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Stochastic fractional potential theory

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Dedicated to Professor Olli Martio on the occasion of his 60th birthday.

Abstract

We present a white noise calculus for d-parameter fractional Brownian motion $B_H(x,\omega)$; $x \in \mathbb{R}^d$, $\omega \in \Omega$ with general d-dimensional Hurst parameter $H = (H_1, \dots, H_d) \in (0,1)^d$. As an illustration we solve the Poisson problem $\Delta U(x) = -W_H(x)$; $x \in D$, $U = 0$ on ∂D, where the potential $W_H(x)$ is d-parameter fractional white noise given by $W_H(x) = \frac{\partial^d B_H(x)}{\partial x_1 \dots \partial x_d}$, and $D \subset \mathbb{R}^d$ is a given bounded smooth domain.

1 Introduction

Recall that a *1-parameter fractional Brownian motion* (fBm) *with Hurst parameter* $H \in (0,1)$ is a Gaussian stochastic process $B_H(t) = B_H(t,\omega)$; $t \in \mathbb{R}$, $\omega \in \Omega$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t^{(H)}, P)$ with the mean

(1.1) $$E[B_H(t)] = B_H(0) = 0 \qquad \text{for all } t \in \mathbb{R}$$

and covariance

(1.2) $$E[B_H(s)B_H(t)] = \tfrac{1}{2}\big\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\big\} \qquad \text{for all } s,t \in \mathbb{R}\ ,$$

where E denotes expectation with respect to P. Note that if $H = \frac{1}{2}$ then $B_H(t)$ coincides with the classical Brownian motion.

For any $H \in (0,1)$ the process $B_H(t)$ is *H-self-similar*, in the sense that the law of $\{B_H(\alpha t)\}_{t\in\mathbb{R}}$ is the same as the law of $\{\alpha^H B_H(t)\}_{t\in\mathbb{R}}$ for all $\alpha > 0$.

One of the reasons of the interest of fractional Brownian motion is that it can be used to model random phenomena with memory.

For example, if $\frac{1}{2} < H < 1$ then $B_H(t)$ has a *long range dependence*, in the sense that

(1.3) $$\sum_{n=1}^{\infty} E[B_H(1)(B_H(n+1) - B_H(n))] = \infty\ .$$

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In this case the process is *persistent*, in the sense that high values have a tendency to be followed by an increase and low values by a decrease. This type of behavior is often observed in the levels of rivers, the characters of solar activity, the widths of consecutive annual rings and in the values of log returns in finance.

Similarly, if $0 < H < \frac{1}{2}$ then

$$E[B_H(1)(B_H(n+1) - B_H(n))] < 0 \tag{1.4}$$

and the process is *anti-persistent*, in the sense that high values have a tendency to be followed by a decrease and low values by an increase. This feature makes the process natural for turbulence modeling. Indeed, fractional Brownian motion was first introduced by Kolmogorov in 1940 (see [Ko]), in connection with turbulence studies. In 1968 the process was reintroduced by Mandelbrot and van Ness [MvN], who gave the process its current name and suggested a number of applications.

For more information on 1-parameter fractional Brownian motion we refer to the book by Shiryaev [S] and the references therein.

There is a natural generalization of fBm to the multi-parameter case:

Fix a parameter dimension $d \in \mathbb{N}$ and a *Hurst parameter* $H = (H_1, H_2, \ldots, H_d) \in (0,1)^d$. Then we define the d-parameter fractional Brownian motion (or fractional Brownian *field*) $B_H(x_1, \ldots, x_d)$; $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ as the Gaussian process (field) with mean

$$E[B_H(x)] = B_H(0) = 0 \qquad \text{for all } \; x \in \mathbb{R}^d \tag{1.5}$$

and covariance

$$E[B_H(x)B_H(y)] = (\tfrac{1}{2})^d \prod_{i=1}^{d} (|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}) \qquad \text{for all } \; x, y \in \mathbb{R}^d. \tag{1.6}$$

These stochastic processes have been suggested in the modeling of the shape of mountain ranges ($d = 2$), the density of clouds ($d = 3$) and many other quantities. We refer to [AF] and [M] for more examples of modeling by multi-parameter fBm.

A stochastic calculus for 1-parameter fBm based on the Wick-Itô integral was constructed by [DHP] in the case $\frac{1}{2} < H < 1$. This was generalized to a fractional white noise calculus in [HØ], still for the case $\frac{1}{2} < H < 1$. Subsequently this 1-dimensional theory was extended (with certain restrictions) to be valid for all Hurst coefficients $H \in (0,1)$ by [EvdH].

A multi-parameter fractional white noise calculus was developed in [H1], [H2] and subsequently in [HØZ1] and [ØZ], where it was used to solve certain stochastic partial differential equations driven by multi-parameter fractional white noise $W_H(x)$. However, the presentation in all these papers was based on the assumption that $H = (H_1, \ldots, H_d) \in (\frac{1}{2}, 1)^d$.

The purpose of this paper is to give a survey of the multi-parameter fractional white noise theory valid for all Hurst parameters $H \in (0,1)^d$ as presented in [HØZ2]. Such a theory is constructed by making a synthesis of the 1-parameter approach of [EvdH] and the multi-parameter approach of [H1], [H2], [HØZ1] and [ØZ]. The theory is illustrated by solving explicitly the stochastic fractional Poisson equation

$$\Delta U(x) = -W_H(x) \; ; \qquad x \in D \subset \mathbb{R}^d \tag{1.7}$$

$$u(x) = 0 \; ; \qquad x \in \partial D \tag{1.8}$$

where D is a given bounded domain in $\mathbb{R}^d$ with smooth boundary ∂D and $W_H(x) = \frac{\partial^d B_H(x)}{\partial x_1 \ldots \partial x_d}$ is d-parameter fractional white noise.

2 Multiparameter fractional Brownian motion

We start by recalling the standard white noise construction of multiparameter *classical* Brownian motion $B(x)$; $x \in \mathbb{R}^d$. We refer to [HKPS], [HØUZ] and [Ku] for more details. Our presentation here will follow the presentation in [HØZ2] closely.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ and let $\Omega := \mathcal{S}'(\mathbb{R}^d)$ be its dual, usually called *the space of tempered distributions*. By the Bochner-Minlos theorem there exists a probability measure μ on the Borel σ-algebra $\mathcal{B}(\Omega)$ such that

$$\int_\Omega e^{i\langle\omega,f\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|f\|^2}; \qquad f \in \mathcal{S}(\mathbb{R}^d) \tag{2.1}$$

where $\langle\omega, f\rangle = \omega(f)$ denotes the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R}^d)$ applied to $f \in \mathcal{S}(\mathbb{R}^d)$ and $\|f\|^2 = \int_{\mathbb{R}^d} |f(x)|^2 dx = \|f\|^2_{L^2(\mathbb{R}^d)}$. From (2.1) one can deduce that

$$E_\mu[\langle\omega, f\rangle] = 0 \qquad \text{for all } \ f \in \mathcal{S}(\mathbb{R}^d) \tag{2.2}$$

where E_μ denotes the expectation with respect to μ. Moreover, we have the isometry

$$E_\mu[\langle\omega, f\rangle\langle\omega, g\rangle] = (f, g)_{L^2(\mathbb{R}^d)} \ ; \qquad f, g \in \mathcal{S}(\mathbb{R}^d) \ . \tag{2.3}$$

Using this isometry we can extend the definition of $\langle\omega, f\rangle \in L^2(\mu)$ from $\mathcal{S}(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ as follows:

$$\langle\omega, f\rangle = \lim_{n\to\infty} \langle\omega, f_n\rangle \quad (\text{limit in } L^2(\mu))$$

when $f_n \in \mathcal{S}(\mathbb{R}^d)$, $f_n \to f \in L^2(\mathbb{R}^d)$ (limit in $L^2(\mathbb{R}^d)$).

In particular, we can now define, for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,

$$\widetilde{B}(x) = \widetilde{B}(x, \omega) = \langle\omega, \mathcal{X}_{[0,x]}(\cdot)\rangle \ ; \qquad \omega \in \Omega \tag{2.4}$$

where

$$\mathcal{X}_{[0,x]}(y) = \prod_{i=1}^{d} \mathcal{X}_{[0,x_i]}(y_i) \qquad \text{for } y = (y_1, \dots, y_d) \in \mathbb{R}^d \tag{2.5}$$

and

$$\mathcal{X}_{[0,x_i]}(y_i) = \begin{cases} 1 & \text{if} \quad 0 \le y_i \le x_i \\ -1 & \text{if} \quad x_i \le y_i \le 0, \text{ except } x_i = y_i = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

By Kolmogorov's continuity theorem the process $\{\widetilde{B}(x)\}$ has a continuous version which we will denote by $\{B(x)\}$. By (2.1)–(2.3) it follows that $\{B(x)\}$ is a Gaussian process with mean

$$E[B(x)] = B(0) = 0 \tag{2.7}$$

and covariance (using (2.3))

$$E[B(x)B(y)] = (\mathcal{X}_{[0,x]}, \mathcal{X}_{[0,y]})_{L^2(\mathbb{R}^d)} = \begin{cases} \prod_{i=1}^{d} x_i \wedge y_i & \text{if} \quad x_i, y_i \geq 0 \text{ for all } i \\ \prod_{i=1}^{d} (-x_i) \wedge (-y_i) & \text{if} \quad x_i, y_i \leq 0 \text{ for all } i \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

Therefore $\{B(x)\}_{x\in\mathbb{R}^d}$ is a d-parameter Brownian motion.

We now use this Brownian motion to construct d-parameter *fractional* Brownian motion $B_H(x)$ for all Hurst parameters $H = (H_1, \dots, H_d) \in (0,1)^d$. We do this by extending the procedure of [EvdH] to the d-dimensional case, as follows:

For $0 < H_j < 1$ put

$$K_j = k_j\Big[2\Gamma(H_j - \tfrac{1}{2}) \cos\Big(\frac{\pi}{2}(H_j - \tfrac{1}{2})\Big)\Big]^{-1}, \qquad k_j = \sin(\pi H_j)\Gamma(2H_j + 1) \tag{2.9}$$

and if $g \in \mathcal{S}(\mathbb{R}^d)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, define $m_j g(\cdot) : \mathbb{R}^d \to \mathbb{R}$ by

$$m_j g(x) = \begin{cases} K_j \int\limits_{\mathbb{R}} \frac{g(x - t\varepsilon^{(j)}) - g(x)}{|t|^{\frac{3}{2} - H_j}} dt & \text{if} \quad 0 < H_j < \frac{1}{2} \\ g(x) & \text{if} \quad H_j = \frac{1}{2} \\ K_j \int\limits_{\mathbb{R}} \frac{g(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d) dt}{|x_j - t|^{\frac{3}{2} - H_j}} & \text{if} \quad \frac{1}{2} < H_j < 1 \end{cases} \tag{2.10}$$

where

$$\varepsilon^{(j)} = (0, 0, \dots, 1, \dots, 0)\ , \quad \text{the } j\text{'th unit vector.} \tag{2.11}$$

Then define

$$M_H f(x) = m_1(m_2(\dots(m_{d-1}(m_d f))\dots))(x)\ ; \qquad f \in \mathcal{S}(\mathbb{R}^d)\ . \tag{2.12}$$

Note that if $f(x) = f_1(x_1) \dots f_d(x_d) =: (f_1 \otimes \dots \otimes f_d)(x)$ is a tensor product, then

$$M_H f(x) = \prod_{j=1}^{d} (M_{H_j} f_j)(x_j) \tag{2.13}$$

where

$$M_{H_j} f_j(x_j) = \begin{cases} K_j \int\limits_{\mathbb{R}} \frac{f_j(x_j - t) - f_j(x_j)}{|t|^{\frac{3}{2} - H_j}} dt & ; \quad 0 < H_j < \frac{1}{2} \\ f_j(x_j) & ; \quad H_j = \frac{1}{2} \\ K_j \int\limits_{\mathbb{R}} \frac{f_j(t) dt}{|t - x_j|^{\frac{3}{2} - H_j}} & ; \quad \frac{1}{2} < H_j < 1 \end{cases} \tag{2.14}$$

Therefore, if

$$\mathcal{F}g(\xi) := \hat{g}(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} g(x) dx\ ; \qquad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d\ ,$$

denotes the Fourier transform of g, we have by (2.13)

$$\widehat{M_H f}(\xi) = \prod_{j=1}^{d} \widehat{M_{H_j} f_j}(\xi_j) = \prod_{j=1}^{d} k_j |\xi_j|^{\frac{1}{2}-H_j} \hat{f}_j(\xi_j) \tag{2.15}$$

and

$$\widehat{M_H^{-1} f}(\xi) = \Big(\prod_{j=1}^{d} k_j |\xi_j|^{\frac{1}{2}-H_j} \Big)^{-1} \hat{f}_j(\xi) \; .$$

M_H maps $\mathcal{S}(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$. For more information see [EvdH, Appendix].

We now construct d-parameter fractional Brownian motion $B_H(x)$ with Hurst parameter $H = (H_1, \dots, H_d) \in (0,1)^d$ as follows:

First define

$$\widetilde{B}_H(x) = \widetilde{B}_H(x, \omega) = \langle \omega, M_H(\mathcal{X}_{[0,x]}(\cdot)) \rangle \tag{2.16}$$

with $\mathcal{X}_{[0,x]}(\cdot)$ as in (2.5)–(2.6). Then $\widetilde{B}_H(x)$ is a Gaussian process with mean

$$E[\widetilde{B}_H(x)] = \widetilde{B}_H(0) = 0 \tag{2.17}$$

and covariance (using (2.13) and [EvdH, (1.13)])

$$\begin{aligned} E[\widetilde{B}_H(x)\widetilde{B}_H(y)] &= \int_{\mathbb{R}^d} M_H(\mathcal{X}_{[0,x]}(z)) M_H(\mathcal{X}_{[0,y]}(z)) dz \\ &= \int_{\mathbb{R}^d} \prod_{i=1}^{d} M_{H_i} \mathcal{X}_{[0,x_i]}(z_i) \cdot \prod_{j=1}^{d} M_{H_j} \mathcal{X}_{[0,y_j]}(z_j) dz_1 \dots dz_d \\ &= \prod_{j=1}^{d} \int_{\mathbb{R}} M_{H_j} \mathcal{X}_{[0,x_j]}(t) \cdot M_{H_j} \mathcal{X}_{[0,y_j]}(t) dt \\ &= (\tfrac{1}{2})^d \prod_{j=1}^{d} \{ |x_j|^{2H_j} + |y_j|^{2H_j} - |x_j - y_j|^{2H_j} \} \; ; \qquad x, y \in \mathbb{R}^d \; . \end{aligned} \tag{2.18}$$

By Kolmogorov's continuity theorem we get that $\{\widetilde{B}_H(x)\}$ has a continuous version, which we denote by $\{B_H(x)\}$. From (2.17), (2.18) we conclude that $B_H(x)$ is a d-parameter fractional Brownian motion with Hurst parameter $H = (H_1, \dots, H_d) \in (0,1)^d$.

If f is a simple (deterministic) function of the form

$$f(x) = \sum_{j=1}^{N} a_j \mathcal{X}_{[0,y^{(j)}]}(x) \; ; \qquad x \in \mathbb{R}^d$$

for some $a_j \in \mathbb{R}$, $y^{(j)} \in \mathbb{R}^d$ and $N \in \mathbb{N}$, then we define its integral with respect to B_H by

$$\int_{\mathbb{R}^d} f(x) dB_H(x) = \sum_{j=1}^{N} a_j B_H(y^{(j)}) \; .$$

Note that by (2.16) this coincides with $\langle \omega, M_H f \rangle$, and we have the isometry

$$E\Big[\Big(\int_{\mathbb{R}^d} f(x)dB_H(x)\Big)^2\Big] = E[\langle \omega, M_H f\rangle^2] = \|M_H f\|^2_{L^2(\mathbb{R}^d)} .$$

By linearity and completeness we can therefore extend the definition of this integral to all $g \in L^2_H(\mathbb{R}^d)$, where

$$L^2_H(\mathbb{R}^d) = \{g : \mathbb{R}^d \to \mathbb{R}; \|g\|_{L^2_H(\mathbb{R}^d)} := \|M_H g\|_{L^2(\mathbb{R}^d)} < \infty\} . \tag{2.19}$$

Then it follows from (2.16) that

$$\langle \omega, M_H g\rangle = \int_{\mathbb{R}^d} g(x)dB_H(x) \qquad \text{for all } \; g \in L^2_H(\mathbb{R}^d) . \tag{2.20}$$

Moreover, if $f, g \in L^2_H(\mathbb{R}^d)$ then we have the isometry

$$\begin{aligned} E\Big[\Big(\int_{\mathbb{R}^d} f(x)dB_H(x)\Big)\Big(\int_{\mathbb{R}^d} g(x)dB_H(x)\Big)\Big] &= E[\langle \omega, M_H f\rangle\langle \omega, M_H g\rangle] \\ &= (M_H f, M_H g)_{L^2(\mathbb{R}^d)} = (f, g)_{L^2_H(\mathbb{R}^d)} . \end{aligned} \tag{2.21}$$

3 Multiparameter fractional white noise calculus

With the processes $B_H(x)$ constructed in Section 2 as a starting point we proceed to develop a d-parameter white noise theory as in [HØZ1] and [ØZ], but modified according to the 1-parameter approach in [EvdH].

Let

$$h_n(t) = (-1)^n e^{\frac{t^2}{2}} \frac{d^n}{dt^n}\big(e^{-\frac{t^2}{2}}\big) ; \qquad n = 0, 1, 2, \dots ; \;\; t \in \mathbb{R}$$

be the *Hermite polynomials* and let

$$\tilde{h}_n(t) = \pi^{-\frac{1}{4}}\big((n-1)!\big)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}\, t) e^{-\frac{t^2}{2}} ; \qquad n = 1, 2, \dots ; \;\; t \in \mathbb{R} \tag{3.1}$$

be the *Hermite functions.*

If $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ (with $\mathbb{N} = \{1, 2, \dots\}$) and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ define

$$\eta_\alpha(x) = \tilde{h}_{\alpha_1}(x_1) \dots \tilde{h}_{\alpha_d}(x_d) = (\tilde{h}_{\alpha_1} \otimes \dots \otimes \tilde{h}_{\alpha_d})(x) \tag{3.2}$$

and

$$e_\alpha(x) = \big(M^{-1}_{H_{\alpha_1}} \tilde{h}_{\alpha_1}\big)(x_1) \dots \big(M^{-1}_{H_{\alpha_d}} \tilde{h}_{\alpha_d}\big)(x_d) = (M_H^{-1} \eta_\alpha)(x) . \tag{3.3}$$

Let $\{\alpha^{(i)}\}_{i=1}^\infty$ be a fixed ordering of $\mathbb{N}^d$ with the property that, with $|\alpha^{(i)}| = \alpha^{(i)}_1 + \dots + \alpha^{(i)}_d$,

$$i < j \Rightarrow |\alpha^{(i)}| \le |\alpha^{(j)}| . \tag{3.4}$$

Note that this implies that there exists a constant $C < \infty$ such that

$$|\alpha^{(k)}| \le C\, k \qquad \text{for all } \; k . \tag{3.5}$$

With a slight abuse of notation let us write

$$\eta_n(x) := \eta_{\alpha^{(n)}}(x) = M_H e_n(x) \tag{3.6}$$

and

$$e_n(x) := e_{\alpha^{(n)}}(x) = M_H^{-1}\eta_n(x) \; ; \qquad n = 1, 2, \ldots \tag{3.7}$$

Now let $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$ denote the set of all finite sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $m = 1, 2, \ldots$ Then if $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{J}$ we define

$$\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(\langle \omega, \eta_1 \rangle) \ldots h_{\alpha_m}(\langle \omega, \eta_m \rangle) \; . \tag{3.8}$$

In particular, note that by (2.19) we have

$$\begin{aligned} \mathcal{H}_{\varepsilon^{(i)}}(\omega) &= h_1(\langle \omega, \eta_i \rangle) = \langle \omega, \eta_i \rangle = \int_{\mathbb{R}^d} \eta_i(x) dB(x) \\ &= \int_{\mathbb{R}^d} M_H e_i(x) dB(x) = \langle \omega, M_H e_i \rangle = \int_{\mathbb{R}^d} e_i(x) dB_H(x) \; ; \qquad i = 1, 2, \ldots \end{aligned} \tag{3.9}$$

We recall the following well-known result:

Theorem 3.1 (The chaos expansion theorem)
Every $F \in L^2(\mu)$ can be written on the form

$$F(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathcal{H}_\alpha(\omega) \tag{3.10}$$

where $c_\alpha \in \mathbb{R}$. Moreover, we have the isometry

$$\|F\|^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{J}} \alpha! \, c_\alpha^2 \tag{3.11}$$

where $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_m!$ if $\alpha = (\alpha_1, \ldots, \alpha_m)$.

Note that if $f \in \mathcal{S}(\mathbb{R}^d)$ then $M_H f \in L^2(\mathbb{R}^d)$. Moreover, if $f, g \in \mathcal{S}(\mathbb{R}^d)$ then

$$(g, M_H f)_{L^2(\mathbb{R}^d)} = (\hat{g}, \widehat{M_H f})_{L^2(\mathbb{R}^d)} = (M_H g, f)_{L^2(\mathbb{R}^d)} \; . \tag{3.12}$$

Therefore, since the action of $\omega \in \Omega = \mathcal{S}'(\mathbb{R}^d)$ extends to $L^2(\mathbb{R}^d)$ by (2.3), we can extend the definition of the operator M_H from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ by setting

$$\langle M_H \omega, f \rangle = \langle \omega, M_H f \rangle \; ; \qquad f \in \mathcal{S}(\mathbb{R}), \;\; \omega \in \mathcal{S}'(\mathbb{R}) \; . \tag{3.13}$$

We now define

$$L^2_H(\mu) = \{G : \Omega \to \mathbb{R}; G \circ M_H \in L^2(\mu)\} \tag{3.14}$$

and

$$\|G\|^2_{L^2_H(\mu)} = \|G \circ M_H\|^2_{L^2(\mu)} \qquad \text{for } \; G \in L^2_H(\mu) \; . \tag{3.15}$$

Example 3.2 The chaos expansion of classical Brownian motion $B(x) \in L^2(\mu)$ is

$$B(x) = \langle \omega, \mathcal{X}_{[0,x]} \rangle = \sum_{k=1}^{\infty} (\mathcal{X}_{[0,x]}, \eta_k)_{L^2(\mathbb{R}^d)} \langle \omega, \eta_k \rangle = \sum_{k=1}^{\infty} \Big(\int_{-\infty}^{x} \eta_k(y) dy \Big) \cdot \mathcal{H}_{\varepsilon^{(k)}}(\omega) \, , \tag{3.16}$$

where in general we put

$$\int_{-\infty}^{x} g(y) dy = \int_{-\infty}^{x_d} \cdots \int_{-\infty}^{x_1} g(y) dy_1 \dots dy_d \, ; \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d \, . \tag{3.17}$$

Hence by (2.16) the chaos expansion of fractional Brownian motion $B_H(x) \in L^2_H(\mu)$ is

$$\begin{aligned} B_H(x) &= \langle \omega, M_H \mathcal{X}_{[0,x]} \rangle = \langle M_H \omega, \mathcal{X}_{[0,x]} \rangle = \sum_{k=1}^{\infty} (\mathcal{X}_{[0,x]}, e_k)_{L^2_H(\mathbb{R}^d)} \langle M_H \omega, e_k \rangle \\ &= \sum_{k=1}^{\infty} (M_H \mathcal{X}_{[0,x]}, \eta_k)_{L^2(\mathbb{R}^d)} \langle \omega, \eta_k \rangle = \sum_{k=1}^{\infty} (\mathcal{X}_{[0,x]}, M_H \eta_k)_{L^2(\mathbb{R}^d)} \mathcal{H}_{\varepsilon^{(k)}}(\omega) \end{aligned} \tag{3.18}$$

$$= \sum_{k=1}^{\infty} \Big(\int_{-\infty}^{x} M_H \eta_k(y) dy \Big) \mathcal{H}_{\varepsilon^{(k)}}(\omega) \, . \tag{3.19}$$

Similarly, if $f \in L^2_H(\mathbb{R}^d)$ then by (2.19)

$$\int_{\mathbb{R}} f(x) dB_H(x) = \langle \omega, M_H f \rangle = \langle M_H \omega, f \rangle = \sum_{k=1}^{\infty} (M_H \eta_k, f)_{L^2(\mathbb{R}^d)} \mathcal{H}_{\varepsilon^{(k)}}(\omega) \, . \tag{3.20}$$

Next we define the *d-parameter Hida test function and distribution spaces* $(\mathcal{S})$ and $(\mathcal{S})^*$, respectively:

Definition 3.3
a) For $k = 1, 2, \dots$ let $(\mathcal{S})^{(k)}$ be the set of $G \in L^2(\mu)$ with expansion

$$G(\omega) = \sum_{\alpha} c_\alpha \mathcal{H}_\alpha(\omega)$$

such that

$$\|G\|^2_{(\mathcal{S})^{(k)}} := \sum_{\alpha} \alpha! \, c_\alpha^2 (2\mathbb{N})^{\alpha k} < \infty \tag{3.21}$$

where

$$(2\mathbb{N})^\beta = (2 \cdot 1)^{\beta_1} (2 \cdot 2)^{\beta_2} \cdots (2m)^{\beta_m} \qquad \text{if} \quad \beta = (\beta_1, \dots, \beta_m) \in \mathcal{J} \tag{3.22}$$

The space of *Hida test functions*, $(\mathcal{S})$, is defined by

$$(\mathcal{S}) = \bigcap_{k=1}^{\infty} (\mathcal{S})^{(k)}, \quad \text{equipped with the projective topology.} \tag{3.23}$$

b) For $q = 1, 2, \ldots$ let $(\mathcal{S})^{(-q)}$ be the set of all formal expansions

$$G = \sum_{\alpha} c_\alpha \mathcal{H}_\alpha(\omega)$$

such that

$$\|G\|_{(\mathcal{S})^{(-q)}} := \sum_{\alpha} \alpha!\, c_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty \,. \tag{3.24}$$

The space of *Hida distributions*, $(\mathcal{S})^*$, is defined by

$$(\mathcal{S})^* = \bigcup_{q=1}^{\infty} (\mathcal{S})^{(-q)}, \quad \text{equipped with the inductive topology.} \tag{3.25}$$

Note that with this definition we have

$$(\mathcal{S}) \subset L^2(\mu) \subset (\mathcal{S})^* \,. \tag{3.26}$$

Example 3.4 Define *fractional white noise*, $W_H(x)$, by

$$W_H(x) = \sum_{k=1}^{\infty} M_H \eta_k(x) \mathcal{H}_{\varepsilon^{(k)}}(\omega) \,; \qquad x \in \mathbb{R}^d \,. \tag{3.27}$$

Then $W_H(x) \in (\mathcal{S})^*$ because in this case, by (3.3) and (3.5),

$$\begin{aligned}
\sum_{\alpha} \alpha! c_\alpha^2 (2\mathbb{N})^{-q\alpha} &= \sum_{k=1}^{\infty} (M_H \eta_k)^2(x) (2\mathbb{N})^{-q\varepsilon^{(k)}} \\
&= \sum_{k=1}^{\infty} (M_{H_{\alpha_1^{(k)}}} \tilde{h}_{\alpha_1^{(k)}})^2(x_1) \ldots (M_{H_{\alpha_d^{(k)}}} \tilde{h}_{\alpha_d^{(k)}})^2(x_d)(2k)^{-q} \\
&\le \sum_{k=1}^{\infty} C_1 \Big(\prod_{j=1}^{d} (\alpha_j^{(k)})^{\frac{2}{3} - \frac{H_{\alpha_j^{(k)}}}{2}} \Big) (2k)^{-q} \le C_1 \sum_{k=1}^{\infty} (2k)^{\frac{2d}{3} - q} < \infty
\end{aligned}$$

for $q > \frac{2d}{3} + 1$ (C_1 is a constant). Here we have used the estimate

$$|M_{H_j} \tilde{h}_n(t)| \le C_2 n^{\frac{2}{3} - \frac{H_j}{2}} \quad \text{for all } t \ (C_2 \text{ constant}) \tag{3.28}$$

from Section 3 of [EvdH].

Note that from (3.27) and (3.19) we have that

$$\frac{\partial^d}{\partial x_1 \ldots \partial x_d} B_H(x) = W_H(x) \qquad (\text{in } (\mathcal{S})^*) \text{ for all } x \in \mathbb{R}^d \,. \tag{3.29}$$

This justifies the name *fractional white noise* for the process $W_H(x)$.

We now define the Wick product just as in [HØUZ], [HØZ1] and [ØZ]:

Definition 3.5 *Let* $F = \sum_{\alpha \in \mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega)$ *and* $G = \sum_{\beta \in \mathcal{J}} b_\beta \mathcal{H}_\beta(\omega)$ *be elements of* $(\mathcal{S})^*$. *Then we define their* Wick product, $(F \diamond G)(\omega)$, *by*

$$(F \diamond G)(\omega) = \sum_{\alpha, \beta \in \mathcal{J}} a_\alpha b_\beta \mathcal{H}_{\alpha+\beta}(\omega) = \sum_{\gamma \in \mathcal{J}} \Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) \mathcal{H}_\gamma(\omega) \,. \tag{3.30}$$

Remark It is not hard to prove that $F \diamond G \in (\mathcal{S})^*$. Moreover, if $F, G \in (\mathcal{S})$ then $F \diamond G \in (\mathcal{S})$ also.

Example 3.6
a) Let $f, g \in L^2_H(\mathbb{R}^d)$. Then by (3.20)

$$\begin{aligned}
\Big(\int_{\mathbb{R}^d} f dB_H\Big) \diamond \Big(\int_{\mathbb{R}^d} g dB_H\Big) &= \sum_{j,k=1}^{\infty} (M_H\eta_j, f)_{L^2(\mathbb{R}^d)} (M_H\eta_k, g)_{L^2(\mathbb{R}^d)} \mathcal{H}_{\varepsilon^{(j)}+\varepsilon^{(k)}} \\
&= \Big(\sum_{j=1}^{\infty} (M_H\eta_j, f)_{L^2(\mathbb{R}^d)} \mathcal{H}_{\varepsilon^{(j)}}\Big) \cdot \Big(\sum_{k=1}^{\infty} (M_H\eta_k, g)_{L^2(\mathbb{R}^d)} \mathcal{H}_{\varepsilon^{(k)}}\Big) \\
&\quad - \sum_{j=1}^{\infty} (M_H\eta_j, f)_{L^2(\mathbb{R}^d)} (M_H\eta_j, g)_{L^2(\mathbb{R}^d)} \\
&= \Big(\int_{\mathbb{R}^d} f dB_H\Big) \cdot \Big(\int_{\mathbb{R}^d} g dB_H\Big) - \sum_{j=1}^{\infty} (\eta_j, M_H f)_{L^2(\mathbb{R}^d)} (\eta_j, M_H g)_{L^2(\mathbb{R}^d)} \\
&= \Big(\int_{\mathbb{R}^d} f dB_H\Big) \cdot \Big(\int_{\mathbb{R}^d} g dB_H\Big) - (M_H f, M_H g)_{L^2(\mathbb{R}^d)} \\
&= \Big(\int_{\mathbb{R}^d} f dB_H\Big) \cdot \Big(\int_{\mathbb{R}^d} g dB_H\Big) - (f, g)_{L^2_H(\mathbb{R}^d)} \ . \qquad (3.31)
\end{aligned}$$

b) Similarly, by proceeding as in [HØ, Example 3.9] we obtain that if $f \in L^2_H(\mathbb{R}^d)$ then

$$\exp^{\diamond}(\langle \omega, f\rangle) := \sum_{n=1}^{\infty} \frac{1}{n!} \langle \omega, f\rangle^{\diamond n} \qquad (3.32)$$

converges in $(\mathcal{S})^*$ and

$$\exp^{\diamond}(\langle \omega, f\rangle) = \exp(\langle \omega, f\rangle - \tfrac{1}{2}(f, f)_{L^2_M(\mathbb{R}^d)}) \ . \qquad (3.33)$$

We now use multiparamter white noise and the Wick product to define integration of a general class of processes with respect to $B_H(x)$ as follows.

Definition 3.7 *Let $Y(x) : \mathbb{R}^d \to (\mathcal{S})^*$ be such that $Y(x) \diamond W_H(x)$ is integrable in $(\mathcal{S})^*$ with respect to Lebesgue measure in $\mathbb{R}^d$. Then we define*

$$\int_{\mathbb{R}^d} Y(x) dB_H(x) = \int_{\mathbb{R}^d} Y(x) \diamond W_H(x) dx \ .$$

We call this the Wick-Itô integral *with respect to $B_H(x)$.*

Remark If $d = 1$ and $H = \frac{1}{2}$ this integral represents an extension of the Hitsuda-Skorohod integral. See e.g. [HØUZ, Section 2.5] for details.

An example: The stochastic fractional Poisson equation

We now illustrate the use of the theory above by solving the Poisson equation with fractional white noise heat source:

Let $D \subset \mathbb{R}^d$ be a given bounded domain with smooth (C^∞) boundary. We want to find $U(\cdot) : \bar{D} \to (\mathcal{S})^*$ such that

$$\Delta U(x) = -W_H(x) \qquad \text{for } x \in D \tag{3.34}$$

$$U(x) = 0 \qquad \text{for } x \in \partial D \tag{3.35}$$

(where $\Delta = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator) and such that U is continuous on the closure $\bar{D}$ of D.

From classical potential theory we are led to the solution candidate

$$U(x) = \int_D G(x,y) W_H(y) dy = \int_D G(x,y) dB_H(y) \tag{3.36}$$

where G is the classical Green function for the Laplacian.

We first verify that $U(x) \in (\mathcal{S})^*$ for all x. To this end, consider the expansion of $U(x)$:

$$U(x) = \int_D G(x,y) \sum_{k=1}^{\infty} M_H \eta_k(y) \mathcal{H}_{\varepsilon^{(k)}}(\omega) dy$$

$$= \sum_{k=1}^{\infty} a_k(x) \mathcal{H}_{\varepsilon^{(k)}}(\omega), \quad \text{where}$$

$$a_k(x) = \int_D G(x,y) M_H \eta_k(y) dy \ . \tag{3.37}$$

By the estimate (3.28) we have

$$|a_k(x)| \le C_3 k^{\frac{2d}{3}} \int_D G(x,y) dy \le C_4 k^{\frac{2d}{3}}, \tag{3.38}$$

and therefore

$$\sum_{k=1}^{\infty} a_k^2(x) \big(2\mathbb{N}\big)^{-q\varepsilon_k} \le C_4^2 \sum_{k=1}^{\infty} \big(2k\big)^{\frac{4d}{3}} (2k)^{-q} < \infty$$

for $q > \frac{4d}{3} + 1$.

This proves that $U(x) \in (\mathcal{S})^*$ and the same estimate gives that $U : \bar{D} \to (\mathcal{S})^*$ is continuous.

The proof that $\Delta U(x) = -W_H(x)$ is identical to the proof given in [HØZ1, Section 3] and is omitted. We conclude that $U(x)$ given by (3.36) is indeed the solution of (3.34)–(3.35).

Thus we have:

Theorem 3.8 *Let $H = (H_1, \ldots, H_d) \in (0,1)^d$. The stochastic fractional Poisson equation* (3.34)–(3.35) *has a unique solution $U(x) \in (\mathcal{S})^*$ given by*

$$U(x) = \int_D G(x,y) dB_H(y) \ , \tag{3.39}$$

where $G(x,y)$ is the classical Green function for the Laplacian.

In [HØZ2] conditions are given which ensure that $U(x) \in L^2(\mu)$ for all x.

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