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JUMP UNCERTAINTY VERSUS BROWNIAN NOISE IN STOCHASTIC OPTIMAL HARVESTING MODELS*

Nils Chr. Framstad †

Abstract

The problem of irreversibly harvesting from a general one dimensional (Wiener-Poisson) jump diffusion population model is studied. For a wide class of models, including stochastic generalizations of the logistic model earlier studied by Lungu & Øksendal (1997) and Alvarez & Shepp (1998), the optimal strategy is a downwards local time reflection at a trigger level x^*. Both these works find that this trigger level is higher than of the corresponding deterministic problem; we show that this property depends crucially upon the uncertainty being Brownian. Furthermore, we give conditions under which jump uncertainty also increases x^* compared to the deterministic model.

Key words: Optimal harvesting, singular stochastic control, reflected jump diffusion model, behavior towards risk.
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0 Introduction

The problem of optimally harvesting a population has been widely studied. The canonical example is asking how to get the most out of a logistic growth model. This paper is strongly inspired by two papers studying stochastic versions of the logistic growth models. Lungu & Øksendal [1] assume the process to follow the Itô (non-anticipative) stochastic differential equation

$$\mathrm{d}X_t = X_t(K - X_t) \cdot (r\,\mathrm{d}t + \sigma_{\mathrm{L}}\,\mathrm{d}B_t) \tag{1a}$$

if not harvested. This is maybe the most straightforward generalization to a stochastic model, merely adding (a constant factor σ times) white noise to

*Supported by the Research Council of Norway.
†Dept. of Mathematics, University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway. e-mail: ncf@math.uio.no

the (constant, positive) growth factor r; here, B is standard Brownian motion and the positive constant K is the carrying capacity of the environment. In this model, volatility is largest when growth rate is. An alternative model is the Verhulst-Pearl model, where the population if not harvested follows

$$\mathrm{d}X_t = rX_t(K - X_t)\,\mathrm{d}t + \sigma_{\mathrm{VP}} X_t\,\mathrm{d}B_t - X_{t^-}\cdot\mathrm{d}N_t \tag{1b}$$

In this model, studied by e.g. Alvarez & Shepp [2] and later by Myhre [3], *relative* uncertainty is constant. In particular, the uncertainty may drive the population across the carrying capacity K (which is defined as the value where the drift term changes from positive to negative sign). N_t is a Poisson process, i.e. it has unit jumps and constant jump intensity, and the population (or the opportunity to harvest it) will disappear completely and forever at first jump time. This term is only implicit in their model. Yet another model is

$$\mathrm{d}X_t = r(X_t - A)(K - X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \tag{1c}$$

studied by Lande, Engen and Sæther [4] in a less rigorous setting. We merely make a note that this process has negative drift at 0, and should only be used as a model up to the first time the process hits 0. Our model (2) below may be extended to cover the case (1c) if the coefficient β is allowed to be unbounded at 0, and while this case is not covered by our Section 3, Section 4 will still apply.

As a model of a real life economy the shortcomings are severe, as we do assume full information, total control of the amount harvested, no risk aversion (but we do have a (constant) discount rate ρ to cope with intertemporal trade-off) and infinite time horizon. [2] point out that the model does not take into account the value of preserving a species and is therefore objectionable from an environmentalistic point of view (shared by this author). However, the model might still be of interest, and one key result of this paper is that even in this simple model, behavior towards risk depends on how uncertainty is modeled: Both [2] and [1] find that the optimal strategy is a local time downwards reflection at a trigger level x^*, and [4] is also dedicated to problems with this kind of solution. All three works find that $x^* \geq x_0^*$ where the latter is the optimal trigger value in the deterministic case $\sigma = 0$. We shall see that this property (usually) holds in the continuous case, but not necessarily in a jump diffusion model. While it is obvious that that an *uncompensated* jump term may lower x^* – like in the model (1b) above, where the jump to zero intensity has the same effect as an increased discounting rate – we emphasize that we will instead be introducing a pure jump *martingale* to the model. Pure jump martingales may be regarded as a modeling alternative to the Brownian motion, at least if the jumps are small, and we shall see that the phenomena might have qualitatively different implications.

1 The model.

Assume given a filtered probability space $(\Omega, \mathfrak{S}, \{\mathcal{F}_t\}_{t\geq 0}, \mathsf{P})$ satisfying the usual conditions. The population is assumed to be an adapted process X following a stochastic differential equation to be defined in (2) below. X will be driven by a standard Brownian motion B and by a centered integer-valued random measure $\tilde{M}$ governing jumps. Due to mathematical convenience we will allow the jump intensity q to be state-dependent, meaning that we do not at this point follow the usual setup with Poisson random jumps (which will however be useful at the end of Section 3); if population at time t^- is at level x, then the intensity of a jump by a factor $z \in Z$ (Borel set) is $q(x, Z)$, with $q(x, \{0\}) = 0$. The population is now assumed to follow the dynamics

$$\mathrm{d}X_t = X_{t^-} \cdot \Big(\beta(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t + \int_{[-1,\infty)} z\tilde{M}(\mathrm{d}t, \mathrm{d}z)\Big) - \mathrm{d}H_t \tag{2}$$

where H is our control, interpreted as the total amount harvested up to time t, chosen from the class of *admissible controls* $\mathcal{H}$ to be defined in a moment. Let $\mathsf{P}^{t_0,x}$ be the probability law of the time-space process (t, X) starting at $(t_0, X_{t_0}) = (t_0, x)$. Define $\mathcal{H}$ to be the class of $\mathcal{F}_t$-predictable, left-continuous non-decreasing functions such that $\mathsf{P}^{t_0,x}$-a.s. we have $H_{t_0} = 0$ and $X_t \geq 0$ for all $t \geq t_0$. We assume $\mathcal{H} \neq \varnothing$.

It will be useful to split up the jump term: Let $q = \hat{q} + \check{q} + \bar{q}$ where $\hat{q}$ is supported by $\mathbf{R}^+$, $\check{q}$ is supported by $(-1, 0)$ and $\bar{q}$ by point $\{-1\}$. With abuse of notation, we shall denote $q(x, \{-1\})$ by $\bar{q}(x)$.

We make the following regularity assumptions on the coefficients:

ASSUMPTION
Point 0 is a trap for X, i.e., if the solution of (2) is not a priori unique at $x = 0$; apart from this, the coefficients β, σ^2 and q are assumed to admit uniqueness and local existence of a weak solution to (2). We assume that for all $x \geq 0$, $\mathrm{d}z \mapsto q' := \frac{\partial}{\partial x} q(x, \mathrm{d}z)$ and $\mathrm{d}z \mapsto q'' := (\frac{\partial}{\partial x})^2 q(x, \mathrm{d}z)$ are signed measures integrating $z \wedge z^2$. △

Let $\mathsf{E} = \mathsf{E}^{t_0,x}$ be expectation wrt. $\mathsf{P}^{t_0,x}$. We assume one wants to maximize total discounted expected harvest defined as follows:

$$J^H := \mathsf{E}\Big[\int_{[0,\bar{T})} e^{-\rho t}\mathrm{d}H_t\Big] \tag{3}$$

and if it exists, an optimal admissible H^* such that

$$\Phi(t_0, x) = J^{H^*} = \sup_{H\in\mathcal{H}} J^H. \tag{4}$$

In (3), $\rho > 0$ is a constant discount rate. $\bar{T}$ has the interpretation of being the first time the population hits 0, but for technical reasons we assume

$$\bar{T} := \lim_{y \nearrow \infty} \inf\{t > t_0;\ X_t \notin (0, y)\}. \tag{5}$$

By the Markov property, we can without loss of generality assume $t_0 = 0$, and $\Psi := e^{\rho t_0}\Phi(t_0, x)$ will be a function of x only. Clearly, $\Psi(0) = 0$.

2 Sufficient conditions and properties of the value function.

Define

$$\mathsf{Q}\psi(x) := \int \big(\psi(x + zx) - \psi(x) - zx\psi'(x)\big) q(x, \mathrm{d}z) \tag{6a}$$

and

$$\mathsf{L}\psi(x) := -\rho\psi(x) + x\beta(x)\psi'(x) + \frac{1}{2}x^2\sigma^2(x)\psi''(x) + \mathsf{Q}\psi(x). \tag{6b}$$

For functions $\psi \in \mathrm{C}^2$ vanishing at 0 and with sublinear growth (cf. Proposition 2), $(\mathsf{L} - \rho)$ coincides with the generator of the process X when $\mathrm{d}H = 0$. For completeness, we state a verification theorem:

THEOREM 1 (Sufficient conditions).
Suppose we can find a nonnegative $\psi \in \mathrm{C}^2((0, \infty))$ *such that for all* $x > 0$,

$$\max\{\mathsf{L}\psi,\ 1 - \psi'\} \leq 0. \tag{7}$$

Then

$$\psi \geq \Psi. \tag{8}$$

Suppose in addition that $\psi(0) = 0$ *and*

$$\max\{\mathsf{L}\psi,\ 1 - \psi'\} = 0. \tag{9}$$

Define the non-intervention region D *as* $D := \{x \geq 0;\ \psi'(x) > 1\}$ *and assume that there exists an* $x^* >\in (0, \infty)$ *such that*

$$D = [0, x^*). \tag{10}$$

Define the control $\hat{H}$ *by*

$$\hat{H}_{t^+} = L_t^{x^*} + \sum_{s \in [0,t]} \max\big(0, X_s - x^*\big) \tag{11}$$

where L^{x^*} *is the local time of* X *at* x^*. *Then*

$$\psi = \Psi \tag{12}$$

and $H^* := \hat{H}$ *is optimal.*

Proof. The proof is standard; assuming $0 \leq \psi \in \mathrm{C}^2$ and (7) and an arbitrary control H, the Itô formula yields

$$\psi \geq \mathsf{E}[e^{-\rho\tau}\psi(X_\tau)] + \mathsf{E}[\int_{[0,\tau)} e^{-\rho t}\mathrm{d}H_t] \tag{13}$$

and (8) follows. To prove (12), note that using H^* we get equality in (13). Since D is assumed bounded, $\mathsf{E}[e^{-\rho\tau}\psi(X_{\tau+})] \to 0$ and the conclusion follows. □

Remark. Note that (super)optimality holds if we instead of twice continuous differentiability assume $\psi \geq 0$ to be a viscosity (super)solution of (9). △

Let us consider the following extreme cases:

PROPOSITION 2.
If for some $\epsilon > 0$ we have $\beta - \rho \geq \epsilon$ everywhere, then $\Psi = \infty$ except at $x = 0$. If on the other hand $x(\beta(x) - \rho) \leq \rho F$ everywhere for some $F \geq 0$, then $\Psi \leq x + F$, with equality if $F = 0$.

Proof. In the former case, use Itô's formula to get

$$x^* \exp\{-\rho\tau\} = x\exp\{-\rho t_0\}Y_\tau \exp\{\int_{t_0}^{\tau} (\beta(X_t) - \rho)\mathrm{d}s\}. \tag{14}$$

where Y is an exponential martingale with $Y_{t_0} = 1$. Now take expectation and let τ grow. In the latter case, we immediately have superoptimality, and if $\psi(x) = x$ is superoptimal then it is also optimal. □

Note that the Markovian nature of the coefficients is not really needed here.

One frequently finds control problems which has finite value function but no optimal control, and this is no exception: An example is to let $\theta \in (0,1)$ and $A > 0$ and assume $\beta = \frac{x+Ax^\theta}{x+\theta Ax^\theta}\rho$ and for simplicity $\sigma = 0 = q$. Then it is easy to see that no optimal strategy exists, as waiting is always better; however, $\psi(x) = x + Ax^\theta$ is superoptimal. We shall see in Proposition 7 that the condition that $\beta \leq \rho$ at some positive finite x is quite crucial. This is hardly a restrictive assumption in real world population models, though.

A converse to Theorem 1 is:

PROPOSITION 3.
Ψ is a viscosity solution to (9) on each interval I on which Ψ is continuous.

Proof. Also a standard proof; see e.g. Framstad, Øksendal and Sulem [5], proof of Theorem 3.2. □

An easy consequence of Proposition 3 is the following important result:

THEOREM 4.
Suppose $\Psi(x) = x$ on some $I = (0, \bar{x}) \neq \varnothing$. Then $\beta(x) \leq \rho$ on I.

Proof. Ψ must satisfy $0 \geq x(\beta(x) - \rho)$. □

Remark. This shows that if $\rho < \beta(0)$, it is never optimal to harvest the population to extinction. This is not the same as to say that the optimal harvesting strategy will not indirectly lead to extinction; let for example $y > 0$ be a trap of the process if uncontrolled, while for $x < y$ the annihilation intensity $\bar{q}$ is positive. Then if we harvest any amount, the population will almost surely become extinct in finite time. △

Even if the viscosity solution enables us to consider also nonsmooth candidates for the value function, smoothness is certainly a valuable property; indeed, we should expect the value function to be smooth (if finite – Proposition 2 is a counterexample):

PROPOSITION 5.
Assume there is an interval $I \ni x^*$ such that for $I \ni x < x^*$ intervention is not optimal, but for $I \ni x > x^*$ the optimal strategy H^* has $\mathrm{d}H^* = (x - x^*)$, and that Ψ is C^1 on I and C^2 around x^*. Assume continuous coefficients at x^* and $\sigma(x^*) \neq 0$. Then Ψ is C^2 at x as well.

Proof. Approaching from above and from below yields

$$\Psi''(x^{*-}) = \frac{\rho\Psi(x^*) - x^*\beta(x^*) - \mathsf{Q}\Psi(x^*)}{\frac{1}{2}(x^*)^2\sigma^2(x^*)} \geq \Psi''(x^{*+}) = 0. \tag{15}$$

and the claim follows since Ψ' must be nonincreasing at x^*, so $\Psi''(x^{*-}) \leq 0$ as well. □

For a given ψ, we shall define $Q(x) := \mathsf{Q}\psi(x)$. A couple of properties of Q will be useful. First, notice that we can write

$$Q(x) = \int \int_{x+zx}^{x} \int_{y}^{x} f''(w)\,\mathrm{d}w\,\mathrm{d}y\,q(x, \mathrm{d}z), \tag{16a}$$

the first two derivatives are then

$$\begin{aligned} Q'(x) = &\int \int_{x+zx}^{x} \int_{y}^{x} f''(w)\,\mathrm{d}w\,\mathrm{d}y\,q'(x, \mathrm{d}z) \\ &+ \int \big((1+z)(\psi'(x+zx) - \psi'(x)) - zx\psi''(x)\big)\,q(x, \mathrm{d}z) \end{aligned} \tag{16b}$$

and

$$Q''(x) = \int \int_{x+zx}^{x} \int_{y}^{x} f''(w)\,\mathrm{d}w\,\mathrm{d}y\, q''(x, \mathrm{d}z)$$
$$+ 2\int \big((1+z)(\psi'(x+zx) - \psi'(x)) - zx\psi''(x)\big)\, q'(x, \mathrm{d}z)$$
$$+ \int \big((1+z)((1+z)\psi''(x+zx) - \psi''(x)) - (zx\psi''(x))'\big)\, q(x, \mathrm{d}z). \tag{16c}$$

3 Finding an optimal solution

In this section, we shall see that under suitable conditions, there is a value function solving the quasi-variational inequality. In view of Proposition 2, we shall throughout this section assume

$$\beta(x) > \rho \quad \text{for all small enough } x > 0. \tag{17}$$

Then as noted in the Introduction, the dynamics (1c) is not covered by this section.

Assuming a slight bit more than (17) will imply that there is a solution which is increasing and concave near 0:

PROPOSITION 6.
Assume that the coefficients and their first derivatives are bounded at 0. Assume ψ is of sublinear growth at ∞ and not identically 0, that ψ solves $\mathsf{L}\psi = 0$ near 0 and that $\psi(0) = 0$. Then $\psi'(0) = -\psi''(0) = \pm\infty$ if $\rho < \beta(0)$. If $\beta(0) - \rho = 0 < \beta'(0)$, $\beta''(0^+)$ finite and $x \mapsto \int |z| q(x, \mathrm{d}z)$ either is infinite for some sequence $x_n \searrow 0$ or bounded as $x \searrow 0$, then $\psi'(0)\psi''(0) < 0$. Finally, if we allow $\beta(0^+) = +\infty$ but $x\beta'$ bounded, then we also have $\psi'(0)\psi''(0) < 0$.

Proof. To simplify, we note that the following arguments will also exclude unbounded oscillations near 0, by considering arbitrary positive sequences $\{x_n\}$ converging to 0:

Assume $\psi(0) = 0 \neq \psi'(0)$ to avoid the zero solution. In the case $\rho < \beta(0)$, divide the HJB equation $\mathsf{L}\psi = 0$ by x. Assume ψ'' bounded near 0. By l'Hôpital, $\frac{1}{2}x\sigma^2(x)\psi''(x) + \frac{Q(x)}{x}$ tends to 0 and we arrive at the contradiction $0 = (\beta(0) - \rho)\psi'(0)$. So $x\psi''(x) \to \pm\infty$. Then for small enough x and some $N > 0$, $|\psi''(x)| > \frac{N}{x}$ implying $\psi'(0)$ infinite and equal to $-\psi''(0)$. A similar argument also yields the conclusion for the case $\beta(0^+) = +\infty$.

Assume now $\beta(0)-\rho = 0 < \beta'(0)$ and $\beta''(0^+)$ finite. Divide the HJB equation by x^2. Assume first that ψ' and ψ'' are both bounded near 0. Then by (16),

$$\begin{aligned} 0 &= \sigma^2(0)\psi''(0) - \lim \big(x\psi'''(x)\int z\, q(x, \mathrm{d}z)\big) + 2\beta'(0) \lim \frac{-\rho\psi + x\beta\psi'}{x(\beta-\rho)} \\ &= (\rho + \sigma^2(0))\psi''(0) + \psi'(0) - \lim \big(x\psi'''(x)\int z\, q(x, \mathrm{d}z)\big). \end{aligned} \tag{18}$$

If the integral term is infinite, we have a contradiction. If it is bounded, then so is necessarily $x\psi'''$. Either $x\psi'''$ tends to 0, in which case there is nothing more to prove, or ψ'' is unbounded at 0. In that case, assume $\psi'(0)$ finite; Consider once more the HJB equation divided by x^2, to get

$$0 = \lim \big(\rho - \frac{1}{x}\int z\, q(x, \mathrm{d}z)) + \beta\frac{\psi'}{\psi''}\big)\psi''. \tag{19}$$

Now the same argument as above for the integral yields a contradiction and thus $\psi'(0^+)$ is infinite as well. Since ψ' and ψ'' must have have opposite signs, we are done. □

In the continuous case or $\hat{q} = 0$, we will proceed by finding a solution f to the HJB equation $\mathsf{L}f = 0$, paste it C^2 with an affine function at a zero $\tilde{x}$ for f''; if f is concave at 0, then one can choose $\tilde{x}$ as a minimum point of f'. If $f'(\tilde{x}) > 0$, the construction

$$\psi(x) = \frac{f(\min(x^*, x))}{f'(x^*)} + \max\big(0, x - x^*\big). \tag{20}$$

yields a function which solves the integro-differential equation on $[0, \tilde{x}]$ and has derivative ≥ 1 (> 1 iff $x < \tilde{x}$). If there are positive jumps, it is much more difficult to find candidates for the value function, and this is the reason why the next result has a slightly ad hoc formulation. However, if $\hat{q} = 0$ and Proposition 6 applies, then we can in fact find apply the next Proposition. But first, we define $\breve{x}$ and $\bar{x}$ by

$$\breve{x} := \inf\{x > 0;\ \beta(x) \leq \rho\} \tag{21a}$$

$$\bar{x} := \sup\{x > \tilde{x};\ \beta(x) > \rho\}. \tag{21b}$$

Note that the assertion that $\tilde{x} < \infty$ is *not* part of the hypothesis of the next Proposition:

PROPOSITION 7.
Assume (17), and that ψ vanishes at 0, increases at 0, is concave, affine for $x \geq \tilde{x}$ and solves the HJB equation $\mathsf{L}\psi = 0$ for $x \leq \tilde{x}$ (> 0.) If $\breve{x} < \infty$, then $\tilde{x} < \breve{x}$ and $\psi'(\tilde{x}) > 0$ and furthermore $\mathsf{L}\psi \leq 0$ for $x \geq \bar{x}$ ($\leq \infty$).

Proof. By (16a) and concavity, $Q \leq 0$. Concavity also implies $-\rho\psi(x) < x\psi'(x)$ for all $x > 0$ (unless ψ is linear, which is a trivial case,) since ψ vanishes at 0. We therefore have

$$\mathsf{L}\psi(x) < x(\beta(x)-\rho)\psi'(x) + \frac{1}{2}x^2\sigma^2(x)\psi''(x) + \mathsf{Q}\psi(x) \leq x(\beta(x)-\rho)\psi'(x) \tag{22}$$

for all $x \in (0, \tilde{x})$. In particular, if $\breve{x} < \infty$ we cannot have $\breve{x} \leq \tilde{x}$, and therefore $\psi' > 0$ on $(0, \tilde{x}]$ and thus everywhere, and then the last claim also follows. □

We will therefore without loss of generality assume $\psi'(\tilde{x}) = 1$, and the problem is solved if we can show that $\mathsf{L}\psi \leq 0$ on $(\tilde{x}, \bar{x})$. Then we have the main result of this section:

THEOREM 8.
Assume that $\tilde{x} < \infty$:

i) **Optimality, $x(\beta - \rho) + Q$ eventually nonincreasing:**
If Proposition 7 applies and

$$\tilde{x}(\beta(\tilde{x}) - \rho) + Q(\tilde{x}) \geq x(\beta(x) - \rho) + Q(x) \quad \forall x \geq \tilde{x} \tag{23}$$

then we have $\Psi = \psi$ and the optimal continuation region is $D = [0, x^*) = [0, \tilde{x})$.

ii) **Optimality, $x(\beta - \rho) + Q$ eventually concave:**
If Proposition 7 applies and the coefficients are continuous at $\tilde{x}$, and $x\beta + Q$ is concave on $(\tilde{x}, \bar{x})$, then $\mathsf{L}\psi \leq 0$ and thus $\Psi = \psi$ and the optimal continuation region is $D = [0, x^*) = [0, \tilde{x})$. Note that Q is concave at some $x \geq \tilde{x}$ if $x \mapsto \check{q}$ is nonincreasing ($\Rightarrow Q$ nondecreasing!) and convex there, and $\bar{q}$ is convex there.

iii) **Optimality wrt. a possibly modified problem:**
Assume that Proposition 7 applies and that ψ is not C^∞ at $\tilde{x}$. Let the nth derivative $\psi^{(n)}$ be discontinuous at $\tilde{x}$ and assume in addition that we either have $\sigma(\tilde{x}) \neq 0$ and C^{n-2} coefficients, or $\beta(\tilde{x}) - \int z\, q(\tilde{x}, \mathrm{d}z)$ exists and is > 0 and C^{n-1} coefficients. Then $\mathsf{L}\psi \leq 0$ for all small enough $x \geq \tilde{x}$. Therefore, if $\hat{q} = 0$ then we can construct a new *problem with value function ψ and optimal continuation region $[0, \tilde{x})$ by leaving the coefficients unchanged on some right-open interval containing $[0, \tilde{x}]$ and changing β from there on.*

Proof.

i) For $x \geq \tilde{x}$,

$$\begin{aligned} \mathsf{L}\psi(x) &= \mathsf{L}\psi(x) - \mathsf{L}\psi(\tilde{x}) \\ &= -\rho\psi(x) + \rho\psi(\tilde{x}) + x\beta(x) - \tilde{x}\beta(\tilde{x}) + \mathsf{Q}\psi(x) - \mathsf{Q}\psi(\tilde{x}) \\ &= x(\beta(x) - \rho) - \tilde{x}(\beta(\tilde{x}) - \rho) + \mathsf{Q}\psi(x) - \mathsf{Q}\psi(\tilde{x}). \end{aligned} \tag{24}$$

ii) Notice that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{L}\psi(x) = (x(\beta - \rho))'\psi'(x) + \frac{\mathrm{d}}{\mathrm{d}x}\mathsf{Q}\psi(x) \\ + (x\beta + \frac{1}{2}(x^2\sigma^2)')\psi''(x) + \frac{1}{2}x^2\sigma^2\psi'''(x). \end{aligned} \tag{25}$$

By the nonnegativity of the latter term at $\tilde{x}^-$, we know that $\mathsf{L}\psi$ has nonpositive derivative at $\tilde{x}$. Relaxing differentiability, we still have that $\mathsf{L}\psi$ is either decreasing or has zero derivative; concavity on $(\tilde{x}, \bar{x})$ will then grant that $\mathsf{L}\psi \leq 0$. By (16c), we have that if $x \mapsto q$ is convex and nondecreasing at some $x > \tilde{x}$ (where $\psi''' = \psi'' = 0$), then Q is concave there, as claimed.

iii) To prove the last assertion, let n be the smallest number such that the nth derivative $\psi^{(n)}$ is discontinuous at $\tilde{x}$. Then necessarily n is odd and $\psi^{(n)}(\tilde{x}^-) > 0 = \psi^{(n)}(\tilde{x}^+)$. If $\sigma(\tilde{x}) \neq 0$, differentiate $n-2$ times to get

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-2}\mathsf{L}\psi(\tilde{x}^+) - \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-2}\mathsf{L}\psi(\tilde{x}^-) = -\frac{1}{2}\tilde{x}^2\sigma^2(\tilde{x}) \cdot \psi^{(n)}(\tilde{x}^-) \tag{26a}$$

which is < 0 by assumption. If $\sigma(\tilde{x}) = 0$ differentiate instead $n-1$ times to get

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\mathsf{L}\psi(\tilde{x}^+) - \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1}\mathsf{L}\psi(\tilde{x}^-) \\ &= -\tilde{x}\big(\beta(\tilde{x}) - \int z\, q(x, \mathrm{d}z)\big) \cdot \psi^{(n)}(\tilde{x}^-) \end{aligned} \tag{26b}$$

which is < 0 by assumption. Finally, if $\hat{q} = 0$ then we can change β at x without affecting the quasi-variational inequality at values to the left of x.

□

Note that if we remove the assumption $\tilde{x} = \infty$, we can only conclude that ψ is superoptimal.

Just like in [1], real analyticity conditions may be used to express the solution in terms of a series. To do so, let us re-write the process into the familiar

Lévy representation except for the jump to zero term, which we leave as is. For a compensated Poisson measure $\tilde{N}$ with associated Lévy measure λ, X may be represented as

$$\mathrm{d}X_t = X_{t-} \cdot \Big(\beta(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t + \int \eta(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}\bar{M}_t\Big) - \mathrm{d}H_t \tag{27}$$

where $\bar{M}_t = \tilde{M}(t, \{-1\})$. We shall only treat the case $\hat{q} = 0$, i.e. $\eta \in (-1, 0)$. Assume now that within some positive convergence radius, the coefficients may be written as

$$\begin{aligned} \beta(x) &= \sum_{j=0}^{\infty} \beta_j x^j, \quad \sigma^2(x) = \sum_{j=0}^{\infty} \varsigma_j x^j, \\ \eta(x, z) &= \sum_{j=0}^{\infty} \eta_j(z) x^j \quad \text{and} \quad \bar{q}(x) = \sum_{j=0}^{\infty} q_j x^j. \end{aligned} \tag{28}$$

We may now adapt the Frobenius theory to find a solution with inductively determined coefficients. Insert

$$f = x^\theta \sum_{i=0}^{\infty} a_i x^i, \quad a_0 = 1 \tag{29}$$

into the HJB equation to get

$$\begin{aligned} 0 = \frac{1}{2}\Big(\sum_{j=0}^{\infty} \varsigma_j x^j\Big)\Big(\sum_{i=0}^{\infty} i(i-1)a_i x^i\Big) + \Big(\sum_{j=0}^{\infty}(\beta_j + q_j + \varsigma_j)x^j\Big)\Big(\sum_{i=0}^{\infty} i a_i x^i\Big) \\ + \Big(-\rho + \sum_{j=0}^{\infty}(\theta\beta_j + (\theta-1)(q_j + \frac{1}{2}\theta\varsigma_j))x^j\Big)\Big(\sum_{i=0}^{\infty} a_i x^i\Big) \\ + \Big(\sum_{i=0}^{\infty} a_i x^i \int \Big((1+\eta(x,z))^{\theta+i} - 1 - \eta(x,z)(\theta+i)\Big)\lambda(\mathrm{d}z)\Big). \end{aligned} \tag{30}$$

Note first that the function $\eta \mapsto (1+\eta)^{\theta+i}$ is analytic for $\eta > -1$, so for each i the integrand is the composition of analytic functions and may be written as $\sum_{i=0}^{\infty} h_j x^j$. The constant term determines $\theta \in (0, 1]$:

$$0 = -\rho + \theta\beta_0 + (\theta-1)(\bar{q}_0 + \frac{1}{2}\theta\varsigma_0) + \int \Big((1+\eta_0(z))^\theta - 1 - \theta\eta_0(z)\Big)\lambda(\mathrm{d}z) \tag{31}$$

(the right hand side is negative for $\theta = 0$ and equal to $\beta(0) - \rho$ for $\theta = 1$; in particular, if $\theta = 1$ then $a_1 = -(\rho + q_0 + \varsigma_0 + \int \eta_0^2(z)\lambda(\mathrm{d}z))^{-1} \cdot \rho$ ($\cdot a_0$, actually), and in accordance with Proposition 6, f is increasing and concave at 0.

The a_i may now be found inductively. Then we have the following:

THEOREM 9.
Assume (28) – (31) within some positive convergence radius, and that $\hat{q} = 0$. Assume f'' has a zero. Then Theorem 8 point iii) applies.

Proof. Analytic functions are determined by their derivatives, so while the coefficients are C^∞ at x^*, ψ is not, unless identically equal to x which is impossible by (17). □

4 The effect of uncertainty

Having found (under suitable conditions) the value function in Section 3, we shall throughout this section assume that it is optimal to reflect the process downwards at x^* and that the value function Ψ is C^2 there. While this regularity assumption is ad hoc, we know from Section 3 that it covers a wide range of control problems. The author is aware that similar regularity results are recently obtained by Alvarez [6] in the non-jump case.

In the continuous cases studied by [4], [2] and [1] they find that $x^* \geq x_0^*$ (equal to $\operatorname{argmax} x(\beta - \rho)$); [3] later improves the bound to $x^* \geq x_0^* + \sigma_{\mathrm{VP}}/r$ for the case (1b). The relation $x^* \geq x_0^*$ may have the interpretation of one being more careful under uncertainty. It turns out that jump uncertainty may violate this property (at least apparently, see the Closing remarks for an interpretation). However, it holds if the jump intensity is nonincreasing in x. More generally, we have

PROPOSITION 10.
Assume $\Psi'''(x^{*-})$ exists and assume (for simplicity, admits generalizations) coefficients differentiable at x^*. If $Q'(x^*) \geq 0$ (resp. > 0), then $0 \geq$ (resp. $>$) $\beta(x^*) - \rho + x^*\beta'(x^*)$; hence if $x\beta$ concave, then x^* is no smaller (resp. strictly greater) than in the deterministic case. If $\sigma(x^*) = 0$ and $Q'(x^*) \leq 0$ (resp. < 0), then $\beta(x^*) - \rho + x^*\beta'(x^*) \geq 0$ (resp > 0); hence if $x\beta$ concave, then x^* is no larger than (resp. strictly smaller than) in the deterministic case.

Proof. Differentiate the equation $\mathsf{L}\Psi = 0$ and insert x^{*-}:

$$-\big(x^*(\beta(x^*) - \rho)\big)' = \frac{1}{2}(x^*)^2\sigma^2(x^*)\Psi'''(x^{*-}) + Q'(x^*) \geq Q'(x^*) \tag{32}$$

with equality if $\sigma(x^*) = 0$. □

Note that as (32) lower bounds the derivative of $Q + x(\beta - \rho)$ at x^*, it may be potentially difficult to utilize point i) of Theorem 8.

Proposition 10 combined with Theorem 8 point iii) yields a main point of this paper:

META-THEOREM 11 (Behavior towards risk).
There is a wide class of problems for which the optimal solution is to harvest at a lower level than the corresponding deterministic problem. Thus the introduction of (Markov martingale) uncertainty may reduce the optimal population.

While this may appear a bit counterintuitive at first glance, it certainly makes sense: jump intensity decreasing in x may lead to one keeping the population at a higher level to reduce the probability of "disasters", i.e. jumps; on the other hand, the presence of the jump terms may lead us to harvest and reduce X before the jumps do. This is however not a valid argument, since jumps are compensated. The case with only annihilation risk is illustrative:

THEOREM 12 (Annihilation risk only).
Assume that $\hat{q} = \check{q} = \sigma = 0$ and β and $\bar{q}$ continuous and piecewise differentiable. Then x^* is a stationary point of $x(\beta - \rho)/(\rho + \bar{q})$. Furthermore, $x(\beta - \rho)$ is (strictly) increasing/decreasing at x^* iff $\bar{q}$ is. In particular, if $x(\beta - \rho)/\rho$ and $x(\beta - \rho)/(\rho + \bar{q})$ have unique stationary points x_0^*, x^*, respectively, their respective control problems have optimal continuation regions $D_0 = [0, x_0^*)$ and $D = [0, x^*)$ and $x^* \leq x_0^*$ ($< x_0^*$) iff $\bar{q}$ is (strictly) increasing at x^*.

Proof. The Hamilton-Jacobi-Bellman equation $\mathsf{L}\Psi = 0$ is now

$$0 = -\Psi(x) + \frac{x(\beta + \bar{q})}{\rho + \bar{q}}\Psi'(x). \tag{33}$$

Differentiation yields

$$0 = \Psi'(x) \cdot \Big(\frac{x(\beta - \rho)}{\rho + \bar{q}}\Big)' + \frac{x(\beta - \rho)}{\rho + \bar{q}} \cdot \Psi''(x). \tag{34}$$

Now $x(\beta - \rho)/(\rho + \bar{q})$ has stationary point where

$$\frac{(x(\beta - \rho))'}{x(\beta - \rho)} = \frac{\bar{q}'}{(\rho + \bar{q})}. \tag{35}$$

Note that the denominators are both positive. □

Example. If $x\beta$ is concave and $\bar{q}$ is convex, we can solve the problem completely if $x(\beta - \rho)$ is increasing at 0 and has some stationary point, and that $\beta(0) > \rho$ and $\beta'(0^+)$ and $\bar{q}(0^+)$ are both finite. Then it is easy to verify that $x(\beta - \rho)/(\rho + \bar{q})$ also is increasing at 0 and has some stationary point; let x^* be the leftmost. Then for $x \leq x^*$,

$$\psi(x) = x^* \frac{\beta(x^*) + \bar{q}(x^*)}{\rho + \bar{q}(x^*)} \exp\{\int_{x^*}^{x} \frac{\rho + \bar{q}(y)}{y(\beta(y) + \bar{q}(y))} \mathrm{d}y\}. \tag{36}$$

It is easy to verify that the HJB equation holds, that $\psi'' \le 0$ and hence that $\psi' \ge \psi'(x^*) = 1$ and that Theorem 8 point ii) applies. By (35), we can merely check sign $\bar{q}'(x^*)$ to verify if the optimal trigger is higher than in the deterministic case. As a special case, consider the logistic growth model

$$\beta(x) = r(K - x) \tag{37}$$

(with $rK > \rho$) modified with a compensated annihilation term with (convex) intensity $\bar{q}(x) = (q_0 + q_1 x)^+$; $\sigma = q = 0$. Then $x_0^* = (K - \rho/r)/2$. Assume that $\bar{q}(x^*) > 0$, one may verify that

$$x^* = \frac{\rho + q_0}{q_1}\Big(- 1 + \sqrt{1 + 2x_0^* \frac{q_1}{\rho + q_0}}\Big) \tag{38}$$

We see that x^* is strictly decreasing in $(\rho + q_0)/q_1$; therefore, increasing both q_0 and q_1 simultaneously gives no information on whether x^* increases or decreases. We may also find the value function in terms of

$$\ln f(x) = \begin{cases} \frac{\rho+q_0}{rK+q_0} \ln x + (\frac{q_1}{q_1-r} - \frac{\rho+q_0}{rK+q_0}) \ln(rK + q_0 + (q_1 - r)x) & \text{if } q_1 \neq r \\ \frac{\rho+q_0}{rK+q_0} \ln x + \frac{r}{rK+q_0} x & \text{if } q_1 = r. \end{cases} \tag{39}$$

We omit the details. △

5 Closing remarks

We have seen that even in this simple model, the optimal strategy may adapt qualitatively differently to the introduction of a jump martingale compared to the introduction of Brownian noise to the model. The result suggests that one should be careful with respect to how one models uncertainty.

We will attempt to interpret Proposition 10: While a jump martingale term may reduce the optimal trigger x^*, jumps seem to have the same effect as Brownian noise if Q defined by (6a) is nondecreasing at x^*. However, using the Lévy representation (27) instead of the representation (2) for X, we have that κ is nondecreasing at x^* if both $(-\bar{q})$ and $x + \eta x$ are. Note that κ nondecreasing (locally) is precisely what is needed to generalize (locally) the well-known comparison theorems of the continuous Brownian framework. In that case, let us compare two strategies corresponding to processes X and $\hat{X}$ respectively, where $\hat{X}$ corresponds to an initial harvesting amount ϵ while X does not (everything else equal): Then $X \ge \hat{X}$ (unless by harvesting the former), and the loss in the former is bounded by ϵ. $x + \eta x$ nondecreasing corresponds to state after jump being increasing with respect to state after jump. If that fails, then X may jump to a state lower than $\hat{X}$. We therefore interpret Proposition 10 as *a trade-off between noise level on one hand, and* *on the other exposure to risk of falling to a lower level.* Speaking heuristically, the assertion that "risk leads to higher trigger level" is only modified by adding "as long as it does not become risky to increase the trigger level". Speaking even more heuristically, one could interpret the behavior towards Brownian noise as an adaptation towards irreversibility and notice that if one waits for a higher level, one has "better" information in the sense that it is less probable that the process will fall to a level where drift is low; however, if waiting leads to a disaster, maybe one doesn't really want to know.

Having established that introducing a pure jump Markov martingale to a deterministic model may either reduce or increase the optimal trigger x^*, one may want to ask: Introducing jump intensities $\{\ell q\}_{\ell\geq 0}$; for what q is x^* monotone with respect to ℓ? And introducing Brownian volatilities $\{\ell\sigma\}$, for what σ is x^* increasing in ℓ? And, ultimately, for what $\{(\ell\sigma, \ell q)\}$ is x^* monotone in ℓ? These are topics for future research.

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