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Wave equations of the two common sediment acoustic theories

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Two popular theories for wave propagation in sediments, the Biot poroelastic theory and the Viscous Grain Shearing (VGS) theory, have been formulated in terms of dispersion relations. The lack of explicit wave equations for these models could be considered to be a weakness. Hence, providing wave equations that are fully consistent with these theories might confirm that they have a physical basis. In the Biot theory, it is the frequency-dependent viscodynamic operator that poses the problem as there is no equivalent time-domain operator. Previous work has shown that this factor can be approximated well with a simplified square root operator. Drawing on the parallel to the Cole-Davidson dielectric theory for complex media where the same factor appears, and using the equivalent time-domain operator, which is a fractional pseudo-differential operator, enables the formulation of wave equations for all three wave modes of the Biot theory. The same operator turns out to be central for transforming the dispersion relations of the VGS theory to time-space wave equations. In this case, the relation between the dispersion equations and the wave equations is exact. The wave equations of these two theories will enable a closer study of the properties of the medium models.

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1. INTRODUCTION

Acoustic waves are used in underwater navigation, underwater communication, and in seismic exploration as they propagate long distances in the ocean. Acoustic wave propagation in the ocean is greatly influenced by properties of the sea bed or marine sediments. Marine sediments are porous media composed of interlocking solid/mineral granules framework permeated by a fluid in the pores between the granules. The sound speed and attenuation of the propagating acoustic waves are dependent on the physical properties of the granular solid framework's rigidity, porosity, and permeability, see Ref. 1, Chap. 16. Several theoretical models with varying degree of complexity were developed to predict the geoacoustic properties from the measurement of the physical properties of the marine sediments but two models that are widely used are Biot's theory for wave propagation in fluid saturated porous media and Buckingham's grain shearing theory for wave propagation in saturated granular material.[2]

Apart from the difficulties in measuring the geophysical properties, the geoacoustic properties like sound speed and attenuation in sediments with heterogeneities or gradients in porosity also vary significantly from the predictions of the existing geoacoustic models. To solve the laws of poroelasticity in such complex media numerically, the frequency-dependent geophysical parameters and the wave equations need to be appropriately represented.[3] The time-domain representation is essential for modelling pulse propagation in porous media and acoustic characterization of the material/medium from its reflection and transmission characteristics (inverse scattering problem). However, for certain frequency-dependent characterization functions (such as relaxation or creep response) of porous media, the time-domain equivalent is in the form of a convolution memory kernel resulting in fractional derivative operators.[4]

In this article, we derive the wave equations across the whole frequency range for the compressional and shear wave modes of both the Biot and the Grain Shearing family of models in the framework of fractional derivative operators. This paper, as well as its predecessor,[5] can be seen as part of an effort to develop "models with physical and mathematical rigor" for geoacoustic models, which is how Holland and Dosso[6] characterized past work by our group.[7] The article is organized as follows: In Sec. 2, we derive time-domain equivalent for the simplified viscodynamic correction factor and wave equations for three wave modes of the Biot poroelastic model. In Sec. 3, we then derive the wave equations for the viscous grain-shearing model from its relaxation response and show that in the limit they are equivalent to the wave equations for the simpler grain-shearing model.

2. BIOT POROELASTIC MODEL

The poroelastic model of Biot describes wave propagation in a consolidated porous medium that is modelled as an elastic skeletal frame with tubular pores permeated by a fluid through a set of coupled wave equations. The dissipation occurs from the relative motion of the fluid and solid phases and with an assumption of laminar flow. Biot's theory predicts the existence of two compressional wave and one shear wave where the second compressional (slow) wave which is non-existing in non-porous media is the result of the inertia of the pore fluid with respect to motion of the solid. The frequency-independent coefficient of dissipation is obtained from Darcy's law.[8,9]

The assumption of laminar flow (viscous dissipation) breaks down once the characteristic pore size is greater than the thickness of the viscous boundary layer and flow becomes turbulent. The viscous boundary layer thickness (δ_{bl}) is dependent on frequency (ω), fluid density (ρ_f), and viscosity (η), see Ref. 10, Sec. 7.6. This thickness is given as

$$\delta_{bl} = \sqrt{\frac{2\eta}{\rho_f \omega}}. \tag{1}$$

Above a certain frequency, the viscous effects are confined to a thin boundary layer in the vicinity of the pore walls, and the inertial effect dominates, leading to a turbulent fluid flow. To describe the viscous and the inertial effects over the entire frequency range, Biot derived the frequency dependent viscodynamic correction factor for a 2-D parallel duct and a 3D circular pore geometry. For the 2-D parallel duct geometry, the viscodynamic correction factor is equivalent to tanh function ratios[11] while for the 3D circular pore geometry, the viscodynamic correction factor is equivalent to the Bessel function ratios for acoustic wave propagation in a cylindrical tube (without thermal conductivity) derived in Ref. 12, Chap. 2, and is given as follows

$$F(\omega) = \frac{\mathrm{i}z}{4}\frac{J_1(\mathrm{i}z)}{J_2(\mathrm{i}z)}, \tag{2}$$

where J_ν is the Bessel function of the first kind of order ν, $\mathrm{i}z = \mathrm{i}\sqrt{\mathrm{i}\omega\tau_r}$ and τ_r is the characteristic relaxation time and it depends on the radius of the circular pore (R), fluid density and viscosity ($\tau_r = \rho_f R^2/\eta$). The flow is laminar if $\omega\tau_r << 1$ and $F(\omega) = 1$ and turbulent if $\omega\tau_r >> 1$.

Viscous/inertial dissipation is a characteristic relaxation response of an acoustic wave propagating through a porous medium.[13] The circular pore geometry with unconnected tubes is a simplified model of the irregular porous media of the real world. A simpler expression in the form of a square root operator which matches asymptotic characteristics of the viscodynamic operator is just as good and is given as follows

$$F(\omega) = \sqrt{1 + \mathrm{i}\omega c_p \tau_r}, \tag{3}$$

where c_p is the pore shape factor and its value is $1/16$ for the simplified 3D circular pore geometry, see Ref. 10, Eq. 7.95. The correction factor proposed by Johnson[14] is in the same form as the square root approximation given here, but it depends on the physical parameters of the porous volume. The advantage of the square root approximation is that it is of the same form as the Cole-Davidson dielectric relaxation model[15] and it has an equivalent time-domain representation known as the fractional pseudo-differential operator or the shifted time fractional derivative operator as provided in Ref. 16, Eq. 3.13:

$$\mathcal{F}^{-1}\left[\sqrt{1 + \mathrm{i}\omega c_p \tau_r}\right] \Longleftrightarrow \sqrt{c_p\tau_r}\left(\mathrm{D}_t + \frac{1}{\tau_r c_p}\right)^{\frac{1}{2}}. \tag{4}$$

Using the time-domain representation of the simplified viscodynamic operator, the wave equations of the three wave modes of the Biot poroelastic model are obtained from the corresponding dispersion relations as in Ref. 5, Eq. 16 and Eq. 22, and they are given as follows:

1. Shear mode wave equation of the Biot poroelastic model

$$\begin{aligned}\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}u + \frac{\rho_c B_0}{\eta}\frac{1}{\sqrt{c_p\tau_r}}\left(\mathrm{D}_t + \frac{1}{\tau_r c_p}\right)^{\frac{1}{2}}\nabla^2 u \\ - \frac{1}{c_0^2}\left(1 - \frac{\rho_f^2}{\rho\rho_c}\right)\frac{\rho_c B_0}{\eta}\frac{1}{\sqrt{c_p\tau_r}}\left(\mathrm{D}_t + \frac{1}{\tau_r c_p}\right)^{\frac{1}{2}}\frac{\partial^2}{\partial t^2}u = 0,\end{aligned} \tag{5}$$

   where u is the displacement, c_0 is the sound speed, B_0 is the static permeability, ρ_c is the mass coupling density, and ρ is the composite density of the porous volume.

2. Fast compressional mode wave equation of the Biot poroelastic model

$$\begin{aligned}\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}u + C_f\left[\frac{\rho_c}{\rho} + \frac{M}{H} - 2\frac{C}{H}\frac{\rho_f}{\rho}\right]\frac{\rho B_0}{\eta}\frac{1}{\sqrt{c_p\tau_r}}\left(\mathrm{D}_t + \frac{1}{\tau_r c_p}\right)^{\frac{1}{2}}\nabla^2 u \\ - \frac{1}{c_0^2}\left(1 - \frac{\rho_f^2}{\rho\rho_c}\right)\frac{\rho_c B_0}{\eta}\frac{1}{\sqrt{c_p\tau_r}}\left(\mathrm{D}_t + \frac{1}{\tau_r c_p}\right)^{\frac{1}{2}}\frac{\partial^2}{\partial t^2}u = 0,\end{aligned} \tag{6}$$

where M and C are the elastic moduli, H is the plane-wave modulus, and C_f is a fast-mode correction factor.

3. Slow compressional mode wave equation of the Biot poroelastic model

$$\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}u - \left[\frac{M}{H} - \frac{C^2}{H^2}\right]\frac{\rho B_0}{\eta}\frac{1}{\sqrt{c_p \tau_r}}\left(\mathrm{D}_t + \frac{1}{\tau_r c_p}\right)^{\frac{1}{2}}\nabla^2 u + \frac{1}{c_0^2}\frac{1}{C_s}\left[\frac{\rho_c}{\rho} + \frac{M}{H} - 2\frac{C}{H}\frac{\rho_f}{\rho}\right]\frac{\rho B_0}{\eta}\frac{1}{\sqrt{c_p \tau_r}}\left(\mathrm{D}_t + \frac{1}{\tau_r c_p}\right)^{\frac{1}{2}}\frac{\partial^2}{\partial t^2}u = 0, \quad (7)$$

where C_s is the slow mode correction factor.

The derivation and the justification of correction factors C_f and C_s are provided in Ref. 5, Sec. III B, where examples show that the correction is small, i.e., typically $0.9 < C_f \leq 1$, and the correction factors are inverses of each other i.e., $C_f \cdot C_s = 1$.

The shear-mode and the fast compressional mode wave equations, (5) and (6), are equivalent to the tempered half-order fractional Zener model and are associated with an f^2 attenuation at low frequencies and $\sqrt{f}$ attenuation at high frequencies.[17] The slow compressional-mode wave equation, (7), is equivalent to the tempered half-order fractional Maxwell model and it is an almost non-propagating wave that is highly attenuated and difficult to measure, see Ref. 10, Chap. 8.

Biot's poroelastic theory provides a general framework for modelling acoustic wave propagation in porous media but to accurately model the underlying physics of complex media like marine sediments, additional relaxation mechanisms can be incorporated as required. Chotiros and Isakson[18] incorporated a relaxation process that occurs due to squirt flow (i.e., suction and expulsion of water) at the grain-grain contact through complex and frequency dependent frame bulk and shear moduli and the model is called as the extended Biot model. With the additional relaxation mechanism, it was shown that a near linear frequency variation attenuation can be obtained in the intermediate frequency range.[13]

3. GRAIN-SHEARING FAMILY OF MODELS

The justification for the grain shearing family of models is based on an unconsolidated granular medium assumption for a marine sediment.[19,20] In this medium there is elastic behavior from postulated stick-slip, intergranular micro-asperities subject to viscous dissipation in a pore fluid film during the passage of a wave and this gives rise to a time-dependent drag force. The model is statistical due to the random in-time contribution of different micro-asperities to shearing. Averaging different shearing events over an ensemble provides the constitutive relationship of the medium. The physical process of individual grain-grain interaction in the medium model is akin to a strain hardening mechanism and is modelled as a time-dependent Maxwell model with a linear increase in viscosity. The model is inspired from Gittus (Ref. 21, Ch. 8.6) which stated that a power-law response can be obtained from a Maxwell model with a dashpot which increases its viscosity with time. However, Gittus did not provide an exact functional form for the change in viscosity.

As in other similar theories that find power-law responses from time-varying media models, reciprocity is not maintained, i.e. in a mechanical system, the relaxation modulus is a power law and the creep compliance does not relate to it via the correspondence principle of linear viscoelasticity.[7] Likewise, only the current response of the equivalent electrical circuit is a power law and not the voltage response.[22]

The grain-shearing theory predicts one compressional wave and one shear wave that are independent of each other. The shear wave mode of the grain-shearing model medium leads to a power-law relaxation response and the compressional wave mode leads to a power relaxation response with a bulk modulus of the fluid acting as spring parallel to it:

$$G_{\mathrm{GS,S}}(t) = \frac{K_s t^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1, \tag{8}$$

and for the compressional wave mode we get

$$G_{\mathrm{GS,P}}(t) = K_f + \frac{K_p t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{4}{3}\frac{K_s t^{-\alpha}}{\Gamma(1-\alpha)}, \tag{9}$$

where α is a material exponent, $\Gamma(1-\alpha)$ is the gamma function, K_f is the bulk modulus of the fluid, and K_p and K_s are the compressional and shear stress relaxation coefficient with unit [Pa $\cdot$ s$^\alpha$].

In real media, the viscosity cannot increase indefinitely with time, but may saturate to a fixed value. Taking the dissipation of the pore fluid into account, the saturation was modelled with a dashpot in series with the time-dependent Maxwell model and the corresponding model is called the Viscous Grain Shearing model (VGS).[20] In the VGS model medium, the power law relaxation response of GS model is tempered by an exponential factor for both the shear wave mode:

$$G_{\mathrm{VGS,S}}(t) = \frac{K_s t^{-\alpha}}{\Gamma(1-\alpha)} \mathrm{e}^{-t/\tau_s}, \quad 0 < \alpha < 1, \tag{10}$$

and the compressional wave mode:

$$G_{\mathrm{VGS,P}}(t) = K_f + \frac{K_p t^{-\alpha}}{\Gamma(1-\alpha)} \mathrm{e}^{-t/\tau_p} + \frac{4}{3}\frac{K_s t^{-\alpha}}{\Gamma(1-\alpha)} \mathrm{e}^{-t/\tau_s}, \tag{11}$$

where τ_p and τ_s are the compressional and shear viscoelastic time constant with unit [s]. These responses have singularities at $t = 0$. The consequences of this were pointed out a long time ago[23] and further analyzed in Ref. 24.

These relaxation responses can equivalently be described using the framework of fractional derivative operators. In order to describe the tempered responses, the fractional pseudo-differential operator as in (4) is useful. The details of the following derivations are all given in Ref. 24.

The wave equation for the shear wave mode of the VGS model is obtained from the corresponding dispersion relation of (10) and is equivalent to the tempered fractional diffusion wave equation:

$$\nabla^2 u - \frac{\rho_0}{K_s} \left[\mathrm{D}_t + 1/\tau_s\right]^{1-\alpha} \frac{\partial u}{\partial t} = 0, \tag{12}$$

where u is the displacement and ρ_0 is the density.

Similarly, the wave equation for the compressional wave mode of the VGS model is obtained from the corresponding dispersion relation of (11) which is equivalent to the tempered fractional Kelvin-Voigt wave equation[24] and is given as

$$\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{K_p}{K_f}\left[\mathrm{D}_t + 1/\tau_p\right]^{\alpha-1} \frac{\partial}{\partial t}\nabla^2 u + \frac{4}{3}\frac{K_s}{K_f}\left[\mathrm{D}_t + 1/\tau_s\right]^{\alpha-1} \frac{\partial}{\partial t}\nabla^2 u = 0. \tag{13}$$

The wave equation for the VGS compressional mode is associated with an f^2 attenuation at low frequencies and a power-law attenuation with near linear frequency variation for high frequencies, whereas the VGS shear mode undergoes an $\sqrt{f}$ attenuation at low frequencies and a near-linear frequency attenuation at high frequencies. The asymptotic versions of (12) and (13), i.e., the low- and high-frequency versions, were given in Ref 10, Chap. 8.2, but not the exact wave equations given above.

The compressional and shear wave equations of the GS model can be obtained as a limiting case (i.e., $t \to 0$) of the fractional pseudo-differential operator which can be expanded using an infinite binomial series of fractional derivatives (see Ref. 16, Eq. B.18), yielding:

$$[\mathrm{D}_t + 1/\tau_s]^{1-\alpha} = \sum_{n=0}^{\infty} \binom{1-\alpha}{n} \tau_s^{-n} \mathrm{D}_t^{1-\alpha-n}. \tag{14}$$

When $t/\tau_s \to 0$, i.e. $n = 0$, this infinite series simplifies to

$$\lim_{t/\tau_s \to 0} [D_t + 1/\tau_s]^{1-\alpha} = D_t^{1-\alpha}. \tag{15}$$

Hence, in the limit the wave equation in (12) simplifies to a fractional diffusion equation (i.e., shear wave mode of the GS model):

$$\nabla^2 u - \frac{\rho_0}{K_s} \frac{\partial^{2-\alpha} u}{\partial t^{2-\alpha}} = 0. \tag{16}$$

Likewise, the wave equation in (13) simplifies to a fractional Kelvin-Voigt model (i.e., compressional wave mode of GS model):

$$\nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \left[\frac{K_p + \frac{4}{3} K_s}{K_f} \right] \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u = 0. \tag{17}$$

The wave equations for the GS compressional and shear modes are associated with a power-law attenuation with near linear frequency variation across the whole frequency range.

4. CONCLUSION

The contribution of this paper is the derivation of the wave equations for each of the wave modes of the approximate Biot poroelastic model and the exact VGS model across the whole frequency range using the concept of the fractional pseudo-differential operator first applied to sediment acoustic models in Ref. 5. The wave equations for the GS model are obtained as a limiting case of the VGS model. The tempered power-law wave equations are equivalent to equations discussed in the fractional calculus literature[16,25,26] and it enables one to draw on already existing literature for analysis of properties of the solutions.

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