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Price Operators Analysis in L_p-spaces

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Abstract

An integral type representation and various extension theorems for monotone linear operators in L_p-spaces are considered in relation to market prices modeling. In particular a characterization of the existence of a risk-neutral probability measure is provided in terms of the given prices. An evaluation of the density of the risk-neutral probability measure with respect to the underlying applied one is also provided.

Key-words: risk-neutral probability measure, price operators, Hölder equality, Hahn-Banach extension theorem, König sandwich theorem.

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0 Introduction.

According to the concept of "fair" market (market with "no arbitrage" - cf. [2], for example), the prices are considered to satisfy the equation

$$X_s = E^0(R^{-1}X_t/\mathfrak{A}_s), \qquad t > s,$$

in the course of time, where X_t is the price at time t and E^0 is the expectation with respect to the probability measure P^0 for the market events. The conditional expectation is with respect to the σ-algebra $\mathfrak{A}_s$ of the events up to time s. The discount factor R^{-1} is due to the *risk less return* $R > 0$, in time t for 1-unit of capital invested in a certain "risk less" financial operation at time s. We can think of $X = X_t$ as a possible future *payoff* at time t for the capital $x(X) = X_s$ invested at time s. In other terms $x(X)$ can be considered as the *price* that must be paid at time s in order to get the corresponding future *gain* X at time t. In particular it is $x(R) = 1$.

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Let us set

$$\mathfrak{B} = \mathfrak{A}_s.$$

With the notation above, we have that the mapping $X \Longrightarrow x(X)$ given by

$$x(X) = E^0(R^{-1}X/\mathfrak{B}) \tag{0.1}$$

is a *linear operator* which is *monotone*, i.e.

$$x(X_1) \geq x(X_2) \geq 0, \tag{0.2}$$

for the elements $X_1 \geq X_2 \geq 0$ in the domain of the operator. We remark that for a *linear* operator, the domain of which is a *linear sub-space* of L_p, the above property is equivalent to

$$x(X) \geq 0, \qquad X \geq 0. \tag{0.3}$$

In fact, one has

$$x(X_1) - x(X_2) = x(X_1 - X_2) \geq 0, \qquad X_1 - X_2 \geq 0, \quad X_2 \geq 0.$$

Moreover the operator x in (0.1) is $\mathfrak{B}$-*homogenous* in the sense that its range consists of all the $\mathfrak{B}$-measurable variables $x(X)$ such that

$$x(\lambda X) = \lambda x(X) \tag{0.4}$$

with respect to any $\mathfrak{B}$-measurable multiplicator $\lambda \geq 0$.

Naturally, all *achievable* payoffs, $X \geq 0$, constitute a convex cone L^+ and, therefore, we can look at the prices

$$x(X), \qquad X \in L^+, \tag{0.5}$$

in the standard framework of the theory of linear operators on convex cones. Thus we refer to x in (0.5) as the *price operator*. Note that having the linear operator $x(X)$, $X \in L^+$, on the convex cone L^+, we can always deal with its unique linear extension, here denoted by the same symbol,

$$x(X), \qquad X \in L, \tag{0.6}$$

on the linear space

$$L := L^+ - L^+$$

consisting of all the differences

$$X = X_1 - X_2, \qquad X_1, X_2 \in L^+.$$

This extension $x(X)$, $X \in L$, is defined by

$$x(X) := x(X_1) - x(X_2). \tag{0.7}$$

With respect to the above extension we note that, in a market model where *short-selling* is available, the negative value $x = -|x|$ actually indicates a loan of price $|x|$.

Returning to the representation (0.1) of the price operator x, we stress that the "*true*" probability P^0 for the future events of the market is *unknown* in practice. Hence, in modeling the prices using the probability space

$$(\Omega, \mathfrak{A}, P),$$

we meet the problem of analysing the *underlying applied* probability measure $P(A)$, $A \in \mathfrak{A}$, in relation to *some* probability measure $P^0(A)$, $A \in \mathfrak{A}$, associated with the market prices through the representation (0.1). For a variety of market models, the very existence of an *equivalent* probability measure P^0:

$$P^0 \sim P, \tag{0.8}$$

is the subject of various versions of the "fundamental theorem of asset pricing" - cf. [2], [9], for example. The probability measure P^0 is usually called *risk-neutral* or *martingale measure.*

If we think of P^0 as the unknown *true* probability measure, then it is preferable to use in the practice a probability measure P which is somehow "close" to P^0. For example, P can be chosen such that there exists a risk-neutral probability measure P^0 with density

$$f(\omega) = \frac{P^0(d\omega)}{P(d\omega)}, \qquad \omega \in \Omega, \tag{0.9}$$

laying in some *pre-considered* upper and lower bounds M and m:

$$0 < m \leq f \leq M < \infty. \tag{0.10}$$

See also [8], for example.

With the above motivation, we derive a series of results for $\mathfrak{B}$-homogeneous monotone linear operators in a *separable* L_p-space

$$L_p := L_p(\Omega, \mathfrak{A}, P), \qquad 1 \leq p < \infty,$$

of the $\mathfrak{A}$-measurable random variables $X = X(\omega)$, $\omega \in \Omega$, with norm

$$\|X\| = (E|X|^p)^{1/p}, \qquad X \in L_p.$$

The notion of $\mathfrak{B}$-homogeneous - cf. (0.4) is related to the σ-algebra $\mathfrak{B}$ which is an arbitrary sub-σ-algebra of $\mathfrak{A}$, i.e. $\mathfrak{B} \subseteq \mathfrak{A}$.

The *separable* space $L_p = L_p(\Omega, \mathfrak{A}, P)$ is considered as a *lattice*, where the relation "$\leq$" means the standard point-wise relation "$\leq$ a.e.". For the study of general lattices we refer to [11]. In the lattice framework, we also use the stronger relation "$<$" which means that, in addition to "$\leq$ ", the strict point-wise relation "$<$" holds on some sub-set of Ω of *non-zero* probability measure.

The representation of type (0.1) for the operator x *defines* its $\mathfrak{B}$-homogeneous monotone linear extension via the right-hand side conditional expectation with respect to P^0.

Dealing with $X \in L_p$, we focus on the probability measure P^0 which is *regular* in the sense that the conditional expectation

$$E^0(X/\mathfrak{B}), \qquad X \in L_p, \tag{0.11}$$

is well defined for all $X \in L_p$ and represents the L_p-space elements

$$E^0(X/\mathfrak{B}) \in L_p.$$

Accordingly, the very existence of P^0 with certain required properties is analysed though the study of the corresponding $\mathfrak{B}$-*homogeneous monotone linear extension*

$$x(X), \qquad X \in L_p,$$

of the operator x, initially defined on some *convex sub-cone* $L^+ \subseteq L_p^+$, with

$$L_p^+ = \{X \in L_p : \ X \geq 0\}$$

or on some *linear sub-space* $L \subseteq L_p$ in $L_p = L_p(\Omega, \mathfrak{A}, P)$.

The extensions of linear operators/functionals are an important object of study. In functional analysis, classical examples are the Hahn-Banach extension theorem and its monotone versions. For these and related topics we refer to [7], [10], and [11], where, in particular, the following comments can be found (cf. [7], p. 72): "The Hahn–Banach theorem is certainly one of the most fundamental results in modern analysis, it is one of the best investigated individual theorems and the literature about it covers thousands of pages. [...] It has been proved and reformulated countless times, and yet there is still demand for new versions which allow more effective applications than before. And surprisingly, new and better versions are still found".

In the present paper we suggest several new versions of extension theorems. Among them we also treat the problem of the extension of a $\mathfrak{B}$-homogeneous operator in a way that preserves $\mathfrak{B}$-homogeneity.

The main results on the linear operators extension involve *majorants*

$$M(X), \qquad X \in L_p,$$

which are themselves operators such that

$$M(X) = M(|X|) \geq 0 \tag{0.12}$$

and, moreover, such that

$$M(\lambda X) = \lambda M(X) \tag{0.13}$$

for any constant $\lambda \geq 0$ and

$$M(X_1 + X_2) \leq M(X_1) + M(X_2) \tag{0.14}$$

for any $X_1, X_2 \geq 0$. We shortly refer to this type of operators as *sub-linear* operators.

The *monotone sub-linear* operator $M(X)$, $X \in L_p$, is characterized, in addition to (0.12)-(0.14), by

$$M(X) \leq M(Y), \qquad 0 \leq X \leq Y. \tag{0.15}$$

Theorem 0.1. *Any monotone sub-linear operator $M(X)$, $X \in L_p$, is bounded (continuous), i.e.*

$$\|M(X)\| \leq C\|X\|, \qquad X \in L_p, \tag{0.16}$$

for some constant $C < \infty$.

Proof. If the boundness (0.16) does not hold true, then there would be X_n with $\|X_n\| = 1$ $(n = 1, 2, ...)$, such that

$$n^{-2}\|M(X_n)\| \longrightarrow \infty, \qquad n \to \infty.$$

Then for

$$X = \sum_{n=1}^{\infty} n^{-2}|X_n| \geq 0$$

in L_p, this would imply that

$$M(X) \geq M(n^{-2}X_n) = n^{-2}M(X_n) \geq 0$$

and that

$$\|M(X)\| \geq n^{-2}\|M(X_n)\| \longrightarrow \infty, \qquad n \to \infty,$$

which is absurd since $M(X) \in L_p$. $\square$

Corollary 0.1. *Any monotone linear operator $x(X)$, $X \in L_p$, is bounded (continuous).*

Proof. The monotone sub-linear operator

$$M(X) := x(|X|), \qquad X \in L_p,$$

is bounded and it is a *majorant*, i.e.

$$|x(X)| \leq M(X), \qquad X \in L_p,$$

for the considered operator x. In fact it is

$$\pm x(X) = x(\pm X) \leq x(|X|)$$

for $\pm X \leq |X|$. Hence,

$$\|x(X)\| \leq \|M(X)\| \leq C\|X\|, \qquad X \in L_p.$$

By this we end the proof. $\square$

We apply these results to price modeling with respect to the problem of the existence of the risk-neutral probability measure P^0. For example, our results on the sandwich preserving extension lead to certain criteria for the existence of P^0 in a pre-determined "neighbourhood" of the underlying applied probability measure P - cf. (0.9) and (0.10).

1 Regular monotone operators.

Let $L_p = L_p(\Omega, \mathfrak{A}, P)$ be as in Section 0. We consider an arbitrary $\mathfrak{B}$-*homogenous monotone linear* operator x with

$$x(R) = 1$$

for some $\mathfrak{B}$-measurable random variable $R > 0$ - cf. (0.1)-(0.4). We refer to this operator as being *regular* if it is well defined on the whole space L_p as

$$L_p \ni X \quad \Longrightarrow \quad x(X) \in L_p(\Omega, \mathfrak{B}, P) \tag{1.1}$$

where $\mathfrak{B} \subseteq \mathfrak{A}$ is the σ-algebra involved in the definition of $\mathfrak{B}$-homogeneity. Cf. (0.11).

Theorem 1.1. *Any regular operator admits the representation* (0.1)*:*

$$x(X) = E^0\big(R^{-1}X/\mathfrak{B}\big), \qquad X \in L_p,$$

with respect to the probability measure P^0 *such that*

$$P^0(A) = \int_A f(\omega)P(d\omega), \qquad A \in \mathfrak{A},$$

where $f \in L_q$*, with* $L_q = L_q(\Omega, \mathfrak{A}, P)$*,* $q = p(1-p)^{-1}$ *(i.e.* L_q *is the dual space to* L_p*).*

Proof. Thanks to Corollary 0.1, we have that the monotone linear operator x on L_p is continuous. For any arbitrary $\mathfrak{B}$-measurable probability density $g \in L_q$: $g > 0$ *a.e.* and $Eg = 1$, let us consider the well defined linear functional

$$E^0 X := E\Big(Rx(X)g\Big), \qquad X \in L_p, \tag{1.2}$$

which is continuous since $X_n \to X$ in L_p implies that $Rx(X_n)g \to Rx(X)g$ in L_1. Equation (1.2) represents the expectation with respect to the probability measure

$$P^0(A) := E^0 1_A, \qquad A \in \mathfrak{A}, \tag{1.3}$$

defined by (1.2) (1_A being the indicator of the event A). We remark that

$$E^0 1 = E\big(Rx(1)g\big) = 1,$$

for the $\mathfrak{B}$-homogeneous operator $x(X), X \in L_p$, with

$$Rx(1) = x(R) = 1.$$

(by the choice of R). Moreover we have that

$$E^0\big(1_B[Rx(X)]\big) = E\big(Rx[1_B Rx(X)]g\big) = E\big(Rx(1_B X)g\big) = E^0\big(1_B X\big),$$

for any $B \in \mathfrak{B}$, since

$$x\big(1_B Rx(X)\big) = x\big(x(1_B X)R\big) = x(1_B X),$$

R being $\mathfrak{B}$-measurable. Thus

$$Rx(X) = E^0(X/\mathfrak{B}), \qquad X \in L_p. \tag{1.4}$$

This implies that the representation (0.1) holds true with respect to P^0.

Taking the known representation of linear continuous functionals on an L_p-space into account, we have that

$$E^0 X = E(Xf), \qquad X \in L_p, \tag{1.5}$$

for $f \in L_q$, L_q being the dual space to L_p (i.e. $q = p(p-1)^{-1}$). Here f represents the density of P^0 with respect to P:

$$P^0(A) = E 1_A f = \int_A f(\omega) P(d\omega), \qquad A \in \mathfrak{A}.$$

This completes the proof. □

We remark that the $\mathfrak{B}$-measurable probability density g: $g > 0$ a.e., chosen in (1.2) is related to f as

$$g = E(f/\mathfrak{B}), \tag{1.6}$$

since for all $\mathfrak{B}$-measurable variables X in the L_p-space we have that

$$E(Xf) = E(Rx(X)g) = E(Xx(R)g) = E(Xg)$$

by using (1.5). Thus, considering P^0 and P just on the σ-algebra $\mathfrak{B} \subseteq \mathfrak{A}$, we have

$$g(\omega) = \frac{P^0(d\omega)}{P(d\omega)}, \qquad \omega \in \Omega,$$

as the corresponding density of P^0 with respect to P:

$$P^0(B) = \int_B g(\omega) P(d\omega), \quad B \in \mathfrak{B}. \tag{1.7}$$

Then as a continuation of Theorem 1.1, we have the following result.

Theorem 1.2. *A regular operator admits the representation* (0.1) *in the following equivalent form:*

$$x(X) = E\Big(R^{-1} X \, \frac{f}{g} / \mathfrak{B}\Big), \qquad X \in L_p. \tag{1.8}$$

Proof. The equalities

$$E^0(1_B X) = E(1_B X f) = E(1_B E(Xf/\mathfrak{B}))$$

$$= E\Big(1_B \big[E(Xf/\mathfrak{B}) \, \frac{1}{g}\big] g\Big) = E^0\Big(1_B \big[E(Xf/\mathfrak{B}) \, \frac{1}{g}\big]\Big), \qquad B \in \mathfrak{B},$$

show that

$$E^0(X/\mathfrak{B}) = E\big(Xf/\mathfrak{B}\big) \, \frac{1}{g}, \qquad X \in L_p.$$

And, since the product

$$E\big(Xf/\mathfrak{B}\big) \cdot \frac{1}{g} = E^0\big(X/\mathfrak{B}\big) = Rx(X) \in L_p,$$

is an element of L_p, then we have that

$$E(Xf/\mathfrak{B}) \cdot \frac{1}{g} = E(X \frac{f}{g} / \mathfrak{B}), \qquad X \in L_p. \tag{1.9}$$

This leads to formula (1.8). $\square$

2 Hölder equality.

Let us simplify formula (1.8) by writing "f" instead of "$R^{-1} f/g$", so that (1.8) becomes

$$x(X) = E(Xf/\mathfrak{B}), \quad X \in L_p \qquad (1 \leq p < \infty). \tag{2.1}$$

Theorem 2.1. *The linear operator of the form* (2.1) *is well defined on the whole space* L_p*, where it is bounded (continuous), if and only if the factor* f *belongs to the dual space* L_q*,* $q = p(1-p)^{-1}$*, and*

$$\text{essup } E(|f|^q/\mathfrak{B}) < \infty.$$

Remark. *In fact the following Hölder equality holds true:*

$$(2.2) \quad \sup_{\|X\|\le 1} [E|E(Xf/\mathfrak{B})|^p]^{1/p} = \begin{cases} \text{essup } [E(|f|^q/\mathfrak{B})]^{1/q}, & p > 1, \\ \lim_{q\to\infty} \text{essup}\,[E(|f|^q\mathfrak{B})]^{\frac{1}{q}} = \text{essup } |f|, & p = 1. \end{cases}$$

Cf. [3]. In relation to the equality (2.2) we recall the (conditional) *Hölder inequality*:

$$(2.3) \qquad E(|Xf|/\mathfrak{B}) \le [E(|X|^p/\mathfrak{B})]^{\frac{1}{p}}[E|f|^q/\mathfrak{B})]^{\frac{1}{q}}.$$

(see, e.g., [6]). Relations (2.2) and (2.3) together justify the following property of operators of the form (2.1)

$$(2.4) \qquad |x(X)| \le C[E(|X|^p/\mathfrak{B})]^{\frac{1}{p}}, \qquad X \in L_p,$$

where the *minimal* constant $C < \infty$ for which (2.4) holds is the operator norm

$$(2.5) \qquad \|x\| = \sup_{\|X\|\le 1} \|E(Xf/\mathfrak{B})\|.$$

Note that in the case where $\mathfrak{B}$ is the *trivial* σ-algebra, an operator of type (2.1) is a linear functional on L_p with the properties (2.2)-(2.4) having

$$\sup_{\|X\|\le 1} |E(Xf)| = [E(|f|^q)]^{\frac{1}{q}}.$$

Proof of Theorem 2.1. For a linear continuous operator $x(X), X \in L_p$, in the L_p-space of the form (2.1), the expectation

$$E(x(X)) = E(E(Xf/\mathfrak{B})) = E(Xf), \qquad X \in L_p,$$

represents a linear continuous functional with its representative

$$f \in L_q$$

in the dual L_q-space, $q = p(p-1)^{-1}$. Thus we proceed with arguments that are quite similar to those usually applied to linear continuous functionals - cf. [11], for example. Considering the Hölder's inequality (2.3) for $1 < p < \infty$, let us set

$$\xi \equiv [E(|X|^p/\mathfrak{B})]^{\frac{1}{p}}, \qquad \varphi \equiv [E(|f|^q/\mathfrak{B})]^{\frac{1}{q}}.$$

Although the proof of (2.3) is known, we go here shortly through it in order to get to our Hölder equality (2.2). We see that

$$Xf = 0, \qquad E(|Xf|/\mathfrak{B}) = 0$$

on the sub-set $\{\xi\varphi = 0\} \subseteq \Omega$ belonging to $\mathfrak{B}$. Hence we can focus on

$$\Omega^+ = \{\xi\varphi > 0\}.$$

According to the known elementary inequality

$$\alpha\beta \leq \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q \qquad (\alpha, \beta \geq 0) \tag{2.6}$$

where the equality sign holds if and only if

$$\alpha^p = \beta^q, \qquad q = p(p-1)^{-1},$$

we get

$$\frac{|Xf|}{\xi\varphi} \leq \frac{1}{p}\frac{|X|^p}{\xi^p} + \frac{1}{q}\frac{|f|^q}{\varphi^q}$$

on Ω^+, setting $\alpha = \frac{|X|}{\xi}$, $\beta = |f|\varphi$. The variables on both sides are *integrable* on the set Ω^+ (with respect to the underlying probability measure P), since

$$\frac{1}{\xi^p}E(|X|^p/\mathfrak{B}), \qquad \frac{1}{\varphi^q}E(|f|^q/\mathfrak{B})$$

are integrable. Hence,

$$\frac{E(|Xf|/\mathfrak{B})}{\xi\varphi} \leq \frac{1}{p}\frac{E(|X|^p/\mathfrak{B})}{\xi^p} + \frac{E(|f|^q/\mathfrak{B})}{\varphi^q} = 1$$

on the set Ω^+. This implies the Hölder inequality (2.3), where the equality sign holds if and only if

$$\frac{|X|^p}{\xi^p} = \frac{|f|^q}{\varphi^q} \tag{2.7}$$

for almost all $\omega \in \Omega^+$ - cf. (2.6). The above condition (2.7) can be equivalently characterized as follows:

$$|X| = a|f|^{\frac{1}{p-1}} a.e. \quad \text{or} \quad |f| = b|X|^{\frac{1}{q-1}} a.e. \tag{2.8}$$

for some $\mathfrak{B}$-measurable variables a and b on the set Ω^+. For this we just note that condition (2.8) implies

$$|X|^p = a^p|f|^q, \qquad \xi^p = a^p\varphi^q$$

from which we conclude that

$$\frac{|X|^p}{\xi^p} = \frac{a^p|f|^q}{a^p\varphi^q} = \frac{|f|^q}{\varphi^q}$$

using (2.7). Now, taking

$$X = \begin{cases} f^{q-1} & \text{for } f \geq 0 \\ -|f|^{q-1} & \text{for } f < 0, \end{cases}$$

into account, we have that

$$|X| = |f|^{\frac{1}{p-1}}$$

for $q-1 = (p-1)^{-1}$, and

$$|X|^p = |f|^q = |Xf| = Xf.$$

So, the equality in (2.3) holds in this case:

$$E(Xf/\mathfrak{B}) = [E(|X|^p/\mathfrak{B})]^{\frac{1}{p}}[E(|f|^q/\mathfrak{B})]^{\frac{1}{q}}.$$

The multiplication by a $\mathfrak{B}$-measurable function $\xi^{-1}1_B$ with $\xi = [E(|X|^p/\mathfrak{B})]^{\frac{1}{p}}$ and $B \subseteq \Omega^+$, $B \in \mathfrak{B}$, gives

$$E\Big(\frac{X}{\xi}1_B f/\mathfrak{B}\Big) = \Big[E\Big(|\frac{X}{\xi}|^p 1_B/\mathfrak{B}\Big)\Big]^{1/q} = 1_B\Big[E\big(|f|^q/\mathfrak{B}\big)\Big]^{1/q}.$$

Then it follows that for any arbitrary finite constant $C \leq C_0$, with

$$C_0 := \text{essup } [E(|f|^q/\mathfrak{B})]^{\frac{1}{q}},$$

and $B = \Big\{[E(|f|^q/\mathfrak{B})]^{\frac{1}{q}} \geq C\Big\}$, we have that

$$E\Big|E\big(\frac{X}{\xi}1_B f/\mathfrak{B}\big)\Big|^p \geq C^p E\big(1_B\big) = C^p E\big|\frac{X}{\xi}1_B\big|^p,$$

since $1_B = E(|\frac{X}{\xi}1_B|^p/\mathfrak{B})$. The above relation implies

$$\sup_{\|X\|\leq 1} \|E(Xf/\mathfrak{B})\| \geq C$$

and, therefore,

$$\sup_{\|X\|\leq 1} \|E(Xf/\mathfrak{B})\| \geq C_0 \;=\; \text{essup}[E(|f|^q/\mathfrak{B})]^{\frac{1}{q}}.$$

On the other hand, thanks to Hölder inequality (2.3), we have

$$E\left(|E(Xf/\mathfrak{B})|^p\right) \leq E\big[E(|X|^p/\mathfrak{B})\,[E(|f|^q/\mathfrak{B})]^{p/q}\big] \leq C_0^p E|X|^p$$

which implies

$$\sup_{\|X\|\leq 1} \|E(Xf/\mathfrak{B})\| \leq C_0.$$

For $1 < p < \infty$, the proof is over.

It remains to consider the case $p = 1$. First of all we show that the limit in (2.2) holds true. Let $C_0 = \operatorname{essup} |f|$, then

$$C_0 \geq |f| \quad a.e.$$

and we have

$$C_0 \geq [E(|f|^q/\mathfrak{B})]^{\frac{1}{q}} \quad a.e.$$

for all $q, 1 \leq q < \infty$. Hence,

$$C_0 \geq \operatorname{essup} [E(|f|^q/\mathfrak{B})]^{\frac{1}{q}}, \qquad \text{and} \qquad C_0 \geq \overline{\lim}_{q\to\infty} \operatorname{essup} [E(|f|^q/\mathfrak{B})]^{\frac{1}{q}}.$$

On the other hand, for any constant $C < C_0$, the set $A = \{|f| \geq C\}$ is of positive measure, i.e. $P(A) > 0$. Let

$$\alpha = E(1_A/\mathfrak{B}),$$

and consider the sets

$$B = \{\alpha > 0\} \in \mathfrak{B}, \qquad B^c = \{\alpha = 0\}.$$

We can see that $P(B) > 0$, since

$$0 = E(\alpha 1_{B^c}) = E\left(E(1_A 1_{B^c}/\mathfrak{B})\right) = P(A \cap B^c)$$

and

$$P(B) \geq P(A \cap B) = P(A) > 0.$$

With $P(B) > 0$ and the point-wise convergence

$$\lim_{q\to\infty} \alpha^{\frac{1}{q}}(\omega) = 1_B(\omega), \quad \omega \in \Omega,$$

we have

$$\lim_{q\to\infty} P\left\{\alpha^{\frac{1}{q}} > (1-\varepsilon)1_B\right\} = P(B)$$

for any $\varepsilon > 0$, so there is a set $B_\varepsilon \subseteq B : P(B_\varepsilon) > 0$, such that

$$\alpha\frac{1}{q} \geq (1-\varepsilon)1_{B_\varepsilon}$$

with $q \geq q_\varepsilon$. Consequently, for $\alpha = E(1_A/\mathfrak{B})$ and

$$|f| \geq |f|1_A \geq C1_A,$$

we obtain

$$[E(|f|^q/\mathfrak{B})]^{\frac{1}{q}} \geq C\alpha^{\frac{1}{q}} \geq C(1-\varepsilon)1_{B_\varepsilon}$$

which shows that

$$\lim_{q\to\infty} \text{ essup } [E(|f|^q/\mathfrak{B})]^{\frac{1}{q}} \geq C,$$

for *any* $C \leq C_0 =$ essup $|f|$. This ends the proof of (2.2).

To conclude the proof of Theorem 2.1, it remains to show that Hölder equality holds also for $p = 1$. Obviously, for $C_0 =$ essup $|f|$, we have

$$E\left(|E(Xf/\mathfrak{B})|\right) \leq E\left(E(|Xf|/\mathfrak{B})\right) \leq E[C_0E(|X|/\mathfrak{B})] = C_0E|X|,$$

with $E|X| = \|X\|$ in the L_1-space, so

$$\sup_{\|X\|\leq 1} E|E(Xf)/\mathfrak{B}| \leq \text{ essup } |f|.$$

On the other hand, for any $C < C_0$, let $X = 1_A signf$, where the set $A = \{|f| \geq C\}$ is of measure $P(A) > 0$; then

$$E\left(|E(Xf/\mathfrak{B})|\right) = E\left(E(1_A|f|/\mathfrak{B})\right) \geq CE[1_A] = C\|X\|$$

Hence, for all $X \in L_1$, we have

$$\sup_{\|X\|\leq 1} E\left(|E(Xf)/\mathfrak{B}|\right) \geq C$$

and therefore

$$\sup_{\|X\|\leq 1} E|E((Xf)/\mathfrak{B})| \geq C_0 = \text{ essup } |f|,$$

which ends the proof of Theorem 2.1. □

3 Majorant characterization.

According to the previous results, any $\mathfrak{B}$-homogenous monotone linear operator x on $L_p = L_p(\Omega, \mathfrak{A}, P)$ admits a *standard majorant* of form

$$M(X) := C[E(|X|^p/\mathfrak{B})]^{\frac{1}{p}}, \qquad X \in L_p, \tag{3.1}$$

involving the constant C - with the minimal constant

$$C = \|x\|$$

as the operator norm - cf. (2.4). Namely, we have

$$|x(X)| \leq M(X), \qquad X \in L_p. \tag{3.2}$$

Since the operator x is *monotone*, the majorant condition (3.2) can be equivalently given as

$$x(X) \leq M(Y) \tag{3.3}$$

for $X, Y \in L_p$ such that $X \leq Y$. Indeed, (3.2) implies

$$x(X) \leq M(X), \qquad x(X) \leq x(Y) \leq M(Y),$$

for $X \leq Y$ and (3.3) with $Y = |X|$ justifies that

$$-x(X) = x(-X) \leq M(Y) = M(X).$$

Theorem 3.1. *For an arbitrary linear operator*

$$L_p \ni X \quad \Longrightarrow \quad x(X) \in L_p(\Omega, \mathfrak{B}, P),$$

and the standard majorant of the form (3.1)*, the condition* (3.3) *implies that this operator is monotone and* $\mathfrak{B}$*-homogenous.*

Proof. For the linear operator $x(X), X \in L_p$, on the *linear* space $L_p = L_p(\Omega, \mathfrak{A}, P)$, the condition (3.3) implies that

$$-x(X) = x(-X) \leq M(0) = 0$$

for $X \geq 0$, that is $x(X) \geq 0$. Thus the operator is monotone - cf. (0.2) and (0.3). We recall that the considered operator x is $\mathfrak{B}$-*homogenous*, i.e.

$$x(\lambda X) = \lambda x(X) \tag{3.4}$$

for any $\mathfrak{B}$-measurable multiplicator λ - cf. (0.4). Let us consider $X \geq 0$ and $\lambda = 1_B$, for $B \in \mathfrak{B}$. According to (3.3),

$$0 \leq x(1_B X) \leq M(1_B X) = 1_B M(X),$$

with M as in (3.1) and we can see that

$$x(1_B X) = 1_B x(1_B X).$$

Hence, for the unit decomposition

$$1 = \sum_k 1_{B_k}$$

with the *disjoint sets* $B_k \in \mathfrak{B}$: $\sum_k B_k = \Omega$, (with $\sum$ meaning the disjoint union) we obtain

$$\sum_k 1_{B_k} x(X) = \big(\sum_k 1_{B_k}\big) x(X) = x(X)$$

$$= x\Big[\big(\sum_k 1_{B_k}\big) X\Big] = \sum_k x\big(1_B X\big) = \sum_k 1_{B_k} x(1_{B_k} X)$$

which shows that

$$1_{B_k} x(1_{B_k} X) = 1_{B_k} x(X).$$

Therefore, (3.4) holds for $X \geq 0$ and for any simple multiplicator of the form

$$\lambda = \sum_k c_k 1_{B_k}$$

with the constant values

$$\lambda(\omega) = c_k, \quad \omega \in B_k,$$

on the partition sets $B_k \in \mathfrak{B}$: $\sum_k B_k = \Omega$.
Clearly, for any $\mathfrak{B}$-measurable multiplicator λ such that $\lambda X \in L_p$, there are appropriate simple λ_n such that

$$\lambda(\omega) = \lim_{n\to\infty} \lambda_n(\omega), \qquad \omega \in \Omega,$$

point-wise and

$$\lambda X = \lim_{n\to\infty} \lambda_n X$$

in L_p. Then, we can see that

$$x(\lambda X) = \lim_{n\to\infty} x(\lambda_n X)$$

in L_p, since the monotone operator $x(X)$, $X \in L_p$, is continuous. At the same time

$$\lim_{n\to\infty} x(\lambda_n X) = \lim_{n\to\infty} \lambda_n x(X) = \lambda x(X)$$

point-wise. Hence (3.4) holds true for all $X \geq 0$. Consequently it holds for all $X \in L_p$, since $X = X^+ - X^-$ for

$$X^+ := \sup(X, 0), \qquad X^- := \sup(-X, 0),$$

and

$$x(\lambda X) = x(\lambda X^+) - x(\lambda X^-) = \lambda[x(X^+) - x(X^-)] = \lambda x(X).$$

This ends the proof. $\square$

Remark. The given proof holds with respect to *any majorant*

$$M(X) = M(|X|) \geq 0, \qquad X \in L_p,$$

which is $\mathfrak{B}$-*homogenous* in the sense that

$$M(\lambda X) = \lambda M(X) \tag{3.5}$$

for any $\mathfrak{B}$-measurable multiplicator $\lambda \geq 0$. Note that $M(0) = 0$.

With reference to (0.5)-(0.7) and (0.12)-(0.15), we consider the linear operator $x(X)$, $X \in L_p^+$, on the cone

$$L_p^+ = \{X \in L_p : \ X \geq 0\},$$

and the *monotone* sub-linear majorant $M(X)$, $X \in L_p^+$. Then Theorem 3.1 can be strengthened as follows.

Theorem 3.2. *The majorant condition*

$$x(X) \leq M(Y) \tag{3.6}$$

for $X, Y \in L_p^+$*:* $X \leq Y$*, implies that the extension*

$$x(X) = x(X^+) - x(X^-), \qquad X \in L_p,$$

is a $\mathfrak{B}$*-homogenous monotone linear operator, which satisfies the extended majorant condition* (3.3).

Proof. For any $X, Y \in L_p$: $X \leq Y$, we have

$$X^+ = \sup(X, 0) \leq \sup(Y, 0) = Y^+ \leq |Y|,$$

and

$$x(X) = x(X^+) - x(X^-) \leq x(X^+) \leq M(X^+) \leq M(Y^+) \leq M(|Y|) = M(Y).$$

Cf. Theorem 3.1. By this we end the proof. □

4 Monotone version of Hahn–Banach extension theorem.

In the space $L_p = L_p(\Omega, \mathfrak{A}, P)$, we consider linear operators and their majorants having range in the sub-space $L_p(\Omega, \mathfrak{B}, P)$ where $\mathfrak{B}$ is an arbitrary σ-algebra $\mathfrak{B} \subseteq \mathfrak{A}$.

For a *linear* operator x:

$$L \ni X \quad \Longrightarrow \quad x(X) \in L_p(\Omega, \mathfrak{B}, P),$$

defined on an arbitrary *linear sub-space*

$$L \subseteq L_p,$$

let us introduce the *majorant condition*

$$x(X) \leq M(Y) \tag{4.1}$$

for $X \in L$, $Y \in L_p : X \leq Y$, which involves the monotone sub-linear operator $M(Y)$, $Y \in L_p$ ($M(0) = 0$) - cf. (0.12)-(0.15) and (3.3).

Theorem 4.1. *The linear operator $x(X)$, $X \in L$, admits its monotone linear extension*

$$L_p \ni X \quad \Longrightarrow \quad x(X) \in L_p(\Omega, \mathfrak{B}, P)$$

on the whole space L_p, if and only if condition (4.1) *holds for some monotone sub-linear majorant. Moreover, the condition* (4.1) *implies the existence of the majorant preserving extension:*

$$x(X) \leq M(Y) \tag{4.2}$$

for $X, Y \in L_p$: $X \leq Y$.

Proof. Note that the majorant condition (4.1) justifies that the initial operator x is monotone on its initial domain L. If the operator x admits its monotone linear extension $x(X)$, $X \in L_p$, on the whole space L_p, then the condition (4.1) holds for the monotone sub-linear majorant

$$M(Y) := x(|Y|), \quad Y \in L_p.$$

For any monotone sub-linear majorant, with (4.1) in hands, we can proceed as in the sequel in order to get the required extension $x(X)$, $X \in L_p$.

Let us look for a *one-step extension*

$$x(-X + \lambda Y^0) = -x(X) + \lambda x^0 \tag{4.3}$$

on the linear sub-space of all elements

$$-X + \lambda Y^0, \qquad X \in L, \lambda \in \mathbb{R},$$

with an arbitrary element $Y^0 \in L_p$: $Y^0 \notin L$. Since we deal with a separable L_p-space, we can apply the least upper bound

$$a := \sup_{-X' - Y' \leq Y^0} [-x(X') - M(Y')]$$

and the largest lower bound

$$b := \inf_{X'' + Y'' \geq Y^0} [x(X'') + M(Y'')]$$

for $X', X'' \in L$ and $Y', Y'' \in L_p$. We remark that

$$a \leq b,$$

since

$$-X' - Y' \leq Y^0 \leq X'' + Y'', \qquad -X' - X'' \leq Y' + Y'',$$

and therefore

$$-x(X') - x(X'') = x(-X' - X'') \leq M(Y' + Y'') \leq M(Y') + M(Y'')$$

which shows that

$$-x(X') - M(Y') \leq x(X'') + M(Y'')$$

in the definitions of a and b. Let us consider any $\mathfrak{B}$-measurable element $x^0 \in L_p$ such that

$$a \leq x^0 \leq b. \tag{4.4}$$

We shall show that

$$x(-X + \lambda Y^0) \leq M(Y), \qquad -X + \lambda Y^0 \leq Y,$$

for the extension (4.3). Indeed, for $\lambda = 0$, the above majorant condition holds. In the case where $\lambda > 0$ and

$$-X - \lambda Y^0 \leq Y, \qquad -\frac{X}{\lambda} - \frac{Y}{\lambda} \leq Y^0$$

we have

$$-x(X) - \lambda x^0 \leq -x(X) - \lambda a$$
$$\leq -x(X) - \lambda\Big[-x\big(\frac{X}{\lambda}\big) - M\big(\frac{Y}{\lambda}\big)\Big] = M(Y).$$

In the case where $\lambda > 0$ and

$$-X + \lambda Y^0 \leq Y, \qquad \frac{X}{\lambda} + \frac{Y}{\lambda} \geq Y^0$$

we have

$$-x(X) + \lambda x^0 \leq -x(X) + \lambda b$$
$$-x(X) + \lambda\Big[x\big(\frac{X}{\lambda}\big) + M\big(\frac{Y}{\lambda}\big)\Big] = M(Y).$$

Hence, $M(Y)$, $Y \in L_p$, is a majorant both for the extension as well as for the initial operator $x(X)$, $X \in L$ - cf. (4.1).

In a similar way, we can determine the next one-step extension, and going on, with a *countable* number of steps, we are getting the majorant preserving extension $x(X)$, $X \in L^0$, on a linear space $L^0 \subseteq L_p$ *dense* in L_p, i.e.

$$x(X) \leq M(Y)$$

for $X \in L^0$, $Y \in L_p$: $X \leq Y$. Consequently, we have

$$|x(X)| \leq M(X), \qquad X \in L^0,$$

and this shows that the operator $x(X)$, $X \in L^0$, is continuous due to the continuity of the monotone majorant M. Cf. Theorem 0.1. Finally, by the continuity of x and the density of L^0, we extend the linear operator $x(X)$, $X \in L^0$, to the whole L_p-space.

Let us show that this final extension $x(X)$, $X \in L_p$, satisfies the majorant condition (4.2). For $X \leq Y$, considering

$$X = \lim_{n\to\infty} X_n$$

as the limit of $X_n \in L^0$ both in L_p and point-wise for almost all $\omega \in \Omega$, we can see that the preserved majorant condition

$$x(X_n) \leq M(Y_n), \qquad Y_n \equiv \sup(X_n Y),$$

implies

$$x(X) = \lim_{n\to\infty} x(X_n) \leq \lim_{n\to\infty} M(Y_n) = M(Y)$$

for $Y = \lim_{n\to\infty} Y_n$ in L_p. We remark that

$$Y_n = X_n 1_{A_n} + Y 1_{A_n^c}$$

where $\lim_{n\to\infty} X_n 1_{A_n} = 0$, $\lim_{n\to\infty} Y 1_{A_n^c} = Y$ for the ω-sets $A_n = \{X_n > Y\}$ with $\lim_{n\to\infty} P(A_n) = 0$. As it was shown at the begining of the proof of Theorem 3.1, the majorant condition (4.2) justifies that the extension $x(X)$, $X \in L_p$, is monotone. This ends the proof. $\square$

Note, that the majorant condition (4.2) for the *strictly monotone* operator $x(X)$, $X \in L_p$, implies that

$$\inf[M(Y) - x(X)] > 0 \tag{4.5}$$

for X, $Y \in L_p$ such that $Y - X \geq Y^0$, for any arbitrary fixed element $Y^0 > 0$. In fact we have

$$M(Y) - x(X) \geq x(Y) - x(X) = x(Y - X) \geq x(Y^0)$$

with

$$\inf[M(Y) - x(X)] \geq x(Y^0) > 0. \tag{4.6}$$

Given the above observation we can strethen Theorem 4.1 as follows.

Theorem 4.2. *The linear operator $x(X)$, $X \in L$, admits a strictly monotone extension $x(X)$, $X \in L_p$, if and only if*

$$\inf[M(Y) - x(X)] > 0 \tag{4.7}$$

for $X \in L$, $Y \in L_p$ such that $Y - X \geq Y^0$, whatever $Y^0 > 0$ be.

Proof. According to (4.7), for a given Y^0, there is a certain monotone linear extension $x_{Y^0}(X)$, $X \in L_p$:

$$x_{Y^0}(Y^0) > 0,$$

determined at the first step of extension as

$$\begin{aligned} x_{Y^0}(Y^0) &= b \\ &= \inf_{(-X)+Y \geq Y^0} [x(-X) + M(Y)] \\ &= \inf_{Y-X \geq Y^0} [M(Y) - x(X)] > 0; \end{aligned}$$

- cf. (4.4). We note that this step is also applicable for $Y^0 \in L$. Considering the family of all extensions of the above form $x_{Y^0}(X)$, $X \in L_p$, we can see that these operators are *uniformly bounded*, having their norm bounded by some constant C, i.e.

$$||x_{Y^0}|| \leq C.$$

Indeed it is

$$||x_{Y^0}(X)|| \leq ||M(X)|| \leq C\|X\|, \qquad X \in L_p,$$

for

$$|x_{Y^0}(X)| \leq M(X), \qquad X \in L_p.$$

The family of all uniformly bounded linear operators can be treated as a *separable metric space*, with the metric

$$\mu(x', x'') := \sum_{k=1}^{\infty} \frac{1}{k^2} ||x'(X_k) - x''(X_k)||,$$

involving a *complete system* X_k, $k \in \mathbb{N}$, of elements $X_k \in L_p$: $||X_k|| = 1$. The convergence with respect to the above metric, i.e. $\mu(x', x'') \to 0$, is equivalent to the point-wise convergence

$$||x'(X) - x''(X)|| \longrightarrow 0, \qquad \text{for all} \quad X \in L_p.$$

Hence, we can select certain operators

$$x_n(X) := x_{Y_n^0}(X), \qquad X \in L_p \quad (n \in \mathbb{N})$$

which are *dense* in the set of all $x_{Y^0}(X)$, $X \in L_p$, and therefore, for whatever fixed element $Y^0 > 0$,

$$x_{n_m}(Y^0) \longrightarrow x_{Y^0}(Y^0) > 0, \qquad m \to 0,$$

for some subsequence n_m, $m \in \mathbb{N}$. And here, for $Y^0 > 0$ and $x_{n_m}(Y^0) \geq 0$, it is $x_{n_m}(Y^0) > 0$ for all the sufficiently large n_m. Accordingly, we can determine the *strictly monotone* extension by

$$x(X) := \sum_{n=1}^{\infty} c_n x_n(X), \qquad X \in L_p, \tag{4.8}$$

with strictly positive constant coefficients $c_n > 0$ such that

$$\sum_{n=1}^{\infty} c_n = 1.$$

Whatever $Y^0 > 0$ be, for the corresponding elements $x_{n_m}(Y^0)$ which are close enough to the element $x_{Y^0}(Y^0) > 0$ in L_p, we have that $x_{n_m}(Y^0) > 0$ as well, thus we obtain

$$x(Y^0) = \sum_{n=1}^{\infty} c_n x_n(Y^0) > 0.$$

Note that *the monotone linear extension* (4.7) *preserves the majorant:*

$$|x(X)| \leq \sum_{n=1}^{\infty} c_n |x_n(X)| \leq M(X)$$

- cf. (3.2) and (3.3). By this, we end the proof. □

For a version of these results with respect to monotone linear operators on Banach lattices see also [5].

5 Some versions of König sandwich theorem.

As in Section 4, we consider operators in the L_p-space with range in the sub-space $L_p(\Omega, \mathfrak{B}, P)$. Let M be a *monotone sub-linear* operator $M(X)$, $X \in L_p^+$, and let m be a *monotone super-linear* operator $m(X)$, $X \in L_p^+$, i.e.

$$m(\lambda X) = \lambda m(X)$$

for any constant $\lambda \geq 0$ and

$$m(X_1 + X_2) \geq m(X_1) + m(X_2),$$

on the cone

$$L_p^+ = \{X \in L_p : \; X \geq 0\}.$$

We consider M and m as the corresponding *majorant* and *minorant* for a *monotone linear* operator $x(X)$, $X \in L_p^+$, such that

$$m(X) \leq x(X) \leq M(X), \qquad X \in L_p^+. \tag{5.1}$$

For the elements

$$Z + X'' \leq X' + Y$$

of the cone L_p^+, the condition (5.1) implies

$$\begin{aligned} m(Z) + x(X'') &\leq x(Z) + x(X'') \\ &= x(Z + X'') \\ &\leq x(X' + Y) \\ &= x(X') + x(Y) \\ &\leq x(X') + M(Y), \end{aligned}$$

and, in fact, the condition (5.1) is equivalent to the following sandwich relationship

$$m(Z) + x(X'') \leq x(X') + M(Y) \tag{5.2}$$

for *all* L_p^+-elements such that

$$Z + X'' \leq X' + Y.$$

Having this in mind, for the monotone linear operator $x(X)$, $X \in L^+$, defined on some convex sub-cone

$$L^+ \subseteq L_p^+$$

- cf. (0.5), we introduce the corresponding sandwich condition (5.2) as

$$m(Z) + x(X'') \leq x(X') + M(Y) \tag{5.3}$$

for all $X', X'' \in L^+$ and $Y, Z \in L_p^+$ such that

$$Z + X'' \leq X' + Y.$$

Theorem 5.1. *The linear operator $x(X)$, $X \in L^+$, admits a monotone linear extension $x(X)$, $X \in L_p^+$, if and only if the sandwhich condition* (5.3) *holds for some majorant and minorant. Moreover,* (5.3) *implies the existence of the sandwich preserving extension $x(X)$, $X \in L_p^+$, which satisfies* (5.2) *as the extended sandwich.*

Proof. For this, we refer to [1] where the known König theorem - cf. [7], is re-proven in the operator case. Here we only recall the key of the proof. Let us consider the *new* majorant

$$\widetilde{M}(Y) := \inf[x(X') + M(Y')]$$

defined by inf over $X' \in L^+$ and $Y' \in L_p^+$ such that

$$X' + Y' \geq Y.$$

For the *monotone* linear operator $x(X)$, $X \in L^+$, the operator $\widetilde{M}(Y)$, $Y \in L_p^+$, is sub-linear and monotone. Obviously,

$$\widetilde{M}(Y) \leq M(Y), \qquad Y \in L_p^+, \tag{5.4}$$

and

$$\widetilde{M}(X) \leq x(X), \qquad X \in L^+.$$

On the other hand, condition (5.3) says that

$$x(X) \leq x(X') + M(Y'), \qquad X \leq X' + Y',$$

and this implies that

$$\widetilde{M}(X) \geq x(X), \qquad X \in L^+.$$

Hence, we have

$$\widetilde{M}(X) = x(X), \qquad X \in L^+, \tag{5.5}$$

Also, let us take the *new* minorant

$$\widetilde{m}(Z) := \sup[m(Z') + x(X')]$$

into account defined by sup over $X' \in L^+$ and $Z' \in L_p^+$ such that

$$X' + Z' \leq Z.$$

The operator $\widetilde{m}(Z)$, $Z \in L_p^+$, is a monotone super-linear operator. Obviously,

$$\widetilde{m}(Z) \geq m(Z), \qquad Z \in L_p^+, \tag{5.6}$$

and

$$\widetilde{m}(X) \geq x(X), \qquad X \in L^+.$$

On the other hand, condition (5.3) says that

$$m(Z') + x(X') \leq x(X), \qquad Z' + X' \leq X,$$

and this implies

$$\widetilde{m}(X) \leq x(X), \qquad X \in L^+.$$

Hence, we have

$$\widetilde{m}(X) = x(X), \qquad X \in L^+. \tag{5.7}$$

Now, we have just to apply the following version of König sandwich theorem: *there is a monotone linear operator $x(X)$, $X \in L_p^+$, which satisfies the sandwich condition*

$$\widetilde{m}(X) \leq x(X) \leq \widetilde{M}(X), \qquad X \in L_p^+.$$

Thanks to (5.5)-(5.7), the above operator represents a monotone linear extension of the initial operator $x(X)$, $X \in L^+$, and here according to (5.4)-(5.6), we have

$$m(X) \leq x(X) \leq M(X), \qquad X \in L_p^+,$$

- cf.(5.1) and (5.2). □

Corollary 5.1. *The strictly monotone linear extension $x(X)$, $x \in L_p^+$, exists if and only if the sandwich condition* (5.3) *holds for some strictly positive minorant:*

$$m(X) > 0, \qquad X > 0.$$

We remind that *the linear operator $x(X)$, $X \in L_p^+$, admits its unique linear extension on the whole space $L_p = L_p(\Omega, \mathfrak{A}, P)$ via the formula*

$$x(X) = x(X^+) - x(X^-) \tag{5.8}$$

with

$$X^+ = \sup(X, 0), \qquad X^- = \sup(-X, 0).$$

Cf. (0.5)- (0.7).

Theorem 5.2. *The sandwich condition* (5.3) *with the $\mathfrak{B}$-homogeneous majorant justifies the existence of the* $\mathfrak{B}$-homogeneous *monotone linear extension $x(X)$, $X \in L_p$. Moreover the extension is of the form* (2.1)*:*

$$x(X) = E(Xf/\mathfrak{B}), \qquad X \in L_p^+. \tag{5.9}$$

with the factor f such that

$$0 \leq m \leq f \leq M, \tag{5.10}$$

for the standard majorant and minorant*:*

$$M(X) = E(XM/\mathfrak{B}), \quad m(X) = E(Xm/\mathfrak{B}), \qquad X \in L_p^+. \tag{5.11}$$

defined by means of the corresponding elements $m \geq 0$ and M:

$$(5.12) \qquad \text{essup } [E(M^q/\mathfrak{B})]^{\frac{1}{q}} < \infty.$$

Proof. We refer to Theorem 1.1, (2.2)-(2.3) and Theorem 3.2. Then it only remains to note that the sandwich condition (5.1) is of the form:

$$E(Xm/\mathfrak{B}) \leq E(Xf/\mathfrak{B}) \leq E(XM/\mathfrak{B}), \qquad X \in L_p^+.$$

so it is

$$E(Xm) \leq E(Xf) \leq E(XM), \qquad X \in L_p^+,$$

and this implies condition (5.10). □

Obviously, the sandwich condition (5.3) with the standard majorant and minorant of the type (5.10) is *necessary* for the existence of the monotone linear extension $x(X)$, $X \in L_p$, of the form (2.1).

Corollary 5.2. *The strictly monotone linear extension $x(X)$, $X \in L_p$, of the form* (2.1), *exists if and only if the sandwich condition* (5.3) *holds for some standard majorant M and minorant m with*

$$m > 0 \quad a.e..$$

6 Application to market prices modeling.

The following results link to the extension theorems of Section 4 and Section 5 with the prices modeling described in Section 0 (cf. (0.1)-(0.11)) for a multi period market. In this case the σ-algebra $\mathfrak{B}$ is given by

$$\mathfrak{B} = \mathfrak{A}_{t-1}.$$

It consists of events preceding the time-period t:

$$t = 1, \ldots, T \qquad (T \in \mathbb{N},\ T > 1).$$

The probability space is $(\Omega, \mathfrak{A}, P)$ with

$$\mathfrak{A} = \mathfrak{A}_T$$

and $\mathfrak{A}_0$ is the *trivial* σ-algebra.

Theorem 6.1. *The regular risk-neutral probability measure*

$$P^0 \sim P$$

exists if and only if, for every time-period t, the corresponding price operator (0.5)*: $x(X)$, $X \in L^+$, admits its regular extension $x(X)$, $X \in L_p$, as a $\mathfrak{B}$-homogeneous strictly monotone linear operator.*

Proof. The representation (0.1) with respect to the equivalent probability measure $P^0 \sim P$ defines the *strictly monotone* extension $x(X)$, $X \in L_p$. On the other hand, having this type of regular extension $x(X)$, $X \in L_p$, in hands, we can define the corresponding probability measure P^0 in a sequence of steps as follows - cf. (1.1)-(1.8). For $t = 1$, we define P^0 on the σ-algebra $\mathfrak{A}_1$ as

$$P^0(A) := E(Rx(1_A)), \qquad A \in \mathfrak{A}_1, \tag{6.1}$$

getting $P^0(A) > 0$ for all $A \in \mathfrak{A}_1$, $P(A) > 0$, since $Rx(1_A) > 0$. And for every following step t, having

$$P^0(A) = \int_A f_{t-1}(\omega)P(d\omega), \qquad A \in \mathfrak{A}_{t-1}, \tag{6.2}$$

on the σ-algebra $\mathfrak{A}_{t-1} = \mathfrak{B}$ with the probability density

$$g = f_{t-1} > 0 \quad a.e. \qquad (Eg = 1),$$

we define

$$P^0(A) := E(Rx(1_A)g), \qquad A \in \mathfrak{A}_t, \tag{6.3}$$

on the σ-algebra $\mathfrak{A}_t$ - cf. (1.2)-(1.5). Clearly, formula (6.3) defines the *extension* of $P^0(A)$, $A \in \mathfrak{A}_{t-1}$, to the σ-algebra $\mathfrak{A}_t$, since (6.3) represents (6.2) for $A \in \mathfrak{A}_{t-1}$:

$$E(Rx(1_A)g) = E(1_A x(R)g) = E(1_A g).$$

And having the (6.3) density $g > 0$ a.e., we preserve the equivalence $P^0 \sim P$. In fact for, $P^0(A) > 0$ for all $A \in \mathfrak{A}_t$, we have $P(A) > 0$, since $Rx(1_A)g > 0$. Clearly, for the final $t = T$, we are getting the *regular risk-neutral probability measure* $P^0 \sim P$ on the σ-algebra $\mathfrak{A}_T = \mathfrak{A}$. □

In addition, the following characterization of

$$P^0(A) = \int_A f(\omega)P(d\omega), \qquad A \in \mathfrak{A}, \tag{6.4}$$

can be given. Cf. also [4].

Theorem 6.2. *In the well defined conditional expectations*

$$E^0(X/\mathfrak{A}_{t-1}) = E(X\frac{f_t}{f_{t-1}}/\mathfrak{A}_{t-1}), \qquad X \in L_p(\Omega, \mathfrak{A}_t, P) \tag{6.5}$$

the conditional probability densities

$$\frac{f_t}{f_{t-1}}: \qquad f_0 = 1, \quad f_t = E(f/\mathfrak{A}_t), \qquad t = 1, \dots, T, \tag{6.6}$$

belong to L_q-space, $q = p(1-p)^{-1}$, with

$$\text{(6.7)} \qquad \operatorname{essup}\left(|\frac{f_t}{f_{t-1}}|^q / \mathfrak{A}_{t-1} \right) < \infty.$$

Proof. The proof just follows from (2.1)-(2.3). □

Thinking of P^0 as the "true" probability measure for the market future events, it seems preferable to be sure that P^0 is somehow close to the applied probability measure P. In particular, the proximity of P^0 to P can be valuated through the conditional probability densities (6.6). Namely, P^0 is closer to P, if these densities are closer to 1. In this line, for every time-period t, considering the corresponding risk less return R and price operator $x(X), X \in L^+$, on the convex cone

$$L^+ \subseteq L_p^+(\Omega, \mathfrak{A}_t, P),$$

we have the following result.

Theorem 6.3. *For the prices $x(X)$, $X \in L^+$, conditioned by the sandwich*

$$\text{(6.8)} \qquad E(R^{-1}Zm/\mathfrak{A}_{t-1}) + x(X'') \le x(X') + E(R^{-1}YM/\mathfrak{A}_{t-1})$$

with $X', X'' \in L^+$ and $Y, Z \in L_p^+(\Omega, \mathfrak{A}_t, P)$ such that

$$Z + X'' \le X' + Y,$$

the $\mathfrak{A}_t$-measurable elements $M_t, m_t \in L_q$ provide the lower and upper bounds:

$$\text{(6.9)} \qquad m_t \le \frac{f_t}{f_{t-1}} \le M_t, \qquad (t = 1, \dots, T).$$

Proof. We just refer to Theorem 5.2. □

We note that through the bounds (6.9), the probability density (6.4) having the form

$$\text{(6.10)} \qquad f = \prod_{t=1}^{T} \frac{f_t}{f_{t-1}}$$

can be evaluated as well:

$$m = \prod_{t=1}^{T} m_t \le f \le \prod_{t=1}^{T} M_t = M$$

- cf. (0.10).

The particular case of lower and upper bounds for the factor f given by positive constants is considered in [1].

References

[1] S. Albeverio, Yu.A. Rozanov: "On multi-period market extension with the uniformly equivalent martingale measure". *Acta Appl. Math.*, **67**, 253–275, 2001.

[2] F. Delbaen, W. Schachermayer: "A general version of fundamental theorem of asset pricing". *Math. Ann.*, **300**, 463–520, 1994.

[3] G. Di Nunno: "Hölder equality for conditional expectations with applications to linear monotone operators". To appear in *Theor. Probability Appl.*, 2003.

[4] G. Di Nunno: "Some versions of the fundamental teorem of asset pricing". Preprint series in pure mathematics, University of Oslo, 13, 2002.

[5] G. Di Nunno, Yu.A. Rozanov: "On monotone versions of Hahn–Banach extension theorem". IMATI–CNR, Thechnical report, 99.11, Milano.

[6] R. Durrett: *Probability: Theory and Examples*. Wadsworth & Brookes, Pacific Grove, 1991.

[7] B. Fuchssteiner, W. Lusky: *Convex Cones*. North-Holland, Amsterdam 1981.

[8] Yu. Kabanov, C. Stricker: "On equivalent martingale measures with bounded densities". *Séminaire de Probabilité, XXXV*, 139–148, *Lecture Notes in Math.*, 1755, Springer, Berlin, 2001.

[9] O. Kreps: "Arbitrage and equilibrium in economics with infinitely many commodities". *Journal of Math. Econom.*, **8**, 15–35, 1981.

[10] H.H. Schaefer: *Banach Lattices and Positive Operators*. Springer, Berlin 1974.

[11] K. Yosida: *Functional Analysis*. (6^{th} ed.) Springer, Berlin 1980.