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3-FOLDS OF $\mathbb{P}^5$ WITH ONE APPARENT 4-TUPLE POINT

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ABSTRACT. In this article we classify all the smooth 3-folds of $\mathbb{P}^5$ with an apparent 4-tuple point provided that the family of its 4-secant lines is an irreducible (first order) congruence. This is sufficient to conclude the classification of all the smooth codimension two varieties of $\mathbb{P}^n$ with one apparent $(n-1)$-point and with irreducible family of $(n-1)$-secant lines.

INTRODUCTION

A congruence of lines in $\mathbb{P}^n$ is a family of lines of dimension $n-1$, and its order is the number of lines passing through a general point of $\mathbb{P}^n$. A codimension two subvariety of $\mathbb{P}^n$ is said to have q apparent $(n-1)$-tuple points if its general projection from a point to a hyperplane has q $(n-1)$-tuple points as singularities.

In an our previous work, [De 00] we proved that the degree of these varieties is bounded by $(n-1)^2$, and we observed that this implies that they cannot be complete intersections. By an A. Holme and M. Schneider's result, [HS85], this implies also that in order to classify the smooth ones, we can stop up to dimension three.

In this paper we give a partial result towards this classification, *i.e.* we restrict ourselves to the case in which the family of the $(n-1)$-secant lines is in fact an irreducible first order congruence.

This article is structured as follows: after giving, in Section 1, the basic definitions, we redo some general results about first order congruences in $\mathbb{P}^n$ given in [De 00] for the sake of completeness. In particular, after giving the central definition of fundamental d-loci, we show how to obtain the degree bound for the fundamental $(n-2)$-locus of a congruence.

In Section 2, we give two general examples of congruences in $\mathbb{P}^n$: the first one is that of linear congruences, *i.e.* the congruences which come out from general linear sections of the Grassmannians; the second one is that of congruences given by the $(n-1)$-secant lines of the varieties given by the degeneracy locus of a general map $\phi \in \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}, \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1))$. Of these two examples we calculate the minimal free resolution of the ideal sheaf of their focal locus and of the congruences themselves. We study these examples because they give us all the congruences of our classification but one (*i.e.* case (4b) of Theorem 0.1).

In Section 3 we prove two multiple point formulae: the 4-tuple point formula for a smooth 3-fold of $\mathbb{P}^5$ and the formula which gives the number of 4-secant lines to a smooth surface of $\mathbb{P}^4$ passing through a general point of the surface itself. Strangely enough, the 4-tuple point formula for the 3-folds—which is actually an application of S. Kleiman's multiple point formulae of maps (see [Kle81])—it seems to have been unknown, at least in modern times (see for example [BSS95]). With

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this formula, E. Mezzetti has been able to exclude the only degree 12 smooth 3-fold of $\mathbb{P}^5$ for which the existence was uncertain (see [Ede94]). I must say that after the proof of this formula, I realized that the same was obtained by S. Kwak in [Kwa01], but with other methods, *i.e.* through the monoidal construction.

The irreducible congruences of order one of $\mathbb{P}^5$ which are given by the 4-secant lines of a smooth 3-fold are classified in Section 4. By the results of Section 2 and what we said above, we obtain the following complete list, where d is the degree of the smooth codimension two variety X, π its sectional genus and, if $\dim X = 3$, S is its general hyperplane section; finally, H and K are the hyperplane and canonical divisor classes, respectively (and X_i is referred to the classification given in [DP95]):

Theorem 0.1. *The smooth codimension two subvarieties of* $\mathbb{P}^n$ *for which the family of their* $(n-1)$*-secant lines is an irreducible first order congruence are*

1. *for* $n = 2$ *a point, and the congruence is a pencil of lines;*
2. *for* $n = 3$ *the twisted cubic, and the congruence is the Veronese surface and has bidegree* $(1, 3)$ *(for more details, see Subsection* 2.2*);*
3. *for* $n = 4$ *we have the following possibilities:*
   - (a) *a (projected) Veronese surface, which is rational, with* $d = 4$, $\pi = 0$*; in this case we have a linear congruence, which has bidegree* $(1, 2)$ *(see Subsection* 2.1*);*
   - (b) *a Bordiga surface, which is rational, with* $d = 6$, $\pi = 3$*; the congruence is smooth and has bidegree* $(1, 8)$ *(see Subsection* 2.2*);*
4. *for* $n = 5$ *we have the following possibilities:*
   - (a) *the Palatini scroll, which is rational with* $d = 7$, $\pi = 4$, $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X) = 1$ *(case* X_6*); in this case we have a linear congruence, which has* 3*-degree* $(1, 3, 2)$ *(see Subsection* 2.1*);*
   - (b) *a non rational scroll,* $\mathbb{P}^1$*-bundle over a minimal* $K3$ *surface of* $\mathbb{P}^8$ *via* $|K + H|$ *(case* X_{11}*);* $d = 9$, $\pi = 8$, $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X) = 2$*; the congruence, has* 3*-degree* $(1, 7, 13)$*;*
   - (c) *a log-general type rational* 3*-fold, linked with a* $(4, 4)$ *complete intersection to a Bordiga* 3*-fold (case* X_{15}*);* $d = 10$, $\pi = 11$, $\chi(\mathcal{O}_S) = 5$, $\chi(\mathcal{O}_X) = 1$*; the congruence is smooth (see Subsection* 2.2*) and has* 3*-degree* $(1, 15, 20)$.

Vice versa the $(n-1)$*-secant lines of any of the above varieties generate a first order congruence.*

We conjecture that the congruences whose pure fundamental locus (see Definition 1.1 below) is a smooth codimension two variety are the families of $(n-1)$-secant lines of the varieties of Theorem 0.1.

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1. Notations, definitions and general results

We will work with schemes and varieties over the complex field $\mathbb{C}$. By *variety* we mean a reduced and irreducible algebraic $\mathbb{C}$-scheme. More information about general results and references about families of lines, focal diagrams and congruences can be found in [De 01] or [De 99]. Besides, we refer to [GH78] for notations about Schubert cycles and to [Ful84] for the definitions and results of intersection theory. Here we recall that a *congruence of lines* of $\mathbb{P}^n$ is a flat family (Λ, B, p) of lines of $\mathbb{P}^n$ obtained by the desingularization of a subvariety B' of dimension $n-1$ of the Grassmannian $\mathbb{G}(1, n)$ of lines of $\mathbb{P}^n$. p is the restriction of the projection $p_1 : B \times \mathbb{P}^n \to B$ to Λ, while we will denote the restriction of $p_2 : B \times \mathbb{P}^n \to \mathbb{P}^n$ by f. $\Lambda_b := p^{-1}(b)$, $(b \in B)$ will be an element of the family and $f(\Lambda_b) =: \Lambda(b)$ is a line of $\mathbb{P}^n$. We can summarise all these notations in the following two diagrams: the first one defines the family

$$\begin{array}{ccccc} \Lambda := \psi^*(\mathcal{H}_{1,n}) & \xrightarrow{\psi^*} & \mathcal{H}_{1,n} & \xrightarrow{p_2} & \mathbb{P}^n \\ p\big\downarrow & & p_1\big\downarrow & & \\ B & \xrightarrow{\psi} & B' \subset \mathbb{G}(1,n), & & \end{array}$$

where $\mathcal{H}_{1,n} \subset \mathbb{G}(1,n) \times \mathbb{P}^n$ is the incidence variety and ψ is the desingularization map, and the second one explains the notation for the elements of the family

$$\begin{array}{ccc} \Lambda_b \subset \Lambda & \xrightarrow{f:=\psi^* p_2} & \mathbb{P}^n \supset \Lambda(b) := f(\Lambda_b) \\ p\big\downarrow & & \\ b \in B. & & \end{array}$$

A point $y \in \mathbb{P}^n$ is called *fundamental* if its fibre has dimension greater than the dimension of the general one. The *fundamental locus* is the set of the fundamental points. The *subscheme of the foci of the first order* $V \subset \Lambda$ is the scheme of ramification points of f. The *locus of the first order foci*, or, simply, the *focal locus*, $\Phi := f(V) \subset \mathbb{P}^n$, is the set of the branch points of f. In this article, as we did in [De 01], we will endow this locus with the scheme structure given by considering it as the scheme-theoretic image of V under f (see, for example, [Har77]).

To a congruence is associated a *sequence of degrees* or $(\nu+1)$-*degree* $(a_0, \ldots, a_\nu)$ if we write

$$[B] = \sum_{i=0}^{\nu} a_i \sigma_{(n-1-i)i}$$

—where we put $\nu := \left[\frac{n-1}{2}\right]$—as a linear combination of Schubert cycles of the Grassmannian; in particular, the *order* a_0 is the number of lines of B passing through a general point of $\mathbb{P}^n$. The fundamental locus is contained in the focal locus and the two loci coincide in the case of a first order congruence, *i.e.* through a focal point there will pass infinitely many lines of the congruence. An important result—independent of order and class—is the following (see also [De 01]):

Proposition 1.1. *On every line $\Lambda(b)$ of the family, the focal locus Φ either coincides with the whole $\Lambda(b)$—in which case $\Lambda(b)$ is called* focal line*—or is a zero dimensional scheme of $\Lambda(b)$ of length at least $n-1$. Moreover, if Λ is a first order congruence, this zero dimensional scheme has length exactly $n-1$.*

Proof. Let $\lambda : \mathcal{T}_{(B\times\mathbb{P}^n/\mathbb{P}^n)|_\Lambda} \to \mathcal{N}_{\Lambda/B\times\mathbb{P}^n}$ be the *global characteristic map* for the family Λ (see [CS89]). From the *focal diagram* (diagram (3) of [CS89]) one gets that the subscheme of the foci of the first order V is the degeneracy locus of λ. If we restrict the map λ to a fibre $\Lambda_b \cong \Lambda(b)$, we obtain the *characteristic map of the family relative to b*:

$$\begin{array}{ccc} \lambda(b) : T_{B,b} \otimes \mathcal{O}_{\Lambda(b)} & \longrightarrow & \mathcal{N}_{\Lambda(b)/\mathbb{P}^n} \\ \cong\big\downarrow & & \cong\big\downarrow \\ \mathcal{O}_{\Lambda(b)}^{n-1} & \longrightarrow & \mathcal{O}_{\Lambda(b)}(1)^{n-1}. \end{array}$$

From the preceding isomorphisms, the map $\lambda(b)$ can be seen as an $(n-1)\times(n-1)$-matrix with linear entries on $\Lambda(b)$; so the focal locus on $\Lambda(b)$ is given by the vanishing of the determinant of this matrix, and our claim follows.

Concerning the first order congruences, we observe that a fundamental point P is a focal point for every line $\Lambda(b)$ which contains it, since the characteristic map relative to it, $\lambda(b)$, drops rank in P. □

From now on we will consider only first order congruences.

A central definition, introduced first in [De 01], is that of *fundamental d-locus*, *i.e.* the subscheme of the fundamental locus of pure dimension d, with $0 \leq d \leq n-2$, which is met by the general line of the congruence. Let us see how these schemes are constructed: the closed set

$$S_d := \{(\Lambda(b), P) \in \Lambda \mid \operatorname{rk}(\mathrm{d}f_{(\Lambda(b),P)}) \leq d\}$$

has a natural subscheme structure, which is defined by a Fitting ideal, *i.e.* the ideal generated by the $(d+1)$-minors of $\mathrm{d}f$ (or, by the Fitting lemma, see [Eis95], by the d-minors of λ), see [Kle77]; in particular, $S_{n-1} = V$. Let us define

$$D_{d+1} := \overline{S_{d+1} \setminus S_d}$$

with the scheme structure induced by S_{d+1}. Finally, we consider the scheme-theoretic image Φ_d of D_{d+1} in $\mathbb{P}^n$ under f. The component of Φ_d of pure dimension d (with the scheme structure induced by Φ_d) which is met by the general line of the congruence is the fundamental d-locus.

These subschemes of the fundamental locus are particularly important, since a first order congruence can be characterised as a component of the set of lines which meet the fundamental d-loci a certain number of times, see the Classification Theorem 3.2 of [De 01].

Definition 1.1. The union of the fundamental d-loci of F is called *pure fundamental locus*, or, in what follows, simply *fundamental locus* and it is denoted by F.

After this, we give the following theorem of [De 00]:

Theorem 1.2. *If Λ is a first order congruence such that the pure fundamental locus F is irreducible and coincides with the fundamental $(n-2)$-locus; then*

$$\frac{n-1}{k} < m < (n-1)^2,$$

where $m := \deg(F)_{\mathrm{red}}$ and k is the geometric multiplicity $(F)_{\mathrm{red}}$ in F.

Idea of the proof. First of all, we have that $n-1 < km$ by degree reasons, since the congruence is given by lines which intersect F in a zero dimensional scheme of length $n-1$.

To prove the other bound, we need of course more work. Let B be our congruence, which has sequence of degrees $(1, a_1, \ldots, a_\nu)$. If Π is a (fixed) general $(n-2)$-plane, we denote by V_Π the scroll given by the lines of the congruence which meet Π. Then by the Schubert calculus one can show that V_Π is a hypersurface of $\mathbb{P}^n$ of degree $1 + a_1$.

Moreover, if ℓ is a line of B not contained in V_Π and P is a point of $V_\Pi \cap \ell$, then P is a focus for B, since at least two lines of the congruence pass through it.

Then, if Π' is another general $(n-2)$-plane of $\mathbb{P}^n$, the complete intersection of the hypersurfaces V_Π and $V_{\Pi'}$ is a (reducible) $(n-2)$-dimensional scheme Γ which contains the focal locus F and the $(n-2)$-dimensional scroll Σ given by the lines of the congruence meeting Π and Π', which has degree $1 + 2a_1 + a_2$ (actually, $1 + a_1$ in the case of $\mathbb{P}^3$; since this case can be treated analogously, we will suppose from now on that $n > 3$).

In fact, if a point P of $V_\Pi \cap V_{\Pi'}$ does not belong to the scroll Σ, it belongs to the fundamental locus. Indeed in this case

$$P \in \ell \cap \ell', \text{ where } \ell \in G_\Pi, \ \ell \in G_{\Pi'}, \text{ and } \ell \neq \ell'$$

—where G_Π and $G_{\Pi'}$ denote the subvarieties of the Grassmannian corresponding to the two scrolls V_Π and $V_{\Pi'}$. Since Λ is a first order congruence and P belongs to two of the lines of Λ, it belongs to infinitely many ones.

Reciprocally, if $P \in F$ is a general focal point, the set of the lines of B through P, χ_P, is a cone of dimension (at least) two, so its intersection with Π and Π' is not empty and therefore $P \in V_\Pi \cap V_{\Pi'}$.

The degree of the scroll follows from the Schubert calculus.

Claim. The following formulae hold:

$$(n-1)h = 1 + a_1, \tag{1}$$

$$(1+a_1)^2 \geq h^2 m + 1 + 2a_1 + a_2, \tag{2}$$

where we denoted by h the algebraic multiplicity of $(F)_{\text{red}}$ on V_Π.

Let us start proving relation (1). If we take a line $\Lambda(b)$ of the congruence not contained in $F \cap V_\Pi$, then, intersecting $\Lambda(b)$ with V_Π, we obtain a zero dimensional scheme of length $1+a_1$, since this is the degree of V_Π. $V_\Pi \cap \Lambda(b)$ contains $(F)_{\text{red}} \cap \Lambda(b)$ with intersection multiplicity h, so the relation (1) is proved.

We recall that the degree of Γ is $(1+a_1)^2$, and it contains F and the scroll Σ. Actually, the fundamental $(n-2)$-locus has geometric multiplicity in Γ equal to h^2. The proof of this fact is the following: the intersection multiplicity $i(((F)_{\text{red}}, V_\Pi \cdot V_{\Pi'}, \mathbb{P}^n)$ of F in $V_\Pi \cdot V_{\Pi'}$ is equal to the geometric multiplicity of $(F)_{\text{red}}$ in Γ, but $i((F)_{\text{red}}, V_\Pi \cdot V_{\Pi'}, \mathbb{P}^n) = h^2$.

Finally, as we seen, the scroll Σ has degree $1 + 2a_1 + a_2$, so we get formula (2).

Now, if we substitute formula (1) in formula (2), we obtain

$$(n-1)^2 h^2 - h^2 m - 1 - 2a_1 - a_2 \geq 0, \tag{3}$$

and since $-1 - 2a_1 - a_2 < 0$, we deduce $m < (n-1)^2$.

□

A fundamental consequence of the preceding theorem is the following:

Theorem 1.3. *If we have a first order congruence of $\mathbb{P}^n$ such that the fundamental locus F satisfies the hypothesis of the preceding theorem, then F cannot be a complete intersection. If moreover F is smooth, then $n \leq 5$.*

Proof. In fact, by the preceding theorem $\deg(F) < (n-1)^2$, therefore if it were a complete intersection, it would be contained in a hypersurface V of degree less than $n-1$, and so every $(n-1)$-secant line of F would be contained in V.

If F is smooth, since by [HS85] we know that Hartshorne's conjecture is true in codimension two up to degree $(n-1)(n+5)$, then $\dim(F) \leq 3$. □

2. General examples of first order congruences

We give now two examples of first order congruences of lines of $\mathbb{P}^n$. Actually, these examples gives us all the congruences of Theorem 0.1 but case 4b.

2.1. Linear sections of $\mathbb{G}(1,n)$.

First of all, we will analyse the congruences which come out from linear sections of the Grassmannian $\mathbb{G}(1,n)$, *i.e.* we will consider the so called, classically, *linear congruences.*

We recall that the Schubert cycle which corresponds to a hyperplane section of (the projective embedding of) the Grassmannian is σ_1, so the following technical lemma gives us the formula for the general intersection of these special Schubert cycles:

Lemma 2.1. *If $\ell \leq n-1$ and we set $k := \left[\frac{\ell}{2}\right]$, the following formula holds:*

$$\sigma_1^\ell = \sum_{i=0}^{k} \left(\binom{\ell-1}{i} - \binom{\ell-1}{i-2} \right) \sigma_{(\ell-i)i} \tag{4}$$

—with the convention that $\binom{\ell}{h} = 0$ if $h < 0$.

Proof. Let us prove the lemma by induction: for $\ell = 1$ it is obvious. Let us suppose it is true for $\ell - 1$; then, by inductive hypothesis

$$\sigma_1^{\ell-1} = \sum_{i=0}^{k'} \left(\binom{\ell-2}{i} - \binom{\ell-2}{i-2} \right) \sigma_{(\ell-1-i)i},$$

where $k' := \left[\frac{\ell-1}{2}\right]$. By Pieri's formula, we have

$$\sigma_{(\ell-1-i)i} \cdot \sigma_1 = \begin{cases} \sigma_{(\ell-i)i} & \text{if } \ell-1-i = i \\ \sigma_{(\ell-i)i} + \sigma_{(\ell-1-i)(i+1)} & \text{otherwise,} \end{cases}$$

i.e. if $\ell - 1 \neq 2k' + 1$ and $i \neq k'$, we obtain

$$\binom{\ell-2}{i} - \binom{\ell-2}{i-2} + \binom{\ell-2}{i-1} - \binom{\ell-2}{i-3} = \binom{\ell-1}{i} - \binom{\ell-1}{i-2}$$

while if $i = k' = \frac{\ell-2}{2}$,

$$\binom{2k'}{k'} - \binom{2k'}{k'-2} = \binom{2k'+1}{k'+1} - \binom{2k'+1}{k'-1},$$

which follows from the formula

$$\binom{\ell-1}{i} - \binom{\ell-1}{i-2} = \binom{\ell}{i} \cdot \frac{\ell-2i+1}{\ell-i+1}. \tag{5}$$

□

Theorem 2.2. *If Λ is a congruence with sequence of degrees $(a_0, \ldots, a_\nu)$ then B, as a subvariety of the Plücker embedding of the Grassmannian, has degree*

$$\deg(B) = \sum_{i=0}^{\nu} a_i \left(\binom{n}{i} \cdot \frac{n-2i+1}{n-i+1} \right).$$

Proof. It is a corollary of formulae (4) and (5). □

Corollary 2.3. *An $(n-1)$-linear section B of the Grassmannian of lines of $\mathbb{P}^n$ generates a first order congruence Λ with sequence of degrees*

$$(\nu+1)\text{-}\deg(\Lambda) = \left(1, \ldots, \left(\binom{n-2}{i} - \binom{n-2}{i-2} \right), \ldots, \left(\binom{n-2}{\nu} - \binom{n-2}{\nu-2} \right) \right);$$

in particular, as a subvariety of the Plücker embedding of the Grassmannian, this is a smooth congruence of degree

$$\deg(B) = \sum_{j=0}^{\nu} \left(\binom{n-2}{j} - \binom{n-2}{j-2} \right)^2.$$

This corollary gives us a first non-trivial example of a first order congruence. Some general results about fundamental varieties of these congruences are given in [BM01]; in particular, it is proven that the focal locus is the degeneracy locus F of a general morphism

$$\phi : \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)} \to \Omega_{\mathbb{P}^n}(2) \tag{6}$$

of (coherent) sheaves on $\mathbb{P}^n$ and that F is smooth if $\dim(F) \leq 3$. An improvement of a result of [BM01] is the following:

Proposition 2.4. *If F is the focal locus of a general linear congruence of $\mathbb{P}^n$, then*

(1) *if n is even, F is a rational variety;*
(2) *if n is odd, F has the structure of a scroll over a Pfaffian hypersurface Z of degree $(n+1)/2$ contained in a $\mathbb{P}^{n-2}$.*

Besides,

$$\deg(F) = \frac{n^2-3n+4}{2}.$$

Proof. If we dualise the Eagon-Northcott complex applied to (6), we have the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(1-n) \to \mathcal{T}_{\mathbb{P}^n}(-2) \xrightarrow{{}^t\phi} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)} \to \omega_F^\circ(2) \to 0,$$

where ω_F° is the dualising sheaf of F. Hunting in the sequence, we get

$$H^0(\mathbb{P}^n, \omega_F^\circ(2)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}),$$

and so the map associated to the complete linear system $|K_F + 2H|$,

$$\varphi_{|K_F+2H|} : F \to \mathbb{P}^{n-2}$$

is well defined. The fibre of $\varphi_{|K_F+2H|}$ is given by the solutions a homogeneous linear system of $n+1$ equations in $n+1$ indeterminates. The matrix A associated to this system is antisymmetric; therefore, if n is even, its determinant is zero and has only one or infinitely many (projective) solutions; so, for dimensional reasons, we get that $\varphi_{|K_F+2H|}$ is birational and F is rational.

If instead n is odd, $\det A$ is, in general, not zero, and so its Pfaffian defines a hypersurface Z of degree $(n+1)/2$ and the fibres of $\varphi_{|K_F+2H|}$ are $\mathbb{P}^1$'s. We observe that the Eagon-Northcott gives in fact a locally free resolution of the ideal sheaf of F:

$$0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(1-n) \xrightarrow{\phi(1-n)} \Omega_{\mathbb{P}^n}(3-n) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_F \to 0.$$

From this we get the Hilbert polynomial of F, and in particular, after some computations, the degree. □

Remark. In low dimension and with a general section, we have that (see [BM01] for details) if $n=3$, F is the union of two skew lines, if $n=4$, F is a smooth projected Veronese surface and if $n=5$, F is a (rational) 3-fold of degree seven, which is a scroll over a cubic surface in $\mathbb{P}^3$. It is also known as Palatini scroll (see [Ott92]).

Clearly, we can say more about the congruence B:

Proposition 2.5. *If B is a linear congruence of $\mathbb{P}^n$, then its ideal sheaf has the following resolution:*

$$(7)\quad \begin{aligned} 0 \to \mathcal{S}ym^n\,\mathcal{Q}(2-n) &\to (\mathcal{S}^*)^{\oplus(n-1)} \otimes \mathcal{S}ym^{n-1}\mathcal{Q}(2-n) \to \\ &\to (\wedge^2((\mathcal{S}^*)^{\oplus(n-1)})) \otimes \mathcal{S}ym^{n-2}\mathcal{Q}(2-n) \to \cdots \\ &\cdots \to \wedge^n((\mathcal{S}(2-n)^*)^{\oplus(n-1)}) \to \mathcal{O}_{\mathbb{G}(1,n)} \to \mathcal{O}_B \to 0, \end{aligned}$$

where $\mathcal{Q}$ and $\mathcal{S}$ are the quotient sheaf and the universal subsheaf of $\mathbb{G}(1,n)$, respectively (hence $\mathrm{rk}(\mathcal{Q}) = n-1$ and $\mathrm{rk}(\mathcal{S}) = 2$).

Proof. For proving this, we see that the map ϕ of (6) gives rise to a map

$$f : \mathcal{Q} \to (\mathcal{S}^*)^{\oplus(n-1)}$$

where f is obtained by considering the dual of ϕ twisted by one and then pulled back to the incidence variety and finally pushed it forward to $\mathbb{G}(1,n)$. Now we apply the Eagon-Northcott complex to $f \in \mathcal{H}om_{\mathcal{O}_{\mathbb{G}(1,n)}}(\mathcal{Q}, (\mathcal{S}^*)^{\oplus(n-1)})$ getting—after tensorizing by $\mathcal{O}_{\mathbb{G}(1,n)}(2-n)$—the resolution above. □

2.2. **Matrices of type $(n-1)\times n$ with linear entries.** Let us consider a general morphism $\phi \in \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}, \mathcal{O}_{\mathbb{P}^n}^{\oplus n)}(1))$, whose minors vanish in the expected codimension two. In this case, $F := V(\phi)$—the degeneracy locus of ϕ—is a locally Cohen-Macaulay subscheme, the Eagon-Northcott complex is exact (see [BE75]) and gives a free resolution of our ideal sheaf:

$$0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n-1)}(-n) \xrightarrow{\phi(-n)} \mathcal{O}_{\mathbb{P}^n}^{\oplus n}(1-n) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_F \to 0. \tag{8}$$

Then—for example—from the Hilbert polynomial we get

$$\deg(F) = \binom{n}{2} \tag{9}$$

$$\pi(F) = 1 + \frac{2n-7}{3}\binom{n}{2} \tag{10}$$

where $\pi(F)$ is the sectional genus of F. It is easy to prove that:

Proposition 2.6. *F is rational, and if $n \leq 5$ it is smooth. Besides, the adjunction map $\varphi_{|K_F+H|}$ exhibits F as the blow-up of $\mathbb{P}^{n-2}$ in a scheme Z of degree $\binom{n+1}{2}$ and sectional genus $\frac{n}{6}(2n-5)(n+1)-1$. In particular, if $n=4$, F is a rational sextic which is the blow-up of the plane in 10 points, i.e. a* Bordiga surface.

Proof. The smoothness is a consequence of Bertini type theorems. For proving that F is rational, we can apply the standard argument used in the proof of Proposition 2.4 to get that the fibre of $\varphi_{|K_F+H|}$ is given by the solutions of a homogeneous linear system of n equations in $n+1$ indeterminates, whose matrix has maximal rank if ϕ is general, and then it has only one (projective) solution; so, $\varphi_{|K_F+H|}$ is birational and F is rational. Besides, $\varphi_{|K_F+H|}$ has (at least) one dimensional fibres on the degeneracy locus Z of the map $\Phi : \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus n} \to \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus(n+1)}(1)$, *i.e.* $\varphi_{|K_F+H|}$ gives F as the blow-up of $\mathbb{P}^{n-2}$ in Z. As usual, Eagon-Northcott gives a free resolution of Z:

$$0 \to \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus n}(-n-1) \to \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus(n+1)}(-n) \to \mathcal{O}_{\mathbb{P}^{n-2}} \to \mathcal{O}_Z \to 0,$$

from which we obtain the degree and the sectional genus of Z. □

With this, we prove that

Theorem 2.7. *The $(n-1)$-secant lines of the variety F defined as above form a first order congruence of lines B of $\mathbb{P}^n$. The congruence B is smooth for general ϕ.*

Proof. For proving this, we see that the map ϕ gives rise, as we did in the proof of Proposition 2.5, to a map

$$\varphi : \mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n} \to (\mathcal{S}^*)^{\oplus(n-1)}.$$

We can now apply the Eagon-Northcott complex to the morphism of coherent sheaves on $\mathbb{G}(1,n)$, $\varphi \in \mathcal{H}om_{\mathcal{O}_{\mathbb{G}(1,n)}}(\mathcal{O}_{\mathbb{G}(1,n)}^{\oplus n}, (\mathcal{S}^*)^{\oplus(n-1)})$ getting, since $\mathrm{rk}(\mathcal{S}) = 2$

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{G}(1,n)}^{\oplus\binom{n}{2}} \to (\mathcal{S}^*)^{\oplus(n-1)\binom{n}{3}} \to \\ \to (\wedge^2((\mathcal{S}^*)^{\oplus(n-1)}))^{\oplus\binom{n}{4}} \to \cdots \\ \cdots \to \wedge^{n-2}((\mathcal{S}^*)^{\oplus(n-1)}) \to \mathcal{O}_{\mathbb{G}(1,n)}(n-1) \to \mathcal{O}_B(n-1) \to 0. \end{aligned} \tag{11}$$

We know that the image of B in the Chow ring of the Grassmannian is (see [GP82]) $c_{n-1}(\operatorname{coker}\varphi)$; then we have to calculate $c_{n-1}(\mathcal{I}_B(n-1))$. We can calculate this Chern class by applying the Giambelli-Thom-Porteous formula: see [FP98]; we recall that this formula gives the class in the Chow ring of a degeneracy locus of a map, as a polynomial in the Chern classes of the two bundles:

$$\begin{aligned}[D_{n-1}(\varphi)] &= \Delta_{n-1}(c)\\ &= c_{n-1}(\mathcal{O}^{\oplus n}_{\mathbb{G}(1,n)} - (\mathcal{S}^*)^{\oplus(n-1)})\\ &= c_{n-1}(\mathcal{S}^{\oplus(n-1)}).\end{aligned}$$

—where $D_{n-1}(\varphi)$ denotes the degeneracy locus of φ and $\Delta_{n-1}(c)$ is a Schur determinant. To calculate this Chern class, fist of all, by the universal exact sequence of the Grassmannian we get

$$c(\mathcal{S}) = c(\mathcal{Q})^{-1}$$

but $c_i(\mathcal{Q}) = \sigma_i$, and then applying the formula which gives the inverse of a Chern class, we obtain (recalling the Giambelli's formula and that $\mathrm{rk}(\mathcal{S}) = 2$) $c(\mathcal{S}) = 1 - \sigma_1 + \sigma_{11}$, and so

$$c(\mathcal{S}^{\oplus(n-1)}) = (1 - \sigma_1 + \sigma_{11})^{n-1}. \tag{12}$$

To see the order of the congruence B is therefore sufficient to find the coefficient of σ_{n-1} in (12); it is not hard to see, by Pieri's formula, that the only way to get σ_{n-1} is from the expansion of σ_1^{n-1}; then, by formula (4) (or by Corollary 2.3) we get that B is a first order congruence.

The fact that the family of the $(n-1)$-secant lines of the variety F is indeed B, can be seen intersecting by a sufficient number of general $h_\lambda := \sum_{i=1}^n \ell_{\lambda,i} A_i$—where $A_1, \dots, A_n$ are the minors of order $(n-1)$ of the matrix $\phi(-n)$—in such a way that for the general $P \in \mathbb{P}^n$ we get a one dimensional scheme, *i.e.* $\lambda = 1, \dots, n-1$. B is smooth as usual by Bertini type theorems. □

3. Two multiple point formulae

We will prove now the 4-tuple point formula for a smooth 3-fold X of $\mathbb{P}^5$ and the formula which gives the number of 4-secant lines to a surface S of $\mathbb{P}^4$ passing through a general point $P \in S$.

In this and in the next section, we will denote by S a general hyperplane section of the 3-fold X, by H and K the hyperplane and the canonical divisors of X. We recall that the basic invariants of X are $d := \deg(X) = H^3$, its sectional genus $\pi = \frac{1}{2}H^2(K + 2H) + 1$, its Euler-Poincaré characteristic $\chi(\mathcal{O}_X)$ and the Euler-Poincaré characteristic of S, $\chi(\mathcal{O}_S)$, thank to the double point formulae for X and S, which can be written as

$$\begin{aligned}K^3 &= -5d^2 + d(2\pi + 25) + 24(\pi - 1) - 36\chi(\mathcal{O}_X) - 24\chi(\mathcal{O}_S),\\ H \cdot K^2 &= \frac{1}{2}d(d+1) - 9(\pi - 1) + 6\chi(\mathcal{O}_X)\end{aligned}$$

(see for example [DP95]).

We start with the 4-tuple point formula. We refer to [Kle82] for the definitions and results used in the proof.

Proposition 3.1. *Let X be a smooth 3-fold of $\mathbb{P}^5$; if $P \in (\mathbb{P}^5 \setminus X)$ is a point through which there is a finite number $q(X)$ of 4-secant lines of X, then the following formula holds:*

$$\begin{aligned}q(X) = \frac{1}{24}d^4 - \frac{1}{4}d^3 + \frac{1}{2}d^2(\frac{11}{12} - \pi) + d(\frac{5}{2}\pi + 2\chi(\mathcal{O}_S) - \frac{9}{4}) +\\ + \frac{1}{2}\pi^2 - \frac{7}{2}\pi + 6\chi(\mathcal{O}_X) - 9\chi(\mathcal{O}_S) + 3.\end{aligned} \tag{13}$$

Proof. Let us consider the projection $\pi_P : X \to \mathbb{P}^4$ from the point P to a hyperplane. π_P is *practically* 4-*generic* (see [Kle82] for the definition): in fact X is smooth, and so π_P is a local complete intersection; besides, if we restrict π_P to the complementary set of the 4-tuple locus—which, by hypothesis, is finite—it becomes a 4-*generic* map (see again [Kle82]). Since the codimension of of π_P is one, we can apply Kleiman's 4-tuple point formula (for example, formula (28) of [Kle82]) even if π_P has $\overline{S_2}$-*singularities*, and from this we obtain formula (13). □

Next, we pass to prove the other formula.

Proposition 3.2. *Let $S \subset \mathbb{P}^4$ be a smooth surface of degree d, sectional genus π, hyperplane and canonical divisors H and K, respectively. Then the number of 4-secant lines h of S passing through a general point $P \in S$ is given by the formula*

$$h = \frac{1}{6}d^3 - \frac{3}{2}d^2 + d(\frac{16}{3} - \pi) + 4\pi + 2\chi(\mathcal{O}_S) - 10. \tag{14}$$

Proof. h is actually equal to the number of triple points of the image of S under the projection from P to a hyperplane. Therefore h can be obtained from the triple point formula for a map f from a smooth surface to $\mathbb{P}^3$: we simply blow-up S in P and then we compose the map which defines the blow-up $g : \mathrm{Bl}_P(S) \to S$ with the projection, *i.e.* $f := \pi_p \circ g$. The triple point formula can be found in [Le 87], and it is, in our situation

$$h = \frac{1}{6}(\tilde{d}(\tilde{d}^2 - 12\tilde{d} + 44) + 4\tilde{K}^2 - 2\tilde{c_2} - 3\tilde{H}\tilde{K}(\tilde{d} - 8)), \tag{15}$$

where $\tilde{H} = g^*H - E$ is the strict transform of H—E is the exceptional divisor of the blow-up—, $\tilde{K} = g^*K + E$ the canonical divisor of $\mathrm{Bl}_P(S)$, $\tilde{d} = \tilde{H}^2$ and $\tilde{c_2}$ its topological Euler-Poincaré characteristic. Clearly, we have that $\tilde{d} = d - 1$, $\tilde{c_2} = c_2 + 1$, where c_2 is the topological Euler-Poincaré characteristic of S, $\tilde{K}^2 = K^2 - 1$ and $\tilde{H}\tilde{K} = 2\pi - d - 1$. Then, if we express the invariants of S in terms of the basic invariants $(d, \pi, \chi(\mathcal{O}_S))$, we get formula (14). □

Remark. We obtained the formulae with the help of S. Katz and A. Strømme's Maple package "Schubert".

4. Congruences of $\mathbb{P}^5$

In this section we study the irreducible first order congruences B which are given by the families of the 4-secant lines of smooth 3-folds $X \subset \mathbb{P}^5$ (with the notations for its invariants given in Section 3), proving Theorem 0.1. We need the following preliminary result:

Lemma 4.1. *The following formula holds:*

$$\begin{aligned} 0 = {} & \frac{1}{8}d^4 - \frac{23}{12}d^3 - d^2(\pi - \frac{83}{8}) - d(\frac{355}{12} - 11\pi - 2\chi(\mathcal{O}_S)) + \\ & + \frac{1}{2}\pi^2 - \frac{57}{2}\pi - 17\chi(\mathcal{O}_S) + 53. \end{aligned} \tag{16}$$

Proof. Formula (16) is formula (1), which is, in our situation,

$$4h = 1 + a_1 \tag{17}$$

(with the notations of the proof of Theorem 1.2). Now, h is the algebraic multiplicity of the fundamental locus on the variety V_Π of the lines of B which meet a general 3-plane Π, *i.e.* if we fix $P \in X$, there are h lines of B though it which meet Π also. The hyperplane $\overline{P\Pi}$ intersects X in a smooth surface S, and the h lines are exactly the ones which are 4-secants to S. So h is given by formula (14).

By definition, a_1 is the number of lines of B contained in a hyperplane H and which meet a line $\ell \subset H$, *i.e.* it is the degree of the hypersurface of the 4-secant lines of S. But this is formula can be easily deduced from [Le 90] and it is

$$(18)\quad a1 = \frac{1}{8}d^4 - \frac{5}{4}d^3 + d^2(\frac{35}{8} - \pi) + d(+7\pi + 2\chi(\mathcal{O}_S) - \frac{33}{4}) + \frac{1}{2}\pi^2 - \frac{25}{2}\pi - 9\chi(\mathcal{O}_S) + 12.$$

Substituting formulae (14) and (18) in (17), we get formula (16). □

Proof of Theorem 0.1. By Theorem 1.3, it is enough to consider the cases of $\mathbb{P}^n$ with $n \leq 5$.

By Theorem 1.2 (and by Castelnuovo's bound) we get that in $\mathbb{P}^3$ the only congruence is the one given by the secant lines of the twisted cubic.

In $\mathbb{P}^4$, again by Theorem 1.2, we obtain that the surfaces X we are looking for have to satisfy $4 \leq \deg(X) \leq 8$, and since the smooth surfaces are classified up to degree ten, see [DP95], if we apply the triple point formula (15) to them, we get only the cases of the theorem.

Passing to the next case, since the smooth 3-folds of $\mathbb{P}^5$ are classified up to degree 12 (see [BSS95]), we can check which of them have an apparent 4-tuple point, and it turns out that are the ones of the list of the theorem. The three congruences are indeed irreducible, since they satisfy formula (16) (and therefore there cannot exist a component of order zero).

Next, from Theorem 1.2 we have to exclude the cases 13, 14 and 15. To do this, we calculate the possible invariants of these 3-folds (see for example [BSS95]) and then we request that they have to satisfy $q(X) = 1$ in formula (13), and equation (16); it turns out that there cannot exist 3-folds with these conditions.

Finally we can calculate the multidegree for the three congruences of $\mathbb{P}^5$ (the cases of $\mathbb{P}^3$ and $\mathbb{P}^4$ can be easily deduced from Section 2; a_1 is from formula (18); a_2 is instead the number of 4-secant lines contained in a 3-dimensional linear space G, *i.e.* the number of 4-secant lines of the smooth curve $G \cap X$, and this formula is in [Le 82]:

$$(19)\qquad a2 = \frac{1}{12}d^4 - d^3 + \frac{53}{12}d^2 - \frac{17}{2}d + 6 - \frac{1}{2}\pi d^2 + \frac{7}{2}d\pi - \frac{13}{2}\pi + \frac{1}{2}\pi^2.$$

□

Remark. We performed the calculations in the last proof with the help of a simple program in Maple.

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