Multiparameter fractional Brownian motion and quasi-linear stochastic partial differential equations

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Abstract

We develop a multiparameter white noise theory for fractional Brownian motion with Hurst multiparameter $H = (H_1, \ldots, H_d) \in (\frac{1}{2}, 1)^d$. The theory is used to solve the linear and a quasi-linear heat equation driven by multiparameter fractional white noise. It is proved that for some values of H (depending on the dimension) the solution has a jointly continuous version in t and x.

1 Introduction

Recall that if $0 < H < 1$ then the *(1-parameter) fractional Brownian motion* with Hurst parameter H is the Gaussian process $B_H(t) = B_H(t,\omega)$; $t \in \mathbf{R}$, $\omega \in \Omega$ satisfying

$$B_H(0) = E[B_H(t)] = 0 \qquad \text{for all } t \in \mathbf{R} \tag{1.1}$$

and

$$E[B_H(s)B_H(t)] = \tfrac{1}{2}\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\} \qquad \text{for all } s,t \in \mathbf{R}\,. \tag{1.2}$$

Here E denotes the expectation with respect to the probability law P for $\{B_H(t,\omega)\}_{t\in\mathbf{R},\omega\in\Omega}$, where $(\Omega, \mathcal{F})$ is a measurable space.

If $H = \frac{1}{2}$ then $B_H(t)$ coincides with the standard Brownian motion $B(t)$. Much of the recent interest in fractional Brownian motion stems from its property that if $H > \frac{1}{2}$ then $B_H(t)$ has a *long range dependence*, in the sense that

$$\sum_{n=1}^{\infty} E[B_H(1)(B_H(n+1) - B_H(n))] = \infty\,.$$

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Moreover, for any $H \in (0,1)$ and $\alpha > 0$ the law of $\{B_H(\alpha t)\}_{t\in\mathbf{R}}$ is the same as the law of $\{\alpha^H B_H(t)\}_{t\in\mathbf{R}}$, i.e. $B_H(t)$ is H-self-similar.

For more information on 1-parameter fractional Brownian motion see e.g. [MV], [NVV] and the references therein.

Recently a stochastic calculus based on Itô-type of integration with respect to $B_H(t)$ has been constructed for $H > \frac{1}{2}$ [DHP]. Subsequently a corresponding fractional white noise theory has been developed [HØ], and this has been used to study the corresponding fractional models in mathematical finance [HØ], [HØS].

As in [H1], [H2] and [HØZ] we define *d-parameter fractional Brownian motion* $B_H(x)$; $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ with *Hurst parameter* $H = (H_1, \ldots, H_d) \in (0,1)^d$ as a Gaussian process on $\mathbf{R}^d$ with mean

$$E[B_H(x)] = 0 \qquad \text{for all } x \in \mathbf{R}^d \tag{1.3}$$

and covariance

$$E[B_H(x)B_H(y)] = (\tfrac{1}{2})^d \prod_{i=1}^{d} (|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}) \tag{1.4}$$

We also assume that

$$B_H(0) = 0 \quad \text{a.s.} \tag{1.5}$$

From now on we will assume that

$$\tfrac{1}{2} < H_i < 1 \qquad \text{for } i = 1, \ldots, d\,. \tag{1.6}$$

The purpose of this paper is to extend the fractional white noise theory to the multiparameter case and use this theory to study the linear and quasilinear heat equation with a fractional white noise force.

2 Multiparameter fractional white noise

In this section we outline how the multiparameter white noise theory for standard Brownian motion (see e.g. [HKPS], [HØUZ] or [K]) can be extended to fractional Brownian motion. In the 1-parameter case such an extension was presented in [HØ]. The following outline will follow the introduction in [HØZ] closely.

Fix a parameter dimension $d \in \mathbf{N}$ and a *Hurst parameter*

$$H = (H_1, \ldots, H_d) \in (\tfrac{1}{2}, 1)^d\,. \tag{2.1}$$

Define

$$\varphi(x,y) = \varphi_H(x,y) = \prod_{i=1}^{d} H_i(2H_i - 1)|x_i - y_i|^{2H_i - 2} \tag{2.2}$$

for $x=(x_1,\dots,x_d)\in\mathbf{R}^d$, $y=(y_1,\dots,y_d)\in\mathbf{R}^d$.

Let $L^2_\varphi(\mathbf{R}^d)$ be the space of measurable functions $f\colon\mathbf{R}^d\to\mathbf{R}$ satisfying

$$|f|^2_\varphi\colon=\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(x)f(y)\varphi(x,y)dx\,dy<\infty \tag{2.3}$$

where $dx=dx_1\dots dx_d$ and $dy=dy_1\dots dy_d$ denotes Lebesgue measure.

Then $L^2_\varphi(\mathbf{R}^d)$ is a separable Hilbert space with the inner product

$$(f,g)_\varphi=\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(x)g(y)\varphi(x,y)dx\,dy\ ;\qquad f,g\in L^2_\varphi(\mathbf{R}^d)\ . \tag{2.4}$$

In fact, we have (see [HØ, Lemma 2.1] for the case $d=1$):

Lemma 2.1 *For $f\in L^2_\varphi(\mathbf{R}^d)$ and $u=(u_1,\dots,u_d)\in\mathbf{R}^d$ define*

$$\Gamma_\varphi f(u)=\int_{u_1}^\infty\cdots\int_{u_d}^\infty f(x_1,\dots,x_d)\prod_{i=1}^d c_{H_i}(x_i-u_i)^{H_i-3/2}dx_1\dots dx_d\ , \tag{2.5}$$

where

$$c_{H_i}=\sqrt{\frac{H_i(2H_i-1)\cdot\Gamma(\frac{3}{2}-H_i)}{\Gamma(H_i-\frac{1}{2})\cdot\Gamma(2-2H_i)}}\ ;\qquad i=1,\dots,d\ . \tag{2.6}$$

Then Γ_φ is an isometry from $L^2_\varphi(\mathbf{R}^d)$ into $L^2(\mathbf{R}^d)$.

Proof. For $f,g\in L^2_\varphi(\mathbf{R}^d)$ we have

$$\begin{aligned}
&(\Gamma_\varphi(f),\Gamma_\varphi(g))_{L^2(\mathbf{R}^d)}\\
&=\int_{\mathbf{R}^d}\Big(\int_{u_i}^\infty\cdots\int_{u_d}^\infty f(x)\prod_{i=1}^d c_{H_i}(x_i-u_i)^{H_i-3/2}dx\Big)\\
&\quad\cdot\Big(\int_{u_1}^\infty\cdots\int_{u_d}^\infty g(y)\prod_{i=1}^d c_{H_i}(y_i-u_i)^{H_i-3/2}dy\Big)du_1\dots du_d\\
&=\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(x)g(y)\Big(\prod_{i=1}^d\int_{-\infty}^{x_i\wedge y_i} c^2_{H_i}(x_i-u_i)^{H_i-3/2}(y_i-u_i)^{H_i-3/2}du_i\Big)dx\,dy\\
&=\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(x)g(y)\varphi(x,y)dx\,dy\ ,
\end{aligned}$$

where we have used the fact that (see e.g. [GN, p. 404])

$$\int_{-\infty}^{x_i \wedge y_i} c_{H_i}^2 (x_i - u_i)^{H_i - 3/2} (y_i - u_i)^{H_i - 3/2} du_i = H_i(2H_i - 1)|x_i - y_i|^{2H_i - 2} . \tag{2.7}$$

□

Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbf{R}^d$. The dual of $\mathcal{S}(\mathbf{R}^d)$, the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbf{R}^d)$. The functional

$$f \to \exp(-\tfrac{1}{2}|f|_\varphi^2) \; ; \qquad f \in \mathcal{S}(\mathbf{R}^d)$$

is positive definite on $\mathcal{S}(\mathbf{R}^d)$, so by the Bochner-Minlos theorem there exists a probability measure μ_φ on $\mathcal{S}'(\mathbf{R}^d)$ such that

$$\int_{\mathcal{S}'(\mathbf{R}^d)} e^{i\langle \omega, f\rangle} d\mu_\varphi(\omega) = e^{-\frac{1}{2}|f|_\varphi^2} \; ; \qquad f \in \mathcal{S}(\mathbf{R}^d) \tag{2.8}$$

where $\langle \omega, f\rangle$ denotes the action of $\omega \in \Omega \colon = \mathcal{S}'(\mathbf{R}^d)$ on $f \in \mathcal{S}(\mathbf{R}^d)$. From (2.8) one can deduce that if $f_n \in \mathcal{S}(\mathbf{R}^d)$ and $f_n \to f$ in $L_\varphi^2(\mathbf{R}^d)$ then

$$\langle \omega, f\rangle \colon = \lim_{n\to\infty} \langle \omega, f_n\rangle \qquad \text{exists in } \; L^2(\mu_\varphi) \tag{2.9}$$

and defines a Gaussian random variable. Moreover,

$$E[\langle \cdot, f\rangle] = 0 \tag{2.10}$$

and

$$E[\langle \cdot, f\rangle \langle \cdot, g\rangle] = (f, g)_\varphi \qquad \text{for } \; f, g \in L_\varphi^2(\mathbf{R}^d) \; . \tag{2.11}$$

Here, and in the following, $E[\cdot] = E_{\mu_\varphi}[\cdot]$ denotes the expectation with respect to μ_φ.

In particular, we may define

$$\widetilde{B}_H(x) = \langle \omega, \mathcal{X}_{[0,x]}(\cdot)\rangle \; ; \qquad x = (x_1, \ldots, x_d) \in \mathbf{R}^d \tag{2.12}$$

where

$$\mathcal{X}_{[0,x]}(y) = \prod_{i=1}^d \mathcal{X}_{[0,x_i]}(y_i) \qquad \text{for } \; y = (y_1, \ldots, y_d) \in \mathbf{R}^d$$

and

$$\mathcal{X}_{[0,x_i]}(y_i) = \begin{cases} 1 & \text{if } \; 0 \le y_i \le x_i \\ -1 & \text{if } \; x_i \le y_i \le 0, \text{ except } \; x_i = y_i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Using (2.10)–(2.11) and Kolmogorov's criterion, we see that $\widetilde{B}_H(x);\ x \in \mathbf{R}^d$ is a Gaussian process and it has a continuous version. Furthermore, we see that

$$E[B_H(x)] = 0$$

and

$$E[B_H(x)B_H(y)] = (\tfrac{1}{2})^d \prod_{i=1}^{d} (|x_i|^{2H_i} + |y|^{2H_i} - |x_i - y_i|^{2H_i}) \ . \tag{2.13}$$

Therefore $B_H(x);\ x \in \mathbf{R}^d$ is a d-parameter fractional Brownian motion with Hurst parameter $H = (H_1, \dots, H_d) \in (\frac{1}{2}, 1)^d$ (see (1.3)–(1.5)). It is this version of $B_H(x)$ we will use from now on.

Let $f \in L^2_\varphi(\mathbf{R}^d)$. The *stochastic integral* of f with respect to the fractional Brownian motion $B_H(x)$ is the Gaussian random variable on Ω defined by

$$\int_{\mathbf{R}^d} f(x)dB_H(x) = \int_{\mathbf{R}^d} f(x)dB_H(x,\omega) = \langle \omega, f \rangle \ . \tag{2.14}$$

Note that this is a natural definition from the point of view of Riemann sums:

If f_n is a simple integrand of the form

$$f_n(x) = \sum_{j=1}^{N_n} a_j^{(n)} \mathcal{X}_{(-\infty, y_j)}(x)$$

then (2.13) gives

$$\int_{\mathbf{R}^d} f_n(x)dB_H(x) = \langle \omega, f_n \rangle = \sum_{j=1}^{N_n} a_j^{(n)} B_H(y_j)$$

and if $f_n \to f$ in $L^2_\varphi(\mathbf{R}^d)$ then by (2.9) we have, as desired, that

$$\int_{\mathbf{R}^d} f_n(x)dB_H(x) = \langle \omega, f_n \rangle \to \langle \omega, f \rangle = \int_{\mathbf{R}^d} f(x)dB_H(x) \ .$$

Note that from (2.14) and (2.11) we have the *fractional Ito isometry*

$$E\Big[\Big(\int_{\mathbf{R}^d} f(x)dB_H(x)\Big)^2\Big] = |f|^2_\varphi \qquad \text{for } f \in L^2_\varphi(\mathbf{R}^d) \ . \tag{2.15}$$

As in [HØZ] we now proceed in analogy with [HØUZ] (as done in [HØ] in the 1-parameter case) to obtain a multiparameter fractional chaos expansion:

Let

$$h_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n}(e^{-t^2/2}) \; ; \qquad t \in \mathbf{R} \, , \quad n = 0, 1, 2, \dots$$

be the standard Hermite polynomials and let

$$\widetilde{h}_n(t) = \pi^{-1/4}((n-1)!)^{-1/2} h_{n-1}(\sqrt{2}\, t) e^{-t^2/2} \; ; \qquad n = 1, 2, \dots \tag{2.16}$$

be the *Hermite functions*. Let $\mathbf{N} = \{1, 2, \dots\}$. For $\alpha \in \mathbf{N}^d$ let $\eta_\alpha(x) = \prod_{i=1}^{d} \widetilde{h}_{\alpha_i}(x_i)$. Then $\{\eta_\alpha\}_{\alpha \in \mathbf{N}^d}$ constitutes an orthonormal basis of $L^2(\mathbf{R}^d)$. Therefore

$$e_\alpha(x) \colon = \Gamma_\varphi^{-1}(\eta_\alpha)(x) \; ; \qquad \alpha \in \mathbf{N}^d \, , \quad x \in \mathbf{R}^d$$

constitutes an orthonormal basis of $L^2_\varphi(\mathbf{R}^d)$. From now on we let $\{\alpha^{(i)}\}_{i=1}^{\infty}$ be a fixed ordering of $\mathbf{N}^d$ with the property that

$$i < j \Rightarrow |\alpha^{(i)}| \leq |\alpha^{(j)}|$$

and we write

$$e_n(x) \colon = e_{\alpha^{(n)}}(x) \, . \qquad \text{(See (2.2.7) in [HØUZ])} \tag{2.17}$$

Then just as in [HØ, Lemma 3.1] we can prove

Lemma 2.2 *There exists a locally bounded function $C(x)$ on $\mathbf{R}^d$ such that*

$$\left| \int_{\mathbf{R}^d} e_n(y) \varphi(x, y) dy \right| \leq C(x) \prod_{i=1}^{d} (\alpha_i^{(n)})^{1/6}$$

Let $\mathcal{J} = (\mathbf{N}_0^{\mathbf{N}})_c$ denote the set of all (finite) multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_i \in \mathbf{N}_0 \colon = \mathbf{N} \cup \{0\}$. Then if $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{J}$ we define

$$\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(\langle \omega, e_1 \rangle) \cdots h_{\alpha_m}(\langle \omega, e_m \rangle) \, . \tag{2.18}$$

In particular, if we put

$$\varepsilon^{(i)} = (0, 0, \dots, 1) \qquad \text{(the } i\text{'th unit vector)}$$

then by (2.14) we get

$$\mathcal{H}_{\varepsilon^{(i)}}(\omega) = h_1(\langle \omega, e_i \rangle) = \langle \omega, e_i \rangle = \int_{\mathbf{R}^d} e_i(x) dB_H(x) \, . \tag{2.19}$$

As is well-known in a more general context (see e.g. [J, Theorem 2.6]) we have the following Wiener-Itô chaos expansion theorem (see also [DHP] and [HØ]):

Theorem 2.3 *Let $F \in L^2(\mu_\varphi)$. Then there exist constants $c_\alpha \in \mathbf{R}$ for $\alpha \in \mathcal{J}$, such that*

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega) \qquad (\text{convergence in } L^2(\mu_\varphi)) . \tag{2.20}$$

Moreover, we have the isometry,

$$\|F\|^2_{L^2(\mu_\varphi)} = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 \tag{2.21}$$

where $\alpha! = \alpha_1!\alpha_2! \dots \alpha_m!$ if $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{J}$.

Example 2.4 *If $F(\omega) = \langle \omega, f\rangle$ for some $f \in L^2_\varphi(\mathbf{R}^d)$, then F has the expansion*

$$F(\omega) = \Big\langle \omega, \sum_{i=1}^\infty (f, e_i)_\varphi e_i \Big\rangle = \sum_{i=1}^\infty (f, e_i)_\varphi \mathcal{H}_{\varepsilon^{(i)}}(\omega) . \tag{2.22}$$

In particular, for d-parameter fractional Brownian motion we get, by (2.12),

$$\begin{aligned} B_H(x) &= \langle \omega, \mathcal{X}_{[0,x]}(\cdot)\rangle = \sum_{i=1}^\infty (\mathcal{X}_{[0,x]}, e_i)_\varphi \mathcal{H}_{\varepsilon^{(i)}}(\omega) \\ &= \sum_{i=1}^\infty \Big[\int_0^x \Big(\int_{\mathbf{R}^d} e_i(v)\varphi(u,v)dv \Big) du \Big] \mathcal{H}_{\varepsilon^{(i)}}(\omega) , \end{aligned} \tag{2.23}$$

where $\int_0^x = \int_0^{x_d} \cdots \int_0^{x_1}$ *and* $\int_0^{x_i} = -\int_{x_i}^0$ *if* $x_i < 0$.

Next we proceed as in [HØUZ] to define the *multiparameter fractional Hida test function space* $(\mathcal{S})_H$ and *distribution space* $(\mathcal{S})^*_H$:

Definition 2.5 a) *(The multiparameter fractional Hida test function spaces) For $k \in \mathbf{N}$ define $(\mathcal{S})_{H,k}$ to be the space of all*

$$\psi(\omega) = \sum_{\alpha \in \mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega) \in L^2(\mu_\varphi) \tag{2.24}$$

such that

$$\|\psi\|^2_{H,k} := \sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha^2 (2\mathbf{N})^{k\alpha} < \infty \tag{2.25}$$

where

$$(2\mathbf{N})^\gamma = \prod_j (2j)^{\gamma_j} \qquad \text{if} \quad \gamma = (\gamma_1, \dots, \gamma_m) \in \mathcal{J} .$$

Define $(\mathcal{S})_H = \bigcap_{k=1}^{\infty} (\mathcal{S})_{H,k}$ *with the projective topology.*

b) *(The multiparameter fractional Hida distribution spaces)*
For $q \in \mathbf{N}$ *let* $(\mathcal{S})^*_{H,-q}$ *be the space of all formal expansions*

$$G(\omega) = \sum_{\beta \in \mathcal{J}} b_\beta \mathcal{H}_\alpha(\omega) \tag{2.26}$$

such that

$$\|G\|^2_{H,-q} : = \sum_{\beta \in \mathcal{J}} \beta! b^2_\beta (2\mathbf{N})^{-q\beta} < \infty \ . \tag{2.27}$$

Define

$$(\mathcal{S})^*_H = \bigcup_{q=1}^{\infty} (\mathcal{S})^*_{H,-q}$$

with the inductive topology. Then $(\mathcal{S})^*_H$ *becomes the dual of* $(\mathcal{S})_H$ *when the action of* $G \in (\mathcal{S})^*_H$ *given by (2.26) on* $\psi \in (\mathcal{S})_H$ *given by (2.24) is defined by*

$$\langle\langle G, \psi \rangle\rangle = \sum_{\alpha \in \mathcal{J}} \alpha! a_\alpha b_\alpha \ . \tag{2.28}$$

Example 2.6 (Multiparameter fractional white noise)
Define, for $y \in \mathbf{R}^d$

$$W_H(y) = \sum_{i=1}^{\infty} \Big[\int_{\mathbf{R}^d} e_i(v) \varphi(y, v) dv \Big] \mathcal{H}_{\varepsilon^{(i)}}(\omega) \tag{2.29}$$

Then as in [HØ, Example 3.6] we obtain that $W_H(y) \in (\mathcal{S})^*_H$ for all y. Moreover, $W_H(y)$ is integrable in $(\mathcal{S})^*_H$ for $0 \le y_i \le x_i$; $i = 1, \ldots, d$, and

$$\int_0^x W_H(y) dy = \sum_{i=1}^{\infty} \Big[\int_0^x \Big(\int_{\mathbf{R}^d} e_i(v) \varphi(y, v) dv \Big) dy \Big] \mathcal{H}_{\varepsilon^{(i)}}(\omega) = B_H(x) \ , \tag{2.30}$$

by (2.23). Therefore $B_H(x)$ is differentiable with respect to x in $(\mathcal{S})^*_H$ and we have

$$\frac{\partial^d}{\partial x_1 \ldots \partial x_d} B_H(x) = W_H(x) \qquad \text{in} \quad (\mathcal{S})^*_H \ . \tag{2.31}$$

This justifies the name (multiparameter) *fractional white noise* for $W_H(x)$.

The Wick product is defined just as in [HØUZ] and [HØ]:

Definition 2.7 *Suppose* $F(\omega) = \sum_{\alpha\in\mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega)$ *and* $G(\omega) = \sum_{\beta\in\mathcal{J}} b_\beta \mathcal{H}_\beta(\omega)$ *both belong to* $(\mathcal{S})_H^*$. *Then we define their* Wick product $(F \diamond G)(\omega)$ *by*

$$(F \diamond G)(\omega) = \sum_{\alpha,\beta\in\mathcal{J}} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega) = \sum_{\gamma\in\mathcal{J}} \Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) \mathcal{H}_\gamma(\omega) \; . \tag{2.32}$$

Example 2.8 **a)** ([HØ, Example 3.9]) If $f, g \in L^2_\varphi(\mathbf{R}^d)$ then

$$\Big(\int_{\mathbf{R}^d} f\, dB_H \Big) \diamond \Big(\int_{\mathbf{R}^d} g\, dB_H \Big) = \Big(\int_{\mathbf{R}^d} f\, dB_H \Big) \cdot \Big(\int_{\mathbf{R}^d} g\, dB_H \Big) - (f, g)_\varphi \; . \tag{2.33}$$

b) ([HØ, Example 3.10]) If $f \in L^2_\varphi(\mathbf{R}^d)$ then

$$\exp^\diamond(\langle \omega, f \rangle) \colon = \sum_{n=1}^\infty \frac{1}{n!} \langle \omega, f \rangle^{\diamond n}$$

converges in $(\mathcal{S})_H^*$ and is given by

$$\exp^\diamond(\langle \omega, f \rangle) = \exp(\langle \omega, f \rangle - \tfrac{1}{2} |f|_\varphi^2) \; . \tag{2.34}$$

We now use multiparameter fractional white noise to define integration with respect to multiparameter fractional Brownian motion, just as in [HØ, Definition 3.11] for the 1-parameter case:

Definition 2.9 *Suppose* $Y : \mathbf{R}^d \to (\mathcal{S})_H^*$ *is a given function such that* $Y(x) \diamond W_H(x)$ *is integrable in* $(\mathcal{S})_H^*$ *for* $x \in \mathbf{R}^d$. *Then we define the multiparameter fractional stochastic integral (of Itô type) of* $Y(x)$ *by*

$$\int_{\mathbf{R}^d} Y(x) dB_H(x) = \int_{\mathbf{R}^d} Y(x) \diamond W_H(x) dx \; . \tag{2.35}$$

Remark 2.10 If $H = \frac{1}{2}$ this definition gives an extension of the Itô-Skorohod integral. See [HØUZ, Section 2.5] for more details.

3 The linear heat equation driven by fractional white noise

In this section we illustrate the theory above by applying it to the linear stochastic fractional heat equation

$$\frac{\partial U}{\partial t}(t, x) = \tfrac{1}{2} \Delta U(t, x) + W_H(t, x) \; ; \qquad t \in (0, \infty), \quad x \in D \subset \mathbf{R}^n \tag{3.1}$$

$$U(0, x) = 0 \; ; \qquad x \in D \tag{3.2}$$

$$U(t, x) = 0 \; ; \qquad t \geq 0 \; , \quad x \in \partial D \tag{3.3}$$

Here $W_H(t,x)$ is the fractional white noise with Hurst parameter $H = (H_0, H_1, \ldots, H_n) \in (\frac{1}{2}, 1)^{n+1}$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $D \subset \mathbf{R}^n$ is a bounded open set with smooth boundary ∂D, $0 \le T \le \infty$ is a constant. We are looking for a solution $U : [0,\infty) \times \bar{D} \to (\mathcal{S})_H^*$ which is continuously differentiable in (t,x) and twice continuously differentiable in x, i.e. belongs to $C^{1,2}((0,\infty) \times D; (\mathcal{S})_H^*)$, and which satisfies (3.1) in the strong sense (as an $(\mathcal{S})_H^*$-valued function).

Based on the corresponding solution in the deterministic case (with $W_H(t,x)$ replaced by a bounded deterministic function) it is natural to guess that the solution will be

$$U(t,x) = \int_0^t \int_D W_H(s,y) G_{t-s}(x,y) dy\, ds \tag{3.4}$$

where $G_{t-s}(x,y)$ is the Green function for the heat operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta$. It is well-known [D] that G is smooth in $(0,T) \times D$ and that

$$G_u(x,y) \sim u^{-n/2} \exp\left(- \frac{|x-y|^2}{\delta u} \right) \qquad \text{in } (0,\infty) \times D \ , \tag{3.5}$$

where the notation $X \sim Y$ means that

$$\frac{1}{C} X \le Y \le CX \qquad \text{in } (0,\infty) \times D \ ,$$

for some positive constant $C < \infty$ depending only on D.

We use this to verify that $U(t,x) \in \mathcal{S}_H^*$ for all $(t,x) \in [0,\infty) \times \bar{D}$:

Using (2.29) we see that the expansion of $U(t,x)$ is

$$\begin{aligned} U(t,x) &= \int_0^t \int_D G_{t-s}(x,y) \sum_{k=1}^{\infty} \left[\int_{\mathbf{R}^n} e_k(v) \varphi(y,v) dv \right] \mathcal{H}_{\varepsilon^{(k)}}(\omega) dy\, ds \\ &= \sum_{k=1}^{\infty} b_k(t,x) \mathcal{H}_{\varepsilon^{(k)}}(\omega) \ , \end{aligned} \tag{3.6}$$

where

$$b_k(t,x) = b_{\varepsilon^{(k)}}(t,x) = \int_0^t \int_D G_{t-s}(x,y) \left[\int_{\mathbf{R}^n} e_k(v) \varphi(y,v) dv \right] dy\, ds \tag{3.7}$$

In the following C denote constants, not necessarily the same from place to place. From

Lemma 2.2 and (3.7) we obtain that

$$
\begin{aligned}
(3.8)\qquad |b_k(t,x)| &\le C\prod_{i=1}^{d}(\alpha_i^{(k)})^{1/6}\int_0^t\int_D G_{t-s}(x,y)dy\,ds \\
&\le C\prod_{i=1}^{d}(\alpha_i^{(k)})^{1/6}\int_0^t\Big(\int_{\mathbf{R}^n} s^{-n/2}\exp\Big(-\frac{y^2}{\delta s}\Big)dy\Big)ds \\
&\overset{y=\sqrt{\delta s}\,z}{\le} C\prod_{i=1}^{d}(\alpha_i^{(k)})^{1/6}\int_0^t\Big(\int_{\mathbf{R}^n} s^{-n/2}\exp(-z^2)(\delta s)^{n/2}dz\Big)ds \\
&= C\prod_{i=1}^{d}(\alpha_i^{(k)})^{1/6}t\ .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\sum_{k=1}^{\infty} b_k^2(t,x)(2\mathbf{N})^{-q\varepsilon^{(k)}} \\
(3.9)\qquad &\le C(t)\sum_{k=1}^{\infty}\prod_{i=1}^{d}(\alpha_i^{(k)})^{1/3}(2k)^{-q}\le\sum_{k=1}^{\infty}k^{d/3}(2k)^{-q}<\infty\qquad\text{for }\ q>\frac{d+3}{3}\ .
\end{aligned}
$$

Here we used the fact $|\alpha^{(k)}|\le k$, which is the consequence of the special order. Hence $U(t,x)\in(\mathcal{S})^*_{H,-q}$ for all $q>\frac{d+3}{3}$, for all t,x.

In fact, this estimate also shows that $U(t,x)$ is uniformly continuous as a function from $[0,T]\times\bar{D}$ into $(\mathcal{S})^*_H$ for any $T<\infty$ and that $U(t,x)$ satisfies (3.2) and (3.3). Moreover, by the properties of $G_{t-s}(x,y)$ we get from (3.4) that

$$
\begin{aligned}
\frac{\partial U}{\partial t}(t,x)-\Delta U(t,x) &= \int_0^t\int_D W_H(s,y)\Big(\frac{\partial}{\partial t}-\Delta\Big)G_{t-s}(x,y)dy\,ds+W_H(t,x) \\
(3.10)\qquad &= W_H(t,x)\ ,\quad\text{so }\ U(t,x)\text{ satisfies (3.1) also}\ .
\end{aligned}
$$

In the standard white noise case ($H_i=\frac{1}{2}$ for all i) the same solution formula (3.4) holds. In this case we see that the solution $U(t,x)$ belongs to $L^2(\mu)$ (μ being the standard white noise measure) iff

$$
(3.11)\qquad E_\mu[U^2(t,x)]=\int_0^t\int_D G^2_{t-s}(x,y)dy\,ds<\infty\ .
$$

Now, if $D \subset (-\frac{1}{2}R, \frac{1}{2}R)^n$ and we put $F = [-R, R]^n$,

$$\int_0^t \int_D G_{t-s}^2(x,y) ds\, dy \sim \int_0^t \int_D s^{-n} \exp\left(-\frac{2y^2}{\delta s}\right) dy\, ds$$

$$\sim \int_0^t \left(\int_{F/\sqrt{s}} s^{-n/2} \exp\left(-\frac{2z^2}{\delta}\right) dz \right) ds \; .$$

Hence

$$E_\mu[U^2(t,x)] < \infty \Longleftrightarrow n = 1 \; . \tag{3.12}$$

Next, consider the fractional case $\frac{1}{2} < H_i < 1$ for all i. Then

$$E_{\mu_\varphi}[U^2(t,x)] = \int_0^t \int_0^t \int_D \int_D G_{t-r}(x,y) G_{t-s}(x,z) \varphi(r,s,y,z) dr\, ds\, dy\, dz$$

$$\sim \int_0^t \int_0^t \int_D \int_D r^{-n/2} s^{-n/2} \exp\left(-\frac{|x-y|^2}{\delta r}\right) \exp\left(-\frac{|x-z|^2}{\delta s}\right)$$

$$\cdot |r-s|^{2H_0-2} \prod_{i=1}^n |y_i - z_i|^{2H_i-2} dy_1 \ldots dy_n dz_1 \ldots dz_n\, dr\, ds \; . \tag{3.13}$$

Choose $1 < q < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. By the Hölder inequality we have

$$\prod_{i=1}^n \int_{\frac{1}{2}R}^{\frac{1}{2}R} \int_{\frac{1}{2}R}^{\frac{1}{2}R} \exp\left(-\frac{|x_i - y_i|^2}{\delta r} - \frac{|x_i - z_i|^2}{\delta s}\right) |y_i - z_i|^{2H_i - 2} dy_i\, dz_i$$

$$\leq \prod_{i=1}^n \int_{\frac{1}{2}R}^{\frac{1}{2}R} \exp\left(-\frac{|x_i - y_i|^2}{\delta r}\right) dy_i \left\{ \left[\int_{\frac{1}{2}R}^{\frac{1}{2}R} \exp\left(-\frac{p|x_i - z_i|^2}{\delta s}\right) dz_i \right]^{1/p} \right.$$

$$\left. \cdot \left[\int_{\frac{1}{2}R}^{\frac{1}{2}R} |y_i - z_i|^{q(2H_i-2)} dz_i \right]^{1/q} \right\}$$

$$\sim \left(\frac{r}{p}\right)^{n/2} \left[\left(\frac{s}{p}\right)^{n/2} \right]^{1/p} \qquad \text{if} \quad q(2H_i - 2) > -1 \; . \tag{3.14}$$

Substituted into (3.13) this gives

$$E_{\mu_\varphi}[U^2(t,x)] \leq C(p) \int_0^t \int_0^t (s)^{-\frac{n}{2}(1-\frac{1}{p})} |r-s|^{2H_0-2} dr\, ds \tag{3.15}$$

$$< \infty \qquad \text{if} \quad n < \frac{2p}{p-1} \; .$$

Combined with the requirement $q(2H_i - 2) > -1$ we obtain from this that

$$E_{\mu_\varphi}[U^2(t,x)] < \infty \qquad \text{if } \; n < \frac{1}{1-H_i} \; \text{ for } \; 1 \le i \le n \; .$$

We summarize what we have proved:

Theorem 3.1 **a)** *For any space dimension n there is a unique strong solution $U(t,x) : [0,\infty) \times D \to (\mathcal{S})^*_H$ of the fractional heat equation (3.1)–(3.3). The solution is given by*

$$U(t,x) = \int_0^t \int_D W_H(s,y) G_{t-s}(x,y) dy\, ds \; . \tag{3.16}$$

It belongs to $C^{1,2}((0,\infty) \times D \to (S)^*_H) \cap C([0,\infty) \times \bar{D} \to (S)^*_H)$.

b) *If $H = (H_0, H_1, \dots, H_n) \in (\frac{1}{2}, 1)^{n+1}$ and*

$$H_i > 1 - \frac{1}{n} \qquad \textit{for} \quad i = 1, 2, \dots, n \tag{3.17}$$

then $U(t,x) \in L^2(\mu_\varphi)$ for all $t \ge 0$, $x \in \bar{D}$.

c) *In particular, for all $H \in (\frac{1}{2}, 1)^{d+1}$ we have*

$$U(t,x) \in L^2(\mu_\varphi) \qquad \textit{if} \;\; n \le 2 \; . \tag{3.18}$$

Remark 3.2 Note that condition (3.17) is sharp at $H_i = \frac{1}{2}$, in the sense that if we let $H_i \to \frac{1}{2}$ for $i = 1, \dots, n$ then (3.17) reduces to the condition $n = 1$ which we found for the standard white noise case (3.12).

Remark 3.3 In [H1] (and more generally in [H2]) the heat equation with a *fractional white noise potential* is studied:

$$\frac{\partial u}{\partial t}(t,x) = \tfrac{1}{2}\Delta u(t,x) + u(t,x) \diamond W_H(t,x) \; ; \qquad x \in \mathbf{R}^n \, , \quad t > 0 \; .$$

There it is shown that if $H = (H_0, H_1, \dots, H_n)$ with $H_i \in (\frac{1}{2}, 1)$ for $i = 0, 1, \dots, n$ and

$$H_1 + H_2 + \cdots + H_n > n - \frac{2}{2H_0 - 1}$$

then $u(t,x) \in L^2(\mu_\varphi)$ for all t, x.

4 The quasilinear stochastic fractional heat equation

Let $f : \mathbf{R} \to \mathbf{R}$ be a function satisfying

$$|f(x) - f(y)| \le L|x - y| \qquad \text{for all } x, y \in \mathbf{R} \tag{4.1}$$

$$|f(x)| \le M(1 + |x|) \qquad \text{for all } x \in \mathbf{R}\,, \tag{4.2}$$

where L and M are constants.

In this section we consider the following quasi-linear generalization of equation (3.1)–(3.3):

$$\frac{\partial U}{\partial t}(t,x) = \tfrac{1}{2}\Delta U(t,x) + f(U(t,x)) + W_H(t,x)\,; \qquad t > 0\,,\ x \in \mathbf{R}^n \tag{4.3}$$

$$U(0,x) = U_0(x)\,; \qquad x \in \mathbf{R}^n \tag{4.4}$$

where $U_0(x)$ is a given bounded deterministic function on $\mathbf{R}^n$.

We say that $U(t,x)$ is a solution of (4.3)–(4.4) if

$$\begin{aligned} &\int_{\mathbf{R}^n} U(t,x)\varphi(x)dx - \int_{\mathbf{R}^n} U_0(x)\varphi(x)dx \\ &\quad = \tfrac{1}{2}\int_0^t \int_{\mathbf{R}^n} U(s,x)\Delta\varphi(x)dx\,ds + \int_0^t \int_{\mathbf{R}^n} f(U(s,x))\varphi(x)dx\,ds \\ &\qquad + \int_0^t \int_{\mathbf{R}^n} \varphi(x)dB_H(s,x) \end{aligned} \tag{4.5}$$

for all $\varphi \in C_0^\infty(\mathbf{R}^n)$.

As in Walsh [W] we can show that $U(t,x)$ solves (4.5) if and only if it satisfies the following integral equation

$$\begin{aligned} U(t,x) &= \int_{\mathbf{R}^n} U_0(y)G_t(x,y)dy + \int_0^t \int_{\mathbf{R}^n} f(U(s,y))G_{t-s}(x,y)dy\,ds \\ &\quad + \int_0^t \int_{\mathbf{R}^n} G_{t-s}(x,y)dB_H(s,y)\,, \end{aligned} \tag{4.6}$$

where

$$G_{t-s}(x,y) = (2\pi(t-s))^{-n/2}\exp\left(-\frac{|x-y|^2}{2(t-s)}\right); \qquad s < t\,,\ x \in \mathbf{R}^n \tag{4.7}$$

is the Green function for the heat operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta$ in $(0,\infty) \times \mathbf{R}^n$.

For the proof of our main result, we need the following two lemmas. Let $0 < \alpha < 1$. Define, for $u > 0$,

$$g(u, y) = \int_{\mathbf{R}} |y - z|^{-\alpha} \frac{1}{\sqrt{u}} exp(-\frac{z^2}{2u}) dz \tag{4.8}$$

Lemma 4.1 *Assume $p > \frac{1}{1-\alpha}$. Then $g(u, y) \leq C(1 + u^{-\frac{1}{2}(1-\frac{1}{p})})$, where C is a constant independent of y and u.*

Proof. In the proof, we will use C to denote a generic constant independent of y and u. First,note that

$$g(u, y) = \int_{|z-y|\leq 1} |y - z|^{-\alpha} \frac{1}{\sqrt{u}} exp(-\frac{z^2}{2u}) dz + \int_{|z-y|>1} |y - z|^{-\alpha} \frac{1}{\sqrt{u}} exp(-\frac{z^2}{2u}) dz$$

By Hölder inequality,

$$g(u, y) \leq C\Big\{1 + \Big[\int_{|z-y|\leq 1} |y - z|^{-\alpha \frac{p}{p-1}} dz\Big]^{\frac{p-1}{p}} \Big[\int_{|z-y|\leq 1} \frac{1}{u^{\frac{1}{2}p}} \exp\Big(-\frac{pz^2}{2u}\Big) dz\Big]^{\frac{1}{p}}\Big\}$$

$$\leq C(1 + u^{-\frac{1}{2}(1-\frac{1}{p})}) \tag{4.9}$$

□

Let $F(y_1, y_2, \ldots, y_n)$ denote a function on $\mathbf{R}^n$.

Lemma 4.2 *Let $h = (h_1, h_2, \ldots, h_n)$ with $h_i \geq 0$, $1 \leq i \leq n$. Asume that F and all its partial derivatives of first order are integrable with respect to the Lebesgue measure. Then*

$$\int_{\mathbf{R}^n} |F(y - h) - F(y)| dy \leq \sum_{i=1}^{n} \Big(\int_{\mathbf{R}^n} \Big|\frac{\partial F}{\partial y_i}(y_1, y_2, \ldots, y_n)\Big| dy\Big) h_i \tag{4.10}$$

Proof. Observe that

$$\begin{aligned} F(y - h) - F(y) &= \sum_{i=1}^{n} (F(y_1, \ldots, y_{i-1}, y_i - h_i, y_{i+1} - h_{i+1}, \ldots, y_n - h_n) \\ &\quad - F(y_1, \ldots, y_{i-1}, y_i, y_{i+1} - h_{i+1}, \ldots, y_n - h_n)) \\ &= \sum_{i=1}^{n} \int_{y_i - h_i}^{y_i} -\frac{\partial F}{\partial y_i}(y_1, \ldots, y_{i-1}, z, y_{i+1} - h_{i+1}, \ldots, y_n - h_n) dz \qquad (4.11) \end{aligned}$$

Integrating the equation (4.11), we get

$$
\begin{aligned}
&\int_{\mathbf{R}^n} |F(y-h) - F(y)| dy \\
&\leq \sum_{i=1}^{n} \int_{\mathbf{R}^{n-1}} dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_n \\
&\quad \cdot \int_{\mathbf{R}} dy_i \int_{y_i - h_i}^{y_i} \Big| \frac{\partial F}{\partial y_i} \Big| (y_1, \cdots, y_{i-1}, z, y_{i+1} - h_{i+1}, \cdots, y_n - h_n) dz \\
&= \sum_{i=1}^{n} \int_{\mathbf{R}^{n-1}} dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_n \\
&\quad \cdot \int_{\mathbf{R}} dz \Big| \frac{\partial F}{\partial y_i} \Big| (y_1, \cdots, y_{i-1}, z, y_{i+1} - h_{i+1}, \cdots, y_n - h_n) \int_{z}^{z+h_i} dy_i \\
&= \sum_{i=1}^{n} \Big(\int_{\mathbf{R}^n} \Big| \frac{\partial F}{\partial y_i} (y_1, y_2, \ldots, y_n) \Big| dy \Big) h_i
\end{aligned}
$$

□

Our main result is the following:

Theorem 4.3 *Let* $H = (H_0, H_1, \ldots, H_n) \in (\frac{1}{2}, 1)^{n+1}$ *with*

$$
H_i > 1 - \frac{1}{n} \qquad \text{for } \; i = 1, 2, \ldots, n \; .
$$

Then there exists a unique $L^2(\mu_\varphi)$*-valued random field solution* $U(t,x)$*;* $t \geq 0$*,* $x \in \mathbf{R}^n$ *of (4.3)–(4.4). Moreover, the solution has a jointly continuous version in* (t,x) *if* $H_0 > \frac{3}{4}$*.*

Proof. Define

$$
V(t,x) = \int_0^t \int_{\mathbf{R}^n} G_{t-s}(x,y) dB_H(s,y) \; . \tag{4.12}
$$

Dividing R into regions $\{z; |z-y| \leq 1\}$ and $\{z; |z-y| > 1\}$, we see that a slight modification of the arguments in Section 4 gives that $E_{\mu_\varphi}[V^2(t,x)] < \infty$, so $V(t,x)$ exists as an ordinary random field. The existence of the solution now follows by usual Picard iteration: Define

$$
U_0(t,x) = U_0(x) \tag{4.13}
$$

and iteratively

$$U_{j+1}(t,x) = \int\limits_{\mathbf{R}^n} U_0(y) G_t(x,y)dy + \int\limits_0^t \int\limits_{\mathbf{R}^n} f(U_j(s,y)) G_{t-s}(x,y)dy\,ds + V(t,x)\ ; \qquad j = 0,1,2,\dots \tag{4.14}$$

Then by (4.2) $U_j(t,x) \in L^2_{\mu_\varphi}$ for all j. We have

$$U_{j+1}(t,x) - U_j(t,x) = \int\limits_0^t \int\limits_{\mathbf{R}^n} [f(U_j(s,y)) - f(U_{j-1}(s,y))] G_{t-s}(x,y)dy\,ds$$

and therefore by (4.1), if $t \in [0,T]$,

$$\begin{aligned}
&E_{\mu_\varphi}[|U_{j+1}(t,x) - U_j(t,x)|^2] \\
&\quad \le L E_{\mu_\varphi}\Big[\Big(\int\limits_0^t \int\limits_{\mathbf{R}^n} |U_j(s,y) - U_{j-1}(s,y)| G_{t-s}(x,y)dy\,ds\Big)^2\Big] \\
&\le L\Big(\int\limits_0^t \int\limits_{\mathbf{R}^n} G_{t-s}(x,y)dy\,ds\Big) E_{\mu_\varphi}\Big[\int\limits_0^t \int\limits_{\mathbf{R}^n} |U_j(s,y) - U_{j-1}(s,y)|^2 G_{t-s}(x,y)dy\,ds\Big] \\
&\quad \le C_T \int\limits_0^t \sup_y E[|U_j(s,y) - U_{j-1}(s,y)|^2]ds \\
&\quad \le \cdots \le C_T^j \int\limits_0^t \int\limits_0^{s_1} \cdots \int\limits_0^{s_{j-1}} \sup_y E[|U_1(s,y) - U_0(s,y)|^2] ds ds_{j-1} \dots ds_1 \\
&\quad \le A_T C_T^j \frac{T^j}{(j)!} \qquad \text{for some constants } A_T,\ C_T\ .
\end{aligned}$$

It follows that the sequence $\{U_j(t,x)\}_{j=1}^\infty$ of random fields converges in $L^2(\mu_\varphi)$ to a random field $U(t,x)$. Letting $k \to \infty$ in (4.10) we see that $U(t,x)$ is a solution of (4.3)–(4.4). The uniqueness follows from the Gronwall inequality. It is not difficult to see that both

$$\int\limits_{\mathbf{R}^n} U_o(y) G_t(x,y)dy \qquad \text{and} \qquad \int\limits_0^t \int\limits_{\mathbf{R}^n} f(U(s,y)) G_{t-s}(x,y)dy\,ds$$

are jointly continuous in (t,x). So to finish the proof of the theorem it suffices to prove that $V(t,x)$ has a jointly continuous version.

To this end, consider for $h \in \mathbf{R}$

$$V(t+h,x)-V(t,x)=\int_t^{t+h}\int_{\mathbf{R}^d} G_{t+h-s}(x,y)dB_H(s,y)$$

$$+\int_0^t\int_{\mathbf{R}^n}(G_{t+h-s}(x,y)-G_{t-s}(x,y))dB_H(s,y) \tag{4.15}$$

By the estimate in (3.15) it follows that

$$E\Big[\Big|\int_t^{t+h}\int_{\mathbf{R}^n} G_{t+h-s}(x,y)dB_H(s,y)\Big|^2\Big]\le C\int_t^{t+h}(u-t)^{2H_0-2}du$$

$$\le C\,h^{2H_0-1}\,. \tag{4.16}$$

To estimate the second term on the right hand side of (4.15), we use (2.15) and proceed as follows:

$$E\Big[|\int_0^t\int_{\mathbf{R}^n}(G_{t+h-s}(x,y)-G_{t-s}(x,y))dB_H(s,y)|^2\Big]$$

$$\le C\int_{\mathbf{R}}\int_{\mathbf{R}}\mathcal{X}_{[0,t]}(r)\mathcal{X}_{[0,t]}(s)|r-s|^{2H_0-2}$$

$$\cdot\Big[\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\Big\{(t+h-r)^{-n/2}\exp\Big(-\frac{|x-z|^2}{2(t+h-r)}\Big)$$

$$-(t-r)^{-n/2}\exp\Big(-\frac{|x-z|^2}{2(t-r)}\Big)\Big\}$$

$$\cdot\Big\{(t+h-s)^{-n/2}\exp\Big(-\frac{|x-y|^2}{2(t+h-s)}\Big)$$

$$-(t-s)^{-n/2}\exp\Big(-\frac{|x-y|^2}{2(t-r)}\Big)\Big\}$$

$$\cdot\prod_{i=1}^n|y_i-z_i|^{2H_i-2}dy\,dz\Big]dr\,ds \tag{4.17}$$

$$\le C\int_{\mathbf{R}}\int_{\mathbf{R}}\mathcal{X}_{[0,t]}(r)\mathcal{X}_{[0,t]}(s)|r-s|^{2H_0-2}$$

$$\cdot\Big[\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\Big\{(r+h)^{-n/2}\exp\Big(-\frac{|z|^2}{2(r+h)}\Big)-r^{-n/2}\exp\Big(-\frac{|z|^2}{2r}\Big)\Big\}$$

$$\cdot\Big\{(s+h)^{-n/2}\exp\Big(-\frac{|y|^2}{2(s+h)}\Big)-s^{-n/2}\exp\Big(-\frac{|y|^2}{2s}\Big)\Big\}$$

$$\cdot\prod_{i=1}^n|y_i-z_i|^{2H_i-2}dy\,dz\Big]dr\,ds \tag{4.18}$$

From (4.17) to (4.18), we first perform the change of variables: $x-y=y'$, $x-z=z'$, $t-r=r'$, $t-s=s'$ and then we change the name of y',z',r',s' back to y,z,r,s again for simplicitiy. (4.18) is further less than

$$
\begin{aligned}
& C\int_0^t ds\int_0^s dr(s-r)^{2H_0-2} \\
& \cdot\Big[\int_{\mathbf{R}^n} dy\int_s^{s+h}(-\frac{n}{2}v^{-\frac{n}{2}-1}\exp\Big(-\frac{|y|^2}{2v}\Big)+\frac{1}{2}v^{-\frac{n}{2}-2}|y|^2\exp\Big(-\frac{|y|^2}{2v}\Big))dv \\
& \cdot\Big\{(r+h)^{-n/2}\exp\Big(-\frac{|z|^2}{2(r+h)}\Big)-r^{-n/2}\exp\Big(-\frac{|z|^2}{2r}\Big)\Big\} \\
& \cdot\prod_{i=1}^n|y_i-z_i|^{2H_i-2}dz\Big] \\
\le\ & C\int_0^t ds\int_0^s dr(s-r)^{2H_0-2} \\
& \cdot\Big[\int_s^{s+h}dv\int_{\mathbf{R}^n}dy(\frac{n}{2}v^{-\frac{n}{2}-1}\exp\Big(-\frac{|y|^2}{2v}\Big)+\tfrac{1}{2}v^{-\frac{n}{2}-2}|y|^2\exp\Big(-\frac{|y|^2}{2v}\Big)) \\
& \cdot\int_{\mathbf{R}^n}\Big\{(r+h)^{-n/2}\exp\Big(-\frac{|z|^2}{2(r+h)}\Big)+r^{-n/2}\exp\Big(-\frac{|z|^2}{2r}\Big)\Big\} \\
& \cdot\prod_{i=1}^n|y_i-z_i|^{2H_i-2}dz\Big] \qquad (4.19)
\end{aligned}
$$

Choose $p>1$ such that

$$\frac{1}{2H_i-1}<p<\frac{n}{n-2}\qquad \text{for } \ i=1,2,\dots,n\,.$$

This is possible since $H_i>1-\frac{1}{n}$ for $i=1,2,\dots,n$. Then

$$\frac{2p}{p-1}>d\qquad\text{and}\qquad\frac{p}{p-1}(2H_i-2)>-1,\qquad i=1,2,\dots,n$$

Now applying Lemma 4.1 repeatedly to this choice of p and to $\alpha=2-2H_i$, we get

$$
\begin{aligned}
(4.19) &\le C\int_0^t ds\int_0^s dr(s-r)^{2H_0-2}\cdot\int_s^{s+h}dv\frac{1}{v}\Big(1+Cr^{-\frac{1}{2}(1-\frac{1}{p})}\Big)^n \\
&\le C\int_0^t ds\int_0^s dr\int_s^{s+h}dv\frac{1}{v}\Big(1+r^{-\frac{n}{2}(1-\frac{1}{p})}\Big)(s-r)^{2H_0-2} \qquad (4.20)
\end{aligned}
$$

Choose β such that $2-2H_0<\beta<1$. It follows that (4.20) is dominated by

$$C\int_0^t ds\int_0^s dr\frac{1}{s^{1-\beta}}\int_s^{s+h} dv\frac{1}{v^\beta}\Big(1+r^{-\frac{n}{2}(1-\frac{1}{p})}\Big)(s-r)^{2H_0-2}$$

$$\le Ch^{1-\beta}\int_0^t ds\int_0^s dr\frac{1}{s^{1-\beta}}\,r^{-\frac{n}{2}(1-\frac{1}{p})}(s-r)^{2H_0-2}$$

$$= Ch^{1-\beta}\int_0^t ds\frac{1}{s^{1-\beta}}\Big[\int_0^{\frac{s}{2}} r^{-\frac{n}{2}(1-\frac{1}{p})}(s-r)^{2H_0-2}dr+\int_{\frac{s}{2}}^{s} r^{-\frac{n}{2}(1-\frac{1}{p})}(s-r)^{2H_0-2}dr\Big]$$

$$\le Ch^{1-\beta}\int_0^t \frac{1}{s^{1-\beta}}s^{1-\frac{n}{2}(1-\frac{1}{p})-(2-2H_0)}ds\le Ch^{1-\beta}\,. \tag{4.21}$$

On the other hand, for $k\in\mathbf{R}^n$ we have

$$V(t,x+k)-V(t,x)=\int_0^t\int_{\mathbf{R}^n}(G_{t-s}(x+k,y)-G_{t-s}(x,y))dB_H(s,y)$$

Hence, by (4.7),

$$E[\,|V(t,x+k)-V(t,x)|^2]$$
$$\le C\int_0^t\int_0^t|r-s|^{2H_0-2}$$
$$\cdot\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\Big\{(t-r)^{-n/2}\Big(\exp\Big(-\frac{|x+k-y|^2}{2(t-r)}\Big)-\exp\Big(-\frac{|x-y|^2}{2(t-r)}\Big)\Big)\Big\}$$
$$\cdot\Big\{(t-s)^{-n/2}\Big(\exp\Big(-\frac{|x+k-z|^2}{2(t-s)}\Big)-\exp\Big(-\frac{|x-z|^2}{2(t-s)}\Big)\Big)\Big\}$$
$$\cdot\prod_{i=1}^n|y_i-z_i|^{2H_i-2}dy\,dz\,dr\,ds$$
$$\le C\int_0^t\int_0^t|r-s|^{2H_0-2}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\Big\{r^{-n/2}\Big(\exp\Big(-\frac{|y+k|^2}{2r}\Big)-\exp\Big(-\frac{|y|^2}{2r}\Big)\Big)\Big\}$$
$$\cdot\Big\{s^{-n/2}\Big(\exp\Big(-\frac{|z+k|^2}{2s}\Big)-\exp\Big(-\frac{|z|^2}{2s}\Big)\Big)\Big\}\prod_{i=1}^n|y_i-z_i|^{2H_i-2}dy\,dz\,dr\,ds$$
$$\le C\int_0^t ds\int_0^s dr(s-r)^{2H_0-2}\int_{\mathbf{R}^d}dy\Big|s^{-n/2}\Big(\exp\Big(-\frac{|y+k|^2}{2s}\Big)-\exp\Big(-\frac{|y|^2}{2s}\Big)\Big)\Big|$$
$$\cdot\int_{\mathbf{R}^n}dz\Big\{r^{-n/2}\Big(\exp\Big(-\frac{|z+k|^2}{2r}\Big)-\exp\Big(-\frac{|z|^2}{2r}\Big)\Big)\Big\}\prod_{i=1}^n|y_i-z_i|^{2H_i-2} \tag{4.22}$$

Applying Lemma 4.1 and Lemma 4.2 we get

$$(4.22) \le C \int_0^t ds \int_0^s dr (s-r)^{2H_0-2} \left(1 + r^{-\frac{1}{2}(1-\frac{1}{p})}\right)^n$$

$$\cdot \sum_{i=1}^n |k_i| \int_{\mathbf{R}^n} s^{-n/2-1} \exp\left(-\frac{|y|^2}{2s}\right) |y_i| dy$$

$$\le C|k| \int_0^t ds \int_0^s dr \frac{1}{s^{\frac{1}{2}}} (s-r)^{2H_0-2} r^{-\frac{n}{2}(1-\frac{1}{p})}$$

$$\le C|k| \int_0^t ds \frac{1}{s^{\frac{1}{2}}} \Big[\int_0^{\frac{s}{2}} dr (s-r)^{2H_0-2} r^{-\frac{n}{2}(1-\frac{1}{p})} + \int_{\frac{s}{2}}^s dr (s-r)^{2H_0-2} r^{-\frac{n}{2}(1-\frac{1}{p})} \Big]$$

$$\le C|k| \int_0^t s^{2H_0 - \frac{n}{2}(1-\frac{1}{p}) - \frac{3}{2}} ds \le C|k|, \qquad \text{if} \quad H_0 > \tfrac{3}{4}\ . \tag{4.23}$$

Combining the estimates (4.16), (4.21) and (4.23) we get, for some $\beta < 1$,

$$E[\,|V(t+h, x+h) - V(t,x)|^2] \le C[h^{1-\beta} + |k|\,]\ .$$

Since $V(t+h, x+k) - V(t,x)$ is a Gaussian random variable with mean zero, it follows that for any $m \ge 1$

$$E[\,|V(t+h,x+k) - V(t,x)|^{2m}] \le C_m E[\,|V(t+h, x+k) - V(t,x)|^2]^m$$
$$\le C_m [h^{1-\beta} + |k|\,]^m \le C_m [h^{1-\beta} + |k|]^m \quad \text{if } m \text{ is big enough}\ .$$

Hence by Kolmogorov's theorem we conclude that $V(t,x)$ admits a jointly continuous version. □

References

[BG] F.E. Benth and H. Gjessing: A non-linear parabolic equation with noise. A reduction method. To appear in Potential Analysis.

[D] E.B. Davis: Heat Kernels and Spectral Theory. Cambridge University Press 1989.

[DHP] T. E. Duncan, Y. Hu and B. Pasik-Duncan: Stochastic calculus for fractional Brownian motion. I. Theory. To appear in SIAM Journal of Control and Optimization.

[GN] G. Gripenberg and I. Norros: On the prediction of fractional Brownian motion. *J. Appl. Prob.* **33** (1996), 400–410.

[H1] Y. Hu: Heat equation with fractional white noise potentials. Preprint 1998.

[H2] Y. Hu: A class of stochastic partial differential equations driven by fractional white noise. Preprint 1999.

[HKPS] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: White Noise Analysis. Kluwer 1993.

[HØ] Y. Hu and B. Øksendal: Fractional white noise calculus and applications to finance. Preprint, Dept. of Mathematics, University of Oslo 1999.

[HØUZ] H. Holden, B. Øksendal, J. Ubøe and T. Zhang: Stochastic Partial Differential Equations. Birkhäuser 1996.

[HØS] Y. Hu, B. Øksendal and A. Sulem: Optimal portfolio in a fractional Black & Scholes market. Preprint, Dept. of Mathematics, University of Oslo 1999.

[HØZ] Y. Hu, B. Øksendal and T. Zhang: Stochastic partial differential equations driven by multiparameter fractional white noise. Preprint, Dept. of Mathematics, University of Oslo 1999.

[J] S. Janson: Gaussian Hilbert Spaces. Cambridge University Press 1997.

[K] H.-H. Kuo: White Noise Distribution Theory. CRC Press 1996.

[MV] B.B. Mandelbrot and J.W. Van Ness: Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** (1968), 422–437.

[NVV] I. Norros, E. Valkeila and J. Virtamo: An elementary approach to a Girsanov formula and other analytic results on fractional Brownian motions. *Bernoulli* **5** (1999), 571–587.

[W] J.B. Walsh: An introduction to stochastic partial differential equations. In R. Carmona, H. Kesten and J.B. Walsh (editors): École d'Été de Probabilités de Saint-Flour XIV–1984. Springer LNM 1180, pp. 265–437.