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OPTIMAL CONSUMPTION UNDER PARTIAL OBSERVATIONS FOR A STOCHASTIC SYSTEM WITH DELAY

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Abstract. We investigate a class of optimal consumption problems where the wealth $X(t)$ at time t is given by a stochastic differential delay equation with a parameter. Not only the present value $X(t)$, but also $X(t-\delta)$ and some sliding average of previous values affects the growth at time t. Two cases are considered: 1) The parameter is a given deterministic function, giving a stochastic control problem with complete observations. 2) The parameter is an unobserved random variable, giving a stochastic control problem with partial observations. In this case, filtering theory is used to reduce the problem to a completely observed one.

In both cases, due to the delay, the resulting dynamic programming problems are in general infinite dimensional. Because of the specific structure of the dependence of the past that we consider, we are able to reduce the problem to finite dimensions. A verification theorem of variational inequality type is proved and applied to solve explicitly the control problems. (Explicit formulas for the value functions and the optimal consumption rates are given.)

1. Introduction

We consider a model for utility maximization from consumption and terminal wealth where the wealth process is given by a stochastic differential delay equation with a parameter. The wealth may be thought of as the value of an investment in a financial market. However, we do not specify an underlying market with tradeable assets. We consider two cases. In the first case the parameter is a known deterministic function. This means that we have complete information about the wealth process. In the second case we assume that the parameter is an unknown random variable with a given distribution, observable only indirictly through some auxiliary process. In this case we have only partial information.

In both cases the introduction of delay allows us to take into account the fact that it may take some time before new market information affects the value of our investment. Also the decisions we make regarding consumption may be based on both present and past values of the wealth. The second case captures in addition the fact that we do not always have complete information about all the parameters in a mathematical model in finance.

1.1. **The model.** We are given a complete probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ with a filtration satisfying the usual assumptions, and an $\{\mathcal{F}_t\}$-Brownian motion $W(t) \in \mathbb{R}$. Suppose the wealth $X(t)$ of a person with consumption rate $c(\,\cdot\,) \geq 0$ satisfies the stochastic differential delay equation (SDDE)

$$\begin{aligned} dX(t) &= \mu \cdot \left[X(t) + \nu e^{\lambda\delta} Y(t)\right] dt + \nu Z(t)\, dt - c(t)\, dt \\ &\quad + \alpha(t)\left[X(t) + \nu e^{\lambda\delta} Y(t)\right] dW(t), \quad t \geq 0, \end{aligned} \tag{1.1}$$

$$X(s) = \varphi(s), \quad -\delta \leq s \leq 0, \tag{1.2}$$

where

$$Y(t) := \int_{-\delta}^{0} e^{\lambda s} X(t+s)\, ds \quad \text{and} \quad Z(t) := X(t-\delta). \tag{1.3}$$

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Here, $\delta > 0$ is the constant *delay*, the initial path $\varphi \in C([-\delta, 0])$, the set of continuous real functions on $[-\delta, 0]$, $c(t)$ is an $\{\mathcal{F}_t\}$-adapted process, $\alpha(t)$ is a deterministic function with

$$\int_0^T \frac{dt}{\alpha(t)^2} < \infty,$$

ν, λ are real constants, and μ is a parameter to be further specified below. The solution of (1.1) with initial path (1.2) and consumption rate c is denoted by $X^{\varphi,c}(t)$ if it exists. See [13] or [14] for conditions for existence and uniqueness of solutions of such equations.

For $t \geq 0$ we denote by $X_t(\,\cdot\,)$ the function defined by

$$X_t(s) = X(t+s), \qquad -\delta \leq s \leq 0,$$

i.e. X_t is the segment of the path of X from $t-\delta$ to t.

Let

$$U_1 : [0,T] \times [0,\infty) \to \mathbb{R}$$

be a utility function which is continuous in both variables and increasing, concave in the second variable. Also we assume that

$$\mathbb{E}^{s,\varphi,c}\left[\int_0^T |U_1(s+t, c(t))|\, dt\right] < \infty \quad \text{for all } s, \varphi, c$$

where $\mathbb{E}^{s,\varphi,c}$ denotes expectation with respect to the probability law $Q^{s,\varphi,c}$ of the time-space process $(s+t, X^{\varphi,c}(t))$. Also let

$$U_2 : C([-\delta, 0]) \to \mathbb{R}$$

be a utility functional such that

$$\mathbb{E}^{t,\varphi,c}\big[U_2(X_T)\big] < \infty \quad \text{for all } \varphi, c.$$

Suppose the expected total discounted utility $J^c(t,\varphi)$ corresponding to the consumption rate c is given by

$$J^c(s,\varphi) = \mathbb{E}^{s,\varphi,c}\left[\int_0^T U_1(s+t, c(t))\, dt + U_2(X_T)\right]. \tag{1.4}$$

We now formulate two control problems.

Let $\mathcal{A}$ denote the class of consumption rate processes $c(t)$ that are $\{\mathcal{F}_t\}$-adapted and nonnegative.

Problem 1.1 (Optimal consumption with complete observations)**.** Find the value function $\Phi(s,\varphi)$ and an optimal consumption rate $c^* \in \mathcal{A}$ such that

$$\Phi(s,\varphi) = \sup\big\{J^c(s,\varphi)\,;\, c \in \mathcal{A}\big\} = J^{c^*}(s,\varphi) \tag{1.5}$$

when the system is given by (1.1)–(1.2) and the parameter μ is a given deterministic function $\mu(t)$.

Now assume the parameter μ is a Gaussian random variable with $\mu \sim \mathcal{N}(\bar{\mu}, \sigma)$ independent of $\{W(t)\}_{t\geq 0}$. The value of μ is unknown, but observable through the process $\xi(t)$ defined by

$$d\xi(t) = \mu\, dt + \alpha(t)\, dW(t), \quad t \geq 0, \tag{1.6}$$

$$\xi(0) = 0. \tag{1.7}$$

Let

$$\mathcal{G}_t := \sigma\big(\,\xi(s)\,;\, 0 \leq s \leq t\,\big), \tag{1.8}$$

and note that $\mathcal{G}_t \subset \mathcal{F}_t$.

Let $\mathcal{A}_p$ denote the class of consumption rate processes $c(t)$ that are $\{\mathcal{G}_t\}$-adapted and nonnegative.

Problem 1.2 (Optimal consumption with partial observations)**.** Find the value function $\Phi(s,\varphi)$ and an optimal consunption rate $c^* \in \mathcal{A}_p$ such that

$$\Phi(s,\varphi) = \sup\big\{J^c(s,\varphi)\,;\, c \in \mathcal{A}_p\big\} = J^{c^*}(s,\varphi) \tag{1.9}$$

when the system and observations are given by (1.1)–(1.2) and (1.6)–(1.7).

In general, these problems are infinite dimensional due to the infinite dimensionality of the space of initial data. The purpose of the paper is to show that for a certain class of systems (1.1) the value functions and the optimal consumption rates for both problems depend on the initial path φ only through the three linear functionals

$$x = x(\varphi) := \varphi(0), \tag{1.10}$$

$$y = y(\varphi) := \int_{-\delta}^{0} e^{\lambda s}\varphi(s)\, ds, \tag{1.11}$$

$$z = z(\varphi) := \varphi(-\delta). \tag{1.12}$$

If this is the case we can write for Problem 1.1

$$\Phi(s,\varphi) = V(s,x,y,z) \qquad \text{where } V : \mathbb{R}^4 \to \mathbb{R} \tag{1.13}$$

with an additional dimension for Problem 1.2 due to the observations.

The idea of making an assumption like (1.13) was used by Kolmanovskii and Maizenberg in [8] (also see [9]) and by Elsanousi, Øksendal and Sulem in [4], where respectively, a nonsingular and a singular stochastic control problem for a certain linear delay system was solved. We use the idea to state and prove a dynamic programming verification theorem (Theorem 2.1) in Section 2. In Sections 3 and 4 we use the theorem to solve explicitly Problems 1.1 and 1.2 for specific choices of utilities U_1 and U_2 having the same kind of dependence of the past as the value function. In the solution of Problem 1.2 we also use filtering theory.

Koivo [7] and Lindquist [11, 12] also treat stochastic control problems for systems with delay where the dependence of the past is of the same kind as we consider. Their approach differ from ours in that they allow only additive noises in the system, and quadratic performance functionals. Also their methods differ from ours.

For stochastic systems without delay, control problems of this type have been studied in [10] and [1]. For a general introduction to optimal consumption problems the reader should consult the book [6] by Karatzas and Shreve and the references therein.

2. Dynamic programming

In this section we state and prove a verification theorem for the cases we consider. Let the controlled system be of the form

$$\begin{aligned} dX(t) &= b_1\big(t, X(t), Y(t), Z(t), R(t), c(t, X(t), R(t), Y(t))\big)\, dt \\ &\qquad + \sigma_1\big(t, X(t), Y(t), Z(t), R(t), c(t, X(t), R(t), Y(t))\big)\, dW(t), \quad t \geq 0 \\ X(s) &= \varphi(s), \quad -\delta \leq s \leq 0, \quad \varphi \in C([-\delta, 0]), \\ dR(t) &= b_2\big(t, R(t)\big)\, dt + \sigma_2\big(t, R(t)\big)\, dW(t), \quad R(0) = r. \end{aligned}$$

As usual we write $X^{\varphi,c}$ and R^r to emphasize the initial conditions and the control used. The control should belong to the class $\mathcal{A}$ of nonnegative consumption rate processes adapted to the same filtration as the driving Brownian motion W. We let $\mathbb{E}^{s,\varphi,r,c}$ denote expectation with respect to the law of the process $(s+t, X^{\varphi,c}(t), R^r(t))$.

Let $f \in C^{1,2,2,1}(\mathbb{R}^4)$ and define

$$G(t) = f(s+t, X^{\varphi}(t), R(t), y(X_t^{\varphi})). \tag{2.1}$$

where

$$y(\eta) = \int_{-\delta}^{0} e^{\lambda s}\eta(s)\, ds, \quad \eta \in C([-\delta, 0]), \quad (\lambda \text{ constant}). \tag{2.2}$$

Then the following Ito's and Dynkin's formulas are needed.

Lemma 2.1 (The Ito formula).

$$dG(t) = Lf\,dt + \frac{\partial f}{\partial x}\cdot\sigma_1(u,x,y,z,r,c)\,dW(t) + \frac{\partial f}{\partial r}\cdot\sigma_2(u,r)\,dW(t) + \frac{\partial f}{\partial y}\cdot[x - e^{-\lambda\delta}z - \lambda y]\,dt$$

where

$$\begin{aligned} Lf &= L_c f(u,x,r,y,z) \\ &= \frac{\partial f}{\partial u} + b_1(u,x,y,z,r,c)\frac{\partial f}{\partial x} + b_2(u,r)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma_1^2(u,x,y,z,r,c)\frac{\partial^2 f}{\partial x^2} \\ &\quad + \sigma_1(u,x,y,z,r,c)\sigma_2(u,r)\frac{\partial^2 f}{\partial x\partial r} + \frac{1}{2}\sigma_2^2(u,r)\frac{\partial^2 f}{\partial r^2}, \end{aligned}$$

which is evaluated at

$$u = s+t, \quad x = X(s), \quad r = R(s), \quad y = y(X_s), \quad z = z(X_s) = X(s-\delta).$$

Proof. Note that if $\eta \in C([-\delta,T])$ then $\eta_t \in C([-\delta,0])$ for each $t\in[0,T]$. By (2.2) we get, using the Leibniz formula,

$$\begin{aligned} \frac{d}{dt}(y(\eta_t)) &= \frac{d}{dt}\left[\int_{-\delta}^{0} e^{\lambda s}\eta(t+s)\,ds\right] \\ &= \frac{d}{dt}\left[\int_{t-\delta}^{t} e^{\lambda(r-t)}\eta(r)\,dr\right] \qquad (r = t+s) \\ &= \eta(t) - e^{-\lambda\delta}\eta(t-\delta) - \lambda\int_{t-\delta}^{t} e^{\lambda(r-t)}\eta(r)\,dr \\ &= x(\eta_t) - e^{-\lambda\delta}z(\eta_t) - \lambda y(\eta_t). \end{aligned}$$

Now since $G(t) = f(s+t, X^\varphi(t), R(t), y(X_t^\varphi))$, the result follows from the classical Ito formula. □

Remark 2.1. Note that with $\eta = X$ the calculation in the lemma gives the following useful formula for dY:

$$dY(t) = \left[X(t) - e^{-\lambda\delta}Z(t) - \lambda Y(t)\right]dt. \tag{2.3}$$

From Lemma 2.1 we get

Lemma 2.2 (The Dynkin formula). Let $f\in C_0^{1,2,2,1}(\mathbb{R}^4)$. Then for $t\geq 0$ we have

$$\begin{aligned} &\mathbb{E}^{s,\varphi,r}\left[f(s+t,X(t),R(t),y(X_t))\right] \\ &\quad = f(s,\varphi(0),r,y(\varphi)) + \mathbb{E}^{s,\varphi,r}\left[\int_0^t\left(Lf + \frac{\partial f}{\partial y}\cdot[x - e^{-\lambda\delta}z - \lambda y]\right)dv\right], \end{aligned} \tag{2.4}$$

where $Lf(u,x,r,y)$ and the other functions in the bracket are evaluated at

$$u = s+v, \quad x = X(v), \quad r = R(v), \quad y = y(X_v), \quad z = z(X_v) = X(v-\delta).$$

Proof. The result easily follows from the Ito formula. □

Definition 2.1. For $f = f(t,x,r,y)$ in $C_0^{1,2,2,1}(\mathbb{R}^4)$ we define the operator $\mathcal{L}$ by

$$\mathcal{L}f := Lf + \frac{\partial f}{\partial y}\cdot[x - e^{-\lambda\delta}z - \lambda y].$$

Theorem 2.1 (Verification theorem). *Assume that we have found a function $V = V(s,x,r,y)$ in $C^{1,2,2,1}(S^o)\cap C(\bar{S})$, where $S\subset\mathbb{R}^4$, S^o denotes the interior and $\bar{S}$ the closure of S, and a consumption rate $c^* = c^*(s,x,r,y)$, satisfying for all $z\in\mathbb{R}$, $c\in\mathcal{A}$ and $(s,x,r,y)\in S^o$, the conditions*

C1 $\mathcal{L}_{c^*}V(s,x,r,y,z) + F_1(s,c^*(s,x,r,y)) = 0,$
C2 $\mathcal{L}_c V(s,x,r,y,z) + F_1(s,c(s,x,r,y)) \leq 0,$
C3 $\lim_{t\to T} V_z(t,X(t),R(t),Y(t)) = F_0(X(T),R(t),Y(T))\cdot\mathbf{1}_{\{T<\infty\}}$ *a.s.* $Q^{s,\varphi,r,c}$,

C4 $\{V(\tau, X(\tau), R(\tau), Y(\tau))\}_{\tau \leq T}$ *is uniformly* $Q^{s,\varphi,r,c}$ *-integrable,*
where F_0 *and* F_1 *are bounded functions and*

$$T = \inf\{v > 0\,;\, (s+v, X(v), R(v), Y(v)) \notin S^o\}.$$

Define

$$J^c(s,r,\varphi) := \mathbb{E}^{s,\varphi,r,c}\left[\int_0^T F_1\Big(s+v, c(v, X(v), R(v), Y(v))\Big)\, dv \right.$$
$$\left. + F_0(X(T), R(T), Y(T)) \cdot \mathbf{1}_{\{T<\infty\}}\right].$$

Then

(a) *the consumption rate* c^* *is optimal for the problem*

$$\Phi(s,r,\varphi) := \sup_{c\in\mathcal{A}} J^c(s,r,\varphi).$$

(b) *The value function* $\Phi(s,r,\varphi) = V(s, x(\varphi), r, y(\varphi))$ *for all* (s,r,φ).

Proof. The proof is derived by the same method as in the proof of the non delay case. For example, see Theorem 11.2 in [15]. For completeness we give the details: Assume that $V \in C^{1,2,2,1}(S^o) \cap C(\bar{S})$ satisfies conditions (C2)–(C4). Then by the Dynkin formula we have

$$\begin{aligned}
(2.5)\quad & \mathbb{E}^{s,\varphi,r,c}\big[V(s+T_M, X^{\varphi,c}(T_M), R^r(T_M), y(X^{\varphi,c}_{T_M}(\,\cdot\,)))\big] \\
&= V(s,\varphi(0), r, y(\varphi)) + \mathbb{E}^{s,\varphi,r,c}\left[\int_0^{T_M} \mathcal{L}V\, dt\right] \\
&\leq V(s,\varphi(0), r, y(\varphi)) - \mathbb{E}^{s,\varphi,r,c}\left[\int_0^{T_M} F_1(s+v, c(v, X(v), R(v), Y(v)))\, dv\right],
\end{aligned}$$

where

$$T_M = T \wedge M \wedge \inf\big\{t > 0\,;\, \|(X^{\varphi,c}(t), R^r(t), Y^{\varphi,c}(t))\| \geq M\big\}.$$

Hence

$$\begin{aligned}
V(s,\varphi(0), r, y(\varphi)) &\geq \mathbb{E}^{s,\varphi,r,c}\left[\int_0^{T_M} F_1(s+v, c(v, X(v), R(v), Y(v))\, dv\right] \\
&\quad + V(s+T_M, X^{\varphi,c}(T_M), R^r(T_M), y(X^{\varphi,c}_{T_M}(\,\cdot\,))) \\
&\to \mathbb{E}^{s,\varphi,r,c}\left[\int_0^{T} F_1(s+v, c(v, X(v), R(v), Y(v))\, dv \right. \\
&\quad \left. + F_0(X(T), R(T), Y(T)) \cdot \mathbf{1}_{\{T<\infty\}}\right] = J^c(s,r,\varphi)
\end{aligned}$$

as $M \to \infty$. That means

$$V(s,\varphi(0), r, y(\varphi)) \geq J^c(s,r,\varphi).$$

Next if c is such that (C1) holds, then the calculations above give equality and the proof is complete. □

3. The complete observations case

We make the following choices of U_1 and U_2, to be used throughout this section.

$$U_1(t, c(\varphi)) = e^{-\rho t}\frac{c(\varphi)^\gamma}{\gamma} = e^{-\rho t}\frac{c(t,x,y)^\gamma}{\gamma}, \tag{3.1}$$

$$U_2(X_T) = U_2(X(T), Y(T)) = e^{-\rho T}\frac{(X(T) + \nu e^{\lambda\delta} Y(T))^\gamma}{\gamma}, \tag{3.2}$$

where $\rho > 0$ and $\gamma \in (0,1)$ are given constants. With these choices of utility functions, Problem 1.1 may be restated as

Problem 3.1. Find the value function $\Phi(s,\varphi)$ and an optimal consumption rate $c^* \in \mathcal{A}$ such that

$$\Phi(s,\varphi) = \sup_{c\in\mathcal{A}} \mathbb{E}^{s,\varphi,c}\left[\int_0^T e^{-\rho(s+t)}\frac{c(t,X(t),Y(t))^\gamma}{\gamma}\,dt + e^{-\rho T}\frac{(X(T)+\nu e^{\lambda\delta}Y(T))^\gamma}{\gamma}\right] \tag{3.3}$$

when the system is given by

$$\begin{aligned} dX(t) &= \mu(t)\big[X(t)+\nu e^{\lambda\delta}Y(t)\big]\,dt + \nu Z(t)\,dt - c(t)\,dt \\ &\quad + \alpha(t)\big[X(t)+\nu e^{\lambda\delta}Y(t)\big]\,dW(t), \quad t\ge 0, \end{aligned} \tag{3.4}$$

$$X(s) = \varphi(s), \quad -\delta \le s \le 0, \tag{3.5}$$

As mentioned in the introduction, we assume that the value function is of the form

$$\Phi(s,\varphi) = V(s,x,y,z) \quad \text{where } V:\mathbb{R}^4\to\mathbb{R},$$

and that the consumption rate is of the form

$$c(\varphi) = c(t,x,y).$$

We shall seek a functional V satisfying the conditions of Theorem 2.1 in the form

$$V(s,x,y,z) = K(s)e^{-\rho s}\psi(x,y) \tag{3.6}$$

for some real functions ψ and K. Plugging this into condition (C1) and recalling (3.3), we get

$$\begin{aligned} e^{\rho s}\mathcal{L}_c V(s,x,y,z) + \frac{c(s,x,y)^\gamma}{\gamma} &= K'(s)\psi(x,y) - \rho K(s)\psi(x,y) \\ &\quad + K(s)\frac{\partial\psi}{\partial y}[x - e^{-\lambda\delta}z - \lambda y] + K(s)\Big[\mu(s)\big(x+\nu e^{\lambda\delta}y\big) + \nu z - c(s,x,y)\Big]\frac{\partial\psi}{\partial x} \\ &\quad + \frac{1}{2}K(s)\alpha(s)^2\big(x+\nu e^{\lambda\delta}y\big)^2\frac{\partial^2\psi}{\partial x^2} + \frac{c(s,x,y)^\gamma}{\gamma} = 0, \end{aligned} \tag{3.7}$$

or equivalently,

$$\begin{aligned} K'(s)\psi(x,y) &+ K(s)\Big\{-\rho\psi(x,y) + (x-\lambda y)\frac{\partial\psi}{\partial y} + \mu(s)\big(x+\nu e^{\lambda\delta}y\big)\frac{\partial\psi}{\partial x} \\ &- c(s,x,y)\frac{\partial\psi}{\partial x} + \frac{1}{2}\alpha(s)^2\big(x+\nu e^{\lambda\delta}y\big)^2\frac{\partial^2\psi}{\partial x^2}\Big\} + \frac{c(s,x,y)^\gamma}{\gamma} \\ &+ K(s)z\Big\{\nu\frac{\partial\psi}{\partial x} - e^{-\lambda\delta}\frac{\partial\psi}{\partial y}\Big\} = 0, \end{aligned} \tag{3.8}$$

with boundary condition

$$V(T,X(T),Y(T)) = e^{-\rho T}\frac{\big(X(T)+\nu e^{\lambda\delta}Y(T)\big)^\gamma}{\gamma}. \tag{3.9}$$

The maximum in (3.7) is obtained when

$$\frac{\partial}{\partial c}\left(\mathcal{L}_c V(s,x,y,z) + e^{-\rho s}\frac{c^\gamma}{\gamma}\right) = 0,$$

i.e. when

$$c(s,x,y) = c^*(s,x,y) = \left(K(s)\frac{\partial\psi}{\partial x}\right)^{\frac{1}{\gamma-1}}.$$

We require that (3.7) holds with this value of c. Now,

$$\mathcal{L}_{c^*}V(s,x,y,z) + e^{-\rho s}\frac{{c^*}^\gamma}{\gamma} = 0 \quad \text{for all } z \tag{3.10}$$

if equation (3.8) holds, that is if

$$K(s)z\Big\{\nu\frac{\partial\psi}{\partial x} - e^{-\lambda\delta}\frac{\partial\psi}{\partial y}\Big\} = 0 \tag{3.11}$$

and

$$K'(s)\psi(x,y)+K(s)\Big\{-\rho\psi(x,y)+(x-\lambda y)\frac{\partial\psi}{\partial y}+\mu(s)\big(x+\nu e^{\lambda\delta}y\big)\frac{\partial\psi}{\partial x} - c^*\frac{\partial\psi}{\partial x}+\frac{1}{2}\alpha(s)^2\big(x+\nu e^{\lambda\delta}y\big)^2\frac{\partial^2\psi}{\partial x^2}\Big\}+\frac{{c^*}^\gamma}{\gamma}=0. \tag{3.12}$$

Equations (3.11) and (3.12) hold if

$$\psi(x,y)=g(w) \tag{3.13}$$

for some function $g:\mathbb{R}\to\mathbb{R}$ where

$$w=w(x,y)=x+\nu e^{\lambda\delta}y. \tag{3.14}$$

Substituting (3.13)–(3.14) into (3.12) we get

$$K'(s)g(w)+K(s)\Big\{-\rho g(w)+(x-\lambda y)\nu e^{\lambda\delta}g'(w)+\mu(s)\big(x+\nu e^{\lambda\delta}y\big)g'(w) - c^*g'(w)+\frac{1}{2}\alpha(s)^2\big(x+\nu e^{\lambda\delta}y\big)^2g''(w)\Big\}+\frac{{c^*}^\gamma}{\gamma}=0. \tag{3.15}$$

Equation (3.15) has a solution depending only on s and w if

$$\nu=-\lambda e^{-\lambda\delta},\quad \lambda<0. \tag{3.16}$$

Using this, equation (3.15) becomes

$$K'(s)g(w)+K(s)\Big\{-\rho g(w)+\big(\mu(s)-\lambda\big)wg'(w) - c^*g'(w)+\frac{1}{2}\alpha(s)^2w^2g''(w)\Big\}+\frac{{c^*}^\gamma}{\gamma}=0. \tag{3.17}$$

Now we guess that

$$g(w)=w^\gamma. \tag{3.18}$$

With this g the optimal consumption rate c^* takes the form

$$c^*(x,y)=\Big(K(s)\frac{\partial\psi}{\partial x}\Big)^{\frac{1}{\gamma-1}}=\big(K(s)g'(w)\big)^{\frac{1}{\gamma-1}}=\big(\gamma K(s)\big)^{\frac{1}{\gamma-1}}w. \tag{3.19}$$

Plugging (3.18) and (3.19) into (3.17) we get

$$K'(s)w^\gamma+K(s)\Big\{-\rho w^\gamma+\big(\mu(s)-\lambda\big)\gamma w^\gamma - \big(\gamma K(s)\big)^{\frac{1}{\gamma-1}}\gamma w^\gamma+\frac{1}{2}\alpha(s)^2\gamma(\gamma-1)w^\gamma\Big\}+\frac{1}{\gamma}\big(\gamma K(s)\big)^{\frac{\gamma}{\gamma-1}}w^\gamma=0. \tag{3.20}$$

or equivalently,

$$K'(s)+K(s)\Big\{-\rho+\big(\mu(s)-\lambda\big)\gamma+\frac{1}{2}\alpha(s)^2\gamma(\gamma-1)\Big\}+\frac{1-\gamma}{\gamma}\gamma^{\frac{\gamma}{\gamma-1}}K(s)^{\frac{\gamma}{\gamma-1}}=0. \tag{3.21}$$

With

$$Q(s)=\rho-\gamma(\mu(s)-\lambda)-\frac{1}{2}\alpha(s)^2\gamma(\gamma-1)\quad\text{and}\quad\beta=\frac{\gamma-1}{\gamma}\gamma^{\frac{\gamma}{\gamma-1}}, \tag{3.22}$$

we write equation (3.21) as

$$K'(s)=Q(s)K(s)+\beta K(s)^{\frac{\gamma}{\gamma-1}}. \tag{3.23}$$

From (3.6), (3.9), and (3.18) we get the terminal condition

$$K(T)=\frac{1}{\gamma}. \tag{3.24}$$

We solve for $K(s)$ as follows. Multiply by $K(s)^{-\frac{\gamma}{\gamma-1}}$ on both sides of (3.23) to get

$$K(s)^{-\frac{\gamma}{\gamma-1}}K'(s) = Q(s)K(s)^{1-\frac{\gamma}{\gamma-1}} + \beta.$$

Now write this as

$$(1-\gamma)\,\frac{d}{ds}\left[K(s)^{\frac{1}{1-\gamma}}\right] = Q(s)\,K(s)^{\frac{1}{1-\gamma}} + \beta,$$

and put $m(s) = K(s)^{\frac{1}{1-\gamma}}$ and $k = (1-\gamma)^{-1}$. Then

$$m'(s) = kQ(s)m(s) + k\beta, \quad m(T) = K(T)^{\frac{1}{1-\gamma}} = \gamma^{\frac{1}{\gamma-1}}.$$

Using the integrating factor $\exp\left(k\int_s^T Q(p)\,dp\right)$ we get

$$m(s) = m(T)\exp\left(-k\int_s^T Q(p)\,dp\right) - k\beta\int_s^T \exp\left(k\int_r^s Q(u)\,du\right)dr,$$

and so

$$K(s) = \left[K(T)^{\frac{1}{1-\gamma}}\exp\left(-k\int_s^T Q(p)\,dp\right) - k\beta\int_s^T \exp\left(k\int_p^s Q(u)\,du\right)dp\right]^{1-\gamma}, \tag{3.25}$$

or

$$K(s) = \frac{1}{\gamma}\left[\exp\left(\tfrac{1}{\gamma-1}\int_s^T Q(p)\,dp\right) + \int_s^T \exp\left(\tfrac{1}{1-\gamma}\int_p^s Q(u)\,du\right)dp\right]^{1-\gamma}, \tag{3.26}$$

where $Q(\cdot)$ is defined in (3.22). We summarize what we have found in the following theorem.

Theorem 3.1. *Let the total discounted expected utility of the consumption and terminal wealth be given by*

$$J^c(s,\varphi) = \mathbb{E}^{s,\varphi,c}\left[\int_0^T e^{-\rho(s+t)}\frac{c(t,X(t),Y(t))^\gamma}{\gamma}\,dt + e^{-\rho T}\frac{(X(T)+\nu e^{\lambda\delta}Y(T))^\gamma}{\gamma}\right]$$

where $\rho > 0$ and $\gamma \in (0,1)$ are given constants and ν is given by (3.16). Then the value function of Problem 3.1 is

$$\Phi(s,\varphi) = K(s)e^{-\rho s}(\varphi(0) - \lambda y(\varphi))^\gamma, \qquad (\lambda < 0)$$

and the optimal consumption rate is

$$c^*(s,\varphi) = \left[\gamma K(s)\right]^{\frac{1}{\gamma-1}}(\varphi(0) - \lambda y(\varphi)),$$

where $K(s)$ is given by (3.26).

Remark 3.1. If we let the delay δ approach 0 then $Y(t) \to 0$, $Z(t) \to X(t)$ and, assuming that (3.16) holds, $\nu \to -\lambda$. The system $X(t)$ given by (3.4)–(3.5) approaches the limit $X_0(t)$ given by

$$dX_0(t) = X_0(t)[(\mu(t) - \lambda)\,dt + \alpha(t)\,dB(t)] - c(t)\,dt, \quad X_0(0) = x.$$

The corresponding problem without delay

$$\Phi_0(s,x) := \sup_{c\in\mathcal{A}}\mathbb{E}^{s,x}\left[\int_0^T e^{-\rho(t+s)}\frac{c(t,X_0(t))^\gamma}{\gamma}\,dt + e^{-\rho T}\frac{X_0(T)^\gamma}{\gamma}\right] \tag{3.27}$$

will then be the limit of $\Phi(s,\varphi) = \Phi_\delta(s,\varphi)$ as $\delta \to 0^+$. The value function is

$$\Phi_0(s,x) = K(s)e^{-\rho s}x^\gamma, \qquad (\lambda < 0)$$

and the optimal consumption rate is

$$c_0^*(s,x) = \left[\gamma K(s)\right]^{\frac{1}{\gamma-1}}x,$$

where $K(s)$ is given by (3.26). This problem is already covered by the optimal control theory for Markov diffusion processes as described in Chapter XI of [15] or Chapter VI of [5].

Remark 3.2. Define
$$H(t) := X(t) + \nu e^{\lambda\delta} Y(t), \qquad \text{for } t > 0.$$
Calculating the differential of this using (3.4) and (2.3) we get
$$\begin{aligned} dH(t) &= dX(t) + \nu e^{\lambda\delta}\, dY(t) \\ &= \mu[X(t) + \nu e^{\lambda\delta} Y(t)]\, dt + \nu Z(t)\, dt - c(t)\, dt \\ &\quad + \alpha(t)[X(t) + \nu e^{\lambda\delta} Y(t)]\, dW(t) - \nu e^{\lambda\delta}[X(t) - e^{-\lambda\delta} Z(t) - \lambda Y(t)]\, dt \\ &= [(\mu(t) + \nu e^{\lambda\delta}) X(t) + (\mu(t)\nu e^{\lambda\delta} - \lambda\nu e^{\lambda\delta}) Y(t)]\, dt \\ &\quad + (\nu - \nu) Z(t) - c(t)\, dt + \alpha(t)[X(t) + \nu e^{\lambda\delta} Y(t)]\, dW(t). \end{aligned}$$
If we assume that condition (3.16) holds, then we get
$$dH(t) = H(t)[(\mu(t) - \lambda)\, dt + \alpha(t)\, dW(t)] - c(t)\, dt$$
with initial condition
$$H(0) = X(0) - \lambda Y(0) = \varphi(0) - \lambda y(\varphi) =: h.$$
This is consumption from a geometric Brownian motion without delay. The corresponding optimization problem is
$$\tilde{\Phi}(s,h) := \sup_{c\in\mathcal{A}} \mathbb{E}^{s,h}\left[\int_0^T e^{-\rho(t+s)} \frac{c(t,H(t))^\gamma}{\gamma}\, dt + e^{-\rho T}\frac{H(T)^\gamma}{\gamma}\right] \tag{3.28}$$
Comparing with the previous remark and with Theorem 3.1, we see that the control problem with delay (Problem 3.1) is finite-dimensional and can be reduced to the non-delay problem above.

4. The partial observations case

To use our verification theorem to solve Problem 1.2, we need to have all processes in the system consisting of (1.1)–(1.2) and (1.6)–(1.7) adapted to the same filtration. This is accomplished as follows. From filtering theory we get the following result (see e.g. [15] or [2]).

Lemma 4.1. Put $\hat{\mu}(t) = \mathbb{E}[\mu|\mathcal{G}_t]$ and define the innovation process $B(t)$ by
$$dB(t) = \frac{1}{\alpha(t)}\big(\mu - \hat{\mu}(t)\big)\, dt + dW(t), \quad B(0) = 0. \tag{4.1}$$
Then $B(t)$ is a $\{\mathcal{G}_t\}$-Brownian motion. In fact, $\{B(s)\,;\, 0 \le s \le t\}$ generates the same filtration $\{\mathcal{G}_t\}$ as $\{\xi(s)\,;\, 0 \le s \le t\}$.

From the definition of $B(t)$ we immediately get
$$d\xi(t) = \mu\, dt + \alpha(t)\, dW(t) = \hat{\mu}(t)\, dt + \alpha(t)\, dB(t). \tag{4.2}$$
Using this, we write the system (1.1)–(1.2) as
$$\begin{aligned} dX(t) &= \big[X(t) + \nu e^{\lambda\delta} Y(t)\big]\big[\hat{\mu}(t)\, dt + \alpha(t)\, dB(t)\big] \\ &\qquad + \nu Z(t)\, dt - c(t)\, dt, \quad t \ge 0, \end{aligned} \tag{4.3}$$
$$X(s) = \varphi(s), \quad -\delta \le s \le 0, \tag{4.4}$$
Now we are in a situation with complete information since all terms on the right hand side of (4.3) are $\{\mathcal{G}_t\}$-adapted. But the situation is more complicated than in the previous section. Instead of a given deterministic function $\mu(t)$ we have the process $\hat{\mu}(t) = \hat{\mu}(t,\omega)$. The dynamics of $\hat{\mu}(t)$ is given by the following result, which is a special case of the Kalman filter (see [15], example 6.13).

Lemma 4.2. With $\bar{\mu} = \mathbb{E}[\mu]$, $\sigma = \mathbb{E}[(\mu - \bar{\mu})^2]$, $dR(t) = \alpha(t)^{-2}\, d\xi(t)$, and $dP(t) = \alpha(t)^{-2}\, dt$ we have
$$\hat{\mu}(t) = \frac{\bar{\mu} + \sigma R(t)}{1 + \sigma P(t)}. \tag{4.5}$$

Plugging this into (4.3), the system with complete observations becomes three-dimensional, as follows:

$$\begin{aligned}
(4.6)\qquad dX(t) &= \big[X(t) + \nu e^{\lambda\delta} Y(t)\big]\left(\frac{\bar\mu + \sigma R(t)}{1+\sigma P(t)}\,dt + \alpha(t)\,dB(t)\right)\\
&\quad + \nu Z(t)\,dt - c(t)\,dt, \quad t \geq 0,\\
(4.7)\qquad X(s) &= \varphi(s), \quad -\delta \leq s \leq 0,\\
(4.8)\qquad dR(t) &= \frac{1}{\alpha(t)^2}\,\frac{\bar\mu + \sigma R(t)}{1+\sigma P(t)}\,dt + \frac{1}{\alpha(t)}\,dB(t), \quad R(0) = r,\\
(4.9)\qquad dP(t) &= \frac{1}{\alpha(t)^2}\,dt, \quad P(0) = p.
\end{aligned}$$

Using the integrating factor $\exp\left(-\int_0^t \frac{\sigma\,du}{\alpha(u)^2(1-\sigma P(u))}\right)$, we may solve explicitly for $R(t)$ and $P(t)$. From (4.8) we see that

$$\begin{aligned}
(4.10)\qquad R(t) &= -\frac{\bar\mu}{\sigma} + \left(\frac{\bar\mu}{\sigma} + r\right)\exp\left(\int_0^t \frac{\sigma\,du}{\alpha(u)^2(1-\sigma P(u))}\right)\\
&\quad + \int_0^t \exp\left(\int_s^t \frac{\sigma\,du}{\alpha(u)^2(1-\sigma P(u))}\right)\frac{1}{\alpha(s)}\,dB(s).
\end{aligned}$$

Integrating (4.9) gives

$$(4.11)\qquad P(t) = p + \int_0^t \frac{ds}{\alpha(s)^2}.$$

Since $\hat\mu(0) = \bar\mu$, it follows from (4.5) that $r = p\bar\mu$. This fact, together with (4.11), shows that the system (4.6)–(4.9) is only two-dimensional. Let us write the system as

$$\begin{aligned}
(4.12)\qquad dX(t) &= \big[X(t) + \nu e^{\lambda\delta} Y(t)\big]\big[\hat\mu(t)\,dt + \alpha(t)\,dB(t)\big]\\
&\quad + \nu Z(t)\,dt - c(t)\,dt, \quad t \geq 0,\\
(4.13)\qquad X(s) &= \varphi(s), \quad -\delta \leq s \leq 0,\\
(4.14)\qquad dR(t) &= \frac{1}{\alpha(t)^2}\,\hat\mu(t)\,dt + \frac{1}{\alpha(t)}\,dB(t), \quad R(0) = r,
\end{aligned}$$

where $\hat\mu(t)$ is given by (4.5).

Now we make the following choices of U_1 and U_2 to be used in the rest of this section.

$$\begin{aligned}
(4.15)\qquad U_1(t, c(\varphi, r)) &= e^{-\rho t}\,\frac{c(t,x,r,y)^\gamma}{\gamma},\\
(4.16)\qquad U_2(X_T, R(T)) &= e^{-\rho T} R(T)\,\frac{(X(T) + \nu e^{\lambda\delta} Y(T))^\gamma}{\gamma},
\end{aligned}$$

where $\rho > 0$ and $\gamma \in (0,1)$ are given constants. We restate Problem 1.2 as

Problem 4.1 (Optimal consumption with complete observations)**.** Find $\Phi(s,r,\varphi)$ and $c^* \in \mathcal{A}_p$ such that

$$\begin{aligned}
(4.17)\qquad \Phi(s,r,\varphi) = \sup_{c\in\mathcal{A}_p} \mathbb{E}^{s,\varphi,r,c}\Bigg[\int_0^T & e^{-\rho(s+t)}\,\frac{c(t, X(t), R(t), Y(t))^\gamma}{\gamma}\,dt\\
&+ e^{-\rho T} R(T)\,\frac{(X(T) + \nu e^{\lambda\delta} Y(T))^\gamma}{\gamma}\Bigg]
\end{aligned}$$

when the system is given by (4.12)–(4.14)

Using the same idea as before, we assume that the value functional is of the form

$$\Phi(s,r,\varphi) = V(s,x,r,y,z) \quad \text{where } V : \mathbb{R}^5 \to \mathbb{R},$$

and that the consumption rate is of the form

$$c(s,\varphi,r) = c(s,x,r,y).$$

We shall seek a function V satisfying the conditions of Theorem 2.1 in the form

$$V(s,x,r,y,z) = K(s)e^{-\rho s}\psi(x,r,y) \tag{4.18}$$

for some real functions ψ and K. Plugging this into condition (C1) and recalling (4.17), we get

$$\begin{aligned} e^{\rho s}\mathcal{L}_c V(s,x,r,y,z) + \frac{c(s,x,r,y)^\gamma}{\gamma} &= K'(s)\psi(x,r,y) - \rho K(s)\psi(x,r,y) \\ &+ \Big[\hat{\mu}(s)\big(x+\nu e^{\lambda\delta}y\big) + \nu z - c(s,x,r,y)\Big]K(s)\frac{\partial\psi}{\partial x} + \frac{\hat{\mu}(s)}{\alpha(s)^2}K(s)\frac{\partial\psi}{\partial r} \\ &+ [x - e^{-\lambda\delta}z - \lambda y]K(s)\frac{\partial\psi}{\partial y} + \frac{\alpha(s)^2}{2}\big(x+\nu e^{\lambda\delta}y\big)^2K(s)\frac{\partial^2\psi}{\partial x^2} \\ &+ \big(x+\nu e^{\lambda\delta}y\big)K(s)\frac{\partial^2\psi}{\partial x\partial r} + \frac{1}{2\alpha(s)^2}K(s)\frac{\partial^2\psi}{\partial r^2} + \frac{c(s,x,r,y)^\gamma}{\gamma} = 0, \end{aligned} \tag{4.19}$$

or equivalently,

$$\begin{aligned} K'(s)\psi(x,r,y) &+ K(s)\Big\{-\rho\psi(x,r,y) + \hat{\mu}(s)\big(x+\nu e^{\lambda\delta}y\big)\frac{\partial\psi}{\partial x} - c(s,x,r,y)\frac{\partial\psi}{\partial x} \\ &+ (x-\lambda y)\frac{\partial\psi}{\partial y} + \frac{\hat{\mu}(s)}{\alpha(s)^2}\frac{\partial\psi}{\partial r} + \frac{\alpha(s)^2}{2}\big(x+\nu e^{\lambda\delta}y\big)^2\frac{\partial^2\psi}{\partial x^2} \\ &+ \big(x+\nu e^{\lambda\delta}y\big)\frac{\partial^2\psi}{\partial x\partial r} + \frac{1}{2\alpha(s)^2}\frac{\partial^2\psi}{\partial r^2}\Big\} + \frac{c(s,x,r,y)^\gamma}{\gamma} \\ &+ K(s)z\Big\{\nu\frac{\partial\psi}{\partial x} - e^{-\lambda\delta}\frac{\partial\psi}{\partial y}\Big\} = 0, \end{aligned} \tag{4.20}$$

with boundary condition

$$V(T,X(T),R(T),Y(T)) = e^{-\rho T}R(T)\frac{\big(X(T)+\nu e^{\lambda\delta}Y(T)\big)^\gamma}{\gamma}. \tag{4.21}$$

The maximum in (4.19) is obtained when

$$\frac{\partial}{\partial c}\left(\mathcal{L}_c V(s,x,r,y,z) + e^{-\rho s}\frac{c^\gamma}{\gamma}\right) = 0,$$

i.e. when

$$c = c^* = \left(K(s)\frac{\partial\psi}{\partial x}\right)^{\frac{1}{\gamma-1}}.$$

We require that (4.19) holds with this value of c. Now,

$$\mathcal{L}_{c^*}V(s,x,r,y,z) + e^{-\rho s}\frac{{c^*}^\gamma}{\gamma} = 0 \quad \text{for all } z \tag{4.22}$$

if equation (4.20) holds, that is if

$$K(s)z\Big\{\nu\frac{\partial\psi}{\partial x} - e^{-\lambda\delta}\frac{\partial\psi}{\partial y}\Big\} = 0 \tag{4.23}$$

and

$$\begin{aligned} K'(s)\psi(x,r,y) &+ K(s)\Big\{-\rho\psi(x,r,y) + \hat{\mu}(s)\big(x+\nu e^{\lambda\delta}y\big)\frac{\partial\psi}{\partial x} - c^*\frac{\partial\psi}{\partial x} \\ &+ (x-\lambda y)\frac{\partial\psi}{\partial y} + \frac{\hat{\mu}(s)}{\alpha(s)^2}\frac{\partial\psi}{\partial r} + \frac{\alpha(s)^2}{2}\big(x+\nu e^{\lambda\delta}y\big)^2\frac{\partial^2\psi}{\partial x^2} \\ &+ \big(x+\nu e^{\lambda\delta}y\big)\frac{\partial^2\psi}{\partial x\partial r} + \frac{1}{2\alpha(s)^2}\frac{\partial^2\psi}{\partial r^2}\Big\} + \frac{{c^*}^\gamma}{\gamma} = 0. \end{aligned} \tag{4.24}$$

Equations (4.23) and (4.24) hold if

$$\psi(x,r,y) = r^{\gamma-1}g(w) \tag{4.25}$$

for some function $g : \mathbb{R} \to \mathbb{R}$ where

$$w = w(x,y) = x + \nu e^{\lambda\delta} y. \tag{4.26}$$

Substituting (4.25)–(4.26) into (4.24) we get

$$\begin{aligned} K'(s)r^{\gamma-1}g(w) + K(s)\Big\{ &-\rho r^{\gamma-1}g(w) + \hat{\mu}(s)\big(x+\nu e^{\lambda\delta}y\big)r^{\gamma-1}g'(w) - c^* r^{\gamma-1}g'(w) \\ &+ (x-\lambda y)\nu e^{\lambda\delta}r^{\gamma-1}g'(w) + \frac{\hat{\mu}(s)}{\alpha(s)^2}(\gamma-1)r^{\gamma-2}g(w) + \frac{\alpha(s)^2}{2}\big(x+\nu e^{\lambda\delta}y\big)^2 r^{\gamma-1}g''(w) \\ &+ \big(x+\nu e^{\lambda\delta}y\big)(\gamma-1)r^{\gamma-2}g'(w) + \frac{1}{2\alpha(s)^2}(\gamma-1)(\gamma-2)r^{\gamma-3}g(w)\Big\} + \frac{c^{*\gamma}}{\gamma} = 0. \end{aligned} \tag{4.27}$$

Equation (4.27) has a solution depending only on s, r, and w if

$$\nu = -\lambda e^{-\lambda\delta}, \quad \lambda < 0. \tag{4.28}$$

Using this, equation (4.27) becomes

$$\begin{aligned} K'(s)r^{\gamma-1}g(w) + K(s)\Big\{ &-\rho r^{\gamma-1}g(w) + \hat{\mu}(s)wr^{\gamma-1}g'(w) - c^* r^{\gamma-1}g'(w) \\ &- \lambda w r^{\gamma-1}g'(w) + \frac{\hat{\mu}(s)}{\alpha(s)^2}(\gamma-1)r^{\gamma-2}g(w) + \frac{\alpha(s)^2}{2}w^2 r^{\gamma-1}g''(w) \\ &+ w(\gamma-1)r^{\gamma-2}g'(w) + \frac{1}{2\alpha(s)^2}(\gamma-1)(\gamma-2)r^{\gamma-3}g(w)\Big\} + \frac{c^{*\gamma}}{\gamma} = 0. \end{aligned} \tag{4.29}$$

Now we guess that

$$g(w) = w^\gamma. \tag{4.30}$$

With this g the optimal consumption rate c^* takes the form

$$c^* = \left(K(s)\frac{\partial\psi}{\partial x}\right)^{\frac{1}{\gamma-1}} = \big(K(s)r^{\gamma-1}g'(w)\big)^{\frac{1}{\gamma-1}} = r\big(\gamma K(s)\big)^{\frac{1}{\gamma-1}}w. \tag{4.31}$$

Plugging (4.30) and (4.31) into (4.29) we get

$$\begin{aligned} K'(s)r^{\gamma-1}w^\gamma + K(s)\Big\{ &-\rho r^{\gamma-1}w^\gamma + \hat{\mu}(s)r^{\gamma-1}\gamma w^\gamma - r\big(\gamma K(s)\big)^{\frac{1}{\gamma-1}}r^{\gamma-1}\gamma w^\gamma \\ &- \lambda r^{\gamma-1}\gamma w^\gamma + \frac{\hat{\mu}(s)}{\alpha(s)^2}(\gamma-1)r^{\gamma-2}w^\gamma + \frac{\alpha(s)^2}{2}\gamma(\gamma-1)r^{\gamma-1}w^\gamma + \gamma(\gamma-1)r^{\gamma-2}w^\gamma \\ &+ \frac{1}{2\alpha(s)^2}(\gamma-1)(\gamma-2)r^{\gamma-3}w^\gamma\Big\} + \frac{1}{\gamma}r^\gamma\big(\gamma K(s)\big)^{\frac{\gamma}{\gamma-1}}w^\gamma = 0. \end{aligned} \tag{4.32}$$

or equivalently, dividing by $r^{\gamma-1}w^\gamma$,

$$\begin{aligned} K'(s) + K(s)\Big\{ &-\rho + \frac{\hat{\mu}(s)}{\alpha(s)^2}\left(\alpha(s)^2\gamma + \frac{\gamma-1}{r}\right) - \lambda\gamma \\ &+ \gamma(\gamma-1)\left(\frac{\alpha(s)^2}{2} + \frac{1}{r}\right) + \frac{(\gamma-1)(\gamma-2)}{2\alpha(s)^2 r^2}\Big\} + (1-\gamma)\gamma^{\frac{1}{\gamma-1}}rK(s)^{\frac{\gamma}{\gamma-1}} = 0. \end{aligned} \tag{4.33}$$

Now we solve for $K(s)$ in the same way as in the complete observations case. With

$$\begin{aligned} Q(s) = Q(s,r) = \rho &- \frac{\hat{\mu}(s)}{\alpha(s)^2}\left(\alpha(s)^2\gamma + \frac{\gamma-1}{r}\right) + \lambda\gamma \\ &- \gamma(\gamma-1)\left(\frac{\alpha(s)^2}{2} + \frac{1}{r}\right) - \frac{(\gamma-1)(\gamma-2)}{2\alpha(s)^2 r^2}, \end{aligned} \tag{4.34}$$

$$\beta = \beta(r) = (\gamma-1)\gamma^{\frac{1}{\gamma-1}}r, \tag{4.35}$$

we write equation (4.33) as

$$K'(s) = Q(s)K(s) + \beta K(s)^{\frac{\gamma}{\gamma-1}} \tag{4.36}$$

which is (3.23) with r-dependent coefficients. From (4.18), (4.21), and (4.30) we get the terminal condition

$$K(T) = \frac{1}{\gamma} R(T)^{2-\gamma}. \tag{4.37}$$

With $k = (1-\gamma)^{-1}$ the solution is

$$K(s) = \left[K(T)^{\frac{1}{1-\gamma}} \exp\left(-k \int_s^T Q(p)\,dp \right) - k\beta \int_s^T \exp\left(k \int_p^s Q(u)\,du \right) dp \right]^{1-\gamma}, \tag{4.38}$$

or

$$K(s) = \frac{1}{\gamma} \left[R(T)^{\frac{2-\gamma}{1-\gamma}} \exp\left(\frac{1}{\gamma-1} \int_s^T Q(p)\,dp \right) + r \int_s^T \exp\left(\frac{1}{1-\gamma} \int_p^s Q(u)\,du \right) dp \right]^{1-\gamma}, \tag{4.39}$$

where β and $Q(\cdot)$ are given by (4.35)–(4.34). We summarize what we have found in the following theorem.

Theorem 4.1. *Let the total discounted expected utility of the consumption and terminal wealth be given by*

$$J^c(s,r,\varphi) = \mathbb{E}^{s,\varphi,r,c}\left[\int_0^T e^{-\rho(s+t)} \frac{c(t,X(t),R(t),Y(t))^\gamma}{\gamma}\,dt + e^{-\rho T} R(T) \frac{(X(T)+\nu e^{\lambda\delta} Y(T))^\gamma}{\gamma} \right]$$

where $\rho > 0$ and $\gamma \in (0,1)$ are given constants and ν given by (4.28). Then the value function of Problem 4.1 is

$$\Phi(s,r,\varphi) = K(s) e^{-\rho s} r^{\gamma-1} \cdot (\varphi(0) - \lambda y(\varphi))^\gamma, \quad (\lambda < 0),$$

and the optimal consumption rate is

$$c^*(s,r,\varphi) = \big(\gamma K(s)\big)^{\frac{1}{\gamma-1}} r \cdot (\varphi(0) - \lambda y(\varphi)),$$

where $K(s)$ is given by (4.39).

Remark 4.1. If we let the delay δ approach 0 and assume that (4.28) holds, then the system $\big(X(t), R(t)\big)$ given by (4.12)–(4.14) approaches the limit $\big(X_0(t), R(t)\big)$ given by

$$\begin{aligned} dX_0(t) &= X_0(t)\big[(\hat\mu(t) - \lambda)\,dt + \alpha(t)\,dB(t)\big] - c(t), \quad X_0(0) = x, \\ dR(t) &= \frac{1}{\alpha(t)^2}\,\hat\mu(t)\,dt + \frac{1}{\alpha(t)}\,dB(t), \quad R(0) = r, \end{aligned}$$

where $\hat\mu(t)$ is given by (4.5). The corresponding problem without delay

$$\Phi_0(s,r,x) := \sup_{c\in\mathcal{A}_p} \mathbb{E}^{s,x,r,c}\left[\int_0^T e^{-\rho(s+t)} \frac{c(t,X(t),R(t))^\gamma}{\gamma}\,dt + e^{-\rho T} R(T) \frac{X(T)^\gamma}{\gamma} \right]$$

will then be the limit of $\Phi(s,\varphi) = \Phi_\delta(s,\varphi)$ as $\delta \to 0$. The value function is

$$\Phi_0(s,r,x) = K(s) e^{-\rho s} r^{\gamma-1} x^\gamma, \quad (\lambda < 0),$$

and the optimal consumption rate is

$$c_0^*(s,r,x) = \big(\gamma K(s)\big)^{\frac{1}{\gamma-1}} r x,$$

where $K(s)$ is given by (4.39).

Remark 4.2. As in Remark 3.2 we may define

$$H(t) := X(t) + \nu e^{\lambda\delta} Y(t), \qquad \text{for } t > 0.$$

Calculating the differential of this using (1.1) and (2.3), and assuming (4.28) holds, we get

$$dH(t) = H(t)[(\mu - \lambda)\,dt + \alpha(t)\,dW(t)] - c(t)\,dt \tag{4.40}$$

with initial condition

$$H(0) = X(0) - \lambda Y(0) = \varphi(0) - \lambda y(\varphi) =: h.$$

Recall that μ is observed through

$$d\xi(t) = \mu\,dt + \alpha(t)\,dW(t), \quad \xi(0) = 0,$$

so we may write (4.40) as

$$dH(t) = H(t)[d\xi(t) - \lambda\,dt] - c(t)\,dt.$$

This shows that

$$\mathcal{F}_t^H \subset \mathcal{G}_t \subset \mathcal{F}_t^X. \tag{4.41}$$

Using the same method as before, we get the system

$$dH(t) = H(t)[(\hat{\mu}(t) - \lambda)\,dt + \alpha(t)\,dB(t)] - c(t)\,dt, \quad H(0) = h, \tag{4.42}$$

$$dR(t) = \frac{1}{\alpha(t)^2}\,\hat{\mu}(t)\,dt + \frac{1}{\alpha(t)}\,dB(t), \quad R(0) = r. \tag{4.43}$$

Note that there is no delay in this system. Now Problem 4.1 takes the form

Problem 4.2 (A non-delay problem)**.** Find $\Phi(s, h, r)$ and $c^* \in \mathcal{A}_p$ such that

$$\Phi(s, r, h) = \sup_{c \in \mathcal{A}_p} \mathbb{E}^{s,h,r,c}\left[\int_0^T e^{-\rho(s+t)}\,\frac{c(t, H(t), R(t))^\gamma}{\gamma}\,dt + e^{-\rho T} R(T)\frac{(H(T))^\gamma}{\gamma}\right] \tag{4.44}$$

when the system is given by (4.42)–(4.43)

The verification theorem shows that Problems 4.1 and 4.2 have the same solution. But the calculation above shows that Problem 4.1 can not be reduced to Problem 4.2. The inclusions (4.41) may be strict, which means that the admissibility condition does not hold for the system (4.42)–(4.43). The supremum in (4.44) should be taken over a class $\tilde{\mathcal{A}}_p \subset \mathcal{A}_p$ of $\mathcal{F}_t^H$-adapted controls.

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