Compatibility of Fundamental Matrices for Complete Viewing Graphs

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Abstract

This paper studies the problem of recovering cameras from a set of fundamental matrices. A set of fundamental matrices is said to be compatible if a set of cameras exists for which they are the fundamental matrices. We focus on the complete graph, where fundamental matrices for each pair of cameras are given. Previous work has established necessary and sufficient conditions for compatibility as rank and eigenvalue conditions on the n-view fundamental matrix obtained by concatenating the individual fundamental matrices. In this work, we show that the eigenvalue condition is redundant in the generic and collinear cases. We provide explicit homogeneous polynomials that describe necessary and sufficient conditions for compatibility in terms of the fundamental matrices and their epipoles. In this direction, we find that quadruple-wise compatibility is enough to ensure global compatibility for any number of cameras. We demonstrate that for four cameras, compatibility is generically described by triple-wise conditions and one additional equation involving all fundamental matrices.

Introduction

The problem of finding camera matrices that correspond to a given set of fundamental matrices is crucial in 3D reconstructions from 2D images. Typically, multiview structure-from-motion pipelines start by estimating fundamental matrices from point correspondences, with early methods for such estimations dating back to the 1990s and new methods still being developed today [21, 25, 26, 29]. However, these methods usually only estimate a subset of all possible fundamental matrices between cameras. To describe this incomplete set of fundamental matrices, viewing graphs are often used [17].

In this paper, we focus on understanding the conditions under which a reconstruction of n cameras can be obtained given complete knowledge of $\binom{n}{2}$ fundamental matrices, but we also give a result for general graphs at the end. Here, a camera refers to a full-rank 3×4 matrix, and the fundamental matrix of two cameras P_1 and P_2 with distinct kernels is a 3×3 rank-2 matrix that encodes all point correspondences between them. For any given rank-2 3×3 matrix F^{12}, there exists a pair of cameras P_1 and P_2 for which F^{12} is the fundamental matrix, this pair is unique up to global projective transformation. However, for a set of $\binom{n}{2}$ rank-2 3×3 matrices F^{ij}, where $n > 2$, it is not always guaranteed that there exist cameras $P_1, \dots, P_n$ such that F^{ij} is the fundamental matrix of P_i and P_j for each i, j. Following the notation of [12] we say that the set F^{ij} is compatible if such cameras do exist. Note that some recent literature uses the term consistent instead [15].

Finding necessary and sufficient conditions for compatibility of fundamental matrices has practical applications as well as theoretical ones. [15] proposes an algorithm for projective structure-from-motion that employs their necessary and sufficient condition for compatibility. The algorithm is designed to handle collections of measured fundamental matrices, both complete and partial, and aims to find camera matrices that minimize a global algebraic error for the given set of matrices. As for theoretical purposes, [5, 6, 11] uses necessary and sufficient conditions for compatibility to give a classification of critical configurations.

In the case of $n = 3$, a classical result [12, Section 15.4] provides triple-wise constraints on F^{12}, F^{13}, F^{23} in terms of the fundamental matrices and their epipoles, where the i-th epipole in the j-th image is defined as $e_j^i := \ker F^{ij}$. For non-collinear cameras, [15, Theorem 1] provides necessary and sufficient conditions for compatibility for any n. These conditions rely on the eigenvalues and rank of the n-view fundamental matrix, which is obtained by stacking all fundamental matrices into a $3n \times 3n$ matrix. In the follow-up work, [9, Theorem 2] arrives at a similar condition in the collinear case. Both methods rely on fixing a correct scaling of each matrix and are therefore not projectively well-defined, nor are the conditions expressed in terms of the fundamental matrices and their epipoles, as in the $n = 3$ case.

The contributions of this paper include giving explicit homogeneous polynomials that provide necessary and sufficient conditions for the compatibility of fundamental matrices in the case of complete graphs. This is done in Section 3. Specifically, for the case of $n = 4$, we establish that a set of six fundamental matrices admits a reconstruction of camera matrices with linearly independent centers only if the triple-wise constraints and an additional polynomial equation involving all six fundamental matrices and their epipoles are satisfied. We also demonstrate, using the computer algebra system `Macaulay2` [10], that the eigenvalue conditions from [9, 15] are superfluous in the generic case and in the case where all epipoles in each image coincide. In Section 2 we introduce the fundamental action, a key tool in simplifying the problem of finding compatibility conditions. Section 4 presents a necessary and sufficient condition for compatibility for any viewing graph via a cycle condition, similar to cycle-based formulations of parallel rigidity that appear in the calibrated case.

We approach compatibility of fundamental matrices from an algebraic point of view, i.e., we aim to describe constraints through algebraic equations and polynomial equations using techniques and software from applied algebraic geometry. This approach to questions in computer vision has a long standing tradition [1, 7, 13, 16, 27].

Related work

History. The problem of determining whether a set of fundamental matrices is compatible has a curious history. [12] provided a necessary and sufficient triple-wise condition for the compatibility of three fundamental matrices F^{12}, F^{13} and F^{23} arising from three cameras with non-collinear centers. In 2007, the paper [11, Theorem 2.2] claimed that this condition was sufficient for compatibility even in the case of cameras with collinear centers, a claim that we show to be false in Example 3.3. During the next decade, few advances were made in understanding compatibility. Over time, a belief seemed to develop that triple-wise compatibility was enough to ensure global compatibility. In fact, articles such as [24, Section 2.1] claimed this to be true, based on a faulty proof provided in [22]. In 2018, [28, Section 3.3] pointed out that the proof in [22] fails in some cases, but still agreed that the result holds for complete graphs. Example 3.5 shows that this is not the case by providing a counterexample.

Essential matrices. In the context of uncalibrated cameras, which are defined as full-rank 3×4 matrices, this work, as well as [15], provide necessary and sufficient conditions for compatibility of fundamental matrices. However, camera matrices are often assumed to be calibrated, represented in the form of $[R|t]$ for a rotation matrix R and a translation vector t. The corresponding fundamental matrices are called essential matrices. In [14], the authors build upon their previous work and provide a necessary and sufficient condition for compatibility of essential matrices, in terms of the n-view essential matrix obtained by stacking all essential matrices into a larger matrix. This condition is then used to recover a consistent set of essential matrices, given a partial set of measured essential matrices. In [18], Martyushev provides a necessary and sufficient condition for compatibility of three essential matrices.

Solvability. There has been extensive research on the topic of solvability of viewing graphs in computer vision, as evidenced by various studies such as [3, 4, 17, 19, 22, 27, 28]. A viewing graph is considered solvable if, given a generic set of cameras, their fundamental matrices have a unique solution in terms of cameras up to global projective transformation. Recently, [3] proposed a new formulation of solvability and developed an effective algorithm for testing it, which was able to resolve some open questions from previous studies, such as [28].

The primary distinction between solvability and compatibility lies in the fact that, in the latter, the existence of cameras that correspond to a set of fundamental matrices is not assumed to exist. Moreover, compatibility has mostly been studied for graphs where each possible fundamental matrix is given, whereas papers on solvability study viewing graphs without such restrictions.

Furthermore, solvability has been investigated in the case of calibrated cameras, where it is known that the solvable graphs are precisely those that are parallel rigid [20, 23].

1. Preliminaries

In this section we recall established notation and results. We refer the reader to [8] for the basics on algebraic geometry and [12] for the application of algebra in 3D reconstruction problems.

We work over the real numbers, although all results in this paper either directly hold in the complex case or can be reformulated to do so. Where slight adjustments have to be made over the complex numbers, we make a remark.

Let $\mathbb{R}^n$ denote the set of real vectors with n coordinates, we call this affine space. Let $\mathbb{P}^{n-1}$ denote its projectivization. We write $\mathbb{R}^{n\times m}$ to denote the set of real $n \times m$ matrices, and we write $\mathbb{P}^{n\times m}$ to denote the set of real projective $n \times m$ matrices.

We define a rational map,

$$\psi : \mathbb{P}^{3\times 4} \times \mathbb{P}^{3\times 4} \dashrightarrow \mathbb{P}^{3\times 3} \tag{1}$$

as follows. Given a pair of 3×4 matrices P_1 and P_2 (defined up to scale), let $\mathbf{x}$ and $\mathbf{y}$ be two 3×1 vectors. The determinant

$$\det \begin{bmatrix} P_1 & \mathbf{x} & 0 \\ P_2 & 0 & \mathbf{y} \end{bmatrix}, \tag{2}$$

is a bilinear polynomial with in $\mathbf{x}$ and $\mathbf{y}$, meaning there is a matrix F^{12} (defined up to scale) such that (2) can be written as $\mathbf{x}^T F^{12}\mathbf{y}$. We define $\psi(P_1, P_2)$ to be this 3×3 matrix. This map is undefined, i.e. $\psi(P_1, P_2) = 0$, precisely when $\ker P_1 \cap \ker P_2 \neq \{0\}$.

We refer to rank-2 3×3 matrices as *fundamental matrices* (either in $\mathbb{R}^{3\times 3}$ or $\mathbb{P}^{3\times 3}$) and we refer to rank-3 3×4 matrices as cameras (either in $\mathbb{R}^{3\times 4}$ or $\mathbb{P}^{3\times 4}$). The center of a camera P is its kernel $\ker P$. Before we list a set of well-known results, partly found in [12, Section 9], we recall that GL_n denotes the set of invertible $n \times n$ matrices and that PGL_n is its projectivization.

Proposition 1.1.

1. *$\psi(P_1, P_2)$ is of rank at most 2, and it attains this rank if P_1, P_2 are cameras with distinct centers;*
2. *for any fundamental matrix F^{12}, there exist two cameras P_1, P_2 such that F^{12} is their fundamental matrix. All other cameras C_1, C_2 with fundamental matrix F^{12} satisfy $C_1 = P_1 H, C_2 = P_2 H$ for some $H \in \mathrm{PGL}_4$;*
3. *$\psi(P_2, P_1) = \psi(P_1, P_2)^T$;*
4. *if F^{12} is the fundamental matrix of P_1, P_2, then $\ker F^{12} = P_2 \ker(P_1)$;*
5. *for cameras P_1, P_2, we have $F^{12} = \psi(P_1, P_2)$ if and only if $P_1^T F^{12} P_2$ is a skew-symmetric matrix.*

We say that a set of fundamental matrices $\{F^{ij}\}$ is compatible if there are cameras $P_1, \ldots, P_n$ such that $F^{ij} = \psi(P_i, P_j)$. The cameras $P_1, \ldots, P_n$ are called a solution to F^{ij}. We mostly focus on complete viewing graphs, i.e. when $\{F^{ij}\}$ contains all $\binom{n}{2}$ fundamental matrices for n indices. However, in Section 4, we provide a result that holds not only in this setting, but for any viewing graph.

We define the i-th epipole e_j^i in the j-th image to be an affine representative of $\ker F^{ij}$. By Proposition 1.1 4., e_j^i is the image of the i-th camera center taken by the j-th camera.

Lemma 1.2 ([12, Section 15.4])**.** *Let $\{F^{12}, F^{13}, F^{23}\}$ be compatible. There is a unique solution if and only if the two epipoles in each image are distinct.*

Although fundamental matrices and epipoles are only defined up to scale, i.e. as elements in projective space, we always assume for convenience that we are given affine representatives of them and that the representatives of fundamental matrices satisfy $(F^{ij})^T = F^{ji}$, unless otherwise is specified.

Given a fixed set of fundamental matrices F^{ij}, we point out that there is a rather simple method of finding possible solutions in terms of cameras by first using F^{12} to recover P_1, P_2 and then using Lemma 1.2 with matrices $\{F^{12}, F^{1i}, F^{2i}\}$ to recover the remaining P_i (a detailed algorithm can be found in [11, Section 6.1]). Finding explicit equations in terms of the fundamental matrices and epipoles for compatability is however more difficult, and is the subject of this paper.

2. The Fundamental Action

In this section, we formally introduce the fundamental action, a key tool in simplifying the problem of finding compatibility conditions. GL_3^n (or equivalently PGL_3^n) acts on a set of fundamental matrices $\{F^{ij}\}$ by

$$\{F^{ij}\} \mapsto \{H_i^T F^{ij} H_j\}. \tag{3}$$

We call this the fundamental action of GL_3^n. The main appeal of this action is that we can use it to simplify a set of fundamental matrices, without affecting compatibility.

Proposition 2.1. *Let $\{F^{ij}\}$ be a set of fundamental matrices. Let P_i be a solution to $\{F^{ij}\}$. For any $(H_1, \ldots, H_n, H) \in \mathrm{PGL}_3^n \times \mathrm{PGL}_4$, we have,*

$$\psi(H_i^{-1} P_i H, H_j^{-1} P_j H) = H_i^T \psi(P_i, P_j) H_j. \tag{4}$$

In particular, $\{F^{ij}\}$ is compatible if and only if $\{G^{ij}\}$ is compatible, where $G^{ij} := H_i^T F^{ij} H_j$.

Proof. It is a standard fact that the action of $H \in \mathrm{PGL}_4$ in Equation (4) does not change the fundamental matrix, so we may set $H = I$. Consider the following equality up to scaling,

$$\det \begin{bmatrix} H_i^{-1} P_i & x_i & 0 \\ H_j^{-1} P_j & 0 & x_j \end{bmatrix} = \det \begin{bmatrix} P_i & H_i x_i & 0 \\ P_j & 0 & H_j x_j \end{bmatrix}. \tag{5}$$

Writing these expressions in terms of fundamental matrices, we get exactly Equation (4). □

The fundamental action gives rise to an equivalence relation. For compatible fundamental matrices, the equivalence classes turn out to be the equivalence classes of n points in $\mathbb{P}^3$ under PGL_4.

Proposition 2.2. *Let $\{F^{ij}\}$ and $\{G^{ij}\}$ be two sets of compatible fundamental matrices. They are equivalent under fundamental action if and only if they have solutions whose camera centers are equivalent under PGL_4.*

Remark 2.3. *All proofs that are not in the main body appear in the Supplementary Material.*

In our study, quantities of the form $\mathbf{e}_{sijt} := (e_i^s)^T F^{ij} e_j^t$, called epipolar numbers, are important (see Theorems 3.2 and 3.6). The epipolar numbers are invariant under the fundamental action:

Lemma 2.4. *Let $\{F^{ij}\}$ be a set of fundamental matrices with epipoles $\{e^i_j\}$. Let $H_i \in \mathrm{GL}^n_3$ and consider the fundamental matrices $G^{ij} := H_i^T F^{ij} H_j$, whose epipoles are $h^i_j = H_j^{-1} e^i_j$. Then*

$$(e^s_i)^T F^{ij} e^t_j = (h^s_i)^T G^{ij} h^t_j. \qquad (6)$$

Proof. The equality follows directly by the definitions of G^{ij} and h^i_j. □

We have the following geometrical interpretation of the epipolar numbers.

Lemma 2.5. *Let $\{F^{ij}\}$ be set of compatible fundamental matrices that include F^{si}, F^{ij} and F^{jt}. We have $\mathbf{e}_{sijt} = 0$ if and only if the centers c_s, c_i, c_j and c_t of any solution are coplanar.*

It follows from the lemma that putting any of the two indices s, i, j, t equal, the epipolar number is zero. In particular, $\mathbf{e}_{sijs}$ is always zero for compatible fundamental matrices, because three centers are always in a plane.

3. Compatibility for Complete Graphs

We begin by giving our main results for complete graphs, that is, the case where all the fundamental matrices are known. The main contribution of this paper is providing explicit, algebraic conditions for compatibility expressed in terms of the fundamental matrices and their epipoles for any number of views. Let K_n denote the complete graph on n nodes.

In Section 3.1, we deal with K_3 graphs and recall the triple-wise conditions. We also state a result for the collinear case. In Section 3.2 we find necessary and sufficient constraints for compatibility in the case of K_4. In Section 3.3 we prove that quadruple-wise compatbility implies global compatibility. Finally, in Section 3.4 we state that the eigenvalue condition from the theorem of Kasten et. al. is redundant in the generic and collinear cases.

Due to limitations on the length of the paper, most proofs are moved to the Supplementary Material. In this section we provide a sketch of proof for the main theorem (Theorem 3.6) to display the techniques we use.

Remark 3.1. *In this section, we work only with real numbers, because it allows us to give polynomials equations using the standard inner product and norm on $\mathbb{R}^3$. However, all of our statements in Section 3.1 and Section 3.2 can be extended to the complex numbers as explained in the Supplementary Material.*

3.1. K_3

The case of three fundamental matrices is fairly straightforward. We have two possible configurations for the three camera centers; they either all lie on a line, or they do not.

Theorem 3.2 ([12, Section 15.4]). *Let F^{12}, F^{13}, F^{23} be fundamental matrices. There exist non-collinear cameras P_1, P_2, P_3 such that $F^{ij} = \psi(P_i, P_j)$ if and only if*

$$e^2_1 \neq e^3_1, \quad e^1_2 \neq e^3_2, \quad e^1_3 \neq e^2_3, \qquad (7)$$

and

$$(e^3_1)^T F^{12} e^3_2 = (e^2_1)^T F^{13} e^2_3 = (e^1_2)^T F^{23} e^1_3 = 0. \qquad (8)$$

The conditions of Equation (8) are called the <u>triple-wise conditions.</u>

If P_1, P_2, P_3 are cameras with collinear centers, then it follows that $P_i(\ker P_j) = P_i(\ker P_k)$ for all distinct i, j, k. This implies that for the corresponding fundamental matrices F^{12}, F^{13}, F^{23}, we have $e^i_j = e^k_j$ for all distinct i, j, k. However, contrary to what is claimed in [11], the conditions in Equation (8) are not enough in this case:

Example 3.3. Consider the fundamental matrices:

$$F^{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad F^{13} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\ F^{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad (9)$$

with epipoles:

$$e^2_1 = [1,0,0], \quad e^1_2 = [1,0,0], \quad e^1_3 = [1,0,0], \\ e^3_1 = [1,0,0], \quad e^3_2 = [1,0,0], \quad e^2_3 = [1,0,0]. \qquad (10)$$

These six matrices satisfy the conditions in Equation (8). However, no solution of cameras P_1, P_2, P_3 exist for which F^{12}, F^{13}, F^{23} are the fundamental matrices. This can be checked for instance via the algorithm described at the end of Section 1. ◇

To the best of our knowledge, the following result does not appear in the literature.

Proposition 3.4. *Let F^{12}, F^{13}, F^{23} be fundamental matrices. There exist collinear cameras P_1, P_2, P_3 such that $F^{ij} = \psi(P_i, P_j)$ if and only if*

$$e^2_1 = e^3_1, \quad e^1_2 = e^3_2, \quad e^1_3 = e^2_3, \qquad (11)$$

and

$$(F^{12})^T [e^2_1]_\times F^{13} = F^{23}. \qquad (12)$$

3.2. K_4

We start this section with a counterexample to the previous belief that triple-wise compatibility is enough to ensure full compatibility.

Example 3.5. Consider the fundamental matrices:

$$F^{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad F^{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$F^{14} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F^{23} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{13}$$
$$F^{24} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F^{34} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with epipoles:

$$\begin{aligned} &e_1^2 = [1,0,0], \quad e_1^3 = [0,1,0], \quad e_1^4 = [0,0,1], \\ &e_2^1 = [1,0,0], \quad e_2^3 = [0,1,0], \quad e_2^4 = [0,0,1], \\ &e_3^1 = [1,0,0], \quad e_3^2 = [0,1,0], \quad e_3^4 = [0,0,1], \\ &e_4^1 = [1,0,0], \quad e_4^2 = [0,1,0], \quad e_4^3 = [0,0,1]. \end{aligned} \tag{14}$$

It can easily be verified that these six matrices satisfy the conditions in Theorem 3.2. Nonetheless, no solution exists. Any attempt to find four cameras will end up matching at most five of the six fundamental matrices. We will soon see that this is because the sextuple does not satisfy the conditions in Theorem 3.6. ◇

Before we get to the main results, we list the possible configurations of camera centers in the case of four cameras (six fundamental matrices). By Proposition 2.2, these correspond to the equivalence classes of compatible fundamental matrices. Each of these will be recognizable from the epipoles e_i^j:

Case 1: Cameras are in generic position, meaning no plane contains all four centers. Epipoles are in generic position, meaning in each image, the three epipoles do not lie on a line.

Case 2: All camera centers lie in the same plane, but no three lie on a line. In each image, the three epipoles are distinct and lie on a line.

Case 3: Precisely three camera centers lie on a line. In the three corresponding images, the epipoles corresponding to the other two cameras are equal, with the third one different from these two. In the final image, the three epipoles are distinct and lie on a line.

Case 4: All four camera centers lie on a line. In each image, the three epipoles coincide.

These are the only possible configurations of four cameras, so any compatible sextuple $\{F^{ij}\}$ must have its epipoles in one of the configurations above. If we have, for instance, collinear epipoles in one image, but not all, the fundamental matrices can not be compatible. In all four cases above, the configuration of the epipoles together with the triple-wise conditions alone is not enough to ensure compatibility; we need additional constraints. We cover all cases in sequence. We recall the epipolar numbers: $\mathbf{e}_{sijt} = (e_i^s)^T F^{ij} e_j^t$.

Theorem 3.6 (Case 1)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices such that the three epipoles in each image do not lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold and*

$$\begin{aligned} &\mathbf{e}_{4123}\mathbf{e}_{2134}\mathbf{e}_{3142}\mathbf{e}_{4231}\mathbf{e}_{1243}\mathbf{e}_{2341} \\ =&\mathbf{e}_{3124}\mathbf{e}_{4132}\mathbf{e}_{2143}\mathbf{e}_{1234}\mathbf{e}_{3241}\mathbf{e}_{1342}. \end{aligned} \tag{15}$$

Remark 3.7. *The condition that the epipoles in each image do not lie on a line is equivalent to all epipolar number $\mathbf{e}_{ijkl}$ being non-zero for distinct i, j, k, l.*

Sketch of Proof. The triple-wise conditions are necessary for compatibility, so we assume that they are satisfied and prove that in this case compatibility is equivalent to Equation (15) being satisfied. We begin by simplifying the problem. By Proposition 2.1, changing coordinates in the images via the fundamental action does not affect the compatibility. In each image, the three epipoles are linearly independent, so we can take them to be:

$$\begin{aligned} &h_1^2 = [1,0,0], \quad h_1^3 = [0,1,0], \quad h_1^4 = [0,0,1], \\ &h_2^1 = [1,0,0], \quad h_2^3 = [0,1,0], \quad h_2^4 = [0,0,1], \\ &h_3^1 = [1,0,0], \quad h_3^2 = [0,1,0], \quad h_3^4 = [0,0,1], \\ &h_4^1 = [1,0,0], \quad h_4^2 = [0,1,0], \quad h_4^3 = [0,0,1]. \end{aligned} \tag{16}$$

We denote by G^{ij} the fundamental matrices we get after this fundamental action. Since we assumed that F^{ij} satisfies the triple-wise conditions, so do G^{ij}. In other words, we have $(h_i^k)^T G^{ij} h_j^k = 0$ for all i, j, k. It follows that the six matrixes G^{ij} must be on the form:

$$G^{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{12} \\ 0 & y_{12} & 0 \end{bmatrix}, \quad G^{13} = \begin{bmatrix} 0 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & y_{13} & 0 \end{bmatrix},$$
$$G^{14} = \begin{bmatrix} 0 & 0 & x_{14} \\ 0 & y_{14} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G^{23} = \begin{bmatrix} 0 & 0 & x_{23} \\ 0 & 0 & 0 \\ y_{23} & 0 & 0 \end{bmatrix}, \tag{17}$$
$$G^{24} = \begin{bmatrix} 0 & 0 & x_{24} \\ y_{24} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G^{34} = \begin{bmatrix} 0 & x_{34} & 0 \\ y_{34} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This sextuple is compatible if and only if there exists a reconstruction consisting of 4 cameras P_i. We are free to choose coordinates in $\mathbb{P}^3$ without affecting compatibility, so we take the four camera centers (assuming cameras exist) to be the four unit vectors. If P_i is a solution to G^{ij}, then recall that $h_i^j = P_i(\ker(P_j))$ for each epipole. Therefore, if $\{G^{ij}\}$ has a reconstruction $\{P_i\}$, it must be on the form:

$$\begin{aligned} P_1 &= \begin{bmatrix} 0 & \alpha_1^1 & 0 & 0 \\ 0 & 0 & \alpha_1^2 & 0 \\ 0 & 0 & 0 & \alpha_1^3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} \alpha_2^1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2^2 & 0 \\ 0 & 0 & 0 & \alpha_2^3 \end{bmatrix}, \\ P_3 &= \begin{bmatrix} \alpha_3^1 & 0 & 0 & 0 \\ 0 & \alpha_3^2 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3^3 \end{bmatrix}, \quad P_4 = \begin{bmatrix} \alpha_4^1 & 0 & 0 & 0 \\ 0 & \alpha_4^2 & 0 & 0 \\ 0 & 0 & \alpha_4^3 & 0 \end{bmatrix}, \end{aligned} \tag{18}$$

where α_i^j are scalars. Computing the fundamental matrices of these four cameras, and setting them equal to the G^{ij}, we get the following six equations:

$$\begin{aligned} x_{12}\alpha_1^2\alpha_2^3 = y_{12}\alpha_1^3\alpha_2^2, &\quad x_{13}\alpha_1^1\alpha_3^3 = y_{13}\alpha_1^3\alpha_3^2, \\ x_{14}\alpha_1^1\alpha_4^3 = y_{14}\alpha_1^2\alpha_4^2, &\quad x_{23}\alpha_2^1\alpha_3^3 = y_{23}\alpha_2^3\alpha_3^1, \\ x_{24}\alpha_2^1\alpha_4^3 = y_{24}\alpha_2^2\alpha_4^1, &\quad x_{34}\alpha_3^1\alpha_4^2 = y_{34}\alpha_3^2\alpha_4^1. \end{aligned} \tag{19}$$

Since the fundamental matrices are rank-2 and the cameras rank-3, all the α_i^j, as well as the x_{ij} and y_{ij} are non-zero. Eliminating the variables α_i^j, we are left with a single polynomial,

$$x_{12}y_{13}x_{14}x_{23}y_{24}x_{34} - y_{12}x_{13}y_{14}y_{23}x_{24}y_{34} = 0. \tag{20}$$

We can express the x_{ij} and y_{ij} in terms of G^{ij} and h_i^j. For instance, $x_{12} = (h_1^3)^T G^{12} h_1^4$, and by Lemma 2.4, $x_{12} = (e_1^3)^T F^{12} e_1^4$. Making these substitutions for all the x_{ij} and y_{ij}, we get Equation (15), thus completing the proof. □

The proofs of the results in the other 3 cases follow a similar pattern. These proofs, as well as the full proof of Theorem 3.6, can be found in the Supplementary Material.

In Cases 2, 3 and 4, the three epipoles in each image lie on a line. This is equivalent to all epipolar numbers $\mathbf{e}_{ijkl}$ being zero for distinct i, j, k, l.

Theorem 3.8 (Case 2)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices whose epipoles in each image are distinct and lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold,*

$$\begin{aligned} \langle F^{jk}e_k^i, F^{jl}e_l^i\rangle\langle F^{kj}e_j^i, F^{kl}e_l^i\rangle\langle F^{lj}e_j^i, F^{lk}e_k^i\rangle + \\ + \|F^{lj}e_j^i\|^2\|F^{jk}e_k^i\|^2\|F^{kl}e_l^i\|^2 = 0, \end{aligned} \tag{21}$$

for all distinct i, j, k, l satisfying $j < k < l$, and for $\boldsymbol{x}_i = F^{ij}e_j^l$ with $j < k < l$, we have

$$\begin{aligned} &-\frac{e_2^3F^{24}\boldsymbol{x}_4}{\boldsymbol{x}_2F^{24}e_4^1}\frac{\boldsymbol{x}_1F^{12}\boldsymbol{x}_2}{\boldsymbol{x}_1F^{12}e_2^3} + \frac{e_3^2F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}\frac{\boldsymbol{x}_1F^{13}\boldsymbol{x}_3}{\boldsymbol{x}_1F^{13}e_3^2} + \\ &-\frac{\boldsymbol{x}_3F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1} + \frac{e_3^2F^{34}\boldsymbol{x}_4}{e_1^2F^{14}\boldsymbol{x}_4}\frac{e_1^2F^{13}\boldsymbol{x}_3}{\boldsymbol{x}_1F^{13}e_3^2}\frac{\boldsymbol{x}_1F^{14}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1} + \\ &+\frac{\boldsymbol{x}_2F^{24}\boldsymbol{x}_4}{\boldsymbol{x}_2F^{24}e_4^1} + \frac{e_3^1F^{34}\boldsymbol{x}_4}{\boldsymbol{x}_3F^{34}e_4^1}\frac{\boldsymbol{x}_2F^{23}\boldsymbol{x}_3}{\boldsymbol{x}_2F^{23}e_3^1} = 0. \end{aligned} \tag{22}$$

Remark 3.9. *As equations (22) and (25) are already oversaturated with sub/superscript, we are omitting the transpose symbol from these equations. It is to be understood that the 3-vectors $\boldsymbol{x}_i$ and e_i^j are column-vectors when directly right of a fundamental matrix, and row-vectors when to the left.*

Theorem 3.10 (Case 3)**.** *Let $\{F^{ij}\}$ be a sextuple of fundamental matrices such that*

$$e_1^2 = e_1^3 \neq e_1^4,\; e_2^1 = e_2^3 \neq e_2^4,\; e_3^1 = e_3^2 \neq e_3^4, \tag{23}$$

and e_4^1, e_4^2, e_4^3 are distinct and lie on a line. Then $\{F^{ij}\}$ is compatible if and only if the triple-wise conditions hold,

$$\begin{aligned} \langle F^{12}e_2^4, F^{13}e_3^4\rangle\langle F^{21}e_1^4, F^{23}e_3^4\rangle\langle F^{31}e_1^4, F^{32}e_2^4\rangle + \\ + \|F^{12}e_2^4\|^2\|F^{23}e_3^4\|^2\|F^{31}e_1^4\|^2 = 0, \end{aligned} \tag{24}$$

and for $\boldsymbol{x}_i = F^{ij}e_j^l$ with $l > k > j$, we have

$$\frac{e_2^4F^{23}\boldsymbol{x}_3}{\boldsymbol{x}_2F^{23}e_3^4}\frac{\boldsymbol{x}_1F^{12}\boldsymbol{x}_2}{\boldsymbol{x}_1F^{12}e_2^4} + \frac{\boldsymbol{x}_1F^{13}\boldsymbol{x}_3}{\boldsymbol{x}_1F^{13}e_3^4} - \frac{\boldsymbol{x}_2F^{23}\boldsymbol{x}_3}{\boldsymbol{x}_2F^{23}e_3^4} = 0. \tag{25}$$

Remark 3.11. *All polynomials equations in Theorems 3.6, 3.8 and 3.10 are homogeneous in every fundamental matrix and epipole.*

We discuss Case 4 in the next subsection.

3.3. K_n

For the case of more than 4 cameras, it turns out that quadruple-wise compatibility is sufficient to ensure global compatibility.

Theorem 3.12. *Let $\{F^{ij}\}$ be a complete set of $\binom{n}{2}$, $n \geq 4$, fundamental matrices such that for all i, j, k, l, the sextuple $F^{ij}, F^{ik}, F^{jk}, F^{il}, F^{jl}, F^{kl}$ is compatible. Then $\{F^{ij}\}$ is compatible.*

Moreover, if all epipoles in each image coincide, then triple-wise compatibility implies that $\{F^{ij}\}$ is compatible. The reconstruction in this case will be a set of cameras whose centers all lie on a line.

Remark 3.13. *In the Supplementary Material, we state a more general version of this theorem that doesn't require every single sextuple to be compatible.*

We then get necessary and sufficient polynomial constraints for Case 4 by the second half of Theorem 3.12 and Proposition 3.4.

While uniqueness is not the focus of this paper, we give the following useful theorem on the complete graph:

Proposition 3.14. *A compatible set of $\binom{n}{2}$ fundamental matrices has a unique solution up to action by* $\mathrm{PGL}(4)$ *unless all the epipoles in each image are equal.*

3.4. n-view matrices

The compatibility of $\binom{n}{2}$ fundamental matrices F^{ij} was also studied in [9, 15] and we recall their results below. Given a set of $\binom{n}{2}$ fundamental matrices F^{ij}, the n-view fundamental matrix is the $3n \times 3n$ symmetric matrix

$$\mathbf{F} := \begin{bmatrix} 0 & F^{12} & \cdots & F^{1n} \\ F^{21} & 0 & \cdots & F^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F^{n1} & F^{n2} & \cdots & 0 \end{bmatrix}. \tag{26}$$

Theorem 3.15 (Theorem 1 of [15], Theorem 2 of [9])**.** *Let $\{F^{ij}\}$ be a complete set of $\binom{n}{2}$ real fundamental matrices, where $n \geq 3$. Then $\{F^{ij}\}$ is compatible with a solution of real cameras whose centers are not all collinear if and only if there exist non-zero scalars $\lambda_{ij} = \lambda_{ji}$ such that:*

1. *the m-view fundamental matrix $\mathbf{F} = (\lambda_{ij}F^{ij})_{ij}$ is rank-6 and has exactly three positive and three negative eigenvalues;*
2. *the $3 \times 3m$ and $3m \times 3$ block rows and block columns of $\mathbf{F}$ are all of rank 3.*

Further, $\{F^{ij}\}$ is compatible with a solution of real cameras whose centers are all collinear if and only if there exist non-zero scalars $\lambda_{ij} = \lambda_{ji}$ such that:

1. *the m-view fundamental matrix $\mathbf{F} = (\lambda_{ij}F^{ij})_{ij}$ is rank-4 and has exactly two positive and two negative eigenvalues;*
2. *the $3 \times 3m$ and $3m \times 3$ block rows and block columns of $\mathbf{F}$ are all of rank 2.*

Our work regarding the K_3 and K_4 cases can be used to improve on this result by showing that the eigenvalue condition can often be dropped.

Theorem 3.16. *In the collinear case of Theorem 3.15, the eigenvalue condition can be dropped. In the non-collinear case, the eigenvalue condition can be dropped if in each image, no three epipoles lie on a line.*

4. The Cycle Theorem

Although the focus of this paper has been on complete graphs, in this section we state the cycle theorem, which holds for all graphs. We use this theorem to give an alternative derivation of necessary conditions for compatibility from Section 3. We consider sets of fundamental matrices $\{F^{ij}\}$, where the index pairs (ij) are a subset of all $\binom{n}{2}$ possible ones. Let $\mathcal{G} = (V, E)$ denote the corresponding graph, where V is the set of indices and E the set of pairs of indices for which there is a fundamental matrix in our set. The definitions of compatibility and solution extends naturally to this setting.

The theorem below gives a necessary and sufficient condition for when a set of fundamental matrices are compatible using the cycle condition for any graph $\mathcal{G}$. Recall that a directed cycle C of a graph is a closed path, i.e. a path that starts and ends at the same vertex. Let $E(C)$ denote its directed edges.

Theorem 4.1. *Let $\{F^{ij}\}$ be a set of fundamental matrices with corresponding graph $\mathcal{G}$. $\{F^{ij}\}$ is compatible if and only if there are matrices $H_i \in \mathrm{GL}_3$ and scalars $\lambda_{ij} = \lambda_{ji} \neq 0$ such that $G^{ij} := \lambda_{ij} H_i^T F^{ij} H_j$ satisfy*

$$\sum_{(ij)\in E(C)} G^{ij} = 0, \text{ for each directed cycle } C \text{ of } \mathcal{G}. \tag{27}$$

In particular, any set of 3×3 rank-2 matrices G^{ij} satisfying the cycle condition Equation (27) are the fundamental matrices of some set of cameras.

This theorem is very similar to the result [2, Proposition 5], which appears in the context of parallel rigidity and is relevant for the solvability of essential matrices.

Observe that the cycles of length two in Equation (27) imply that G^{ij} are skew-symmetric.

Proof. We prove direction $\Rightarrow$ here and prove direction $\Leftarrow$ in the Supplementary Material, from which the last part of the statement follows.

Let $\{F^{ij}\}$ be a compatible set of fundamental matrices, with a solution of cameras P_i. By right action of $H \in \mathrm{GL}_4$, we may assume that the centers of these cameras has a non-zero last coordinate. Then the the first three vectors must be linearly independent and the cameras can be written $[H_i|v^{(i)}]$, where $H_i \in \mathrm{GL}_3$ and $v^{(i)} \in \mathbb{R}^3$. By left multiplication with H_i^{-1}, we may further assume that all cameras are of the form $C_i = [I|t^{(i)}]$, where $t^{(i)} \in \mathbb{R}^3$. Write

$$[t^{(i)}]_\times = \begin{bmatrix} 0 & -t_3^{(i)} & t_2^{(i)} \\ t_3^{(i)} & 0 & -t_1^{(i)} \\ -t_2^{(i)} & t_1^{(i)} & 0 \end{bmatrix}. \tag{28}$$

One can then check that the fundamental matrix of C_i and C_j is

$$[t^{(j)}]_\times - [t^{(i)}]_\times = [t^{(j)} - t^{(i)}]_\times \in \mathbb{R}^{3\times 3}, \tag{29}$$

and we call these skew-symmetric matrices G^{ij}. Note that G^{ij} are scalings of $H_i^T F^{ij} H_j$. If we sum G^{ij} for (ij) in a cycle C, we must get 0 by Equation (29). This proves direction $\Rightarrow$. □

For the rest of this section, we apply the cycle theorem to find conditions that must hold for compatible fundamental matrices. For instance, let G^{12}, G^{13} and G^{23} be fundamental matrices satisfying the cycle condition. By the 2-cycles, we can write $G^{ij} = [g^{ij}]_\times$ for some $g^{ij} \in \mathbb{R}^3$. Letting $\{h^i_j\}$ be the epipoles of $\{G^{ij}\}$ defined as

$$h^i_j := (g^{ij}_1, g^{ij}_2, g^{ij}_3)^T, \tag{30}$$

one can check that

$$(h^1_2)^T G^{23} h^1_3 = \det[g^{12}\ g^{23}\ g^{31}]. \tag{31}$$

Therefore, $g^{12} + g^{23} + g^{31} = 0$ implies $(h^1_2)^T G^{23} h^1_3 = 0$. Now if F^{12}, F^{13} and F^{23} are compatible fundamental matrices, then by the cycle theorem there is a scaling and fundamental action such that $G^{ij} = \lambda_{ij} H_i^T F^{ij} H_j$ satisfy the cycle condition. This means that F^{ij} must satisfy $e^1_2 F^{23} e^1_3 = 0$, hence giving us Equation (8).

We next sketch an argument for why the n-view fundamental matrix $\mathbf{F}$ (see Section 3) for compatible $\{F^{ij}\}$ is at most rank 6 given appropiate scalings. For the sake of simplicity assume $n = 4$, but note that the below principle directly extends to any n. Let $\{G^{ij}\}$ be six fundamental matrices satisfying the cycle condition. Consider the 4-view matrix $\mathbf{G} = (G^{ij})_{ij}$. Subtracting the first row of $\mathbf{G}$ from the other rows, we have

$$\mathbf{G} = \begin{bmatrix} 0 & G^{12} & G^{13} & G^{14} \\ G^{21} & 0 & G^{23} & G^{24} \\ G^{31} & G^{32} & 0 & G^{34} \\ G^{41} & G^{42} & G^{43} & 0 \end{bmatrix} \tag{32}$$

$$\sim \begin{bmatrix} 0 & G^{12} & G^{13} & G^{14} \\ -G^{12} & -G^{12} & -G^{12} & -G^{12} \\ -G^{13} & -G^{13} & -G^{13} & -G^{13} \\ -G^{14} & -G^{14} & -G^{14} & -G^{14} \end{bmatrix}, \tag{33}$$

where $\sim$ denotes equivalence under Gaussian eliminiation. The rank of the first three rows of Equation (33) is at most 3, and the rank of the last nine rows is the rank of the first three columns of Equation (33), which is at most 3. In total, the matrix is of rank at most 6. Now if $\{F^{ij}\}$ is a set of compatible fundamental matrices, there is a scaling and fundamental action such that $G^{ij} = \lambda_{ij} H_i^T F^{ij} H_j$ satisfy the cycle condition. Define the n-view fundamental matrix $\mathbf{F} = (\lambda_{ij} F^{ij})_{ij}$. Since the rank of a matrix is invariant under conjugation, the above shows that $\operatorname{rank} \mathbf{F} \leq 6$.

Finally, we use the cycle theorem to give alternative proof that Equation (15) is necessary to ensure compatibility. Let $\{G^{ij}\}$ be 6 skew-symmetric matrices. Again, write $G^{ij} = [g^{ij}]_\times$ and let $\lambda_{ij} = \lambda_{ji} \neq 0$ be scalars such that $\lambda_{ij} G^{ij}$ satisfy the cycle condition. Then

$$\lambda_{kl} g^{kl} = -\lambda_{jk} g^{jk} - \lambda_{ij} g^{ij} - \lambda_{li} g^{li}, \tag{34}$$

for all indices $i, j, k, l \in \{1, 2, 3, 4\}$ and it follows that

$$\begin{aligned} &\det[\lambda_{ij} g^{ij}\ \lambda_{jk} g^{jk}\ \lambda_{kl} g^{kl}] \\ &= \det[\lambda_{ij} g^{ij}\ \lambda_{jk} g^{jk}\ -\lambda_{li} g^{li}] \\ &= -\det[\lambda_{li} g^{li}\ \lambda_{ij} g^{ij}\ \lambda_{jk} g^{jk}]. \end{aligned} \tag{35}$$

Factoring out the constants, and with h^i_j defined as in Equation (30), we get

$$\lambda_{ij}\lambda_{jk}\lambda_{ki}(h^i_j)^T G^{jk} h^l_k = -\lambda_{li}\lambda_{ij}\lambda_{jk}(h^l_i)^T G^{ij} h^k_j. \tag{36}$$

Assuming that all epipolar numbers $(h^i_j)^T G^{jk} h^l_k$ are nonzero, and recalling that λ_{ij} are non-zero, we find

$$\frac{(h^i_j)^T G^{jk} h^l_k}{(h^l_i)^T G^{ij} h^k_j} = -\frac{\lambda_{li}}{\lambda_{ki}}. \tag{37}$$

Further, using $\lambda_{ij} = \lambda_{ji}$,

$$\frac{\lambda_{31}}{\lambda_{21}} \frac{\lambda_{12}}{\lambda_{32}} \frac{\lambda_{23}}{\lambda_{43}} \frac{\lambda_{34}}{\lambda_{24}} \frac{\lambda_{24}}{\lambda_{14}} \frac{\lambda_{41}}{\lambda_{31}} = 1. \tag{38}$$

Combining Equations (37) and (38), we get Equation (15) for $\{\lambda_{ij} G^{ij}\}$. Now if we start with a set of six compatible fundamental matrices $\{F^{ij}\}$, then by Theorem 4.1, there is a fundamental action such that $G^{ij} = H_i^T F^{ij} H_j$ are skew-symmetric and there are scalars λ_{ij} making the cycle condition hold for $\{\lambda_{ij} G^{ij}\}$. Then Equation (15) holds for $\{G^{ij}\}$ and by the invariance of the epipolar numbers under fundamental action, we get Equation (15) for $\{F^{ij}\}$.

5. Conclusion

This paper provided explicit polynomial contraints as necessary and sufficient conditions for $\binom{n}{2}$ fundamental matrices to be compatible. These polynomials were expressed in terms of the fundamental matrices and their epipoles, and are projectively well-defined, i.e. homogeneous. As a consequence of our work, the previously established necessary and sufficient condition [15] can be simplified by dropping the eigenvalue condition. Our main tool was to define and use the fundamental action of sets of fundamental matrices. In the final section, we gave a necessary and sufficient condition for compatibility that applied not only to complete graphs, but to any viewing graph. We used it to give an alternative derivation of necessary conditions for compatibility.

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