Compatability of Fundamental Matrices for Complete Graphs
Supplementary Material

Martin Bråtelund
University of Oslo
Moltke Moes vei 35, 0851 Oslo, Norway
mabraate@math.uio.no

Felix Rydell
KTH Royal Institute of Technology
Lindstedtsvägen 25, Stockholm, Sweden
felixry@kth.se

Overview

In this Supplementary Material we prove all the mathematical results from the main body of the paper, whose proofs were left out. For the convenience of the reader, we explain some concepts from applied algebraic geometry that are helpful to understand the Supplementary Material in Appendix A. Results that also appear in the main body of the paper are restated and are given the same number. Additional results not stated in the main body are numbered independently.

In Appendix B, we prove the statements of Section 2. We prove our results on compatibility for complete graphs in Appendix C. In Appendix D we prove the cycle theorem.

A. Algebraic Geometry Preliminaries

For the proofs in Appendix C, it is helpful to understand saturation and elimination of ideals. We refer the reader to [2] for a detailed study of these topics. Consider a field k and its polynomial ring $k[x] = k[x_1, \dots, x_m]$; the set of all polynomials with coefficients in k. That $k[x]$ is a ring means that addition and multiplication of polynomials satisfy a certain set of axioms that we don't list here. An ideal I of a ring R is an additive subgroup that is closed under multiplication of elements in R.

Let $f_1, \dots, f_s \in k[x]$ be polynomials. They generate an ideal of $k[x]$ as follows:

$$\langle f_1, \dots, f_s \rangle := \Big\{ \sum g_i f_i : g_i \in k[x] \Big\} \subseteq k[x]. \quad (1)$$

From the geometric point of view, an ideal in a polynomial ring defines a variety $\mathcal{V}$ as the zero set of all polynomials in the ideal. In other words,

$$\mathcal{V}(I) := \big\{ x \in k^m : f(x) = 0 \; \forall f \in I \big\}. \quad (2)$$

The Zariski closure of a set $U \subseteq k^m$ is the smallest variety X that contains U. We write $\overline{U}$ for the Zariski closure of U.

The goal of saturation is to remove unwanted components from a variety. Let I, J be ideals. The saturation of I with respect to J is

$$I : J^\infty := \big\{ f \in k[x] : \forall g \in J, \; \exists N \in \mathbb{N} \text{ such that } f g^N \in I \big\}. \quad (3)$$

Theorem A.1 ([2, p. 203])**.** *Let $\mathcal{V}(J), \mathcal{V}(I)$ be two varieties over any field k. Then*

$$\overline{\mathcal{V}(I) \setminus \mathcal{V}(J)} \subseteq \mathcal{V}(I : J^\infty). \quad (4)$$

The elimination of variables $x_1, \dots, x_l$ from an ideal $I \subseteq k[x]$ is the intersection

$$I \cap k[x_{l+1}, \dots, x_m]. \quad (5)$$

Given $(x_1, \dots, x_n) \in \mathcal{V}(I)$, we have that $(x_{l+1}, \dots, x_n) \in \mathcal{V}(I \cap k[x_{l+1}, \dots, x_m])$, because any f in Equation (5) also lies in I. In this way, elimination of variables gives us conditions on the projection of $\mathcal{V}(I)$ away from the first l coordinates.

In Appendix C, we use the symbolic programming language `Macaulay2` [4] to symbolically saturate ideals and eliminate variables in the ring $\mathbb{Q}[x]$. In our study, all polynomials have rational coefficients, i.e. are elements of $\mathbb{Q}[x]$. However, our varieties lie in real space. For saturation and elimination, it may matter in which ring the operations are performed in. In `Macaulay2` all such operations happen inside $\mathbb{Q}[x]$, and we therefore prove the following lemma for clarity.

Lemma A.2. *Let I, J be ideals in $\mathbb{R}[x]$ generated by elements of $\mathbb{Q}[x]$. Write $I_Q, J_Q \subseteq \mathbb{Q}[x]$ for the ideals defined as the intersections $I \cap \mathbb{Q}[x]$, $J \cap \mathbb{Q}[x]$, respectively. If $y \in \mathbb{R}^m$ lies in $\mathcal{V}(I) \setminus \mathcal{V}(J)$, then $f(y) = 0$ for every f in the saturation $I_Q : J_Q^\infty$ performed inside the ring $\mathbb{Q}[x]$.*

Hence saturation in $\mathbb{Q}[x]$ tell us something also for the real numbers. The statement and proof works the same if $\mathbb{R}[x]$ is replaced by $\mathbb{C}[x]$.

Proof. By Theorem A.1, $y \in \mathcal{V}(I : J^\infty)$. It suffices to show that $I_Q : J_Q^\infty \subseteq I : J^\infty$, since then we have $\mathcal{V}(I : J^\infty) \subseteq \mathcal{V}(I_Q : J_Q^\infty)$ over the real numbers. Let $f \in I_Q : J_Q^\infty$. Then $f \in \mathbb{Q}[x]$ and for every $g \in J_Q$, there is an N such that $fg^N \in I_Q$. Let $g_1, \dots, g_k \in \mathbb{Q}[x]$ generate I and I_Q. Let N_i denote an integer such that $fg_i^{N_i} \in I_Q$. Now take any $g \in J$. We can write $g = \sum_{i=1}^k h_i g_i$ for some $h_i \in \mathbb{R}[x]$. There is an integer N depending on k and N_i such that each term of g^N is divisible by some $g_i^{N_i}$ and $fg^N \in I_Q$. For such N, we can write $g^N = \sum_{i=1}^k h_i' g_i^{N_i}$ for some $h_i' \in \mathbb{R}[x]$. Then, since $fg_i^{N_i} \in I_Q$, we must have that $fg^N \in I$. This shows that inclusion $I_Q : J_Q^\infty \subseteq I : J^\infty$ and we are done. □

In the main body of the text, the term rational map was used, which we now define. A variety $\mathcal{V}$ is called irreducible if it cannot be written as a union of two proper varieties, meaning that for two subvarieties X, Y of $\mathcal{V}$, the equality $\mathcal{V} = X \cup Y$ implies $\mathcal{V} = X$ or $\mathcal{V} = Y$. A rational map f between projective varieties X and Y, with X irreducible, is defined on a Zariski open set of X, which is a set that can be written $X \setminus Y$ for a proper subvariety $Y \subseteq X$. A rational map between X and Y is written

$$f : X \dashrightarrow Y. \tag{6}$$

B. The Fundamental Action

Proposition 2.2. *Let $\{F^{ij}\}$ and $\{G^{ij}\}$ be two sets of compatible fundamental matrices. They are equivalent under fundamental action if and only if they have solutions whose camera centers are equivalent under* PGL_4.

For the proof we need the following lemma:

Lemma B.1 ([5, Result 22.1])**.** *Let P and P' be two camera matrices with the same center. Then there exists $H \in \mathrm{PGL}_3$ such that $P' = HP$.*

Proof of Proposition 2.2.

$\Rightarrow$) Let $G^{ij} = H_i^T F^{ij} H_j$. If $P_1, \dots, P_n$ is a solution to $\{F^{ij}\}$, then by Proposition 2.1, $H_1^{-1}P_1, \dots, H_n^{-1}P_n$ is a solution to G^{ij}, which have the same centers as $P_1, \dots, P_n$.

$\Leftarrow$) Let $P_1, \dots, P_n$ be a solution to $\{F^{ij}\}$ with centers c_i and $P_1', \dots, P_n'$ a solution to $\{G^{ij}\}$ with centers c_i' such that $c_i' = H^{-1}c_i$ for some $H \in \mathrm{PGL}_4$. By Lemma B.1, there are $H_i \in \mathrm{PGL}_3$ such that $P_i' = H_i P_i H$, since P_i' and $P_i H$ have the same center $H^{-1}c_i$. Then by Proposition 2.1, $\{F^{ij}\}$ and $\{G^{ij}\}$ are equivalent under fundamental action. □

Lemma 2.5. *Let $\{F^{ij}\}$ be set of compatible fundamental matrices that include F^{si}, F^{ij} and F^{jt}. We have $\mathbf{e}_{sijt} = 0$ if and only if the centers c_s, c_i, c_j and c_t of any solution are coplanar.*

The back-projected line of an image point x for a camera P is the line in $\mathbb{P}^3$ of all points that are projected by P to x. This line contains the center of P.

Proof. Let $P_1, \dots, P_n$ be a solution to $\{F^{ij}\}$. Let $L_{i,s}$ be the back-projected line of e_i^s and $L_{j,t}$ the back-projected line of e_j^t. Then $e_i^s F^{ij} e_j^t = 0$ means precisely that the back-projected lines $L_{i,s}$ and $L_{j,t}$ meet in a point. Therefore, $L_{i,s}$ and $L_{j,t}$ together span a plane unless they are the same line. In either case, all centers lie in this span, since $L_{i,s}$ contains c_i and c_s, and $L_{j,t}$ contains c_j and c_t. The other direction follows similarly. □

C. Compatibility for Complete Graphs

C.1. K_3

Proposition 3.4. *Let F^{12}, F^{13}, F^{23} be fundamental matrices. There exist collinear cameras P_1, P_2, P_3 such that $F^{ij} = \psi(P_i, P_j)$ if and only if*

$$e_1^2 = e_1^3, \quad e_2^1 = e_2^3, \quad e_3^1 = e_3^2, \tag{7}$$

and

$$(F^{21})^T [e_1^2]_\times F^{13} = F^{23}. \tag{8}$$

Remark C.1. *When we in the proofs below write "it can be verified that" or "it can be checked that" in relation to the shape of fundamental matrices, we have checked this fact in* `Macaulay2`.

Proof. Recall that the epipole e_j^i equals $P_j(\ker(P_i))$. It follows that if a solution to F^{12}, F^{13}, F^{23} consists of collinear cameras, then Equation (7) must be satisfied. Conversely, if Equation (7) is satisfied, any solution must consist of collinear camera centers.

We begin by simplifying the problem using the fundamental action. Let

$$H_i = \begin{bmatrix} e_i^k & \mathbf{x}_i & \mathbf{y}_i \end{bmatrix}, \tag{9}$$

for any $k \neq i$ and $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^3$ such that the determinant is non-zero, meaning H_i is invertible. We get a new triple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{10}$$

Write h_j^i for the epipoles of G^{ij}. By the fact that $H_j^{-1}e_j^i$ spans $\ker G^{ij}$ we have $h_j^i = H_j^{-1}e_j^i$ (up to scaling). By construction of H_j, we then have:

$$\begin{aligned} h_1^2 &= [1,0,0], & h_2^1 &= [1,0,0], & h_3^1 &= [1,0,0], \\ h_1^3 &= [1,0,0], & h_2^3 &= [1,0,0], & h_3^2 &= [1,0,0]. \end{aligned} \tag{11}$$

Since the epipoles span kernels of G^{ij}, we conclude that G^{ij} take the following form

$$\begin{aligned} G^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{12} & b_{12} \\ 0 & c_{12} & d_{12} \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{13} & b_{13} \\ 0 & c_{13} & d_{13} \end{bmatrix}, \\ G^{23} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{23} & b_{23} \\ 0 & c_{23} & d_{23} \end{bmatrix}, \end{aligned} \tag{12}$$

for some $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$ making them rank-2.

We next find conditions on triplets of cameras P_1, P_2, P_3 with collinear centers whose fundamental matrices are of the form given by Equation (12). We may up to PGL_4 action assume that the center of P_1 is $[1, 0, 0, 0]$, the center of P_2 is $[0, 1, 0, 0]$ and the center of P_3 is $[1, 1, 0, 0]$. Fix P_1 to be

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{13}$$

Using the fact that $e_j^i = P_j(\ker P_i)$, we find that P_2 and P_3 must take the following form:

$$P_2 = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}, P_3 = \begin{bmatrix} 1 & -1 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}. \tag{14}$$

One can check that the two right-most elements of the first rows of P_2 and P_3 do not affect the fundamental matrices. In particular, if G^{ij} are compatible, then one solution must be

$$P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & \alpha_3 & \alpha_4 \end{bmatrix}, P_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & \beta_3 & \beta_4 \end{bmatrix}, \tag{15}$$

for α_i and β_i such that $\alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$ and $\beta_1\beta_4 - \beta_2\beta_3 \neq 0$. Given such cameras, the fundamental matrices are calculated as

$$\begin{aligned} \psi(P_1, P_2) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_3 & \alpha_1 \\ 0 & -\alpha_4 & \alpha_2 \end{bmatrix}, \\ \psi(P_1, P_3) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\beta_3 & \beta_1 \\ 0 & -\beta_4 & \beta_2 \end{bmatrix}, \\ \psi(P_2, P_3) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha_4\beta_2+\alpha_2\beta_4 & \alpha_4\beta_1-\alpha_2\beta_3 \\ 0 & \alpha_3\beta_2-\alpha_1\beta_4 & -\alpha_3\beta_1+\alpha_1\beta_3 \end{bmatrix}. \end{aligned} \tag{16}$$

Define the $\star$ operator on 2×2 matrices as

$$\begin{aligned} &\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \star \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} \\ &:= \begin{bmatrix} v_3w_1 - v_1w_3 & v_3w_2 - v_1w_4 \\ v_4w_1 - v_2w_3 & v_4w_2 - v_2w_4 \end{bmatrix} \\ &= \begin{bmatrix} v_3 & v_1 \\ v_4 & v_2 \end{bmatrix} \begin{bmatrix} w_1 & w_2 \\ -w_3 & -w_4 \end{bmatrix}. \end{aligned} \tag{17}$$

Then, by Equation (16), G^{ij} on the form Equation (16) are compatible if and only if (up to scaling) we have

$$\begin{aligned} &\begin{bmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{bmatrix} \star \begin{bmatrix} a_{13} & b_{13} \\ c_{13} & d_{13} \end{bmatrix} \\ &= \begin{bmatrix} a_{23} & b_{23} \\ c_{23} & d_{23} \end{bmatrix}. \end{aligned} \tag{18}$$

By the construction of our fundamental action, we have

$$\begin{aligned} a_{ij} &= [0,1,0]G^{ij}[0,1,0]^T = \mathbf{x}_i^T F^{ij} \mathbf{x}_j, \\ b_{ij} &= [0,1,0]G^{ij}[0,0,1]^T = \mathbf{x}_i^T F^{ij} \mathbf{y}_j, \\ c_{ij} &= [0,0,1]G^{ij}[0,1,0]^T = \mathbf{y}_i^T F^{ij} \mathbf{x}_j, \\ d_{ij} &= [0,0,1]G^{ij}[0,0,1]^T = \mathbf{y}_i^T F^{ij} \mathbf{y}_j. \end{aligned} \tag{19}$$

In the below, and throughout this section, we skip the transpose notation and write for instance $\mathbf{x}_i F^{ij} \mathbf{x}_j$ instead of $\mathbf{x}_i^T F^{ij} \mathbf{x}_j$. We get

$$\begin{aligned} &\begin{bmatrix} \mathbf{x}_1 F^{12} \mathbf{x}_2 & \mathbf{x}_1 F^{12} \mathbf{y}_2 \\ \mathbf{y}_1 F^{12} \mathbf{x}_2 & \mathbf{y}_1 F^{12} \mathbf{y}_2 \end{bmatrix} \star \begin{bmatrix} \mathbf{x}_1 F^{13} \mathbf{x}_3 & \mathbf{x}_1 F^{13} \mathbf{y}_3 \\ \mathbf{y}_1 F^{13} \mathbf{x}_3 & \mathbf{y}_1 F^{13} \mathbf{y}_3 \end{bmatrix} \\ &\qquad = \begin{bmatrix} \mathbf{x}_2 F^{23} \mathbf{x}_3 & \mathbf{x}_2 F^{23} \mathbf{y}_3 \\ \mathbf{y}_2 F^{23} \mathbf{x}_3 & \mathbf{y}_2 F^{23} \mathbf{y}_3 \end{bmatrix}. \end{aligned} \tag{20}$$

However,

$$\begin{bmatrix} \mathbf{x}_i F^{ij} \mathbf{x}_j & \mathbf{x}_i F^{ij} \mathbf{y}_j \\ \mathbf{y}_i F^{ij} \mathbf{x}_j & \mathbf{y}_i F^{ij} \mathbf{y}_j \end{bmatrix} = \begin{bmatrix} \mathbf{x}_i^T \\ \mathbf{y}_i^T \end{bmatrix} F^{ij} \begin{bmatrix} \mathbf{x}_j & \mathbf{y}_j \end{bmatrix}, \tag{21}$$

and therefore,

$$\begin{aligned} &\begin{bmatrix} \mathbf{x}_1 F^{12} \mathbf{x}_2 & \mathbf{x}_1 F^{12} \mathbf{y}_2 \\ \mathbf{y}_1 F^{12} \mathbf{x}_2 & \mathbf{y}_1 F^{12} \mathbf{y}_2 \end{bmatrix} \star \begin{bmatrix} \mathbf{x}_1 F^{13} \mathbf{x}_3 & \mathbf{x}_1 F^{13} \mathbf{y}_3 \\ \mathbf{y}_1 F^{13} \mathbf{x}_3 & \mathbf{y}_1 F^{13} \mathbf{y}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}_2^T \\ \mathbf{y}_2^T \end{bmatrix} F^{21} \begin{bmatrix} \mathbf{y}_1 & \mathbf{x}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ -\mathbf{y}_1^T \end{bmatrix} F^{13} \begin{bmatrix} \mathbf{x}_3 & \mathbf{y}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}_2^T \\ \mathbf{y}_2^T \end{bmatrix} F^{23} \begin{bmatrix} \mathbf{x}_3 & \mathbf{y}_3 \end{bmatrix}. \end{aligned} \tag{22}$$

Since this holds for generic choices of $\mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_3, \mathbf{y}_3$, we conclude that, projectively,

$$F^{21} \begin{bmatrix} \mathbf{y}_1 & \mathbf{x}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ -\mathbf{y}_1^T \end{bmatrix} F^{13} = F^{23}, \tag{23}$$

for all $\mathbf{x}_1, \mathbf{y}_1$ such that $[e_1^2 \; \mathbf{x}_1 \; \mathbf{y}_1]$ is invertible. Further,

$$\begin{bmatrix} \mathbf{y}_1 & \mathbf{x}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^T \\ -\mathbf{y}_1^T \end{bmatrix} \tag{24}$$

is skew-symmetric and equals $[\ell]_\times$ for $\ell = \mathbf{x}_1 \times \mathbf{y}_1 \in \mathbb{R}^3$. Then choosing $\mathbf{x}_1, \mathbf{y}_1$ such that $\ell = e_1^2$, we have over the real numbers that $[e_1^2 \; \mathbf{x}_1 \; \mathbf{y}_1]$ is full-rank. In other words,

$$F^{21}[e_1^2]_\times F^{13} = F^{23} \tag{25}$$

is a necessary and sufficient condition for compatibility. □

Remark C.2. *In the complex setting, it does not always suffice to put $\ell = e_1^2$, because it could be the case that $(e_1^2)^T e_1^2 = 0$. Then ℓ should be any vector such that $\ell^T e_1^2 \neq 0$.*

C.2. K_4

Remark C.3. *`Macaulay2` code for the elimination used in the proofs of this section is attached.*

Theorem 3.6 (Case 1)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices such that the three epipoles in each image do not lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold and*

$$\begin{aligned}&\mathbf{e}_{4123}\mathbf{e}_{2134}\mathbf{e}_{3142}\mathbf{e}_{4231}\mathbf{e}_{1243}\mathbf{e}_{2341}\\ =&\mathbf{e}_{3124}\mathbf{e}_{4132}\mathbf{e}_{2143}\mathbf{e}_{1234}\mathbf{e}_{3241}\mathbf{e}_{1342}.\end{aligned} \tag{26}$$

Proof. The triple-wise conditions are clearly necessary for compatibility, so we assume that they are satisfied and prove that in this case compatibility is equivalent to Equation (26) being satisfied. We begin by simplifying the problem. Let

$$H_i = \begin{bmatrix} e_i^j & e_i^k & e_i^l \end{bmatrix}. \tag{27}$$

This 3×3 matrix is of full-rank and takes the three coordinate points to the three epipoles in the i-th image. Using this as our fundamental action, we get a new sextuple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{28}$$

Since the fundamental action preserves compatibility, the sextuple $\{G^{ij}\}$ is compatible if and only if $\{F^{ij}\}$ is. Note that the epipoles of G^{ij}, denoted by h_j^i, are:

$$\begin{aligned}
h_1^2 &= [1,0,0], & h_1^3 &= [0,1,0], & h_1^4 &= [0,0,1],\\
h_2^1 &= [1,0,0], & h_2^3 &= [0,1,0], & h_2^4 &= [0,0,1],\\
h_3^1 &= [1,0,0], & h_3^2 &= [0,1,0], & h_3^4 &= [0,0,1],\\
h_4^1 &= [1,0,0], & h_4^2 &= [0,1,0], & h_4^3 &= [0,0,1].
\end{aligned} \tag{29}$$

Moreover, since G^{ij} satisfy the triple-wise conditions (we assumed F^{ij} did, and these are preserved under fundamental action), it follows that the six matrices must be on the form:

$$\begin{aligned}
G^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{12} \\ 0 & y_{12} & 0 \end{bmatrix}, & G^{13} &= \begin{bmatrix} 0 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & y_{13} & 0 \end{bmatrix},\\
G^{14} &= \begin{bmatrix} 0 & 0 & x_{14} \\ 0 & y_{14} & 0 \\ 0 & 0 & 0 \end{bmatrix}, & G^{23} &= \begin{bmatrix} 0 & 0 & x_{23} \\ 0 & 0 & 0 \\ y_{23} & 0 & 0 \end{bmatrix},\\
G^{24} &= \begin{bmatrix} 0 & 0 & x_{24} \\ y_{24} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & G^{34} &= \begin{bmatrix} 0 & x_{34} & 0 \\ y_{34} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{30}$$

The sextuple $\{G^{ij}\}$ is compatible if and only if there exists a reconstruction consisting of 4 cameras P_i. Since the epipoles do not lie on a line, any such reconstruction must have 4 linearly independent centers. We are free to choose coordinates in $\mathbb{P}^3$ without affecting compatibility, so we take the four camera centers (assuming cameras exist) to be the four unit vectors. Furthermore, we know that the epipoles satisfy

$$h_i^j = P_i(\ker(P_j)). \tag{31}$$

So if $\{G^{ij}\}$ has a reconstruction $\{P_i\}$, it must be on the form:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0 & \alpha_1^1 & 0 & 0 \\ 0 & 0 & \alpha_1^2 & 0 \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, & P_2 &= \begin{bmatrix} \alpha_2^1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2^2 & 0 \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix},\\
P_3 &= \begin{bmatrix} \alpha_3^1 & 0 & 0 & 0 \\ 0 & \alpha_3^2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, & P_4 &= \begin{bmatrix} \alpha_4^1 & 0 & 0 & 0 \\ 0 & \alpha_4^2 & 0 & 0 \\ 0 & 0 & \alpha_4^3 & 0 \end{bmatrix},
\end{aligned} \tag{32}$$

where α_i^j are scalars. Since the fundamental matrices are of rank-2 and the cameras are rank-3, all the α_i^j, as well as the x_{ij} and y_{ij} are non-zero. Computing the fundamental matrices of these four cameras, and setting them equal to the G^{ij}, we get the following six equations:

$$\begin{aligned}
x_{12}\alpha_1^2\alpha_2^3 &= y_{12}\alpha_1^3\alpha_2^2, & x_{13}\alpha_1^1\alpha_3^3 &= y_{13}\alpha_1^3\alpha_3^2,\\
x_{14}\alpha_1^1\alpha_4^3 &= y_{14}\alpha_1^2\alpha_4^2, & x_{23}\alpha_2^1\alpha_3^3 &= y_{23}\alpha_2^3\alpha_3^1,\\
x_{24}\alpha_2^1\alpha_4^3 &= y_{24}\alpha_2^2\alpha_4^1, & x_{34}\alpha_3^1\alpha_4^2 &= y_{34}\alpha_3^2\alpha_4^1.
\end{aligned} \tag{33}$$

Eliminating the variables α_i^j, we are left with a single polynomial,

$$x_{12}y_{13}x_{14}x_{23}y_{24}x_{34} - y_{12}x_{13}y_{14}y_{23}x_{24}y_{34} = 0. \tag{34}$$

This tells us that Equation (33) implies Equation (34), and we are left to argue that if x_{ij}, y_{ij} are non-zero numbers such that Equation (34) holds, then there are non-zero α_i^j such that Equation (33) holds. Note that we can assume $\alpha_1^j = 1$ by PGL_4 action and that $\alpha_i^1 = 1$ by scaling. Writing $\lambda_{ij} = x_{ij}/y_{ij}$, we then aim to find non-zero α_i^j such that

$$\begin{aligned}
\lambda_{12}\alpha_2^3 &= \alpha_2^2, & \lambda_{13}\alpha_3^3 &= \alpha_3^2, & \lambda_{14}\alpha_4^3 &= \alpha_4^2,\\
\lambda_{23}\alpha_3^3 &= \alpha_2^3, & \lambda_{24}\alpha_4^3 &= \alpha_2^2, & \lambda_{34}\alpha_4^2 &= \alpha_3^2.
\end{aligned} \tag{35}$$

It is clear that we can find non-zero α_i^j that solve the first five equations. However, this is enough because using $\lambda_{12}\lambda_{14}\lambda_{23}\lambda_{34} = \lambda_{13}\lambda_{24}$, the sixth equation $\lambda_{34}\alpha_4^2 = \alpha_3^2$ is implied by the other five through substitution.

It follows that the set $\{G^{ij}\}$ is compatible if and only if Equation (34) is satisfied. Finally, we can express the x_{ij} and y_{ij} in terms of F^{ij} and e_i^j, for instance we have

$$\begin{aligned}
x_{12} &= (h_1^3)^T G^{12} h_2^4\\
&= (h_1^3)^T H_1^T F^{12} H_2 h_2^4\\
&= (e_1^3)^T F^{12} e_2^4.
\end{aligned} \tag{36}$$

Making these substitutions for all the x_{ij} and y_{ij}, we get Equation (26). $\square$

Theorem 3.8 (Case 2)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices whose epipoles in each image are distinct and lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold,*

$$\begin{aligned}\langle F^{jk}e_k^i, F^{jl}e_l^i\rangle\langle F^{kj}e_j^i, F^{kl}e_l^i\rangle\langle F^{lj}e_j^i, F^{lk}e_k^i\rangle+\\ +\|F^{lj}e_j^i\|^2\|F^{jk}e_k^i\|^2\|F^{kl}e_l^i\|^2=0,\end{aligned}\tag{37}$$

for all distinct i,j,k,l satisfying $l>k>j$, and for $\boldsymbol{x}_i = F^{ij}e_j^l$ with $l>k>j$, we have

$$\begin{aligned}-\frac{e_2^3F^{24}\boldsymbol{x}_4}{\boldsymbol{x}_2F^{24}e_4^1}\frac{\boldsymbol{x}_1F^{12}\boldsymbol{x}_2}{\boldsymbol{x}_1F^{12}e_2^3}+\frac{e_3^2F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}\frac{\boldsymbol{x}_1F^{13}\boldsymbol{x}_3}{\boldsymbol{x}_1F^{13}e_3^2}+\\ -\frac{\boldsymbol{x}_3F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}+\frac{e_3^2F^{34}\boldsymbol{x}_4}{e_1^2F^{14}\boldsymbol{x}_4}\frac{e_1^2F^{13}\boldsymbol{x}_3}{\boldsymbol{x}_1F^{13}e_3^2}\frac{\boldsymbol{x}_1F^{14}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}+\\ +\frac{\boldsymbol{x}_2F^{24}\boldsymbol{x}_4}{\boldsymbol{x}_2F^{24}e_4^1}+\frac{e_3^1F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}\frac{\boldsymbol{x}_2F^{23}\boldsymbol{x}_3}{\boldsymbol{x}_2F^{23}e_3^1}=0.\end{aligned}\tag{38}$$

Proof. Like in the previous proof, we begin by assuming the triple-wise conditions are satisfied. The three epipoles in each image lie on a line and therefore we fix a scaling such that for each i we have $e_i^l = e_i^j + e_i^k$, where $l>k>j$. Let

$$H_i = \begin{bmatrix} e_i^j & e_i^k & \mathbf{x}_i \end{bmatrix}. \tag{39}$$

Note that $(e_i^j)^T\mathbf{x}_i$ and $(e_i^k)^T\mathbf{x}_i$ for $\mathbf{x}_i$ in the statement are both zero, so H_i is of full-rank. Using this as our fundamental action, we get a new sextuple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{40}$$

Since the fundamental action preserves compatibility, the sextuple $\{G^{ij}\}$ is compatible if and only if $\{F^{ij}\}$ is. Note that the epipoles of G^{ij} are as follows:

$$\begin{aligned} h_1^2&=[1,0,0], & h_1^3&=[0,1,0], & h_1^4&=[1,1,0],\\ h_2^1&=[1,0,0], & h_2^3&=[0,1,0], & h_2^4&=[1,1,0],\\ h_3^1&=[1,0,0], & h_3^2&=[0,1,0], & h_3^4&=[1,1,0],\\ h_4^1&=[1,0,0], & h_4^2&=[0,1,0], & h_4^3&=[1,1,0]. \end{aligned}\tag{41}$$

With these epipoles and the fact that the G^{ij} satisfy the triple-wise conditions (we assumed F^{ij} did, and these are preserved under fundamental action), it follows that the six matrices must be on the form:

$$\begin{aligned} G^{12}&=\begin{bmatrix}0&0&0\\0&0&x_{12}\\0&y_{12}&z_{12}\end{bmatrix}, & G^{13}&=\begin{bmatrix}0&0&x_{13}\\0&0&0\\0&y_{13}&z_{13}\end{bmatrix},\\ G^{14}&=\begin{bmatrix}0&0&x_{14}\\0&0&-x_{14}\\0&y_{14}&z_{14}\end{bmatrix}, & G^{23}&=\begin{bmatrix}0&0&x_{23}\\0&0&0\\y_{23}&0&z_{23}\end{bmatrix},\\ G^{24}&=\begin{bmatrix}0&0&x_{24}\\0&0&-x_{24}\\y_{24}&0&z_{24}\end{bmatrix}, & G^{34}&=\begin{bmatrix}0&0&x_{34}\\0&0&-x_{34}\\y_{34}&-y_{34}&z_{34}\end{bmatrix}. \end{aligned}\tag{42}$$

The sextuple $\{G^{ij}\}$ is compatible if and only if there exists a reconstruction consisting of 4 cameras P_i with centers that lie in a plane, but no three collinear, since the three epipoles are collinear in each image. We are free to choose coordinates in $\mathbb{P}^3$ without changing the fundamental matrices, so we take the four camera centers (assuming they exist) to be $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, and $[1,1,1,0]$. Furthermore, by the definition of the epipole, we know that the epipoles satisfy

$$h_i^j = P_i(\ker(P_j)). \tag{43}$$

So if $\{G^{ij}\}$ has a reconstruction $\{P_i\}$, it must be on the form:

$$\begin{aligned} P_1&=\begin{bmatrix}0&1&0&\alpha_1^1\\0&0&1&\alpha_1^2\\0&0&0&\alpha_1^3\end{bmatrix}, & P_2&=\begin{bmatrix}1&0&0&\alpha_2^1\\0&0&1&\alpha_2^2\\0&0&0&\alpha_2^3\end{bmatrix},\\ P_3&=\begin{bmatrix}1&0&0&\alpha_3^1\\0&1&0&\alpha_3^2\\0&0&0&\alpha_3^3\end{bmatrix}, & P_4&=\begin{bmatrix}1&0&-1&\alpha_4^1\\0&1&-1&\alpha_4^2\\0&0&0&\alpha_4^3\end{bmatrix}, \end{aligned}\tag{44}$$

where the α_i^j are scalars. Since the fundamental matrices are of rank 2 and the cameras of rank 3, the four scalars α_i^3, as well as all the x_{ij} and y_{ij} are non-zero. Computing the fundamental matrices of these four cameras, and setting them equal to the G^{ij}, we get the following set of equations:

$$\begin{aligned} \frac{x_{12}}{y_{12}}&=-\frac{\alpha_1^3}{\alpha_2^3}, & \frac{x_{13}}{y_{13}}&=-\frac{\alpha_1^3}{\alpha_3^3}, & \frac{x_{14}}{y_{14}}&=-\frac{\alpha_1^3}{\alpha_4^3},\\ \frac{x_{23}}{y_{23}}&=-\frac{\alpha_2^3}{\alpha_3^3}, & \frac{x_{24}}{y_{24}}&=-\frac{\alpha_2^3}{\alpha_4^3}, & \frac{x_{34}}{y_{34}}&=-\frac{\alpha_3^3}{\alpha_4^3}, \end{aligned}\tag{45}$$

and

$$\begin{aligned} \frac{z_{12}}{y_{12}}&=\frac{\alpha_1^2-\alpha_2^2}{\alpha_2^3}, & \frac{z_{14}}{y_{14}}&=\frac{\alpha_1^1-\alpha_1^2-\alpha_4^2}{\alpha_4^3},\\ \frac{z_{13}}{y_{13}}&=\frac{\alpha_1^1-\alpha_3^2}{\alpha_3^3}, & \frac{z_{24}}{y_{24}}&=\frac{\alpha_2^1-\alpha_2^2-\alpha_4^1}{\alpha_4^3},\\ \frac{z_{23}}{y_{23}}&=\frac{\alpha_2^1-\alpha_3^1}{\alpha_3^3}, & \frac{z_{34}}{y_{34}}&=\frac{\alpha_3^1+\alpha_4^2-\alpha_3^2-\alpha_4^1}{\alpha_4^3}. \end{aligned}\tag{46}$$

Eliminating the α_i^j from these equations gives us the following constraints:

$$x_{jk}x_{kl}y_{jl}+y_{jk}y_{kl}x_{jl}=0 \quad \forall j<k<l, \tag{47}$$

and

$$\frac{x_{24}}{y_{24}}\frac{z_{12}}{y_{12}}-\frac{x_{34}}{y_{34}}\frac{z_{13}}{y_{13}}+\frac{x_{34}}{y_{34}}\frac{z_{23}}{y_{23}}-\frac{z_{14}}{y_{14}}+\frac{z_{24}}{y_{24}}-\frac{z_{34}}{y_{34}}=0. \tag{48}$$

As in the proof of Theorem 3.6, the fundamental matrices are compatible if and only if Equations (47) and (48) are satisfied. Let k be the smallest index satisfying $k \neq i, j$, then we can write

$$\begin{aligned} x_{ij} &= e_i^k F^{ij} \mathbf{x}_j, \\ y_{ij} &= \mathbf{x}_i F^{ij} e_j^k, \\ z_{ij} &= \mathbf{x}_i F^{ij} \mathbf{x}_j. \end{aligned} \tag{49}$$

With the substitution $\mathbf{x}_i = F^{ij} e_j^l$ in Equation (47), we get:

$$\begin{aligned} &y_{jl} x_{jk} x_{kl} + x_{jl} y_{jk} y_{kl} \\ =&(\mathbf{x}_j F^{jl} e_l^i)(e_j^i F^{jk} \mathbf{x}_k)(e_k^i F^{kl} \mathbf{x}_l) \\ +&(e_j^i F^{jl} \mathbf{x}_l)(\mathbf{x}_j F^{jk} e_k^i)(\mathbf{x}_k F^{kl} e_l^i) \\ =&(e_k^i F^{kj} F^{jl} e_l^i)(e_j^i F^{jk} F^{kl} e_l^i)(e_j^i F^{jl} F^{lk} e_k^i)+ \\ +&(e_j^i F^{jl} F^{lj} e_j^i)(e_k^i F^{kj} F^{jk} e_k^i)(e_l^i F^{lk} F^{kl} e_l^i), \\ =&\langle F^{jk} e_k^i, F^{jl} e_l^i \rangle \langle F^{kj} e_j^i, F^{kl} e_l^i \rangle \langle F^{lj} e_j^i, F^{lk} e_k^i \rangle + \\ +&\|F^{lj} e_j^i\|^2 \|F^{jk} e_k^i\|^2 \|F^{kl} e_l^i\|^2 = 0, \end{aligned} \tag{50}$$

hence we arrive at Equation (37). In Equation (48), we use Equation (47) to substitute

$$-\frac{1}{y_{14}} = \frac{x_{13} x_{34}}{x_{14} y_{13} y_{34}} \tag{51}$$

and then plug in $\mathbf{x}_i = F^{ij} e_j^l$ (we do this step to get a homogeneous equation in every fundamental matrix and epipole). This gives us Equation (38). $\square$

Remark C.4. *In the complex setting, we cannot always put $\boldsymbol{x}_i = F^{ij} e_j^l$ in Theorem 3.8, because there is no longer any guarantee that this makes H_i invertible. For fixed complex F^{ij}, one can check if they are compatible in Case 2 instead by choosing any $\boldsymbol{x}_i$ that make H_i invertible. The same principle applies in Case 3.*

Theorem 3.10 (Case 3)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices such that*

$$e_1^2 = e_1^3 \neq e_1^4,\ e_2^1 = e_2^3 \neq e_2^4,\ e_3^1 = e_3^2 \neq e_3^4, \tag{52}$$

and e_4^1, e_4^2, e_4^3 are distinct and lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold,

$$\begin{aligned} \langle F^{12} e_2^4, F^{13} e_3^4 \rangle \langle F^{21} e_1^4, F^{23} e_3^4 \rangle \langle F^{31} e_1^4, F^{32} e_2^4 \rangle + \\ + \|F^{12} e_2^4\|^2 \|F^{23} e_3^4\|^2 \|F^{31} e_1^4\|^2 = 0, \end{aligned} \tag{53}$$

and for $\boldsymbol{x}_i = F^{ij} e_j^l$ with $l > k > j$, we have

$$\frac{e_2^4 F^{23} \boldsymbol{x}_3}{\boldsymbol{x}_2 F^{23} e_3^4} \frac{\boldsymbol{x}_1 F^{12} \boldsymbol{x}_2}{\boldsymbol{x}_1 F^{12} e_2^4} + \frac{\boldsymbol{x}_1 F^{13} \boldsymbol{x}_3}{\boldsymbol{x}_1 F^{13} e_3^4} - \frac{\boldsymbol{x}_2 F^{23} \boldsymbol{x}_3}{\boldsymbol{x}_2 F^{23} e_3^4} = 0. \tag{54}$$

Proof. Like in the two previous proofs, we begin by assuming the triple-wise conditions are satisfied, since we know them to be necessary. Fix a scaling such that $e_4^3 = e_4^1 + e_4^2$. Let

$$H_i = \begin{bmatrix} e_i^j & e_i^l & \mathbf{x}_i \end{bmatrix}, \quad H_4 = \begin{bmatrix} e_4^1 & e_4^2 & \mathbf{x}_4 \end{bmatrix} \tag{55}$$

for $i = 1, 2, 3$ and $l > k > j$, and

$$G^{ij} = H_i^T F^{ij} H_j. \tag{56}$$

Let $\mathbf{x}_i = F^{ij} e_j^l$ with $l > k > j$ for $i = 1, 2, 3$. Since all epipolar numbers are zero in this case, $(e_i^j)^T \mathbf{x}_i$ and $(e_i^l)^T \mathbf{x}_i$ are both zero, H_i is of full-rank for $i = 1, 2, 3$. Let $\mathbf{x}_4$ be such that H_4 is full-rank. The fundamental matrices G^{ij} are compatible if and only if F^{ij} are. Note that the epipoles of G^{ij} are:

$$\begin{aligned} h_1^2 &= [1,0,0], & h_1^3 &= [1,0,0], & h_1^4 &= [0,1,0], \\ h_2^1 &= [1,0,0], & h_2^3 &= [1,0,0], & h_2^4 &= [0,1,0], \\ h_3^1 &= [1,0,0], & h_3^2 &= [1,0,0], & h_3^4 &= [0,1,0], \\ h_4^1 &= [1,0,0], & h_4^2 &= [0,1,0], & h_4^3 &= [1,1,0]. \end{aligned} \tag{57}$$

With these epipoles and the fact that the G^{ij} satisfy the triple-wise conditions (preserved under fundamental action), it follows that the six matrices must be on the form:

$$\begin{aligned} G^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{12} \\ 0 & y_{12} & z_{12} \end{bmatrix}, & G^{13} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{13} \\ 0 & y_{13} & z_{13} \end{bmatrix}, \\ G^{14} &= \begin{bmatrix} 0 & 0 & x_{14} \\ 0 & 0 & 0 \\ 0 & y_{14} & z_{14} \end{bmatrix}, & G^{23} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{23} \\ 0 & y_{23} & z_{23} \end{bmatrix}, \\ G^{24} &= \begin{bmatrix} 0 & 0 & x_{24} \\ 0 & 0 & 0 \\ y_{24} & 0 & z_{24} \end{bmatrix}, & G^{34} &= \begin{bmatrix} 0 & 0 & x_{34} \\ 0 & 0 & 0 \\ y_{34} & -y_{34} & z_{34} \end{bmatrix}. \end{aligned} \tag{58}$$

We have seen in the proof of Proposition 3.4 that regarding the triple G^{12}, G^{13}, G^{23}, we must have (up to scale)

$$G^{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -y_{12} x_{13} \\ 0 & x_{12} y_{13} & x_{12} z_{13} - x_{13} z_{12} \end{bmatrix}.$$

The sextuple $\{G^{ij}\}$ is compatible if and only if there exists a reconstruction consisting of 4 cameras P_i with the centers of P_1, P_2, P_3 lying on a line that does not contain the center of P_4. To see this, note that the three epipoles in each image are collinear, implying that any reconstruction must consist of cameras with coplanar centers. Furthermore, since two epipoles coincide in the first three images, the centers of P_1, P_2, P_3 must lie on a line. We are free to choose coordinates in $\mathbb{P}^3$ without changing the fundamental matrices, so we take the four camera centers (assuming they exist) to be $[1,0,0,0]$, $[0,1,0,0]$, $[1,1,0,0]$, and $[0,0,1,0]$. We recall that the epipoles satisfy

$$h_i^j = P_i(\ker(P_j)). \tag{59}$$

So if $\{G^{ij}\}$ has a reconstruction $\{P_i\}$, it must be on the form:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0 & 1 & 0 & \alpha_1^1 \\ 0 & 0 & \beta_1 & \alpha_1^2 \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 0 & \alpha_2^1 \\ 0 & 0 & \beta_2 & \alpha_2^2 \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix}, \\
P_3 &= \begin{bmatrix} 1 & -1 & 0 & \alpha_3^1 \\ 0 & 0 & \beta_3 & \alpha_3^2 \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & 0 & 0 & \alpha_4^1 \\ 0 & 1 & 0 & \alpha_4^2 \\ 0 & 0 & 0 & \alpha_4^3 \end{bmatrix}.
\end{aligned} \tag{60}$$

where the β_i, α_i^j are scalars. Since the fundamental matrices are rank-2 and the cameras rank-3, the four scalars α_i^3, as well as all the β_i, x_{ij} and y_{ij} are non-zero. Computing the fundamental matrices of these four cameras, and setting them equal to the G^{ij}, we get after elimination the following two equations:

$$\begin{aligned}
x_{12}x_{23}y_{13} + x_{13}y_{12}y_{23} = 0, \\
\frac{x_{23}}{y_{23}}\frac{z_{12}}{y_{12}} + \frac{z_{13}}{y_{13}} - \frac{z_{23}}{y_{23}} = 0.
\end{aligned} \tag{61}$$

As in the proof of Theorem 3.6, the fundamental matrices are compatible if and only if Equation (61) is satisfied. As in Theorem 3.8, we can write Equation (61) in terms of $\mathbf{x}_i$, the fundamental matrices and their epipoles. Indeed, we get

$$\frac{e_1^4 F^{12}\mathbf{x}_2}{\mathbf{x}_1 F^{12} e_2^4}\frac{e_2^4 F^{23}\mathbf{x}_3}{\mathbf{x}_2 F^{23} e_3^4} + \frac{e_1^4 F^{13}\mathbf{x}_3}{\mathbf{x}_1 F^{13} e_3^4} = 0, \tag{62}$$

and

$$\frac{e_2^4 F^{23}\mathbf{x}_3}{\mathbf{x}_2 F^{23} e_3^4}\frac{\mathbf{x}_1 F^{12}\mathbf{x}_2}{\mathbf{x}_1 F^{12} e_2^4} + \frac{\mathbf{x}_1 F^{13}\mathbf{x}_3}{\mathbf{x}_1 F^{13} e_3^4} - \frac{\mathbf{x}_2 F^{23}\mathbf{x}_3}{\mathbf{x}_2 F^{23} e_3^4} = 0. \tag{63}$$

Setting $\mathbf{x}_i = F^{ij} e_j^l$ for $i = 1, 2, 3$ and $l > k > j$, Equations (62) and (63) become Equations (53) and (54), the conditions of the statement. □

C.3. K_n

Theorem 3.12. *Let $\{F^{ij}\}$ be a complete set of $\binom{n}{2}$, $n \geq 4$, fundamental matrices such that the sextuple $F^{12}, F^{13}, F^{14}, F^{23}, F^{24}, F^{34}$ is compatible with a solution of cameras P_1, P_2, P_3, P_4 such that the line spanned by the centers of P_1, P_2 do not contain the centers of P_3, P_4. If each sextuple of fundamental matrices corresponding to indices $\{1, 2, 3, i\}$ and $\{1, 2, 4, i\}$ for $i \geq 5$ are compatible, then $\{F^{ij}\}$ is compatible.*

Moreover, if all epipoles in each image coincide, then triple-wise compatibility implies that $\{F^{ij}\}$ is compatible. The reconstruction in this case will be a set of cameras whose centers all lie on a line.

Proof of Theorem 3.12. We start with the collinear case. As in the proof of Proposition 3.4, it suffices to prove the statement for fundamental matrices

$$G^{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{ij} & b_{ij} \\ 0 & c_{ij} & d_{ij} \end{bmatrix}. \tag{64}$$

By the compatibility of $\{G^{1i}, G^{1j}, G^{ij}\}$, we have by Proposition 3.4 that

$$G^{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_{1i}a_{1j} - a_{1i}c_{1j} & c_{1i}b_{1j} - a_{1i}d_{1j} \\ 0 & d_{1i}a_{1j} - b_{1i}c_{1j} & d_{1i}b_{1j} - b_{1i}d_{1j} \end{bmatrix}, \tag{65}$$

for all $i, j \neq 1$. It can be verified that the following cameras P_i form a reconstruction of these fundamental matrices:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
P_i &= \begin{bmatrix} i & 1 & 0 & 0 \\ 0 & 0 & b_{1i} & d_{1i} \\ 0 & 0 & -a_{1i} & -c_{1i} \end{bmatrix}, \forall i \neq 1.
\end{aligned} \tag{66}$$

Hence the $\binom{n}{2}$-tuple is compatible whenever each triple is compatible. We also observe that all cameras have a center lying on the line $[\lambda_1, \lambda_2, 0, 0]$.

Now assume that in some image, not all epipoles coincide. We prove the theorem for the case $n = 5$ and note that the principle extends to any n.

Consider the sextuple $S_{1234} = \{F^{12}, F^{13}, F^{14}, F^{23}, F^{24}, F^{34}\}$. Let P_1, P_2, P_3, P_4 be a solution. Let P_1', P_2', P_3', P_5 be a solution to $S_{1235} = \{F^{12}, F^{13}, F^{15}, F^{23}, F^{25}, F^{35}\}$. By Lemma 1.2, we have that P_1, P_2, P_3 and P_1', P_2', P_3' differ by PGL_4, and we may therefore take them to be equal.

It remains to prove that F^{45} is the fundamental matrix of P_4, P_5. For this we note that either 1) P_1, P_2, P_5 or 2) P_1, P_3, P_5 are not collinear cameras, since P_1, P_2, P_3 are not collinear. In the first case 1), consider the tuple $S_{1245} = \{F^{12}, F^{14}, F^{15}, F^{24}, F^{25}, F^{45}\}$ with solution $P_1'', P_2'', P_4'', P_5''$. By Lemma 1.2, the overlap between S_{1235} and S_{1245} imply that we can via PGL_4 action assume $P_1'' = P_1, P_2'' = P_2, P_5'' = P_5$, and the overlap between S_{1234} and S_{1245} imply that we can also assume $P_4'' = P_4$, since P_1, P_2, P_4 are not collinear. But since F^{45} is the fundamental matrix of P_4'', P_5'' we conclude that it is also the fundamental matrix of P_4, P_5. In the second case 2) the argument is analogous when we consider S_{1345} instead of S_{1245}. □

Proposition 3.14. *A compatible set of $\binom{n}{2}$ fundamental matrices has a unique solution unless all the epipoles in each image are equal.*

Proof. Since the set of fundamental matrices is compatible, and the epipoles in each image are not all equal, we know that there exists a reconstruction consisting of n cameras, not all lying on a line. It follows from the Sylvester-Gallai theorem [1, Chapter 11] that there will always be at least two cameras P_1, P_2 such that the line spanned by their camera centers does not contain any other camera centers.

By Lemma 1.2, a triple of compatible fundamental matrices has a unique solution if the two epipoles in each image are distinct, or equivalently if their reconstruction consists of three non-collinear cameras. Up to projective transformation, we can uniquely recover P_1, P_2 from F^{12}, which fixes coordinates in $\mathbb{P}^3$. All other cameras P_i are then uniquely determined by the triple F^{12}, F^{1i}, F^{2i}. Since this uniquely determines all cameras (up to global projective transformation), the fundamental matrices F^{ij} can only have one solution. $\square$

C.4. n-view matrices

We recall the following theorem from [3, 6].

Theorem 3.15. *Let $\{F^{ij}\}$ be a complete set of $\binom{n}{2}$ real fundamental matrices, where $n \geq 3$. Then $\{F^{ij}\}$ is compatible with a solution of real cameras whose centers are not all collinear if and only if there exist non-zero scalars $\lambda_{ij} = \lambda_{ji}$ such that:*

1. *the m-view fundamental matrix $\mathbf{F} = (\lambda_{ij} F^{ij})_{ij}$ is rank-6 and has exactly three positive and three negative eigenvalues;*
2. *the $3 \times 3m$ and $3m \times 3$ block rows and block columns of $\mathbf{F}$ are all of rank 3.*

Further, $\{F^{ij}\}$ is compatible with a solution of real cameras whose centers are all collinear if and only if there exist non-zero scalars $\lambda_{ij} = \lambda_{ji}$ such that:

1. *the m-view fundamental matrix $\mathbf{F} = (\lambda_{ij} F^{ij})_{ij}$ is rank-4 and has exactly two positive and two negative eigenvalues;*
2. *the $3 \times 3m$ and $3m \times 3$ block rows and block columns of $\mathbf{F}$ are all of rank 2.*

Theorem 3.16. *In the collinear case of Theorem 3.15, the eigenvalue condition can be dropped. In the non-collinear case, the eigenvalue condition can be dropped if in each image, no three epipoles lie on a line.*

Proof of Theorem 3.16. The structure of the proof is as follows. We prove in detail the when $n = 3$ and sketch $n = 4$ for Case 1. The `Macaulay2` code used in all these settings is attached. Then, we use Theorem 3.12 to argue that the general setting is implied by these case studies.

We start with $n = 3$ in the collinear setting. Let F^{ij} be three fundamental matrices for which there exists a scaling λ such that

$$\begin{bmatrix} 0 & F^{12} & F^{13} \\ F^{21} & 0 & \lambda F^{23} \\ F^{31} & \lambda F^{32} & 0 \end{bmatrix} \tag{67}$$

is rank-4 and the 3×6 and 6×3 block rows and colums are rank-2. Note that we don't need to scale F^{12} and F^{21} or F^{13} and F^{31} in the same way, because scaling each row and each column does not change the rank of the 3-view matrix, so we may choose their scalings to be 1 without loss of generality. By the latter condition, F^{12} and F^{13} must have the same epipoles. We can say even more, namely that

$$e_1^2 = e_1^3, \quad e_2^1 = e_2^3, \quad e_3^1 = e_3^2. \tag{68}$$

As in the proof of Proposition 3.4, this assumption allows us to assume via fundamental action F^{ij} take the form G^{ij} of Equation (12). We work in the polynomial ring $R = \mathbb{Q}[a_{ij}, b_{ij}, c_{ij}, d_{ij}, \lambda]$, where $1 \leq i < j \leq 3$ consider the following 3-view matrix:

$$\mathbf{G}(\lambda) := \begin{bmatrix} 0 & G^{12} & G^{13} \\ G^{21} & 0 & \lambda G^{23} \\ G^{31} & \lambda G^{32} & 0 \end{bmatrix}. \tag{69}$$

The rank of $\mathbf{G}(\lambda)$ is at most 4 if and only if all 5×5 minors of $\mathbf{G}(\lambda)$ vanish and we therefore consider the ideal I_{minors} in R defined by the 5×5 minors of $\mathbf{G}(\lambda)$. Since we don't want solutions with $\lambda = 0$ or $\text{rank} G^{ij} < 2$, we saturate I_{minors} with respect to the ideals $I_\lambda = \langle \lambda \rangle$ and $I_{ij} = \langle a_{ij} d_{ij} - b_{ij} c_{ij} \rangle$. After this is done in `Macaulay2`, we get a new ideal I_{rank} in R with nine generators.

Write $G^{ij'}$ for the matrices we get by removing the first row and column from G^{ij}. Recall that G^{ij} on the form Equation (12) are compatible if and only if they are rank-2 and up to scaling, $G^{12'} \star G^{13'} = G^{23'}$, i.e. Equation (18) holds. To get rid of the scale ambiguity, we divide both sides of Equation (18), by, say, the top left entry. We get

$$\begin{aligned} &\begin{bmatrix} 1 & \frac{c_{12} b_{13} - a_{12} d_{13}}{c_{12} a_{13} - a_{12} c_{13}} \\ \frac{d_{12} a_{13} - b_{12} c_{13}}{c_{12} a_{13} - a_{12} c_{13}} & \frac{d_{12} b_{13} - b_{12} d_{13}}{c_{12} a_{13} - a_{12} c_{13}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & b_{23}/a_{23} \\ c_{23}/a_{23} & d_{23}/a_{23} \end{bmatrix}. \end{aligned} \tag{70}$$

This equality gives us three polynomial equalities (after clearing the denominators), and for each choice of entry in the 2×2 matrices, we get another three equations. We let J_{red} in R be the ideal generated by these twelve equations. It is not hard to check that rank-2 matrices $G^{ij'}$ satisfy Equation (18) (up to scale) if and only if they satisfy the equations of J_{red}, and for brevity we leave the details to the reader. Note that this ideal is reducible, as shown by the command `primaryDecomposition` in `Macaulay2`. One component consists of rank-deficient tuples $G^{ij'}$ and we call the other component $J_\star$. In particular, any tuple of rank-2 matrices $G^{ij'}$ satisfy Equation (18) (up to scale) if and only if they satisfy the conditions of $J_\star$.

By Lemma A.2, if G^{ij} are rank-2, on the form Equation (12), and there exists $\lambda \neq 0$ with $\mathbf{G}(\lambda)$ rank-4, then the entries of G^{ij} satisfy the equations of I_{rank}. In `Macaulay2` we see that the ideals I_{rank} and $J_\star$ are equal. It follows that G^{ij} satisfy the equations of $J_\star$. By the above, this implies that G^{ij} are compatible, showing that the eigenvalue condition was not needed for compatibility.

For $n = 3$ in the non-collinear setting, we choose a fundamental action

$$\begin{aligned} H_1 &= \begin{bmatrix} e_1^2 \; e_1^3 \; \mathbf{x}_1 \end{bmatrix}, H_2 = \begin{bmatrix} e_2^1 \; e_2^3 \; \mathbf{x}_2 \end{bmatrix}, \\ H_3 &= \begin{bmatrix} e_3^1 \; e_3^2 \; \mathbf{x}_3 \end{bmatrix}, \end{aligned} \tag{71}$$

for $\mathbf{x}_i$ making H_i full-rank. Using this as our fundamental action, we get a new sextuple of fundamental matrices

$$G^{ij} = H_i^T F^{ij} H_j. \tag{72}$$

The sextuple $\{G^{ij}\}$ is compatible if and only if $\{F^{ij}\}$ is. Note that the epipoles of G^{ij}, denoted by h_j^i, are:

$$\begin{aligned} h_1^2 &= [1,0,0], \quad h_1^3 = [0,1,0], \\ h_2^1 &= [1,0,0], \quad h_2^3 = [0,1,0], \\ h_3^1 &= [1,0,0], \quad h_3^2 = [0,1,0]. \end{aligned} \tag{73}$$

The three matrices must be on the form:

$$\begin{aligned} G^{12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_{12} & y_{12} \\ 0 & z_{12} & w_{12} \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & x_{13} & y_{13} \\ 0 & 0 & 0 \\ 0 & y_{13} & z_{13} \end{bmatrix}, \\ G^{23} &= \begin{bmatrix} x_{23} & 0 & y_{23} \\ 0 & 0 & 0 \\ z_{23} & 0 & w_{23} \end{bmatrix}. \end{aligned} \tag{74}$$

We work in the polynomial ring $R = \mathbb{Q}[x_{ij}, y_{ij}, z_{ij}, w_{ij}, \lambda]$, where $1 \leq i < j \leq 3$ consider the following 3-view matrix:

$$\mathbf{G}(\lambda) := \begin{bmatrix} 0 & G^{12} & G^{13} \\ G^{21} & 0 & \lambda G^{23} \\ G^{31} & \lambda G^{32} & 0 \end{bmatrix}. \tag{75}$$

The corresponding I_{rank}, defined analogously to the collinear case, equals $\langle x_{12}, x_{13}, x_{23} \rangle$. This means $\mathbf{G}(\lambda)$ being rank-6 for a $\lambda \neq 0$ implies $x_{12} = 0, x_{13} = 0, x_{23} = 0$.

As in the proof of Theorem 3.6, if there is a solution of cameras P_i with non-collinear centers to Equation (74), then we may choose them to be

$$\begin{aligned} P_1 &= \begin{bmatrix} 0 & \alpha_1^1 & 0 & * \\ 0 & 0 & \alpha_1^2 & * \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} \alpha_2^1 & 0 & 0 & * \\ 0 & 0 & \alpha_2^2 & * \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} \alpha_3^1 & 0 & 0 & * \\ 0 & \alpha_3^2 & 0 & * \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, \end{aligned} \tag{76}$$

where α_i^j are non-zero scalars, and $*$ are some other (possible zero) scalars. Computing the fundamental matrices of these four cameras, one can check that by the degrees of freedom of the cameras established in Equation (76), any triple of fundamental matrices on the form Equation (74) with $x_{ij} = 0$ has a solution with cameras on the form Equation (76). It follows that if there is a non-zero scalar λ for which Equation (75) is rank-6, then the triples of fundamental matrices G^{ij} are compatible, which is sufficient.

In the setting of $n = 4$ in Case 1, we use the same ideas and therefore only sketch the proofs. Start with a 4-view matrix $\mathbf{F}$ that is rank-6 and with block rows and columns of rank-3 as in Theorem 3.15. Then take any sub 3-view matrix $\mathbf{F}'$. It is at most rank-6. However, since the epipoles in each image are all distinct, all its block rows and columns must be rank-3. This is only possible if $\mathbf{F}'$ is at least rank-6. Now we can apply the above to see that the three fundamental matrices of this 3-view matrix are compatible. In other words, we have triple-wise compatibility. Then we can assume the fundamental matrices to be of the form Equation (30) and look at the ideal generated by the 7×7 minors given such matrices with indeterminate entries. Here we scale G^{23}, G^{24}, G^{34} with $\lambda_1, \lambda_2, \lambda_3$, respectively. After saturation of λ_i and rank-deficienly loci, and after elimination of λ_i, we get in each case an ideal that we call I_{rank}. This ideal in each case describes the same conditions as the ideal generated by Equation (34). This means that the rank condition implies compatibility.

Now we move on to general values of n. First, in the general collinear case, let F^{ij} be fundamental matrices for which there are scalars λ_{ij} such that the n-view matrix $\mathbf{F} = (\lambda_{ij} F^{ij})$ is rank-4 and whose $3 \times 3n$ and $3n \times 3$ block rows and columns are rank-2. By Theorem 3.12, it suffices to show triple-wise compatibility. Take any 3-view submatrix $\mathbf{F}'$. It is at most rank-4 and its block rows and columns at most rank-2. But since the fundamental matrices are rank-2, the block rows and columns must be at least rank-2 and it follows that the 3-view matrix itself is at least rank-4. Therefore triple-wise compatibility follows from an earlier step of this proof. By similar logic, if the n-view matrix $\mathbf{F}$ instead is rank-6 with block rows and columns of rank 3, then this also applies for any sub 4-view matrix $\mathbf{F}'$, since we assumed that any three epipoles in each image do not lie on a line. In particular, we are then in Case 1 and by the above, we have quadruple-wise compatibility. By Theorem 3.12, this suffices. □

D. The Cycle Theorem

In order to prove the cycle theorem we need a lemma.

Lemma D.1. *Let $\mathcal{G}$ be a connected graph and T any spanning tree subgraph. Then there is a sequence $T^i \subseteq \mathcal{G}$ such that*

$$T = T^0 \subseteq \cdots \subseteq T^k = \mathcal{G}, \tag{77}$$

where T^{i+1} contains exactly one more edge than T^i and this edge is part of a cycle of T^{i+1}.

Proof. We get T^{k-1} from T^k by removing an edge of T^k that is not in T. We repeat this process until we reach T^0. Assume that the edge removed from T^{i+1} is not part of a cycle of T^{i+1}. Then T^i would have to be disconnected. This implies that T cannot be connected, which is a contradiction. ☐

Theorem 4.1. *Let $\{F^{ij}\}$ be a set of fundamental matrices with corresponding graph $\mathcal{G}$. $\{F^{ij}\}$ is compatible if and only if there are matrices $H_i \in \mathrm{GL}_3$ and scalars $\lambda_{ij} = \lambda_{ji} \neq 0$ such that $G^{ij} := \lambda_{ij} H_i^T F^{ij} H_j$ satisfy*

$$\sum_{(ij)\in E(C)} G^{ij} = 0, \text{ for each directed cycle } C \text{ of } \mathcal{G}. \quad (78)$$

In particular, any set of 3×3 rank-2 matrices G^{ij} satisfying the cycle condition Equation (78) are the fundamental matrices of some set of cameras.

Proof. We proved direction $\Rightarrow$ in the main body of the paper.

$\Leftarrow$) We find a set of cameras C_i such that $\psi(C_i, C_j)$ equals G^{ij} for every edge of $\mathcal{G}$. Since F^{ij} and G^{ij} are equivalent under fundamental action, this is enough. We may without restriction assume that $\mathcal{G}$ is connected with m nodes. Since G^{ij} are skew-symmetric and rank-2, there are non-zero $g^{ij} \in \mathbb{R}^3$ such that $G^{ij} = [g^{ij}]_\times$. The cycle condition is then equivalent to

$$\sum_{(ij)\in E(C)} g^{ij} = 0, \text{ for each directed cycle } C \text{ of } \mathcal{G}. \quad (79)$$

Let T be a spanning tree subgraph of $\mathcal{G}$.

Fix $i = 1$ and let $t^{(1)} = 0 \in \mathbb{R}^3$. To any node v in T, there is a unique path with no repeated vertices from 1 to v in T, since T is a tree. Let $\sigma_{u,v} = \{u = i_1, i_2, \dots, i_k = v\}$ denote the unique path between two vertices u, v of T. For $i > 1$, define

$$t^{(v)} := \sum_{(ij)\in\sigma_{1,v}} g^{ij}. \quad (80)$$

This gives us cameras $C_i = [I|t^{(i)}]$ for each $i = 1, \dots, m$. We must check that $G^{ij} = \psi(C_i, C_j)$ for every edge of $\mathcal{G}$. Recall that for cameras on this form, $\psi(C_i, C_j) = [t^{(j)} - t^{(i)}]_\times$. If (ij) is an edge of T, then $t^{(j)} - t^{(i)} = g^{ij}$ by construction, which shows $G^{ij} = \psi(C_i, C_j)$. For (ij) that are not edges of T, we proceed as follows. Consider the sequence T^i of Lemma D.1. We proceed via induction to show that $G^{ij} = \psi(C_i, C_j)$ for every edge of T^l for any l. The base case $T^0 = T$ is already done. Assume that C_i satisfy $G^{ij} = \psi(C_i, C_j)$ for all edges of T^l. In T^{l+1}, there is precisely one new edge (ij) and that edge is part of a cycle C of T^{l+1}. Using Equation (80), we get after some cancellation for some vertex u of the cycle that

$$\psi(C_i, C_j) = [t^{(j)} - t^{(i)}]_\times \quad (81)$$
$$= \sum_{(st)\in\sigma_{u,j}} [g^{st}]_\times - \sum_{(st)\in\sigma_{u,i}} [g^{st}]_\times. \quad (82)$$

Since G^{ij} are skew-symmetric by the conditions of the 2-cycles, $g^{ji} = -g^{ij}$. Therefore we get

$$\psi(C_i, C_j) = \sum_{(st)\in\sigma_{i,j}} [g^{st}]_\times. \quad (83)$$

However, by the cycle condition for the cycle C, this equals $[g^{ij}]_\times$, which shows $G^{ij} = \psi(C_i, C_j)$ for every edge in T^{l+1} and completes the induction. ☐

References

[1] Martin Aigner and Günter M Ziegler. Proofs from the book. Berlin. Germany, 1:2, 1999. 7

[2] David Cox, John Little, Donal O'Shea, and Moss Sweedler. Ideals, varieties, and algorithms. American Mathematical Monthly, 101(6):582–586, 1994. 1

[3] Amnon Geifman, Yoni Kasten, Meirav Galun, and Ronen Basri. Averaging essential and fundamental matrices in collinear camera settings. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition, pages 6021–6030, 2020. 8

[4] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/, 2020. 1

[5] Richard I. Hartley and Andrew Zisserman. Multiple View Geometry in Computer Vision. Cambridge University Press, ISBN: 0521540518, second edition, 2004. 2

[6] Yoni Kasten, Amnon Geifman, Meirav Galun, and Ronen Basri. Gpsfm: Global projective sfm using algebraic constraints on multi-view fundamental matrices. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition, pages 3264–3272, 2019. 8