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Critical configurations for two projective views, a new approach

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ABSTRACT

This article develops new techniques to classify critical configurations for 3D scene reconstruction from images taken by unknown cameras. Generally, all information can be uniquely recovered if enough images and image points are provided, but there are certain cases where unique recovery is impossible; these are called *critical configurations*. In this paper, we use an algebraic approach to study the critical configurations for two projective cameras. We show that all critical configurations lie on quadric surfaces, and classify exactly which quadrics constitute a critical configuration. The paper also describes the relation between the different reconstructions when unique reconstruction is impossible.

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1. Introduction

In computer vision, one of the main problems is that of *structure from motion*, where given a set of 2-dimensional images the task is to reconstruct a scene in 3-space and find the camera positions in this scene. Over time, many techniques have been developed for solving these problems for varying camera models and under different assumptions on the space and image points Maybank (1993); Hartley and Zisserman (2004). In general, with enough images and enough points in each image, one can uniquely recover all information about the original scene. However, there are also some configurations of points and images where a unique recovery is never possible. These are called *critical configurations*.

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Much work has been done to understand critical configurations for various settings Buchanan (1988); Hartley (2000); Kahl et al. (2001); Hartley and Kahl (2002); Bertolini and Turrini (2007); Hartley and Kahl (2007); Bertolini et al. (2020); Buchanan (1992), with results dating as far back as 1941 Krames (1941). While interesting from a purely theoretical viewpoint, critical configurations also play a part in practical applications. Even though critical configurations are rare in real-life reconstruction problems when enough data is available (due to noise), it has been shown that as the configurations approach the critical ones, reconstruction algorithms tend to become less and less stable Luong and Faugeras (1994); Hartley and Kahl (2007); Bertolini et al. (2007).

Our new techniques confirm the main result of Hartley and Kahl (2007) for two cameras, and provide proofs for many unproven assertions in Hartley and Kahl (2007) (see Section 4). Moreover, they yield explicit maps between different critical configurations that have the same images (see Section 5). We also give a complete description of all critical configurations in the case of a single camera (see Section 3). Our companion paper Brätelund (2021) uses the techniques developed in this article to correct the classification of Hartley and Kahl (2007) for three cameras. We plan to use these new techniques to classify critical configurations for any number of views, as well as using them in other, more complicated scenarios (e.g., cameras observing lines Breiding et al. (2022); Buchanan (1992), or rolling-shutter cameras Albl et al. (2016) in future work.

The main result of this paper is the classification of the critical configurations for two views in Theorem 4.11.

2. Background

We refer the reader to Hartley and Zisserman (2004) for the basics on computer vision and multi-view geometry.

Let \mathbb{C} denote the complex numbers, and let \mathbb{P}^n denote the projective space over the vector space \mathbb{C}^{n+1} . Projection from a point $p \in \mathbb{P}^3$ is a linear map

$$p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2.$$

We refer to such a projection and its projection center p as a *camera* and its *camera center* (following established terminology, we use the words *camera* and *view* interchangeably). Following this theme, we refer to points in \mathbb{P}^3 as *space points* and points in \mathbb{P}^2 as *image points*. Similarly, \mathbb{P}^2 will be referred to as an *image*.

Once a basis is chosen in \mathbb{P}^3 and \mathbb{P}^2 , a camera can be represented by a 3×4 matrix of full rank called the *camera matrix*. The camera center is then given as the kernel of the matrix. For the most part, we make no distinction between a camera and its camera matrix, referring to both simply as cameras. We use the *real projective pinhole camera model*, meaning that we require a camera matrix to be of full rank and to have only real entries.

Remark 2.1. Throughout the paper, whenever we talk about cameras it is to be understood that a choice of basis has been made, both on the images and 3-space.

Since the map P is not defined at the camera center p , it is not a morphism. This problem can be mended by taking the blow-up. Let $\widetilde{\mathbb{P}}^3$ be the blow-up of \mathbb{P}^3 in the camera center of P . We then get the following diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{P}}^3 & \xrightarrow{P} & \mathbb{P}^2 \\ \pi \downarrow & \nearrow P & \\ \mathbb{P}^3 & & \end{array}$$

where π denotes the blow-down of $\widetilde{\mathbb{P}}^3$. This gives a morphism from $\widetilde{\mathbb{P}}^3$ to \mathbb{P}^2 . For ease of notation, we retain the symbol P and the names *camera* and *camera center*, although one should note that in $\widetilde{\mathbb{P}}^3$ the camera center is no longer a point, but an exceptional divisor.

Definition 2.2. Given an n -tuple of cameras $\mathbf{P} = (P_1, \dots, P_n)$, with camera centers p_1, \dots, p_n , let $\widetilde{\mathbb{P}^3}$ denote the blow-up of \mathbb{P}^3 in the n camera centers. We define the *joint camera map* to be the map

$$\begin{aligned} \widetilde{\phi}_{\mathbf{P}} = P_1 \times \dots \times P_n: \widetilde{\mathbb{P}^3} &\rightarrow (\mathbb{P}^2)^n, \\ x &\mapsto (P_1(x), \dots, P_n(x)). \end{aligned}$$

Remark 2.3. Throughout the paper, we assume that all camera centers are distinct.

This again gives a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{P}^3} & \xrightarrow{\widetilde{\phi}_{\mathbf{P}}} & (\mathbb{P}^2)^n \\ \pi \downarrow & \nearrow \phi_{\mathbf{P}} & \\ \mathbb{P}^3 & & \end{array}$$

The reason we use the blow-up $\widetilde{\mathbb{P}^3}$ rather than \mathbb{P}^3 is to turn the cameras (and hence the joint camera map) into morphisms rather than rational maps. This ensures that the image of the joint camera map is Zariski closed, turning it into a projective variety.

Definition 2.4. We denote the image of the joint camera map $\widetilde{\phi}_{\mathbf{P}}$ as the *multi-view variety* of P_1, \dots, P_n . The set of all homogeneous polynomials vanishing on $\text{Im}(\widetilde{\phi}_{\mathbf{P}})$ is an ideal that we denote as the *multi-view ideal*.

Notation. Most works on computer vision do not use the blow-up, defining cameras/the joint camera map as rational maps rather than morphisms. Hence, they tend to define the multi-view variety as the *closure* of the image rather than as the image itself. While our definition of the multi-view variety seems different, it is equivalent to that used in other works, like Agarwal et al. (2021).

Notation. While the multi-view variety is always irreducible, we use the term *variety* to also include reducible algebraic sets.

Definition 2.5. Given a set of points $S \subset (\mathbb{P}^2)^n$, a *reconstruction* of S is an n -tuple of cameras $\mathbf{P} = (P_1, \dots, P_n)$ and a set of points $X \subset \widetilde{\mathbb{P}^3}$ such that $S = \widetilde{\phi}_{\mathbf{P}}(X)$ where $\widetilde{\phi}_{\mathbf{P}}$ is the joint camera map of the cameras P_1, \dots, P_n .

Definition 2.6. Given a configuration of cameras and points (P_1, \dots, P_n, X) , we refer to $\widetilde{\phi}_{\mathbf{P}}(X) \subset (\mathbb{P}^2)^n$ as the *images* of (P_1, \dots, P_n, X) .

Given a set of image points $S \subset (\mathbb{P}^2)^n$ as well as a reconstruction (P_1, \dots, P_n, X) of S , note that any scaling, rotation, translation, or more generally, any real projective transformation of (P_1, \dots, P_n, X) does not change the images, giving rise to a large family of reconstructions of S . However, we are not interested in differentiating between these reconstructions.

Definition 2.7. Given a set of points $S \subset (\mathbb{P}^2)^n$, let (\mathbf{P}, X) and (\mathbf{Q}, Y) be two reconstructions of S , let \overline{X} and \overline{Y} denote the blow-downs of X and Y respectively and let P'_i and Q'_i be the matrix representation of P_i and Q_i respectively. The two reconstructions of S are considered *equivalent* if there exists an element $A \in \text{PGL}(4)$, such that

$$\begin{aligned} A(\overline{X}) &= \overline{Y}, \\ P'_i A^{-1} &= Q'_i, \quad \forall i. \end{aligned}$$

From now on, whenever we talk about a configuration of cameras and points, it is to be understood as unique up to such an isomorphism/action of $\text{PGL}(4)$, and two configurations are considered different only if they are not isomorphic/do not lie in the same orbit under this group action. As such, we consider a reconstruction to be unique if it is unique up to action by $\text{PGL}(4)$.

Definition 2.8. Given a configuration of cameras and points (P_1, \dots, P_n, X) , a *conjugate configuration* is a configuration (Q_1, \dots, Q_n, Y) , nonequivalent to the first, such that $\tilde{\phi}_{\mathbf{P}}(X) = \tilde{\phi}_{\mathbf{Q}}(Y)$. Pairs of points $(x, y) \in X \times Y$ are called *conjugate points* if $\tilde{\phi}_{\mathbf{P}}(x) = \tilde{\phi}_{\mathbf{Q}}(y)$.

Definition 2.9. A configuration of cameras (P_1, \dots, P_n) and points $X \subset \mathbb{P}^3$ is said to be a *critical configuration* if it has at least one conjugate configuration. A critical configuration (P_1, \dots, P_n, X) is said to be *maximal* if there exists no critical configuration (P_1, \dots, P_n, X') such that $X \subsetneq X'$.

Hence, a configuration is critical if and only if the images it produces do not have a unique reconstruction.

Remark 2.10. Various definitions of critical configurations exist. For instance, Krames (1941) considers the cone with two cameras on the same generator to be critical, while it fails to be critical by our definition. We use a definition similar to the one in Hartley and Kahl (2007), except we are working in \mathbb{P}^3 . If one considers the blow-down, our definition matches the results in Hartley and Kahl (2007).

Definition 2.11. Let \mathbf{P} and \mathbf{Q} be two n -tuples of cameras, let $\tilde{\mathbb{P}}^3_{\mathbf{P}}$ and $\tilde{\mathbb{P}}^3_{\mathbf{Q}}$ denote the blow-up of \mathbb{P}^3 in the camera centers of \mathbf{P} and \mathbf{Q} respectively. Projecting the fiber product

$$\tilde{\mathbb{P}}^3_{\mathbf{P}} \times_{\mathbb{P}^2} \tilde{\mathbb{P}}^3_{\mathbf{Q}} = \left\{ (x, y) \in \tilde{\mathbb{P}}^3_{\mathbf{P}} \times \tilde{\mathbb{P}}^3_{\mathbf{Q}} \mid \tilde{\phi}_{\mathbf{P}}(x) = \tilde{\phi}_{\mathbf{Q}}(y) \right\}$$

to the first coordinate gives a variety, X . We call X the *set of critical points of \mathbf{P} with respect to \mathbf{Q}* .

This definition is motivated by the following fact:

Proposition 2.12. Let \mathbf{P} and \mathbf{Q} be two (different) n -tuples of cameras, and let X and Y be their respective sets of critical points. Then (\mathbf{P}, X) is a critical configuration, with (\mathbf{Q}, Y) as its conjugate. Furthermore, (\mathbf{P}, X) is maximal with respect to \mathbf{Q} in the sense that if there exists a critical configuration (\mathbf{P}, X') with $X \subsetneq X'$ then its conjugate consists of cameras different from \mathbf{Q} .

Proof. It follows from Definition 2.11 that for each point $x \in X$, we have a conjugate point $y \in Y$. Hence the two configurations have the same images, so they are both critical configurations, conjugate to one another.

The (partial) maximality follows from the fact that if we add a point x_0 to X that does not lie in the set of critical points, there is (by Definition 2.11) no point $y_0 \in \tilde{\mathbb{P}}^3_{\mathbf{Q}}$ such that $\tilde{\phi}_{\mathbf{P}}(x_0) = \tilde{\phi}_{\mathbf{Q}}(y_0)$. Hence (\mathbf{P}, X) will no longer be critical. \square

For two different pairs of cameras, the sets of critical points turn out to be the quadric surfaces $\tilde{S}_{\mathbf{P}}, \tilde{S}_{\mathbf{Q}}$ described in Section 4.

The goal of this paper is to classify all maximal critical configurations for three cameras. The reason we focus primarily on the maximal ones is that every critical configuration is contained in a maximal one and (when working with more than one camera) the converse is true as well, any subconfiguration of a critical configuration is itself critical.

We conclude this section with a final, useful property of critical configurations, namely that the only property of the cameras we need to consider when exploring critical configurations is the position of their camera centers (i.e. change of coordinates in the images does not affect criticality).

Proposition 2.13 (Hartley and Kahl, 2007, Proposition 3.7). Let (P_1, \dots, P_n) be n cameras with centers p_1, \dots, p_n , and let (P_1, \dots, P_n, X) be a critical configuration.

If (P'_1, \dots, P'_n) is a set of cameras sharing the same camera centers, the configuration (P'_1, \dots, P'_n, X) is critical as well.

Proof. Since P_i and P'_i share the same camera center and the camera center determines the map uniquely up to a choice of coordinates, there exists some $H_i \in \text{PGL}(3)$ such that $P'_i = H_i P_i$. Let (Q_1, \dots, Q_n, Y) be a conjugate to (P_1, \dots, P_n, X) . Then $(H_1 Q_1, \dots, H_n Q_n, Y)$ is a conjugate to $(H_1 P_1, \dots, H_n P_n, X) = (P'_1, \dots, P'_n, X)$, so this configuration is critical as well. \square

3. The one-view case

Reconstruction of a 3D-scene from the image of one projective camera is generally considered impossible, so most papers start with the two-view case. Still, for the sake of completeness, we give a summary of the critical configurations for one camera.

Let P be a camera, and let p be its camera center, we then have the joint camera map:

$$\widetilde{\phi}_P: \widetilde{\mathbb{P}^3} \rightarrow \mathbb{P}^2$$

For any point $x \in \mathbb{P}^2$, the fiber over x is a line through p , so no point can be uniquely recovered. From this, one might assume that every configuration with one camera is critical. However, this is only the case if our configuration consists of sufficiently many points.

Theorem 3.1. *A configuration of one point and one camera is never critical. A configuration of one camera and $n > 1$ points is critical if and only if the camera center along with the n points span a space of dimension less than n .*

Proof. For the first part, note that up to a projective transformation, there exists only one configuration of one point and one camera. In other words, any configuration of one point and one camera can be taken to any other such configuration by simply changing coordinates. By Definition 2.7 this makes them equivalent, which means that only one reconstruction exists.

The same turns out to be the case if the configuration is such that the camera center along with the $n > 1$ points span a space of dimension n . In \mathbb{P}^3 , one can never span a space of dimension greater than 3, so this implies that $n \leq 3$. Furthermore, if the n points along with the camera center span a space of dimension n , then the points and camera center lie in general position. However, for $n \leq 4$ points (fewer than 3 points + one camera center) there exists only one configuration (up to action with $\text{PGL}(4)$) where all points are in general position. This means (by Definition 2.7) that all reconstructions are equivalent.

Now it only remains to show that a configuration is critical if the camera center along with the $n > 1$ points span a space of dimension less than n . Indeed, if the points along with the camera center span a space of dimension less than n , the image points span a space of dimension $m < n - 1$. Then there are at least two nonequivalent reconstructions; one reconstruction where the points span a space of dimension m not containing the camera center and one where they span a space of dimension $m + 1$ which contains the camera center. \square

4. The two-view case

4.1. The multi-view ideal

We start the study of the case of two cameras P_1 and P_2 by understanding their multi-view variety. We assume, here and throughout the rest of the paper, that all cameras have distinct centers. The two cameras define the joint camera map:

$$\widetilde{\phi}_P: \widetilde{\mathbb{P}^3} \rightarrow \mathbb{P}^2 \times \mathbb{P}^2.$$

Proposition 4.1. For two cameras (P_1, P_2) , the joint camera map $\widetilde{\phi}_P$ takes the line spanned by the two camera centers to a point, and is an embedding everywhere else.

Proof. For $x = (x_1, x_2) \in \mathbb{P}^2 \times \mathbb{P}^2$, the preimage of x is given by

$$\widetilde{\phi}_P^{-1}(x) = l_1 \cap l_2,$$

where l_i is the line $P_i^{-1}(x_i)$. The line l_i passes through the camera center p_i , so the intersection of the two lines is a single point unless they are both equal to the line spanned by the camera centers. \square

This means that the multi-view variety $\text{Im}(\widetilde{\phi}_P)$ is an irreducible singular 3-fold in $\mathbb{P}^2 \times \mathbb{P}^2$. It is described by a single bilinear polynomial F_P , which we call *the fundamental form* of P_1, P_2 .

The fundamental form is well-studied in the literature and is often represented by a 3×3 matrix of rank 2 called the *fundamental matrix*. See (Hartley and Zisserman, 2004, section 9.2) for a geometric construction of the fundamental matrix. We use the construction in Agarwal et al. (2021) where the fundamental form is given as the determinant of a 6×6 matrix.

4.2. The multi-view variety

Proposition 4.2 (*The fundamental form*). (Hartley and Zisserman, 2004, Sections 9.2 and 17.1) For two cameras P_1, P_2 , the multi-view variety $\text{Im}(\widetilde{\phi}_P) \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the vanishing locus of a single, bilinear, rank 2 form F_P , called the *fundamental form* (or *fundamental matrix*).

$$F_P(\mathbf{x}, \mathbf{y}) = \det \begin{bmatrix} P_i & \mathbf{x} & \mathbf{0} \\ P_j & \mathbf{0} & \mathbf{y} \end{bmatrix},$$

where \mathbf{x} and \mathbf{y} are the variables in the first and second image respectively.

Proof. By Proposition 4.1, the multi-view variety for two cameras is an irreducible, 3-fold. It follows that the multi-view ideal is generated by a single polynomial. Let (\mathbf{x}, \mathbf{y}) be a generic point in the multi-view variety, then there exists a point $\mathbf{X} \in \mathbb{P}^3$ such that $P_1(\mathbf{X}) = \lambda_1 \mathbf{x}$ and $P_2(\mathbf{X}) = \lambda_2 \mathbf{y}$, then

$$\begin{bmatrix} P_i & \mathbf{x} & \mathbf{0} \\ P_j & \mathbf{0} & \mathbf{y} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ -\lambda_1 \\ -\lambda_2 \end{bmatrix} = 0.$$

Since the matrix has a non-zero kernel, the determinant F_P has to vanish on (\mathbf{x}, \mathbf{y}) . Now we need only show that the determinant is irreducible to prove that $F_P(\mathbf{x}, \mathbf{y})$ generates the multi-view ideal. Irreducibility follows from the fact that the polynomial is of rank 2 (a reducible polynomial is always of rank 1) which in turn follows from the fact that F_P satisfies

$$F_P(e_{P_1}^2, -) = F_P(-, e_{P_2}^1) = 0, \tag{1}$$

where

$$e_{P_i}^j = P_i(p_j). \quad \square \tag{2}$$

Remark 4.3. Recall that the entries in the camera matrix are real, this means that the fundamental form always has real coefficients.

Definition 4.4. The *epipoles* $e_{P_i}^j$ are the image points we get by mapping the j -th camera center to the i -th image

$$e_{P_i}^j = P_i(p_j).$$

The fundamental form satisfies

$$F_P(e_{P_1}^2, -) = F_P(-, e_{P_2}^1) = 0,$$

in other words, it vanishes in either epipole. This means that the fundamental form is of rank 2 (also follows from $\text{Im}(\tilde{\phi}_P)$ being singular), so for each pair of cameras, we get a bilinear form of rank 2. The following result states that the converse is also true, i.e. that any bilinear form of rank 2 is the fundamental form for some pair of cameras.

Theorem 4.5. *There is a 1 : 1 correspondence between bilinear forms of rank two, and pairs of cameras P_1, P_2 (up to action by $\text{PGL}(4)$).*

Proof. By Proposition 4.2, the fundamental form of two cameras is a real bilinear form of rank 2. The converse follows from Theorem 9.13. in Hartley and Zisserman (2004). \square

With these results, we can move on to classifying all the critical configurations for two views. We start with a special type of critical configuration.

4.3. Trivial critical configurations

Definition 4.6. A configuration (P_1, P_2, X) is said to be a *non-trivial critical configuration* if it has a conjugate configuration (Q_1, Q_2, Y) satisfying

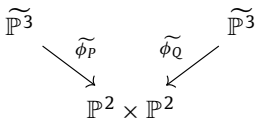
$$F_P \neq F_Q.$$

Critical configurations not satisfying this property exist, they are called trivial. If (P_1, P_2, X) is a trivial critical configuration, then all its conjugates (Q_1, Q_2, Y) have the same fundamental form as the cameras P_1, P_2 . By Theorem 4.5 this means that, after a change of coordinates, $Q_1 = P_1$ and $Q_2 = P_2$. Since the cameras are the same, Proposition 4.1 tells us that the sets X and Y are equal, with the exception of any point lying on the line spanned by the two camera centers. It is a well-known fact that no number of cameras can differentiate between points lying on a line containing all the camera centers, hence the name “trivial”.

The focus of this paper is the non-trivial critical configurations. A classification of the trivial critical configurations for any number of views can be found in Hartley and Kahl (2007) Section 4, or in Hartley and Zisserman (2004) Chapter 22.

4.4. Critical configurations for two views

Let us consider a non-trivial critical configuration (P_1, P_2, X) . Since it is critical, there exists a conjugate configuration (Q_1, Q_2, Y) giving the same images in $\mathbb{P}^2 \times \mathbb{P}^2$. The two sets of cameras define two joint-camera maps $\tilde{\phi}_P$ and $\tilde{\phi}_Q$.



Now, since the configuration is critical, we have that $\tilde{\phi}_P(X) = \tilde{\phi}_Q(Y)$. As such, the two sets X and Y both map (with their respective maps) into the intersection of the two multi-view varieties $\text{Im}(\tilde{\phi}_P) \cap \text{Im}(\tilde{\phi}_Q)$. Taking the preimage of this intersection under $\tilde{\phi}_P$, we get a variety S in $\widetilde{\mathbb{P}^3}$ which needs to contain all the points in X . Moreover, if (P_1, P_2, X) is maximal, then $X = S$. As such, classifying all non-trivial maximal critical configurations can be done by classifying all possible intersections between two multi-view varieties, and then examining what these intersections pull back to in $\widetilde{\mathbb{P}^3}$.

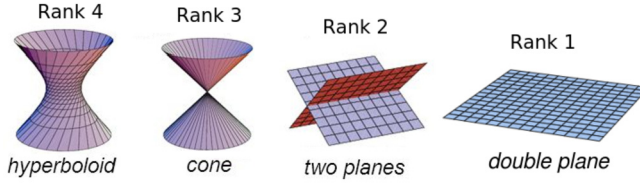


Fig. 1. All types of real ruled quadrics.

This is made even simpler by the fact that the pullback of $\text{Im}(\tilde{\phi}_P) \cap \text{Im}(\tilde{\phi}_Q)$ is just the variety we get by pulling back the fundamental form F_Q (the fundamental form F_P pulls back to the zero polynomial).

The pullback of a bilinear form describes the strict (or proper) transform of a quadric surface¹. It follows that X and Y lie on the strict transform of two quadric surfaces (quadrics) which we denote by S_P and S_Q respectively. Let \tilde{S}_P and \tilde{S}_Q denote their strict transforms. These quadrics are given by the following equations:

$$\begin{aligned} S_P(x) &= F_Q(P_1(x), P_2(x)), \\ S_Q(x) &= F_P(Q_1(x), Q_2(x)). \end{aligned} \tag{3}$$

The two quadrics have the following properties:

Lemma 4.7 (Lemma 5.10 in Hartley and Kahl (2007)).

1. The quadric S_P contains the camera centers p_1, p_2 .
2. The quadric S_P is ruled (contains real lines).

Proof. The first item follows from the nature of the pullback, whereas the second is because we are working with forms of rank 2. Detailed proof is given in Hartley and Kahl (2007). □

There are only four quadrics containing lines (up to choice of coordinates), these are illustrated in Fig. 1.

The discussion so far can be summarized as follows:

Theorem 4.8 (Lemma 5.10 in Hartley and Kahl (2007)). Let (P_1, P_2, X) be a non-trivial critical configuration. Then there exists a ruled quadric S_P passing through the camera centers p_1, p_2 , whose strict transform contains the points X .

Theorem 4.8 tells us that all non-trivial critical configurations have their points and camera centers lying on a ruled quadric. The converse, however, is not always true. For instance, we will soon see that cameras and camera centers all lying on a cone is not critical if the cone contains the line spanned by the two camera centers (see Fig. 2). Let us instead give a partial converse:

Lemma 4.9. Let (P_1, P_2, X) be a configuration of cameras and points such that X is contained in the strict transform \tilde{S}_P of a ruled quadric S_P that passes through both the camera centers. Then for each real bilinear form F_Q of rank 2 such that:

$$S_P(x) = F_Q(P_1(x), P_2(x)),$$

there exists a conjugate configuration to (P_1, P_2, X) .

¹ This is a surface over the complex numbers. The real points on this surface will generally also form a surface, but we will later see that there is one exception, namely when the surface is the union of two complex conjugate planes, and only their line of intersection is real

Proof. Assume such a bilinear form F_Q exists. By Theorem 4.5 there exists a pair of cameras (Q_1, Q_2) such that F_Q is their fundamental form. Since X lies on \widetilde{S}_p , we have

$$\widetilde{\phi}_P(X) \subset \text{Im}(\widetilde{\phi}_Q).$$

Then for every point $x \in X$ we can find a point y such that

$$\widetilde{\phi}_P(x) = \widetilde{\phi}_Q(y).$$

Let Y be the set of these points y . Then (Q_1, Q_2, Y) is a conjugate to (P_1, P_2, X) . This can be repeated for each bilinear form of rank 2, giving unique conjugate configurations. \square

The problem of determining which configurations are critical is now reduced to finding out which quadrics are the pullbacks of real bilinear forms of rank 2.

Let \mathbf{F} denote the space of bilinear forms on $\mathbb{P}^2 \times \mathbb{P}^2$. Since all such forms can be represented by a 3×3 matrix (up to scaling) we have that \mathbf{F} is isomorphic to \mathbb{P}^8 . The fundamental form F_p of the pair (P_1, P_2) is an element in \mathbf{F} .

Lemma 4.10. *There is a 1 : 1 correspondence between the set of real quadrics in \mathbb{P}^3 passing through p_1, p_2 , and the set of real lines in \mathbf{F} passing through F_p .*

Proof. Let $L \subset \mathbf{F}$ be a line through F_p , and let $F_0 \neq F_p$ be some point on L . Every point $F \in L$ can be written as $\alpha F_0 + \beta F_p$ for some $[\alpha : \beta] \in \mathbb{P}^1$. But then we have for $F \neq F_p$:

$$\begin{aligned} F(P_1(x), P_2(x)) &= \alpha F_0(P_1(x), P_2(x)) + \beta F_p(P_1(x), P_2(x)), \\ &= \alpha F_0(P_1(x), P_2(x)) + \beta \cdot 0, \\ &= F_0(P_1(x), P_2(x)). \end{aligned}$$

Hence, the equation $F(P_1(x), P_2(x))$ describes the same quadric for all points $F \neq F_p$ on L .

Next, a quadric S passing through p_1 and p_2 is fixed by a set of 7 points $\{x_1, \dots, x_7\}$ on S in generic position. Demanding that a bilinear form pulls back to a quadric passing through a specific point is one linear constraint in \mathbf{F} . So with seven generic points, there is exactly one line L through F_p such that the forms on this line pull back to S . \square

Using this 1 : 1 correspondence and Lemma 4.9, the problem has been reduced to determining which quadrics correspond to lines in \mathbf{F} containing at least one viable form of rank 2. Let \mathbf{F}_2 denote the Zariski closure of the rank 2 locus. Since \mathbf{F}_2 is a hypersurface of degree 3, a generic line L will contain two forms of rank 2 in addition to F_p . There are also other possibilities, listed in the tables below (the underlying computations for these tables can be found in the appendix).

We start with the cases where the line L corresponding to S_p has a finite number of intersections with the rank 2 locus \mathbf{F}_2 .

Intersection points	S_p
All three intersection points are distinct real points	A smooth quadric, cameras not on a line
Two intersection points F_p and F_Q , $L \cap \mathbf{F}_2$ has multiplicity 2 at F_p	A cone, two cameras not on a line, neither camera on a vertex
Two intersection points F_p and F_Q , $L \cap \mathbf{F}_2$ has multiplicity 2 at F_Q	A smooth quadric, cameras lie on a line

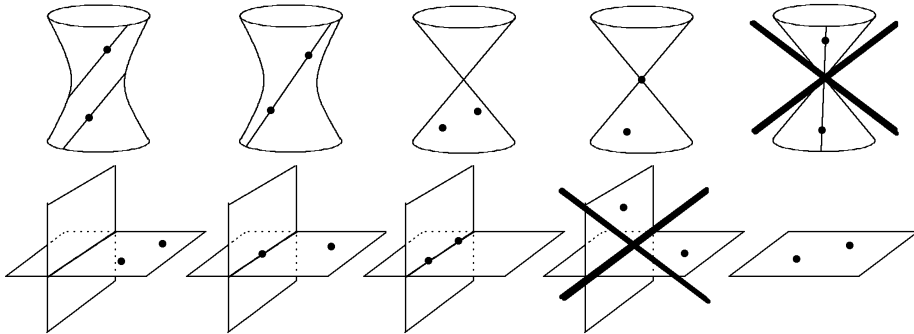


Fig. 2. Illustration of the blow-downs of the non-trivial critical configurations for two views. The two marked with crosses are not critical.

These are the cases where we have at least one real rank 2 form $F \in L$ different from F_p . There are, however, some cases where there are no viable forms:

Intersection points	S_p
The two other intersection points are complex conjugates	A smooth non-ruled quadric
The two other intersection points are of rank 1	Union of two planes, cameras in different planes
The two intersection points are equal to F_p	A cone, both cameras on a line, neither camera at the vertex

In the case where L is contained in \mathbf{F}_2 , all the forms on L are of rank 2 (with the possible exception of at most 2 that can be of rank 1). As such, rather than looking at where the intersections are, we look at the epipoles of the forms in L :

Epipoles	S_p
All forms have different epipoles	Two planes, cameras lying in same plane
All forms share the same right epipole, the left epipoles trace a line ^a	Two planes, one camera on the intersection of the planes
All forms share the same right epipole, the left epipoles trace a conic ^a	Cone, one camera at the vertex
All forms share the same right and left epipole	Two (possibly complex) planes, both cameras lying on the intersection of the planes
All forms share the same right and left epipole AND the two rank one forms on L coincide	Double plane (as a set it is equal to a plane, but every point has multiplicity 2)

^a The statement also holds if we swap “left” and “right”.

With this, we have a classification of all maximal critical configurations for two views:

Theorem 4.11. *A configuration (P_1, P_2, X) is non-trivially critical if and only if there exists a real quadric S_p containing the camera centers p_1, p_2 and whose strict transform contains X , and S_p is one of the quadrics in Table 1. (Illustrated in Fig. 2).*

Recall that by Definition 2.7, we required two conjugate configurations to not be projectively equivalent. Yet by Theorem 4.11, most critical configurations have conjugates that are of the same type. Now, while there is indeed some $A \in \text{PGL}(4)$ taking any smooth quadric S_p to any other smooth quadric S_q , we will soon see that the map taking a point in S_p to its conjugate on S_q certainly does not lie in $\text{PGL}(4)$. In the final section, we give a description of the map taking a point to its conjugate, to make it clear that it is not a projective transformation.

Table 1

A list of all possible non-trivial critical configurations and their conjugates, as well as the number of conjugates for each configuration.

Quadric S_P	Conjugate quadric	Conjugates
Smooth quadric, cameras not on a line	Same	2
Smooth quadric, cameras on a line	Cone, cameras not on a line	1
Cone, cameras not on a line	Smooth quadric, cameras on a line	1
Cone, one camera at vertex, other one not	Same	∞
Two planes, cameras in the same plane	Same	∞
Two planes, one camera on the singular line	Same	∞
Two planes, cameras on the singular line	Same	∞
A double plane, cameras in the plane	Same	∞

5. Maps between quadrics

5.1. Epipolar lines

Before we can describe the map taking a point to its conjugate, we need to point out a certain pair of lines on S_P . Given two pairs of cameras (P_1, P_2) and (Q_1, Q_2) , let \widetilde{S}_P be the pullback of F_Q using $\widetilde{\phi}_P$, and define

$$\widetilde{g}_{P_1}^2 = P_1^{-1}(e_{Q_1}^2) = P_1^{-1}(Q_1(q_2)), \tag{4}$$

$$\widetilde{g}_{P_2}^1 = P_2^{-1}(e_{Q_2}^1) = P_2^{-1}(Q_2(q_1)). \tag{5}$$

The blowdown of $\widetilde{g}_{P_1}^2$ and $\widetilde{g}_{P_2}^1$ are two lines on S_P , we denote them by $g_{P_i}^j$. Since they are the pullback of the epipoles from the other set of cameras, so we call them *epipolar lines*.

Epipolar lines are key in understanding the relation between points on \widetilde{S}_P and points on its conjugate \widetilde{S}_Q . They also play an important role in the study of critical configurations for more than 2 cameras, so let us give a brief analysis of these lines.

Lemma 5.1 (Lemma 5.10, Definition 5.11 in Hartley and Kahl (2007)).

1. The line $g_{P_i}^j$ lies on S_P and passes through p_i .
2. Any point lying on both $g_{P_i}^j$ and $g_{P_j}^i$ is a singular point on S_P .
3. Any point in the singular locus of S_P that lies on one of the lines also lies on the other.
4. If S_P is the union of two planes, $g_{P_i}^j$ and $g_{P_j}^i$ lie in the same plane.

The first two properties are taken from Lemma 5.10 in Hartley and Kahl (2007), the last two are neither stated nor proven in the paper. Nevertheless, the authors seem to have been aware of all four properties.

Proof.

1-2. See proof of Lemma 5.10 in Hartley and Kahl (2007).

3. For ease of reading, we use matrix notation. As such, S_P , F_Q , and P_i are represented by matrices of dimensions 4×4 , 3×3 and 3×4 respectively. In particular, S_P is represented by the symmetric matrix $P_1^T F_Q P_2 + P_2^T F_Q^T P_1$.

If $x_0 \in g_{P_1}^2$ lies in the singular locus of S_P , then we have

$$\begin{aligned} S_P x_0 &= (P_1^T F_Q P_2 + P_2^T F_Q^T P_1) x_0, \\ &= P_1^T F_Q P_2 x_0 + P_2^T F_Q^T e_{Q_1}^2, \\ &= P_1^T F_Q P_2 x_0. \end{aligned}$$

However, since x_0 lies in the singular locus of S_P , this expression is equal to zero. Since both the camera matrices are of full rank, the only way we can get zero is if $x_0 = p_2$ or if $P_2x_0 = e^1_{Q_2}$. In either case, it follows that x_0 lies on $g^1_{P_2}$ as well.

4. Assume there exist cameras P_1, P_2, Q_1, Q_2 such that S_P is the union of two planes and the epipolar lines $g^j_{P_i}$ lie in different planes. When the quadric S_P consists of two planes, one of the planes, which we denote by Π , will (by Theorem 4.11) contain both camera centers. As such, the only way that the epipolar lines can lie in different planes is if one of the camera centers, say p_2 , lies on the intersection of the two planes and the other does not (if both lie on the intersection, then by 3., the epipolar lines must both be equal to the intersection of the two planes). Recall that the quadric S_P and its conjugate \widetilde{S}_Q (also two planes) are both pullbacks of the surface $\text{Im}(\widetilde{\phi}_P) \cap \text{Im}(\widetilde{\phi}_Q) \subset \mathbb{P}^2 \times \mathbb{P}^2$. The map $\widetilde{\phi}_P$ takes the plane Π to the product of two lines in $\mathbb{P}^2 \times \mathbb{P}^2$. The line in the first image passes through the epipole $e^2_{Q_1}$, whereas the line in the second image does not pass through the epipole $e^1_{Q_2}$ (this is because Π contains one of the epipolar lines but not the other). The problem is now that neither of the planes on S_Q can map to the product of these two lines, since any such plane would have to be both
 - (a) a plane passing through q_2 but not through q_1 (because in the second image, the line does not pass through $e^1_{Q_2}$) and
 - (b) a plane passing through both q_1 and q_2 (because the plane maps to a line in both images), which gives us a contradiction. It follows that there are no P_1, P_2, Q_1, Q_2 such that S_P is the union of two planes and the epipolar lines $g^j_{P_i}$ lie in different planes, so whenever S_P is the union of two planes, the epipolar lines lie in the same plane. \square

Definition 5.2. Let S_P be a quadric surface and let p_1, p_2 be two distinct points on S_P . A pair of lines $g^2_{P_1}, g^1_{P_2}$ is called *permissible* if $g^2_{P_1}, g^1_{P_2}$ satisfy the four conditions in Lemma 5.1.

Proposition 5.3. Let P_1, P_2 be two cameras, and let S_P be a quadric passing through their camera centers. The configuration (P_1, P_2, S_P) is critical if and only if S_P contains a permissible pair of lines. Furthermore, if p_1, p_2 do not both lie in the singular locus of S_P , there is a 1:1 correspondence between permissible pairs of lines and configurations conjugate to (P_1, P_2, S_P) .

Proof. The first part can be proven by comparing the quadrics in Table 1 to the set of quadrics containing the required lines, and noting that they are the same. We leave this to the reader.

The second part is immediate for the three cases where there is a finite number of conjugates since the number of conjugates is equal to the number of pairs of epipolar lines and each conjugate comes with its unique choice of lines. For the remaining quadrics S_P (cone and two planes), there exists a pencil of fundamental forms, where each $F_Q \neq F_P$ satisfies

$$S_P(x) = F_Q(P_1(x), P_2(x)).$$

By the tables on pages 11 and 12, we get that each form in the pencil yields a different pair of lines as long as the camera centers do not both lie in the singular locus of S_P . For $F_Q \neq F_P$, the lines are permissible, while for F_P , the lines coincide (not permissible). Note also that on each of these quadrics, the set of permissible lines forms a set whose Zariski closure is also a pencil (the pair where the two lines coincide lie in the closure, but the pair itself is not permissible). We now have a map from the pencil of fundamental forms to the pencil of permissible lines. Since the map is not constant, it needs to be surjective, and since no two fundamental forms in the pencil share the same left and right kernel, it is also injective. Hence there is a 1:1 correspondence between permissible pairs of lines and configurations conjugate to (P_1, P_2, S_P) . \square

5.2. Maps between quadrics

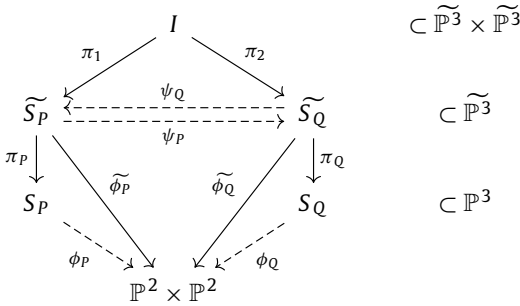
Let us now have a closer look at the map ψ taking a point to its conjugate. Given two pairs of cameras P_1, P_2 and Q_1, Q_2 with camera centers p_1, p_2 and q_1, q_2 respectively, let the quadrics S_P and S_Q and the epipolar lines g be defined as before (Equations (3), (4), and (5)). Let

$$\pi_P : \widetilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^3$$

be the blow-up of \mathbb{P}^3 in the two camera centers p_1, p_2 and let \widetilde{S}_P denote the strict transform of S_P , and similarly for Q . Define the incidence variety:

$$I = \{(x, y) \in \widetilde{S}_P \times \widetilde{S}_Q \mid \widetilde{\phi}_P(x) = \widetilde{\phi}_Q(y)\},$$

where ϕ is the joint-camera map. If we can understand I , we will know the exact relation between points on one quadric and the other. We have the following the commutative diagram:



We study I by studying the fibers of the projection map π_1 . For any point $x \in \widetilde{S}_P$, we have

$$\pi_1^{-1}(x) = l_1 \cap l_2,$$

where l_i is the line $Q_i^{-1}(P_i(x))$. We will rarely refer to this formula explicitly, but it is the foundation for the analysis of the fibers.

Lemma 5.4.

1. If \widetilde{S}_P is smooth, the map π_1 is an isomorphism.
2. The map ψ_P taking a point $x \in \widetilde{S}_P$ to its conjugate is a birational map, defined everywhere except the intersection $g_{p_1}^2 \cap g_{p_2}^1$.
3. The quadric S_P is singular if and only if the quadric S_Q contains the line spanned by the camera centers q_1, q_2 .
4. For any point $x \in \widetilde{g}_{p_1}^2$ or $\widetilde{g}_{p_2}^1$ (but not both), the conjugate $\psi_P(x)$ is a point on the exceptional divisor we get when blowing up q_2 or q_1 respectively. In particular, on the blow-down $S_P \subset \mathbb{P}^3$, the conjugate to any point $x \in g_{p_1}^2$ or $x \in g_{p_2}^1$ (apart from the camera centers themselves) is the camera center q_2 or q_1 respectively.

Proof.

1. As long as x does not lie in $\widetilde{g}_{p_1}^2 \cap \widetilde{g}_{p_2}^1$, the two lines l_i do not coincide, so the fiber $\pi_1^{-1}(x)$ consists of a single point. By Lemma 5.1, any point lying on this intersection is a singular point on S_P , so if S_P is smooth, then $\widetilde{g}_{p_1}^2$ and $\widetilde{g}_{p_2}^1$ do not intersect. Then every fiber is a singleton, meaning that π_1 is injective. Furthermore, for each point $x \in \widetilde{S}_P$ there is at least one point $y \in \widetilde{S}_Q$ such that $\widetilde{\phi}_P(x) = \widetilde{\phi}_Q(y)$, so π_1 is surjective as well.
2. As mentioned above, $\pi_1^{-1}(x)$ is a singleton if x does not lie in $\widetilde{g}_{p_1}^2 \cap \widetilde{g}_{p_2}^1$. If it does, on the other hand, the fiber is a line $L \in I$ such that $\pi_2(L)$ is the strict transform of the line spanned by the camera centers q_1, q_2 , so in this case the conjugate is not unique. Since the same is true for the map taking a point $y \in \widetilde{S}_Q$ to its conjugate, the map is birational.
3. If S_P is singular, the epipolar line $g_{p_1}^2$ passes through some singular point $x \in \widetilde{S}_P$ (as do all other lines, see Fig. 1). By Lemma 5.1, the other epipolar line $g_{p_2}^1$ passes through the same point. Since

the two epipolar lines intersect, the quadric \widetilde{S}_Q must contain the line spanned by the camera centers.

Conversely, if S_Q contains the line spanned by the camera centers, the conjugate to any point on the strict transform of this line is the intersection $\widetilde{g}_{P_1}^2 \cap \widetilde{g}_{P_2}^1$ on \widetilde{S}_P . By Lemma 5.1, the blow-down of these points are always singular points on S_P .

- Let x be a point lying on the epipolar line $\widetilde{g}_{P_1}^2$, but not on $\widetilde{g}_{P_2}^1$. We have $P_1(x) = e_{Q_1}^2$ and $P_2(x) \neq e_{Q_2}^1$. The conjugate to x is then

$$\psi(x) = l_1 \cap l_2 = Q_1^{-1}(P_1(x)) \cap Q_2^{-1}(P_2(x)).$$

The line l_1 is here the line spanned by the camera centers q_1, q_2 , while l_2 is a different line, passing through q_2 . Their intersection must then be the camera center q_2 . The proof for the other epipolar line is the same. \square

Remark 5.5. When the two pairs cameras P_1, P_2 and Q_1, Q_2 are known, one can explicitly compute the map ψ_P as a rational function. For some representatives of a general point $X \in S_P$ and its conjugate $\psi_P(X) \in S_Q$, we have

$$\underbrace{\begin{pmatrix} Q_1 & P_1 X & 0 \\ Q_2 & 0 & P_2 X \end{pmatrix}}_A \cdot \begin{pmatrix} \psi_Q(X) \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} P_1 & Q_1 \psi_Q(X) & 0 \\ P_2 & 0 & Q_2 \psi_Q(X) \end{pmatrix} \cdot \begin{pmatrix} X \\ 1 \\ 1 \end{pmatrix} = 0.$$

Using the adjugate of A , we can get an expression for ϕ_Q as a rational function in the homogeneous coordinates of X .

Let us now give two results on how ϕ_P acts on the curves on S_P :

Definition 5.6. Let S_P be a smooth quadric or cone and let C_P be a curve on S_P . We say that C_P is of type (a, b, c_1, c_2) , where c_i is the multiplicity of C_P in the camera center p_i and where a and b is:

- If S_P is smooth, a is the number of times C_P intersects a generic line in the same family as the epipolar lines $g_{P_i}^j$, and b is the number of times it intersects the lines in the other family, (meaning (a, b) is the bidegree of C_P).
- If S_P is a cone, a is the number of times C_P intersects each line outside of the vertex, and b the number of times it intersects each line.

Proposition 5.7 (Hartley and Kahl, 2007, Lemma 8.32). *Let S_P be a smooth quadric or a cone, and let $C_P \subset S_P$ be a curve of type (a, b, c_1, c_2) , such that C_P does not contain either of the epipolar lines. Then the conjugate curve $C_Q \subset S_Q$ is of type $(a, a + b - c_1 - c_2, a - c_2, a - c_1)$.*

An illustration of how ψ_P acts on the lines on a quadric/cone can be found in Figs. 3 and 4.

Definition 5.8. Let S_P be two planes with two camera centers not both lying on the intersection and let C_P be a curve on S_P . We say that C_P is of type (a, b, c_0, c_1, c_2) , where a is the degree of the curve in the plane with the epipolar lines, b the degree of the curve in the other plane, c_0 is the multiplicity of C_P in the intersection of the epipolar lines $g_{P_i}^j$ and c_1, c_2 is the multiplicity in the camera centers p_1, p_2 respectively.

Proposition 5.9. *Let S_P be two planes with two camera centers not both lying on the intersection, and let $C_P \subset S_P$ be a curve of type (a, b, c_0, c_1, c_2) such that C_P does not contain either of the epipolar lines or the line spanned by the camera centers. Then the conjugate curve $C_Q \subset S_Q$ is of type $(2a - c_0 - c_1 - c_2, b, a - c_1 - c_2, a - c_0 - c_2, a - c_0 - c_1)$.*

The goal of the remainder of this section will be to give a rigorous proof of Lemma 8.32 in Hartley and Kahl (2007) and to generalize this result to also hold for singular quadrics (that is, to prove Propositions 5.7 and 5.9). While the method described in Remark 5.5 gives us an explicit expression for ψ_P , using this to prove Propositions 5.7 and 5.9 is somewhat difficult. We instead switch our approach to a more classical one, using intersection theory. The map ψ_P taking a point on \widetilde{S}_P to its conjugate can be described by describing the pullback of the hyperplane sections of \widetilde{S}_Q . For such hyperplane sections H , the pullbacks $\psi_P^{-1}(H)$ will be curves on \widetilde{S}_P . Finding the class of $\psi_P^{-1}(H)$ in the Chow ring will give us the information we need about ψ_P . We refer the reader to Eisenbud and Harris (2016) (chapter 2 in particular) for the basics on intersection theory used in the rest of the section.

By Lemma 5.4, the map ψ is defined everywhere except the intersection of the two epipolar lines, so if we assume that they do not intersect, we get the diagram below:

$$\begin{array}{ccc} \widetilde{\mathbb{P}}^3_P \supset \widetilde{S}_P & \begin{array}{c} \xleftarrow{\psi_Q} \\ \xrightarrow{\psi_P} \end{array} & \widetilde{S}_Q \subset \widetilde{\mathbb{P}}^3_Q \\ \pi_P \downarrow & & \downarrow \pi_Q \\ \mathbb{P}^3 \supset S_P & \begin{array}{c} \xleftarrow{\text{dashed}} \\ \xrightarrow{\text{dashed}} \end{array} & S_Q \subset \mathbb{P}^3 \end{array}$$

If either pair of epipolar lines DO intersect, however (this happens whenever the quadrics are not smooth), then the map ψ is not a morphism, but rather a rational map as it is not defined on the intersection. This can be mended by first blowing up the intersection of the epipolar lines, let

$$\pi_{g_P} : \widetilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^3$$

denote this blow-up, and let π_P , like before, be the blow-up in the two camera centers p_i . Let \overline{S}_P be the strict transform of S_P after the first blow-up, and let $\widetilde{\overline{S}}_P$ be the strict transform of \overline{S}_P after the second. And similarly for Q . We then get the following diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{P}}^3_P \supset \widetilde{S}_P & \begin{array}{c} \xleftarrow{\psi_Q} \\ \xrightarrow{\psi_P} \end{array} & \widetilde{S}_Q \subset \widetilde{\mathbb{P}}^3_Q \\ \pi_P \downarrow & & \downarrow \pi_Q \\ \widetilde{\mathbb{P}}^3 \supset \overline{S}_P & \begin{array}{c} \xleftarrow{\text{dashed}} \\ \xrightarrow{\text{dashed}} \end{array} & \overline{S}_Q \subset \overline{\mathbb{P}}^3 \\ \pi_{g_P} \downarrow & & \downarrow \pi_{g_Q} \\ \mathbb{P}^3 \supset S_P & \begin{array}{c} \xleftarrow{\text{dashed}} \\ \xrightarrow{\text{dashed}} \end{array} & S_Q \subset \mathbb{P}^3 \end{array}$$

We can now cover the different cases one by one, but first, let us give a result which will be helpful to determine the intersection multiplicities of curves on \widetilde{S}_P :

Proposition 5.10 (Eisenbud and Harris, 2016, Proposition 2.19). *Let S be a smooth projective surface and $\pi : \widetilde{S} \rightarrow S$ the blow-up of S at a point p ; let $E \in A^1(S)$ be the class of the exceptional divisor.*

1. $A(\widetilde{S}) = A(S) \oplus \mathbb{Z}E$ as abelian groups.
2. $\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha\beta)$ for any $\alpha, \beta \in A^1(S)$.
3. $E \cdot \pi^*\alpha = 0$ for any $\alpha \in A^1(S)$.
4. $E^2 = -1$.

5.2.1. Smooth quadric

We start by describing the map ψ_P in the case where the quadric S_P is smooth, i.e. a hyperboloid of one sheet. Let L_2 be the class of the total transform of a line in the same family as $g_{P_1}^j$, and let L_1 be the class of the total transform of a line in the other family. Let E_i be the class of the exceptional divisor we get when blowing up p_i .

On S_p , two lines from the same family do not intersect, whereas two lines in different families intersect once. By Proposition 5.10 (2) this is also the case on \widetilde{S}_p , by (3) L_i will not intersect E_i , and by (4), $E_i^2 = -1$. We get the following intersection multiplicity table:

\cdot	L_1	L_2	E_1	E_2
L_1	0	1	0	0
L_2	1	0	0	0
E_1	0	0	-1	0
E_2	0	0	0	-1

Next, let H_Q be the pullback of a hyperplane section from the conjugate configuration \widetilde{S}_Q . We want to find the class of H_Q on \widetilde{S}_p .

First, let us consider what the map ψ_p does to lines in each family. Let l be a generic curve in L_2 , in each image, $P_i(l)$ is a line, furthermore, since l does not intersect $g_{p_i}^j$, $P_i(l)$ is a line not passing through the epipole $e_{p_i}^j \in \mathbb{P}^2$. It follows that $\psi(l)$ is a curve appearing as the intersection of \widetilde{S}_Q and two planes (the preimages $Q_i^{-1}(P_i(l))$), each passing through exactly one camera center. In other words: $\psi(l)$ is a line, which means it intersects a generic hyperplane once. For a generic curve l' in L_1 , $P_i(l')$ is a line passing through the epipole, hence $\psi(l')$ is a curve appearing as the intersection of \widetilde{S}_Q and two planes, both passing through the camera centers, in other words, a conic curve. This means $\psi(l')$ will intersect a generic hyperplane twice.

Furthermore, by Lemma 5.4 (4), the lines $g_{p_i}^j$, belonging to the class $L_2 - E_i$, map to camera centers on S_Q . These will not intersect a generic hyperplane. Finally, H_Q has self-intersection 2. This gives us the following equations:

$$H_Q L_2 = 1, \quad H_Q L_1 = 2, \quad H_Q (L_2 - E_i) = 0, \quad H_Q^2 = 2.$$

Let $H_Q = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 E_1 + \alpha_4 E_2$. Using the intersection multiplicity table, we solve equations above for α_i and get $H_Q = L_1 + 2L_2 - E_1 - E_2$. In other words, the hyperplane sections on \widetilde{S}_Q pull back to curves of bidegree (1, 2) passing through both camera centers.

5.2.2. Cone

Next, let S_p be a cone. Since the two epipolar lines intersect in the vertex, we first blow up \mathbb{P}^3 in the vertex of the cone and then in the camera centers (see diagram above). Let L be the class of the total transform (with the second blow-up) of the strict transform (with the first blow-up) of a line on S_p , let E_0 be the class of the exceptional divisor which is the blow-up of the vertex and let E_i be the class of the exceptional divisor we get when blowing up p_i .

Since the cone is singular, we can not apply Proposition 5.10 to the first blow-up, we need a separate argument. Blowing up the vertex of the cone, and taking the strict transform of the lines from S_p , we get curves L which no longer intersect one another, so $L^2 = 0$, they do, however, intersect the exceptional divisor E_0 , so $LE_0 = 1$. Moreover, a hyperplane section is of class $H = 2L + E_0$, blowing up the cone, the resulting surface still has degree 2, so $H^2 = 2$, it follows that $E_0^2 = -2$. When we next blow up the two camera centers we are blowing up a smooth surface, so Proposition 5.10 applies, we get the following intersection multiplicity table:

\cdot	L	E_0	E_1	E_2
L	0	1	0	0
E_0	1	-2	0	0
E_1	0	0	-1	0
E_2	0	0	0	-1

Again, let H_Q be the pullback of a hyperplane from the conjugate configuration \widetilde{S}_Q . By arguments similar to the smooth quadric case we get the following equations:

$$H_Q L = 1, \quad H_Q(L - E_i) = 0, \quad H_Q^2 = 2.$$

Furthermore, if neither camera center on S_P is on the vertex, the conjugate quadric S_Q is a smooth quadric with both cameras lying on the same line, the preimage of this line (under ψ_P) is E_0 , hence $H_Q E_0 = 1$. On the other hand, if one of the cameras, say p_1 , lies on the vertex, S_Q is a cone and the strict transform $E_0 - E_1$ collapses to the vertex on S_Q , meaning $H_Q(E_0 - E_1) = 0$, so $H_Q E_0 = 1$ in this case also.

It follows that $H_Q = 3L + E_0 - E_1 - E_2$ in both cases.

5.2.3. Two planes, at most one camera center on the intersection

Let S_P be the union of two planes with at most one camera center lying on the intersection. In this case, S_Q is reducible as well and ψ_P takes the plane with the camera centers p_i to the plane with the camera centers q_i and the plane with no camera centers to the plane with no camera centers. As such we can consider the planes separately. The map ψ_P restricted to the plane with no camera centers is an isomorphism, so we need only consider how ψ acts on the plane with the camera centers.

Since the two epipolar lines $g_{p_i}^j$ intersect in a point, we blow up this point, and then the camera centers to get a morphism ψ_P . Let L be the class of the total transform of a line in the plane containing the camera centers, let E_i be the class of the exceptional divisor we get when blowing up p_i , and let E_0 be the class of the exceptional divisor we get when blowing up the intersection of the epipolar lines. The plane is smooth, so by Proposition 5.10 we get the following intersection multiplicity table:

	L	E_0	E_1	E_2
L	1	0	0	0
E_0	0	-1	0	0
E_1	0	0	-1	0
E_2	0	0	0	-1

Again, let H_Q be the pullback of a hyperplane from the conjugate configuration \widetilde{S}_Q . The strict transform of the line spanned by the two camera centers on S_P is mapped to the intersection of the epipolar lines on S_Q , similarly, the two epipolar lines map to camera centers. There are three corresponding lines on S_Q whose strict transform maps to points on S_P . Lastly, since we are working with a single plane, H_Q has self-intersection 1 this time. This gives us the equations:

$$H_Q(L - E_1 - E_2) = 0, \quad H_Q(L - E_0 - E_1) = 0, \quad H_Q(L - E_0 - E_2) = 0,$$

$$H_Q E_i = 1, \quad H_Q^2 = 1.$$

It follows that $H_Q = 2L - E_0 - E_1 - E_2$.

5.2.4. One or two planes, both cameras in singular locus

Finally, there is the case where S_P (and hence S_Q) is either a double plane or two planes with both cameras lying on their intersection. In this case, the map ψ_P is not defined on the line spanned by the two camera centers p_1, p_2 as any point on this line is conjugate to any point on the line spanned by q_1, q_2 . Outside this line, ψ_P acts simply as a linear transformation on each of the two (or one) planes. Note however that in the case of two planes the two linear transformations need not coincide on the intersection, for instance, two intersecting lines, one from each plane, might be taken to two disjoint lines.

Remark 5.11. In the case where S_P is a double plane, ψ_P acts as a linear transformation on S_P outside the line spanned by the camera centers. As such, it is in many ways similar to the trivial critical configurations, although it is, by Definition 4.6, non-trivial.

With all these results in place, we can now prove Propositions 5.7 and 5.9. Recall that Definition 5.6 defines the type of a curve to be (a, b, c_1, c_2) where c_1 and c_2 is the multiplicity in each

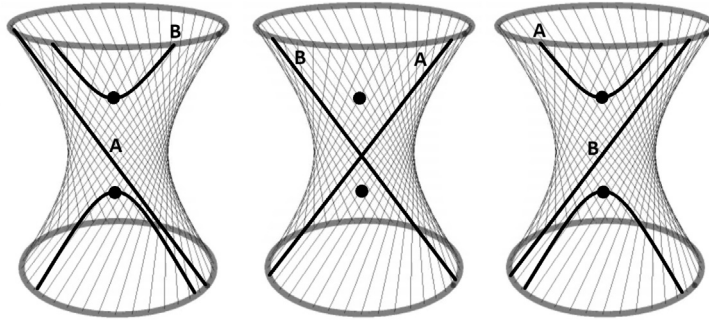


Fig. 3. Illustration of the map taking a point to its conjugate. The smooth quadric has two conjugates (original in the middle), the left is the one we get if we take the lines in the “A” family to be the epipolar lines, whereas the one on the right is the one we get if we choose “B”. In both cases the lines in the family with the epipolar lines are preserved, whereas the lines in the other family are mapped to conic curves.

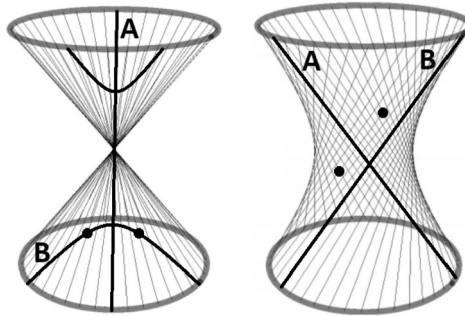


Fig. 4. Illustration of the map taking a point to its conjugate. The line spanned by the camera centers (right) maps to the vertex on the cone. The other lines in this family map to conics. Lines in the other family are preserved.

camera center, and where a and b is the number of times it intersects the epipolar lines and the other lines respectively (or in the case of a cone, the number of times it intersects a generic line outside the vertex, and the number of times it does so in total respectively). In other words, a curve of type (a, b, c_1, c_2) belongs to the class $aL_1 + bL_2 - c_1E_1 - c_2E_2$ on a smooth quadric, whereas on the cone, it belongs to the class $(a + b)L + aE_0 - c_1E_1 - c_2E_2$.

Proof of Proposition 5.7. Let $C_P \subset S_P$ be a curve of type (a, b, c_1, c_2) and let $C_Q \subset S_Q$ be of type (a', b', c'_1, c'_2) . By Lemma 5.4 (4), the pullback of the camera center $q_2 \in S_Q$ is the epipolar line $g_{P_1}^2$, which belongs to the class $L_2 - E_1$. C_P intersects this line $a - c_1$ times. It follows that $c'_2 = a - c_1$, similarly, $c'_1 = a - c_2$.

Assume now that S_P is smooth. A hyperplane section on S_Q is of type $(1, 1, 0, 0)$, we have shown above that the pullback of such a section is of type $(1, 2, 1, 1)$ on S_P . Certain hyperplane sections of S_Q are reducible, reducing into two components of type $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$, with each component moving in a pencil. Their pullback should do the same. The only way for a $(1, 2, 1, 1)$ -curve to reduce into two such components is as $(0, 1, 0, 0)$ and $(1, 1, 1, 1)$, since bidegree $(0, 2)$ would mean it is reducible, and the bidegree $(0, 1)$ -component passing through a camera center would mean it can not move. This means that the pullback of the $(1, 0, 0, 0)$ component is of type $(1, 1, 1, 1)$ (since this is the component intersecting the epipolar lines, it should correspond to the component through the camera centers on the other side), similarly, the pullback of the $(0, 1, 0, 0)$ -component is of type

$(0, 1, 0, 0)$. It follows that $a' = a$ and $b' = a + b - c_1 - c_2$, meaning C_Q is of type $(a, a + b - c_1 - c_2, a - c_2, a - c_1)$.

The proof in the case where S_P is a cone follows a similar pattern: in this case, a hyperplane section $(1, 1, 0, 0)$ on S_Q pulls back to a curve in the class $3L + E_0 - E_1 - E_2$, that is, a curve of type $(1, 2, 1, 1)$. Regardless of whether S_Q is a cone or smooth quadric, there are (like in the previous case) hyperplane sections of S_Q which reduce to two irreducible components, each free to move in a pencil. Similarly, their pullback must be reducible into two such components. The only way a curve in $3L + E_0 - E_1 - E_2$ reduces in such a way is as L and $2L + E_0 - E_1 - E_2$ since $3L$ and $2L$ would both be further reducible, and the curve in class L passing through a camera center would prevent it from moving. L is of type $(0, 1, 0, 0)$ and $2L + E_0 - E_1 - E_2$ is of type $(1, 1, 1, 1)$. Following the same argument as in the smooth quadric case, C_Q is of type $(a, a + b - c_1 - c_2, a - c_2, a - c_1)$ in this case also. \square

Proof of Proposition 5.9. Let $C_P \subset S_P$ be a curve of type (a, b, c_0, c_1, c_2) and let $C_Q \subset S_Q$ be of type $(a', b', c'_0, c'_1, c'_2)$. By Lemma 5.4 (4) the pullback of the camera center $q_2 \in S_Q$ is the epipolar line $g_{p_1}^2$, which belongs to the class $L - E_0 - E_1$. C_P intersects this line $a - c_0 - c_1$ times. It follows that $c'_2 = a - c_0 - c_1$, similarly, we get $c'_1 = a - c_0 - c_2$. Moreover, by Lemma 5.4 (3) the pullback of the intersection of the epipolar lines, is the lines spanned by the two camera centers (class $L - E_1 - E_2$), it follows that $c'_0 = a - c_1 - c_2$.

A generic line in the plane containing the camera centers on S_Q pulls back to a curve in the class $2L - E_0 - E_1 - E_2$ on S_P , C_P intersects this $2a - c_0 - c_1 - c_2$ times, it follows that $a' = 2a - c_0 - c_1 - c_2$. Lastly, ϕ_P is an isomorphism on the final plane, so $b' = b$. This means that C_Q is of type $(2a - c_0 - c_1 - c_2, b, a - c_1 - c_2, a - c_0 - c_2, a - c_0 - c_1)$. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A

A.1. Proof behind tables in Section 4.4

Let P_1, P_2 be two cameras with camera centers p_1, p_2 , and fundamental form F_P . Let \mathbf{F} denote the space of bilinear forms on $\mathbb{P}^2 \times \mathbb{P}^2$. By Lemma 4.10, there is an isomorphism between the set of quadrics passing through p_1, p_2 , and lines $L \subset \mathbf{F}$ that pass through F_P . The set of bilinear forms that are not of full rank is denoted by \mathbf{F}_2 . There are several different ways a line l through F_P can intersect \mathbf{F}_2 , namely:

When we have a finite number of intersections:

- Three distinct, real intersection points (one of which is F_P).
- Three distinct intersection points, two of which are complex conjugates.
- Two distinct intersection points of rank 2, F_P has multiplicity 2.
- Two distinct intersection points of rank 2, the one that is not F_P has multiplicity 2.
- Only one intersection point: F_P with multiplicity 3.
- Two distinct intersection points, F_P and one which has rank 1. The latter will have multiplicity 2.

Then there are the ones where L lies in \mathbf{F}_2 ; here we look at the kernels, and at the number of rank 1 forms on L :

- Two distinct, real, rank 1 forms (implies all bilinear forms share the same left- and right kernel).
- Two distinct, complex, rank 1 forms (implies all bilinear forms share the same left- and right kernel).
- One form of rank 1, which has multiplicity 2 (implies all bilinear forms share the same left- and right kernel)
- One form of rank 1, which has multiplicity 1 (implies all bilinear forms share the same left- OR right kernel, but never both)
- No forms of rank 1, all bilinear forms share the same left- or right kernel
- No forms of rank 1, all bilinear forms have distinct kernels.

This gives us a total of 12 different configurations to check. For each of these, we want to know what kind of quadric/camera configuration it corresponds to in \mathbb{P}^3 .

Let $\text{PGL}(3, \mathbb{R})$ be the projective general linear group of degree 3 (the group of real, invertible 3×3 matrices up to scale). This group acts on \mathbf{F} with an action that can be represented by matrix multiplication. Let $\text{PGL}(3)_{F_P}$ be the subgroup of $\text{PGL}(3)$ that fixes F_P . This gives us a group that acts on the \mathbb{P}^7 of lines passing through F_P . This group will never take a line with one of the 12 configurations above to a line with a different configuration. For instance, a real invertible matrix can not take three distinct real points, to anything other than three distinct real points. In particular, this means that the 12 configurations above all lie in distinct orbits under this group action.

Recall the isomorphism between the set of quadrics through p_1, p_2 and the set of lines through F_P . It gives us a similar group action on the set of quadrics passing through the camera centers p_1, p_2 , namely the subgroup of $\text{PGL}(4)$ that fixes the two camera centers. Here too, the camera/quadric configurations fall into 12 different orbits (not listed). Now we only need to check one representative from each orbit to find the exact correspondences.

By Proposition 2.13, the only property of the cameras that matters when considering critical configurations is their center. Furthermore, one pair of distinct points in \mathbb{P}^3 is no different from any other pair. This means that when we make computations for the two view case, we are free to pick any two cameras P_1, P_2 with distinct centers. We use

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The fundamental matrix of these two cameras is

$$F_P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now we can go ahead and check the 12 possible configurations listed earlier. This will be done by picking a fundamental matrix F_0 such that the line l spanned by F_P and F_0 intersects \mathbf{F}_2 the way we want and then checking what quadric it corresponds to. The full results are given in Table 2.

Table 2
Possible intersections between l and F_2 , and their corresponding quadrics.

Matrix F_0	Intersections between l and F_2	Configuration in \mathbb{P}^3
$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Three distinct, real intersection points, (one of which is F_p)	Smooth ruled quadric, camera centers not on same line
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Three distinct intersection points, two of which are complex conjugates.	Smooth, non-ruled quadric
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Two distinct intersection points, F_p has multiplicity 2	A cone, cameras on different lines, no camera at vertex
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	Two distinct intersection points, the one that is not F_p has multiplicity 2	Smooth quadric, camera centers lie on the same line
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	Only one intersection point: F_p with multiplicity 3	Cone, both cameras lie on the same line, neither lies at the vertex
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Two distinct intersection points, F_p and one which has rank 1. The latter has multiplicity 2	Union of two planes, camera centers in different planes
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	l lies in F_2 , and contains two distinct, real, rank 1 forms ^a	Union of two planes, camera centers lie on the intersection of the planes
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	l lies in F_2 , and contains two distinct, complex, rank 1 forms ^a	Union of two complex conjugate planes, both cameras on intersection.
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	l lies in F_2 , and contains one form of rank 1, which has multiplicity 2 ^a	A single plane with multiplicity 2
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	l lies in F_2 , and contains one form of rank 1, which has multiplicity 1 ^b	Union of two planes, one camera on the intersection
$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	l lies in F_2 , and contains no forms of rank 1, all bilinear forms share the same left or right kernel	Cone, one camera at the vertex
$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	l lies in F_2 , and contains no forms of rank 1, all bilinear forms have distinct kernels.	Union of two planes, cameras in same plane, neither at the intersection

^a This implies all bilinear forms share the same left and right kernel.

^b This implies all bilinear forms share the same left or right kernel.

References

Agarwal, Sameer, Pryhuber, Andrew, Thomas, Rekha R., 2021. Ideals of the multiview variety. *IEEE Trans. Pattern Anal. Mach. Intell.* 43 (4), 1279–1292. <https://doi.org/10.1109/TPAMI.2019.2950631>.

Aibl, Cenek, Sugimoto, Akihiro, Pajdla, Tomas, 2016. Degeneracies in rolling shutter sfm. In: *Leibe, Bastian, Matas, Jiri, Sebe, Nicu, Welling, Max (Eds.), Computer Vision – ECCV 2016*. Springer International Publishing, Cham. ISBN 978-3-319-46454-1, pp. 36–51.

Bertolini, Marina, Turrini, Cristina, 2007. Critical configurations for 1-view in projections from $\mathbb{P}^k \rightarrow \mathbb{P}^2$. *J. Math. Imaging Vis.* 27, 277. <https://doi.org/10.1007/s10851-007-0649-6>.

Bertolini, Marina, Turrini, Cristina, Besana, GianMario, 2007. Instability of projective reconstruction of dynamic scenes near critical configurations. In: *2007 IEEE 11th International Conference on Computer Vision*, pp. 1–7.

Bertolini, Marina, Besana, GianMario, Notari, Roberto, Turrini, Cristina, 2020. Critical loci in computer vision and matrices dropping rank in codimension one. *J. Pure Appl. Algebra* 224 (12), 106439.

Brätelund, Martin, 2021. Critical configurations for three projective views. *arXiv e-prints*. arXiv:2112.05478. <https://doi.org/10.48550/arXiv.2112.05478>.

Breiding, Paul, Rydell, Felix, Shehu, Elima, Torres, Angélica, 2022. Line multiview varieties. *arXiv e-prints*. arXiv:2203.01694. <https://doi.org/10.48550/arXiv.2203.01694>.

- Buchanan, Thomas, 1988. The twisted cubic and camera calibration. *Comput. Vis. Graph. Image Process.* (ISSN 0734-189X) 42 (1), 130–132. [https://doi.org/10.1016/0734-189X\(88\)90146-6](https://doi.org/10.1016/0734-189X(88)90146-6). <https://www.sciencedirect.com/science/article/pii/0734189X88901466>.
- Buchanan, Thomas, 1992. Critical sets for 3d reconstruction using lines. In: Sandini, G. (Ed.), *Computer Vision – ECCV'92*. Springer Berlin Heidelberg, Berlin, Heidelberg. ISBN 978-3-540-47069-4, pp. 730–738.
- Eisenbud, David, Harris, Joe, 2016. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press.
- Hartley, Richard, Kahl, Fredrik, 2007. Critical configurations for projective reconstruction from multiple views. *Int. J. Comput. Vis.* 71 (1), 5–47. <https://doi.org/10.1007/s11263-005-4796-1>.
- Hartley, Richard, Kahl, Fredrik, 2002. Critical curves and surfaces for euclidean reconstruction. In: Heyden, Anders, Sparr, Gunnar, Nielsen, Mads, Johansen, Peter (Eds.), *Computer Vision – ECCV 2002*. Springer Berlin Heidelberg, Berlin, Heidelberg. ISBN 978-3-540-47967-3, pp. 447–462.
- Hartley, Richard, Zisserman, Andrew, 2004. *Multiple View Geometry in Computer Vision*, 2 edition. Cambridge University Press.
- Hartley, Richard I., 2000. Ambiguous configurations for 3-view projective reconstruction. In: *Computer Vision - ECCV 2000*. Springer Berlin Heidelberg, Berlin, Heidelberg. ISBN 978-3-540-45054-2, pp. 922–935.
- Kahl, F., Hartley, R., Astrom, K., 2001. Critical configurations for n-view projective reconstruction. In: *Proceedings of the 2001 IEEE Computer Society Conference on Computer Vision and Pattern Recognition. CVPR 2001*, vol. 2, pp. II-158–II-164.
- Krames, J., 1941. Zur Ermittlung eines Objektes aus zwei Perspektiven. (Ein Beitrag zur Theorie der "gefährlichen Örter"). *Monatshefte Math. Phys.* 49, 327–354.
- Luong, Q.T., Faugeras, O.D., 1994. A stability analysis of the fundamental matrix. In: Eklundh, Jan-Olof (Ed.), *Computer Vision – ECCV '94*. Springer Berlin Heidelberg, Berlin, Heidelberg. ISBN 978-3-540-48398-4, pp. 577–588.
- Maybank, S., 1993. *Theory of Reconstruction from Image Motion*. Springer Series in Information Sciences. Springer Berlin Heidelberg. ISBN 978-3-642-77559-8.