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PORTFOLIO SEPARATION WITHOUT STOCHASTIC CALCULUS (ALMOST) *

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Abstract

While it is common knowledge that portfolio separation in a continuous-time lognormal market is due to the basic properties of the normal distribution, the usual exposition found in text books relies on dynamic programming and therefore invokes Itô stochastic calculus. Khanna & Kulldorff (1999) gives a rigorous proof which essentially reduces to the elementary properties assuming a risk free asset exists, an assumption we drop. Further simplifications are given, and generalizations to (symmetric and non-symmetric) α-stable driving noise.

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0 Introduction

This paper concerns the portfolio optimization problem for a small agent in a frictionless continuous-time market where the assets are geometric stable-like additive (i.e. independent increment) processes. The concept of portfolio separtion should be well known; from Tobin (1958), generalizations have gone either in the direction of characterizing the preferences which admit separation (Cass & Stiglitz (1970), discrete time, if the utility function is smooth) or a characterization in terms of distributions (Ross (1978), discrete time.) In a complete lognormal diffusion market, two fund separation was obtained by Merton (1971) by means of dynamic programming. Instead of minimizing variance given mean, Khanna & Kulldorff (1999) choose to maximize mean given the variance, and are by remarkably simple methods able to remove the risk aversion and completeness assumption and also allow for "no short sale" constraints on a subset of the portfolios, as well as incomplete markets; they do however assume the existence of a risk free asset. This paper will remove this latter assumption and a few others, and allow for α-stable laws as well. A reference discrete reference is Fama (1965) (the symmetric case only.) We shall see that there are cases admitting separation if all noise sources have the same skewness and short sale is disallowed.

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The outline of the paper is as follows: Section 1 gives a heuristic exposition of the lognormal diffusion market, semple enough for undergraduate level if one assumes a rough understanding of continuous-time stochastic processes. In Section 2 we make the argument rigorous under quite general conditions. Going through through the key properties, we will see that part of the results carry over to the case where noise is α-stable: Two fund separation in the presence of risk-free asset is given in Section 3. Other generalizations, including cases of one fund separation, are given in Section 4. Finally we will see that we may have 2 fund separation if components are dependent in the non-Gaussian case as well; although this is a substantial generalization of Theorem 3.1, it is left to the end to keep the exposition simple.

Throughout the paper, boldface symbols denote vectors. We will also suppress time-dependence of the parameters and choice variables. We will also implicite assume all distributional properties to hold *jointly* in the variables.

1 The geometric Brownian market.

Consider consumption-portfolio optimization in a market with n risky assets $\{S_i\}$ each satisfying the (nonanticipative) stochastic differential equation

$$\mathrm{d}S_i(t) = S_i(t)[\mu_i\,\mathrm{d}t + \boldsymbol{\sigma}_i^{\mathsf{T}}\,\mathrm{d}\mathbf{X}(t)] \tag{1}$$

where in this Section $\mathbf{X}$ is a Wiener process, i.e. continuous-time random walk with multinormal independent increments and covariance matrix R. We shall treat both the case where a safe money market exists and the case where it does not; assume for the moment that there is an asset $S_0(t) = 1$ for all t, i.e. we assume that we work with discounted figures. (With reference to the title: If one only knows the non-discounted asset dynamics, then one needs to invoke the Itô formula without second order term.) We form a portfolio from the assets: If at time t one has $\xi_i(t)$ units of asset i, then the market value is $\sum \xi_i(t)S_i(t)$. In discrete time, the self financing condition says that the change in the market value of the portfolio should come from changes in the prices of the assets, i.e. that change in wealth should be $\sum \xi_i(t)\big(S_i(t+h) - S_i(t)\big)$ for $h > 0$. The analogous requirement is taken as a definition of a self-financing portfolio in our continuous-time setting, i.e. wealth fluctuating as $\sum \xi_i(t)\,\mathrm{d}S_i(t)$; however, from our wealth we will also deduct an amount for consumption. Letting $C(t)$ be (discounted) cumulative net consumption up to time t and assume that the portfolio is self-financing apart from the consumption, we have that wealth at time t is $Y(t) = \sum \xi_i S_i(t)$, and developing according to

$$\begin{aligned}\mathrm{d}Y(t) &= \sum_{i=0}^{n} \xi_i(t)\,\mathrm{d}S_i(t) - \mathrm{d}C(t) \\ &= \sum_{i=1}^{n} \xi_i(t)S_i(t)[\mu_i\,\mathrm{d}t + \boldsymbol{\sigma}_i^{\mathsf{T}}\,\mathrm{d}\mathbf{X}(t)] - \mathrm{d}C(t) = \mathbf{u}^{\mathsf{T}}[\boldsymbol{\mu}\,\mathrm{d}t + \Sigma\,\mathrm{d}\mathbf{X}(t)] - \mathrm{d}C(t)\end{aligned} \tag{2}$$

where Σ has rows $\{\sigma_i^{\mathsf{T}}\}$; write $M := \Sigma R \Sigma^{\mathsf{T}}$ for the volatility matrix. Our control $\mathbf{u}$ represents the *value* invested in each risky asset, and the amount invested in the safe asset is then equal to $Y - \mathbf{u}^{\mathsf{T}}\mathbf{1}$.

The usual argument is now to minimize variance for given mean, and this will lead to two fund separation. However, variance minimization assumes implicitely risk aversion, which is in fact unneccessary if the agents prefer more to less, as pointed out by Khanna & Kulldorff (1999). We shall instead maximize mean for given variance. In the next Section we will make the argument rigorous, but let us for the moment proceed heuristically. Up to now, we have not specified whether the agent is free to choose $\mathbf{u}$. Let us assume that $\mathbf{u}$ is required to belong to some given closed set U. The mean-variance optimization problem now becomes:

$$\max_{\mathbf{u}\in U} \mathbf{u}^\mathsf{T}\boldsymbol{\mu} \quad \text{subject to } \mathbf{u}^\mathsf{T} M\mathbf{u} = Q.$$

Let us say that a $Q \geq 0$ is *attainable* if there is some $\mathbf{u} \in U$ such that $\mathbf{u}^\mathsf{T} M\mathbf{u} = Q$. Since we might have a constrained market, absence of arbitrage does not imply that the volatility matrix is invertible; arbitrage-freeness is simply the condition that $Q = 0$ implies $\mathbf{u}^\mathsf{T}\boldsymbol{\mu} = 0$. A particulary convenient class of constraints is if U is a cone with vertex at $\mathbf{0}$, which may be obtained by forbidding short sale on some subportfolios, some of or all of the assets, the latter corresponding to U being the first orthant – or even on some subportfolio, corresponding to a cone with vertex at $\mathbf{0}$.) To push it even further, if U is merely a family of half-lines from the origin, then we have, with one exception, at worst two fund separation: if there is no arbitrage, find the $\mathbf{f}^* \in U$ which solves the maximization problem for $Q = 1$ and just scale. The exception is if the market does admit arbitrage: Since we have not assumed risk aversion, a non-risk averse agent may require three funds, namely the arbitrage, the bank, and the "best risky fund" $\mathbf{f}^*$.

Other interesting U's are the ones generated by linear constraints. If $U = \{\mathbf{u};\ \mathbf{u}^\mathsf{T} A \leq \boldsymbol{\zeta}\}$ (componentwise inequality), then the first order Kuhn-Tucker condition for optimality is $\lambda_0 M\mathbf{u} = \boldsymbol{\mu} - A\boldsymbol{\lambda}$ and if $\lambda_0 \neq 0$ and the volatility matrix is invertible, we immediately have $(2 + \text{rank}(A))$ fund separation. Furthermore, $M^{-1}\boldsymbol{\lambda}/\lambda_0$ has interpretations both as shadow prices on the constraints and positions in the funds needed to satisfy them. In particular, we immediately recover the classical case of two fund separation assuming that there is no risk-free asset, by taking $A = \mathbf{1}$ and $\zeta = 1$; while two fund separation then is a consequence of requiring $\mathbf{u}^\mathsf{T}\mathbf{1} = 1$, the cases with inequality are also worth mentioning, as they allow for different interest rates for borrowing and lending (see Example 4.7). We merely mention that further generalizations are straightforward, the maybe most noteworthy is that we may allow for non-traded income whos law does not depend on the asset prices, or even driven by $\mathbf{X}$. Note that $\boldsymbol{\zeta}$ does not affect the funds (except for degeneracies) and may then be individual.

Mean-variance efficiency can easily seen to be optimal (under conditions we will make precise in the next Section) through the following argument: Consider an arbitrary strategy $(C, \mathbf{u})$. Let $Q = \mathbf{u}^\mathsf{T} M\mathbf{u}$ and let $\mathbf{u}^*$ maximize drift given that volatility is Q. The additional drift is then immediately consumed, so that we form a consumption process C^* such that $\mathrm{d}C^* = \mathrm{d}C + (\mathbf{u}^* - \mathbf{u})^\mathsf{T}\boldsymbol{\mu}\,\mathrm{d}t$. Then there exists a standard Brownian motion X such that

$$\mathbf{u}^{*\mathsf{T}}\Sigma\,\mathrm{d}\mathbf{X} \;\sim\; Q\,\mathrm{d}X \;\sim\; \mathbf{u}^\mathsf{T}\Sigma\,\mathrm{d}\mathbf{X} \tag{3a}$$

where "$\sim$" denotes conincidence in law. Therefore the process Y^* corresponding to the strategy $(C^*, \mathbf{u}^*)$ satisfies

$$\mathrm{d}Y^* = \mathbf{u}^{*\mathsf{T}}\boldsymbol{\mu}\,\mathrm{d}t + \mathbf{u}^{*\mathsf{T}}\Sigma\,\mathrm{d}\mathbf{X} - \mathrm{d}C^* \;\sim\; \mathbf{u}^{*\mathsf{T}}\boldsymbol{\mu}\,\mathrm{d}t - (\mathbf{u}^* - \mathbf{u})^\mathsf{T}\boldsymbol{\mu}\,\mathrm{d}t + \mathbf{u}^\mathsf{T}\Sigma\,\mathrm{d}\mathbf{X} - \mathrm{d}C = \mathrm{d}Y, \tag{3b}$$

i.e. one can have more consumption and the same wealth, up to coinciding probability law.

2 The assumptions needed, and α-stable random variables.

We shall see what assumptions we need for the argument of the previous Section. But first, let us note that we do not assume existence of an optimal strategy. In return, we have to stick to a slightly weaker concept of portfolio separation:

2.1 Definition (m fund separation).

We shall say that we have m *fund separation* if there exist m funds independent of wealth such that for each admissible (consumption, portfolio) pair there is one which is preferred and whose portfolio consists of the m funds. △

2.2 Assumptions.

- **Predictability.** The strategies should be predictable (that is, non-anticipative) and admit unique (weak) solution to (2).

- **Greed.** Preferences are assumed to form a partial ordering on the (wealth, consumption) pairs such that (Y^*, C^*) is preferred to (Y, C) if

$$(Y^*, C^*) \sim (Y, C + \int^{\cdot} c \,\mathrm{d}t) \quad \text{for some predictable } c \geq 0.$$

- **Consumption** must not covariate with $\mathbf{X}$, i.e., in terms of Itô differentials, we must have $\mathrm{d}C\,\mathrm{d}\mathbf{X} = \mathbf{0}$ for (3) to hold. In many applications one may want to restrict the class of admissible consumption strategies even further, for example assuming finite variation, lower boundedness or nonnegativity; we then have to assume that for any admissible C then any $C + \int^{\cdot} c\,\mathrm{d}t$ for predictable nonnegative c is also admissible.

- **Existence** of some admissible strategy.

- **Probability distributions** must permit the construction (3). In Section 1, we used the property that the zero-mean normal distributions constitute a one parameter family which is closed not only under convolution, but under arbitrary linear combinations as well. The *symmetric α-stable distributions* share this property. If we only require *positive* linear combinations, then also the skew stable distributions may be treated. Unfortunately, dependence between non-Gaussian stable variables is not characterized nicely through a covariance matrix, as used in Section 1. Through Sections 3 and 4 we will therefore assume that $\mathbf{X}$ has independent components, corresponding to formally taking R to be the identity, while we will treat dependent components in Section 5.

The coinciding law argument (3) can now be repeated in an entirely rigorous manner; under these quite mild regularity assumptions, every agent is a mean-variance optimizer in the Gaussian case. The non-Gaussian stable random variables have infinite variance, but the main principle will work nevertheless. For completeness, let us recall the basic properties:

2.3 Independent α-stable random variables.

A stable real r.v. Z is one for which for two independent copies Z_1 and Z_2 of Z, and any two positive numbers a_1 and a_2, there exist numbers $a \geq 0$ and a_0 such that $a_0 + a_1 Z_1 + a_2 Z_2 \sim aZ$. The law is *strictly stable* if one can take $a_0 = 0$. The stable laws form a four parameter family $S_\alpha(\sigma, \beta, \mu)$. The number a always satisfies $a^\alpha = a_0^\alpha + a_1^\alpha$ for some *index of stability* $\alpha \in (0, 2]$ (unique unless Z is constant). $\sigma > 0$ (except $= 0$ for the constant) is called the *scale parameter*; Z/σ has scale parameter 1. $\mu \in \mathbf{R}$ is called the *location parameter*; $Z - \mu$ has location parameter zero. $\beta \in [-1, 1]$ the *skewness parameter*; $a_0 + a_1 Z$ has skewness $\beta \operatorname{sign} a_1$. Only the Gaussian is independent of β and will be taken to have $\beta = 0$ by convention. Then the law is symmetric around μ (i.e. $Z - \mu \sim \mu - Z$) iff $\beta = 0$; beware, however, that μ equals the expectation only iff $\alpha > 1$; in fact, $\mathrm{E}[|Z|^p] = \infty$ iff $p \geq \alpha$ except for the Gaussian. Indeed, if $\alpha < 1 = |\beta|$, the r.v. is supported by the half-line $(-\infty, \mu)$ if $\beta = -1$ and (μ, ∞) if $\beta = 1$. Different parametrizations do exist, we refer to chapter 1 of Samorodnitsky & Taqqu (1994).

For our purposes, we can and will assume all nondegenerate real r.v.'s to have unity scale parameter and be located at 0. Now fix a common α; then an arbitrary linear combination of independent α-stable r.v.'s is α-stable. Indeed, if Z_i are independent $S_\alpha(1, \beta_i, 0)$, then the scalar product $\mathbf{v}^\mathsf{T}\mathbf{Z}$ is distributed

$$S_\alpha\big(\|v\|_\alpha,\ \frac{\sum v_i^{<\alpha>}\beta_i}{\|v\|_\alpha^\alpha},\ -\frac{2}{\pi}\beta \sum v_i \log|v_i| \cdot \chi_{\{\alpha=1\}}\big), \tag{4}$$

where the *signed power* $v^{<\alpha>}$ equals $|v|^\alpha \operatorname{sign} v$ and $\|v\|_\alpha := (\sum |v_i|^\alpha)^{1/\alpha}$ is only a quasi-norm for $\alpha < 1$. In particular, except for the skew 1-stable case where the "scaling parameter" actually does more than just "scaling", a stable r.v. minus its location is strictly stable.

In Section 1, we assumed the driving noise $\mathbf{X}$ in (1) to be a Gaussian additive process. Let us make the following generalization (weakened further in Section 5): $\mathbf{X}$ is assumed to be a vector of i.i.d. stable additive processes with zero location and unity scale parameter. That is, the h-time increments of any component X_i are i.i.d. $S_\alpha(h^{1/\alpha}, \beta, 0)$. We remark that by (4) an $S_\alpha(1, \beta_0, 0)$ r.v. with $|\beta_0| < \beta$ may be written as linear combination of two independent $S_\alpha(1, \beta, 0)$ variables; however, by the assumptions we will make on the portfolio constraints this apparent generalization is of little interest.

The stable distributions are precisely the ones obtainable from the generalized central limit theorem. For this reason, they are frequently considered heavy-tailed alternatives to the Gaussian. A careful note is appropriate though: While the solution to the geometric SDE (1) is lognormal if $\mathbf{X}$ follows the normal distribution, the solution is *not* log-stable in the non-Gaussian stable case. The empirical works by Mandelbrot and Fama on estimating α in log returns (see the collection Mandelbrot (1997), chapters E1, E14 and E16) do not justify (1). Works have been done on stable absolute returns as well, though: Mantegna (1991) estimates α-values ranging from 1.00 ± 0.04 to 1.40 ± 0.04 for daily differences in Sectorial indices over ten or fifteen year periods on the Milan stock exchange, and 1.16 ± 0.02 for the M.I.B. index, assuming symmetric stability (i.e. $\mu = \beta = 0$). Janici et al. (1997) treat the problem of pricing options on assets following (1) with symmetric noise, to explain the smile effect.

Warning.

The geometric process (1) will change sign (unless totally skewed to the right or $\alpha = 2$), and will therefore violate limited liability. We emphasize that we do not assume limited liability, and we arguably have a theoretical shortcoming to the model if intended to model stocks. On the other hand, it makes the model better suited for insurance liabilities. The non-Gaussian stable laws have tails asymptotically like $|z|^{-\alpha}$, a property they share with the loggamma and Pareto distributions. The α-stable laws may therefore be an alternative to these for $\alpha < 2$. In fact, from Kagan (1997) we have data indicating that the magnitude of earthquakes (hence possibly related insurance claims) have sufficiently heavy power tails.

We will also need a convention on what happens when at changes of sign, as we are soon to allow nonnegativity constraints on the value invested in a risky asset. In an insurance market, a major negative jump corresponds to a claim against the portfolio, and has to be paid out immediately from the insurer's holdings – and then the insurance contract continues to develop with the same dynamics as before (recall that the states $\{S_i\}$ are not present in the wealth dynamics (2)). Small fluctuations, on the other hand, may be interpreted as diffusion-alike changes in the value of the contract.

Arguably, this interpretation is somewhat troublesome if there is no risk free asset and, say, consumption required to be nonnegative; imposing a nonnegative amount invested in each asset with nowhere to borrow makes the problem ill-posed at first time wealth is negative, which it will be for some parameter values. On the other hand, if the problem considered is assumed terminated at the first time no admissible strategy exists, our analysis will still be valid. Furthermore, let us note that C is *net* consumption, including income from other sources independent of (the α-stable part of) the market, for example from labour or stochastic sources not following the particular stable law, and may therefore cover losses from this particular asset market. △

With the above reservation, we shall refer to nonnegativity constraints on the amount invested in an asset as *forbidding short sale*.

3 Portfolio separation, α-stable case, risk free asset exists.

The main arguments of Sections 1 and 2 carry over, and will frequently give rise to portfolio separation. For technical reasons, we may want to use $\mathbf{v} := \Sigma^{\mathsf{T}}\mathbf{u}$ to rewrite the wealth dynamics into

$$\mathrm{d}Y(t) = \mathbf{u}^{\mathsf{T}}\boldsymbol{\mu}\,\mathrm{d}t + \mathbf{v}^{\mathsf{T}}\,\mathrm{d}\mathbf{X}(t) - \mathrm{d}C(t)$$

and pose restrictions in terms of $V := \Sigma^{\mathsf{T}}U$ on $\mathbf{v}$ instead of U. If

$$\text{Either} \quad \beta = 0 \quad \text{or both} \quad \alpha \neq 1 \text{ and } V \text{ contained in the first orthant,} \tag{5}$$

then all admissible portfolios will preserve skewness and the location-scale optimization problem

$$\sup_{\mathbf{u}\in U} \mathbf{u}^{\mathsf{T}}\boldsymbol{\mu} \quad \text{subject to} \quad \|\mathbf{v}\|_{\alpha} = Q \tag{6}$$

is sufficient to grant coinciding law. Assuming there is no arbitrage, we may choose $\mathbf{f}$ to maximize $\mathbf{f}^\mathsf{T}\boldsymbol{\mu}$ subject to $\mathbf{f} \in U$ and $\|\Sigma^\mathsf{T}\mathbf{f}\|_\alpha = 1$ to get:

3.1 Theorem: Two fund separation.

Consider problem (6) with U being a closed family of half-lines from $\mathbf{0}$ and such that (5) holds. Then $\mathbf{u}^* = Q\mathbf{f}$ is optimal for all attainable $Q > 0$. □

For a generalization to the case where $\mathbf{X}$ may have dependent components, see Section 5. We make a few remarks: First, a third fund may be needed if there is arbitrage, just as in the Gaussian case. Second, invertible Σ does not in general imply absence of arbitrage, as the X_i may be a.s. positive (if $\alpha < 1$ and $\beta = 1$), in which case no greedy agent will want to minimize scale given location (see Definition 4.3). In other words, drift maximization is more crucial here than in the Gaussian setup; indeed, since the sample paths are no longer continuous, agents with, say, concave utility function except convex in a bounded interval will not necessarily blow up volatility to immediately reach the boundary of that wealth interval. Third, let us remark some consequences for $\alpha \leq 1$: Assume for simplicity that Σ is the identity. First, if $\alpha = 1$ and V is a union of orthants, then one shall only invest in one asset (the one with highest drift, or highest negative drift if negative position allowed.) If $\alpha < 1$ then the same holds if V is the entire space, or if V is the first orthant and at least one of the $\boldsymbol{\theta}_i$ are nonnegative (the "unit ball" is not a convex set.) Note also that symmetry is crucial for $\alpha = 1$; we shall treat the skew 1-stable case separately in the next Section.

Arguably, using $\mathbf{v}$ instead of $\mathbf{u}$ disguises the problem of preserving skewness. An alternative to assuming V contained in the first orthant, is assuming Σ having only nonnegative entries, and U contained in the first orthant, i.e. forbidding short sale. This is the reason why we cannot allow different skewnesses and write everything in terms of differences as mentioned in 2.3, even though we do not (yet) assume the market to be complete.

4 Portfolio separation under inequality constraints.

In this Section, we shall assume Σ invertible and thus square; if necessary, we may complete the market by introducing ficional "dummy stocks" in which investment is forbidden through imposing the zero position in (8). Writing $\boldsymbol{\theta} = \Sigma^{-1}\boldsymbol{\mu}$, the dynamics become

$$\mathrm{d}Y = \mathbf{v}^\mathsf{T}[\boldsymbol{\theta}\,\mathrm{d}t + \mathrm{d}\mathbf{X}] - \mathrm{d}C. \tag{7}$$

We then consider linear constraints of the form

$$V = \{\mathbf{v};\quad \mathbf{v}^\mathsf{T}\Sigma^{-1}A \leq \boldsymbol{\zeta}\} \qquad \text{(componentwise inequality)} \tag{8}$$

and note that this may or may not be interesting to the original setup.

As we saw above, the $\alpha \leq 1$-laws may exhibit a non-diversification behavior. Let us first treat the 1-stable case: If $\beta = 0$, we essentially have a linear programming problem, which may separate into fewer vectors. For $\beta \neq 0$ (not covered by Theorem 3.1) we have the following:

4.1 Theorem: Skew 1-stable case.

Assume $\alpha = 1$, $\beta \neq 0$ and that V is the first orthant. Then we have 2 fund separation .

Proof. The Lagrangian to be maximized wrt. $v_i \geq 0$ is $L = \mathbf{v}^\mathsf{T}\boldsymbol{\theta} - \frac{2\beta}{\pi}\sum_i \big(v_i \log v_i\big) - \lambda \mathbf{v}^\mathsf{T}\mathbf{1}$, which if $\beta > 0$ has a unique maximum at

$$\mathbf{v} = \exp\{-1 - \frac{\pi}{2\beta}\lambda\} \cdot \mathbf{f} = \frac{Q}{\mathbf{f}^\mathsf{T}\mathbf{1}}\mathbf{f} \quad \text{where } f_i = \exp\{\frac{\pi}{2\beta}\theta_i\}. \tag{9}$$

For $\beta < 0$, L is convex in v_i and the optimal is to invest all in the asset with highest θ_i. □

We remark that the argument depends on the constraint being an inequality in $\mathbf{v}^\mathsf{T}\mathbf{1}$. Other constraints will yield an i-dependent factor in front of the corresponding multiplier in the exponent. The approach does only admit generalizations if the constraints can be spanned by a low number of vectors which consist of only zeros and ones.

As the 1-norm constraint is linear on the first orthant, it may define a plane parallell to a linear constraint defining V; a particular degenerate case is if $\mathbf{1}$ is an eigenvector of Σ (i.e. all assets equally volatile) and there is no safe asset, i.e. that $\mathbf{u}^\mathsf{T}\mathbf{1} = Y$ applies as a constraint; then the no short sale constraint is the same for $\mathbf{v}$ as for the original $\mathbf{u}$, and we have the following:

4.2 Corollary: One fund separation if no risk free asset.

Let $\alpha = 1$ and Σ be a constant times the identity. Suppose there is no risk free asset, and that short sale is forbidden. If $\beta \neq 0$, then all agents have the same portfolio weights, the portfolio given by (9) with the only attainable volatility $\bar{Q}$; if $\beta = 0$, then one will invest only in the asset with highest drift, hence the same holds if this asset is unique. □

If $\alpha < 1$, we can only obtain one fund separation among agents which share some particular attitude towards risk. A risk averse agent is usually thought of as someone who will reject a zero mean noise term independent of everything; here, however, mean does not exist:

4.3 Definition: Scale minimization.

Fix (α, β). An agent minimizing the scale parameter given skewness and drift, is then said to be a *scale minimizer.* △

Warning.

The definition should be interpreted (α, β)-wise – a scale minimizer for $\alpha < 1$ and $\beta = 1$, will reject an arbitrage! For $\beta = 0$ the condition simply means replacing "zero mean" by "symmetric around 0", while for $\beta < 0$ the condition is heuristically "weaker"; for $\beta = -1$ (and $\alpha > 1$), it is reasonable to assume scale minimization, as other agents will give away arbitrages to the market. △

In other words, assuming (5), a scale minimizer may instead of (6) be assumed to solve

$$\inf_{\mathbf{v}\in V} \|\mathbf{v}\|_\alpha \quad \text{subject to} \quad \mathbf{v}^\mathsf{T}\boldsymbol{\theta} = D. \tag{10}$$

Since the condition $\mathbf{v}^{\mathsf{T}}\boldsymbol{\theta} = D$ is linear, it may be contained in the prescribed V. Solving (10) for $\mathbf{v}_* = \mathbf{v}_*(D)$ for all D (including the negative!) for which it exists, we find that the solution $\mathbf{v}^* = \mathbf{v}^*(Q)$ of (6) can be chosen as $\mathbf{v}^*(Q) = \mathbf{v}_*(D) + \mathbf{p}(Q)$ with $\mathbf{p}(Q) \perp \boldsymbol{\theta}$ being "pure scale". The canonical example is the well-known "one fund separation" in a market with i.i.d. lognormal assets, which holds only among diversifyers, i.e. variance minimizers.

Scale minimization implies the following:

4.4 Lemma: Non-diversification under scale minimization, $\alpha \leq 1$.

Let V satisfy (5) and (8), and assume $\alpha \leq 1$. Then no scale minimizer will choose interior solution except possibly $v_i = 0$.

Proof. Suppose the contrary, and form the Lagrangian $L = \sum |v_i|^\alpha - \mathbf{v}^{\mathsf{T}}(\lambda_0\boldsymbol{\theta} + \Sigma^{-1}A\boldsymbol{\lambda})$. Then $(\partial/\partial v_i)^2 L \leq 0$ except at 0. □

Then it easily follows:

4.5 Theorem: One fund separation under scale minimization, $\alpha < 1$.

Assume $\alpha \leq 1$. Define V by the absence of risk free asset, intersected with the first orthant if implied by (5). A scale minimizer may then choose the zero position in all assets except the one with highest drift. □

We note that the one fund separation results presented in this Section do not satisfy the necessary conditions of Ross (1978), Theorem 1, which implicitely assumes finite conditional mean. We furthermore note that by a Hamilton-Jacobi-Bellman argument, it is not too difficult to see that one frequently will want to hold an infinite position when $\alpha \leq 1$, unless the appropriate constraints apply.

For $\alpha \in (1, 2)$, risk averse agents will be scale minimizers. Considering the Kuhn-Tucker condition associated to (10), we have separation properties under certain restrictive assumptions which nevertheless generalize the familiar Gaussian setup:

4.6 Theorem: The symmetric $1 + \frac{1}{\text{odd}}$-stable case.

Assume that $\beta = 0$, that $\frac{1}{\alpha-1}$ a natural odd number and that V is given by (8). Then a risk averse agent has $m + 2$ fund separation, where m is the number of independent vectors in expanding the power

$$v_i^* = \left((\Sigma^{-1}A\boldsymbol{\lambda})_i\right)^{1/(\alpha-1)} \tag{11}$$

and solving to get $\mathbf{u}^{*\mathsf{T}} = \mathbf{v}^{*\mathsf{T}}\Sigma^{-1} = \sum_{j=1}^m \bar{\lambda}_j(\mathbf{f}_j^{\mathsf{T}}\Sigma^{-1})$. □

Again, we make some remarks: If $\frac{1}{\alpha-1}$ is even, then the Kuhn-Tucker condition cannot determine the sign of the coordinates v_i. *Imposing* nonnegativity or nonpositivity will require at least as many funds as assets in (11) – a number possibly larger than the original number of assets due to the augmentation with the "dummy" assets. However, in quite a few interesting cases, say, we do not need that $\frac{1}{\alpha-1}$ is odd: If all drift terms are nonnegative and V is such that given any admissible portfolios with a negative position in certain subset of the asset, then forming a portfolio with the zero position in these assets (but not changing the others) is still admissible, then a risk averse agent will do so if this subset is independent of the rest. In those cases, all the v_i automatically have the same sign, and we may allow for even $\frac{1}{\alpha-1}$.

4.7 Example.

In the Gaussian case, we made a remark on different interest rates for lending and borrowing . This may be covered by introducing a constraint $\mathbf{u}^\top\mathbf{1} = K$ and for each K, the alternative drift condition $\mathbf{u}^\top(\boldsymbol{\mu} - r(K)\mathbf{1}) = D$, i.e. $\mathbf{u}^\top\boldsymbol{\mu} = D + Kr$. Note that the funds do not depend on the right hand sides, i.e. $\boldsymbol{\zeta}$, and therefore not on the chosen K, even though r may do. Assume complete unconstrained market with Σ being the identity. The first order constraint is then $u_i^{<\alpha-1>} = \lambda + \lambda_0\mu_i$. If $\frac{1}{\alpha-1}$ odd natural number, we have strong $\frac{\alpha}{\alpha-1}+1$ fund separation over risk averse agents. △

Instead of letting $r(K)$ be as in the examble, we may disallow for certain K's (let r be infinite there); in particular, we have covered the case where there is no borrowing – or where no risk free asset exists (admit only the particular value $K = Y$). We summarize a genaralization of this classical result:

4.8 Corollary: $\frac{\alpha}{\alpha-1}$ fund separation if no risk free asset.

Assume that $\frac{1}{\alpha-1}$ *is an odd natural number and that the only constraint is* $\mathbf{u}^\top\mathbf{1} = Y$. *Then we have* $\frac{\alpha}{\alpha-1}$ *fund separation.* □

5 Non-Gaussian α-stable noise with dependent components.

We shall generalize Theorem 3.1 to the case where $\mathbf{X}$ is merely an α-stable vector process, possibly with dependent components. We already know the Gaussian case, so assume $\alpha < 2$. For the results needed in this Section, we refer to Samorodnitsky & Taqqu (1994), Chapter 2, especially Example 2.3.4 and Theorem 2.4.3: If $\mathbf{Z}$ is an α-stable d-vector, its characteristic function is determined by a finite *spectral measure* Γ living on the unit sphere in $\mathbf{R}^d$, and a constant vector $\bar{\mathbf{z}}$ which in our model enters via the drift term in (2) and therefore is $\mathbf{0}$. Then $\mathbf{v}^\top\mathbf{Z}$ is $S_\alpha(\sigma, \beta, \mu)$ with

$$\sigma = \sigma_\mathbf{v} = \Big(\int |\mathbf{v}^\top\mathbf{s}|^\alpha \,\Gamma(d\mathbf{s})\Big)^{1/\alpha}, \tag{12a}$$

$$\beta = \beta_\mathbf{v} = \sigma^{-\alpha}\int (\mathbf{v}^\top\mathbf{s})^{<\alpha>}\,\Gamma(d\mathbf{s}), \tag{12b}$$

$$\mu = \mu_\mathbf{v} = -\frac{2}{\pi}\int \mathbf{v}^\top\mathbf{s}\cdot\log|\mathbf{v}^\top\mathbf{s}|\,\Gamma(d\mathbf{s})\cdot\chi_{\{\alpha=1\}}, \tag{12c}$$

and in the symmetric case (i.e. $Z \sim -Z$, implying Γ symmetric and thus $\beta_\mathbf{v} = \mu_\mathbf{v} = 0\ \forall\mathbf{v}$), the law of $\mathbf{v}^\top\mathbf{Z}$ depends only on (α and) $\sigma_\mathbf{v}$ via (12a). If we in the non-symmetric case ad hoc assume $\alpha \neq 1$ and V such that β is constant (analogous to (5)), then the law of $\mathbf{v}^\top\mathbf{Z}$ is determined by $\sigma_\mathbf{v}$ (and α). Indeed, $\{\mathbf{v};\ \beta_\mathbf{v} = B\}$ is for fixed B a family of half-lines from $\mathbf{0}$. Then we can choose $\mathbf{f}^*$ to maximize drift given $\sigma_\mathbf{v} = 1$ and put $\mathbf{u}^* = Q\mathbf{f}^*$ just as in Theorem 3.1. We have thus:

5.1 Theorem: Two fund separation, dependent components.

Assume the dynamics follows (2)*, with* $\mathbf{X}$ *being a stable* $\mathbf{R}^d$*-valued additive process with spectral measure* Γ*. Let* V *be a family of half-lines from the origin, and such that* $\beta_{\mathbf{v}}$ *given by* (12b) *is constant on* V*; if* $\alpha = 1$*, assume in addition* $\mathbf{X}$ *symmetric. Then we have two fund separation (three if there is arbitrage.)*

6 Concluding remarks.

We make a remark to the assumption on the preferences: In the Gaussian case, it may be replaced by the assumption that given consumptions coinciding in law, then Y^* is preferred to Y if $Y^* \geq \tilde{Y}$ a.s. for some $\tilde{Y} \sim Y$. Then we can still maximize drift given variance, and apply to well-known comparison theorems (see Ikeda and Watanabe (1977), Theorem 1.1 for the result, and their Section 2 for an application to stochastic control.) However, in the discontinuous case $\alpha < 2$, this breaks down, as wealth can with positive probability jump by a factor less than -1 (unless we have totally skewed laws or choose the zero portfolio), and the elegant setup of Khanna & Kulldorff is then crucial, more than in their continuous setting.

The coefficients may be time-dependent or even stochastic if independent of everything. That includes the stability index α, in which case $\mathbf{X}$ will be a so-called *stable-like* process. However, as Theorem 4.6 only admits discrete values, this generalization is most interesting in the setting of Theorem 3.1. Note however that all X_i are still supposed to have the same α and β; we can adapt the theory to the X_i having different stability indices and skewnesses if Σ is invertible, by separating the market into independent groups of assets, each with common α and β.

Khanna & Kulldorff note that also the number of stocks (i.e. the constrains U and V in our setting) may vary in time (even in a random manner, if independent of the driving noise); this is of course correct if the investor is allowed to sell an asset which is about to disappear. Note that as the stable laws have no atoms, we will almost surely not face the modelling dilemma occurring when the process jumps to 0 and disappearing at the same time.

While a Lévy motion has no fixed discontinuity times, it is possible to admit this generalization; assume that at a stopping time τ independent of everything else we have with positive probability of disontinuity, and the conditional distributions of $\{S_i(\tau^+) - S_i(\tau)\}$ given jump is stable with common skewness and index of stability. Then the optimal portfolio at time τ should be location-dispertion efficient, as in discrete time.

Finally, we note that we do not really need that $\mathbf{X}$ itself is stable, only all linear combinations $\{\mathbf{v}^\mathsf{T}\, d\mathbf{X}\}_{\mathbf{v}\in V}$. However, the cases lost are few (though examples do exist) and not the most interesting, as α must be < 1 and $\mathbf{X}$ cannot be infinitely divisible (i.e. cannot be a Lévy process), and the linear combinations cannot all be strictly stable nor all symmetric. See Samorodnitsky & Taqqu (1994), Theorem 2.1.5 and Section 2.2.

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