4.4}$$

and to the boundary conditions

$$x(0) = x_0 > 0 \tag{4.5}$$

$$x(\tau) \geq 0 \tag{4.6}$$

$$y(0) = y_0 > 0 \tag{4.7}$$

$$y(\tau) = 0. \tag{4.8}$$

To solve Problem 1.3, we first tried to apply Theorem 2.1 directly, as in Chapter 3. However, this did not work out, as we were unable to find a way to determine the sign of $p_2(t)$. Thus, we changed our strategy to the following:

In Chapter 3 we solved Problem 1.3 given the extra condition that the amount of food for the livestock was always sufficient. This extra condition could also be formulated as

$$x(\tau^*) > 0, \forall \tau^*.$$

We observe that $R_1(\mathbf{x}(\tau^*), \tau^*) = x(\tau^*)$ and $R_2(\mathbf{x}(\tau^*), \tau^*) = y(\tau^*)$ in Problem 1.3. In other words, the extra condition from Chapter 3 can also be stated as

$$R_1(x(\tau^*), \tau^*) > 0, \forall \tau^*.$$

In this chapter, we will use the results from Chapter 3 to find a candidate for an optimal solution to Problem 1.3 when $R_1(x(\tau^*), \tau^*) > 0$. Then, we will discuss how we can make sure that we always have $R_1(x(\tau^*), \tau^*) > 0$. In the last part of the chapter, we show the existence of an optimal solution using Theorem 2.2, just as we did in Chapter 3.

4.2 Finding candidates for an optimal solution

Let us formalize the discussion above by using Theorem 2.1 as in Chapter 3.2. We assume that we have an optimal solution $(\mathbf{x}^*, c^*, \tau^*)$ for Problem 1.3.

<u>We first state the conditions in the theorem for Problem 1.3:</u>

(1) $(p_0, \gamma_1, \gamma_2) \neq (0, 0, 0)$;

(2) $c^*(t)$ maximizes the Hamiltonian, $H(\mathbf{x}^*(t), c(t), \mathbf{p}(t), t)$, for $c \in [0, K]$, i.e.,

$$H(\mathbf{x}^*(t), c^*(t), \mathbf{p}(t), t) \geq H(\mathbf{x}^*(t), c(t), \mathbf{p}(t), t), \forall c \in [0, K]. \tag{4.9}$$

In this case, the Hamiltonian is

$$H(\mathbf{x}(t), c(t), \mathbf{p}(t), t) = p_0 u(c(t))e^{-\delta t} + p_1(t)[g(t)x(t) - kNh(t)y(t)] + p_2(t)[h(t)y(t) - c(t)]; \tag{4.10}$$

(3) Except at the points of discontinuity of $c^*(t)$, we have:

$$p_1'(t) = \frac{-\partial H^*}{\partial x} \tag{4.11}$$

$$p_2'(t) = \frac{-\partial H^*}{\partial y} \tag{4.12}$$

where

$$\frac{\partial H^*}{\partial x} = \frac{\partial H(\mathbf{x}^*(t), c^*(t), \mathbf{p}(t), t)}{\partial x};$$

(4) $p_0 = 1$ or $p_0 = 0$;

(5) The following transversality conditions are satisfied:

$$p_1(\tau^*) = \gamma_1 \quad (4.13)$$
$$p_2(\tau^*) = \gamma_2. \quad (4.14)$$

where we have

$$\gamma_1 \geq 0\ (= 0 \text{ if } R_1(\mathbf{x}^*(\tau^*), \tau^*) > 0)$$

and γ_2 is a constant (also possibly negative);

(6)

$$H(\mathbf{x}^*(\tau^*), c^*(\tau^*), \mathbf{p}(\tau^*), \tau^*) = 0 \text{ if } \tau^* \in (0, T) \quad (4.15)$$
$$\geq 0 \text{ if } \tau^* = T \quad (4.16)$$

If $\tau^* = 0$, we have

$$\sup_{c \in U} H(\mathbf{x}_0, c, \mathbf{p}(\tau^*), \tau^*) \leq 0. \quad (4.17)$$

As in Chapter 3, we use the conditions above to determine $p_0, \mathbf{p}(t), c^*, \tau^*, \gamma_1, \gamma_2$.

We observe that solving the differential equations (4.3), (4.4) with the initial conditions (4.5), (4.7), we obtain, using standard methods:

$$y^*(t) = e^{H(t)} \left(y_0 - \int_0^t e^{-H(s)} c^*(s) ds \right) \quad (4.18)$$

and

$$x^*(t) = e^{G(t)} \left(x_0 - kN \int_0^t e^{-G(s)} h(s) y^*(s) ds \right), \quad (4.19)$$

where $H(t) = \int_0^t h(s)ds$ and $G(t) = \int_0^t g(s)ds$. From condition (3), we see that

$$p_1'(t) = -p_1(t)g(t),$$

and we get

$$p_1(t) = C_1 e^{-G(t)}, \quad (4.20)$$

where C_1 is a constant.

Moving on to p_2, we see that

$$p_2'(t) = h(t)[kNp_1(t) - p_2(t)]$$

and usingstandard methods and (4.20), we get

$$p_2(t) = kNC_1 e^{-H(t)} \int_0^t h(s) e^{H(s)-G(s)} ds + C_2 e^{-H(t)}, \quad (4.21)$$

where C_2 is a constant.

To determine the constants C_1 and C_2, we use the transversality conditions in (5),

$$\begin{aligned} p_1(\tau^*) &= \gamma_1 \geq 0 \; (= 0 \text{ if } x^*(\tau^*) > 0) \\ p_2(\tau^*) &= \gamma_2. \end{aligned}$$

From $p_1(\tau^*) = \gamma_1$ we get

$$\gamma_1 = C_1 e^{-G(\tau^*)}$$

so

$$C_1 = \gamma_1 e^{G(\tau^*)}.$$

From $p_2(\tau^*) = \gamma_2$,

$$\gamma_2 = kN\gamma_1 e^{G(\tau^*)-H(\tau^*)} \int_0^{\tau^*} h(s)e^{H(s)-G(s)}ds + C_2 e^{-H(\tau^*)}$$

and thus we have

$$C_2 = e^{H(\tau^*)}\gamma_2 - kN\gamma_1 e^{G(\tau^*)} \int_0^{\tau^*} h(s)e^{H(s)-G(s)}ds.$$

Hence,

$$p_1(t) = \gamma_1 e^{G(\tau^*)-G(t)} \tag{4.22}$$

$$p_2(t) = e^{-H(t)}\left[\gamma_2 e^{H(\tau^*)} - kN\gamma_1 e^{G(\tau^*)} \int_t^{\tau^*} h(s)e^{H(s)-G(s)}ds\right]. \tag{4.23}$$

Let us now assume $x(\tau^*) > 0, \forall \tau^*$. Then, we know that $\gamma_1 = 0$, and hence, $C_1 = 0$. Then

$$C_2 = \gamma_2 e^{H(\tau^*)}$$

and we get

$$p_1(t) = 0 \tag{4.24}$$

$$p_2(t) = \gamma_2 e^{H(\tau^*)-H(t)}. \tag{4.25}$$

We see that Problem 1.3 is reduced to Problem 3.1, which we discussed and solved in Chapter 3. We observe that equation (4.25) "is" equation (3.14), where γ_2 replaces γ. Replacing (3.14) with (4.25), we conclude that $\gamma_2 \geq 0$ by the same argument that we used to show that $\gamma \geq 0$ in Chapter 3.2.

From (2), we know that $c^*(t)$ maximizes

$$p_0 u(c(t))e^{-\delta t} + p_1(t)[g(t)x^*(t) - kNh(t)y^*(t)] + p_2(t)[h(t)y^*(t) - c(t)].$$

In other words, c^* maximizes

$$p_0 u(c)e^{-\delta t} - p_2(t)c,$$

which is exactly the function we maximize in Chapter 3.2, and hence, by the same strategy,

$$c^*(t) = (u')^{-1}(e^{H(\tau^*)-H(t)+\delta t}) \tag{4.26}$$

when $p_0 = 1$, and

$$c^*(t) = 0 \tag{4.27}$$

when $p_0 = 0$. As before, it follows that $\gamma_2 > 0$.

We observe that c^* and y^* are the same as in Chapter 3, and we use the double boundary condition on y^* to obtain the equation

$$y_0 = \int_0^{\tau^*} e^{-H(s)} c^*(s) ds$$

which we recognize as (3.20) from Chapter 3.2. Thus, we have $\underline{p_0 = 1}$ by the same argument as in Chapter 3.2.

As in Chapter 3, we assume that $\tau^* \in (0, T)$. Considering condition (6), we get

$$\begin{aligned} u(c^*(\tau^*))e^{-\delta\tau^*} &+ p_1(\tau^*)[g(\tau^*)x^*(\tau^*) - kNh(\tau^*)y^*(\tau^*)] \\ &+ p_2(\tau^*)[h(\tau^*)y^*(\tau^*) - c^*(\tau^*)] = 0 \end{aligned} \tag{4.28}$$

Using that $p_1(\tau^*) = \gamma_1 = 0, p_2(\tau^*) = \gamma_2, y^*(\tau^*) = 0$, we simplify (4.28) to

$$u(c^*(\tau^*))e^{-\delta\tau^*} - \gamma_2 c^*(\tau^*) = 0. \tag{4.29}$$

We recognize this equation as (3.22) from Chapter 3.2.

To sum up, so far we know that

$$\boxed{c^*(t) = (u')^{-1}(e^{H(\tau^*)-H(t)+\delta t})}$$

$$\boxed{y^*(t) = e^{H(t)} \left(y_0 - \int_0^t e^{-H(s)} c^*(s) ds \right)}$$

$$\boxed{x^*(t) = e^{G(t)} \left(x_0 - kN \int_0^t e^{-G(s)} h(s) y^*(s) ds \right),}$$

and that we have the following equations to determine γ_2, τ^*:

$$\boxed{\begin{array}{ccc} u(c^*(\tau^*))e^{-\delta\tau^*} - \gamma_2 c^*(\tau^*) & = & 0 \\ c^*(\tau^*) & = & (u')^{-1}(\gamma e^{\delta\tau^*}) \\ y_0 & = & \int_0^{\tau^*} e^{-H(s)} c^*(s) ds. \end{array}}$$

Replacing γ_2 with γ, this is exactly the system we used to determine γ, τ^* in Chapter 3.2. Hence, we can determine γ_2, τ^* by the exact same argument as we used in

Chapter 3.2, and find a candidate for the optimal solution.

In short: We can find a candidate for an optimal solution of Problem 1.3 as long as we know that $x(\tau^*) > 0, \forall \tau^*$.

The question is: How can we make sure that we always have $x(\tau^*) > 0$? Let us once again consider the expression for x:

$$x^*(t) = e^{G(t)} \left(x_0 - kN \int_0^t e^{-G(s)} h(s) y^*(s) ds \right).$$

We need, for all t,

$$x_0 - kNe \int_0^t e^{-G(s)} h(s) y^*(s) ds > 0 \tag{4.30}$$

We want to find a condition on x_0 which ensures that (4.30) is true. First observe that if $h(t) < 0, \forall t$, we have $x^*(t) > 0, \forall t$. Let us consider the cases where h is not necessarily negative for all t. Rearranging (4.30), it is obvious that we need

$$x_0 > kNe \int_0^t e^{-G(s)} h(s) y^*(s) ds. \tag{4.31}$$

Remark 4.1 We observe that given the latter inequality, we could also assume that we have a given initial value x_0, and then find out how many herders that could use the same pasture when the livestock follows this specific dynamics. This approach would require that the integral be different from zero. △

To make the conditions stated above a bit more concrete, we consider Examples 3.1 and 3.2 from Chapter 3.2.1, where we had $u(c) = \ln(c)$, once again:

Example 4.1 (Continuation of Example 3.1.) In Example 3.1, we assumed $h(t) = -\delta$. As $\delta > 0$, we will always have $x^*(t) > 0$ for this h. ♡

But, we remember that a negative natural growth rate for the population was an undesirable scenario for the herder. However, in Example 3.2, we considered a case where $h(t) = \alpha > 0$, $\forall t$:

Example 4.2 (Continuation of Example 3.2.) We assumed $y_0 = 1000$, $\delta = 0.05, \alpha = 0.02$. We remember that we obtained

$$\tau^* = 97.351$$

$$\gamma = 0.00283$$

$$c^*(t) = 50.42e^{-0.03t}$$

and

$$y^*(t) = 1008.54e^{-0.03t} - 8.54e^{0.02t}.$$

We let $g(t) = \delta = 0.05, \forall t$, and insert the expression for y^* in (4.30). We then get

$$x_0 > kN0.02 \int_0^t e^{-0.05s} y^*(s) ds$$

$\Leftrightarrow$

$$x_0 > 0.02kN \left[\frac{A}{0.08}(1 - e^{-0.08t}) + \frac{B}{0.03}(1 - e^{-0.03t}) \right]$$

where $A = 1008.54, B = -8.54$. We set

$$\phi(t) = \frac{A}{0.03}(1 - e^{-0.03t}) + \frac{B}{0.05}(1 - e^{-0.05t})$$

Differentiating, we get

$$\phi'(t) = Ae^{-0.08t} + Be^{-0.03t}.$$

We set

$$\phi'(t) = 0,$$

and solving the equation above, we obtain

$$t = 95.43$$

This is a maximum. Therefore, we must have

$$x_0 > 0.02kN\phi(95.43)$$

or

$$x_0 > 246.64kN.$$

♡

4.3 Existence of an optimal solution

Again, we show the existence of an optimal solution using Theorem 2.2. As in Chapter 3.3, condition (a) is obviously satisfied, and so is (c) and (d), as we assume $U = [0, K]$, $0 < K < \infty$, and that there exists a positive constant $d < \infty$ such that $\| \mathbf{x}(t) \| \leq d$.

Condition (b) is satisfied by the following proposition:

Proposition 4.1 *The set*

$$N(\mathbf{x}, U, t) := \left\{ (u(c(t))e^{-\delta t} + \beta, g(t)x(t) - kh(t)Y(t), h(t)y(t) - c(t)) : \beta \leq 0, c \in U \right\}$$

is convex in $(\mathbf{x}, t)$.

Proof: Fix $(\mathbf{x}, t)$ and let $\lambda \in [0, 1], \beta_1, \beta_2 \leq 0$ and $c_1, c_2 \in U$. Let

$$\mathbf{w_1} = (u(c_1(t))e^{-\delta t} + \beta_1, g(t)x(t) - kh(t)Y(t), h(t)y(t) - c_1(t))$$

and

$$\mathbf{w_2} = (u(c_2(t))e^{-\delta t} + \beta_2, g(t)x(t) - kh(t)Y(t), h(t)y(t) - c_2(t)).$$

We clearly have $\mathbf{w_1}, \mathbf{w_2} \in N(\mathbf{x}, U, t)$. We want to show that, for $\lambda \in (0, 1)$,

$$\mathbf{w_3} = \lambda \mathbf{w_1} + (1 - \lambda)\mathbf{w_2} \in N(\mathbf{x}, U, t).$$

Let $\theta_1, \theta_2, \theta_3$ denote the components of the vector $\mathbf{w_3}$. We want to show that $\theta_1, \theta_2, \theta_3$ have the same shape as the first, second and third components of $\mathbf{w_1}, \mathbf{w_2}$, respectively. First, observe that θ_2 does not depend on β or c, so θ_2 will clearly have the desired shape. We further observe that the first and third components of $\mathbf{w_1}, \mathbf{w_2}$ are equal to the first and second components of $\mathbf{v_1}, \mathbf{v_2}$ from Section 3.3. We see that $\theta_1 = z_1$ and $\theta_3 = z_2$, where z_1, z_2 are the components of the vector $\mathbf{v_3}$ in Section 3.3. As shown in Section 3.3, z_1, z_2 have the desired shape, and we can construct c_3, β_3 such that $\mathbf{v_3} \in N(\mathbf{x}, U, t)$. Hence, $N(\mathbf{x}, U, t)$ is a convex set. □

Hence, there exists an optimal solution of Problem 1.3 and this solution is the candidate we have described in Chapter 4.2.

5 Epilogue

Through Chapters 3 and 4, we have described a solution to Problem 1.3, and we have considered some examples where we found the solution by computation. It might be a natural question to ask whether a controlled outphasing strategy is to prefer above a strategy with infinite planning horizon. In the following, we will give a short discussion on this, based on the examples we considered in Chapters 3 and 4.

Let us first simplify the individual optimization problem of Brekke et al. from Chapter 1.1 by making the same simplifications as we did to obtain Problem 1.3 from Problem 1.2. In other words, we are considering the following deterministic optimization problem, where the conditions on h, g, c are the same as in Chapter 1.2:

Problem 5.1 (Brekke et al.'s problem, simplified version) *We want to maximize*

$$J(c) = \int_0^\infty \ln(c(t))e^{-\delta t}dt \tag{5.1}$$

subject to the differential equations

$$x'(t) = g(t)x(t) - kNh(t)y(t) \tag{5.2}$$

$$y'(t) = h(t)y(t) - c(t) \tag{5.3}$$

and the boundary conditions

$$x(0) = x_0 > 0 \tag{5.4}$$

$$y(0) = y_0 > 0. \tag{5.5}$$

As we recall from Chapter 1.1, the optimal control for Problem 5.1 is

$$c^*(t) = \delta y(t).$$

Using this in equation (5.3), we get

$$y'(t) - (h(t) - \delta)y(t) = 0,$$

and using standard methods to solve this differential equation,

$$y^*(t) = y_0 e^{H(t)-\delta t}. \tag{5.6}$$

Hence

$$c^*(t) = \delta y_0 e^{H(t)-\delta t}. \tag{5.7}$$

Let us now consider the optimal value functions of Problem 1.3 and Problem 5.1, respectively:

$$J(c^*, \tau^*) = \int_0^{\tau^*} u(c^*(t)) e^{-\delta t} dt \tag{5.8}$$

and

$$J(c^*) = \int_0^{\infty} \ln(c^*(t)) e^{-\delta t} dt. \tag{5.9}$$

We want to compare (5.8) and (5.9) with the parameters from Examples 3.1 and 3.2. We recall that we let $u(c) = \ln(c)$ in both examples.

Let us first consider the situation from Example 3.1. Here we assume that $h(t) = -\delta$, and further that $y_0 = 1000$ and $\delta = 0.05$. Let us begin with the controlled outphasing case. From the computations in Example 3.1, we obtain

$$\tau^* = 31.446,$$

$$\gamma = 0.0764$$

and

$$c^*(t) = 62.8931 e^{-0.1t}.$$

We compute

$$\begin{aligned} J(c^*, 31.446) &= \int_0^{31.446} \ln\left(62.8931 e^{-0.1t}\right) e^{-0.05t} dt \\ &= \int_0^{31.446} (\ln(62.8931) - 0.1t) e^{-0.05t} dt \\ &= 20 \ln(62.8931)[1 - e^{-0.05*31.446}] \\ &+ 2[31.446 e^{-0.05*31.446} - 20(1 - e^{-0.05*31.446})] \\ &= 46.99. \end{aligned}$$

Turning to the infinite horizon case, we compute

$$\begin{aligned} J(c^*) &= \int_0^{\infty} \ln(50 e^{-0.1t}) e^{-0.05t} dt \\ &= \int_0^{\infty} (\ln(50) - 0.1t) e^{-0.05t} dt \\ &= 20 \ln(50) - 40 \\ &= 38.2405. \end{aligned}$$

Moving on to Example 3.2, we recall that we let $\underline{h = \alpha > 0}$, and that we let $y_0 = 1000, \delta = 0.05, \alpha = 0.02$. In the controlled outphasing case we get, using the formulas from Example 3.2:

$$\tau^* = 97.351,$$

$$\gamma = 0.00283$$

and

$$c^*(t) = 50.42e^{-0.03t}.$$

Again, we compute:

$$J(c^*, \tau^*) = \int_0^{97.351} \ln(50.42e^{-0.03t})e^{-0.05t} dt = 66.3422.$$

In the infinite horizon case, we get

$$J(c^*) = \int_0^{\infty} \ln(50e^{-0.03t})e^{-0.05t} dt = 66.2405.$$

In both cases, we see that we get a larger utility in the controlled outphasing case. Thus, these examples suggest that it is preferable for the herder to choose a controlled outphasing strategy. However, more examples and perhaps also other choices of u should be studied to be able to say anything about this in a more general context.

Bibliography

[1] Kjell Arne Brekke, Bernt Øksendal, and Nils Chr. Stenseth. The dynamic effects of occasional (short-term) environmentally deviating conditions. *Supporting information text to [2]*, 2007.

[2] Kjell Arne Brekke, Bernt Øksendal, and Nils Chr. Stenseth. The effect of climate variations on the dynamics of pasture-livestock interactions under cooperative and noncooperative management. *Proceedings for the National Academy of Science (PNAS)*, 104:14730 – 14734, 2007.

[3] Wendell H. Fleming and Raymond A. Rishel. *Deterministic and Stochastic Optimal Control*. Springer, 1975.

[4] Ioannis Karatzas and Steven E. Shreve. *Methods of Mathematical Finance*. Springer, 1998.

[5] Christian Schaal. Cooperative and non-cooperative management for a grazer-plant system. Thesis presented for the degree of master in mathematics, University of Oslo, 2008.

[6] Atle Seierstad and Knut Sydsæter. *Optimal Control Theory with Economic Applications*. North-Holland, 1987.