Optimal Multi-Dimensional Stochastic Harvesting with Density-dependent Prices

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Abstract

We prove a verification theorem for a class of singular control problems which model optimal harvesting with density-dependent prices or optimal dividend policy with capital-dependent utilities. The result is applied to solve explicitly some examples of such optimal harvesting/optimal dividend problems.

1 Introduction

Price dependence on population size can occur in any of the following ways mentioned below: Small population size may lead to a significant increase in the price of the affected species. For instance, the black rhino population, which is hunted for its horn, has decreased to near extinction. The result of this is that the restocking price of the black rhino has increased to such high levels that most game reserves which were in the past natural habitats for the black rhino cannot afford the current stocking prices. The wildebeest population, on the other hand, is so large in most game reserves that its hunting license price is cheap. Another way in which population density affects the price of the species is the quality of the individual members. For instance in places where wildlife movement is restricted the quality of the environment (and indeed the quality of the animals) depends on population size. Availability of quality vegetation and water depends on whether the game reserve's carrying capacity has been exceeded or not. Restricting animal movement has the disadvantage of allowing for inbreeding and subsequently to weaker species that are prone to suffer from genetically acquired diseases and defects, such as small size etc. The value of the animals is therefore reduced considerably.

Although high population density may lead to lower prices, it plays a very important role on the survival of the species. For instance Wildebeest and Zebra are not fast animals (in terms of running away from predators). If they did not live in colonies of densily populated areas, they would become extinct. Because of the large population in a given colony, these animals are able to reproduce and maintain their population at health levels.

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In this paper we consider the problem of optimal harvesting from a collection of interacting populations (described by a coupled system of stochastic differential equations) when the price per unit for each population is allowed to depend on the sizes of the populations.

2 The main result

We now describe our model in detail. This presentation follows [LØ2] closely. Consider n populations whose sizes or densities $X_1(t), \dots, X_n(t)$ at time t are described by a system of n stochastic differential equations of the form

$$dX_i(t) = b_i(t, X(t))dt + \sum_{j=1}^{m} \sigma_{ij}(t, X(t))dB_j(t); 0 \leq s \leq t \leq T \tag{2.1}$$

$$X_i(s) = x_i \in \mathbb{R} \ ; \qquad 1 \leq i \leq n \ , \tag{2.2}$$

where $B(t) = (B_1(t), \dots, B_m(t))$; $t \geq 0$, $\omega \in \Omega$ is m-dimensional Brownian motion and the differentials (i.e. the corresponding integrals) are interpreted in the Itô sense. We assume that $b = (b_1, \dots, b_n) : \mathbb{R}^{1+n} \to \mathbb{R}^n$ and $\sigma = (\sigma_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} : \mathbb{R}^{1+n} \to \mathbb{R}^{n \times m}$ are given continuous functions. We also assume that the terminal time $T = T(\omega)$ has the form

$$T(\omega) = \inf \left\{ t > s; (t, X(t)) \notin S \right\} \tag{2.3}$$

where $S \subset \mathbb{R}^{1+n}$ is a given set. For simplicity we will assume in this paper that

$$S = (0, \infty) \times U$$

where U is an open, connected set in $\mathbb{R}^n$. We may interprete U as the *survival set* and T is the *time of extinction* or simply the *closing/terminal* time.

We now introduce a *harvesting strategy* for this family of populations:

A *harvesting strategy* γ is a stochastic process $\gamma(t) = \gamma(t, \omega) = (\gamma_1(t, \omega), \dots, \gamma_n(t, \omega) \in \mathbb{R}^n$ with the following properties:

(2.4) For each $t \geq s$ $\gamma(t, \cdot)$ is measurable with respect to the σ-algebra $\mathcal{F}_t$ generated by $\{B(s, \cdot); s \leq t\}$. In other words: $\gamma(t)$ is *$\mathcal{F}_t$-adapted.*

(2.5) $\gamma_i(t, \omega)$ is non-decreasing with respect to t, for a.a. $\omega \in \Omega$ and all $i = 1, \dots, n$

(2.6) $t \to \gamma(t, \omega)$ is right-continuous, for a.a. ω

(2.7) $\gamma(s, \omega) = 0$ for a.a. ω .

Component number i of $\gamma(t, \omega), \gamma_i(t, \omega)$, represents *the total amount harvested from population number i up to time t.*

If we apply a harvesting strategy γ to our family $X(t) = (X_1(t), \dots, X_n(t))$ of populations the harvested family $X^{(\gamma)}(t)$ will satisfy the n-dimensional stochastic differential equation

$$\begin{cases} dX^{(\gamma)}(t) = b(t, X^{(\gamma)}(t))dt + \sigma(t, X^{(\gamma)}(t))dB(t) - d\gamma(t) \ ; \quad s \leq t \leq T \\ X^{(\gamma)}(s^-) = x = (x_1, \dots, x_n) \in \mathbb{R}^n \end{cases} \tag{2.8}$$

We let Γ denote the set of all harvesting strategies γ such that the corresponding system (2.7) has a unique strong solution $X^{(\gamma)}(t)$ which does not explode in the time interval $[s,\infty]$ and such that $X^{(\gamma)}(T) \in \bar{S}$.

Since we do not exclude immediate harvesting at time $t = s$ it is necessary to distinguish between $X^{(\gamma)}(s)$ and $X^{(\gamma)}(s^-)$: $X^{(\gamma)}(s^-)$ is the state right before harvesting starts at time $t = s$, while

$$X^{(\gamma)}(s) = X^{(\gamma)}(s^-) - \Delta\gamma$$

is the state immediately after, if γ consists of an immediate harvest of size $\Delta\gamma$ at $t = s$.

Suppose that *the price per unit of population number* i, when harvested at time t and when the current size/density of the vector $X^{(\gamma)}(t)$ of populations is $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, is given by

$$\pi_i(t,\xi) \; ; \qquad (t,\xi) \in S \; , \quad 1 \le i \le n \; , \tag{2.9}$$

where the $\pi_i : S \to \mathbb{R}$; $1 \le i \le n$, are lower bounded continuous functions. We call such prices *density-dependent* since they depend on ξ. The total expected discounted utility harvested from time s to time T is given by

$$J^{(\gamma)}(s,x) = E^{s,x}\Big[\int\limits_{[s,T]} \pi(t, X^{(\gamma)}(t^-)) \cdot d\gamma(t) \Big] \tag{2.10}$$

where $\pi = (\pi_1, \dots, \pi_n)$, $\pi \cdot d\gamma = \sum_{i=1}^{n} \pi_i d\gamma_i$ and $E^{s,x}$ denotes the expectation with respect to the probability law $Q^{s,x}$ of the time-state process

$$Y^{s,x}(t) = Y^{\gamma,s,x}(t) = (t, X^{(\gamma)}(t)) \; ; \qquad t \ge s \tag{2.11}$$

assuming that $Y^{s,x}(s^-) = x$.

The *optimal harvesting problem* is to find the *value function* $\Phi(s,x)$ and an *optimal harvesting strategy* $\gamma^* \in \Gamma$ such that

$$\Phi(s,x) = \sup_{\gamma \in \Gamma} J^{(\gamma)}(s,x) = J^{(\gamma^*)}(s,x) \; . \tag{2.12}$$

This problem differs from the problems considered in [A1], [A3], [AS], [LØ1] and [LØ2] in that the prices $\pi_i(t,\xi)$ are allowed to be density-dependent. This allows for more realistic models. For example, it is usually the case that if a type of fish, say population number i, becomes more scarce, the price per unit of this fish increases. Conversely, if a type of fish becomes abundant then the price per unit goes down. Thus in this case the price $\pi_i(t,\xi) = \pi_i(t,\xi_1,\dots,\xi_n)$ is a *nonincreasing* function of ξ_i. One can also have situations where $\pi_i(t,\xi)$ depends on all the other population densities $\xi_1, \dots, \xi_n$ in a similar way.

It turns out that if we allow the prices to be density-dependent, a number of new – and perhaps surprising – phenomena occurs. The purpose of this paper is not to give a complete discussion of the situation, but consider some illustrative examples.

Remark Note that we can also give the problem (2.12) an economic interpretation: We can regard $X_i(t)$ as the value at time t of an economic quantity or asset and we can let $\gamma_i(t)$ represent the total amount paid in dividends from asset number i up to time t. Then S can be interpreted as the *solvency set*, T as the *time of bankruptcy* and $\pi_i(t,\xi)$ as the *utility rate* of dividends from asset number i at the state (t,ξ). Then (2.12) becomes the problem of finding the *optimal stream of dividends*. This interpretation is used in [JS] (in the density-independent utility case). See also [LØ2].

In the following H^0 denotes the interior of a set H, $\bar{H}$ denotes its closure.

If $G \subset \mathbb{R}^k$ is an open set we let $C^2(G)$ denote the set of real valued twice continuously differentiable functions on G. We let $C_0^2(G)$ denote the set of functions in $C^2(G)$ with compact support in G.

If we do not apply any harvesting, then the corresponding time-state population process $Y(t) = (t, X(t))$, with $X(t)$ given by (2.1)–(2.2), is an Itô diffusion whose generator coincides on $C_0^2(\mathbb{R}^{1+n})$ with the partial differential operator L given by

$$(2.13) \qquad Lg(s,x) = \frac{\partial g}{\partial s}(s,x) + \sum_{i=1}^{n} b_i(s,x)\frac{\partial g}{\partial x_i}(s,x) + \tfrac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(s,x)\frac{\partial^2 g}{\partial s \partial x}$$

for all functions $g \in C^2(S)$.

The following result is a generalization to the multi-dimensional case of Theorem 1 in [A2] and a generalization to density-dependent prices of Theorem 2.1 in [LØ2]. For completeness we give the proof.

Theorem 2.1. *Assume that*

(2.14) $\quad \pi_i(t,\xi)$ *is* nonincresing *with respect to* $\xi_1,\dots,\xi_n$, *for all* t *and all* $i = 1,2,\dots,n$.

a) *Suppose* $\varphi \geq 0$ *is a function in* $C^2(S)$ *satisfying the following conditions*

(i) $\quad \frac{\partial \varphi}{\partial x_i}(t,x) \geq \pi_i(t,x)$ *for all* $(t,x) \in S$, $i = 1,2,\dots,n$

(ii) $\quad L\varphi(t,x) \leq 0$ *for all* $(t,x) \in S$.

Then

$$(2.15) \qquad \varphi(s,x) \geq \Phi(s,x) \qquad \text{for all } (s,x) \in S\ .$$

b) *Define the* nonintervention region D *by*

$$(2.16) \qquad D = \Big\{(t,x) \in S; \frac{\partial \varphi}{\partial x_i}(t,x) > \pi_i(t,x) \text{ for all } i = 1,\dots,n\Big\}.$$

Suppose that, in addition to (i) and (ii) above,

(iii) $\quad L\varphi(t,x) = 0$ *for all* $(t,x) \in D$

and that there exists a harvesting strategy $\hat{\gamma} \in \Gamma$ *such that the following, (iv)–(vii), hold:*

(iv) $\quad X^{(\hat{\gamma})}(t) \in \bar{D}$ *for all* $t \in [s,T]$

(v) $\left(\frac{\partial\varphi}{\partial x_i}(t,X^{(\hat{\gamma})}(t))-\pi_i(t,X^{(\hat{\gamma})}(t))\right)\cdot d\hat{\gamma}_i^{(c)}(t)=0$ *(i.e.* $\hat{\gamma}_i^{(c)}$ *increases only when* $\frac{\partial\varphi}{\partial x_i}=\pi_i$*);* $1\le i\le n$

and

(vi) $\varphi(t_k,X^{(\hat{\gamma})}(t_k))-\varphi(t_k,X^{(\hat{\gamma})}(t_k^-))=-\pi_i(t_k,X^{(\hat{\gamma})}(t_k^-))\cdot\Delta\hat{\gamma}(t_k)$

at all jumping times $t_k\in[s,T)$ *of* $\hat{\gamma}(t)$*, where*

$$\Delta\hat{\gamma}(t_k)=\hat{\gamma}(t_k)-\hat{\gamma}(t_k^-)$$

and

(vii) $E^{s,x}\big[\varphi(T_R,X^{(\hat{\gamma})}(T_R))\big]\to 0$ *as* $R\to\infty$

where

$$T_R=T\wedge R\wedge\inf\left\{t>s;|X^{(\hat{\gamma})}(t)|\ge R\right\}\ ;\qquad R>0\ .$$

Then

$$\varphi(s,x)=\Phi(s,x)\qquad \textit{for all } (s,x)\in S \tag{2.17}$$

and

$$\gamma^*:=\hat{\gamma}\quad \textit{is an optimal harvesting strategy.}$$

Proof. **a)** Choose $\gamma\in\Gamma$ and $(s,x)\in S$. Then by Itô's formula for semimartingales (the Doléans-Dade-Meyer formula) [P, Th. II.7.33] we have

$$\begin{aligned}E^{s,x}[\varphi(T_R,X^{(\gamma)}(T_R))]&=E^{s,x}[\varphi(s,X^{(\gamma)}(s))]\\&+E^{s,x}\Big[\int_s^{T_R}\frac{\partial\varphi}{\partial t}(t,X^{(\gamma)}(t))dt+\int_{(s,T_R]}\sum_{i=1}^n\frac{\partial\varphi}{\partial x_i}(t,X^{(\gamma)}(t^-))dX_i^{(\gamma)}(t)\\&+\sum_{i,j=1}^n\int_s^{T_R}\tfrac12(\sigma\sigma^T)_{ij}(t,X^{(\gamma)}(t))\frac{\partial^2\varphi}{\partial x_i\partial x_j}(t,X^{(\gamma)}(t))dt\\&+\sum_{s<t_k\le T_R}\Big\{\varphi(t_k,X^{(\gamma)}(t_k))-\varphi(t_k,X^{(\gamma)}(t_k^-))-\sum_{i=1}^n\frac{\partial\varphi}{\partial x_i}(t_k,X^{(\gamma)}(t_k^-))\Delta X_i^{(\gamma)}(t_k)\Big\}\Big],\end{aligned}\tag{2.18}$$

where the sum is taken over all jumping times $t_k\in(s,T_R]$ of $\gamma(t)$ and

$$\Delta X_i^{(\gamma)}(t_k)=X_i^{(\gamma)}(t_k)-X_i^{(\gamma)}(t_k^-)\ .$$

Let $\gamma^{(c)}(t)$ denote the continuous part of $\gamma(t)$, i.e.

$$\gamma^{(c)}(t)=\gamma(t)-\sum_{s\le t_k\le t}\Delta\gamma(t_k)\ .$$

Then, since $\Delta X_i^{(\gamma)}(t_k) = -\Delta\gamma_i(t_k)$ we see that (2.18) can be written

$$E^{s,x}[\varphi(T_R, X^{(\gamma)}(T_R))] = \varphi(s,x)$$
$$+E^{s,x}\Big[\int_s^{T_R}\Big\{\frac{\partial\varphi}{\partial t} + \sum_{i=1}^n b_i \frac{\partial\varphi}{\partial x_i} + \tfrac{1}{2}\sum_{i,j=1}^n (\sigma\sigma^T)_{ij}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\Big\}(t, X^{(\gamma)}(t))dt\Big]$$
$$-E^{s,x}\Big[\int_s^{T_R}\sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(t, X^{(\gamma)}(t))d\gamma_i^{(c)}(t)\Big] + E^{s,x}\Big[\sum_{s\le t_k\le T_R}\Delta\varphi(t_k, X^{(\gamma)}(t_k))\Big] \tag{2.19}$$

where

$$\Delta\varphi(t_k, X^{(\gamma)}(t_k)) = \varphi(t_k, X^{(\gamma)}(t_k)) - \varphi(t_k, X^{(\gamma)}(t_k^-)) .$$

Therefore

$$\varphi(s,x) = E^{s,x}[\varphi(T_R, X^{(\gamma)}(T_R))] - E^{s,x}\Big[\int_s^{T_R} L\varphi(t, X^{(\gamma)}(t))dt\Big]$$
$$+E^{s,x}\Big[\int_s^{T_R}\sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(t, X^{(\gamma)}(t))d\gamma_i^{(c)}(t)\Big]$$
$$-E^{s,x}\Big[\sum_{s\le t_k\le T_R}\Delta\varphi(t_k, X^{(\gamma)}(t_k))\Big]. \tag{2.20}$$

Let $y = y(r)$; $0 \le r \le 1$ be a smooth curve in U from $X^{(\gamma)}(t_k)$ to $X^{(\gamma)}(t_k^-) = X^{(\gamma)}(t_k) + \Delta\gamma(t_k)$. Then

$$-\Delta\varphi(t_k, X^{(\gamma)}(t_k)) = \int_o^1 \nabla\varphi(t_k, y(r))dy(r) . \tag{2.21}$$

We may assume that

$$dy_i(r) \ge 0 \qquad \text{for all } \ i, r .$$

Now suppose that (i) and (ii) hold. Then by (2.20) and (2.21) we have

$$\varphi(s,x) \ge E^{s,x}\Big[\int_s^{T_R}\sum_{i=1}^n \pi_i(t, X^{(\gamma)}(t))d\gamma_i^{(c)}(t)\Big]$$
$$+ E^{s,x}\Big[\sum_{s\le t_k\le T_R}\Big(\int_0^1\sum_{i=1}^n \pi_i(t_k, y(r))dy_i(r)\Big)\Big] \tag{2.22}$$

Since we have assumed that $\pi_i(t,\xi)$ is *nonincreasing* with respect to $\xi_1, \dots, \xi_n$ we have

$$\pi_i(t_k, X^{(\gamma)}(t_k^-)) \le \pi_i(t_k, y(r)) \le \pi_i(t_k, X^{(\gamma)}(t_k))$$

for all i, k and $r \in [0,1]$. Hence

$$\int_0^1 \pi_i(t_k, y(r)) dy_i(r) \geq \pi(t :_k, X^{(\gamma)}(t_k^-)) \cdot \Delta\gamma_i(t_k) \ . \tag{2.23}$$

Combined with (2.22) this gives

$$\begin{aligned} \varphi(s,x) &\geq E^{s,x}\Big[\int_0^{T_R} \pi(t, X^{(\gamma)}(t)) d\gamma^{(c)}(t) + \sum_{s \leq t_k \leq T} \pi(t_k, X^{(\gamma)}(t_k^-)) \cdot \Delta\gamma(t_k)\Big] \\ &= E^{s,x}\Big[\int_{[s,T_R]} \pi(t, X^{(\gamma)}(t^-)) d\gamma(t)\Big]. \end{aligned} \tag{2.24}$$

Letting $R \to \infty$ we obtain $\varphi(s,x) \geq J^{(\gamma)}(s,x)$. Since $\gamma \in \Gamma$ was arbitrary we conclude that (2.15) holds. Hence a) is proved.

b) Next, suppose that (iii)–(vii) also hold. Then if we apply the argument above to $\gamma = \hat{\gamma}$ we get in (2.20) the following:

$$\begin{aligned} \varphi(s,x) &= E^{s,x}[\varphi(T_R, X^{(\hat{\gamma})}(T_R))] \\ &\quad + E^{s,x}\Big[\int_0^{T_R} \pi(t, X^{(\hat{\gamma})}(t)) \cdot d\hat{\gamma}^{(c)}(t) + \sum_{s \leq t_k \leq T_R} \pi(t_k, X^{(\hat{\gamma})}(t_k^-)) \cdot \Delta\hat{\gamma}(t_k)\Big] \\ &= E^{s,x}[\varphi(T_R, X^{(\hat{\gamma})}(T_R))] + E^{s,x}\Big[\int_{[s,T_R]} \pi(t, X^{(\hat{\gamma})}(t)) \cdot d\hat{\gamma}(t)\Big] \\ &\longrightarrow J^{(\hat{\gamma})}(s,x) \qquad \text{as } R \to \infty \ . \end{aligned}$$

Hence $\varphi(s,x) = J^{(\hat{\gamma})}(s,x) \leq \Phi(s,x)$. Combining this with (2.14) from a) we get the conclusion (2.16) of part b). This completes the proof of Theorem 2.1. □

If we specialize to the 1-dimensional case with just one population $X^{(\gamma)}(t)$ given by

$$\begin{cases} dX^{(\gamma)}(t) = b(t, X^{(\gamma)}(t))dt + \sigma(t, X^{(\gamma)}(t))dB(t) - d\gamma(t) \ ; & t \geq s \\ X^{(\gamma)}(s^-) = x \in \mathbb{R} \end{cases} \tag{2.25}$$

then Theorem 2.1a) gets the form (see also [A2, Lemma 1])

Corollary 2.2. *Assume that*

$$\xi \to \pi(t,\xi);\ \xi \in \mathbb{R} \quad \textit{is nonincreasing for all} \ \ t \in [0,T] \tag{2.26}$$

$$\varphi(t,x) \geq 0 \qquad \textit{is a function in} \ \ C^2(S) \ \textit{such that} \tag{2.27}$$

$$\frac{\partial \varphi}{\partial x}(t,x) \geq \pi(t,x) \qquad \textit{for all} \ \ (t,x) \in S \tag{2.28}$$

and

$$L\varphi(t,x) \le 0 \qquad \textit{for all } (t,x) \in S\ . \tag{2.29}$$

Then

$$\varphi(s,x) \ge \Phi(s,x) \qquad \textit{for all } (s,x) \in S\ . \tag{2.30}$$

3 Examples

In this section we apply Theorem 2.1 or Corollary 2.2 to some special cases.

Example 3.1. Suppose $X^{(\gamma)}(t)$ is given by

$$\begin{cases} dX^{(\gamma)}(t) = \mu\, dt + \sigma\, dB(t) - d\gamma(t)\ ; \quad t \ge s \\ X^{(\gamma)}(s) = x > 0 \end{cases} \tag{3.1}$$

where $\mu > 0$ and $\sigma \ne 0$ are constants.

We want to maximize the total discounted value of the harvest, given by

$$J^{(\gamma)}(s,x) = E^{s,x}\Big[\int\limits_{[s,T)} e^{-\rho t} g(X^{(\gamma)}(t^-)) d\gamma(t) \Big] \tag{3.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given nonincreasing function (the density-dependent price) and

$$T = \inf\big\{ t > s; X^{(\gamma)}(t) \le 0 \big\} \tag{3.3}$$

is the time of extinction, i.e. $S = \{(t,x); x > 0\}$. The case with g *constant* was solved in [JS]. Then it is optimal to do nothing if the population is below a certain treshold $x^* > 0$ and then harvest according to *local time* of the downward reflected process $\bar{X}(t)$ at $\bar{X}(t) = x^*$.

Now consider the case when

$$g(x) = x^{-1/2}, \qquad \text{i.e.} \quad \pi(t,x) = e^{-\rho t} x^{-1/2}; \quad x > 0\ . \tag{3.4}$$

Then the price increases as the population size x decreases, so (2.24) holds. Suppose we apply the "*take the money and run*"-strategy $\overset{\circ}{\gamma}$. This strategy empties the whole population immediately. It can be described by

$$\overset{\circ}{\gamma}(s) = X(s^-) = x\ . \tag{3.5}$$

Such a strategy gives the harvest value

$$J^{(\overset{\circ}{\gamma})}(s,x) = e^{-\rho s} x^{-1/2} x = e^{-\rho s}\sqrt{x}\ ; \qquad x > 0\ . \tag{3.6}$$

However, it is unlikely that this is the best strategy because it does not take into account that the price increases as the population size goes down. So we try the following "chattering policy", denoted by $\widetilde{\gamma} = \widetilde{\gamma}^{(m,\eta)}$, where m is a fixed natural number and $\eta > 0$:

At the times

$$t_k = \Big(s + \frac{k}{m}\eta\Big) \wedge T \; ; \qquad k = 1, 2, \dots, m \tag{3.7}$$

we harvest an amount $\Delta\widetilde{\gamma}(t_k)$ which is the fraction $\frac{1}{m}$ of the current population. This gives the expected harvest value

$$J^{(\tilde{\gamma}(m,\eta))}(s,x) = E^{s,x}\Big[\sum_{k=1}^{m} e^{-\rho t_k}\big[(X^{(\tilde{\gamma})}(t_k^-))^+\big]^{-1/2}\Big]\Delta\widetilde{\gamma}(t_k) \; , \tag{3.8}$$

where we have used the notation

$$x^+ = \max(x, 0) \; ; \qquad x \in \mathbb{R} \; .$$

This can be written

$$J^{(\tilde{\gamma}(m,\eta))}(s,x) = E^{s,x}\Big[\sum_{k=1}^{m} e^{-\rho t_k}\big[(x - \widetilde{\gamma}(t_k^-))^+\big]^{-1/2}\Big]\Delta\widetilde{\gamma}(t_k) \; . \tag{3.9}$$

Now let $\eta \to 0$. Then all the t_k's converge to s and we get

$$\begin{aligned} J^{(\tilde{\gamma}(m,0))}(s,x) :&= \lim_{\eta\to 0} J^{(\tilde{\gamma}(m,\eta))}(s,x) = e^{-\rho s}\sum_{k=1}^{m}\Big(x - \frac{k}{m}x\Big)^{-1/2}\frac{1}{m}x \\ &= e^{-\rho s}\sum_{k=1}^{m} h(x_k)\Delta x_k \; , \end{aligned} \tag{3.10}$$

where $h(y) = (x-y)^{-1/2}$, $x_k = \frac{k}{m}x$ and $\Delta x_k = x_{k+1} - x_k = \frac{x}{m}$.

Now if $\varepsilon > 0$ is given we can find a natural number m such that

$$\Big| \int_0^x (x-y)^{-1/2}dy - \sum_{k=1}^{m} h(x_k)\Delta x_k \Big| < \varepsilon \; . \tag{3.11}$$

Therefore, by choosing m and η properly we can obtain that

$$\Big| J^{(\tilde{\gamma}(m,\eta))}(s,x) - e^{-\rho s}\int_0^x (x-y)^{-1/2}dy \Big| < \varepsilon \; . \tag{3.12}$$

We conclude that

$$\sup_{\gamma} J^{(\gamma)}(s,x) \le e^{-\rho s}\int_0^x (x-y)^{-1/2}dy = e^{-\rho s}2\sqrt{x} \; . \tag{3.13}$$

We call this "chattering policy" of applying $\widetilde{\gamma}^{m,\eta)}$ in the limit as $\eta \to 0$ and $m \to \infty$ the *policy of immediate chattering down to 0.* (This limit does not exist as a strategy in Γ.) From (3.13) we conclude that

$$\Phi(s,x) \ge 2e^{-\rho s}\sqrt{x} \; . \tag{3.14}$$

On the other hand, let us check if the function

$$\varphi(s,x) := 2e^{-\rho s}\sqrt{x} \tag{3.15}$$

satisfies the conditions (2.26)–(2.28) of Corollary 2.2: Condition (2.26) holds trivially, and since

$$\frac{\partial\varphi}{\partial x}(s,x) = e^{-\rho s}x^{-1/2} = \pi(s,x)\ , \qquad (2.27)\text{ holds .}$$

Now

$$L = \frac{\partial}{\partial s} + \mu\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}$$

and therefore

$$\begin{aligned} L\varphi(s,x) &= 2e^{-\rho s}\big[-\rho x^{1/2} + \mu\cdot\tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}\sigma^2\tfrac{1}{2}(-\tfrac{1}{2})x^{-3/2}\big] \\ &= -2\rho e^{-\rho s}x^{-3/2}\Big[x^2 - \frac{\mu}{2\rho}x + \frac{\sigma^2}{8\rho}\Big]. \end{aligned}$$

So (2.28) holds if $\mu^2 \le 2\rho\sigma^2$. By Corollary 2.2 we conclude that $\varphi = \Phi$ in this case.

We have proved part a) of the following result:

Theorem 3.2. *Let $X^{(\gamma)}(t)$ and T be given by (3.1) and (3.3), respectively.*

a) *Assume that*

$$\mu^2 \le 2\rho\sigma^2\ . \tag{3.16}$$

Then

$$\Phi(s,x) := \sup_{\gamma\in\Gamma} E^{s,x}\Big[\int\limits_{[s,T)} e^{-\rho t}\{X^{(\gamma)}(t^-)\}^{-1/2}d\gamma(t)\Big] = 2e^{-\rho s}\sqrt{x}\ .$$

This value is achieved in the limit if we apply the strategy $\widetilde{\gamma}^{(m,\eta)}$ above with $\eta\to 0$ and $m\to\infty$, i.e. by applying the policy of immediate chattering down to 0.

b) *Assume that*

$$\mu^2 > 2\rho\sigma^2 \tag{3.17}$$

Then the value function has the form

$$\Phi(s,x) = \begin{cases} e^{-\rho s}C(e^{\lambda_1 x} - e^{\lambda_2 x})\ ; & 0\le x < x^* \\ e^{-\rho s}(2\sqrt{x} - 2\sqrt{x^*} + A)\ ; & x^*\le x \end{cases} \tag{3.18}$$

for some constants $C>0$, $A>0$ and $x^*>0$, where

$$\lambda_1 = \sigma^{-2}\big[-\mu + \sqrt{\mu^2+2\rho\sigma^2}\,\big] > 0\ , \quad \lambda_2 = \sigma^{-2}\big[-\mu - \sqrt{\mu^2+2\rho\sigma^2}\,\big] < 0\ . \tag{3.19}$$

The corresponding optimal policy is the following:

(3.20) *If $x > x^*$ it is optimal to apply immediate chattering from x down to x^*.*

(3.21) *if $0 < x < x^*$ it is optimal to apply the harvesting equal to the* local time of the downward reflected process $\bar{X}(t)$ *at x^*.*

Proof of b). First note that if we apply the policy of immediate chattering from x down to x^*, where $0 < x^* < x$, then the value of the harvested quantity is

$$(3.22) \qquad e^{-\rho s} \int_0^{x-x^*} (x-y)^{-1/2} dy = e^{-\rho s} \int_{x^*}^{x} u^{-1/2} du = 2e^{-\rho s}\left(\sqrt{x} - \sqrt{x^*}\right) .$$

This follows by the argument (3.7)–(3.12) above.

To verify (3.18)–(3.21) note that λ_1, λ_2 are the roots of the quadratic equation

$$(3.23) \qquad -\rho + \mu\lambda + \tfrac{1}{2}\sigma^2\lambda^2 = 0 .$$

Hence, with $\varphi(s,x)$ defined to be the right hand side of (3.18) we have

$$(3.24) \qquad L\varphi(s,x) = 0 \qquad \text{for } \; x < x^*$$

and

$$(3.25) \qquad \varphi(s,0) = 0 .$$

We now require that φ is C^2 at $x = x^*$. This gives the 3 equations

$$(3.26) \qquad C(e^{\lambda_1 x^*} - e^{\lambda_2 x^*}) = A$$

$$(3.27) \qquad C(\lambda_1 e^{\lambda_1 x^*} - \lambda_2 e^{\lambda_2 x^*}) = (x^*)^{-1/2}$$

$$(3.28) \qquad C(\lambda_1^2 e^{\lambda_1 x^*} - \lambda_2^2 e^{\lambda_2 x^*}) = -\tfrac{1}{2}(x^*)^{-3/2}$$

Dividing (3.27) by (3.28) we get the equation

$$(3.29) \qquad \frac{\lambda_1 e^{\lambda_1 x^*} - \lambda_2 e^{\lambda_2 x^*}}{\lambda_1^2 e^{\lambda_1 x^*} - \lambda_2^2 e^{\lambda_2 x^*}} = -2x^* .$$

Since the left hand side of (3.29) goes to $(\lambda_1 + \lambda_2)^{-1} < 0$ as $x^* \to 0^+$ and goes to $\lambda_1^{-1} > 0$ as $x^* \to \infty$ we see by the intermediate value theorem that there exists $x^* > 0$ satisfying this equation.

With this value of x^* we define C by (3.27) and then we define A by (3.26). Then we have proved the existence of a solution $C > 0$, $A > 0$, $x^* > 0$ of the system (3.26)–(3.28). With this choice of C, A, x^* the function $\varphi(s,x)$ becomes a C^2 function and one can verify that φ satisfies conditions (i), (ii) of Theorem 2.1 (the details are left to the reader). Hence

$$(3.30) \qquad \varphi(s,x) \geq \Phi(s,x) \qquad \text{for all } \; s, x .$$

Moreover, the nonintervention region D given by (2.16) is seen to be

$$D = \{(s,x); 0 < x < x^*\} .$$

Hence by (3.24) we know that (iii) holds.

Moreover, if $x \leq x^*$ it is well-known that the local time $\hat{\gamma}$ at x^* of the downward reflected process $\bar{X}(t)$ at x^* satisfies (iv)–(vi). (See e.g. [LØ1] for more details.) And (vii) follows from (3.25). By Theorem 2.1 b) we conclude that if $x \le x^*$ then $\gamma^* := \hat{\gamma}$ is optimal and $\varphi(s,x) = \Phi(s,x)$. Finally, if $x > x^*$ then it follows by (3.22) that immediate chattering from x down to x^* gives the value $2e^{-\rho s}\big(\sqrt{x} - \sqrt{x^*}\big) + \Phi(s,x^*)$. Hence

$$\Phi(s,x) \ge 2e^{-\rho s}\big(\sqrt{x} - \sqrt{x^*}\big) + \Phi(s,x^*) \qquad \text{for } \ x > x^* \ .$$

Combined with (3.30) this proves taht

$$\varphi(s,x) = \Phi(s,x) \qquad \text{for all } \ s, x$$

and the proof of b) is complete.

Example 3.3. The Brownian motion example is perhaps not so good in biology contexts, since Brownian motion is a poor model for population growth. Instead, let us consider a standard model for a population (in the sense that it can be generated from a classic birth-death-process), like the logistic diffusion considered in [AS]. That is, let us consider the problem

$$\Phi(0,x) = V(x) := \sup_{\gamma\in\Gamma} E^x \int_{[0,T)} e^{-\rho t} X^{-1/2}(t^-) d\gamma(t) \tag{3.31}$$

subject to

$$dX(t) = \mu X(t)(1 - K^{-1}X(t))dt + \sigma X(t)dB(t) - d\gamma(t), \qquad X(0^-) = x > 0 \ , \tag{3.32}$$

where $\mu > 0$, $K^{-1} > 0$, and $\sigma > 0$ are known constants, $B(t)$ denotes a Brownian motion in $\mathbb{R}$, and $T = \inf\{t \ge 0 : X(t) \le 0\}$ denotes the extinction time. We define the mapping $H : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as

$$H(x) = \int_0^x y^{-1/2} dy = 2\sqrt{x} \ . \tag{3.33}$$

The generator A of $X(t)$ is given by

$$A = \tfrac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x(1 - K^{-1}x)\frac{d}{dx}$$

and we find that

$$G(x) := ((A-\rho)H)(x) = \sqrt{x}\left[\mu - 2\rho - \sigma^2/4 - \mu K^{-1}x\right] \ . \tag{3.34}$$

Thus, if $\mu \le 2\rho + \sigma^2/4$ then by the same argument as in Example 3.2 we see that the optimal policy is *immediate chattering down to 0*. We then have $T = 0$, and the value reads as

$$V(x) = 2\sqrt{x} \ . \tag{3.35}$$

However, if $\mu > 2\rho + \sigma^2/4$, then we see that the mapping $G(x)$ satisfies the conditions of Theorem 2 in [A2] and, therefore we find that there is a unique threshold x^* satisfying the condition

$$x^*\psi''(x^*) + \tfrac{1}{2}\psi'(x^*) = 0 \ , \tag{3.36}$$

where $\psi(x)$ denotes the increasing fundamental solution of the ordinary differential equation $((A-\rho)u)(x) = 0$, that is, $\psi(x) = x^\theta M(\theta, 2\theta + \frac{2\mu}{\sigma^2}, \frac{2\mu K^{-1}}{\sigma^2} x)$, where $\theta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\mu}{\sigma^2})^2 + \frac{2r}{\sigma^2}}$, and M denotes the confluent hypergeometric function. In this case, the value reads as

$$V(x) = \begin{cases} 2(\sqrt{x} - \sqrt{x^*}) + \sqrt{x^*}(\mu(1 - K^{-1}x^*) - \sigma^2/4)/r, & x \geq x^* \\ \frac{\psi(x)}{\sqrt{x^*}\psi'(x^*)} , & x < x^*. \end{cases} \tag{3.37}$$

Especially, the value is a solution of the variational inequality

$$\min\{((\rho - A)V)(x), V'(x) - x^{-1/2}\} = 0.$$

We summarize this as follows:

Theorem 3.4. **a)** *Assume that*

$$\mu \leq 2\rho + \sigma^2/4 \; . \tag{3.38}$$

Then the value function $V(x)$ of problem (3.31) is

$$V(x) = 2\sqrt{x} \; . \tag{3.39}$$

This value is obtained by immediate chattering down to 0.

b) *Assume that*

$$\mu > 2\rho + \sigma^2/4 \; . \tag{3.40}$$

Then $V(x)$ is given by (3.37). The corresponding optimal policy is immediate chattering from x down to x^* if $x > x^*$, and local time at x^* of the downward reflected process $\bar{X}(t)$ at x^* if $x < x^*$, where x^* is given by (3.36).

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