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A remark on non-depletion of a natural resource under intertemporal preferences*

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Abstract

For a wide class of models concerning the optimal harvesting of a natural resource, an expected profit maximizer will not deplete the resource completely if its relative growth rate is strictly greater than the discount rate. This well-known principle is extended to preferences with durability in consumption, and which are risk averse (or linear) sufficiently close to zero, as long as immediate depletion yields finite utility.

Key words: Optimal harvesting, jump diffusion model, intertemporal preferences, discount rate criterion for non-depletion.

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Introduction.

It is well known that if the relative growth rate of a resource is (uniformly) less than of money (i.e. the economic discount rate), then a profit maximizer will deplete the resource completely and immediately if this is possible and costless. Thus, from a conservationist point of view, high interest rate is a «bad» as it represent less value of saving for future times and may lead to the extinction of populations and entire species and the irrecoverable loss of non-renewable natural resources. There are many generalizations to models with uncertainty, see e.g. [LØ], [AS] and [A], all of whom find that when maximizing expected total discounted harvest, the presence of uncertainty will lead one to wait for a higher population (i.e. the opposite effect of the discounting term). However, they do all consider Brownian (Gaussian) noise, and as pointed out by the author in [F], this choice of probability distribution is crucial as introducing qualitatively different zero-mean noises may in fact lead to downwards reflection at populations lower than in ; however, the non-depletion criterion remains, as it is shown that one will not immediately deplete the population completely if relative growth rate at zero exceeds the (constant) discount rate, just as in the Gaussian (or deterministic) case.

The object of this paper is to show that the same criterion will imply that complete depletion cannot be optimal, under a quite general class of preferences. It turns out that in this respect, linear utility still is the «worst case» among the risk averse, which is not à priori obvious as consumption now is certain while future consumption is not. We assume a setting where a single harvester completely and cost free controls the irreversible harvesting of the resource and posesses the relevant information on population size and relative growth rate at 0.

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The preferences.

With preferences represented by a utility function of the consumption rate, there might not be any substitute to consumption at a given time (i.e. a non-degenerate time interval I). For example, if utility at zero is $-\infty$, then not consuming in I will yield the worst possible outcome. In a more realistic setting, a positive portion consumed should keep the agent satisfied at least for some short time interval, and consumption at two close points in time should be considered close substitutes. A theory for such kinds of preferences was developed by [HHK] in the deterministic case, and by [HH] under uncertainty. In this paper, we shall assume that the direct utility rate does not directly depend on consumption from harvesting, though this may be obtained as a limiting case. Instead, we shall consider utility depending on a process R representing present and (decaying) memory of past consumption. For our purposes, we choose to model R such that if $t > \tau$, where τ is last time of consumption, then

$$R(t) = F(t,\tau) \cdot R(\tau^+).$$

Thus, consumption at time τ is «forgotten» according to the decay function F, which is assumed to be of the form $F(t,\tau) = \exp\{-\int_\tau^t \rho(s)\,\mathrm{d}s\}$. The motivation of the exponential form is that we want R to behave in some sense regularly also at times τ_1 for which there is no consumption (i.e. R continuous at τ_1); if $t > \tau_1 > \tau$, we require

$$R(t) = F(t,\tau_1) \cdot R(\tau_1^+) = F(t,\tau_1) \cdot F(\tau_1,\tau) \cdot R(\tau^+),$$

which together with the assumptions that F is continuous and satisfies $F(t,t) = 1$ imply that F represents exponential decay. The decay coefficient ρ will usually be thought of as positive; we shall assume weaker conditions, see below. This far, we have not yet specified how consumption affects R. The intuitive is to assume that R increases as the amount harvested less the amount forgotten, i.e. $\mathrm{d}R = -\rho R\,\mathrm{d}t + \mathrm{d}H$. However, in some settings it is more convenient to assume $\mathrm{d}R = -\rho R\,\mathrm{d}t + \rho\,\mathrm{d}H$; if ρ then tends to ∞, then $R(t)$ will tend to $\mathrm{d}H/\mathrm{d}t$ if the latter exists, i.e., we obtain the classical case as a limit as memory fades infinitely fast. For a flexible formulation, which also may incorporate future *technological* development in harvesting the resource, we shall assume that for some k, we have

$$\mathrm{d}R(t) = -\rho(t)R(t)\,\mathrm{d}t + k(t)\,\mathrm{d}H(t) \tag{1}$$

with initial value $k(0)r$. Of course, one may want to harvest immediately; assuming $H(0^-) = 0$ without loss of generality, we have $R(0^+) = k(0)(r + H(0^+))$. Note that if k is a constant times the efficiency of the technology, then the quantity $\gamma := k'/k$ to appear below, has a nice economic interpretation as the *relative improvement rate of the technology*. We assume that

$$\rho,\ k \text{ and } \gamma := \frac{k'}{k} \text{ all continuous at } 0, \text{ with } k(0) > 0. \tag{2}$$

We then assume that direct utility rate at time t is $\Upsilon(t, R(t))$, where Υ defined on $(\mathbf{R}^+)^2$ satisfies

$$t \mapsto \Upsilon(t,r) \text{ is continuous for small enough } t \geq 0 \tag{3}$$

$$r \mapsto \Upsilon(t,r) \text{ is a difference between convex functions.} \tag{4}$$

We write Υ' and Υ'' for the almost everywhere defined two first derivatives with respect to r.

The optimal control problem.

Consider an agent who wants to maximize expected total utility from harvesting from a population X. It turns out – see the «Concluding remarks» – that X may be a quite general process. To simplify, we shall assume that X follows the Itô stochastic differential equation

$$\mathrm{d}X(t) = X(t^-)\cdot\Big(\beta(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B_t + \int \eta(X(t^-),z)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\Big) - \mathrm{d}H(t) \qquad (5)$$

with initial value x. Here, B is a standard Brownian motion, $\tilde{N}$ is compensated Poisson measure with Lévy measure q (i.e., $\mathrm{d}\tilde{N} = \mathrm{d}N - \mathrm{d}q\,\mathrm{d}t$ where $N([0,t],A)$ measures the number of jumps whose amplitude is in the (Borel) set $A\setminus\{0\}$.) We assume

$$\beta \text{ lower semicontinuous at } 0,\ \eta \text{ bounded near } 0 \text{ and } \lim_{x\searrow 0} x\sigma^2(x) = 0 \qquad (6)$$

H is our control, the total harvested amount up to and including time t; of course, the harvested process should remain nonnegative, so we assume

$$\eta \geq -1; \quad \text{no } H \text{ s.t. } X(t^-) - (H(t^+) - H(t^-)) < 0 \text{ is admissible;} \quad 0 \text{ is a trap for } X. \qquad (7)$$

À priori, the agent should be permitted to choose among a possibly quite large class of predictable (i.e. non-anticipating left continuous) non-decreasing harvesting processes satisfying (7). However, as the purpose of the paper is to give sufficient conditions that it is not optimal to harvest everything at once, restricting the class of strategies will be convenient and represents no loss of generality. Specifically, we shall consider the strategies H_τ where «most of» the population is harvested immediately, and after a short deterministic time τ, the rest is harvested. Therefore it suffices to assume that at time 0 one harvests $H_\tau(0^+) = x - \xi$ (where $\xi \geq 0$ is small); we then let $X = X^\xi$ (starting from $X(0^+) = \xi$) evolve until time τ, when we harvest $X^\xi(\tau)$. Our purpose is to give conditions under which the strategy $H_0 = \dagger$ described by choosing $\tau = 0$ or equivalently $\xi = 0$, cannot be optimal. The performance to be maximised is supposed to be

$$J(H) = J_{\bar{T}}(H) := \mathsf{E}^x \int_0^{\bar{T}} \Upsilon(t, R(t))\,\mathrm{d}t,$$

where $\bar{T}$ is a finite or infinite deterministic horizon. We want to use a weak optimality criterion, namely «sporadically catching up,» i.e., H^* is SCU-optimal if for all admissible H,

$$\limsup_{T\to\bar{T}} \left(J_T(H^*) - J_T(H)\right) \geq 0.$$

Also, we will consider the stronger «overtaking» criterion, i.e. H^* is OT-optimal if for all admissible H,

$$J_T(H^*) - J_T(H) \geq 0 \quad \text{for all large enough } T < \bar{T}.$$

We shall assume that

$$|J_T(\dagger)| < \infty \quad \forall T < \bar{T} \qquad (8)$$

(if not, there is nothing to prove.) On the technical side, we assume that if ρ explodes in a non-integrable manner, then

$$\int_0^T F^2(t,0)|\Upsilon''(t, F(t,0)(r+x)k(0))|\,\mathrm{d}t < \infty \quad \forall T < \bar{T}. \qquad (9)$$

Non-optimality of immediate total depletion.

Let us state the result in a very general form:

THEOREM.
Assume (1) – (9). *Consider*

$$K(T) := (\beta(0)+\rho(0)+\gamma(0))\int_0^T F(t,0)\cdot\Upsilon'(t,(z+x)k(0)F(t,0))\,\mathrm{d}t - \Upsilon'(0,(z+x)k(0)).$$

If $K(T)>0$ for all large enough $T<\bar{T}$, then $H_0=\dagger$ is not OT-optimal. If K is also bounded away from 0 for all large enough $T<\bar{T}$, then $H_0=\dagger$ is not SCU-optimal.

Proof. With the above described strategy H_τ, we have for each $T<\bar{T}$,

$$\begin{aligned} J_T(H_\tau) = &\int_0^\tau \Upsilon(t,(r+x-\xi)k(0)F(t,0))\,\mathrm{d}t \\ &+\int_\tau^T \mathsf{E}\big[\Upsilon(t,F(t,0)\{(r+x-\xi)k(0)+X^\xi(\tau)k(\tau)F(0,\tau)\})\big]\,\mathrm{d}t \end{aligned}$$

where X^ξ means the process when started at $X(0^+)=\xi$. Assume first that Υ is C^2 in r. Differentiating with respect to τ and evaluating at $\tau=0^+$ yields:

$$\begin{aligned} \frac{\partial}{\partial\tau}J_T(H_\tau)\Big|_{\tau=0^+} = &\ \Upsilon(0,(r+x-\xi)k(0)) - \Upsilon(0,(r+x)k(0)) \\ &+k(0)\int_0^T \Big[\xi\big(\beta(\xi)+\rho(0)+\gamma(0)\big)F(t,0)\cdot\Upsilon'(t,F(t,0)(r+x)k(0)) \\ &\qquad + \xi^2\sigma^2(\xi)F^2(t,0)k(0)\cdot\Upsilon''(t,F(t,0)(r+x)k(0)) \\ &\qquad + \int\Big(\Upsilon(t,F(t,0)(r+x-\xi\eta(\xi,z))) - \Upsilon(t,F(t,0)(r+x)) \\ &\qquad\quad - \xi\eta(\xi,z)k(0)F(t,0)\Upsilon'(t,(r+x)F(t,0))\Big)\,q(\mathrm{d}z)\Big]\,\mathrm{d}t. \end{aligned} \tag{10}$$

If Υ merely satisfies (4), we approximate with r-smooth functions and (10) still holds, noting that the integrands are defined $\mathrm{d}t$-almost everywhere. Now (10) vanishes for $\xi=0$, so we divide by ξ and pass to the limit inferior using (6), to get $k(0)K(T)$. Now let T grow, and the conclusion follows. □

COROLLARY.
Assume $T=\infty$ and $\Upsilon(t,r)=e^{-\delta t}u(r)$ with u concave on $(0,(r+x)k(0))$, and ρ constant and $>-\delta$. If $\beta(0)>\delta-\gamma(0)$, then $K(T)$ is bounded away from 0 for all T large enough and thus $\dagger$ is not SCU-optimal. In particular, if k is constant at 0, such as in the particular cases $k\equiv 1$ and $k\equiv\rho$, we recover the classical condition $\beta(0)>\delta$.

Proof. Observe that the coefficient in front of the integral is positive, and that by concavity, $F(t,0)\Upsilon(t,(r+x)k(0)F(t,0)) \geq e^{-(\rho+\delta)t}u'((r+x)k(0))$. Substitute this and calculate explicitely the overestimate, and note that it is increasing at infinity. □

This also exhibits linear utility as «worst case». But even an expected profit maximizer will not want to deplete the population immediately if growth at 0 exceeds the interest rate less the «relative technological improvement rate» γ, just as in the case where R equals the harvest rate itself. Rather than using concavity, we may sometimes want a stronger result for a given utility function:

Example. In the setting of the Corollary, assume u to be CRRA, i.e. $u'(x) = x^{-p}$ for some $p \geq 0$. Then if $\delta \leq \rho(p-1)$, then $K(T)$ will tend to $+\infty$ as T does; if not, $K(\infty) > 0$ if $\beta(0) > \delta - \gamma(0) - p\rho$, an improved estimate. △

Concluding remarks.

We remark that all coefficients may be time-dependent, even stochastic if independent of B and $\tilde{N}$; obviously, under the appropriate uniformity conditions, non-depletion at time zero extends to non-depletion at all future times as well. This may be extended even further; we may achieve a similar (under)estimate on K if the process is regular enough to admit our use of the Itô formula (i.e. semimartingale), and we have appropriate uniform bounds on the coefficients at $x = 0$. In this way, we can allow for many interacting populations and processes with memory or time-dependent coefficients.

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