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On Stochastic Derivative

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Abstract

The stochastic integral representation for an arbitrary random variable in a standard L_2-space is considered in a case of a general L_2-continuous martingale as integrator. In relation to this, a certain *stochastic derivative* is defined. Through this derivative it can be seen whether the random variable admits the above type integral representation. In any case, it is shown that this derivative determines the integrand in the stochastic integral which serves as the *best* L_2*-approximation* to the random variable considered. For a general *Levy process* as integrator some specification of the suggested stochastic derivative is given; in this way, for Wiener process, the known Clark-Ocone formula is derived.

Key-words: non-anticipating integration, stochastic derivative, integral representation, Levy processes, Clark-Ocone formula.

Some preliminaries. We are to recall the Ito type non-anticipating integration scheme in the L_2-space

$$H = L_2(\Omega, \mathfrak{A}, P)$$

of real random variables ξ:

$$\|\xi\| = (E|\xi|^2)^{1/2},$$

involving as integrator a general H*-continuous* martingale η_t, $0 \leq t \leq T$, with respect to an arbitrary filtration

$$\mathfrak{A}_t, \qquad 0 \leq t \leq T.$$

The integrands are considered as elements of a certain *functional L_2-space* of measurable stochastic functions

$$\varphi = \varphi(\omega, t)\,, \qquad (\omega, t) \in \Omega \times (0, T]$$

with a norm

$$\|\varphi\|_{L_2} = \Big(\iint_{\Omega\times(0,T]} |\varphi|^2 \, P(d\omega)\times d[\eta]_t(\omega)\Big)^{1/2} = \Big(E\int_0^T |\varphi|^2 \, d[\eta]_t\Big)^{1/2}$$

given by means of a product type measure

$$P(d\omega) \times d[\eta]_t(\omega)$$

associated with a stochastic function $[\eta]_t$, $0 \le t \le T$, having monotone right-continuous trajectories such that

$$E(\Delta[\eta]|\mathfrak{A}_t) = E(|\Delta\eta|^2|\mathfrak{A}_t)$$

for the increments $\Delta[\eta]$ and $\Delta\eta$ on intervals

$$\Delta = (t, t + \Delta t] \;\subseteq\; (0, T]\,.$$

In particular, for the *Levy process* η_t, $0 \le t \le T$, as integrator ($E\eta_t = 0$, $E\eta_t^2 = \sigma^2 t$), the deterministic function

$$[\eta]_t = \sigma^2 t, \qquad 0 \le t \le T,$$

is applicable.

For *simple functions* φ^h:

$$\varphi^h = \sum_{\Delta} \varphi^h \, 1_{\Delta}(s), \qquad 0 \le s \le T,$$

having their permanent $\mathfrak{A}_t$-measurable values $\varphi^h \in H$ on the *h-partition* intervals

$$\Delta = (t, t + \Delta t] \; : \qquad \sum \Delta = (0, T] \qquad (\Delta t \le h),$$

the stochastic integrals are defined as

$$\int_0^T \varphi^h \, d\eta_s \;\stackrel{\text{def}}{=}\; \sum_{\Delta} \varphi^h \cdot \Delta\eta$$

with summation over the partition intervals; here, it is assumed that

$$E\Big(\varphi^h \Delta\eta\Big)^2 = E\Big(|\varphi^h|^2 \cdot E(|\Delta\eta|^2|\mathfrak{A}_t)\Big) =$$

$$E\Big(|\varphi^h|^2 \cdot E(\Delta[\eta]|\mathfrak{A}_t)\Big) = E\int_\Delta |\varphi^h|^2\, d[\eta]_s < \infty\,,$$

which gives

$$E\Big(\int_0^T \varphi^h\, d\eta_s\Big)^2 = E\int_0^T |\varphi^h|^2 d[\eta]_s\,.$$

And, in general, the *integrands* φ are *identified* as limits

$$\varphi = \lim_{h\to 0} \varphi^h \tag{1}$$

in the involved functional L_2-space: $\|\varphi - \varphi^h\|_{L_2} \longrightarrow 0$, for appropriate *simple functions* φ^h; the corresponding *stochastic integrals* are defined as limits

$$\int_0^T \varphi\, d\eta_s = \lim_{h\to 0}\int_0^T \varphi^h\, d\eta_s \tag{2}$$

in H, with

$$\|\int_0^T \varphi\, d\eta_s\| = \|\varphi\|_{L_2}\,.$$

According to the simple functions structure, the integrands can be characterized in the above functional L_2-space as functions φ on the product $\Omega \times (0, T]$ which are measurable with respect to the σ-algebra generated by all rectangles of form $A \times (t, t + \Delta t]$, with $A \in \mathfrak{A}_t$ (note, the above rectangles constitute the so-called *semi-ring* and their indicators constitute a *complete system* in the L_2-subspace of functions, measurable with respect to the σ-algebra generated). In a case of Levy process as integrator, this characterization can be simplified by identification of the integrands as the stochastic functions φ, having $\mathfrak{A}_t$-measurable values φ_t, $0 \le t \le T$:

$$\int_0^T \|\varphi\|^2 dt < \infty\,.$$

Also, to characterize the functional L_2-subspace of all integrands, one can consider the complete system of a particular form integrands

$$\varphi \cdot 1_{(\sigma,\tau]}(t), \qquad 0 \le t \le T,$$

having $\mathfrak{A}_\sigma$-measurable values φ on random intervals $(\sigma, \tau]$ with *stopping times* σ, τ $(0 \le \sigma < \tau \le T)$:

$$\|\varphi\, 1_{(\sigma,\tau]}\|_{L_2}^2 = E\left[|\varphi|^2\big([\eta]_\tau - [\eta]_\sigma\big)\right] < \infty\,,$$

$$\int_0^T \varphi 1_{(\sigma,\tau]}\, d\eta_s \;=\; \varphi\,(\eta_\tau - \eta_\sigma)\,.$$

Note, in the (1)-(2) integration scheme any h-partitions $(h \to 0)$ can be applied thanks to the H-continuity of the integrator:

$$\|\Delta\eta\| \longrightarrow 0, \qquad \Delta t \to 0,$$

for the increments $\Delta\eta = \eta_{t+\Delta t} - \eta_t$ in H; in particular one always can apply the *monotone h-partitions*, having *increasing* sets of partition points (with $h \to 0$) which altogether represent some *dense* set $\{t\}$ on the considered interval $(0, T]$.

The following questions seem to be of general interest. Whether a random variable

$$\xi \;\in\; H$$

admits representation by the (2)-type stochastic integral, and, in any case, how the best integral approximation to ξ:

$$\hat{\xi} \;=\; \int_0^T \varphi\, d\eta_s \tag{3}$$

can be determined; here $\hat{\xi}$ is meant to be the projection of ξ onto the subspace $H(\eta)$ of all stochastic integrals with the considered integrator η_t, $0 \le t \le T$. To be more precise the latter question is how the above integrand φ can be determined through the corresponding (1)-type simple integrands φ^h.

And, regarding these questions, it also seems of interest to consider a more general case with the subspace $H(\eta)$ of random variables $\hat{\xi}$ in H, admitting the *stochastic integral representation* of a form

$$\hat{\xi} \;=\; \sum_{k=1}^n \int_0^T \varphi_k\, d\eta_s^k \tag{4}$$

- with respect to some system of the *orthogonal* martingales

$$\eta_t^k, \qquad 0 \le t \le T \quad (k = 1, \dots, n)$$

as integrators (including the case $n = \infty$).

For illustration, we refer to the known Black-Scholes type Markets where for the *pre-considered desirable* gain ξ, the corresponding *achievable* gain $\hat{\xi}$, as the "best" approximation to ξ, should be determined through the (4)-type representation - cf. [3], [4].

We can answer to the above questions as follows.

Stochastic derivatives and L_2-approximations with stochastic integrals. With no loss of generality, we can assume that the considered H-continuous martingale η_t, $0 \le t \le T$, does not degenerate, having *non-zero* increments

$$\Delta\eta \;=\; \eta_{t+\Delta t} - \eta_t$$

on all intervals

$$\Delta \;=\; (t, t+\Delta t] \;\subseteq\; (0,T]\,.$$

For the random variable $\xi \in H$, let us define its *stochastic derivative* $D\xi$ with respect to the integrator η_t, $0 \le t \le T$, as

$$D\xi \;\stackrel{\text{def}}{=}\; \lim_{h\to 0} E\Big(\xi \frac{\Delta\eta}{\|\Delta\eta\|_t^2} |\mathfrak{A}_t\Big) \tag{5}$$

- to be more precise,

$$D\xi = \lim_{h\to 0} \sum_{\Delta} E\left(\xi \frac{\Delta\eta}{\|\Delta\eta\|_t^2} |\mathfrak{A}_t\right) 1_\Delta(s), \qquad 0 \le s \le T,$$

being as the (1)-type limit $\varphi = D\xi$ of the simple functions φ^h with their values

$$\varphi^h \;=\; E\Big(\xi \frac{\Delta\eta}{\|\Delta\eta\|_t^2} |\mathfrak{A}_t\Big) \tag{6}$$

on the *h-partition* intervals $\Delta = (t, t+\Delta t]$, where

$$\|\Delta\eta\|_t^2 \;=\; E\Big(|\Delta\eta|^2 |\mathfrak{A}_t\Big)$$

- cf. [1], [2], [8].

Theorem. *Stochastic derivative* $(5)-(6)$ *is well defined for any* $\xi \in H$, *and* ξ *admits unique integral representation*

$$\xi \;=\; \xi^0 + \int_0^T D\xi \, d\eta_s \tag{7}$$

through its derivative $D\xi$ and the corresponding $\xi^0 \in H$:

$$D\xi^0 \;=\; 0.$$

Proof. With the monotone h-partitions, for the subspace

$$H(\eta) \;\subseteq\; H$$

of all (3)-type stochastic integrals, we have

$$H(\eta) = \lim_{h\to 0} \sum \oplus\, H(\Delta\eta) \tag{8}$$

as the limit of the indicated *orthogonal sums* with their components $H(\Delta\eta)$ as subspaces of the corresponding variables in H of a form

$$\psi \cdot \Delta\eta\,,$$

with the $\mathfrak{A}_t$-measurable multiplicators ψ for the increments $\Delta\eta$ on the h-partition intervals $\Delta = (t, t+\Delta t]$. A projection of ξ onto $H(\Delta\eta)$ is

$$\varphi^h \cdot \Delta\eta\,,$$

with the multiplicator $\psi = \varphi^h$:

$$\varphi^h = E\Big(\xi\,\frac{\Delta\eta}{\|\Delta\eta\|_t^2)}|\mathfrak{A}_t\Big)$$

- cf. (6). Indeed,

$$E|\varphi^h\cdot\Delta\eta|^2 < \infty$$

since

$$|\varphi^h|^2 E(|\Delta\eta|^2|\mathfrak{A}_t) \le E(\xi^2|\mathfrak{A}_t)\,,$$

and the following relation

$$E\big((\xi-\varphi^h\Delta\eta)(\psi\Delta\eta)|\mathfrak{A}_t\big) = \psi E(\xi\Delta\eta|\mathfrak{A}_t) - \psi\varphi^h E(|\Delta\eta|^2|\mathfrak{A}_t) = 0$$

implies the orthogonality condition

$$E\big(\xi-\varphi^h\Delta\eta\big)\big(\psi\Delta\eta\big) = 0\,.$$

Hence, projections of ξ onto the (8) pre-limit orthogonal sums are

$$\sum_\Delta \varphi^h\Delta\eta = \int_0^T \varphi^h\,d\eta_s\,,$$

where the integrands φ^h are the simple functions with the values $\psi = \varphi^h$ on the intervals $\Delta = (t, t + \Delta t]$, and these simple functions are exactly the same as in the limit formula (5)-(6). Of course, the (3)-form projection $\hat{\xi}$ of ξ onto the subspace $H(\eta)$ of all integrals is represented by some particular integral being a limit

$$\hat{\xi} = \int_0^T \varphi \, d\eta_s = \lim_{h \to 0} \int_0^T \varphi^h \, d\eta_s$$

in H, and here the integrand φ is the (1)-type limit of the simple functions φ^h, according to

$$\| \int_0^T \varphi \, d\eta_s - \int_0^T \varphi^h \, d\eta_s \| \ = \ \| \varphi - \varphi^h \|_{L_2} \, .$$

Thus in representaiton (7) with the integrand $\varphi = D\xi$, the difference

$$\xi^0 \ = \ \xi - \int_0^T \varphi \, d\eta_s$$

is orthogonal to $H(\eta)$ and according to what was already shown, $D\xi^0 = 0$. The proof is over.

We are to stress that representation (7) leads to the (3)-type integral approximation to ξ as

$$\hat{\xi} \ = \ \int_0^T D\xi \, d\eta_s.$$

A more general result is as follows.

Corollary. *For the orthogonal martingales*

$$\eta_t^k \, , \qquad 0 \le t \le T, \qquad (k = 1, 2, \dots)$$

and the subspace $H(\eta)$ of the (4)-type variables in H, the projection $\hat{\xi}$ of ξ onto $H(\eta)$ is

$$\hat{\xi} = \sum_{k=1}^{\infty} \int_0^T \varphi_k \, d\eta_s^k \, ,$$

with the integrands

$$\text{(9)} \qquad \varphi_k = \lim_{h \to 0} E\Big(\xi \frac{\Delta \eta^k}{\| \Delta \eta^k \|_t^2} | \mathfrak{A}_t \Big), \qquad (k = 1, 2, \dots)$$

as the stochastic derivatives with respect to the corresponding integrators - cf. (5)-(6).

Of course, in particular situations the suggested stochastic derivative admits particular specifications. For illustration we consider the following examples.

Stochastic derivatives with respect to Levy processes as integrators. As usual, let's assume that the filtration $\mathfrak{A}_t$, $0 \le t \le T$, is generated by the very integrator η_t, $0 \le t \le T$ (note, in this case, the filtration is *continuous*).

Example (derivatives with respect to Wiener process). Let η_t, $0 \le t \le T$ be Wiener process with a diffusion coefficient σ^2.

A tipical simple situation can be as follows: the random variable ξ is Normal (jointly with η_t, $0 \le t \le T$), having its *correlation*

$$E\,\xi\eta_t, \qquad 0 \le t \le T,$$

with the integrator; then the stochastic derivative can be specified as

$$D\xi \;=\; \frac{1}{\sigma^2}\frac{d}{dt}E\,\xi\eta_t, \qquad 0 \le t \le T \quad \text{(a.e.)}. \tag{10}$$

Indeed, the projection $\hat{\xi}$ on the subspace $H(\eta)$ admits representation (3) with the *deterministic integrand* $\varphi = D\xi$ and

$$E\xi\eta_t \;=\; E\Big(\int_0^T \varphi d\eta_s \cdot \eta_t\Big) \;=\; \sigma^2 \int_0^t \varphi\, ds, \qquad 0 \le t \le T.$$

In another typical situation, the random variable

$$\xi \;=\; F(\eta_{t_1}, \dots, \eta_{t_n})$$

is a function of the variables $\eta_{t_1}, \dots, \eta_{t_n}$:

$$0 \;=\; t_0 \;<\; t_1 \;<\; \dots \;<\; t_n \;\le\; T.$$

Here, ξ can be treated as the corresponding function

$$\xi \;=\; f(\Delta\eta_{t_1}, \dots, \Delta\eta_{t_n}), \tag{11}$$

of the increments

$$\Delta\eta_{t_i} \;=\; \eta_{t_i} - \eta_{t_{i-1}} \quad (i = 1, \dots, n)\,,$$

for

$$f(x_1,\dots,x_n) \stackrel{\text{def}}{=} F(x_1,\dots,\sum_{i=1}^n x_i), \qquad (x_1,\dots,x_n)\in\mathbb{R}^n.$$

Suppose $f(x_1,\dots,x_n)$ is a *smooth* function of $(x_1,\dots,x_n)\in\mathbb{R}^n$, such that its derivatives of order k $(k=0,1,2)$ satisfy *majorant conditions* of a form

$$|\frac{\partial^k}{\partial x_i^k} f| \;\le\; C\prod_{j=1}^n e^{\epsilon|x_j|^2}$$

for any $\epsilon>0$ and an appropriate constant C. Then the stochastic derivative can be specified as follows:

$$(12)\quad D\xi \;=\; \sum_{i=1}^n E\Big(\frac{\partial}{\partial x_i} f(\Delta\eta_{t_1},\dots,\Delta\eta_{t_n})|\mathfrak{A}_s\Big)\, 1_{(t_{i-1},t_i]}(s), \qquad 0\le s\le T.$$

A proof requires a few elementary steps.

First of all, for the intervals $\Delta = (t,t+\Delta t]$ from the monotone h-partitions, $\sum\Delta=(0,T]$: $\Delta t\le h$, such that

$$t_{i-1} \;<\; t \;<\; t+\Delta t \;\le\; t_i\,,$$

let us consider the difference

$$\Delta f \;=\; f(\dots,\Delta\eta_{t_i},\dots) \;-\; f(\dots,\Delta\eta_{t_i}-\Delta\eta,\dots)\,.$$

We see that $\Delta\eta=\eta_{t+\Delta t}-\eta_t$ is *independent* of the events of the σ-algebra $\mathfrak{A}_t$ and the variable $f(\dots,\Delta\eta_{t_i}-\Delta\eta,\dots)$, taken alltogether, and therefore

$$E[f(\dots,\Delta\eta_{t_i}-\Delta\eta,\dots)\Delta\eta|\mathfrak{A}_t] \;=\; E[f(\dots,\Delta\eta_{t_i}-\Delta\eta,\dots)|\mathfrak{A}_t]E\Delta\eta \;=\; 0\,.$$

Hence, with $\|\Delta\eta\|_t^2=\|\Delta\eta\|^2=\sigma^2\Delta t$, we have

$$E\Big(\xi\frac{\Delta\eta}{\|\Delta\eta\|_t^2}|\mathfrak{A}_t\Big) \;=\; E\Big(\Delta f\frac{\Delta\eta}{\|\Delta\eta\|^2}|\mathfrak{A}_t\Big)\,.$$

Now, we apply Taylor approximations as follows

$$\Delta f-\frac{\partial}{\partial x_i}f(\dots,\Delta\eta_{t_i}-\Delta\eta,\dots)\cdot\Delta\eta=\frac{\partial^2}{\partial x_i^2}f[\dots,(\Delta\eta_{t_i}-\Delta\eta)+\theta\Delta\eta,\dots](\Delta\eta)^2$$

and

$$\frac{\partial}{\partial x_i}f(\dots,\Delta\eta_{t_i},\dots)-\frac{\partial}{\partial x_i}f(\dots,\Delta\eta_{t_i}-\Delta\eta,\dots)=\frac{\partial^2}{\partial x_i^2}f[\dots,(\Delta\eta_{t_i}-\Delta\eta)+\theta\Delta\eta,\dots]\cdot\Delta\eta$$

where $0 \leq \theta \leq 1$. Then, thanks to the majorant conditions, we see that

$$\|\Delta f \Delta\eta - \frac{\partial}{\partial x_i} f(\dots, \Delta\eta_{t_i} - \Delta\eta, \dots)(\Delta\eta)^2\| \;\leq\; C\|e^{\epsilon|\Delta\eta|^2}|\Delta\eta|^3\| = \mathrm{O}(h^{\frac{3}{2}})$$

and

$$\|\frac{\partial}{\partial x_i} f(\dots, \Delta\eta_{t_i}, \dots) - \frac{\partial}{\partial x_i} f(\dots, \Delta\eta_{t_i} - \Delta\eta, \dots)\| \leq C\|e^{\epsilon|\Delta\eta|^2}|\Delta\eta|\| = \mathrm{O}(h^{\frac{1}{2}}).$$

Hence, we have

$$\lim_{h \to 0} \|E\Big(\xi \frac{\Delta\eta}{\|\Delta\eta\|_t^2} |\mathfrak{A}_t\Big) - E\Big(\frac{\partial}{\partial x_i} f(\dots, \Delta\eta_{t_i}, \dots)|\mathfrak{A}_t\Big)\| = 0.$$

The next step is to consider the stochastic function in H

$$\varphi := \sum_{i=1}^{n} E\Big(\frac{\partial}{\partial x_i} f(\Delta\eta_{t_1}, \dots, \Delta\eta_{t_n})|\mathfrak{A}_s\Big) 1_{(t_{i-1}, t_i]}(s) \qquad 0 \leq s \leq T,$$

which is uniformly H-continuous on the open intervals (t_{i-1}, t_i), with

$$\int_0^T \|\varphi\|^2 \, d[\eta]_s \;=\; \|\varphi\|_{L_2}^2 \;<\; \infty.$$

Let us write φ^h for the φ values at the end points t of the h-partition intervals $\Delta = (t, t + \Delta t]$; clearly, the corresponding *simple functions* φ^h with the above permanent values on the intervals Δ converge to the function φ in the sense that

$$\lim_{h \to 0} \|\varphi^h - \varphi\|_{L_2} \;=\; 0$$

- cf. (1). As it was shown in the first step, for every partition point t, we have

$$\lim_{h \to 0} E\Big(\xi \frac{\Delta\eta}{\|\Delta\eta\|_t^2} |\mathfrak{A}_t\Big) = E\Big(\frac{\partial}{\partial x_i} f(\dots, \Delta\eta_{t_i}, \dots)|\mathfrak{A}_t\Big),$$

and this shows that the above simple functions φ^h are exactly the same as (5)-(6); thus

$$D\xi \;=\; \varphi$$

- cf. (12). The proof is over.

Clearly, coming back from $f(\Delta\eta_{t_1}, \dots, \Delta\eta_{t_n})$ to $F(\eta_{t_1}, \dots, \eta_{t_n})$, we are just to modify representation (12) as

$$(13) \quad DF(\eta_{t_1}, \dots, \eta_{t_n}) = \sum_{i=1}^{n} E\Big(\frac{\partial}{\partial x_i} F(\eta_{t_1}, \dots, \eta_{t_n})|\mathfrak{A}_s\Big) 1_{(0, t_i]}(s), \; 0 \leq s \leq T,$$

which gives the known *Clark-Ocone formula* for the integrand $\varphi = D\xi$ in the stochastic integral representation

$$\xi \;=\; \int_0^T \varphi d\eta_s$$

-cf. [6], [7].

Example (derivatives with respect to "jumping" Levy processes). Let η_t, $0 \le t \le T$, be the "jumping" process with homogeneous independent increments:

$$Ee^{iu\Delta\eta} \;=\; \exp\Big\{\Delta t \int_{-\infty}^{\infty} (e^{iux} - 1 - iux) G(dx)\Big\}$$

for the increments $\Delta\eta$ on intervals $\Delta = (t, t+\Delta t] \subseteq (0,T]$. Suppose the "jump" measure $G(dx)$ has moments

$$\sigma_k \;=\; \int_{-\infty}^{\infty} x^k G(dx), \qquad k = 1, 2, \dots \tag{14}$$

that is $\Delta\eta$ has *semi-invariants* $\sigma_k \Delta t$, $k = 1, 2, \dots$, with $\sigma_1 = 0$ for $E\Delta\eta = \sigma_1 \Delta t = 0$. Similar to (11), let us consider

$$\xi \;=\; f(\Delta\eta_{t_1}, \dots, \Delta\eta_{t_n})$$

for certain kind *analitical* functions $f(x_1, \dots, x_n)$ of $(x_1, \dots, x_n) \in \mathbb{R}^n$, in particular, satisfying *majorant conditions* of the polynomial type:

$$|\frac{\partial^k}{\partial x_i^k} f| \;\le\; C \prod_{j=1}^{n} (1 + |x_j|^{m_{j,k}}), \qquad (k = 0, 1, \dots).$$

Then the stochastic derivative admits the following specification:

$$D\xi = \sum_{k=1}^{\infty} \frac{\sigma_{k+1}}{\sigma_2} \frac{1}{k!} \sum_{i=1}^{n} E\Big(\frac{\partial^k}{\partial x_i^k} f(\Delta\eta_{t_1}, \dots, \Delta\eta_{t_n}) | \mathfrak{A}_s\Big) 1_{(t_{i-1}, t_i]}(s), \; 0 \le s \le T. \tag{15}$$

To show it, we can apply the same elementary technique as in the case of Wiener process - cf. (11), etc. At first, let f be *polynomial.* Then, for every monotone h-partition point $t : t_{i-1} < t < t + \Delta t \le t_i$, considering a finite Taylor expansion

$$\Delta f \;=\; f(\dots, \Delta\eta_{t_i}, \dots) - f(\dots, \Delta\eta_{t_i} - \Delta\eta, \dots) \;=$$

$$=\; \sum_k \frac{\partial^k}{\partial x_i^k} f(\dots, \Delta\eta_{t_i} - \Delta\eta, \dots) \frac{(\Delta\eta)^k}{k!},$$

we see that

$$\lim_{h\to 0} E\Big(\Delta f \frac{\Delta\eta}{\|\Delta\eta\|^2} |\mathfrak{A}_t\Big) \;=$$

$$= \lim_{h\to 0} \sum_k \frac{E(\Delta\eta)^{k+1}}{\|\Delta\eta\|^2} \frac{1}{k!} E\Big(\frac{\partial^k}{\partial x_i^k} f(\ldots, \Delta\eta_{t_i} - \Delta\eta, \ldots,)|\mathfrak{A}_t\Big) \;=$$

$$= \sum_k \frac{\sigma_{k+1}}{\sigma_2} \frac{1}{k!} E\Big(\frac{\partial^k}{\partial x_i^k} f(\ldots, \Delta\eta_{t_i}, \ldots)|\mathfrak{A}_t\Big)$$

- thanks to a general relation between moments and semi-invarians:

$$E(\Delta\eta)^{k+1} \;=\; \sum_{q=1}^{k+1} (\Delta t)^q \sum_{k_1+\cdots+k_q=k+1} \frac{(k+1)!}{k_1!\ldots k_q!} \prod_{j=1}^{q} \sigma_{k_j} \,,$$

with the internal sum over all integer solutions of the equation

$$\sum_{j=1}^{q} k_j \;=\; k+1 \qquad (k_j \geq 2),$$

- cf. [5], plus the fact that, according to the majorant conditions

$$\|\frac{\partial^k}{\partial x_i^k} f(\ldots, \Delta\eta_{t_i}, \ldots) - \frac{\partial^k}{\partial x_i^k} f(\ldots, \Delta\eta_{t_i} - \Delta\eta, \ldots)\| \;\leq$$

$$\leq\; C\|(1 + |\Delta\eta|^{m_{i,k+1}})\,|\Delta\eta|\,\| \;=\; \mathrm{O}(h^{\frac{1}{2}})\,.$$

Note, representation (15) holds for the analytical function f such that it satisfies the applied majorant conditions and, for every fixed point t:

$$t_{i-1} < t < t_i \qquad (i = 1, \ldots, n),$$

the series

$$\sum_{k=1}^{\infty} \frac{E(\Delta\eta)^{k+1}}{\|\Delta\eta\|^2} \frac{1}{k!} \,\|\frac{\partial^k}{\partial x_i^k} f(\ldots, \Delta\eta_{t_i} - \Delta\eta, \ldots)\| \tag{16}$$

converge uniformly with respect to $\Delta t \to 0$.

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