

# On amenability and co-amenability of algebraic quantum groups and their corepresentations

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## Abstract

We introduce and study several amenability properties for unitary corepresentations and  $*$ -representations of algebraic quantum groups, which may be used to characterize amenability or co-amenability of such groups. As a background for this study, we also investigate the involved tensor  $C^*$ -categories.

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## 1 Introduction

The concept of amenability plays an important role in the theory of locally compact groups and in the theory of operator algebras (see [25] and references therein). Quite naturally, the concept of amenability and its companion, co-amenability, have been introduced and studied by several authors in various settings related to quantum groups (see [32, 13, 28, 1, 2, 3, 9, 23, 24, 5, 6, 7], in chronological order).

In this paper, we introduce several concepts of “amenability” for unitary corepresentations of analytic extensions of algebraic quantum groups (as defined by J. Kustermans and A. van Daele [20]) : *co-amenability* (a notion inspired by results in [5, 6]), *amenability* (inspired by the concept of amenability of a unitary representation of a locally compact group introduced by M. Bekka [8]) and the *weak containment property* (inspired by the classical characterization of the amenability of a group in terms of weak containment).

We present several equivalent formulations of these properties, and use them to characterize amenability or co-amenability of algebraic quantum groups. After some preliminaries in Section 2, we begin the paper with a categorical interlude in Section 3 where we show how the category of non-degenerate  $*$ -representations of the universal  $C^*$ -algebraic quantum group associated to an algebraic quantum group [19] and the category of unitary corepresentations of the analytic extension of the dual quantum group (with opposite co-product) are naturally isomorphic as tensor  $C^*$ -categories. This result enables to transfer all notions introduced for unitary corepresentations to non-degenerate  $*$ -representations and vice-versa. We also derive the absorbing property of the fundamental multiplicative unitary and of the regular representation. In Section 4, we introduce the conjugate corepresentation and the Hilbert-Schmidt corepresentation associated with a unitary corepresentation. Section 5 is devoted to co-amenability, Section 6 to amenability and Section 7 to the weak containment property, where we also consider briefly property (T) for algebraic quantum groups. In Section 8, we gather some remarks on the relationship between these amenability concepts. Finally, in Section 9, we specify our study of amenability to the setting of algebraic quantum groups of discrete type, where it is possible to exploit the structural properties of these quantum groups to push our analysis further.

Every vector space will be over the ground field  $\mathbb{C}$ . Given a set  $V$ ,  $\iota_V$  denotes the identity map on it (but we simply write  $\iota$  when there is no danger of confusion). If  $\mathcal{H}$  is a Hilbert space, then  $B(\mathcal{H})$  (resp.  $B_0(\mathcal{H})$ ) denotes the algebra of all bounded (resp. compact) linear operators acting on  $\mathcal{H}$ . If  $\mathcal{B}$  is a  $*$ -algebra,  $M(\mathcal{B})$  denotes the multiplier algebra of  $\mathcal{B}$ . If  $\mathcal{B}$  is unital, we denote its unit by  $I_{\mathcal{B}}$ , or by  $I$  when this causes no confusion. In this case, we set  $\mathcal{U}(\mathcal{B})$  for the unitary group of  $\mathcal{B}$ . We denote by  $S(\mathcal{B})$  the state space of the  $C^*$ -algebra  $\mathcal{B}$ . As usual  $\otimes$  denotes tensor product; depending on the context, it may be the tensor product of vector spaces, the Hilbert space tensor product or the minimal (that is, spatial) tensor product of  $C^*$ -algebras,  $\bar{\otimes}$  being used for tensor products

in the von Neumann algebra setting. However, we often use  $\odot$  to stress that we are dealing with an algebraic tensor product. If  $V, W$  are vector spaces,  $\chi : V \otimes W \rightarrow W \otimes V$  is the flip map sending  $v \otimes w$  to  $w \otimes v$  ( $v \in V, w \in W$ ); if  $\mathcal{H}$  is a Hilbert space then  $\Sigma$  is the flip map on  $\mathcal{H} \otimes \mathcal{H}$ . We use the leg-numbering notation as introduced in [1].

## 2 Preliminaries

Throughout this paper,  $(\mathcal{A}, \Delta)$  denotes an algebraic quantum group in the sense of [30], see also [31, 20], where  $\Delta : \mathcal{A} \rightarrow M(\mathcal{A} \odot \mathcal{A})$  is the co-product map. We follow notation and use terminology from these papers. Hence,  $S$  denotes the antipode of  $(\mathcal{A}, \Delta)$ ,  $\varepsilon$  its counit and  $\varphi$  is a fixed faithful left Haar functional. This functional  $\varphi$  is not necessarily tracial (or central). However, there is a unique bijective homomorphism  $\rho : \mathcal{A} \rightarrow \mathcal{A}$  such that  $\varphi(ab) = \varphi(b\rho(a))$ , for all  $a, b \in \mathcal{A}$ . Moreover,  $\rho(\rho(a^*)^*) = a$ .

The pair  $(\mathcal{A}_r, \Delta_r)$  denotes the associated analytic extension (which is a reduced locally compact quantum group in the sense of [21]),  $\pi_r : \mathcal{A} \rightarrow \mathcal{A}_r \subset B(\mathcal{H})$  is the (left) regular representation of  $\mathcal{A}$  acting on the GNS Hilbert space  $\mathcal{H}$  of  $\varphi$ ,  $\Lambda : \mathcal{A} \rightarrow \mathcal{H}$  is the canonical injection,  $W \in M(\mathcal{A}_r \otimes B_0(\mathcal{H}))$  is the associated multiplicative unitary,  $\mathcal{M} = \mathcal{A}_r'' = \pi_r(\mathcal{A})''$  is the von Neumann algebra generated by  $\pi_r(\mathcal{A})$  and  $R$  is the anti-unitary antipode (which is defined on  $\mathcal{M}$ ).

We denote by  $(\hat{\mathcal{A}}, \hat{\Delta})$  the dual algebraic quantum group and by  $(\hat{\mathcal{A}}_r, \hat{\Delta}_r)$  the associated analytic extension. We recall that  $\hat{\mathcal{A}}$  is the subspace of the algebraic dual of  $\mathcal{A}$ , consisting of all functionals  $a\varphi$ , where  $a \in \mathcal{A}$ . Here,  $(a\varphi)(b) = \varphi(ba)$ , and similarly,  $(\varphi a)(b) = \varphi(ab)$ ,  $a, b \in \mathcal{A}$ . Since  $\varphi a = \rho(a)\varphi$ , we have  $\hat{\mathcal{A}} = \{\varphi a \mid a \in \mathcal{A}\}$ .

A right-invariant positive linear functional  $\hat{\psi}$  is defined on  $\hat{\mathcal{A}}$  by setting  $\hat{\psi}(\hat{a}) = \varepsilon(a)$ , for all  $a \in \mathcal{A}$ . Here  $\hat{a} = a\varphi$ . Since the linear map,  $\mathcal{A} \rightarrow \hat{\mathcal{A}}$ ,  $a \mapsto \hat{a}$ , is a bijection (by faithfulness of  $\varphi$ ), the functional  $\hat{\psi}$  is well defined. Further, we have  $\hat{\psi}(\hat{b}^*\hat{a}) = \varphi(b^*a)$ , for all  $a, b \in \mathcal{A}$ .

As shown in [20], one may assume that the regular representation  $\hat{\pi}_r$  of  $\hat{\mathcal{A}}$  also acts on  $\mathcal{H}$ . Accordingly, we identify  $\hat{\mathcal{A}}_r$  with the C\*-algebra generated by  $\hat{\pi}_r(\hat{\mathcal{A}})$  and set  $\hat{\mathcal{M}} = \hat{\mathcal{A}}_r''$ . A useful fact is that both  $\mathcal{M}$  and  $\hat{\mathcal{M}}$  act standardly on  $\mathcal{H}$ .

We will work quite often with the ‘‘opposite’’ dual quantum group  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$ . Note that when we add *op* as a subscript to a co-product map, we mean by this the opposite co-product; that is, the one obtained after ‘‘flipping’’ the original map. One reason for working with  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  is that it corresponds to the dual of  $(\mathcal{A}_r, \Delta_r)$  as defined in [21]. Further, the multiplicative unitary associated to  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  is simply given by  $\hat{W} = \Sigma W^* \Sigma$ , which fits with the usual notation for multiplicative unitaries and their duals (cf. [1]).

We denote by  $(\mathcal{A}_u, \Delta_u)$  the universal (locally compact) C\*-algebraic quantum group associated to  $(\mathcal{A}, \Delta)$ , as introduced by J. Kustermans [19]. We recall

here some details of his construction.

The  $C^*$ -algebra  $\mathcal{A}_u$  is the completion of  $\mathcal{A}$  with respect to the  $C^*$ -norm  $\|\cdot\|_u$  on  $\mathcal{A}$  defined by

$$\|a\|_u = \sup\{\|\Phi(a)\| \mid \Phi \text{ is a } *- \text{homomorphism from } \mathcal{A} \text{ into some } C^* \text{-algebra}\}$$

(The non-trivial fact that this expression gives a well-defined norm on  $\mathcal{A}$  is shown in [19]). The  $C^*$ -algebra  $\mathcal{A}_u$  has then the universal property that one may extend from  $\mathcal{A}$  to  $\mathcal{A}_u$  any  $*$ -homomorphism from  $\mathcal{A}$  into some  $C^*$ -algebra.

The definition of  $\Delta_u$  relies on the following proposition [19, Proposition 3.8], which we restate here as we will need it in the sequel.

**Proposition 2.1.** *Consider  $C^*$ -algebras  $C_1, C_2$  and  $*$ -homomorphisms  $\phi_1$  from  $\mathcal{A}$  into  $M(C_1)$  and  $\phi_2$  from  $\mathcal{A}$  into  $M(C_2)$  such that  $\phi_1(\mathcal{A})C_1$  is dense in  $C_1$  and  $\phi_2(\mathcal{A})C_2$  is dense in  $C_2$ . Then there exists a unique  $*$ -homomorphism  $\phi$  from  $\mathcal{A}$  into  $M(C_1 \otimes C_2)$  such that*

$$(\phi_1(a_1) \otimes \phi_2(a_2))\phi(a) = (\phi_1 \odot \phi_2)((a_1 \otimes a_2)\Delta(a))$$

and

$$\phi(a)(\phi_1(a_1) \otimes \phi_2(a_2)) = (\phi_1 \odot \phi_2)(\Delta(a)(a_1 \otimes a_2))$$

for every  $a_1, a_2 \in \mathcal{A}$ . We have moreover that  $\phi(\mathcal{A})(C_1 \otimes C_2)$  is dense in  $C_1 \otimes C_2$ .

Now, let  $\pi_u$  denote the identity mapping from  $\mathcal{A}$  into  $\mathcal{A}_u$ . Hence,  $\pi_u$  is an injective  $*$ -homomorphism from  $\mathcal{A}$  to  $\mathcal{A}_u$  such that  $\pi_u(\mathcal{A})$  is dense in  $\mathcal{A}_u$ , so  $\pi_u(\mathcal{A})\mathcal{A}_u$  is dense in  $\mathcal{A}_u$  (as  $\mathcal{A}^2 = \mathcal{A}$ ). By applying the above proposition with  $\phi_1 = \phi_2 = \pi_u$  and exploiting the universal property of  $\mathcal{A}_u$ , one obtains that there exists a unique non-degenerate  $*$ -homomorphism  $\Delta_u : \mathcal{A}_u \rightarrow M(\mathcal{A}_u \otimes \mathcal{A}_u)$  such that

$$(\pi_u \odot \pi_u)(x)\Delta_u(\pi_u(a)) = (\pi_u \odot \pi_u)(x\Delta(a))$$

and

$$\Delta_u(\pi_u(a))(\pi_u \odot \pi_u)(x) = (\pi_u \odot \pi_u)(\Delta(a)x)$$

for all  $a \in \mathcal{A}$  and  $x \in \mathcal{A} \odot \mathcal{A}$ .

Being a  $*$ -homomorphism from  $\mathcal{A}$  onto  $\mathbb{C}$ , the co-unit  $\varepsilon$  of  $(\mathcal{A}, \Delta)$  extends to a  $*$ -homomorphism  $\varepsilon_u$  from  $\mathcal{A}_u$  onto  $\mathbb{C}$ , which is easily seen to satisfy the co-unit property for  $(\mathcal{A}_u, \Delta_u)$ . Of course, we identify implicitly here  $\mathcal{A}$  with its canonical copy  $\pi_u(\mathcal{A})$  inside  $\mathcal{A}_u$ . Note that sometimes we add  $u$  as an index to denote the extension to  $\mathcal{A}_u$  of a  $*$ -homomorphism from  $\mathcal{A}$  into some  $C^*$ -algebra, and sometimes just use the same symbol to denote the extension when there is no danger of confusion. For example, we get a canonical map  $\pi_r : \mathcal{A}_u \rightarrow \mathcal{A}_r$  which is the extension of  $\pi_r : \mathcal{A} \rightarrow \mathcal{A}_r$ .

Let now  $(\mathcal{A}, \Delta)$  be an algebraic quantum group of *compact* type, that is,  $\mathcal{A}$  has a unit  $I$ . It is immediate that  $(\mathcal{A}_r, \Delta_r)$  is a compact quantum group in the

sense of Woronowicz [33, 34], with Haar state  $\varphi_r$  given by the restriction of the vector state  $\omega_{\Lambda(I)}$  to  $\mathcal{A}_r$ . The unique dense Hopf  $*$ -subalgebra [5] of  $(\mathcal{A}_r, \Delta_r)$  may be identified with  $(\mathcal{A}, \Delta, \varepsilon, S)$  ( via the Hopf  $*$ -algebra isomorphism  $\pi_r$ ). Using this identification, we may introduce the remarkable family  $(f_z)_{z \in \mathbb{C}}$  of multiplicative linear functionals on  $\mathcal{A}$  constructed by Woronowicz (see [33, 34]).

Some of the properties of this family are:  $f_0 = \varepsilon$ ;  $f_z * f_{z'} = f_{z+z'}$ , where  $\omega * \eta = (\omega \otimes \eta)\Delta$ ,  $\omega, \eta \in A'$ ; the maps  $a \rightarrow f_z * a = (\iota \otimes f_z)\Delta(a)$  and  $a \rightarrow a * f_z = (f_z \otimes \iota)\Delta(a)$  are automorphisms of  $\mathcal{A}$ ; we have  $f_z^* = f_{-\bar{z}}$  and  $f_z \circ S = f_{-z}$ ; we have  $\varphi(ab) = \varphi(b(f_1 * a * f_1))$  and  $S^2(a) = f_{-1} * a * f_1$ .

It follows from [18, Theorem 2.12] that  $M(\hat{\mathcal{A}})$ , the multiplier algebra of  $\mathcal{A}$ , may be concretely realized as the subspace of the algebraic dual of  $\mathcal{A}$  consisting of elements  $\theta$  such that  $(\theta \odot \iota)\Delta(a)$  and  $(\iota \odot \theta)\Delta(a)$  belong to  $\mathcal{A}$  for every  $a \in \mathcal{A}$ . Hence, in the compact case, we have  $f_z \in M(\hat{\mathcal{A}})$  for all  $z$ .

We also recall that the following three conditions are equivalent:

$$\varphi \text{ is tracial; } f_z = \varepsilon \text{ for all } z \in \mathbb{C}; f_1 = \varepsilon.$$

The following description of algebraic quantum groups of *discrete* type, that is, those which are dual to algebraic quantum groups of compact type, will be useful.

**Proposition 2.2.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group of compact type and let  $(U^\alpha)_{\alpha \in A}$  denote a complete set of pairwise inequivalent irreducible unitary corepresentations of the compact quantum group  $(\mathcal{A}_r, \Delta_r)$ . Note that we have  $U^\alpha \in \mathcal{A} \odot M_{d_\alpha}(\mathbb{C})$  for some  $d_\alpha < \infty$ , when identifying  $\mathcal{A}$  as the dense Hopf  $*$ -algebra of  $\mathcal{A}_r$ . Write each  $U^\alpha$  as a matrix  $(u_{ij}^\alpha)$  over  $\mathcal{A}$  and recall that the set  $\text{Span} \{ (u_{ij}^\alpha) | 1 \leq i, j \leq d_\alpha, \alpha \in A \}$  is a linear basis for  $\mathcal{A}$ . Let*

$$M_\alpha = \sum_{i=1}^{d_\alpha} f_{-1}(u_{ii}^\alpha) = \sum_{i=1}^{d_\alpha} f_1(u_{ii}^\alpha)$$

denote the quantum dimension of  $U^\alpha$ .

Further, set  $\hat{\mathcal{A}}_\alpha = \text{Span} \{ (\hat{u}_{ij}^\alpha) | 1 \leq i, j \leq d_\alpha \}$  and define

$$p_\alpha = M_\alpha \sum_{i,j=1}^{d_\alpha} f_1(u_{ji}^\alpha) \hat{u}_{ij}^\alpha \in \hat{\mathcal{A}}_\alpha.$$

Then

(1)

$$(i) \quad \hat{u}_{ij}^\alpha \hat{u}_{kl}^\beta = \frac{\delta_{\alpha\beta}}{M_\alpha} f_{-1}(u_{kj}^\alpha) \hat{u}_{il}^\alpha,$$

$$(ii) \quad (\hat{u}_{ij}^\alpha)^* = \hat{u}_{ji}^\alpha,$$

$$(iii) \quad p_\alpha \hat{u}_{kl}^\beta = \delta_{\alpha\beta} \hat{u}_{kl}^\beta = \hat{u}_{kl}^\beta p_\alpha,$$

where  $1 \leq i, j \leq d_\alpha, 1 \leq k, l \leq d_\beta, \alpha, \beta \in A$ .

- (2) The set  $\{ (\hat{u}_{ij}^\alpha) | 1 \leq i, j \leq d_\alpha, \alpha \in A \}$  is a linear basis for  $\hat{\mathcal{A}}_\alpha$ .
- (3) Each  $\hat{\mathcal{A}}_\alpha$  is a  $*$ -subalgebra of  $\hat{\mathcal{A}}$ , which is unital with unit  $p_\alpha$ . As a  $*$ -algebra,  $\hat{\mathcal{A}}_\alpha$  is isomorphic to the matrix algebra  $M_{d_\alpha}(\mathbb{C})$ .
- (4)  $\hat{\mathcal{A}} = \bigoplus_\alpha \hat{\mathcal{A}}_\alpha$  (algebraic direct sum).
- (5) For each  $\alpha \in A$ , let  $Tr_\alpha$  denote the canonical trace on  $\hat{\mathcal{A}}_\alpha = M_{d_\alpha}(\mathbb{C})$  satisfying  $Tr_\alpha(p_\alpha) = d_\alpha$ . Then

$$\hat{\psi}(x) = \bigoplus_\alpha Tr_\alpha(p_\alpha x f_{-1}), \quad x \in \hat{\mathcal{A}}.$$

*Proof.* This result is essentially known (see [11, p. 722]), but we will need the explicit description presented here in the sequel. We give a proof for the sake of completeness. It relies on the so-called orthogonality relations for the  $U_\alpha$ 's established in [33, 34].

- (1) With obvious index notation, we have

$$\begin{aligned} (\hat{u}_{ij}^\alpha \hat{u}_{kl}^\beta)((u_{pq}^\gamma)^*) &= \sum_r \varphi((u_{pr}^\gamma)^* u_{ij}^\alpha) \varphi((u_{rq}^\gamma)^* u_{kl}^\beta) \\ &= (1/M_\alpha)(1/M_\beta) \sum_r \delta_{\alpha\gamma} f_{-1}(u_{ip}^\alpha) \delta_{rj} \delta_{\beta\gamma} f_{-1}(u_{kr}^\beta) \delta_{ql} \\ &= (1/M_\alpha)(1/M_\beta) \delta_{\alpha\gamma} \delta_{\beta\gamma} \delta_{ql} f_{-1}(u_{ip}^\alpha) f_{-1}(u_{kj}^\beta) \\ &= (1/M_\alpha^2) \delta_{\alpha\beta} \delta_{\alpha\gamma} \delta_{ql} f_{-1}(u_{ip}^\alpha) f_{-1}(u_{kj}^\alpha) \\ &= (1/M_\alpha) \delta_{\alpha\beta} f_{-1}(u_{kj}^\alpha) \varphi((u_{pq}^\gamma)^* u_{il}^\alpha) \\ &= ((1/M_\alpha) \delta_{\alpha\beta} f_{-1}(u_{kj}^\alpha) \hat{u}_{il}^\alpha)((u_{pq}^\gamma)^*), \end{aligned}$$

and (i) follows. Concerning (ii), we have

$$(\hat{u}_{ij}^\alpha)^* = (S(u_{ij}^\alpha)^*)^\wedge = \hat{u}_{ji}^\alpha,$$

using [20, Lemma 7.14]. Further, using (i), we get

$$\begin{aligned} p_\alpha \hat{u}_{kl}^\beta &= M_\alpha \sum_{i,j} f_1(u_{ji}^\alpha) \hat{u}_{ij}^\alpha \hat{u}_{kl}^\beta \\ &= M_\alpha (1/M_\alpha) \delta_{\alpha\beta} \sum_{i,j} f_1(u_{ji}^\alpha) f_{-1}(u_{kj}^\alpha) \hat{u}_{il}^\alpha \\ &= \delta_{\alpha\beta} \sum_i \delta_{ik} \hat{u}_{il}^\alpha = \delta_{\alpha\beta} \hat{u}_{kl}^\beta, \end{aligned}$$

which is equal to  $\hat{u}_{kl}^\beta p_\alpha$  by a similar computation. Hence, (iii) is proved.

- (2) As  $\{ u_{ij}^\alpha | 1 \leq i, j \leq d_\alpha, \alpha \in A \}$  is a linear basis for  $\mathcal{A}_\alpha$  and the map  $a \rightarrow \hat{a}$  is a linear isomorphism between  $\mathcal{A}$  and  $\hat{\mathcal{A}}$ , (2) is clear.

(3) The first sentence is an obvious consequence of (1). Now,  $\hat{\mathcal{A}}_\alpha$  may be seen as a finite dimensional  $\mathbb{C}^*$ -algebra (using the faithfulness of the  $*$ -homomorphism  $\hat{\pi}_r$ ). Hence, to show that  $\hat{\mathcal{A}}_\alpha$  is isomorphic to  $M_{d_\alpha}(\mathbb{C})$ , it is enough to show that if  $b = \sum_{i,j} b_{ij} \hat{u}_{ij}^\alpha$ ,  $b_{ij} \in \mathbb{C}$ , is an element of the center of  $\hat{\mathcal{A}}^\alpha$ , then  $b = \lambda p_\alpha$ , that is,

$$b_{ij} = \lambda f_1(u_{ji}^\alpha) \text{ for some } \lambda \in \mathbb{C}, 1 \leq i, j \leq d_\alpha.$$

Now, using (1)(i), one sees immediately that  $\hat{u}_{kl}^\alpha b = b \hat{u}_{kl}^\alpha$  hold for all  $k$  and  $l$  if, and only if,

$$\sum_{i,j} b_{ij} f_{-1}(u_{kj}^\alpha) \hat{u}_{il}^\alpha = \sum_{i,j} b_{ij} f_{-1}(u_{il}^\alpha) \hat{u}_{kj}^\alpha$$

for all  $k, l$ , which in turn is equivalent to

$$(*) \quad \delta_{rk} \sum_i b_{is} f_{-1}(u_{il}^\alpha) = \delta_{sl} \sum_j b_{rj} f_{-1}(u_{kj}^\alpha), \quad \forall k, l, r, s.$$

We introduce now the two complex matrices  $B = (b_{ij})$  and  $C = (c_{ij})$ , where  $c_{ij} = f_{-1}(u_{ji}^\alpha)$ . Then (\*) may be rewritten as

$$(**) \quad \delta_{rk} d_{ls} = \delta_{sl} e_{rk}, \quad \forall k, l, r, s,$$

where  $d_{ls} = \sum_i c_{li} b_{is}$  and  $e_{rk} = \sum_j b_{rj} c_{jk}$ . From (\*\*), we clearly get

$$(BC)_{sl} = 0 = (CB)_{sl}, \quad s \neq l; \quad (BC)_{ll} = (BC)_{kk},$$

hence that  $BC = CB$  is a complex multiple  $\lambda$  of the identity matrix. But  $C$  is invertible, with inverse  $C^{-1} = (c'_{ij})$ , where  $c'_{ij} = f_1(u_{ji}^\alpha)$ . Indeed, we have

$$\sum_j c_{ij} c'_{jk} = \sum_j f_{-1}(u_{ji}^\alpha) f_1(u_{kj}^\alpha) = \varepsilon(u_{ki}^\alpha) = \delta_{ik}.$$

Therefore, we can conclude that  $B = \lambda C^{-1}$ , that is,  $b_{ij} = \lambda f_1(u_{ji}^\alpha)$ . This establishes (3).

(4) is an easy consequence of the previous assertions.

(5) Fix now  $\alpha \in A$  and define a linear functional  $\tau$  on  $\hat{\mathcal{A}}_\alpha$  by

$$\tau(x) = (1/M_\alpha) \hat{\psi}(x f_1), \quad x \in \hat{\mathcal{A}}_\alpha.$$

To show (4), a moment's thought makes it clear that it is enough to show that  $\tau = Tr_\alpha$ . Due to the uniqueness property of  $Tr_\alpha$ , we only have to show that

$$(a) \quad \tau \text{ is central}; \quad (b) \quad \tau(p_\alpha) = d_\alpha.$$

To show (a), we have to show

$$(a') \quad \hat{\psi}(xy f_1) = \hat{\psi}(yx f_1), \quad x, y \in \hat{\mathcal{A}}^\alpha.$$

Now, let  $\hat{\rho}$  denote the automorphism of  $\hat{\mathcal{A}}$  satisfying  $\hat{\psi}(\hat{a}\hat{b}) = \hat{\psi}(\hat{b}\hat{\rho}(\hat{a}))$  for all  $a, b \in \mathcal{A}$ . Then we get  $\hat{\psi}(yx f_1) = \hat{\psi}(x f_1 \hat{\rho}(y))$ , so (a') follows if  $y f_1 = f_1 \hat{\rho}(y)$

hold for all  $y \in \hat{\mathcal{A}}$ , that is, if  $\hat{\rho}(y) = f_{-1}yf_1$ ,  $y \in \hat{\mathcal{A}}$ . This follows from Lemma 2.3.

To show (b), we first observe that

$$\begin{aligned} (\hat{u}_{ij}^\alpha f_1)(u_{kl}^\beta)^* &= \sum_r \varphi((u_{kr}^\beta)^* u_{ij}^\alpha) f_1((u_{rl}^\beta)^*) \\ &= \delta_{\alpha\beta} (1/M_\alpha) \sum_k f_1((u_{rl}^\beta)^*) \delta_{rk} f_{-1}(u_{ik}^\alpha) = \delta_{\alpha\beta} (1/M_\alpha) f_1((u_{jl}^\beta)^*) f_{-1}(u_{ik}^\alpha). \end{aligned}$$

Using this, we show that  $p_\alpha f_1 = M_\alpha \sum_i \hat{u}_{ii}^\alpha$ . Indeed, we have

$$\begin{aligned} (p_\alpha f_1)((u_{kl}^\beta)^*) &= M_\alpha \left( \sum_{i,j} f_1(u_{ji}^\alpha) \hat{u}_{ij}^\alpha f_1 \right) ((u_{kl}^\beta)^*) \\ &= \delta_{\alpha\beta} \sum_{i,j} f_1(u_{ji}^\alpha) f_1((u_{jl}^\beta)^*) f_{-1}(u_{ik}^\alpha) \\ &= \delta_{\alpha\beta} \sum_j \delta_{jk} f_1((u_{jl}^\beta)^*) = \delta_{\alpha\beta} f_1((u_{kl}^\beta)^*) \\ &= \delta_{\alpha\beta} f_1(S(u_{lk}^\beta)) = \delta_{\alpha\beta} f_{-1}(u_{lk}^\beta), \end{aligned}$$

while

$$\begin{aligned} M_\alpha \sum_i \hat{u}_{ii}^\alpha ((u_{kl}^\beta)^*) &= M_\alpha \sum_i \varphi((u_{kl}^\beta)^* u_{ii}^\alpha) \\ &= \delta_{\alpha\beta} \sum_i \delta_{li} f_{-1}(u_{ik}^\beta) = \delta_{\alpha\beta} f_{-1}(u_{lk}^\beta). \end{aligned}$$

But then we get

$$\begin{aligned} \tau(p_\alpha) &= (1/M_\alpha) \hat{\psi}(p_\alpha f_1) = \sum_i \hat{\psi}(\hat{u}_{ii}^\alpha) \\ &= \sum_i \varepsilon(u_{ii}^\alpha) = d_\alpha, \end{aligned}$$

and (b) is shown. This finishes the proof of (5) and of the proposition.  $\square$

**Lemma 2.3.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group of compact type. Let  $\hat{\rho}$  denote the automorphism of  $\hat{\mathcal{A}}$  satisfying  $\hat{\psi}(\hat{a}\hat{b}) = \hat{\psi}(\hat{b}\hat{\rho}(\hat{a}))$  for all  $a, b \in \mathcal{A}$ . Then*

$$\hat{\rho}(\hat{a}) = f_{-1}\hat{a}f_1, \quad a \in \mathcal{A}.$$

*Proof.* Being of compact type,  $(\mathcal{A}, \Delta)$  is unimodular, that is, the modular element  $\delta$  of  $\mathcal{A}$  is trivial. Hence, it follows from [7, Lemma 2.2] that  $\hat{\rho}(\hat{a}) = (S^2(a))^\wedge$  for all  $a \in \mathcal{A}$ . Therefore, we have

$$\begin{aligned} \hat{\rho}(\hat{a})(b) &= (S^2(a))^\wedge(b) = \varphi(bS^2(a)) \\ &= \varphi S^{-2}(bS^2(a)) = \varphi(S^{-2}(b)a) = \hat{a}(S^{-2}(b)) \\ &= \hat{a}(f_1 * b * f_{-1}) = \hat{a}((f_{-1} \odot \iota \odot f_1)(\Delta \odot \iota)\Delta(b)) \\ &= (f_{-1} \odot \hat{a} \odot f_1)(\Delta \odot \iota)\Delta(b) = (f_{-1}\hat{a}f_1)(b) \end{aligned}$$

for all  $a, b \in \mathcal{A}$ , which proves the assertion.  $\square$



### 3 Categorical interlude

In this section we introduce  $\text{Rep}(\mathcal{A}_u, \Delta_u)$ ,  $\text{Rep}(\mathcal{A}, \Delta)$  and  $\text{Corep}(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  as concrete tensor  $C^*$ -categories, and describe explicitly isomorphisms (of tensor  $C^*$ -categories) between them. We also briefly mention some related categories. Finally, we establish the absorption property for the regular representation with respect to tensor product.

#### 3.1 $C^*$ -tensor categories associated with algebraic quantum groups

We refer to [15, 22] for terminology concerning tensor  $C^*$ -tensor categories. Let  $(\mathcal{A}, \Delta)$  denote an algebraic quantum group. We begin with the category  $\mathcal{R} = \text{Rep}(\mathcal{A}_u, \Delta_u)$  and explain how it may be organized as a concrete tensor  $C^*$ -category with irreducible unit. The objects in  $\mathcal{R}$  are the  $*$ -representations  $\pi$  of  $\mathcal{A}_u$  acting on a Hilbert space  $H_\pi$  satisfying the non-degenerateness (denseness) condition  $\overline{\pi(\mathcal{A}_u)\mathcal{H}_\pi} = \mathcal{H}_\pi$ . The family of arrows (or morphisms) between two objects  $\pi$  and  $\pi'$  is given by

$$\text{Mor}(\pi, \pi') = \{T \in B(\mathcal{H}_\pi, \mathcal{H}_{\pi'}) \mid T\pi(a) = \pi'(a)T, \forall a \in \mathcal{A}\}.$$

The element  $1_\pi \in \text{Mor}(\pi, \pi)$  is given by the identity on  $\mathcal{H}_\pi$ . The adjoint of an element  $T \in \text{Mor}(\pi, \pi')$  is given by its Hilbert space adjoint  $T^* \in \text{Mor}(\pi', \pi)$ , so we clearly have  $\|T^* T\| = \|T\|^2$ . The tensor product  $\pi \times \pi'$  of two objects  $\pi$  and  $\pi'$  is defined as  $\pi \times \pi' = (\pi \otimes \pi')\Delta_u$ , while on the arrows we have the usual tensor product of operators. The unit in the tensor category is given by  $\varepsilon_u$ . Note that this unit is irreducible, since  $\text{Mor}(\varepsilon_u, \varepsilon_u) = \mathbb{C}$ . It is clear that this category has natural subobjects and that one may form direct sums in an obvious way.

Next, we introduce the closely related category  $\mathcal{R}_{\text{alg}} = \text{Rep}(\mathcal{A}, \Delta)$ . The objects in  $\mathcal{R}_{\text{alg}}$  are now the  $*$ -representations  $\pi$  of  $\mathcal{A}$  acting on a Hilbert space  $H_\pi$  satisfying the non-degenerateness (denseness) condition  $\overline{\pi(\mathcal{A})\mathcal{H}_\pi} = \mathcal{H}_\pi$ . Arrows and adjoints are defined in a similar way as above. To define the tensor product of objects, we have to appeal to Proposition 2.1. Let  $\phi_1$  and  $\phi_2$  be objects in  $\mathcal{R}_{\text{alg}}$ , and consider  $\phi_1$  (resp.  $\phi_2$ ) as a  $*$ -homomorphism from  $\mathcal{A}$  into  $M(B_0(\mathcal{H}_{\phi_1}))$  (resp.  $M(B_0(\mathcal{H}_{\phi_2}))$ ). As  $\phi_1$  and  $\phi_2$  are non-degenerate (by assumption), the proposition applies and produces a unique  $*$ -homomorphism  $\phi_1 \times \phi_2$  from  $\mathcal{A}$  into  $M(B_0(\mathcal{H}_{\phi_1}) \otimes B_0(\mathcal{H}_{\phi_2})) = M(B_0(\mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2})) = B(\mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2})$  such that

$$(\phi_1(a_1) \otimes \phi_2(a_2))(\phi_1 \times \phi_2)(a) = (\phi_1 \odot \phi_2)((a_1 \otimes a_2)\Delta(a))$$

and

$$(\phi_1 \times \phi_2)(a)(\phi_1(a_1) \otimes \phi_2(a_2)) = (\phi_1 \odot \phi_2)(\Delta(a)(a_1 \otimes a_2))$$

for every  $a_1, a_2 \in \mathcal{A}$ . Moreover, we have that  $(\phi_1 \times \phi_2)(\mathcal{A})(B_0(\mathcal{H}_{\phi_1}) \otimes B_0(\mathcal{H}_{\phi_2}))$  is dense in  $B_0(\mathcal{H}_{\phi_1}) \otimes B_0(\mathcal{H}_{\phi_2})$ . It follows easily that  $\phi_1 \times \phi_2$ , when regarded

as a  $*$ -homomorphism from  $\mathcal{A}$  into  $B(\mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2})$ , satisfies the non-degeneracy (denseness) condition required for qualifying it as an object in  $\mathcal{R}_{alg}$ . Finally, the unit in the tensor category is of course  $\varepsilon$ .

Not surprisingly, the following proposition holds:

**Proposition 3.1.** *Define  $P : \mathcal{R} \rightarrow \mathcal{R}_{alg}$  on objects by  $P(\pi) = \pi \circ \pi_u$ , and let  $P$  act trivially on arrows. Then  $P$  is an isomorphism of tensor  $C^*$ -categories.*

*Proof.* The only non-trivial fact in this assertion is perhaps to show that  $P$  preserves tensor products.

Let  $\pi_1, \pi_2$  be objects in  $\mathcal{R}$ , and set  $\phi_1 = P(\pi_1), \phi_2 = P(\pi_2), \phi = P(\pi_1 \times \pi_2)$ . We have to show that  $\phi = \phi_1 \times \phi_2$ .

Now, let  $a, a_1, a_2 \in \mathcal{A}$ . Then we have

$$\begin{aligned} & \phi(a)(\phi_1(a_1) \otimes \phi_2(a_2)) \\ &= (\pi_1 \times \pi_2)\pi_u(a)(\pi_1\pi_u(a_1) \otimes \pi_2\pi_u(a_2)) \\ &= (\pi_1 \otimes \pi_2)(\Delta_u\pi_u(a))(\pi_1 \otimes \pi_2)(\pi_u(a_1) \otimes \pi_u(a_2)) \\ &= (\pi_1 \otimes \pi_2)(\Delta_u\pi_u(a)(\pi_u(a_1) \otimes \pi_u(a_2))) \\ &= (\pi_1 \odot \pi_2)(\pi_u \odot \pi_u)(\Delta(a)(a_1 \otimes a_2)) \\ &= (\phi_1 \odot \phi_2)(\Delta(a)(a_1 \otimes a_2)). \end{aligned}$$

In the same way, one shows that

$$(\phi_1(a_1) \otimes \phi_2(a_2))\phi(a) = (\phi_1 \odot \phi_2)((a_1 \otimes a_2)\Delta(a)).$$

From the uniqueness property of  $\phi_1 \times \phi_2$ , we may then conclude that  $\phi = \phi_1 \times \phi_2$ , as desired.  $\square$

Another category  $\mathcal{C} = \text{Corep}(\hat{\mathcal{A}}_r, \hat{\Delta}_{r, \text{op}})$  which may be organized as a concrete tensor  $C^*$ -category is defined as follows. The objects in  $\mathcal{C}$  consist of unitary elements  $U$  lying in  $(M(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U)))$  for some Hilbert space  $\mathcal{H}_U$  and satisfying the corepresentation property

$$(\hat{\Delta}_{r, \text{op}} \otimes \iota)U = U_{13} U_{23}.$$

For objects  $U$  and  $V$  in  $\mathcal{C}$ , we set

$$\text{Mor}(U, V) = \{T \in B(\mathcal{H}_U, \mathcal{H}_V) \mid T(\omega \bar{\otimes} \iota)(U) = (\omega \bar{\otimes} \iota)(V)T, \forall \omega \in \hat{\mathcal{M}}_*\}.$$

The element  $1_U \in \text{Mor}(U, U)$  is given by the identity on  $\mathcal{H}_U$ . The adjoint of an element  $T \in \text{Mor}(U, V)$  is given by its Hilbert space adjoint  $T^* \in \text{Mor}(V, U)$ , so we have  $\|T^* T\| = \|T\|^2$ . The monoidal structure on the objects is determined by setting

$$U \times V = V_{13} U_{12} \in M(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U) \otimes B_0(\mathcal{H}_V)) = M(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U \otimes \mathcal{H}_V)).$$

(Clearly  $U \times V$  is unitary. Moreover,

$$\begin{aligned} (\hat{\Delta}_{r,\text{op}} \otimes \iota)(U \times V) &= (\hat{\Delta}_{r,\text{op}} \otimes \iota \otimes \iota)(V_{13} U_{12}) \\ &= (\hat{\Delta}_{r,\text{op}} \otimes \iota \otimes \iota)(V_{13})(\hat{\Delta}_{r,\text{op}} \otimes \iota \otimes \iota)(U_{12}) \\ &= V_{14} V_{24} U_{13} U_{23} = V_{14} U_{13} V_{24} U_{23} = (U \times V)_{13}(U \times V)_{23}, \end{aligned}$$

so  $U \times V \in \mathcal{C}$ .) The reason for “reversing” the “natural” tensor product will be evident from Theorem 3.3. Again, on the arrows, we just set  $T \times S = T \otimes S$ .

The unit in the tensor category is given by  $I \otimes 1 \in M(\hat{\mathcal{A}}_r \otimes \mathbb{C})$ , where  $I$  denotes the unit of  $M(\hat{\mathcal{A}}_r)$ . As we clearly have  $\text{Mor}(I \otimes 1, I \otimes 1) = \mathbb{C}$ , this unit is irreducible.

For objects  $U, V$  in  $\mathcal{C}$ , we say that  $U$  is (unitarily) *equivalent* to  $V$ , and write  $U \simeq V$ , whenever there exists a unitary  $T \in \text{Mor}(U, V)$ .

One may clearly introduce several related tensor  $\mathbb{C}^*$ -categories, such as  $\text{Rep}(\mathcal{A}_u, \Delta_{u,\text{op}})$ ,  $\text{Corep}(\mathcal{A}_u, \Delta_u)$ ,  $\text{Corep}(\mathcal{A}_r, \Delta_r)$  and  $\text{Corep}(\mathcal{A}_r, \Delta_{r,\text{op}})$ , along the same lines. Note that in the sequel we always refer to the monoidal structure defined in the same way as above. Nevertheless, it should be noted that there are relations between the possible choices of monoidal structure in these tensor categories. For example, if one considers  $\text{Corep}(\mathcal{A}_r, \Delta_r)$  with monoidal structure given by  $X \bar{\times} Y = X_{12} Y_{13}$  and  $\text{Corep}(\mathcal{A}_r, \Delta_{r,\text{op}})$  with  $U \times V = V_{13} U_{12}$ , then it is easy to check that the map  $X \rightarrow X^*$  gives an isomorphism between these two tensor categories (acting trivially on arrows).

### 3.2 From corepresentations to representations and back

We now recall some results from [7]. First, there exists an injective, not necessarily  $*$ -preserving, homomorphism  $Q_r : \mathcal{A} \rightarrow \hat{\mathcal{A}}_r^*$  determined by

$$Q_r(a)(\hat{\pi}_r(\hat{b})) = \hat{b}(S^{-1}(a)) = \varphi(S^{-1}(a)b), \quad \forall a, b \in \mathcal{A}.$$

In fact, there exists an injective homomorphism  $Q : \mathcal{A} \rightarrow \hat{\mathcal{M}}_*$  satisfying  $Q(a)|_{\hat{\mathcal{A}}_r} = Q_r(a)$  for all  $a \in \mathcal{A}$ , and such that  $Q(\mathcal{A})$  is norm-dense in  $\hat{\mathcal{M}}_*$ . For all  $a \in \mathcal{A}$ , we have  $Q(a) = \omega_{\Lambda(a), \Lambda(c)}$  (restricted to  $\hat{\mathcal{M}}$ ), where  $c \in \mathcal{A}$  is chosen such that  $\hat{c} \widehat{S(a^*)} = \widehat{S(a^*)}$  (such a choice is always possible).

We will need the following lemma.

**Lemma 3.2.** *Let  $a \in \mathcal{A}$  and assume that  $c \in \mathcal{A}$  satisfies  $\hat{c} \widehat{S(a^*)} = \widehat{S(a^*)}$ .*

*Then we have*

$$\Delta(a) = (\iota \odot \iota \odot \varphi)((I \otimes I \otimes c^*)(\iota \odot \Delta)\Delta(a)).$$

*Proof.* Recall from [7] that

$$\widehat{cS(a^*)} = \sum_i \varphi(q_i) \hat{p}_i,$$

where  $c \otimes S(a^*) = \sum_i \Delta(p_i)(q_i \otimes I)$  for some  $p_i, q_i \in \mathcal{A}$ , ( $i = 1 \dots n$ ).

Note also that the inversion formula from [20, p. 1024] gives then

$$\sum_i q_i \otimes p_i = ((S^{-1} \odot \iota) \Delta S(a^*))(c \otimes I).$$

Hence, we have

$$\begin{aligned} (\widehat{cS(a^*)})(b) &= \sum_i \varphi(q_i) \hat{p}_i(b) \\ &= \sum_i \varphi(q_i) \varphi(bp_i) = (\varphi \odot \varphi)((I \otimes b)(\sum_i q_i \otimes p_i)) \\ &= (\varphi b)((\varphi \odot \iota)(\sum_i q_i \otimes p_i)) = (\varphi b)((\varphi \odot \iota)((S^{-1} \odot \iota) \Delta S(a^*))(c \otimes I)) \end{aligned}$$

while

$$\widehat{S(a^*)}(b) = \varphi(bS(a^*)) = (\varphi b)(S(a^*))$$

for all  $b \in \mathcal{A}$ .

From the assumption and the fact that  $\{\varphi b \mid b \in \mathcal{A}\}$  separates points in  $\mathcal{A}$  (since  $\varphi$  is faithful on  $\mathcal{A}$ ), we get

$$\begin{aligned} S(a^*) &= (\varphi \odot \iota)((S^{-1} \odot \iota) \Delta S(a^*))(c \otimes I) = (c\varphi \otimes \iota)(S^{-1} \odot \iota) \Delta S(a^*) \\ &= (\iota \odot c\varphi)(S \odot \iota) \Delta(a^*) = S(\iota \odot c\varphi) \Delta(a^*), \end{aligned}$$

where we have used that  $\Delta S = \chi(S \odot S) \Delta$ . Therefore, we have

$$a^* = (\iota \odot c\varphi) \Delta(a^*) = (\iota \odot \varphi)(\Delta(a^*)(I \otimes c)),$$

so

$$a = (\iota \odot \varphi)((I \otimes c^*) \Delta(a)),$$

as  $\iota \odot \varphi$  is  $*$ -preserving.

This implies that

$$\begin{aligned} \Delta(a) &= \Delta(\iota \odot \varphi)((I \otimes c^*) \Delta(a)) \\ &= (\iota \odot \iota \odot \varphi)(\Delta \odot \iota)((I \otimes c^*) \Delta(a)) \end{aligned}$$

$$\begin{aligned}
&= (\iota \odot \iota \odot \varphi)((I \otimes I \otimes c^*)(\Delta \odot \iota)\Delta(a)) \\
&= (\iota \odot \iota \odot \varphi)((I \otimes I \otimes c^*)(\iota \odot \Delta)\Delta(a)),
\end{aligned}$$

as asserted. □

The following result teaches us that  $\mathcal{C}$  and  $\mathcal{R}$  are isomorphic as tensor  $C^*$ -categories.

**Theorem 3.3.** *Define  $F : \mathcal{C} \rightarrow \mathcal{R}$  on objects by  $F(U) = \pi_U$ , where  $\pi_U$  is determined by*

$$\pi_U(a) = (Q_r(a) \otimes \iota)U, \quad \forall a \in \mathcal{A},$$

and let  $F$  act identically on arrows.

Define  $G : \mathcal{R} \rightarrow \mathcal{C}$  on objects by  $G(\pi) = U_\pi$ , where  $U_\pi$  is determined by

$$U_\pi(\Lambda(a) \otimes \pi(b)v) = \sum_{i=1}^n \Lambda(a_i) \otimes \pi(b_i)v$$

for all  $a, b \in \mathcal{A}, v \in H_\pi$ , and the  $a_i$ 's and  $b_i$ 's are elements in  $\mathcal{A}$  chosen as to satisfy  $\Delta(a)(b \otimes I) = \sum_{i=1}^n b_i \otimes a_i$ . Let  $G$  act identically on arrows.

Then  $F$  and  $G$  are covariant monoidal (tensor preserving) functors which are adjoint- and unit-preserving, and satisfy  $GF = \text{id}$ ,  $FG = \text{id}$ .

We also have  $F(\hat{W}) = \pi_r$  and  $F(I \otimes 1) = \varepsilon_u$ .

*Proof.* The fact that  $F$  and  $G$  are well defined on objects is established in [7], where it is also shown that  $GF = \text{id}$ ,  $FG = \text{id}$ ,  $F(\hat{W}) = \pi_r$  and  $F(I \otimes 1) = \varepsilon_u$ .

We now check that  $F$  and  $G$  are well defined on arrows. Let  $U, V$  be unitary corepresentations of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$ , and let  $T \in \text{Mor}(U, V)$ . Then, for all  $a \in \mathcal{A}$ , we have

$$T(Q_r(a) \otimes \iota)(U) = (Q_r(a) \otimes \iota)(V)T,$$

hence,  $T\pi_U(a) = \pi_V(a)T$ . As  $\mathcal{A}$  is norm-dense in  $\mathcal{A}_u$ , we get  $T\pi_U(x) = \pi_V(x)T$  for all  $x \in \mathcal{A}_u$ , that is  $T \in \text{Mor}(\pi_U, \pi_V)$ . Conversely, if  $T \in \text{Mor}(\pi_U, \pi_V)$ , then, using that  $Q(\mathcal{A})$  is norm-dense in  $\hat{\mathcal{M}}_*$ , one readily sees that  $T \in \text{Mor}(U, V)$ .

It is obvious that  $F$  and  $G$  are adjoint- and unit-preserving. To show that  $F$  and  $G$  are monoidal, that is, preserve tensor products, it is enough to show that  $F(U \times V) = F(U) \times F(V)$ , that is,  $\pi_{U \times V} = \pi_U \times \pi_V$ , where  $U$  and  $V$  are unitary corepresentations of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$ .

Let  $a, b, f \in \mathcal{A}, \xi \in \mathcal{H}_U, \eta \in \mathcal{H}_V$ . Then we have

$$[(\pi_U \times \pi_V)(a)](\pi_U(b)\xi \otimes \pi_V(f)\eta) = [(\pi_U \times \pi_V)(a)](\pi_U(b) \odot \pi_V(f))(\xi \otimes \eta)$$

$$= (\pi_U \odot \pi_V)(\Delta(a)(b \otimes f))(\xi \otimes \eta).$$

On the other hand, choose  $c \in A$  such that  $\hat{c} \widehat{S(a^*)} = \widehat{S(a^*)}$ , so we have  $Q_r(a) = \omega_{\Lambda(a), \Lambda(c)}$  (restricted to  $\hat{\mathcal{A}}_r$ ). Then

$$\pi_{U \times V}(a) = (Q_r(a) \otimes \iota \otimes \iota)(U \times V) = (\omega_{\Lambda(a), \Lambda(c)} \otimes \iota \otimes \iota)(V_{13}U_{12}).$$

Now, choose  $h_k, g_k \in \mathcal{A}$ , ( $k = 1 \dots n$ ), such that  $\Delta(a)(b \otimes I) = \sum_k h_k \otimes g_k$ .

Further, for each  $k$ , choose  $f_l, g_k^l \in \mathcal{A}$  ( $l = 1 \dots n_k$ ), such that

$$\Delta(g_k)(f \otimes I) = \sum_l f_l \otimes g_k^l.$$

We then get

$$\begin{aligned} & ([(\pi_{U \times V})(a)](\pi_U(b)\xi \otimes \pi_V(f)\eta), \xi' \otimes \eta') \\ &= (V_{13}U_{12}(\Lambda(a) \otimes \pi_U(b)\xi \otimes \pi_V(f)\eta), \Lambda(c) \otimes \xi' \otimes \eta') \end{aligned}$$

$$= \sum_k (V_{13}(\Lambda(g_k) \otimes \pi_U(h_k)\xi \otimes \pi_V(f)\eta), \Lambda(c) \otimes \xi' \otimes \eta') \quad (\text{using the relation between } U \text{ and } \pi_U)$$

$$= \sum_{k,l} (\Lambda(g_k^l) \otimes \pi_U(h_k)\xi \otimes \pi_V(f_l)\eta), \Lambda(c) \otimes \xi' \otimes \eta') \quad (\text{using the relation between } V \text{ and } \pi_V)$$

$$= \sum_{k,l} \varphi(c^* g_k^l) ((\pi_U \odot \pi_V)(h_k \otimes f_l)(\xi \otimes \eta), \xi' \otimes \eta'),$$

for all  $\xi' \in \mathcal{H}_U, \eta' \in \mathcal{H}_V$ .

Therefore, it suffices to show that

$$\Delta(a)(b \otimes f) = \sum_{k,l} \varphi(c^* g_k^l)(h_k \otimes f_l).$$

Now, we have

$$\Delta(a)(b \otimes f) = \Delta(a)(b \otimes I)(I \otimes f),$$

while

$$\begin{aligned} \sum_{k,l} \varphi(c^* g_k^l)(h_k \otimes f_l) &= \sum_k (\iota \odot \iota \odot \varphi c^*)(h_k \otimes (\sum_l f_l \otimes g_k^l)) \\ &= \sum_k (\iota \odot \iota \odot \varphi c^*)(h_k \otimes \Delta(g_k)(f \otimes I)) \\ &= \sum_k h_k \otimes (\iota \odot \varphi)((I \otimes c^*)\Delta(g_k)(f \otimes I)) \\ &= (\sum_k h_k \otimes (\iota \odot \varphi)((I \otimes c^*)\Delta(g_k)))(I \otimes f). \end{aligned}$$

Hence, this reduces to showing

$$\Delta(a)(b \otimes I) = \sum_k h_k \otimes (\iota \circ \varphi)((I \otimes c^*)\Delta(g_k)).$$

Now, using the previous lemma, we have

$$\begin{aligned} \Delta(a)(b \otimes I) &= ((\iota \circ \iota \circ \varphi)((I \otimes I \otimes c^*)(\iota \circ \Delta)\Delta(a)))(b \otimes I) \\ &= (\iota \circ \iota \circ \varphi)((I \otimes I \otimes c^*)(\iota \circ \Delta)\Delta(a)(b \otimes I \otimes I)) \\ &= (\iota \circ \iota \circ \varphi)((I \otimes I \otimes c^*)(\iota \circ \Delta)(\Delta(a)(b \otimes I))) \\ &= (\iota \circ \iota \circ \varphi)((I \otimes I \otimes c^*)(\iota \circ \Delta)(\sum_k h_k \otimes g_k)) \\ &= (\iota \circ \iota \circ \varphi)((I \otimes I \otimes c^*)(\sum_k h_k \otimes \Delta(g_k))) \\ &= (\iota \circ \iota \circ \varphi)(\sum_k h_k \otimes (I \otimes c^*)\Delta(g_k)) \\ &= \sum_k h_k \otimes (\iota \circ \varphi)((I \otimes c^*)\Delta(g_k)), \end{aligned}$$

which finishes the proof.  $\square$

We may dualize this result by using Pontryagin's duality for algebraic quantum groups [31, 20]. Attached to  $(\hat{\mathcal{A}}, \hat{\Delta})$ , we can first associate an injective homomorphism  $\hat{Q}_r : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}_r^* \simeq \mathcal{A}_r^*$  which is determined by

$$\hat{Q}_r(\hat{a})(\pi_r(b)) = \hat{a}(S(b)) = \varphi(S(b)a)$$

for all  $a, b \in \mathcal{A}$ . Then we get a functor  $\hat{F} : \text{Corep}(\mathcal{A}_r, \Delta_r) \rightarrow \text{Rep}(\hat{\mathcal{A}}_u, \hat{\Delta}_{u,\text{op}})$  determined on objects by

$$\hat{F}(U)(\hat{a}) = (\hat{Q}_r(\hat{a}) \otimes \iota)U, \quad \hat{a} \in \hat{\mathcal{A}},$$

and acting trivially on arrows, which is an isomorphism of tensor  $C^*$ -categories and satisfies  $F(W) = \hat{\pi}_r, \hat{F}(I \otimes 1) = \hat{\varepsilon}_u$ . We will write  $\hat{\pi}_U$  for  $\hat{F}(U)$  in the sequel.

### 3.3 The absorbing property for $\pi_r$ and $\hat{W}$

We show that  $\pi_r$  and  $\hat{W}$  have an absorbing property with respect to tensoring, which is analogous to Fell's classical result for the regular representation of a group [14].

**Proposition 3.4.** *Let  $U$  be a unitary corepresentation of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  and  $I_U = I \otimes I_{\mathcal{H}_U}$  be the trivial unitary corepresentation of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$ . Then  $U \times \hat{W}$  and  $I_U \times \hat{W}$  are equivalent objects in  $\mathcal{C}$ .*

*Proof.* Set  $T = \chi U^* \in M(B_0(\mathcal{H}_U) \otimes \hat{\mathcal{A}}_r) \subset B(\mathcal{H}_U \otimes \mathcal{H})$ . It suffices to check that this unitary satisfies the relation

$$T(\omega \bar{\otimes} \iota)(U \times \hat{W}) = (\omega \bar{\otimes} \iota)(I_U \times \hat{W})T, \quad \forall \omega \in \hat{\mathcal{M}}_* .$$

After some manipulations, this relation reduces to

$$(\omega \bar{\otimes} \iota \bar{\otimes} \iota)(U_{32}^* \hat{W}_{13} U_{12}) = (\omega \bar{\otimes} \iota \bar{\otimes} \iota)(\hat{W}_{13} U_{32}^*).$$

Now, using the fact that  $\hat{W}$  is a multiplicative unitary, we get

$$\hat{W}_{13} U_{12} = U_{32} \hat{W}_{13} U_{32}^* \in M(\hat{\mathcal{A}}_r \otimes B_0(\mathcal{H}_U \otimes \mathcal{H})).$$

Thus, we have  $U_{32}^* \hat{W}_{13} U_{12} = \hat{W}_{13} U_{32}^*$ , and the result clearly follows.  $\square$

Combining this result with Theorem 3.3, one gets at once that  $\pi \times \pi_r$  is equivalent to  $I_\pi \times \pi_r$  for every  $\pi \in \text{Rep}(\mathcal{A}_u, \Delta_u)$ , where  $I_\pi \in \text{Rep}(\mathcal{A}_u, \Delta_u)$  is given by  $I_\pi(a) = \varepsilon_u(a) I_{\mathcal{H}_\pi}$ ,  $\forall a \in \mathcal{A}_u$ .

By duality, we also have a similar result for  $\hat{\pi}_r$  and  $W$ .

## 4 Conjugate and Hilbert-Schmidt corepresentations

In this section, we define the conjugate and the Hilbert-Schmidt corepresentations associated with a unitary corepresentation. Such objects play an important role in the classical representation theory for groups and we will need these concepts in later sections.

### 4.1 Conjugate corepresentations

Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . Let  $\overline{\mathcal{H}}_U$  be any Hilbert space such that there exists an anti-unitary map  $J : \mathcal{H}_U \rightarrow \overline{\mathcal{H}}_U$ . Define then  $j : B(\mathcal{H}_U) \rightarrow B(\overline{\mathcal{H}}_U)$  by  $j(x) = Jx^*J^*$ ,  $\forall x \in B(\mathcal{H}_U)$ . Then  $j$  is linear, unital, normal, isometric,  $*$ -preserving and anti-multiplicative, with inverse  $j^{-1}(\bar{x}) = J^*\bar{x}^*J$ ,  $\bar{x} \in B(\overline{\mathcal{H}}_U)$ . Note that  $j(B_0(\mathcal{H}_U)) = B_0(\overline{\mathcal{H}}_U)$ .

We may then define

$$\overline{U} = (R \otimes j)U \in M(\mathcal{A}_r \otimes B_0(\overline{\mathcal{H}}_U)).$$

**Proposition 4.1.**  $\overline{U}$  is a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ , with  $\mathcal{H}_{\overline{U}} = \overline{\mathcal{H}}_U$ .

*Proof.* We have

$$\overline{U}^* \overline{U} = ((R \otimes j)U)^*((R \otimes j)U) = (R \otimes j)(UU^*) = (R \otimes j)(I) = I_{M(\mathcal{A}_r)} \otimes I_{\mathcal{H}_U} ,$$



and similarly

$$\overline{UU^*} = I_{M(\mathcal{A}_r)} \otimes I_{\overline{\mathcal{H}}_U}.$$

Furthermore,

$$\begin{aligned} (\Delta_r \otimes \iota)\overline{U} &= (\Delta_r R \otimes j)U = (\chi(R \otimes R) \otimes j)(\Delta_r \otimes \iota)U = (\chi(R \otimes R) \otimes j)(U_{13}U_{23}) \\ &= (\chi(R \otimes R) \otimes j)(U_{23})(\chi(R \otimes R) \otimes j)(U_{13}) = ((R \otimes j)U)_{13}((R \otimes j)U)_{23} = \overline{U}_{13}\overline{U}_{23}. \end{aligned}$$

□

*Remark.* We clearly have  $\overline{\overline{U}} \simeq U$ .

Assume now that  $(\mathcal{A}, \Delta)$  is of compact type and let  $U$  be an irreducible unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ , which is then necessarily finite-dimensional [33, 34].

We will show that the conjugate of  $U$ , as defined above, agrees with the conjugate of  $U$  as defined by J. Roberts and L. Tuset [27]. We first recall their definition.

Let  $\tilde{U} \in \mathcal{A} \otimes B(\mathcal{H}_U)$  be given by  $\tilde{U} = (* \circ \tilde{j})U$ , where  $\tilde{j}(x) = \tilde{J}x\tilde{J}^{-1}$  for all  $x \in B(\mathcal{H}_U)$ , and  $\tilde{J} : \mathcal{H}_U \rightarrow \overline{\mathcal{H}}_U$  is any anti-linear invertible operator such that  $\tilde{J}^*\tilde{J}$  intertwines  $U$  and  $(S^2 \circ \iota)U$ . Recall here that  $S^2(a) = f_{-1} * a * f_1$ ,  $a \in \mathcal{A}$ . Then  $(\Delta \circ \iota)\tilde{U} = \tilde{U}_{13}\tilde{U}_{23}$  holds as  $\Delta$  is  $*$ -preserving and  $\tilde{j}$  is multiplicative. The fact that  $\tilde{U}$  is unitary is shown in [27]. (Note that  $(S^2 \circ \iota)U$  is the double contragredient representation; it is not unitary.)

**Proposition 4.2.** *Assume that  $(\mathcal{A}, \Delta)$  is of compact type and let  $U$  be an irreducible unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . Let  $\overline{U} = (R \otimes j)U$  denote the conjugate of  $U$  as we have defined it before. Let then  $\tilde{J} : \mathcal{H}_U \rightarrow \overline{\mathcal{H}}_U$  be defined by  $\tilde{J} = ((f_{1/2} \circ j)U)J$ . Then  $\tilde{J}$  is an anti-linear invertible operator such that  $\tilde{J}^*\tilde{J}$  intertwines  $U$  and  $(S^2 \circ \iota)U$ . Further, we have  $\overline{U} = \tilde{U}$ , where  $\tilde{U}$  is defined as above.*

*Proof.* It is easy to check that  $(f_{1/2} \circ j)(U^*)$  is invertible with inverse given by  $((f_{1/2} \circ j)U^*)^{-1} = (f_{1/2} \circ j)U$ .

We set  $V = I \otimes ((f_{1/2} \circ j)U^*)$ . Then  $(V^*)^{-1} = I \otimes (f_{-1/2} \circ j)(U^*)$ .

Recall from [20] that  $R = S\tau_{i/2} = \tau_{i/2}S$ , where  $\tau_{i/2}(a) = f_{1/2} * a * f_{-1/2}$ .

Therefore we have

$$\begin{aligned} \overline{U} &= (\tau_{i/2} \circ j)(S \circ \iota)U = (\tau_{i/2} \circ j)U^* \\ &= (f_{-1/2} \circ \iota \circ f_{1/2} \circ j)((\Delta \circ \iota)\Delta) \circ \iota)U^* \\ &= (f_{-1/2} \circ \iota \circ f_{1/2} \circ j)((\Delta \circ \iota \circ \iota)(U_{23}^*U_{13}^*)) \\ &= (f_{-1/2} \circ \iota \circ f_{1/2} \circ j)(I \otimes I \otimes U^*)(\Delta \circ \iota \circ \iota)(U_{13}^*) \end{aligned}$$

$$\begin{aligned}
&= ((f_{-1/2} \odot \iota \odot f_{1/2} \odot j)(\Delta \odot \iota \odot \iota)(U_{13}^*))V \\
&= ((f_{-1/2} \odot \iota \odot j)(\Delta \odot \iota)U^*)V \\
&= ((f_{-1/2} \odot \iota \odot j)(U_{23}^*U_{13}^*))V \\
&= ((f_{-1/2} \odot \iota \odot j)(U_{13}^*)(f_{-1/2} \odot \iota \odot j)(U_{23}^*))V \\
&= (I \otimes (f_{-1/2} \odot j)(U^*))(\iota \odot j)(U^*)V \\
&= (V^*)^{-1}(\iota \odot j)(U^*)V.
\end{aligned}$$

From this, we get

$$\bar{U} = (* \odot \tilde{j})U = (* \odot \tilde{J} \cdot \tilde{J}^{-1})U = \bar{U},$$

(using that  $(f_{1/2} \odot j)U = ((f_{-1/2} \odot j)U)^{-1} = ((f_{1/2} \odot j)(U^*))^{*-1}$ , the first equality here relying on

$$\begin{aligned}
I_{\mathcal{H}_U} &= (\varepsilon \odot \iota)U = (f_{1/2} \odot f_{-1/2} \odot \iota)(\Delta \odot \iota)U = (f_{1/2} \odot f_{-1/2} \odot \iota)(U_{13}U_{23}) \\
&= ((f_{1/2} \odot \iota)U)((f_{-1/2} \odot \iota)U).
\end{aligned}$$

Finally, we check that  $\tilde{J}^* \tilde{J}$  intertwines  $U$  and  $(S^2 \odot \iota)U$ . Observe first that

$$\begin{aligned}
\tilde{J}^* \tilde{J} &= J^*((f_{1/2} \odot j)U)^*((f_{1/2} \odot j)U)J = J^*((f_{-1/2} \odot j)U^*)((f_{1/2} \odot j)U)J \\
&= J^*((f_{-1/2} \odot j)U)^{-1}((f_{1/2} \odot j)U)J = J^*((f_{1/2} \odot j)U)((f_{1/2} \odot j)U)J \\
&= J^*((f_1 \odot j)U)J = ((f_1 \odot \iota)U)^* = F_U^* = F_U,
\end{aligned}$$

where  $F_U = (f_1 \odot \iota)U$ .

Now, inserting  $a = (\iota \odot \omega)U$  in  $S^2(a) = f_{-1} * a * f_1$ , we get

$$\begin{aligned}
(S^2 \odot \omega)U &= S^2((\iota \odot \omega)U) = f_{-1} * (\iota \odot \omega)U * f_1 \\
&= (f_1 \odot \iota \odot f_{-1} \odot \omega)[(\Delta \odot \iota)\Delta] \odot \iota U \\
&= (f_1 \odot \iota \odot f_{-1} \odot \omega)(\Delta \odot \iota \odot \iota)(\Delta \odot \iota)U,
\end{aligned}$$

for all  $\omega \in B(\mathcal{H}_U)^*$ . Therefore, we have

$$(S^2 \odot \iota)U = (f_1 \odot \iota \odot f_{-1} \odot \iota)(\Delta \odot \iota \odot \iota)(U_{13}U_{23}) = [I \otimes (f_1 \odot \iota)U]U[I \otimes (f_{-1} \odot \iota)U],$$

hence  $(S^2 \odot \iota)U = (I \otimes F_U)U(I \otimes F_U^{-1})$ . Thus  $\tilde{J}^* \tilde{J} = F_U$  intertwines  $U$  and  $(S^2 \odot \iota)U$ , as claimed.  $\square$

## 4.2 Hilbert-Schmidt corepresentations

Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . We introduce the Hilbert-Schmidt corepresentation  $U_{HS}$  associated with  $U$  and show that  $\overline{U} \times U \simeq U_{HS}$ .

We let  $J$  and  $j$  be as in the previous subsection and denote the Hilbert-Schmidt operators acting on  $\mathcal{H}_U$  by  $HS(\mathcal{H}_U)$ . We recall that  $HS(\mathcal{H}_U)$  is a Hilbert space with inner product  $(x, y) = Tr(y^*x)$ ,  $x, y \in HS(\mathcal{H}_U)$ , where  $Tr$  denotes the canonical trace on  $B(\mathcal{H}_U)$ .

We define first a unitary  $\mathcal{V} : \overline{\mathcal{H}}_U \otimes \mathcal{H}_U \rightarrow HS(\mathcal{H}_U)$  by

$$\mathcal{V}(\eta \otimes \xi)(\xi') = (\xi', J^*\eta)_{\mathcal{H}_U} \xi = (\eta, J\xi')_{\overline{\mathcal{H}}_U} \xi, \quad \xi, \xi' \in \mathcal{H}_U, \eta \in \overline{\mathcal{H}}_U.$$

Define then a normal unital  $*$ -isomorphism  $\tilde{\mathcal{V}} : B(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U) \rightarrow B(HS(\mathcal{H}_U))$  by

$$\tilde{\mathcal{V}}(X) = \mathcal{V}X\mathcal{V}^*, \quad X \in B(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U).$$

Note that  $\tilde{\mathcal{V}}(B_0(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U)) = B_0(HS(\mathcal{H}_U))$ . Further, let

$$\iota \otimes \tilde{\mathcal{V}} : M(\mathcal{A}_r \otimes B_0(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U)) \rightarrow M(\mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U)))$$

denote the canonical extension of

$$\iota \otimes \tilde{\mathcal{V}} : \mathcal{A}_r \otimes B_0(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U) \rightarrow \mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U)) \subset M(\mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U))).$$

It is clear that  $\iota \otimes \tilde{\mathcal{V}}$  is a unital  $*$ -isomorphism.

Define then  $U_{HS} \in M(\mathcal{A}_r \otimes B_0(HS(\mathcal{H}_U)))$  by

$$U_{HS} = (\iota \otimes \tilde{\mathcal{V}})(\overline{U} \times U).$$

**Proposition 4.3.**  $U_{HS}$  is a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ , with  $\mathcal{H}_{U_{HS}} = HS(\mathcal{H}_U)$ , which is equivalent to  $\overline{U} \times U$ .

*Proof.*  $U_{HS}$  is unitary as  $\iota \otimes \tilde{\mathcal{V}}$  is a unital  $*$ -isomorphism and  $\overline{U} \times U$  is a unitary. Moreover,

$$(\Delta_r \otimes \iota)U_{HS} = (\iota \otimes \iota \otimes \tilde{\mathcal{V}})(\Delta_r \otimes \iota)(\overline{U} \times U) = (U_{HS})_{13}(U_{HS})_{23}$$

since  $\overline{U} \times U$  satisfies the corepresentation property and  $\iota \otimes \iota \otimes \tilde{\mathcal{V}}$  is multiplicative. Finally, as  $(\omega \otimes \iota)U_{HS}\mathcal{V} = \mathcal{V}(\omega \otimes \iota)(\overline{U} \times U)$ ,  $\omega \in \mathcal{M}_*$ , we see that  $U_{HS}$  is equivalent to  $\overline{U} \times U$  with unitary  $\mathcal{V} \in Mor(U_{HS}, \overline{U} \times U)$ .  $\square$

It will be useful for us later to have another way of looking at  $U_{HS}$ .

Let  $l$  (resp.  $r$ ) :  $B(\mathcal{H}_U) \rightarrow B(HS(\mathcal{H}_U))$  be the normal  $*$ -homomorphism (resp.  $*$ -antihomomorphism) defined by

$$l(x)(y) = xy \quad (\text{resp. } r(x)(y) = yx), \quad x \in B(\mathcal{H}_U), y \in HS(\mathcal{H}_U).$$

It is then straightforward to check that

$$\begin{aligned}\mathcal{V}(I \otimes x)\mathcal{V}^* &= l(x), \quad x \in B(\mathcal{H}_U), \\ \mathcal{V}(z \otimes I)\mathcal{V}^* &= r(j^{-1}(z)), \quad z \in B(\overline{\mathcal{H}}_U).\end{aligned}$$

Using these relations, one easily gets

$$\begin{aligned}(I \otimes \mathcal{V})X_{13}(I \otimes \mathcal{V}^*) &= (\iota \bar{\otimes} l)X, \quad X \in B(\mathcal{H}) \bar{\otimes} B(\mathcal{H}_U), \\ (I \otimes \mathcal{V})Z_{12}(I \otimes \mathcal{V}^*) &= (\iota \bar{\otimes} r j^{-1})Z, \quad Z \in B(\mathcal{H}) \bar{\otimes} B(\overline{\mathcal{H}}_U).\end{aligned}$$

Now, regarding  $U \in \mathcal{M} \bar{\otimes} B(\mathcal{H}_U) \subset B(\mathcal{H}) \bar{\otimes} B(\mathcal{H}_U)$ , we have :

**Proposition 4.4.**  $U_{HS} = (\iota \bar{\otimes} l)U(R \bar{\otimes} r)U$ .

*Proof.* Indeed,

$$\begin{aligned}U_{HS} &= (I \otimes \mathcal{V})U_{13}\overline{U}_{12}(I \otimes \mathcal{V}^*) \\ &= (I \otimes \mathcal{V})U_{13}(I \otimes \mathcal{V}^*)(I \otimes \mathcal{V})\overline{U}_{12}(I \otimes \mathcal{V}^*) \\ &= (\iota \bar{\otimes} l)U(\iota \bar{\otimes} r j^{-1})(R \bar{\otimes} j)U = (\iota \bar{\otimes} l)U(R \bar{\otimes} r)U.\end{aligned}$$

□

*Remark.* One may also associate with  $U$  another Hilbert-Schmidt corepresentation  $U_{HS'}$  of  $(\mathcal{A}_r, \Delta_r)$  on  $HS(\mathcal{H}_U)$  which is given by

$$U_{HS'} = (\iota \otimes \check{\mathcal{V}}\chi)(U \times \overline{U}),$$

where  $\chi$  denotes the flip map from  $B(\mathcal{H}_U \otimes \overline{\mathcal{H}}_U)$  to  $B(\overline{\mathcal{H}}_U \otimes \mathcal{H}_U)$ . One easily checks that  $U_{HS'} \simeq U \times \overline{U} \simeq (\overline{U})_{HS}$ , and that  $U_{HS'} = (R \bar{\otimes} r)U(\iota \bar{\otimes} l)U$ . The two Hilbert-Schmidt corepresentations associated with  $U$  agree when  $\mathcal{A}$  is commutative, but it is unclear whether or when they are equivalent in the non-commutative case.

## 5 Co-amenable unitary corepresentations

Inspired by [5, Theorem 2.5] and [6, Theorem 4.2], we introduce the notion of co-amenable for unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$ .

**Definition 5.1.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group. A unitary corepresentation  $U$  of  $(\mathcal{A}_r, \Delta_r)$  is said to be co-amenable if there exists  $\phi \in S(\mathcal{A}_r)$  such that  $(\phi \otimes \iota)U = I_{\mathcal{H}_U}$ .*

Note that we can equivalently require that  $\phi \in S(B(\mathcal{H}))$  in this definition. The following result shows that this definition is consistent with the notion of co-amenable for algebraic quantum groups. Recall from [6, 7] (see [5] for the compact case) that an algebraic quantum group  $(\mathcal{A}, \Delta)$  is *co-amenable* if its co-unit  $\varepsilon$  is bounded with respect to the norm on  $\mathcal{A}$  given by  $\|a\| = \|\pi_r(a)\|$ ,  $a \in \mathcal{A}$ . Equivalently,  $(\mathcal{A}, \Delta)$  is co-amenable if there exists a bounded linear functional  $\varepsilon_r : \mathcal{A}_r \rightarrow \mathbb{C}$  such that  $(\iota \otimes \varepsilon_r)\Delta_r = (\varepsilon_r \otimes \iota)\Delta_r = \iota$ . The map  $\varepsilon_r$  is then a  $*$ -homomorphism from  $\mathcal{A}_r$  onto  $\mathbb{C}$  and the existence of such a homomorphism characterizes the co-amenable of  $(\mathcal{A}, \Delta)$ .

**Theorem 5.2.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group. Then the following conditions are equivalent:*

- (1)  $(\mathcal{A}, \Delta)$  is co-amenable;
- (2)  $W$  is co-amenable (as a corepresentation);
- (3) all unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are co-amenable.

*Proof.* The equivalence (1)  $\Leftrightarrow$  (2) follows by [6, Theorem 4.2]. The implication (3)  $\Rightarrow$  (2) is obvious. In order to show (1)  $\Rightarrow$  (3), we set  $\phi = \varepsilon_r$  and let  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . Then

$$\begin{aligned} U &= (\iota \otimes \varepsilon_r \otimes \iota)(\Delta_r \otimes \iota)U \\ &= (\iota \otimes \varepsilon_r \otimes \iota)(U_{13}U_{23}) \\ &= U(I \otimes (\varepsilon_r \otimes \iota)U) \end{aligned}$$

(here  $I$  denotes the unit of  $M(\mathcal{A}_r)$ ). Multiplying by  $U^*$  from the left, we get  $I \otimes I_{\mathcal{H}_U} = I \otimes (\varepsilon_r \otimes \iota)U$  and therefore  $(\varepsilon_r \otimes \iota)U = I_{\mathcal{H}_U}$ .  $\square$

The next result may be seen as an analog of Day's classical characterization of the amenability of a group.

**Proposition 5.3.** *let  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . Then the following conditions are equivalent:*

- (1)  $U$  is co-amenable;
- (2) there exists a net  $(v_i)$  of unit vectors in  $\mathcal{H}$  such that

$$\lim_i \|U(v_i \otimes \xi) - v_i \otimes \xi\|_2 = 0, \quad \forall \xi \in \mathcal{H}_U$$

*Proof.* (2)  $\Rightarrow$  (1): By weak\*-compactness of  $S(B(\mathcal{H}))$ , the net of vector states  $(\omega_{v_i})$  has an accumulation point  $\phi$  in  $S(B(\mathcal{H}))$ . Passing to a subnet of  $(v_i)$  if necessary, we may suppose that  $\phi(x) = \lim_i(xv_i, v_i)$ ,  $x \in B(\mathcal{H})$ .

Now, by assumption, we have  $\lim_i \|U(v_i \otimes \xi) - v_i \otimes \xi\|_2 = 0$ , for all  $\xi \in \mathcal{H}_U$ . Thus

$$\begin{aligned} \omega_\xi((\phi \otimes \iota)U) &= \phi((\iota \otimes \omega_\xi)U) = \lim_i((\iota \otimes \omega_\xi)(U)v_i, v_i) \\ &= \lim_i(U(v_i \otimes \xi), v_i \otimes \xi) = \lim_i(v_i \otimes \xi, v_i \otimes \xi) = \omega_\xi(I) \end{aligned}$$

for every  $\xi \in \mathcal{H}_U$ . Since the set of vector states  $\omega_\xi$  separates the elements of  $B(\mathcal{H})$ , it follows that  $(\phi \otimes \iota)U = I$ .

(1)  $\Rightarrow$  (2): Let  $\phi \in S(B(\mathcal{H}))$  be such that  $(\phi \otimes \iota)U = I$ . As  $\mathcal{M}$  acts standardly on  $\mathcal{H}$ , there exists a net  $(v_i)$  of unit vectors in  $\mathcal{H}$  such that  $\phi(x) = \lim_i(xv_i, v_i)$ ,  $x \in \mathcal{M}$ . Then, for all  $v \in \mathcal{H}_U$ ,

$$\begin{aligned} \lim_i(U(v_i \otimes \xi), v_i \otimes \xi) &= \lim_i((\iota \otimes \omega_\xi)(U)v_i, v_i) = \phi((\iota \otimes \omega_\xi)U) = \omega_\xi((\phi \otimes \iota)U) \\ &= \omega_\xi(I) = (\xi, \xi) = \lim_i(v_i \otimes \xi, v_i \otimes \xi). \end{aligned}$$

The conclusion follows easily from this.  $\square$

We may use the results in section 3.1 to transfer the notion of co-amenable from corepresentations to representations : the  $*$ -representation  $\hat{\pi}_U$  of  $\hat{\mathcal{A}}_u$  associated to a unitary corepresentation  $U$  of  $(\mathcal{A}_r, \Delta_r)$  is said to be *co-amenable* if  $U$  is co-amenable. We have for the moment no intrinsic characterization of this notion.

Finally, concerning compact matrix pseudogroups [33], we mention:

**Proposition 5.4.** *Suppose that  $(\mathcal{A}_r, \Delta_r)$  is a compact matrix pseudogroup with fundamental unitary corepresentation  $U$  (so  $U$  is finite-dimensional with matrix elements generating  $\pi_r(\mathcal{A})$  as a  $*$ -algebra). Then  $U$  is co-amenable if, and only if,  $(\mathcal{A}, \Delta)$  is co-amenable.*

*Proof.* This result is merely a restatement of [5, Theorem 2.5]. A sketch of the argument is as follows. Write  $U = \sum_{i,j} u_{ij} \otimes e_{ij}$ , where  $u_{ij} \in \mathcal{A}$  and the  $e_{ij}$ 's form a usual system of matrix units for  $B(\mathcal{H}_U)$ . If  $\phi \in S(\mathcal{A}_r)$ , then  $(\phi \otimes \iota)U = I_{\mathcal{H}_U}$  if, and only if,  $\phi(u_{ij}) = \delta_{ij}$  for all  $i, j$  if, and only if,  $\phi|_{\mathcal{A}} = \varepsilon$ .  $\square$

In fact, more generally, [5, Theorem 2.5] may be restated as saying that if  $(\mathcal{A}, \Delta)$  is of compact type and  $U$  is a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$  such that its matrix elements generate  $\mathcal{A}_r$  as a  $C^*$ -algebra, then  $(\mathcal{A}, \Delta)$  is co-amenable if, and only if,  $U$  is co-amenable.

## 6 Amenable unitary corepresentations

We first recall the following definition due to M. Bekka [8]. A continuous unitary representation  $u$  of a locally compact group  $G$  on a Hilbert space  $\mathcal{H}_u$  is called *amenable* if there exists an invariant ‘‘mean’’ on  $B(\mathcal{H}_u)$ , that is, if there exists  $m_u \in S(B(\mathcal{H}_u))$  such that

$$m_u(u_g x u_g^*) = m_u(x), \quad \forall x \in B(\mathcal{H}_u), \forall g \in G.$$

This definition has no obvious counterpart in the quantum group setting. However, Bekka also introduces a notion of ‘‘topological’’ invariant mean whose existence is equivalent to the amenability of  $u$ , see [8, Theorem 3.5]. Inspired by this result, we introduce the following notion:

**Definition 6.1.** *A unitary corepresentation  $U$  of  $(\mathcal{A}_r, \Delta_r)$  is called left-amenable (resp. right-amenable) if there exists  $m_U$  (resp.  $m'_U$ )  $\in S(B(\mathcal{H}_U))$  such that*

$$m_U((\omega \bar{\otimes} \iota)(U^*(I \otimes x)U)) = \omega(I)m_U(x)$$

$$(\text{resp. } m'_U((\omega \bar{\otimes} \iota)(U(I \otimes x)U^*)) = \omega(I)m'_U(x))$$

for all  $x \in B(\mathcal{H}_U), \omega \in \mathcal{M}_*$ .

The state  $m_U$  (resp.  $m'_U$ ) is called a left-invariant (resp. right-invariant) mean for  $U$ .

*Remarks.*

(i) When  $U$  is the unitary corepresentation of  $(C_0(\Gamma), \Delta)$  associated to a unitary representation  $u$  of a discrete group  $\Gamma$ , one easily checks that  $U$  is left-amenable (resp.  $U$  is right-amenable) if, and only if,  $u$  is amenable. This is a simple consequence of the fact that  $(\delta_\gamma \bar{\otimes} \iota)U = u_\gamma$ , where the delta function at  $\gamma \in \Gamma$ ,  $\delta_\gamma$ , is considered as an element of  $\ell^1(\Gamma)$ , that is, of the predual of  $\ell^\infty(\Gamma)$ . Further, in this case, it is quite obvious that a left- (resp. right-) invariant mean for  $U$  is both left- and right-invariant.

(ii) We don't know whether the existence of a left-invariant mean for  $U$  is equivalent to the existence of a right-invariant one in the general situation. However, we have  $U$  is left- (resp. right-) amenable if, and only if,  $\bar{U}$  is right- (resp. left-) amenable.

Indeed, if  $m_U$  is a left-invariant mean for  $U$ , then  $m_U \circ j^{-1}$  is a right-invariant mean for  $\bar{U}$ . If  $m'_U$  is a right-invariant mean for  $\bar{U}$ , then  $m'_U \circ j$  is a left-invariant mean for  $U$ . The resp. assertions are proven similarly.

(iii) The property of left-amenable (resp. right-amenable) is clearly invariant under unitary equivalence.

(iv) By “linearizing” the concept of amenability, one gets a related, but seemingly independent, notion: a unitary corepresentation  $U$  of  $(\mathcal{A}_r, \Delta_r)$  is said to be *hypertracial* if there exists  $m''_U \in \mathcal{S}(B(\mathcal{H}_U))$  such that

$$m''_U((\omega \bar{\otimes} \iota)(U(I \otimes x))) = m''_U((\omega \bar{\otimes} \iota)((I \otimes x)U)), \quad \forall x \in B(\mathcal{H}_U), \forall \omega \in \mathcal{M}_*. \quad (1)$$

Actually, condition (1) is equivalent to

$$m''_U((\omega \bar{\otimes} \iota)(U)x) = m''_U(x(\omega \bar{\otimes} \iota)(U)) \quad \forall x \in B(\mathcal{H}_U), \forall \omega \in \mathcal{M}_*,$$

which in turn is equivalent to

$$m'_U(\hat{\pi}_U(a)x) = m'_U(x\hat{\pi}_U(a)), \quad \forall x \in B(\mathcal{H}_U), \quad \forall a \in \mathcal{A}_u.$$

Hence, hypertraciality of  $U$  is equivalent to hypertraciality of  $\hat{\pi}_U$  in the sense of [4]. This hypertrace property is easily seen to correspond to left- and right-amenable in the case of a corepresentation arising from a unitary representation of a discrete group.

Now recall from [6, 7] that an algebraic quantum group  $(\mathcal{A}, \Delta)$  is called *amenable* if there exists a left-invariant mean for  $(\mathcal{A}, \Delta)$ , that is, if there exists  $m \in \mathcal{S}(\mathcal{M})$  such that

$$m((\omega \bar{\otimes} \iota)\Delta_r(x)) = \omega(I)m(x), \quad \forall x \in \mathcal{M}, \forall \omega \in \mathcal{M}_*.$$

A right-invariant mean for  $(\mathcal{A}, \Delta)$  is defined similarly. By composing with the anti-unitary antipode  $R$  (which is defined on  $\mathcal{M}$ , see [20, 21]), one easily sees that the existence of a left-invariant mean for  $(\mathcal{A}, \Delta)$  is equivalent to the existence of a right-invariant mean for it. It is then straightforward to check that  $(\mathcal{A}, \Delta_{\text{op}})$  is amenable if and only if  $(\mathcal{A}, \Delta)$  is amenable.

The following result is well known (see [8] for the equivalence between (3), (4) and (5); the equivalence between (1), (2) and (3) is merely classical, as explained in [5, 6]).

**Theorem 6.2.** *Let  $\Gamma$  be a discrete group and let  $(\mathcal{A}, \Delta)$  be the algebraic quantum group associated with  $\mathcal{A} = \mathbb{C}[\Gamma]$ , the group-algebra of  $\Gamma$ , so  $\hat{\mathcal{A}} = C_c(\Gamma)$ . Then the following are equivalent:*

- (1)  $(\hat{\mathcal{A}}, \hat{\Delta})$  is amenable;
- (2)  $(\mathcal{A}, \Delta)$  is co-amenable;
- (3)  $\Gamma$  is amenable;
- (4) the (left-) regular representation  $\lambda$  of  $\Gamma$  is amenable;
- (5) all unitary representations of  $\Gamma$  are amenable.

In the case of an algebraic quantum group, it is known [6] that (1) implies (2). The converse implication is only known to hold when  $(\mathcal{A}, \Delta)$  is compact with a tracial Haar functional [7]. We will give another proof of this result in Section 9. However, amenability of an algebraic quantum group may be characterized through amenability of its corepresentations as follows.

**Theorem 6.3.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group. Then the following conditions are equivalent:*

- (1)  $(\mathcal{A}, \Delta)$  is amenable;
- (2)  $W$  is left-amenable (as a corepresentation);
- (3)  $\overline{W}$  is right-amenable (as a corepresentation);
- (4) all unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are left-amenable.
- (5) all unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are right-amenable.

*Proof.* The implications (4)  $\Rightarrow$  (2) and (5)  $\Rightarrow$  (3) are obvious. The equivalence between (2) and (3) is just a special case of (ii) in our previous remark.

(2)  $\Rightarrow$  (1): Let  $m_W \in S(B(\mathcal{H}))$  be a left-invariant mean for  $W$ , and let  $m$  be the restriction of  $m_W$  to  $\mathcal{M}$ . Then  $m$  is clearly a state, which is left-invariant since

$$m((\omega \bar{\otimes} \iota)\Delta_r(x)) = m_W((\omega \bar{\otimes} \iota)(W^*(I \otimes x)W)) = \omega(I)m_W(x) = \omega(I)m(x),$$

for all  $x \in \mathcal{M}, \omega \in \mathcal{M}_*$ .

(1)  $\Rightarrow$  (4) and (1)  $\Rightarrow$  (5): Let  $m$  be a left-invariant mean for  $(\mathcal{A}, \Delta)$  and let  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . We pick a normal state  $\Omega$  on  $B(\mathcal{H}_U)$  and define  $m_U \in S(B(\mathcal{H}_U))$  by

$$m_U(x) = m((\iota \bar{\otimes} \Omega)(U^*(I \otimes x)U)), \quad x \in B(\mathcal{H}_U).$$



Then we check the validity of the equation expressing the left-invariance property of  $m_U$ : the *l.h.s.* is

$$m_U(x) = m(\iota\bar{\otimes}\Omega)(U^*(I \otimes [(\omega\bar{\otimes}\iota)(U^*(I \otimes x)U)])U)$$

while the *r.h.s.* is equal to

$$\begin{aligned} \omega(I)m_U(x) &= \omega(I)m((\iota\bar{\otimes}\Omega)(U^*(I \otimes x)U)) \\ &= m(\omega\bar{\otimes}\iota)\Delta((\iota\bar{\otimes}\Omega)(U^*(I \otimes x)U)) \text{ (using left-invariance of } m) \\ &= m(\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\iota\bar{\otimes}\Omega)(\Delta\bar{\otimes}\iota)(U^*(I \otimes x)U) \\ &= m(\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\iota\bar{\otimes}\Omega)((\Delta\bar{\otimes}\iota)U)^*(I \otimes I \otimes x)(\Delta\bar{\otimes}\iota)U \\ &= m(\omega\bar{\otimes}\iota)(\iota\bar{\otimes}\iota\bar{\otimes}\Omega)((U_{13}U_{23})^*(I \otimes I \otimes x)U_{13}U_{23}) \\ &= m(\iota\bar{\otimes}\Omega)(\omega\bar{\otimes}\iota\bar{\otimes}\iota)(U_{23}^*U_{13}^*(I \otimes I \otimes x)U_{13}U_{23}). \end{aligned}$$

Hence, the desired conclusion follows from the identity

$$U^*(I \otimes [(\omega\bar{\otimes}\iota)(U^*(I \otimes x)U)])U = (\omega\bar{\otimes}\iota\bar{\otimes}\iota)(U_{23}^*U_{13}^*(I \otimes I \otimes x)U_{13}U_{23})$$

which is easily verified.

Similarly, we define  $m'_U \in S(B(\mathcal{H}_U))$  by

$$m'_U(x) = m \circ R((\iota\bar{\otimes}\Omega)(U(I \otimes x)U^*)), \quad x \in B(\mathcal{H}_U).$$

Then, using the right-invariance of  $m \circ R$ , one now checks that  $m'_U$  is a right-invariant mean for  $U$ .

It clearly follows that all stated conditions are equivalent.  $\square$

We know from [6, Theorem 4.7] that co-amenability of  $(\mathcal{A}, \Delta)$  implies that  $(\hat{\mathcal{A}}, \hat{\Delta})$  is amenable, hence that  $(\hat{\mathcal{A}}, \hat{\Delta}_{op})$  is amenable. By combining this fact with Theorem 5.2 and Theorem 6.3, we see that if all the unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are co-amenable then all the unitary corepresentations of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,op})$  are amenable. This lends some evidence that there might be some correspondence between co-amenable elements in  $\text{Corep}(\mathcal{A}_r, \Delta_r)$  and amenable elements in  $\text{Corep}(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,op})$ .

By using Theorem 3.3, one may clearly transfer the notion of amenability to representations of algebraic quantum groups. Theorem 6.3 may then be reformulated in an obvious manner.

An analog of [8, Theorem 3.6], which characterizes the amenability of a unitary representation of a group, is as follows.

**Proposition 6.4.** *Let  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . Organize  $TC(\mathcal{H}_U)$ , the trace class operators on  $\mathcal{H}_U$ , as a Banach  $\mathcal{M}_*$ -module by means of*

$$Tr((\omega \cdot s)x) = Tr(s(\omega \bar{\otimes} \iota)(U^*(I \otimes x)U)),$$

$$\omega \in \mathcal{M}_*, s \in TC(\mathcal{H}_U), x \in B(\mathcal{H}_U).$$

*Then  $U$  is left-amenable if, and only if, there exists a net  $(s_i)$  in  $TC(\mathcal{H}_U)_1^+$  such that*

$$\lim_i \|\omega \cdot s_i - s_i\|_1 = 0, \quad \forall \omega \in \mathcal{M}_*.$$

*Proof.* The proof is an easy adaptation of the proof of [8, Theorem 3.6]. If  $(s_i)$  is a net as above, then a left-invariant mean for  $U$  is obtained by picking any weak\*-limit point of the net  $(m_i) \subset S(B(\mathcal{H}_U))$  given by  $m_i(\cdot) = Tr(s_i \cdot)$ . Conversely, assume that  $m_U$  is a left-invariant mean for  $U$ . As the normal states are weak\*-dense in  $S(B(\mathcal{H}_U))$ , we may pick a net  $(t_i) \subset TC(\mathcal{H}_U)_1^+$  such that  $m_U$  is weak\*-limit point of the net  $(Tr(t_i \cdot)) \subset S(B(\mathcal{H}_U))$ . Namioka's classical argument [25] gives then the existence of a net  $(s_i)$  with the required properties.  $\square$

One may clearly also obtain a similar characterization of right-amenableity for unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$ .

To illustrate the notion of invariant mean for corepresentations, we now consider the case where  $(\mathcal{A}, \Delta)$  is of compact type. Let then  $U$  be a finite-dimensional unitary representation of  $(\mathcal{A}_r, \Delta_r)$ . As  $(\mathcal{A}, \Delta)$  is amenable, see the paragraph preceding Theorem 4.7 in [6], we deduce from Theorem 6.3 that all unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are left- (and right-) amenable. We shall now describe somewhat more explicitly a left-invariant mean  $m_U$  for  $U$ , following the construction given in the proof of Theorem 6.3.

Identifying  $\mathcal{A}$  with the dense Hopf \*-subalgebra  $\pi_r(\mathcal{A})$  of  $(\mathcal{A}_r, \Delta_r)$ , we may write  $U = \sum_i a_i \otimes b_i \in \mathcal{A} \odot B(\mathcal{H}_U)$  for some  $a_1, \dots, a_N \in \mathcal{A}, b_1, \dots, b_N \in B(\mathcal{H}_U)$ . Recall that a left-invariant mean for  $U$  is provided by

$$m_U(x) = \varphi_r((\iota \bar{\otimes} \Omega)(U^*(I \otimes x)U)), \quad x \in B(\mathcal{H}),$$

where we have the freedom to choose any  $\Omega \in B(\mathcal{H}_U)_{*,1}^+$ . Set  $d_U = \dim \mathcal{H}_U$  and let  $\tau = 1/d_U \text{Tr}$ , denote the normalized trace on  $B(\mathcal{H}_U)$ . Plugging in  $\Omega = \tau$ , we get

$$\begin{aligned} m_U(x) &= \varphi_r((\iota \bar{\otimes} \tau)(U^*(I \otimes x)U)) \\ &= \sum_{i,j} \varphi(\iota \otimes \tau)(a_i^* a_j \otimes b_i^* x b_j) \\ &= \sum \varphi(a_i^* a_j) \tau(b_i^* x b_j) \\ &= 1/d_U \text{Tr} \left( \sum_{i,j} \varphi(a_i^* a_j) b_j b_i^* x \right) \end{aligned}$$

so that  $m_U(\cdot) = \text{Tr}(K_U \cdot)$ , where the density matrix  $K_U \in B(\mathcal{H}_U)$  is given by

$$K_U = 1/d_U(\varphi \odot \iota)(U(\sigma \otimes \iota)U^*), \quad (2)$$

where  $\sigma$  is the automorphism of  $\mathcal{A}$  given by  $\sigma(a) = f_1 * a * f_1$ ,  $a \in \mathcal{A}$ .

We remark that if  $\varphi$  is tracial, then  $f_1 = \varepsilon$ , hence  $\sigma = \iota$  and thereby  $K = I/d_U$ , that is, the left-invariant mean for  $U$  is just  $\tau$ .

Now assume that  $U$  is irreducible. We write  $U = \sum_{i,j} u_{ij} \otimes m_{ji}$ , where  $u_{ij} \in \mathcal{A}$  and the  $m_{ij}$ 's form a system of matrix units for  $B(\mathcal{H}_U)$  such that  $m_{lk}m_{sr} = \delta_{lr}m_{sk}$  and  $m_{lk}^* = m_{kl}$ . Using the orthogonality relation

$$\varphi((u_{km})^*u_{ln}) = (1/M_U)\delta_{mn}f_{-1}(u_{lk}),$$

where  $M_U$  denotes the quantum dimension of  $U$ , we get Then

$$\begin{aligned} \sum_{i,j,k,l} \varphi((u_{ij})^*u_{kl})m_{lk}m_{ji}^* &= \sum_{i,j,k,l} (1/M_U)\delta_{jl}f_{-1}(u_{ki})\delta_{li}m_{jk} \\ &= \sum_{i,j,k} (1/M_U)f_{-1}(u_{ki})m_{ik} \\ &= (d_U/M_U) \sum_{i,k} f_{-1}(u_{ki})m_{ik} \\ &= (d_U/M_U)(f_{-1} \odot \iota)\left(\sum_{i,k} u_{ki} \otimes m_{ik}\right) \\ &= (d_U/M_U)(f_{-1} \odot \iota)(U). \end{aligned}$$

Hence, in this case, we get

$$K_U = (f_{-1} \odot \iota)(U)/M_U.$$

We summarize what we have shown.

**Proposition 6.5.** *Assume that  $(\mathcal{A}, \Delta)$  is of compact type and let  $U$  be a finite-dimensional unitary representation of  $(\mathcal{A}_r, \Delta_r)$ . Let  $d_U$  (resp.  $M_U$ ) denote the usual (resp. quantum) dimension of  $U$ , and let  $\sigma$  be the automorphism of  $\mathcal{A}$  given by  $\sigma(a) = f_1 * a * f_1$ ,  $a \in \mathcal{A}$ . Then a left-invariant mean  $m_U$  for  $U$  is given by  $m_U(\cdot) = \text{Tr}(K_U \cdot)$ , with density matrix  $K_U$  given by*

$$K_U = 1/d_U(\varphi \odot \iota)(U(\sigma \otimes \iota)U^*).$$

If  $U$  is irreducible, then

$$K_U = (f_{-1} \odot \iota)(U)/M_U.$$

## 7 On weak containment

We discuss in this section the notion of weak containment for representations and corepresentations of algebraic quantum groups. We begin by discussing the stronger (and easier) notion of containment.

## 7.1 Strong containment

We recall the following definition.

**Definition 7.1.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and let  $\pi_1, \pi_2$  be two non-degenerate  $*$ -representations of  $\mathcal{A}_u$ . We say that  $\pi_1$  is contained in  $\pi_2$ , and write  $\pi_1 < \pi_2$ , if there exists an isometry  $T \in \text{Mor}(\pi_1, \pi_2)$ .*

Observe that  $K = T(\mathcal{H}_{\pi_1})$  is then a closed invariant subspace for  $\pi_2$ . Therefore, if  $\pi_2$  is irreducible, then any non-degenerate  $*$ -representation  $\pi_1$  of  $\mathcal{A}_u$  contained in  $\pi_2$  is unitarily equivalent to  $\pi_2$ .

The interesting case where  $\pi_1 = \varepsilon_u$  may be characterized as follows.

**Proposition 7.2.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and consider a non-degenerate  $*$ -representation  $\pi$  of  $\mathcal{A}_u$ . Write  $\pi = \pi_U$  for a unique unitary corepresentation  $U$  of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r, \text{op}})$ . The following conditions are equivalent:*

- (1)  $\varepsilon_u < \pi_U$ ;
- (2) there exists a unit vector  $\xi \in \mathcal{H}_U$  such that  $(\iota \bar{\otimes} \omega_\xi)U = I \in M(\hat{\mathcal{A}}_r)$ ;
- (3) there exists a unit vector  $\xi$  in  $H_U$  such that  $U(v \otimes \xi) = v \otimes \xi$ ,  $\forall v \in \mathcal{H}$ .

*Proof.* We first show (1) implies (2). Assume that (1) holds. Then there exists a linear map  $T : \mathbb{C} \rightarrow \mathcal{H}_U$  such that  $T^*T = 1$  and  $T\varepsilon_u(a) = \pi_U(a)T$  for all  $a \in \mathcal{A}_u$ . Consider the unit vector  $\xi = T(1)$  of  $\mathcal{H}_U$ . Then the adjoint  $T^* : \mathcal{H}_U \rightarrow \mathbb{C}$  is given by  $T^*(\eta) = (\eta, \xi)$  for all  $\eta \in \mathcal{H}_U$ . Now, for all  $a \in \mathcal{A} \subset \mathcal{A}_u$ , we have

$$\begin{aligned} Q(a)(I) &= (Q(a) \bar{\otimes} \iota)(I \otimes 1) = \varepsilon(a) = (\varepsilon_u(a)1, 1)_{\mathbb{C}} \\ &= (T^* \pi_U(a)T(1), 1)_{\mathbb{C}} = (\pi_U(a)T(1), T(1))_{\mathcal{H}_U} \\ &= (\pi_U(a)\xi, \xi)_{\mathcal{H}_U} = ((Q(a) \bar{\otimes} \iota)U\xi, \xi)_{\mathcal{H}_U} = Q(a)((\iota \bar{\otimes} \omega_\xi)U). \end{aligned}$$

Since  $Q(\mathcal{A})$  is dense in  $\hat{\mathcal{M}}_*$  it follows that  $I = (\iota \bar{\otimes} \omega_\xi)U$ .

Next, we show that (2) implies (1). Given a unit vector  $\xi$  satisfying (2), we define the linear isometry  $T : \mathbb{C} \rightarrow \mathcal{H}_U$  by  $T(1) = \xi$ . By reversing the above calculations, we see that  $\varepsilon_u(a) = T^* \pi_U(a)T$  holds for all  $a \in \mathcal{A}_u$ . It is easily checked that  $T \in \text{Mor}(\varepsilon_u, \pi_U)$ .

Finally, to prove the equivalence between (2) and (3), observe first that  $(\iota \bar{\otimes} \omega_\xi)U = I$  if and only if  $(\omega_v \bar{\otimes} \omega_\xi)U = 1$  for all unit vectors  $v \in \mathcal{H}$ . Now, for a unit vector  $v \in \mathcal{H}$ , one easily checks that

$$(U(v \otimes \xi), v \otimes \xi) = 1 \Leftrightarrow \|U(v \otimes \xi) - v \otimes \xi\|_2 = 0.$$

Hence, this equivalence is clear, and the proof is finished.  $\square$

Let  $U, V$  be unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$ . We say that  $U$  is (strongly) contained in  $V$ , and write  $U < V$  if there is an isometry  $T \in \text{Mor}(U, V)$ . It is

an easy exercise to check that  $U < V$  if, and only if  $\hat{\pi}_U < \hat{\pi}_V$ . One may then clearly obtain a result similar to Proposition 7.2.

*Example.* Let  $(\mathcal{A}, \Delta)$  be of compact type and  $U$  be an irreducible unitary representation of  $(\mathcal{A}_r, \Delta_r)$ . Let  $\{e_i\}$  denote an orthonormal basis for  $\mathcal{H}_U$  and  $\tilde{J}$  be defined as in Proposition 4.2. Then the isometry  $R$  from  $\mathbb{C}$  into  $\overline{\mathcal{H}}_U \otimes \mathcal{H}_U$  determined by

$$R(1) = \sum_i \tilde{J}^{*-1}(e_i) \otimes e_i$$

satisfies  $R \in Mor(I \otimes 1, \overline{U} \times U)$ , as shown in [27]. Hence,  $I \otimes 1 < \overline{U} \times U$ . It follows that  $\xi = \sum_i \tilde{J}^{*-1}(e_i) \otimes e_i \in \overline{\mathcal{H}}_U \otimes \mathcal{H}_U$  satisfies  $(\overline{U} \times U)(\eta \otimes \xi) = \eta \otimes \xi$  for all  $\eta \in \mathcal{H}$ . Since  $(\mathcal{A}, \Delta)$  is compact by assumption, we know that all unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are left- (and right-) amenable, as pointed out in the previous section. Indeed, we have seen in Proposition 6.5 that a left-invariant mean  $m_U$  for  $U$  is given by

$$m_U(x) = Tr(K_U x), \quad x \in B(\mathcal{H}_U),$$

where  $K_U = \frac{1}{M_U}(f_{-1} \odot \iota)U$ .

Now, let

$$\widetilde{\mathcal{V}}(\xi) = \frac{1}{(M_U)^{1/2}} \mathcal{V}(\xi),$$

where  $\mathcal{V} : \overline{\mathcal{H}}_U \otimes \mathcal{H}_U \rightarrow HS(\mathcal{H}_U)$  is defined as in Section 4. It is then not difficult to check by direct computation that we also have

$$m_U(\cdot) = Tr(\widetilde{\mathcal{V}}(\xi)^* \cdot \widetilde{\mathcal{V}}(\xi)),$$

that is, we have  $\widetilde{\mathcal{V}}(\xi)\widetilde{\mathcal{V}}(\xi)^* = K_U$ .

## 7.2 Weak containment

Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and let  $\pi_1, \pi_2$  be non-degenerate  $*$ -representations of  $\mathcal{A}_u$ . As usual for representations of  $C^*$ -algebras, we say that  $\pi_1$  is *weakly contained* in  $\pi_2$ , and write  $\pi_1 \prec \pi_2$ , if  $\text{Ker } \pi_2 \subset \text{Ker } \pi_1$ . This relation is obviously transitive and reflexive, and, of course,  $\pi_1 < \pi_2$  implies  $\pi_1 \prec \pi_2$ .

**Proposition 7.3.** *With notation as above, we have  $\pi_1 \prec \pi_2$  if and only if there exists a unique surjective  $*$ -homomorphism  $\theta : \pi_2(\mathcal{A}_u) \rightarrow \pi_1(\mathcal{A}_u)$  such that  $\theta\pi_2(a) = \pi_1(a)$ ,  $\forall a \in \mathcal{A}$ .*

*Proof.* This proof is easy and left to the reader. □

An almost immediate consequence of this proposition is the following.

**Corollary 7.4.**  *$\varepsilon_u \prec \pi_r$  if and only if  $(\mathcal{A}, \Delta)$  is co-amenable.*

*Remark.* Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group. Note that  $(\mathcal{A}, \Delta)$  is co-amenable if, and only if,  $\pi \prec \pi_r$  for every non-degenerate  $*$ -representation  $\pi$  of  $\mathcal{A}_u$ . Indeed, if  $(\mathcal{A}, \Delta)$  is co-amenable, then  $\mathcal{A}_u = \mathcal{A}_r$ , that is,  $\pi_r : \mathcal{A}_u \rightarrow \mathcal{A}_r$  is injective (see [6]). Therefore,  $\{0\} = \text{Ker } \pi_r \subset \text{Ker } \pi$ . On the other hand, if  $(\mathcal{A}, \Delta)$  is not co-amenable, then  $\varepsilon_u$  is not weakly contained in  $\pi_r$ .

We also remark that the condition  $\varepsilon_u \prec \pi$  and the condition  $\pi \prec \pi_r$  are generally independent of each other. In fact, if  $\pi = \varepsilon_u$ , then the first is trivially satisfied, while the second holds if and only if  $(\mathcal{A}, \Delta)$  is co-amenable. On the other hand, if  $\pi = \pi_r$ , then second is trivially satisfied, while the first holds if and only if  $(\mathcal{A}, \Delta)$  is co-amenable.

**Definition 7.5.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group. A non-degenerate  $*$ -representation  $\pi$  of  $\mathcal{A}_u$  is said to have the weak containment property (WCP) if  $\varepsilon_u \prec \pi$ , that is,  $\text{Ker } \pi \subset \text{Ker } \varepsilon_u$ .*

Thus  $\pi$  has the WCP if and only if there exists a  $*$ -homomorphism  $\theta : \pi(\mathcal{A}_u) \rightarrow \mathbb{C}$  such that  $\theta\pi(a) = \varepsilon(a)$  for all  $a \in \mathcal{A} \subset \mathcal{A}_u$ .

**Definition 7.6.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group,  $U, V$  be unitary corepresentations of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  and let  $\pi_U, \pi_V$  be the associated  $*$ -representations of  $(\mathcal{A}_u, \Delta_u)$ .*

*We say that  $U$  is weakly contained in  $V$  if  $\pi_U$  is weakly contained in  $\pi_V$ . Moreover, we say that  $U$  has the weak containment property (WCP) if the trivial corepresentation  $I \otimes 1$  is weakly contained in  $U$ , that is, if  $\pi_U$  has the WCP.*

**Corollary 7.7.** *An algebraic quantum group  $(\mathcal{A}, \Delta)$  is co-amenable if and only if  $\hat{W}$ , as a unitary corepresentation of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$ , has the WCP.*

*Proof.* As  $\pi_{\hat{W}} = \pi_r$ , see Theorem 3.3, this is just a reformulation of Corollary 7.4.  $\square$

The weak containment property for unitary corepresentations may be characterized as follows.

**Theorem 7.8.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group,  $U$  be a unitary corepresentation of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  and let  $\pi_U$  be the associated  $*$ -representation of  $\mathcal{A}_u$ . The following conditions are equivalent:*

- (1)  $I \otimes 1 \prec U$ , that is,  $U$  has the WCP ;
- (2) there exists  $\psi \in S(B(\mathcal{H}_U))$  such that  $\psi((\omega \otimes \iota)U) = \omega(I)$ ,  $\forall \omega \in \hat{\mathcal{M}}_*$ ;
- (3) there exists a net  $(\xi_i)$  of unit vectors in  $H_U$  such that

$$\lim_i \|U(v \otimes \xi_i) - v \otimes \xi_i\|_2 = 0 \quad \forall v \in \mathcal{H} ;$$

- (4) there exists a net  $(\xi_i)$  of unit vectors in  $H_U$  such that

$$\lim_i (U(v \otimes \xi_i), v \otimes \xi_i) = 1, \quad \forall v \in \mathcal{H}, \|v\|_2 = 1 .$$

Further, any of these conditions implies that  $U$  is left-amenable, right-amenable and hypertracial.

*Proof.* The equivalence between (3) and (4) is elementary.

(1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (4): Assume that  $I \otimes 1 \prec U$ . By the remark following Definition 7.5 there exists a  $*$ -homomorphism  $\theta : \pi_U(\mathcal{A}_u) \rightarrow \mathbb{C}$  such that  $\theta(\pi_U(x)) = \varepsilon_u(x)$  for all  $x \in \mathcal{A}_u$ . We extend the state  $\theta$  to a state  $\psi$  on  $B(\mathcal{H}_U)$ . Then, for all  $a \in \mathcal{A} \subset \mathcal{A}_u$ , we have

$$\begin{aligned} \psi((Q(a) \bar{\otimes} \iota)U) &= \psi((Q_r(a) \otimes \iota)U) = \psi(\pi_U(a)) \\ &= \varepsilon_u(a) = (Q_r(a) \otimes \iota)(I \otimes 1) = Q_r(a)(I) = Q(a)(I). \end{aligned}$$

Since  $Q(\mathcal{A})$  is dense in  $\hat{\mathcal{M}}_*$ , we get, by continuity,  $\psi((\omega \bar{\otimes} \iota)U) = \omega(I)$ ,  $\forall \omega \in \hat{\mathcal{M}}_*$ , which shows (2).

Further, as  $\varepsilon_u$  is a  $*$ -homomorphism on  $\mathcal{A}_u$ , it is a pure state on  $\mathcal{A}_u$ . From [10, Proposition 3.4.2, ii)], we can then conclude that there exists a net of unit vectors  $(\xi_i) \in \mathcal{H}_U$  such that  $\varepsilon_u(x) = \lim_i (\pi_U(x)\xi_i, \xi_i)$  for all  $x \in \mathcal{A}_u$ . Since  $\varepsilon_u = \psi \circ \pi_U$  as above, this means that  $\psi(y) = \lim_i \omega_{\xi_i}(y)$  for all  $y \in \pi_U(\mathcal{A}_u)$ . As  $(\omega \bar{\otimes} \iota)U \in \pi_U(\mathcal{A}_u)$  (see [7, Theorem 3.3]) and  $\psi((\omega \bar{\otimes} \iota)U) = \omega(I)$ , for all  $\omega \in \hat{\mathcal{M}}_*$ , we get  $\lim_i \omega_{\xi_i}((\omega \bar{\otimes} \iota)U) = 1$  for all unit vectors  $\omega$  in  $\mathcal{H}$ . This just says that  $\lim_i (U(\eta \otimes \xi_i), \eta \otimes \xi_i) = 1$  for all unit vectors  $\eta$  in  $\mathcal{H}$ , hence that (4) holds.

(2)  $\Rightarrow$  (1): Assume that (2) holds, and let  $\psi$  be as in (2). Let  $x \in \text{Ker } \pi_U$ . Choose a sequence  $(a_n)$  in  $\mathcal{A}$  converging to  $x \in \mathcal{A}_u$  with respect to the norm  $\|\cdot\|_u$ . Then, by continuity of  $\pi_U$ , we get  $(Q_r(a_n) \otimes \iota)U = \pi_U(a_n) \rightarrow \pi_U(x) = 0$ . Using the assumption, we have  $\psi((Q(a_n) \bar{\otimes} \iota)U) = Q(a_n)(I)$  for all  $n$ . By continuity of  $\psi$  we therefore get

$$\begin{aligned} \varepsilon_u(a_n) &= Q_r(a_n)(I) = Q(a_n)(I) \\ &= \psi((Q(a_n) \bar{\otimes} \iota)U) = \psi((Q_r(a_n) \otimes \iota)U) \rightarrow 0. \end{aligned}$$

Thus, by continuity of  $\varepsilon_u$ , we get  $\varepsilon_u(x) = \lim_n \varepsilon_u(a_n) = 0$ , so  $x \in \text{Ker } \varepsilon_u$ . Hence, (1) holds.

(4)  $\Rightarrow$  (2): Let  $(\xi_i)_i$  be a net satisfying condition (4). Using Alaoglu's theorem, and passing to a subnet if necessary, there exists a  $\psi \in S(B(H_U))$  such that  $\psi(x) = \lim_i \omega_{\xi_i}(x)$ , for all  $x \in B(H_U)$ . Since  $\hat{M}$  is in standard form on  $H$ , any normal state  $\omega$  on  $\hat{M}$  is of the form  $\omega_v$  for some unit vector  $v \in \mathcal{H}$ . Then  $\psi((\omega_v \bar{\otimes} \iota)U) = \lim_i \omega_{\xi_i}((\omega_v \bar{\otimes} \iota)U) = \lim_i (U(v \otimes \xi_i), v \otimes \xi_i) = 1 = \omega_v(I)$ , so  $\psi$  satisfies condition (2).

Hence, we have established the equivalence between conditions (1)-(4).

Finally, assume that (2) holds and set  $m_U = \psi$ . Let  $\omega \in \hat{\mathcal{M}}_{*,1}^+$ . Then  $m_U((\omega \bar{\otimes} \iota)U) = \omega(I) = 1$ . As  $U$  is a unitary in  $\hat{M} \bar{\otimes} B(\mathcal{H}_U)$ , it follows from the Cauchy-Schwarz inequality that the state  $m_U((\omega \bar{\otimes} \iota)(\cdot))$  on  $\hat{M} \bar{\otimes} B(\mathcal{H}_U)$  is multiplicative at  $U$  and at  $U^*$ . Hence,

$$m_U((\omega \bar{\otimes} \iota)(U^*(I \otimes x)U)) = m_U((\omega \bar{\otimes} \iota)U^*)m_U((\omega \bar{\otimes} \iota)(I \otimes x))m_U((\omega \bar{\otimes} \iota)U)$$

$$= m_U(x) = \omega(I)m_U(x)$$

for all  $x \in B(\mathcal{H}_U)$  and  $\omega \in \hat{\mathcal{M}}_{*,1}^+$ . It easily follows that  $m_U$  is a left-invariant mean for  $U$ . Similarly,  $m_U$  is a right-invariant mean for  $U$ , and it also serves to show that  $U$  is hypertracial. This finishes the proof.  $\square$

Weak containment and WCP for unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are defined in an analogous way, via weak containment and WCP for the associated representations of  $\hat{\mathcal{A}}_u$ . From a conceptual point of view, it is better to work in this setting, and we will often do this in the sequel. All statements concerning WCP for unitary corepresentations of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  have an analogous statement concerning WCP for unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$ . For example, we have the following counterpart to Theorem 7.8.

**Theorem 7.9.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . The following conditions are equivalent:*

- (1)  $I \otimes 1 \prec U$ , that is,  $U$  has the WCP ;
- (2) there exists  $\psi \in S(B(\mathcal{H}_U))$  such that  $\psi((\omega \otimes \iota)U) = \omega(I)$ ,  $\forall \omega \in \mathcal{M}_*$ ;
- (3) there exists a net  $(\xi_i)$  of unit vectors in  $\mathcal{H}_U$  such that

$$\lim_i \|U(v \otimes v_i) - v \otimes v_i\|_2 = 0, \quad \forall v \in \mathcal{H} ;$$

- (4) there exists a net  $(\xi_i)$  of unit vectors in  $\mathcal{H}_U$  such that

$$\lim_i (U(v \otimes \xi_i), v \otimes \xi_i) = 1, \quad \forall v \in \mathcal{H}, \|v\|_2 = 1 ;$$

Further, any of these conditions implies that  $U$  is left-amenable, right-amenable and hypertracial.

We will illustrate in the next section that amenability of  $U$  does not imply in general that  $U$  has the WCP. We now collect some elementary facts about the WCP.

**Proposition 7.10.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and let  $U, V$  be unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$ .*

- (1) If  $U$  has the WCP, then  $\overline{U}$  has also the WCP.
- (2) If  $U$  and  $V$  have the WCP, then  $U \times V$  also has the WCP.
- (3) If  $U$  has the WCP, then  $U_{HS}$  has also the WCP.
- (4) If  $U \times V$  has the WCP, then  $U$  is left-amenable and  $V$  is right-amenable.
- (5) If  $U_{HS}$  has the WCP, then  $\overline{U}$  is left-amenable and  $U$  is right-amenable.

*Proof.* The proof of assertion (1) is an easy exercise, left to the reader. Assertion (2) is a straightforward application of condition (2) in Theorem 7.9. Assertion (3) follows from (1) and (2). To prove assertion (4), assume that  $U \times V$  has



the WCP. The last assertion in Theorem 7.9 tells us then that  $U \times V$  is left- and right-amenable. If now  $M \in S(B(\mathcal{H}_U) \otimes B(\mathcal{H}_V))$  is a left-invariant (resp. right-invariant) mean for  $U \times V$ , then one checks without difficulty that  $m_U(x) = M(x \otimes I_{\mathcal{H}_V})$  (resp.  $m'_V(y) = M(I_{\mathcal{H}_U} \otimes y)$ ),  $x \in B(\mathcal{H}_U)$  (resp.  $y \in B(\mathcal{H}_V)$ ), is a left-invariant (resp. right-invariant) mean for  $U$  (resp.  $V$ ). This shows (4). Finally, assertion (5) follows clearly from (4).  $\square$

*Remark.* Let  $u$  denote a unitary representation of a discrete group  $\Gamma$ . One of the main results of Bekka in [8] is that  $u$  is amenable if, and only if, its associated Hilbert-Schmidt representation weakly contains the trivial representation. An interesting question is whether some quantum group version of this result is true, that is, whether the converse of assertion (5) in Proposition 7.10 holds, at least in some cases. We will return to this question in Section 9.

**Corollary 7.11.** *Let  $U$  be a finite-dimensional unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ . Assume that  $(R \odot l)U = U^*$ . (This is known to hold in the Kac algebra case, cf. [12, Proposition 1.5.1]).*

*Then  $I \otimes 1 < U_{HS}$ , and  $U$  is both left- and right-amenable.*

*Proof.* Write  $U = \sum_i a_i \otimes b_i$ , where  $a_i \in \mathcal{M}$ ,  $b_i \in B(\mathcal{H}_U)$ ,  $i = 1 \dots n$ . We use the notation introduced in Section 4. Using Proposition 4.4 we may write  $U_{HS} = (\iota \odot l)U(R \odot r)U$ . As  $(R \odot r)U = (\iota \odot r)U^*$ , we have  $(R \odot r)U = \sum_j a_j^* \otimes r(b_j^*)$ . Let  $\xi = I_{\mathcal{H}_U} \in HS(\mathcal{H}_U)$ . For any  $\eta \in \mathcal{H}$  we get

$$\begin{aligned} U_{HS}(\eta \otimes \xi) &= ((\iota \odot l)U(R \odot r)U)(\eta \otimes \xi) \\ &= \left( \sum_i a_i \otimes l(b_i) \right) \left( \sum_j a_j^* \otimes r(b_j^*) \right) (\eta \otimes \xi) \\ &= \left( \sum_{ij} a_i a_j^* \otimes l(b_i) r(b_j^*) \right) (\eta \otimes \xi) = \sum_{ij} a_i a_j^* \eta \otimes b_i b_j^* \\ &= \left( \sum_{ij} a_i a_j^* \otimes b_i b_j^* \right) (\eta \otimes \xi) = \eta \otimes \xi \end{aligned}$$

where, in the last equality, we have used the fact that  $\sum_{ij} a_i a_j^* \otimes b_i b_j^* = UU^* = I_{\mathcal{H}} \otimes I_{\mathcal{H}_U}$ . Thus, appealing to Proposition 7.2, we have shown that  $I \otimes 1 < U_{HS}$ . We may then apply Proposition 7.10 (5) and conclude that  $U$  is right-amenable. Finally, as  $\bar{U}$  is also finite-dimensional, we then easily deduce that  $\bar{U}$  is right-amenable, hence that  $U$  is left-amenable.  $\square$

**Proposition 7.12.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and consider its multiplicative unitary  $W$  as a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ .*

*Then  $W$  has the WCP if, and only if,  $W_{HS}$  has the WCP.*

*Proof.* Using assertion (3) of Proposition 7.10, it suffices to show that  $W$  has the WCP whenever  $W_{HS}$  has it. So assume that  $I \otimes 1 \prec W$ . Using the absorbing property of  $W$  (cf. our remark after Proposition 3.4), we obtain that  $W_{HS} =$

$\overline{W} \times W$  is unitarily equivalent to  $I_{\overline{W}} \times W = W_{13}(I_{\mathcal{H}} \otimes I_{\overline{\mathcal{H}}})_{12} = W_{13}$ . According to Theorem 7.9, there exists  $\psi \in S(B(\overline{\mathcal{H}} \otimes \mathcal{H}))$  such that  $\psi((\omega \bar{\otimes} \iota \bar{\otimes} \iota)W_{13}) = \omega(I)$ ,  $\omega \in \hat{\mathcal{M}}_*$ . Define a state  $\psi'$  on  $B(\mathcal{H})$  by  $\psi'(x) = \psi(I_{\overline{\mathcal{H}}} \otimes x)$ ,  $x \in B(\mathcal{H})$ . Then

$$\psi'((\omega \bar{\otimes} \iota)W) = \psi(I_{\overline{\mathcal{H}}} \otimes (\omega \bar{\otimes} \iota)W) = \psi((\omega \bar{\otimes} \iota \bar{\otimes} \iota)W_{13}) = \omega(I).$$

Using Theorem 7.9 again, we deduce that  $W$  has the WCP, as desired.  $\square$

**Corollary 7.13.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group and consider the multiplicative unitary  $\hat{W}$  as a unitary corepresentation of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$ . Then*

*$(\mathcal{A}, \Delta)$  is co-amenable if, and only if,  $\hat{W}_{HS}$  has the WCP.*

*Proof.* We just have to combine the dual version of Proposition 7.12 with Corollary 7.7.  $\square$

Our interest in such a result is that it is presumably easier to establish that  $\hat{W}_{HS}$  has the WCP than to establish that  $\hat{W}$  has the WCP if one wants to show that  $(\mathcal{A}, \Delta)$  is co-amenable.

We conclude this subsection with another proposition involving containment and amenability.

**Proposition 7.14.** *Let  $(\mathcal{A}, \Delta)$  denote an algebraic quantum group and  $U, V$  be unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$ .*

*If  $U$  is left- (resp. right-) amenable and  $U < V$ , then  $V$  is left- (resp. right-) amenable.*

*Proof.* Let  $T \in \text{Mor}(U, V)$  be such that  $T^*T = I$  and assume that  $U$  is left-amenable. Define  $\tilde{T} : B(\mathcal{H}_V) \rightarrow B(\mathcal{H}_U)$  by  $\tilde{T}(x) = T^*xT$ , and note that  $\tilde{T}$  is a normal  $*$ -preserving completely positive unital linear map. Then note that, since  $T \in \text{Mor}(U, V)$  and  $T^*T = I$ , we have

$$\begin{aligned} (\omega \bar{\otimes} \iota)U &= T^*(\omega \bar{\otimes} \iota)VT \\ &= \tilde{T}(\omega \bar{\otimes} \iota)V \\ &= (\omega \bar{\otimes} \iota)(\iota \bar{\otimes} \tilde{T})V \end{aligned}$$

for all  $\omega \in \mathcal{M}_*$  (the last equality can be checked for  $V$  elementary first and then by continuity). As  $\{(\omega \bar{\otimes} \iota) \mid \omega \in \mathcal{M}_*\}$  separate the elements of  $\mathcal{M} \bar{\otimes} B(\mathcal{H}_U)$ , we get  $U = (\iota \bar{\otimes} \tilde{T})V$ .

Now, let  $m_U \in S(B(\mathcal{H}_U))$  be a left-invariant mean for  $U$ , so

$$m_U((\omega \bar{\otimes} \iota)(U^*(I \otimes y)U)) = \omega(I)m_U(y)$$

for all  $y \in B(\mathcal{H}_U)$  and  $\omega \in \mathcal{M}_*$ .

Define  $m_V \in S(B(\mathcal{H}_V))$  by  $m_V = m_U \tilde{T}$ . Then, for  $x \in B(\mathcal{H}_V)$  and  $\omega \in \mathcal{M}_*$ , we get

$$m_V(x) = m_U \tilde{T}(x) = m_U((\omega \bar{\otimes} \iota)(U^*(I \otimes \tilde{T}(x))U)).$$

Since  $(\iota \bar{\otimes} \tilde{T})V = U$  is unitary and  $\iota \bar{\otimes} \tilde{T}$  is completely positive, it follows from a well known result of M.D. Choi, see e.g. [29, 9.2], that  $\iota \bar{\otimes} \tilde{T}$  is multiplicative at  $V$  and  $V^*$ . Hence, we get

$$(\iota \bar{\otimes} \tilde{T})(V^*(I \otimes x)V) = (\iota \bar{\otimes} \tilde{T})V^*(I \otimes \tilde{T}(x))(\iota \bar{\otimes} \tilde{T})V.$$

Thus

$$\begin{aligned} m_V(x) &= m_U((\omega \bar{\otimes} \iota)(\iota \bar{\otimes} \tilde{T})(V^*(I \otimes x)V)) \\ &= m_U(\tilde{T}(\omega \bar{\otimes} \iota)(V^*(I \otimes x)V)) \\ &= m_V((\omega \bar{\otimes} \iota)(V^*(I \otimes x)V)). \end{aligned}$$

So  $m_V$  is a left-invariant mean for  $V$  and  $V$  is left-amenable. The proof of the resp. part of the statement is similar.  $\square$

It would be interesting to know whether this result still holds if one replaces strong containment with weak containment. Bekka has shown [8, Corollary 5.3] that this is true in the classical case.

### 7.3 On property (T)

We introduce a version of Kazhdan's property (T) [17] for algebraic quantum groups. Then, as in the classical case, we show that every compact quantum group has property (T). This implies that none of the non-trivial irreducible corepresentations of a compact quantum group has the WCP. Furthermore, we show that compactness may be characterized by having property (T) together with co-amenableity of the dual quantum group.

**Definition 7.15.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group. We say that  $(\mathcal{A}, \Delta)$  has property (T) if  $I \otimes 1 \prec U \Rightarrow I \otimes 1 \prec U$  for all unitary corepresentations  $U$  of  $(\mathcal{A}_r, \Delta_r)$ , in other words, if  $\hat{\varepsilon}_u \prec \hat{\pi} \Rightarrow \hat{\varepsilon}_u \prec \hat{\pi}$  for all non-degenerate  $*$ -representations  $\hat{\pi}$  of  $\hat{\mathcal{A}}_u$ .*

**Theorem 7.16.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group of compact type. Then  $(\mathcal{A}, \Delta)$  has property (T).*

*Proof.* Let  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$  and assume that  $U$  has the WCP. To show the theorem, we have to show that  $\hat{\varepsilon}_u \prec \hat{\pi}_U$ .

Since  $(\mathcal{A}, \Delta)$  is of compact type,  $(\mathcal{A}_r, \Delta_r)$  is a compact quantum group in the sense of Woronowicz. Its Haar state  $\varphi_r$  is then left- and right-invariant, and it has a unique extension to a normal state on  $\mathcal{M}$  which we also denote by  $\varphi_r$ .

Now, let  $\xi \in \mathcal{H}_U$  and set  $\eta = ((\varphi_r \otimes \iota)U)\xi \in \mathcal{H}_U$ . Then, for all  $v \in \mathcal{H}$ , we have

$$U(v \otimes \eta) = U(I \otimes (\varphi_r \otimes \iota)U)(v \otimes \xi),$$

while

$$v \otimes \eta = (I \otimes (\varphi_r \otimes \iota)U)(v \otimes \xi).$$

But

$$\begin{aligned} I \otimes ((\varphi_r \otimes \iota)U) &= (\varphi_r(\cdot)I \otimes \iota)U \\ &= (\iota \otimes \varphi_r \otimes \iota)(\Delta_r \otimes \iota)U \quad (\text{using invariance of } \varphi_r) \\ &= (\iota \otimes \varphi_r \otimes \iota)(U_{13} U_{23}) = U(I \otimes (\varphi_r \otimes \iota)U). \end{aligned}$$

Hence, we get  $U(v \otimes \eta) = v \otimes \eta$  for all  $v \in \mathcal{H}$ .

Using the dual version of Proposition 7.2, we will then have shown that  $\hat{\varepsilon}_u < \hat{\pi}_U$  if we can show that the vector  $\eta$  may be chosen to be non-zero. This may be seen as follows. Since  $U$  has the WCP, we know from Theorem 7.9 that there exists a state  $\psi$  on  $B(\mathcal{H}_U)$  such that  $\psi((\varphi_r \bar{\otimes} \iota)U) = 1$ . This implies that  $(\varphi_r \bar{\otimes} \iota)U \neq 0$ . Hence, there exists at least one  $\xi \in \mathcal{H}_U$  such that

$$0 \neq ((\varphi_r \bar{\otimes} \iota)U)\xi = ((\varphi_r \otimes \iota)U)\xi = \eta,$$

as desired.  $\square$

**Theorem 7.17.** *Let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group. Then  $(\mathcal{A}, \Delta)$  is of compact type if and only if  $(\mathcal{A}, \Delta)$  has property (T) and  $(\hat{\mathcal{A}}, \hat{\Delta})$  is co-amenable.*

*Proof.* If  $(\mathcal{A}, \Delta)$  is of compact type, then we know from Theorem 7.16 that  $(\mathcal{A}, \Delta)$  has property (T). Further,  $(\hat{\mathcal{A}}, \hat{\Delta})$  is then of discrete type and therefore co-amenable [6].

Conversely, assume that  $(\mathcal{A}, \Delta)$  has property (T) and  $(\hat{\mathcal{A}}, \hat{\Delta})$  is co-amenable. Then, using the dual version of Corollary 7.4, we get  $\hat{\varepsilon}_u \prec \hat{\pi}_W$ , hence  $\hat{\varepsilon}_u < \hat{\pi}_W$ . This means that there exists a  $T : \mathbb{C} \rightarrow \mathcal{H}$  such that  $T^*T = 1$  and  $\hat{\varepsilon}_u(y) = T^* \hat{\pi}_W(y)T$  for all  $y \in \hat{\mathcal{A}}_u$ . Thus we have  $\hat{\varepsilon}_u(y) = (\hat{\pi}_W(y)\eta, \eta)$  for all  $y \in \hat{\mathcal{A}}_u$ , where  $\eta = T(1)$  is a unit vector in  $\mathcal{H}$ .

Let  $\psi$  denote the vector state  $\omega_\eta$  on  $B(\mathcal{H})$ . Then, proceeding as in the proof of Theorem 7.8, (1) implies (2), we get  $\psi(\omega \bar{\otimes} \iota)W = \omega(I_{\mathcal{H}})$  for all  $\omega \in \mathcal{M}_*$ . As  $\psi$  is normal, this gives  $\omega(\iota \bar{\otimes} \psi)W = \omega(I_{\mathcal{H}})$  for all  $\omega \in \mathcal{M}_*$ , hence  $(\iota \bar{\otimes} \psi)W = I_{\mathcal{H}}$ . Since  $\mathcal{A}_r$  is the norm closure of  $\{(\iota \bar{\otimes} \phi)W \mid \phi \in B(\mathcal{H})_*\}$ , we get  $I_{\mathcal{H}} \in \mathcal{A}_r$ , that is  $(\mathcal{A}_r, \Delta_r)$  is compact, as desired.  $\square$

It is clear that a more detailed study of property (T) for algebraic quantum groups would be an interesting task (see [26] for the case of Kac algebras). However, we don't elaborate further on this as it would take us too far apart from our main theme in this paper.

## 8 Amenability vs. co-amenable vs. WCP

Let  $(\mathcal{A}, \Delta)$  denote an algebraic quantum group and  $U$  be a unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$ .

We show that some of the notions introduced in the previous sections concerning  $U$  are different from each other by producing counter examples to the various possible implications. We consider here only left-amenable, as we may obtain similar statements for right-amenable by considering the conjugate of  $U$ .

(1)  $U$  co-amenable does not imply that  $U$  has the WCP.

In fact, pick  $(\mathcal{A}, \Delta)$  of discrete type and such that  $(\hat{\mathcal{A}}, \hat{\Delta})$  is not co-amenable (e.g.  $\mathcal{A} = C_c(\mathbb{F}_2)$ ). Then  $(\mathcal{A}, \Delta)$  is co-amenable since it is of discrete type, cf. [6, Theorem 4.1]. Hence, every corepresentation of it is co-amenable, by Theorem 5.2. In particular  $U$  is co-amenable. But  $U$  has not the WCP since  $(\hat{\mathcal{A}}, \hat{\Delta})$  is not co-amenable, by the dual version of Corollary 7.7.

(2)  $U$  has the WCP does not imply that  $U$  is co-amenable.

Indeed, pick  $(\mathcal{A}, \Delta)$  non co-amenable and of compact type (e. g.  $\mathcal{A} = \mathbb{C}[\mathbb{F}_2]$ ). Again pick  $U = W$ . Now,  $(\hat{\mathcal{A}}, \hat{\Delta}_{\text{op}})$  is co-amenable (being of discrete type). Hence,  $U$  has the WCP, using the dual version of Corollary 7.7. On the other hand,  $U$  is not co-amenable, according to Theorem 5.2.

(3)  $U$  left-amenable does not imply that  $U$  is co-amenable.

Again, pick  $(\mathcal{A}, \Delta)$  non co-amenable and of compact type and let  $U = W$ . Since any compact quantum group is amenable, see the paragraph preceding Theorem 4.7 in [6],  $U$  is left-amenable according to Theorem 6.3. On the other hand, according to Theorem 5.2,  $U$  is not co-amenable.

(4)  $U$  co-amenable does not imply that  $U$  is left-amenable.

Let  $(\mathcal{A}, \Delta)$  be non-amenable and of discrete type. Being co-amenable, all its unitary corepresentations are then co-amenable. However, they cannot all be amenable.

(5)  $U$  left-amenable does not imply that  $U$  has the WCP.

Indeed, let  $\Gamma$  be any non-trivial finite group and let  $\mathcal{A} = C(\Gamma)$ . Let  $\Delta$  be the usual co-product on  $\mathcal{A}$ . Then pick a non-trivial irreducible unitary representation  $u$  of  $\Gamma$  and let  $U$  be the unitary corepresentation of  $(\mathcal{A}_r, \Delta_r)$  associated with  $u$ . Now, it is clear that  $(\mathcal{A}_r, \Delta_r)$  is amenable and has property (T) (since it is compact). Then  $U$  is amenable (by Theorem 6.3), but  $U$  has not the WCP (as remarked at the beginning of subsection 7.3).

*Remark.* Let  $(\mathcal{A}, \Delta)$  be of compact type. As used several times by now,  $(\mathcal{A}, \Delta)$  is then amenable and all the unitary corepresentations of  $(\mathcal{A}_r, \Delta_r)$  are therefore amenable. If  $(\mathcal{A}, \Delta)$  is also co-amenable (e.g. we may take the compact matrix pseudogroup  $\mathcal{A} = \text{SU}_q(2)$ , cf. [3, 5]), all these corepresentations are then also co-amenable. Further, as  $(\mathcal{A}, \Delta)$  has property T, we get that none of the non-trivial irreducible corepresentations of  $(\mathcal{A}_r, \Delta_r)$  satisfies the WCP.

On the other hand,  $(\hat{\mathcal{A}}, \hat{\Delta})$  is always co-amenable since it is of discrete type. Hence, all the unitary corepresentations of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  are co-amenable. If  $(\mathcal{A}, \Delta)$  is also co-amenable, then we know that  $(\hat{\mathcal{A}}, \hat{\Delta})$  is amenable, hence all these corepresentations are then also amenable.

## 9 Amenability and discrete quantum groups

As we pointed out in connection with Proposition 7.10, it would be interesting to know whether the converse of Proposition 7.10 (5) holds, that is, whether the right-amenability of  $U$  implies that  $U_{HS}$  has the WCP. It does in the classical case, and this is one of the major result in [8]. The problem of going from amenability of  $U$  to the WCP for  $U_{HS}$  seems much more delicate in the general case.

We will now present a proof which works for an algebraic quantum group of discrete type having a (compact) dual with a tracial Haar state. As a consequence, we obtain a new proof of the fact that amenability of such a quantum group is equivalent to the co-amenability of its dual, which has been previously established by Ruan [28, Theorem 4.5], see also [7].

For notational reasons, we let  $(\mathcal{A}, \Delta)$  be an algebraic quantum group of compact type and consider its dual  $(\hat{\mathcal{A}}, \hat{\Delta})$  which is then of discrete type. We use the description of  $(\hat{\mathcal{A}}, \hat{\Delta})$  given in Proposition 2.2 and the notation introduced there. We denote by  $\hat{S} = \hat{S}_{\text{op}}$  the antipode of  $(\hat{\mathcal{A}}, \hat{\Delta}_{\text{op}})$ , and by  $\hat{R}$  the anti-unitary antipode of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$  (which is defined on  $\hat{\mathcal{M}}$ ). For each  $\alpha \in A$ , we denote the central minimal projection of  $\hat{\mathcal{M}}$  which is given by  $\hat{\pi}_r(p_\alpha)$  with the same symbol  $p_\alpha$ . Further, we identify  $\hat{\pi}_r(\hat{\mathcal{A}}_\alpha) = p_\alpha \hat{\pi}_r(\hat{\mathcal{A}})$  with  $\hat{\mathcal{A}}_\alpha = M_{d_\alpha}(\mathbb{C})$  and let  $Tr_\alpha$  denote its canonical trace. Finally, we denote the canonical injection from  $\hat{\mathcal{A}}$  into  $\mathcal{H}$  by  $\hat{\Lambda}$ .

Let now  $U$  be a unitary corepresentation of  $(\hat{\mathcal{A}}_r, \hat{\Delta}_{r,\text{op}})$ . We remark that, using the above identifications and the properties of  $p_\alpha$ , one easily deduces that

$$(p_\alpha \otimes I)U = U(p_\alpha \otimes I) \in \hat{\mathcal{A}}_\alpha \odot B(\mathcal{H}_U), \quad U(p_\alpha \otimes y)U^* \in \hat{\mathcal{A}}_\alpha \odot HS(\mathcal{H}_U)$$

for all  $\alpha \in A$  and  $y \in HS(\mathcal{H}_U)$ . We denote by  $T_\alpha$  the trace on  $\hat{\mathcal{A}}_\alpha \odot B(\mathcal{H}_U)$  given by  $T_\alpha = Tr_\alpha \odot Tr$ , where  $Tr$  denotes the canonical trace on  $HS(\mathcal{H}_U)$ . Further, we denote by  $\|\cdot\|_{1,\alpha}$  and  $\|\cdot\|_{2,\alpha}$  the associated norms on  $\hat{\mathcal{A}}_\alpha \odot TC(\mathcal{H}_U)$  and  $\hat{\mathcal{A}}_\alpha \odot HS(\mathcal{H}_U)$ , respectively.

We establish a series of lemmas.

**Lemma 9.1.** *For each  $\alpha \in A$ , set*

$$b_\alpha = M_\alpha \sum_{i,j=1}^{d_\alpha} f_1(u_{ji}^\alpha) u_{ij}^\alpha,$$

so that we have  $\hat{b}_\alpha = p_\alpha$ . The following conditions are equivalent:

- (1)  $U$  has the WCP.
- (2) There exists a net  $(\xi_i)$  of unit vectors in  $\mathcal{H}_U$  such that

$$\lim_i \|U(\hat{\Lambda}(p_\alpha) \otimes \xi_i) - (\hat{\Lambda}(p_\alpha) \otimes \xi_i)\|_2 = 0 \quad \forall \alpha \in A.$$

(3) There exists a state  $\phi$  on  $B(\mathcal{H}_U)$  such that

$$\phi((\omega_{\hat{\Lambda}(p_\alpha)} \bar{\otimes} \iota)U) = \omega_{\hat{\Lambda}(p_\alpha)}(I), \quad \forall \alpha \in A.$$

(4) There exists a state  $\phi$  on  $B(\mathcal{H}_U)$  such that

$$\phi\pi_U(b_\alpha) = M_\alpha^2, \quad \forall \alpha \in A.$$

(5) There exists a state  $\phi$  on  $B(\mathcal{H}_U)$  such that  $\phi\pi_U = \varepsilon_u$ .

*Proof.* (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) follow as in the proof of Theorem 7.8.

(3)  $\Rightarrow$  (4) : Assume that (3) holds and let  $\phi$  be as in (3). We will show that  $\phi$  satisfies (4). Fix  $\alpha \in A$ .

We first observe that  $S(b_\alpha^*) = b_\alpha$ . Indeed,

$$\begin{aligned} S(b_\alpha^*) &= M_\alpha \sum_{i,j} \overline{f_1(u_{ji}^\alpha)} S((u_{ij}^\alpha)^*) \\ &= M_\alpha \sum_{i,j} \overline{f_1(u_{ji}^\alpha)} \sum_{k,l} f_1(u_{jk}^\alpha) f_{-1}(u_{li}^\alpha) u_{kl}^\alpha \\ &= M_\alpha \sum_{i,j,k,l} f_{-1}((u_{ji}^\alpha)^*) f_{-1}(u_{li}^\alpha) f_1(u_{jk}^\alpha) u_{kl}^\alpha \\ &= M_\alpha \sum_{j,k,l} f_{-1}(\sum_i (u_{ji}^\alpha)^* u_{li}^\alpha) f_1(u_{jk}^\alpha) u_{kl}^\alpha \\ &= M_\alpha \sum_{j,k,l} f_{-1}(\delta_{jl} I) f_1(u_{jk}^\alpha) u_{kl}^\alpha = M_\alpha \sum_{k,l} f_1(u_{ik}^\alpha) u_{kl}^\alpha = b_\alpha. \end{aligned}$$

Thus, we have

$$\hat{b}_\alpha(S((b_\alpha^*)^*))^\wedge = p_\alpha p_\alpha = p_\alpha = (S((b_\alpha^*)^*))^\wedge.$$

Therefore, using the result from [7] recalled at the beginning of subsection 3.1, we have

$$Q(b_\alpha) = \omega_{\Lambda(b_\alpha), \Lambda(b_\alpha)} = \omega_{\hat{\Lambda}(p_\alpha)}.$$

Using [7, Theorem 3.2] and the assumption that  $\phi$  satisfies (3), we get

$$\begin{aligned} \phi(\pi_U(b_\alpha)) &= \phi((Q(b_\alpha) \bar{\otimes} \iota)U) \\ &= \phi((\omega_{\hat{\Lambda}(p_\alpha)} \bar{\otimes} \iota)U) = \omega_{\hat{\Lambda}(p_\alpha)}(I) = \hat{\psi}(p_\alpha^* p_\alpha) \\ &= \hat{\psi}(p_\alpha) = \hat{\psi}(\hat{b}_\alpha) = \varepsilon(b_\alpha) \\ &= M_\alpha \sum_{i,j} f_1(u_{ji}^\alpha) \varepsilon(u_{ij}^\alpha) = M_\alpha \sum_i f_1(u_{ii}^\alpha) = M_\alpha^2, \end{aligned}$$

and (4) is proved.

(4)  $\Rightarrow$  (5) : Assume (4) holds and let  $\phi$  be as in (4). Let  $\eta_u$  be the state on  $\mathcal{A}_u$  given by  $\eta_u = \phi\pi_U$  and let  $\eta$  denote the restriction of  $\eta_u$  to  $\mathcal{A}$ . To show that  $\phi$  satisfies (5), that is  $\eta_u = \varepsilon_u$ , it suffices to show that  $\eta = \varepsilon$ .

Fix  $\alpha \in A$ . As  $\sum_{i,j} f_1(u_{ji}^\alpha)u_{ij}^\alpha = \sum_i f_1 * u_{ii}^\alpha$ , we have

$$(*) \quad \phi\left(\sum_i f_1 * u_{ii}^\alpha\right) = M_\alpha = f_1\left(\sum_i u_{ii}^\alpha\right).$$

Set  $d = \sum_i u_{ii}^\alpha$  and observe that we may write (\*) as  $\eta f_1(d) = f_1(d)$ .

Now, set  $X_{ij} = f_{1/2} * u_{ij}^\alpha - f_{1/2}(u_{ij}^\alpha)I \in \mathcal{A}$ . Then

$$\begin{aligned} & \eta\left(\sum_{i,j} X_{ij}^* X_{ij}\right) \\ &= \sum_{i,j} \eta\left((f_{1/2} * u_{ij}^\alpha)^*(f_{1/2} * u_{ij}^\alpha)\right) - 2\operatorname{Re}\overline{f_{1/2}(u_{ij}^\alpha)}\eta(f_{1/2} * u_{ij}^\alpha) + |f_{1/2}(u_{ij}^\alpha)|^2 \\ &= \sum_{i,j} \eta\left(\sum_{k,l} (u_{ik}^\alpha)^* f_{1/2}(u_{kj}^\alpha) u_{il}^\alpha f_{1/2}(u_{lj}^\alpha)\right) \\ &\quad - 2\operatorname{Re}\overline{f_{1/2}(u_{ij}^\alpha)} \sum_k f_{1/2}(u_{kj}^\alpha) \eta(u_{ik}^\alpha) + |f_{1/2}(u_{ij}^\alpha)|^2 \\ &= \sum_{j,k,l} (\delta_{kl} \eta(I) \overline{f_{1/2}(u_{kj}^\alpha)} f_{1/2}(u_{lj}^\alpha)) - 2\operatorname{Re}\left(\sum_{i,j,k} \eta(u_{ik}^\alpha) f_{1/2}(u_{ji}^\alpha) f_{1/2}(u_{kj}^\alpha)\right) + \sum_i f_1(u_{ii}^\alpha) \\ &= \sum_{j,k} f_{1/2}(u_{jk}^\alpha) f_{1/2}(u_{kj}^\alpha) - 2\operatorname{Re}(\eta f_1(u_{ii}^\alpha)) + \sum_i f_1(u_{ii}^\alpha) \\ &= 2 \sum_i f_1(u_{ii}^\alpha) - 2\operatorname{Re}(\eta f_1(u_{ii}^\alpha)) = 2(f_1(d) - \operatorname{Re}(\eta f_1(d))). \end{aligned}$$

Now, as  $\eta f_1(d) = f_1(d) = M_\alpha$  is real, we get

$$\eta\left(\sum_{i,j} X_{ij}^* X_{ij}\right) = 0.$$

Since  $\eta$  is a positive linear functional, this implies that  $\eta(X_{ij}^* X_{ij}) = 0$  for all  $i, j$ . Using the Cauchy-Schwarz inequality, we obtain  $\eta(X_{ij}) = 0$  for all  $i, j$ , that is,

$$\eta(f_{1/2} * u_{ij}^\alpha) = f_{1/2}(u_{ij}^\alpha), \quad \forall i, j,$$

hence

$$\eta(f_{1/2} * a) = f_{1/2}(a), \quad \forall a \in \mathcal{A},$$

by linearity.

For any  $b \in \mathcal{A}$ , we let  $a = f_{-1/2} * b$  and apply the above. this gives

$$\eta(f_{1/2} * f_{1/2} * b) = f_{1/2}(f_{-1/2} * b),$$

that is,  $\eta(b) = \varepsilon(b)$ . Thus, we have shown that  $\eta = \varepsilon$ , as desired.

(5)  $\Rightarrow$  (1) : Assume (5) holds. Then we clearly have  $\operatorname{Ker} \pi_U \subset \operatorname{Ker} \varepsilon_u$ , that is,  $U$  has the WCP.  $\square$



**Lemma 9.2.** *Let  $\alpha \in A$  and set  $p_\beta = \tilde{S}(p_\alpha)$ . Then we have*

$$(\tilde{S} \odot \iota)((p_\alpha \otimes I)U) = U^*(p_\beta \otimes I).$$

*Proof.* From the proof of [21, Proposition 3.4], we know that

$$\tilde{S}((\iota \bar{\otimes} \omega)U) = (\iota \otimes \omega)(U^*), \quad \omega \in B(\mathcal{H}_U)_*.$$

Hence, we get

$$\begin{aligned} (\iota \bar{\otimes} \omega)(\tilde{S} \odot \iota)((p_\alpha \otimes I)U) &= \tilde{S}(p_\alpha(\iota \bar{\otimes} \omega)U) \\ &= \tilde{S}((\iota \bar{\otimes} \omega)U)p_\beta = (\iota \otimes \omega)(U^*)p_\beta \\ &= (\iota \bar{\otimes} \omega)(U^*(p_\beta \otimes I)) \end{aligned}$$

for all  $\omega \in B(\mathcal{H}_U)_*$ .  $\square$

**Lemma 9.3.** *Assume that  $U$  is right-amenable. Then there exists a net  $(y_i)$  in  $\{y \in HS(\mathcal{H}_U) \mid y \geq 0, \|y\|_2 = 1\}$  such that*

$$\lim_i \|U(p_\alpha \otimes y_i)U^* - p_\alpha \otimes y_i\|_{2,\alpha} = 0, \quad \forall \alpha \in A.$$

*Proof.* We begin as in the proof of Proposition 6.4. Let  $m'_U$  be a right-invariant mean for  $U$ , so

$$m'_U((\omega \bar{\otimes} \iota)U(I \otimes x)U^*) = \omega(I)m'_U(x)$$

for all  $x \in B(\mathcal{H}_U)$ ,  $\omega \in \hat{\mathcal{M}}_*$ . As the normal states are weak\*-dense in  $S(B(\mathcal{H}_U))$ , we may pick a net  $(s_i) \subset TC(\mathcal{H}_U)_1^+$  such that  $m'_U$  is a weak\*-limit point of the net  $(Tr(s_i \cdot)) \subset S(B(\mathcal{H}_U))$ .

Now, we define a net  $(y_i)$  in  $\{y \in HS(\mathcal{H}_U) \mid y \geq 0, \|y\|_2 = 1\}$  by setting  $y_i = s_i^{1/2}$  for all  $i$ .

Let  $\alpha \in A$ . Hereafter, we write  $\hat{a}_\alpha$  to denote  $\hat{a}p_\alpha \in \hat{\mathcal{A}}_\alpha$  whenever  $a \in \mathcal{A}$ . Let  $b, b' \in \mathcal{A}$ . Set  $c_\alpha = \hat{\rho}^{-1}(\hat{b}'_\alpha)\hat{b}_\alpha^*$ . Then we have

$$\begin{aligned} &\lim_i \omega_{\hat{\Lambda}(\hat{b}'_\alpha), \hat{\Lambda}(\hat{b}_\alpha)}(I) Tr(xs_i) \\ &= \lim_i Tr((\omega_{\hat{\Lambda}(\hat{b}'_\alpha), \hat{\Lambda}(\hat{b}_\alpha)} \bar{\otimes} \iota)(U(I \otimes x)U^*)s_i) \\ &= \lim_i (\hat{\psi} \odot Tr)((\hat{b}_\alpha^* \otimes y_i)U(I \otimes x)U^*(\hat{b}'_\alpha \otimes y_i)). \end{aligned}$$

Thus

$$\lim_i (\hat{\psi} \odot Tr)(\hat{b}_\alpha^* \hat{b}'_\alpha \otimes xy_i^2 - (\hat{b}_\alpha^* \otimes y_i)U(I \otimes x)U^*(\hat{b}'_\alpha \otimes y_i)) = 0,$$

which gives

$$\lim_i (\hat{\psi} \odot Tr)(c_\alpha \otimes xy_i^2 - (c_\alpha \otimes y_i^2)U(I \otimes x)U^*) = 0.$$

Now, write  $(p_\alpha \otimes I)U = \sum_r a_r \otimes x_r \in \hat{\mathcal{A}}_\alpha \odot B(\mathcal{H}_U)$ .

Then, using that  $Tr$  is a trace at the third step and Proposition 2.2 at the final step, we get that

$$\begin{aligned}
& (\hat{\psi} \odot Tr)(c_\alpha \otimes xy_i^2 - (c_\alpha \otimes y_i^2)U(I \otimes x)U^*) \\
&= (\hat{\psi} \odot Tr)(c_\alpha \otimes xy_i^2) - \sum_{r,s} (\hat{\psi} \odot Tr)(c_\alpha a_r a_s^* \otimes y_i^2 x_r x_s^*) \\
&= (\hat{\psi} \odot Tr)(c_\alpha \otimes xy_i^2) - \sum_{r,s} (\hat{\psi} \odot Tr)(c_\alpha a_r a_s^* \otimes x x_s^* y_i^2 x_r) \\
&= (\hat{\psi} \odot Tr)((c_\alpha \otimes x)(p_\alpha \otimes y_i^2 - \sum_{r,s} (a_r a_s^* \otimes x_s^* y_i^2 x_r)) \\
&= (\hat{\psi} \odot Tr)((c_\alpha \otimes x)(p_\alpha \otimes y_i^2 - \sum_{r,s} a_r p_\alpha a_s^* \otimes x_s^* y_i^2 x_r)) \\
&= (Tr_\alpha \odot Tr)((c_\alpha \otimes x)(p_\alpha f_{-1} \otimes y_i^2 - \sum_{r,s} a_r p_\alpha a_s^* f_{-1} \otimes x_s^* y_i^2 x_r))
\end{aligned}$$

converges to zero. Note that if we let  $b$  and  $b'$  vary in  $\mathcal{A}$ , then  $c_\alpha$  will give all elements in  $\hat{\mathcal{A}}_\alpha$  (using that  $\hat{\mathcal{A}}^2 = \hat{\mathcal{A}}$  and Proposition 2.2). Hence, adapting Namioka's argument [16, Proof of Theorem 2.4.2] by considering the locally convex product space  $\prod\{\hat{\mathcal{A}}_\alpha \otimes TC(\mathcal{H}_U), \alpha \in A\}$  with the product of the  $\|\cdot\|_{1,\alpha}$ -norm topologies, we may in fact assume that

$$\lim_i \|p_\alpha f_{-1} \otimes y_i^2 - \sum_{r,s} a_r p_\alpha a_s^* f_{-1} \otimes x_s^* y_i^2 x_r\|_{1,\alpha} = 0.$$

Now, as  $\hat{\mathcal{A}}_\alpha$  is a matrix algebra, we can apply any linear map on the first tensor factor and still keep convergence in 1-norm. Doing this with  $\tilde{S}(\cdot f_1) \odot \iota$ , we get

$$\lim_i \|p_\beta \otimes y_i^2 - \sum_{r,s} \tilde{S}(a_s^*) p_\beta \tilde{S}(a_r) \otimes x_s^* y_i^2 x_r\|_{1,\alpha} = 0.$$

Now, Lemma 9.2 says that  $(\tilde{S} \odot \iota)(p_\alpha \otimes I)U = U^*(p_\beta \otimes I)$ . Using this, our last equation reads as

$$\lim_i \|p_\beta \otimes y_i^2 - U(p_\beta \otimes y_i^2)U^*\|_{1,\alpha} = 0.$$

Using the Powers-Størmer inequality (see [8, Lemma 4.2]), one gets then

$$\lim_i \|p_\beta \otimes y_i - U(p_\beta \otimes y_i)U^*\|_{2,\alpha} = 0.$$

As  $\alpha \rightarrow \beta$  is a bijection of  $A$ , we are done.  $\square$

**Lemma 9.4.** *Assume that  $U$  is right-amenable and that  $(\mathcal{A}, \Delta)$  has a tracial Haar state. Then there exists a net  $(y_i)$  in  $\{y \in HS(\mathcal{H}_U) \mid y \geq 0, \|y\|_2 = 1\}$  such that*

$$\lim_i \|U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y_i) - \hat{\Lambda}(p_\alpha) \otimes y_i\|_2 = 0 \quad \forall \alpha \in A.$$

*Proof.* Since we assume that  $(\mathcal{A}, \Delta)$  has a tracial Haar state, it is well known that  $(\mathcal{M}, \Delta_r)$  is a compact Kac algebra. Hence,  $(\hat{\mathcal{M}}, \hat{\Delta}_{r, \text{op}})$  is then a discrete Kac algebra and we may identify  $\tilde{R}$  with the extension of  $\hat{S}$  to  $\hat{\mathcal{M}}$ . Recall from Proposition 4.4 that

$$U_{HS} = (\iota \bar{\otimes} l)U(\tilde{R} \bar{\otimes} r)U.$$

Let  $\alpha \in A$ ,  $y \in HS(\mathcal{H}_U)$  and set  $p_\beta = \tilde{R}(p_\alpha)$ . Then

$$U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y) = (\iota \bar{\otimes} l)U(\tilde{R} \bar{\otimes} r)(U)(p_\alpha \otimes I)(\hat{\Lambda}(p_\alpha) \otimes y).$$

Now, using Lemma 9.2, we get

$$\begin{aligned} (\tilde{R} \bar{\otimes} r)(U)(p_\alpha \otimes I) &= (\tilde{R} \bar{\otimes} r)((p_\beta \otimes I)U) \\ &= (\iota \odot r)(\tilde{R} \odot \iota)((p_\beta \otimes I)U) = (\iota \odot r)(U^*(p_\alpha \otimes I)). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y) &= (\iota \bar{\otimes} l)U(\iota \odot r)(U^*(p_\alpha \otimes I))(\hat{\Lambda}(p_\alpha) \otimes y) \\ &= (\iota \odot l)(U(p_\alpha \otimes I))(\iota \odot r)(U^*(p_\alpha \otimes I))(\hat{\Lambda}(p_\alpha) \otimes y) = (\hat{\Lambda} \odot \iota)(U(p_\alpha \otimes y)U^*). \end{aligned}$$

Now, since  $\varphi$  is assumed to be tracial, we have  $f_1 = \varepsilon$ . According to Proposition 2.2, we then have

$$\hat{\psi}(x) = \oplus_\alpha Tr_\alpha(p_\alpha x), \quad x \in \hat{\mathcal{A}}.$$

It follows that the Hilbert space norm on  $\mathcal{H} \otimes HS(\mathcal{H}_U)$  agrees on each subspace  $\hat{\Lambda}(\hat{\mathcal{A}}_\alpha) \odot HS(\mathcal{H}_U)$  with the  $\|\cdot\|_{2, \alpha}$ -norm on  $\hat{\mathcal{A}}_\alpha \odot HS(\mathcal{H}_U)$ . Therefore, choosing the net  $(y_i)$  as the one provided by Lemma 9.3, we get

$$\begin{aligned} \lim_i \|U_{HS}(\hat{\Lambda}(p_\alpha) \otimes y_i) - \hat{\Lambda}(p_\alpha) \otimes y_i\|_2 &= \|(\hat{\Lambda} \odot \iota)(U(p_\alpha \otimes y_i)U^*) - (\hat{\Lambda} \odot \iota)(p_\alpha \otimes y_i)\|_2 \\ &= \lim_i \|U(p_\alpha \otimes y_i)U^* - p_\alpha \otimes y_i\|_{2, \alpha} = 0 \end{aligned}$$

for all  $\alpha \in A$ , which shows the lemma.  $\square$

We are now in position to derive the following analog of [8, Theorem 5.1].

**Theorem 9.5.** *Assume that  $(\mathcal{A}, \Delta)$  is of compact type and has a tracial Haar state. Let  $U$  be a unitary corepresentation of  $(\hat{A}_r, \hat{\Delta}_{r, \text{op}})$ . Then  $U$  is right-amenable if, and only if,  $U_{HS}$  has the WCP.*

*Proof.* Assume that  $U$  is right-amenable. Combining Lemma 9.4 with Lemma 9.1, we deduce that  $U_{HS}$  has the WCP. The converse implication is shown in Proposition 7.10 (5).  $\square$

*Remark.* Let  $(\mathcal{A}, \Delta)$  and  $U$  be as in Theorem 9.5. Recall from 4.2 that we can associate with  $U$  another Hilbert-Schmidt corepresentation  $U_{HS'} \simeq (\overline{U})_{HS}$ . As  $U$  is left-amenable if, and only if,  $\overline{U}$  is right-amenable, we deduce from Theorem 9.5 that  $U$  is left-amenable if, and only if,  $U_{HS'}$  has the WCP.

As an application of Theorem 9.5 we give a new proof of the following result (see [28, Theorem 4.5], [7, Theorem 1.1]).

**Corollary 9.6.** *Assume that  $(\mathcal{A}, \Delta)$  is of compact type and has a tracial Haar state. Then  $(\mathcal{A}, \Delta)$  is co-amenable if, and only if,  $(\hat{\mathcal{A}}, \hat{\Delta})$  is amenable.*

*Proof.* We know from [7, Theorem 4.7] that  $(\hat{\mathcal{A}}, \hat{\Delta})$  is amenable whenever  $(\mathcal{A}, \Delta)$  is co-amenable. Assume now that  $(\hat{\mathcal{A}}, \hat{\Delta})$  is amenable. From Theorem 6.3, we deduce that  $\hat{W}$  is right-amenable. Using Theorem 9.5, we obtain that  $\hat{W}_{HS}$  has the WCP. It follows from Corollary 7.13 that  $(\mathcal{A}, \Delta)$  is co-amenable.  $\square$

The question whether the traciality assumption in Corollary 9.6 may be removed remains elusive. It relies on whether the traciality assumption in Lemma 9.4 may be removed.

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## References

- [1] Baaj, S. and Skandalis, G.: Unitaires multiplicatifs et dualité pour les produits croisés de  $C^*$ -algèbres. *Ann. Sci. Ec. Norm. Sup.* 26 (1993), 425–488.
- [2] Banica, T.: Representations of compact quantum groups and subfactors. *J. Reine Angew. Math.* 509 (1999), 167–198.
- [3] Banica, T.: Fusion rules for representations of compact quantum groups. *Expo. Math.* 17 (1999), 313–338.
- [4] Bédos, E.: Notes on hypertraces and  $C^*$ -algebras. *J. Operator Th.* 34 (1995), 285–306.
- [5] Bédos, E., Murphy, G.J. and Tuset, L.: Co-amenable of compact quantum groups. *J. of Geom. and Phys.* 40 (2001), 130-153.
- [6] Bédos, E., Murphy, G. J. and Tuset, L.: Amenability and co-amenable for algebraic quantum groups. Preprint (2001).
- [7] Bédos, E., Murphy, G. J. and Tuset, L.: Amenability and co-amenable for algebraic quantum groups, II. Preprint(2001).
- [8] Bekka, M. E. B.: Amenable unitary representation of locally compact groups, *Invent. Math.* 100 (1990), 383–401 .
- [9] Blanchard, E.: Déformations de  $C^*$ -Algèbres de Hopf. *Bull. Soc. Math. France* 124 (1996), 141–215.

- [10] Dixmier, J.: Les  $C^*$ -algèbres et leurs représentations, Gauthiers-Villars, Paris (1969).
- [11] Effros, E.G. and Ruan, Z.-J.: Discrete quantum groups. I. The Haar measure. Intern. J. Math. 5 (1994), 681-723.
- [12] Enock, M. and Schwartz, J.M. : Kac algebras and duality of locally compact groups. Springer Verlag, 1992.
- [13] Enock, M. and Schwartz, J.M. : Algebres de Kac moyennables. Pac. J. Math. 125 (1986), 363-379.
- [14] Fell, J.M.G. : Weak containment and induced representations of groups. Can. J. Math. 14 (1962), 237-268.
- [15] Ghez, Lima, R., Roberts, J. E. :  $W^*$ -categories. Pac. J. Math. 120 (1985), 79-109.
- [16] Greenleaf, F. P.: Invariant means on topological groups. Van Nostrand (New York), 1969.
- [17] de la Harpe, P. and Valette, A.: La propriété (T) de Kazhdan pour les groupes localement compacts. Asterisque 175, Soc. Math. de France, 1989.
- [18] Kustermans, J.: Examining the dual of an algebraic quantum group. Preprint Odense Universitet (1997). (arXiv:funct-an/9704006).
- [19] Kustermans, J.: Universal  $C^*$ -algebraic quantum groups arising from algebraic quantum groups. Preprint Odense Universitet (1997). (arXiv:funct-an/9704004).
- [20] Kustermans, J. and Van Daele, A.:  $C^*$ -algebraic quantum groups arising from algebraic quantum groups. Int. J. Math. 8 (1997), 1067-1139 .
- [21] Kustermans, J. and Vaes, S.: Locally compact quantum groups. Ann. Scient. Ec. Norm. Sup. 33 (2000), 837-934.
- [22] Longo, R. and Roberts, J. E. : A theory of dimension.  $K$ -Theory 11 (1997), 103-159.
- [23] Ng, C.- K. : Amenability of Hopf  $C^*$ -algebras. Proceedings of the 17th OT Conference (2000), 269-284.
- [24] Ng, C.- K. : Examples of amenable Kac algebras. Preprint (2000).
- [25] Paterson, A. L.: Amenability. Math. Surveys and Monographs 29, Amer. Math. Soc. (1988).
- [26] Petrescu, S. and Joita M.: Property (T) for Kac algebras. Rev. Roum. Math. Pures Appl. 37 (1992), 163-178.

- [27] Roberts, J. E. and Tuset, L.: On the equality of  $q$ -dimension and intrinsic dimension, *J. Pure Appl. Alg.* 156 (2001), 329–343 .
- [28] Ruan, Z.-J. : Amenability of Hopf von Neumann algebras and Kac algebras. *J. Funct. Anal.* 139 (1996), 466–499.
- [29] Stratila, S. : Modular theory in operator algebras. Abacus Press, Tunbridge Wells, Kent (1981).
- [30] Van Daele, A.: Multiplier Hopf algebras. *Trans. Amer. Math. Soc.* 342 (1994), 917–932.
- [31] Van Daele, A.: An algebraic framework for group duality. *Adv. Math.* 140 (1998), 323–366.
- [32] Voiculescu, D.: Amenability and Katz Algebras. *Algèbres d’Opérateurs et leurs Applications en Physique Mathématique*, N 274, Colloques Internationaux C.N.R.S. (1977), 451–457.
- [33] Woronowicz, S.L.: Compact matrix pseudogroups. *Comm. Math. Phys.* 111 (1987), 613–665.
- [34] Woronowicz, S.L.: Compact quantum groups, in “Symétries Quantiques”, North Holland, Amsterdam (1998), 845–884.

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