

Explicit Representation of the Minimal Variance Portfolio in Markets driven by Lévy processes

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Abstract

In a market driven by a Lévy martingale, we consider a claim ξ . We study the problem of minimal variance hedging and we give an explicit formula for the minimal variance portfolio in terms of Malliavin derivatives. We discuss two types of stochastic (Malliavin) derivatives for ξ : one based on the chaos expansion in terms of iterated integrals with respect to the power jump processes and one based on the chaos expansion in terms of iterated integrals with respect to the Wiener process and the Poisson random measure components. We study the relation between these two expansions, the corresponding two derivatives and the corresponding versions of the Clark-Haussmann-Ocone theorem.

Key words: incomplete markets, minimal variance hedging, Lévy processes, stochastic derivatives, chaos expansions, Clark-Haussmann-Ocone theorem.

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1. Introduction.

Suppose we have a financial market where the bond and stock prices S_0, S_1, \dots, S_K are modeled as follows:

$$(1.1) \quad \text{Bond price:} \quad S_0(t) = 1, \quad 0 \leq t \leq T,$$

$$(1.2) \quad \text{Stock prices:} \quad S_j(t) = \eta_j(t), \quad j = 1, \dots, K,$$

where η_j , $j = 1, \dots, K$, are *independent Lévy martingales of the second order* on a filtered probability space $(\Omega, \mathfrak{A}, \{\mathfrak{A}_t\}_{0 \leq t \leq T}, P)$ (i.e. the processes η_j are Lévy processes and also martingales with finite second moment). Alternatively, to get a more realistic price model, we could assume that the stock prices were given by

$$(1.2') \quad dS_j(t) = \sigma_j(t, \omega) S_j(t^-) d\eta_j(t), \quad 0 \leq t \leq T, \quad j = 1, \dots, K.$$

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However, since this only would require minor changes in the computations, we prefer to work with the simpler model (1.2) throughout this paper.

Suppose we are only allowed to trade in the first $k + 1$ securities S_0, S_1, \dots, S_k , where $k \leq K$. A random variable $\xi \in L_2(\Omega, \mathfrak{A}_T, P)$ (a *claim*) is called *replicable* (or *hedgeable*) in this market if there exists a (*predictable*) *adapted* process $\varphi(t) = (\varphi_1(t), \dots, \varphi_k(t))$, $0 \leq t \leq T$ (a *portfolio*) such that φ is *admissible*, i.e.

$$(1.3) \quad \sum_{j=1}^k E \left[\int_0^T \varphi_j^2(s) ds \right] < \infty,$$

and such that φ *replicates* (or *hedges*) ξ , in the sense that

$$(1.4) \quad \xi = E\xi + \sum_{j=1}^k \int_0^T \varphi_j(s) d\eta_j(s), \quad \text{P-a.e.}$$

where E denotes the expectation with respect to P . In this case $\varphi(t)$, $0 \leq t \leq T$, is called *hedging portfolio*. The market is called *complete* if every $\xi \in L_2(\Omega, \mathfrak{A}_T, P)$ is replicable (hedgeable). The set of all admissible portfolios is denoted by \mathcal{A} .

It is well known that *a market driven by a Lévy martingale is in general not complete*. In view of this it is natural to ask: given a general Lévy martingale market as above and given $\xi \in L_2(\Omega, \mathfrak{A}_T, P)$, how close can we get to ξ by hedging with an admissible portfolio φ ? If we interpret closeness in terms of variance, the problem is to find $\varphi \in \mathcal{A}$ such that

$$(1.5) \quad E \left[\left(\xi - E\xi - \sum_{j=1}^k \int_0^T \varphi_j(s) d\eta_j(s) \right)^2 \right] = \inf_{\psi \in \mathcal{A}} E \left[\left(\xi - E\xi - \sum_{j=1}^k \int_0^T \psi_j(s) d\eta_j(s) \right)^2 \right].$$

This question is often referred to the problem of *minimal variance hedging* for *incomplete* markets in mathematical finance. Here, we are assuming a market modeled by the above type Lévy process $\eta = (\eta_1, \dots, \eta_K)$ and we assume that we are allowed to trade only in (η_1, \dots, η_k) , $k \leq K$. Thus the market can be incomplete for two reasons: the first is due to the random jumps of prices modeled by the Lévy process and the second is due to the trading constraints, $k \leq K$. A portfolio φ that realizes the minimum is called a *minimal variance portfolio*.

This problem has been studied in the general context of martingales and semimartingales by many authors. See e.g. Chan (1999), Di Nunno (2001), Föllmer and Schweizer (1991), Föllmer and Sonderman (1986), Schweizer (2001), and the references therein.

The main contribution of this paper compared to the above papers is that, by specifying our study to Lévy martingales, we are able to give an explicit formula for the minimal variance portfolio $\varphi = (\varphi_1, \dots, \varphi_k)$ in terms of the stochastic derivatives D of the claim ξ . More precisely, we prove that

$$(1.6) \quad \varphi_j(s) = \frac{1}{\mu_j} \left[\sigma_j E[D_{s,j}\xi | \mathfrak{A}_s] + \int_{\mathbb{R}} x E[D_{s,x,j}\xi | \mathfrak{A}_s] \nu_j(dx) \right], \quad j = 1, \dots, k.$$

See Theorem 4.1. Here $\mu_j = \sigma_j^2 + \int_{\mathbb{R}} x^2 \nu_j(dx)$, ν_j being the Lévy measure of η_j and σ_j the coefficient of the continuous (Wiener process) component W_j of η_j , see Section 2. The

symbols $D_{s,j}$ and $D_{s,x,j}$ denote the stochastic (Malliavin) derivatives in the directions of W_j and \tilde{N}_j , where \tilde{N}_j is the compensated Poisson random measure representing the pure jump component of η_j .

This paper is organized as follows. In Section 2 we recall some basic results about Lévy martingales and we relate our problem (1.5), to the the more general problem and its solution studied in Di Nunno (2001). In Section 3 we give a short review of chaos expansions, stochastic derivatives and Clark-Haussmann-Ocone theorems for Lévy processes, mainly based on Nualart and Schoutens (2000), Léon, Solé, Utzet and Vives (2002) and Løkka (2001). In Section 4 we prove our main results on minimal variance hedging and hedgeable claims. Finally, in Section 5, we illustrate our results by giving some explicit examples.

2. Minimal variance portfolio and the general stochastic derivative \mathfrak{D} .

For the standard L_2 -space $H = L_2(\Omega) = L_2(\Omega, \mathfrak{A}, P)$ of real random variables ξ :

$$\|\xi\| = \left(\int_{\Omega} |\xi(\omega)|^2 P(d\omega) \right)^{1/2} = \left(E |\xi|^2 \right)^{1/2},$$

on the *filtered* probability space $(\Omega, \mathfrak{A}, \{\mathfrak{A}_t\}_{0 \leq t \leq T}, P)$, we consider the Itô type non-anticipating integration scheme with integrator as a K -dimensional Lévy process $\eta(t) = (\eta_1(t), \dots, \eta_K(t))$, $0 \leq t \leq T$, with *independent* H -valued components. Recall that a *Lévy process* is a stochastically continuous process with independent, stationary increments and $\eta(0) = 0$ - cf. Protter (1990). Here we assume we are dealing with the existing càdlàg modification of the process. For more information about Lévy processes we refer to Bertoin (1996), Protter (1990) and Sato (1999). In this paper we confine ourselves to the case of Lévy processes $\eta(t)$, $0 \leq t \leq T$, as \mathfrak{A}_t -martingales in H , i.e. $E\eta_j(t) = 0$ and $E\eta_j^2(t) = \mu_j t$, $0 \leq t \leq T$, with stationary independent increments

$$\Delta\eta_j := \eta_j(t + \Delta t) - \eta_j(t)$$

for the intervals $\Delta = (t, t + \Delta t] \subseteq (0, T]$, for all $j = 1, \dots, K$. In short we will name such processes *Lévy martingales of the second order*. Note that $\eta_j(t)$, $0 \leq t \leq T$, is H -*continuous*, i.e. $\|\eta_j(t) - \eta_j(s)\| \rightarrow 0$, $s \rightarrow t$.

From now on we assume the *completed* filtration \mathfrak{A}_t , $0 \leq t \leq T$, to be generated by the integrator itself $\mathfrak{A}_t = \sigma\{\eta_j(s), 0 \leq s \leq t, j = 1, \dots, K\}$ and $\mathfrak{A} = \mathfrak{A}_T$. Note that this filtration is *continuous*.

For the considered type of Lévy processes η_t , $0 \leq t \leq T$, we recall the following representation formula

$$(2.1) \quad \eta_j(t) = \sigma_j W_j(t) + \int_0^t \int_{\mathbb{R}} x \tilde{N}_j(ds, dx), \quad 0 \leq t \leq T \quad (j = 1, \dots, K),$$

where $W_j(t)$ is the standard Wiener process (with diffusion coefficient 1), $\sigma_j > 0$ is a constant and $\tilde{N}_j(dt, dx)$ is the compensated Poisson random measure, i.e. $\tilde{N}_j(dt, dx) = N_j(dt, dx) - \nu_j(dx)dt$ with $\nu_j(dx)$ as the jump measure (with finite second moment) of the Lévy process η_j itself and $N_j(dt, dx)$ as a Poisson random measure such that $EN_j(dt, dx) = \nu_j(dx)dt$.

For a given ξ in H we recall the following representation

$$(2.2) \quad \xi = E\xi + \sum_{j=1}^K \int_0^T \theta_j(s) \sigma_j dW_j(s) + \sum_{j=1}^K \int_0^T \int_{\mathbb{R}} \Psi_j(s, x) \tilde{N}_j(ds, dx)$$

with θ_j and Ψ_j as well defined integrands for the corresponding integrals - cf. Ito (1956), for example. For the sake of completeness we remind the reader that $\theta_j(s)$, $0 \leq s \leq T$, and $\Psi_j(s, x)$, $0 \leq s \leq T$, $x \in \mathbb{R}$, are (predictable) adapted processes with $E[\int_0^T \int_{\mathbb{R}} |\theta_j(s)|^2 ds] < \infty$ and $E[\int_0^T \int_{\mathbb{R}} |\Psi_j(s, x)|^2 \nu_j(dx) ds] < \infty$. We also recall the following Itô isometry:

$$(2.3) \quad E\xi^2 = (E\xi)^2 + \sum_{j=1}^K \int_0^T E[\theta_j^2(s)] \sigma_j^2 ds + \sum_{j=1}^K \int_0^T \int_{\mathbb{R}} E[\Psi_j^2(s, x)] \nu_j(dx) ds.$$

Which claims $\xi \in H$ are the replicable ones? Using (2.2) we can give the following answer.

Theorem 2.1. *The claim ξ in H such that*

$$(2.4) \quad \xi = E\xi + \sum_{j=1}^K \int_0^T \Phi_j(s) dW_j(s) + \sum_{j=1}^K \int_0^T \int_{\mathbb{R}} \Psi_j(s, x) \tilde{N}_j(ds, dx)$$

is hedgeable if and only if the integrands have the following structure

$$(2.5) \quad \Phi_j(s) = \sigma_j \varphi_j(s) \quad \text{and} \quad \Psi_j(s, x) = \varphi_j(s) x, \quad \text{for } 1 \leq j \leq k,$$

and $\Phi_j = \Psi_j = 0$, *for* $k < j \leq K$,

with $0 \leq s \leq T$ *and* $x \in \mathbb{R}$, *for some process* $\varphi = (\varphi_1, \dots, \varphi_k) \in \mathcal{A}$. *If this is the case, then* $\varphi = (\varphi_1, \dots, \varphi_k)$ *is the hedging strategy for* ξ , *i.e.*

$$(2.6) \quad \xi = E\xi + \sum_{j=1}^k \int_0^T \varphi_j(s) d\eta_j(s).$$

Proof. For a hedgeable ξ in H , from (2.4) and (2.6) we obtain

$$\sum_{j=1}^k \int_0^T \varphi_j(s) d\eta_j(s) = \sum_{j=1}^K \left(\int_0^T \Phi_j(s) dW_j(s) + \int_0^T \int_{\mathbb{R}} \Psi_j(s, x) \tilde{N}_j(ds, dx) \right)$$

and substituting the stochastic differential of η_t - cf. (2.1), we have

$$\begin{aligned} & \sum_{j=1}^k \int_0^T \left(\frac{1}{\sigma_j} \Phi_j(s) - \varphi_j(s) \right) \sigma_j dW_j(s) + \sum_{j=1}^k \int_0^T \int_{\mathbb{R}} \left(\Psi_j(s, x) - x \varphi_j(s) \right) \tilde{N}_j(ds, dx) \\ & + \sum_{j=k+1}^K \left(\int_0^T \Phi_j(s) dW_j(s) + \int_0^T \int_{\mathbb{R}} \Psi_j(s, x) \tilde{N}_j(ds, dx) \right) = 0. \end{aligned}$$

Correspondingly we have

$$\begin{aligned} & \sum_{j=1}^k \left(\int_0^T E \left[\left(\frac{1}{\sigma_j} \Phi_j(s) - \varphi_j(s) \right)^2 \right] \sigma_j^2 ds + \int_0^T \int_{\mathbb{R}} E \left[(\Psi_j(s, x) - x\varphi_j(s))^2 \right] \nu_j(dx) ds \right) \\ & + \sum_{j=k+1}^K \left(\int_0^T E [\Phi_j^2(s)] ds + \int_0^T \int_{\mathbb{R}} E [\Psi_j^2(s, x)] \nu_j(dx) ds \right) = 0. \end{aligned}$$

Hence, (2.5) holds true. Conversely, suppose (2.5) holds. Then representation (2.4) becomes as follows

$$\begin{aligned} \xi &= E\xi + \sum_{j=1}^k \left(\int_0^T \varphi_j(s) \sigma_j dW_j(s) + \int_0^T \int_{\mathbb{R}} x \varphi_j(s) \tilde{N}_j(ds, dx) \right) \\ &= E\xi + \sum_{j=1}^k \int_0^T \varphi_j(s) \left(\sigma_j dW_j(s) + \int_{\mathbb{R}} x \tilde{N}_j(ds, dx) \right) = E\xi + \sum_{j=1}^k \int_0^T \varphi_j(s) d\eta_j(s). \end{aligned}$$

Therefore ξ is hedgeable. This ends the proof. \square

Let $L_2(\Omega \times [0, T]) = L_2(\Omega \times [0, T], \mathfrak{A} \times \mathfrak{B}, P(d\omega) \times \lambda(dt))$, with \mathfrak{B} the Borel σ -algebra of $[0, T]$ and $\lambda(dt) = dt$ the Lebesgue measure on $[0, T]$, be the standard functional L_2 -space of random processes ψ with the norm given by

$$\|\psi\|_{L_2} = \left(\iint_{\Omega \times [0, T]} |\psi(\omega, t)|^2 P(d\omega) \times dt \right)^{1/2} = \left(E \int_0^T |\psi(t)|^2 dt \right)^{1/2} = \left(\int_0^T E |\psi(t)|^2 dt \right)^{1/2}.$$

The integrands $\varphi = (\varphi_1, \dots, \varphi_K)$ are considered as elements of $(L_2(\Omega \times [0, T]))^K$. Here the integrands can be characterized as *adapted* processes, i.e. $\varphi(t)$ is \mathfrak{A}_t -measurable for all t , or equivalently, thanks to the direct product measure $P(d\omega) \times dt$ applied, as *predictable* processes, i.e. processes that are measurable with respect to the so-called *predictable σ -algebra*. Actually, integrands can be *identified* as limits

$$(2.7) \quad \varphi = \lim_{h \rightarrow 0} \varphi^h, \quad \text{i.e.} \quad \|\varphi_j - \varphi_j^h\|_{L_2} \longrightarrow 0, \quad h \rightarrow 0, \quad j = 1, \dots, K,$$

of *simple functions* φ_j^h of the following form

$$(2.8) \quad \varphi_j^h(s) = \sum_{\Delta} \varphi_{j,\Delta}^h 1_{\Delta}(s), \quad 0 \leq s \leq T,$$

with $\varphi_{j,\Delta}^h$ as \mathfrak{A}_t -measurable elements in H to the corresponding intervals $\Delta = (t, t + \Delta t]$ of the *finite h -partition* of $(0, T]$: $\sum \Delta = (0, T]$, $\Delta t \leq h$.

In this framework, we are considering the problem of minimal variance hedging - cf. (1.5). In Di Nunno (2001) for the more general setup of *H -continuous martingales with respect to a general filtration* as integrators, an answer to the above question is suggested in terms of a certain general stochastic derivative. See also Di Nunno and Rozanov (1999). Here we recall its definition specifying it to the case of the Lévy martingales we are dealing with.

Definition 2.2. For the random variable ξ in H , the (*non-anticipating*) *stochastic derivative*

$$\mathfrak{D}_{\eta_j} \xi := \varphi_j = \varphi_j(s), \quad 0 \leq s \leq T,$$

with respect to $\eta_j(t)$, $0 \leq t \leq T$, is an element of the functional space $L_2(\Omega \times [0, T])$ defined as the limit in (2.7) where the (2.8)-type simple functions are of the form

$$\varphi_j^h(s) = \sum_{\Delta} E \left[\xi \cdot \frac{\Delta \eta_j}{\mu_j \Delta t} \middle| \mathfrak{A}_t \right] 1_{\Delta}(s), \quad 0 \leq s \leq T.$$

The stochastic function $\mathfrak{D}_{\eta} \xi = (\mathfrak{D}_{\eta_1} \xi, \dots, \mathfrak{D}_{\eta_K} \xi)$ is referred to as the *stochastic derivative with respect to the integrator* $\eta = (\eta_1, \dots, \eta_K)$. The stochastic derivative in Definition 2.2 is proved to be well defined for all $\xi \in H$. See Di Nunno (2001).

Let $H^0 \subseteq H$ be the subspace of all variables ξ characterized by $\mathfrak{D}_{\eta_j} \xi = 0$, $j = 1, \dots, k$, and $H(\eta) \subseteq H$ the subspace of all stochastic integrals with integrator η . We can now formulate a theorem by Di Nunno (2001) on integral representation for random variables in H .

Theorem 2.3. Any ξ in H admits a unique integral representation in terms of its derivative $\mathfrak{D}_{\eta} \xi$:

$$(2.9) \quad \xi = \xi^0 + \sum_{j=1}^k \int_0^T \mathfrak{D}_{\eta_j} \xi(s) d\eta_j(s)$$

with $\xi^0 \in H^0$. Moreover, we have the following orthogonal decomposition:

$$H = H^0 \oplus H(\eta).$$

Note that the non-anticipating stochastic derivative $\mathfrak{D}_{\eta} \xi$ represents the minimal variance portfolio and the simple functions in Definition 2.2 give a method for the approximation.

The main purpose of this paper is to provide a more explicit formula for the integrand $\varphi = \mathfrak{D}_{\eta} \xi$ when η_j , $j = 1, \dots, K$, are independent Lévy martingales. For example, if $\eta = (\eta_1, \dots, \eta_K) = (W_1, \dots, W_K)$ is the Wiener process with independent components, the representation (2.9), for an appropriate ξ , reduces to

$$(2.10) \quad \xi = \xi^0 + \sum_{j=1}^k \int_0^T E [D_{s,j} \xi | \mathfrak{A}_s] dW_j(s).$$

In other words in this case we have $\mathfrak{D}_{W_j} \xi(s) = E [D_{s,j} \xi | \mathfrak{A}_s]$, $0 \leq s \leq T$, where $D_{s,j} \xi$, $0 \leq s \leq T$, is the *stochastic Malliavin derivative* in the direction of the Wiener process W_j , cf. Malliavin (1997), for example. In particular, if $k = K$ this gives the well-known *Clark-Haussmann-Ocone formula* - cf. Nualart (1995), Øksendal (1996), Aase, Øksendal, Privault and Ubøe (2000), Karatzas, Ocone and Li (1991), for example.

3. Chaos expansions and the stochastic derivatives $\mathcal{D}_t^{(m)}$ and $D_{t,x}$.

In this section we review basic terminology and results about chaos expansions with respect to our Lévy martingales η_j , $j = 1, \dots, K$. For simplicity we assume that $K = 1$ in this section and we write

$$(3.1) \quad \eta_1(t) = \eta(t) = \sigma W(t) + \int_0^t \int_{\mathbb{R}} x \tilde{N}(ds, dx).$$

3.A. Chaos expansions in terms of iterated integrals with respect to power jump processes.

We first recall the chaos expansion of Nualart and Schoutens (2000), see also Léon, Solé, Utzet and Vives (2002) and Nualart and Schoutens (2001). Here we assume that for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(3.2) \quad \int_{\mathbb{R} - (-\varepsilon, \varepsilon)} \exp(\delta|x|) \nu(dx) < \infty.$$

Let $\Delta\eta(t) = \eta(t) - \eta(t^-)$ be the jump of η at time t and let

$$(3.3) \quad \eta^{(i)}(t) = \sum_{0 < s \leq t} (\Delta\eta(s))^i, \quad 0 \leq t \leq T, \quad i = 2, 3, \dots$$

be the i -th power jump process, which is another Lévy process. Then

$$E[\eta^{(i)}(t)] = t \int_{\mathbb{R}} x^i \nu(dx) =: m_i t, \quad i = 2, 3, \dots$$

We put $\eta^{(1)}(t) := \eta(t)$, $0 \leq t \leq T$, and $m_1 := 0$. Hence we can define the *compensated power jump process* $\zeta^{(i)}(t)$, $0 \leq t \leq T$, by

$$(3.4) \quad \zeta^{(i)}(t) := \eta^{(i)}(t) - m_i t, \quad i = 1, 2, \dots$$

Finally, let $a_{i,j}$ be real numbers such that the processes

$$(3.5) \quad \gamma^{(i)}(t) := \zeta^{(i)}(t) + a_{i,i-1} \zeta^{(i-1)}(t) + \dots + a_{i,1} \zeta^{(1)}(t)$$

are *strongly orthogonal martingales*, in the sense that the quadratic covariation $[\gamma^{(i)}, \gamma^{(j)}]_t$, $0 \leq t \leq T$, is a uniformly integrable martingale for all $i \neq j$, $i, j = 1, 2, \dots$. Then there are constants $M(i) > 0$ such that $[\gamma^{(i)}, \gamma^{(i)}]_t = M(i)t$ - cf. Nualart and Schoutens (2000), Section 2. For all $n = 1, 2, \dots$, let \mathcal{I}_n denote the set of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N}$ for $i = 1, \dots, n$ and put $\mathcal{I} = \cup_{n=1}^{\infty} \mathcal{I}_n$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}_n$, we write $l(\alpha) = n$ (*length of α*) and we put

$$S_{l(\alpha)} = S_n = \{(t_1, \dots, t_n) \in [0, T]^n : 0 \leq t_1 \leq \dots \leq t_n \leq T\}.$$

We also use the notation $d\gamma^{\otimes \alpha}(t) = d\gamma^{(\alpha_1)}(t_1) \dots d\gamma^{(\alpha_n)}(t_n)$.

We can now formulate the theorem by Nualart and Schoutens (2000) on the chaos expansion of a random variable in terms of iterated integrals with respect to $\gamma^{(i)}$.

Theorem 3.1. (a) *Every $\xi \in H$ can be written as follows*

$$(3.6) \quad \xi = E\xi + \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{I}_n} \int_0^T \dots \int_0^{t_2} f_{\alpha}(t_1, \dots, t_n) d\gamma^{(\alpha_1)}(t_1) \dots d\gamma^{(\alpha_n)}(t_n)$$

$$= E\xi + \sum_{\alpha \in \mathcal{I}} \int_{S_{l(\alpha)}} f_{\alpha}(\mathbf{t}) d\gamma^{\otimes \alpha}(\mathbf{t}),$$

with $f_{\alpha} \in L_2(\mathbb{R}_+^n, \mathcal{B}(\mathbb{R}_+^n), \lambda^n)$.

(b) Moreover, we have the isometry

$$(3.7) \quad \|\xi\|^2 = E\xi^2 = (E\xi)^2 + \sum_{\alpha \in \mathcal{I}} M(\alpha_1) \dots M(\alpha_n) \|f_{\alpha}\|_{S_{l(\alpha)}}^2,$$

where

$$\|f_{\alpha}\|_{S_{l(\alpha)}}^2 = \int_0^T \dots \int_0^{t_2} f_{\alpha}^2(t_1, \dots, t_n) dt_1 \dots dt_n =: \int_{S_{l(\alpha)}} f_{\alpha}^2(\mathbf{t}) dt.$$

Corollary 3.2. Every $\xi \in H$ can be written as

$$(3.8) \quad \xi = E\xi + \sum_{m=1}^{\infty} \int_0^T \varphi^{(m)}(s) d\gamma^{(m)}(s)$$

where $\varphi^{(m)}$, $m = 1, 2, \dots$, are predictable processes.

We now introduce a stochastic Sobolev space $\mathbf{D}_{1,2}$ and the stochastic derivative $\mathcal{D}_t^{(m)}$.

Definition 3.3. Let $\mathbf{D}_{1,2}$ denote the set of $\xi \in H$ the expansion (3.6) of which, satisfies the condition

$$(3.9) \quad \|\xi\|_{\mathbf{D}_{1,2}}^2 := E\xi^2 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathcal{I}_n \\ \alpha_n = m}} M(\alpha_1) \dots M(\alpha_{n-1}) \int_0^T \|f_{\alpha}(\cdot, t)\|_{S_{n-1}}^2 dt < \infty.$$

For $\xi \in \mathbf{D}_{1,2}$ we define its *stochastic derivatives* $\mathcal{D}_t^{(m)}\xi$, $0 \leq t \leq T$, in $L_2(\Omega \times [0, T])$ for $m = 1, 2, \dots$ by

$$(3.10) \quad \mathcal{D}_t^{(m)}\xi = \sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathcal{I}_n \\ \alpha_n = m}} \int_{S_{n-1}} f_{\alpha}(t_1, \dots, t_{n-1}, t) d\gamma^{(\alpha_1)}(t_1) \dots d\gamma^{(\alpha_{n-1})}(t_{n-1})$$

where the summation is taken over all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{I}_n$ with the last non-zero element $\alpha_n = \alpha_{l(\alpha)} = m$.

Remark. This definition differs from the definition in Léon, Solé, Utzet and Vives (2002).

Note that if $\xi \in \mathbf{D}_{1,2}$ then $\mathcal{D}_t^{(m)}\xi \in L_2(\Omega \times [0, T])$ and

$$\|\mathcal{D}_t^{(m)}\xi\|_{L_2(\Omega \times [0, T])} = \sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathcal{I}_n \\ \alpha_n = m}} M(\alpha_1) \dots M(\alpha_{n-1}) \int_0^T \|f_{\alpha}(\cdot, t)\|_{S_{n-1}}^2 dt < \infty.$$

We can now give an explicit description of the integrands $\varphi^{(m)}$ in Corollary 3.2 in terms of the stochastic derivatives. This result corresponds to Theorem 1.6 in Léon, Solé, Utzet and Vives (2002).

Theorem 3.4. (Clark-Haussmann-Ocone theorem I) Let $\xi \in \mathbf{D}_{1,2}$. Then

$$(3.11) \quad \xi = E\xi + \sum_{m=1}^{\infty} \int_0^T E[\mathcal{D}_t^{(m)}\xi | \mathfrak{A}_t] d\gamma^{(m)}(t).$$

Proof. Consider

$$\begin{aligned}
& \sum_{m=1}^{\infty} \int_0^T E[\mathcal{D}_t^{(m)} \xi | \mathfrak{A}_t] d\gamma^{(m)}(t) \\
&= \sum_{m=1}^{\infty} \int_0^T E \left[\sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathcal{I}_n \\ \alpha_n = m}} \int_{S_{n-1}} f_{\alpha}(t_1, \dots, t_{n-1}, t) d\gamma^{(\alpha_1)}(t_1) \dots d\gamma^{(\alpha_{n-1})}(t_{n-1}) | \mathfrak{A}_t \right] d\gamma^{(m)}(t) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathcal{I}_n \\ \alpha_n = m}} \int_0^T \left(\int_{0 \leq t_1 \leq \dots \leq t_{n-1} \leq t} f_{\alpha}(t_1, \dots, t_{n-1}, t) d\gamma^{(\alpha_1)}(t_1) \dots d\gamma^{(\alpha_{n-1})}(t_{n-1}) \right) d\gamma^{(m)}(t) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\substack{\alpha \in \mathcal{I}_n \\ \alpha_n = m}} \int_{S_n} f_{\alpha}(\mathbf{t}) d\gamma^{\otimes \alpha}(\mathbf{t}) = \xi - E\xi
\end{aligned}$$

by Theorem 3.1. □

3.B. Chaos expansions in terms of iterated integrals with respect to Poisson random measures.

Next we recall the chaos expansion presented in Løkka (2001). For simplicity in notation we assume, as in Løkka (2001), that $\sigma = 0$, so that representation (2.1) reduces to

$$\eta(t) = \int_0^t \int_{\mathbb{R}} x \tilde{N}(ds, dx), \quad 0 \leq t \leq T.$$

The chaos expansion is in terms of iterated integrals with respect to the compensated Poisson measure $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$. In the following we let $L_2((\lambda \times \nu)^n)$ be the space of *deterministic* functions $f : ([0, T] \times \mathbb{R})^n \rightarrow \mathbb{R}$ such that

$$\|f\|_{L_2((\lambda \times \nu)^n)}^2 := \int_{([0, T] \times \mathbb{R})^n} f^2(t_1, x_1, \dots, t_n, x_n) dt_1 \nu(dx_1) \dots dt_n \nu(dx_n) < \infty.$$

Recall that $d\lambda(t) = dt$ denotes the Lebesgue measure on $[0, T]$. If f is a real function on $([0, T] \times \mathbb{R})^n$ we define its *symmetrization* \tilde{f} with respect to the variables $(t_1, x_1), \dots, (t_n, x_n)$ by

$$(3.12) \quad \tilde{f}(t_1, x_1, \dots, t_n, x_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, x_{\sigma_1}, \dots, t_{\sigma_n}, x_{\sigma_n})$$

where the sum is taken over all permutations σ of $\{1, \dots, n\}$. We say that f is *symmetric* if $f = \tilde{f}$ and we let $\tilde{L}_2((\lambda \times \nu)^n)$ denote the set of symmetric deterministic functions in $L_2((\lambda \times \nu)^n)$. Moreover, we put

$$G_n = \{(t_1, x_1, \dots, t_n, x_n) : 0 \leq t_1 \leq \dots \leq t_n \leq T; x_i \in \mathbb{R}, i = 1, \dots, n\},$$

and we denote by $L_2(G_n)$ the space of functions g such that

$$\|g\|_{L_2(G_n)}^2 = \int_{G_n} g^2(t_1, x_1, \dots, t_n, x_n) dt_1 \nu(dx_1) \dots dt_n \nu(dx_n) < \infty.$$

Naturally, for $f \in \tilde{L}_2((\lambda \times \nu)^n)$ we have $\|f\|_{L_2((\lambda \times \nu)^n)}^2 = n! \|f\|_{L_2(G_n)}^2$.

If $f \in L_2(G_n)$ we define its n -fold iterated integral by

$$(3.13) \quad J_n(f) := \int_0^T \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} f(t_1, x_1, \dots, t_n, x_n) \tilde{N}(dt_1, dx_1) \dots \tilde{N}(dt_n, dx_n)$$

If $f \in \tilde{L}_2((\lambda \times \nu)^n)$ we define

$$(3.14) \quad I_n(f) = \int_{([0, T] \times \mathbb{R})^n} f(t_1, x_1, \dots, t_n, x_n) \tilde{N}^{\otimes n}(d\mathbf{t}, d\mathbf{x}) = n! J_n(f).$$

Then if $f \in \tilde{L}_2((\lambda \times \nu)^n)$ and $g \in \tilde{L}_2((\lambda \times \nu)^m)$ we have

$$E[I_n(f)I_m(g)] = \begin{cases} 0, & n \neq m \\ n!(f, g)_{L_2((\lambda \times \nu)^n)}, & n = m, \end{cases}$$

where

$$(f, g)_{L_2((\lambda \times \nu)^n)} = \int_{([0, T] \times \mathbb{R})^n} f(t_1, x_1, \dots, t_n, x_n) g(t_1, x_1, \dots, t_n, x_n) (\lambda \times \nu)^{\otimes n}(d\mathbf{t}, d\mathbf{x}).$$

We can now formulate the theorem by Løkka (2001) on chaos expansion in terms of iterated integrals with respect to $\tilde{N}(dt, dx)$.

Theorem 3.5. (a) *Every $\xi \in H$ can be written as*

$$(3.15) \quad \xi = E\xi + \sum_{n=1}^{\infty} I_n(f_n)$$

for a unique sequence $\{f_n\}_{n=1}^{\infty}$ with $f_n \in \tilde{L}_2((\lambda \times \nu)^n)$.

(b) *Moreover, we have the following isometry*

$$(3.16) \quad E\xi^2 = (E\xi)^2 + \sum_{n=1}^{\infty} n! \|f_n\|_{L_2((\lambda \times \nu)^n)}^2.$$

Based on this expansion we define another stochastic Sobolev space $\mathbb{D}_{1,2}$ and the stochastic derivative $D_{t,x}$ as follows.

Definition 3.6. (a) We define $\mathbb{D}_{1,2}$ to be the set of $\xi \in H$ the expansion (3.15) of which satisfies the condition

$$(3.17) \quad \|\xi\|_{\mathbb{D}_{1,2}}^2 := E\xi^2 + \sum_{n=1}^{\infty} nn! \|f_n\|_{L_2((\lambda \times \nu)^n)}^2 < \infty.$$

(b) If $\xi \in \mathbb{D}_{1,2}$ we define its *stochastic derivative* $D_{t,x}\xi \in L_2(\Omega \times [0, T] \times \mathbb{R})$ by

$$(3.18) \quad D_{t,x}\xi = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, x))$$

where $f_n(\cdot, t, x) = f_n(t_1, x_1, \dots, t_{n-1}, x_{n-1}, t, x)$.

The following isometry is obtained in Løkka (2001):

$$(3.19) \quad \|D_{t,x}\xi\|_{L_2(\Omega \times [0,T] \times \mathbb{R})}^2 = \sum_{n=1}^{\infty} nn! \|f_n\|_{L_2((\lambda \times \nu)^n)}^2.$$

Corresponding to Theorem 3.4 we now have

Theorem 3.7. (Clark-Haussmann-Ocone theorem II). *Let $\xi \in \mathbb{D}_{1,2}$. Then*

$$(3.20) \quad \xi = E\xi + \int_0^T \int_{\mathbb{R}} E[D_{t,x}\xi | \mathfrak{A}_t] \tilde{N}(dt, dx).$$

3.C. Relations between the two chaos expansions and the two stochastic derivatives.

For the sake of completeness we point out the links between the chaos expansions and the stochastic derivatives presented so far in this section. Recall that here we are dealing with a pure jump Lévy martingale satisfying condition (3.2).

First of all we see that the compensated power jump process in (3.4) satisfies the equation

$$\zeta^{(i)}(t) = \int_0^t \int_{\mathbb{R}} x^i \tilde{N}(ds, dx), \quad 0 \leq t \leq T, \quad i = 1, 2, \dots,$$

and therefore from (3.5) we get

$$\gamma^{(i)}(t) = \sum_{j=1}^i a_{i,j} \zeta^{(j)}(t) = \sum_{j=1}^i a_{i,j} \int_0^t \int_{\mathbb{R}} x^j \tilde{N}(ds, dx), \quad 0 \leq t \leq T,$$

with $a_{i,i} = 1$, for all $i = 1, 2, \dots$.

By the following equalities we show how the chaos expansions are related. From Theorem 3.1, for any $\xi \in H$, we have

$$\begin{aligned} \xi - E\xi &= \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{I}_n} \int_0^T \cdots \int_0^{t_2} f_{\alpha}(t_1, \dots, t_n) d\gamma^{(\alpha_1)}(t_1) \dots d\gamma^{(\alpha_n)}(t_n) \\ &= \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{I}_n} \int_0^T \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} \sum_{j_1=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} a_{\alpha_1, j_1} \cdots a_{\alpha_n, j_n} x_1^{j_1} \cdots x_n^{j_n} f_{\alpha}(t_1, \dots, t_n) \tilde{N}(dt_1, dx_1) \dots \tilde{N}(dt_n, dx_n) \\ &= \sum_{n=1}^{\infty} \int_0^T \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} g_n(t_1, x_1, \dots, t_n, x_n) \tilde{N}(dt_1, dx_1) \dots \tilde{N}(dt_n, dx_n) \\ &= \sum_{n=1}^{\infty} J_n(g_n) = \sum_{n=1}^{\infty} n! J_n(\tilde{g}_n) = \sum_{n=1}^{\infty} I_n(\tilde{g}_n), \end{aligned}$$

where \tilde{g}_n is the symmetrization (in the sense of (3.12)) of the function g_n defined by

$$(3.21) \quad g_n(t_1, x_1, \dots, t_n, x_n) =$$

$$\begin{cases} \sum_{\alpha \in \mathcal{I}_n} \sum_{j_1=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} a_{\alpha_1, j_1} \cdots a_{\alpha_n, j_n} x_1^{j_1} \cdots x_n^{j_n} f_\alpha(t_1, \dots, t_n), & \text{on } G_n \\ 0, & \text{on } ([0, T] \times \mathbb{R})^n - G_n. \end{cases}$$

Therefore, by uniqueness, $\{f_n\}_{n=1}^\infty$ in (3.15) is given by

$$(3.22) \quad f_n = \tilde{g}_n, \quad n = 1, 2, \dots$$

This gives the relation between the two expansions.

The relationship between the stochastic derivatives now comes immediately. In fact, from Definition 3.6 we have

$$\begin{aligned} E[D_{t,x}\xi|\mathfrak{A}_t] &= \sum_{n=1}^{\infty} n E\left[I_{n-1}(\tilde{g}_n(\cdot, t, x)) | \mathfrak{A}_t \right] = \sum_{n=1}^{\infty} n(n-1)! E\left[J_{n-1}(\tilde{g}_n(\cdot, t, x)) | \mathfrak{A}_t \right] \\ &= \sum_{n=1}^{\infty} \int_0^t \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} g_n(t_1, x_1, \dots, t_{n-1}, x_{n-1}, t, x) \tilde{N}(dt_1, dx_1) \cdots \tilde{N}(dt_{n-1}, dx_{n-1}) \\ &= \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{I}_n} \int_0^t \int_{\mathbb{R}} \cdots \int_0^{t_2} \int_{\mathbb{R}} \sum_{j_1=1}^{\alpha_1} \cdots \sum_{j_n=1}^{\alpha_n} a_{\alpha_1, j_1} \cdots a_{\alpha_n, j_n} x_1^{j_1} \cdots x_{n-1}^{j_{n-1}} x^n \\ &\quad f_\alpha(t_1, \dots, t_{n-1}, t) \tilde{N}(dt_1, dx_1) \cdots \tilde{N}(dt_{n-1}, dx_{n-1}). \end{aligned}$$

Now $d\gamma^{(\alpha_i)}(t) = \sum_{j=1}^{\alpha_i} a_{\alpha_i, j} d\zeta^{(j)} = \int_{\mathbb{R}} \sum_{j=1}^{\alpha_i} a_{\alpha_i, j} x^j \tilde{N}(dt, dx)$ and therefore we get from the above

$$\begin{aligned} E[D_{t,x}\xi|\mathfrak{A}_t] &= \sum_{n=1}^{\infty} \sum_{\alpha \in \mathcal{I}_n} \sum_{j_n=1}^{\alpha_n} \left(\int_0^t \cdots \int_0^{t_2} f_\alpha(t_1, \dots, t_{n-1}, t) d\gamma^{(\alpha_1)}(t_1) \cdots d\gamma^{(\alpha_{n-1})}(t_{n-1}) \right) a_{\alpha_n, j_n} x^{j_n} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{\substack{\alpha \in \mathcal{I}_n \\ \alpha_n = m}} \sum_{j=1}^m a_{m, j} x^j \left(\int_0^t \cdots \int_0^{t_2} f_\alpha(t_1, \dots, t_{n-1}, t) d\gamma^{(\alpha_1)}(t_1) \cdots d\gamma^{(\alpha_{n-1})}(t_{n-1}) \right) \\ &= \sum_{m=1}^{\infty} \sum_{j=1}^m a_{m, j} x^j E[\mathcal{D}_t^{(m)} \xi | \mathfrak{A}_t]. \end{aligned}$$

Thus, we have proved the following

Proposition 3.8. *The relation between the two expansions of Theorem 3.1 and Theorem 3.5 is given by (3.22) and (3.21). For any $\xi \in \mathbf{D}_{1,2} \cap \mathbb{D}_{1,2}$ the stochastic derivatives in Definition 3.3 and Definition 3.6 satisfy the following relation*

$$(3.23) \quad E[D_{t,x}\xi|\mathfrak{A}_t] = \sum_{m=1}^{\infty} \sum_{j=1}^m a_{m, j} x^j E[\mathcal{D}_t^{(m)} \xi | \mathfrak{A}_t].$$

3.D. The case with both a continuous (Wiener process) component and a pure jump (Poisson random measure) component.

In the case of a general Lévy martingale - cf. (3.1), the chaos expansion and the stochastic (Malliavin) derivative for a certain ξ in the corresponding Sobolev type space $\mathbb{D}_{1,2}$ can still

be defined following the same lines as in Ito (1956) and Løkka (2001). But in this case the multiple integrals of the chaos expansion are expressed in terms of mixtures of both the continuous components and the pure jump components. For an explanation we require the following “unified” notation:

$$\begin{aligned} U_1 &= [0, T] & \text{and} & & U_2 &= [0, T] \times \mathbb{R} \\ dQ_1(\cdot) &= dW(\cdot) & \text{and} & & Q_2(\cdot) &= \tilde{N}(\cdot, \cdot) \\ d\langle Q_1 \rangle &= d\lambda & \text{and} & & d\langle Q_2 \rangle &= d\lambda \times d\nu \\ \int_{U_1} g(u^{(1)}) Q_1(du^{(1)}) &= \int_0^t g(s) W(ds) & \text{and} & & \int_{U_2} g(u^{(2)}) Q_2(du^{(2)}) &= \int_0^t \int_{\mathbb{R}} g(s, x) \tilde{N}(ds, dx) \end{aligned}$$

Theorem 3.9. For any $\xi \in H$

$$(3.24) \quad \xi = E\xi + \sum_{n=1}^{\infty} \sum_{j_1, \dots, j_n=1,2} J_n(g_n^{(j_1, \dots, j_n)})$$

for a unique sequence $g_n^{(j_1, \dots, j_n)}$ ($j_1, \dots, j_n = 1, 2; n = 1, 2, \dots$), of deterministic functions in the corresponding L_2 -space $L_2(G_n) = L_2(G_n, \bigotimes_{i=1}^n d\langle Q_{j_i} \rangle)$, where $G_n = \{(u_1^{(j_1)}, \dots, u_n^{(j_n)}) \in \prod_{i=1}^n U_{j_i} : 0 \leq t_1 \leq \dots \leq t_n \leq T\}$ with $u^{(j_i)} = t$, if $j_i = 1$, and $u^{(j_i)} = (t, x)$, if $j_i = 2$, and

$$J_n(g_n^{(j_1, \dots, j_n)}) = \int_{\prod_{i=1}^n U_{j_i}} g_n^{(j_1, \dots, j_n)}(u_1^{(j_1)}, \dots, u_n^{(j_n)}) 1_{G_n}(u_1^{(j_1)}, \dots, u_n^{(j_n)}) Q_{j_1}(du_1^{(j_1)}) \dots Q_{j_n}(du_n^{(j_n)}).$$

Based on the above expansion the stochastic Sobolev space $\mathbb{D}_{1,2}$ and the stochastic derivative D are as follows.

Definition 3.10. (a) $\mathbb{D}_{1,2}$ is the set of all $\xi \in H$ the expansion (3.24) of which satisfies the condition

$$\|\xi\|_{\mathbb{D}_{1,2}}^2 = E\xi^2 + \sum_{n=1}^{\infty} \sum_{j_1, \dots, j_n=1,2} \int_{U_{j_n}} \|g_n^{(j_1, \dots, j_n)}(\cdot, u_n^{(j_n)})\|_{L_2(G_{n-1})}^2 d\langle Q_{j_n} \rangle(u_n^{(j_n)}) < \infty.$$

(b) If $\xi \in \mathbb{D}_{1,2}$, we define its stochastic derivative as follows:

$$D_u^{\{i\}} \xi = \sum_{n=1}^{\infty} \sum_{j_1, \dots, j_{n-1}=1,2} J_{n-1}(g_n^{(j_1, \dots, j_{n-1}, i)}(\cdot, u^{(i)})), \quad i = 1, 2,$$

i.e.

$$D_t \xi = \sum_{n=1}^{\infty} \sum_{j_1, \dots, j_{n-1}=1,2} J_{n-1}(g_n^{(j_1, \dots, j_{n-1}, 1)}(\cdot, t)), \quad i = 1,$$

$$D_{t,x} \xi = \sum_{n=1}^{\infty} \sum_{j_1, \dots, j_{n-1}=1,2} J_{n-1}(g_n^{(j_1, \dots, j_{n-1}, 2)}(\cdot, t, x)), \quad i = 2.$$

Note that if $\xi \in \mathbb{D}_{1,2}$ then

$$\sum_{n=1}^{\infty} \sum_{j_1, \dots, j_n=1,2} \int_{U_{j_n}} \|g_n^{(j_1, \dots, j_n)}(\cdot, u_n^{(j_n)})\|_{L_2(G_{n-1})}^2 d\langle Q_{j_n} \rangle(u_n^{(j_n)}) = \sum_{i=1,2} E \left[\int_{U_i} |D_u^{\{i\}} \xi|^2 d\langle Q_i \rangle(u^{(i)}) \right].$$

It is worth mentioning here that, in the case of the pure jump Lévy martingale, it is possible to define a derivative operator $D_{t,x}$ on $\mathbb{D}_{1,2}$ by a certain difference operator. See Lokka (2001).

To state the last result of this section we consider again the representation (2.1) in the multidimensional setup. The derivative of ξ with respect to the Wiener process W_j will be denoted by $D_{s,j}\xi$, $0 \leq s \leq T$, and the derivative with respect to the j -th compensated Poisson random measure will be $D_{s,x,j}\xi$, $0 \leq s \leq T$, $x \in \mathbb{R}$, here and further on in this paper.

Theorem 3.11. *For any $\xi \in \mathbb{D}_{1,2}$, representation (2.2) can be specified as follows*

$$(3.25) \quad \xi = E\xi + \sum_{j=1}^K \int_0^T E[D_{s,j}\xi | \mathfrak{A}_s] dW_j(s) + \sum_{j=1}^K \int_0^T \int_{\mathbb{R}} E[D_{s,x,j}\xi | \mathfrak{A}_s] \tilde{N}_j(ds, dx).$$

Similarly we can state the multidimensional version of Corollary 3.2 and Theorem 3.4. Note that we use the corresponding multidimensional notation.

Theorem 3.12. (a) *Any $\xi \in H$, can be written as*

$$(3.26) \quad \xi = E\xi + \sum_{j=1}^K \sum_{m=1}^{\infty} \int_0^T \varphi_j^{(m)}(s) d\gamma_j^{(m)}(s).$$

(b) *Moreover any $\xi \in \mathbf{D}_{1,2}$ admits the following specification of (3.26):*

$$(3.27) \quad \xi = E\xi + \sum_{j=1}^K \sum_{m=1}^{\infty} \int_0^T E[\mathcal{D}_{s,j}^{(m)} \xi | \mathfrak{A}_s] d\gamma_j^{(m)}(s).$$

Here $\mathbf{D}_{1,2}$ is the space of all $\xi \in H$ such that

$$\xi = E\xi + \sum_{j=1}^K \sum_{\alpha \in \mathcal{I}} \int_{S_{l(\alpha)}} f_{\alpha,j}(\mathbf{t}) d\gamma_j^{\otimes \alpha}(\mathbf{t}),$$

satisfies

$$\|\xi\|_{\mathbf{D}_{1,2}}^2 := E\xi^2 + \sum_{j=1}^K \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_n) \in \mathcal{I} \\ \alpha_n=m}} M_j(\alpha_1) \dots M_j(\alpha_{n-1}) \int_0^T \|f_{\alpha,j}(\cdot, t)\|_{S_{n-1}}^2 dt < \infty$$

with $[\gamma_j^{(i)}, \gamma_j^{(i)}]_t = M_j(i)t$, $0 \leq t \leq T$, $i = 1, 2, \dots$

4. Minimal variance portfolio and explicit representation formulae.

In the line suggested by Di Nunno (2001) and exploiting representation (2.2) and its specification in Theorem 3.11, we can now give an explicit formula for the minimal variance portfolio. Our result may be regarded as a generalization of the result obtained in Lambertson and Lapeyre (1996), p. 155, for the sum of a Wiener process and a Poisson process.

Theorem 4.1. *For any $\xi \in \mathbb{D}_{1,2}$, the minimal variance portfolio $\varphi = (\varphi_1, \dots, \varphi_k)$ in (1.5):*

$$\hat{\xi} = E\xi + \sum_{j=1}^k \int_0^T \varphi_j(s) d\eta_j(s),$$

admits the following representation

$$(4.1) \quad \varphi_j(s) := \mathfrak{D}_{\eta_j} \xi(s) = \frac{1}{\mu_j} \left[\sigma_j E[D_{s,j} \xi | \mathfrak{A}_s] + \int_{\mathbb{R}} x E[D_{s,x,j} \xi | \mathfrak{A}_s] \nu_j(dx) \right]$$

where $\mu_j = \sigma_j^2 + \int_{\mathbb{R}} x^2 \nu_j(dx)$ in $E\eta_j^2(t) = \mu_j t$, $0 \leq t \leq T$.

Proof. By the Hilbert space argument, $H = H^0 \oplus H(\eta)$, in the line of Theorem 2.3, the following orthogonality condition is satisfied

$$E[(\xi - \hat{\xi})\Theta] = E[\xi^0 \Theta] = 0, \quad \forall \Theta \in H(\eta),$$

where

$$\Theta = \sum_{j=1}^k \int_0^T \theta_j(s) d\eta_j(s) = \sum_{j=1}^k \int_0^T \theta_j(s) \sigma_j dW_j(s) + \sum_{j=1}^k \int_0^T \int_{\mathbb{R}} x \theta_j(s) \tilde{N}_j(dx, ds)$$

for all $\theta = (\theta_1, \dots, \theta_k) \in \mathcal{A}$, admissible portfolios, and $\xi^0 = \xi - \sum_{j=1}^k \int_0^T \mathfrak{D}_{\eta_j} \xi(s) d\eta_j(s)$. From Theorem 3.11 we can write

$$\xi^0 = E\xi + \sum_{j=1}^k \int_0^T \left(\frac{1}{\sigma_j} E[D_{s,j} \xi | \mathfrak{A}_s] - \mathfrak{D}_{\eta_j} \xi(s) \right) \sigma_j dW_j(s) + \sum_{j=1}^k \int_0^T \int_{\mathbb{R}} \left(E[D_{s,x,j} \xi | \mathfrak{A}_s] - x \mathfrak{D}_{\eta_j} \xi(s) \right) \tilde{N}_j(ds, dx)$$

Hence

$$\begin{aligned} E[\xi^0 \Theta] &= E\xi \cdot \sum_{j=1}^k E \left[\int_0^T \theta_j(s) d\eta_j(s) \right] \\ &+ \sum_{i,j=1}^k E \left[\int_0^T \theta_i(s) \sigma_i dW_i(s) \cdot \int_0^T \left(\frac{1}{\sigma_j} E[D_{s,j} \xi | \mathfrak{A}_s] - \mathfrak{D}_{\eta_j} \xi(s) \right) \sigma_j dW_j(s) \right] \\ &+ \sum_{i,j=1}^k E \left[\int_0^T \theta_i(s) \sigma_i dW_i(s) \cdot \int_0^T \int_{\mathbb{R}} \left(E[D_{s,x,j} \xi | \mathfrak{A}_s] - x \mathfrak{D}_{\eta_j} \xi(s) \right) \tilde{N}_j(ds, dx) \right] \\ &+ \sum_{i,j=1}^k E \left[\int_0^T \int_{\mathbb{R}} x \theta_i(s) \tilde{N}_i(dx, ds) \cdot \int_0^T \left(\frac{1}{\sigma_j} E[D_{s,j} \xi | \mathfrak{A}_s] - \mathfrak{D}_{\eta_j} \xi(s) \right) \sigma_j dW_j(s) \right] \\ &+ \sum_{i,j=1}^k E \left[\int_0^T \int_{\mathbb{R}} x \theta_i(s) \tilde{N}_i(dx, ds) \cdot \int_0^T \int_{\mathbb{R}} \left(E[D_{s,x,j} \xi | \mathfrak{A}_s] - x \mathfrak{D}_{\eta_j} \xi(s) \right) \tilde{N}_j(ds, dx) \right]. \end{aligned}$$

Thanks to the independence of η_1, \dots, η_k and the well-known isometry, see for instance p. 62 in Ikeda and Watanabe (1989), we obtain the following equality

$$E[\xi^0 \Theta] = \sum_{j=1}^k E \left[\int_0^T \theta_j(s) \cdot \left(\frac{1}{\sigma_j} E[D_{s,j} \xi | \mathfrak{A}_s] - \mathfrak{D}_{\eta_j} \xi(s) \right) \sigma_j^2 ds \right]$$

$$+ \sum_{j=1}^k E \left[\int_0^T \int_{\mathbb{R}} x \theta_j(s) \cdot \left(E[D_{s,x,j}\xi | \mathfrak{A}_s] - x \mathfrak{D}_{\eta_j} \xi(s) \right) \nu_j(dx) ds \right],$$

which leads to

$$\begin{aligned} E[\xi^0 \Theta] &= \sum_{j=1}^k E \left[\int_0^T \theta_j(s) \cdot \left\{ \sigma_j E[D_{s,j}\xi | \mathfrak{A}_s] \right. \right. \\ &\quad \left. \left. + \int_{\mathbb{R}} x E[D_{s,x,j}\xi | \mathfrak{A}_s] \nu_j(dx) - \mathfrak{D}_{\eta_j} \xi(s) \cdot \left(\sigma_j^2 + \int_{\mathbb{R}} x^2 \nu_j(dx) \right) \right\} ds \right] = 0, \end{aligned}$$

for all Θ in $H(\eta)$, i.e. for all integrands $\theta = (\theta_1, \dots, \theta_k)$. Therefore, the statement follows

$$\varphi_j(s) := \mathfrak{D}_{\eta_j} \xi(s) = \frac{\sigma_j E[D_{s,j}\xi | \mathfrak{A}_s] + \int_{\mathbb{R}} x E[D_{s,x,j}\xi | \mathfrak{A}_s] \nu_j(dx)}{\sigma_j^2 + \int_{\mathbb{R}} x^2 \nu_j(dx)}. \quad \square$$

It is worth noting that in the case of the pure jump Lévy martingale $\eta_j(t) = \int_0^t \int_{\mathbb{R}} x \tilde{N}_j(ds, dx)$, $0 \leq t \leq T$, ($j = 1, \dots, K$) as integrator, representation (4.1) reduces to

$$(4.2) \quad \varphi_j(s) = \mathfrak{D}_{\eta_j} \xi(s) = \frac{1}{\mu_j} \cdot \int_{\mathbb{R}} x E[D_{s,x,j}\xi | \mathfrak{A}_s] \nu_j(dx)$$

with $\mu_j = \int_{\mathbb{R}} x^2 \nu_j(dx)$ in $E\eta_j^2(t) = \mu_j t$, $0 \leq t \leq T$, $j = 1, \dots, k$.

Moreover, for η_t , $0 \leq t \leq T$, as the Wiener process with $\lambda_j = \sigma_j^2$ in $E\eta_j^2(t) = \mu_j t$, $0 \leq t \leq T$, and $k = K$ representation (4.1) reduces to the case of the Clark-Haussmann-Ocone formula - cf. (2.10).

5. Examples.

In the construction of hedging portfolios, it is important to obtain the processes explicitly. As mentioned in the introduction, the financial market will be complete if it is driven by a Wiener process and $k = K$. In this case the Clark-Haussmann-Ocone theorem leads to an explicit representation of the hedging portfolio in terms of a derivative. However, financial markets modeled by Lévy processes are in general not complete. Here the Clark-Haussmann-Ocone formula can be used to give the "closest" hedge to the desired claim under *minimal risk*, in the sense that the variance of the difference between the claim and the hedged wealth is minimal. As we will illustrate in the following examples, the Clark-Haussmann-Ocone formula is a theoretical characterization of the integrands in (2.2) that sometimes shortens the computation of the hedge substantially.

Example 5.a. Suppose the market is modeled by S_0 and K independent Wiener processes $S = \eta = (\eta_1, \dots, \eta_K) = (W_1, \dots, W_K)$ (cf. (1.1),(1.2)) and we are allowed to trade only in S_0 and (η_1, \dots, η_k) , where $k < K$ securities. Then the market is incomplete because of the given constraints.

For any $\xi \in \mathbb{D}_{1,2}$ the minimal variance hedge $\hat{\xi}$ in Theorem 4.1 is

$$\varphi_j(s) = \mathfrak{D}_{W_j} \xi(s) = E[D_{s,j}\xi | \mathfrak{A}_s], \quad 0 \leq s \leq T,$$

($j = 1, \dots, k$) Moreover, since we have

$$\xi = E\xi + \sum_{j=1}^K \int_0^T E[D_{s,j}\xi|\mathfrak{A}_s] dW_j(s)$$

- cf. (2.10), and also

$$\xi = \xi^0 + \hat{\xi} = \xi^0 + \sum_{j=1}^k \int_0^T \varphi_j(s) dW_j(s),$$

then the minimal risk can be given explicitly:

$$\xi^0 = \sum_{j=k+1}^K \int_0^T \varphi_j(s) dW_j(s).$$

Example 5.b. Let $K = 1$ and $\eta(t) = W(t) + \int_0^t \int_{\mathbb{R}} x \tilde{N}(ds, dx)$, $0 \leq t \leq T$, so that $\mu = 1 + \int_{\mathbb{R}} x^2 \nu(dx)$. Suppose we want to hedge the claim

$$\xi = \int_0^T \int_{\mathbb{R}} x^p \tilde{N}(ds, dx) \quad (p = 0, 1, \dots).$$

Then $D_s \xi = 0$ and $D_{s,x} \xi = x^p = E[D_{s,x} \xi | \mathfrak{A}_s]$. So, according to Theorem 4.1, the minimal hedging portfolio is a constant:

$$\varphi(s) = \mathfrak{D}_\eta \xi(s) = \frac{1}{\mu} \int_{\mathbb{R}} x^{p+1} \nu(dx) =: \varphi, \quad 0 \leq s \leq T,$$

and the minimal risk is given by:

$$\xi^0 = \int_0^T \int_{\mathbb{R}} x(x^{p-1} - \varphi) \tilde{N}(ds, dx) - \varphi W(T).$$

Example 5.c. Let $K = 1$ and $\eta(t) = \int_0^t \int_{\mathbb{R}} x \tilde{N}(ds, dx)$, $0 \leq t \leq T$, so that $\mu = \int_{\mathbb{R}} x^2 \nu(dx)$. Suppose $\xi = (\eta(T))^2$. Then, thanks to Lemma 9 in [L], we have

$$D_{s,x} \xi = (\eta(T) + x)^2 - (\eta(T))^2 = x(2\eta(T) + x), \quad 0 \leq s \leq T.$$

So, the minimal variance portfolio is given by

$$\varphi(s) = \mathfrak{D}_\eta \xi(s) = \frac{1}{\mu} \int x^2 (2\eta(s) + x) \nu(dx) = 2\eta(s) + \frac{1}{\mu} \int x^3 \nu(dx), \quad 0 \leq s \leq T.$$

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