Dept. of Math. Univ. of Oslo
Pure Mathematics
ISBN 82–553–1373–7 No. 11
ISSN 0806–2439 April 2003

CHANGE OF RINGS IN DEFORMATION THEORY OF MODULES

RUNAR ILE

Abstract. Given a B-module M and any presentation $B = A/J$, the obstruction theory of M as B-module is determined by the usual obstruction class o_{A} for deforming M as A-module *and* a new obstruction class o_{J}. These two classes give the tool for constructing two obstruction maps which depend on each other and which characterise the hull of the deformation functor. We obtain relations between the obstruction classes by studying a change of rings spectral sequence and by representing certain classes as elements in the Yoneda complex. Calculation of the deformation functor of M as B-module, including the (generalised) Massey products, is thus possible within any A-free 2-presentation of M.

1. Introduction

In this article we study the following functor of infinitesimal deformations.

Definition 1. Let A be a (commutative) flat $\mathcal{O}$-algebra where $\mathcal{O}$ is a local complete Noetherian ring with k as residue field. Let $\overline{A} = A\otimes_{\mathcal{O}}k$ and let M be an $\overline{A}$-module. Define $\mathsf{Art}_{\mathcal{O}}$ as the category of local commutative Artinian $\mathcal{O}$-algebras R with residue field k such that the composition $\mathcal{O} \to R \to k$ equals the canonical map from $\mathcal{O}$ to its residue field. Morphisms are maps of local $\mathcal{O}$-algebras. The *deformation functor* of M is a covariant functor

$$\mathrm{Def}_{\mathrm{M}}^{\mathrm{A}} : \mathsf{Art}_{\mathcal{O}} \longrightarrow \mathsf{Sets}$$

where $\mathrm{Def}_{\mathrm{M}}^{\mathrm{A}}(R)$ is the set of equivalence classes of *deformations* of M to R. A deformation (or flat lifting) of M to R is an $A_R := A\otimes_{\mathcal{O}}R$-module M_R, flat as R-module together with an A_R-linear map $\pi : M_R \to M$ with $\pi\otimes_R k : M_R\otimes_R k \xrightarrow{\simeq} M$. Two deformations are equivalent if they are isomorphic above M. Maps are induced by tensorisation.

Remark 1. One natural choice for $\mathcal{O}$ is as the hull of the deformation functor of $\overline{A}$ as k-algebra, with A the formally versal formal family which in particular is a flat $\mathcal{O}$-algebra with $A\otimes_{\mathcal{O}}k = \overline{A}$. In the case $\mathcal{O} = k$ we have $\overline{A} = A$ and the article might be somewhat easier to read with this assumption.

More generally, let $F : \mathsf{Art}_{\mathcal{O}} \to \mathsf{Sets}$ be a covariant functor with $F(k)$ a one element set. M. Schlessinger [18] formulated a sufficient and necessary set of criteria for the existence of a complete local ring H, called a (pro-representing) *hull*, and a formally versal formal family $\{M_n\}_{n=1}^{\infty}$, a projective system with $M_n \in F(H/\mathfrak{n}\mathfrak{m}_H^{n-1})$ where $\mathfrak{n} = \mathfrak{m}_H^2 + \mathfrak{m}_{\mathcal{O}}$ such that the induced map $\rho : \mathrm{Hom}^{\mathrm{cont.}}_{\mathcal{O}-\mathrm{alg./k}}(\mathrm{H}, -) \to F$ is *formally smooth* and an *isomorphism on the relative Zariski tangent space*. F is

2000 *Mathematics Subject Classification.* Primary 13D10, 14B10; Secondary 13D07.
Key words and phrases. Obstruction, Massey product, spectral sequence, mixed characteristic.
Acknowledgement. The author is indebted to Prof. O. A. Laudal for sharing essential insight and for valuable comments on the manuscript.
This article is based on parts of the author's 2001 Dr. Phil. Thesis at the Dept. of Mathematics, University of Oslo.

called *pro-representable* if ρ is an isomorphism. Most deformation functors have hulls, if the relative Zariski tangent space of F is finite dimensional.

Example 1. Let $\rho : \Pi \to \mathrm{Gl}_n(k)$ be a continuous representation of a profinite group Π satisfying a p-finiteness condition where p is the characteristic of the finite field k. Define the deformation functor $\mathrm{Def}_\rho : \mathsf{Art}_{\mathcal{O}} \to \mathsf{Sets}$ as equivalence classes of liftings $\widetilde{\rho} : \Pi \to \mathrm{Gl}_n(R)$ of ρ. Here $\mathcal{O}$ is a "coefficient ring" with residue field k, typically $\mathcal{O} = W(k)$, the Witt ring of k. If Π is the Galois group of a number field, one obtains what B. Mazur has termed *deformation theory of Galois representations*, cf. [16]. If $A = \mathcal{O}[[\Pi]]$ and $M = k^n$ with the $\overline{A} = k[[\Pi]]$-module structure induced from ρ, this deformation functor is canonically isomorphic to the one in Definition 1, (we have to allow for non-commutative algebras A). By applying the Schlessinger criteria, Mazur proved that Def_ρ in general has a hull, and is pro-representable if ρ is absolutely irreducible, see [15].

Example 2. If $\mathcal{O} = k$ is an algebraically closed field of any characteristic and M is a finitely generated A-module (and A an *algebraic* k-algebra, i.e. the Henselisation of a finitely generated k-algebra at a maximal ideal), locally free on the complement of the closed point, there exist algebraic versal deformations of M [21]. A. Ishii [11] has constructed a filtration of the versal base spaces (with reduced structure) of all reflexive modules (including the decomposable ones) over rational surface singularities and has determined the local deformation relation of the reflexive modules over the rational double points. These mini-versal base spaces are far from being (locally) "coarse" moduli spaces. In particular the deformation functors of these reflexive modules restricted to Artinian rings are not pro-representable. Indeed there is only a finite set of isomorphism classes of reflexive modules of fixed rank over a quotient surface singularity, yet the singular versal base has complicated geometry.

Schlessinger did not provide any *effective* construction of the hull. The only known general method to compute H given M, is via a natural *obstruction class.*

Definition 2. A *small lifting situation* is a surjective map $\pi : R \to S$ in $\mathsf{Art}_{\mathcal{O}}$ where $\ker\pi$ is contained in the socle of R, i.e. $\mathfrak{m}_R \cdot \ker\pi = 0$, and a deformation M_S of M to S.

The obstruction class is then an element $\mathrm{o}_* = \mathrm{o}_*(\pi, M_S) \in \mathrm{H}^2 \otimes \ker\pi$ where H^2 is the second cohomology group of the object M. If $F = \mathrm{Def}_{\mathrm{M}}^{\mathrm{A}}$ then $\mathrm{o}_{\mathrm{A}} = \mathrm{o}_{\mathrm{A}}(\pi, M_S)$ and $\mathrm{H}^2 = \mathrm{Ext}^2_{\mathrm{A}}(\mathrm{M},\mathrm{M})$. The obstruction class is natural with respect to morphisms of the lifting situation. There exists a lifting of M_S to R (or a prolongation of the deformation M_S to the "thicker" Artinian neighbourhood $\mathrm{Spec}\, R$) if and only if this obstruction class is zero. The obstruction class has been constructed for many deformation functors, e.g. [9, 10, 13], for axiomatic approaches see [1, 5, 7].

If F has a hull, there is a universal element $M_1 \in F(H_1)$ where $H_1 = k[\mathrm{H}^{1*}] = k \oplus \mathrm{H}^{1*}$ and H^1 is the relative Zariski tangent space; $\mathrm{H}^1 \cong F(k[\varepsilon])$ (naturally a k-vector space). In the case $F = \mathrm{Def}_{\mathrm{M}}^{\mathrm{A}}$, $\mathrm{H}^1 = \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M},\mathrm{M})$ and M_1 is given by the universal extension

$$M_1 : \quad 0 \longrightarrow M \otimes_k \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M},\mathrm{M})^* \longrightarrow M_1 \xrightarrow{\pi_1} M \longrightarrow 0\,.$$

The construction of H then proceeds through successive "prolongations" of M_1 to thicker Artinian $\mathcal{O}$-algebras through small lifting situations, at each step calculating the obstruction. If this is done correctly, one obtains power series in T^1, contained in $\mathfrak{m}^2_{T^1} + \mathfrak{m}_{\mathcal{O}}$, one (possibly "0") for each generator in T^2 of the relative cotangent space, where T^i is the completion of the free $\mathcal{O}$-algebra which has H^i as relative Zariski tangent space for $i = 1, 2$. This defines an *obstruction map* $\mathrm{o}^* : T^2 \to T^1$, which is naturally compatible with the obstruction class o_* (see Definition 4), such that $H = T^1 \hat{\otimes}_{T^2} \mathcal{O}$.

The existence of an obstruction map is provided by O. A. Laudal rather abstractly for a deformation functor of a small category of algebras in [13, Thm. 4.2.4] (see also V. P. Palamodov [17, Thm. 5.6] (without proof) for compact analytic manifolds) and for $\mathrm{Def}_{\mathrm{M}}^{\mathrm{A}}$ and $\mathcal{O} = k$ with explicit Yoneda-representations of the generalised Massey products in [14]. For an axiomatic existence theorem, see [7, Thm. 2.3.10], it shows that the existence of a natural obstruction class together with a natural action of the tangent space on the set of liftings in a small lifting situation, implies the existence of an obstruction map for F. Once we have an obstruction map, the general Krull dimension estimate

$$\dim_k \mathrm{H}^1 \geq \dim_{\mathrm{Krull}} H - \dim_{\mathrm{Krull}} \mathcal{O} \geq \dim_k \mathrm{H}^1 - \dim_k \mathrm{H}^2$$

follows. (See also [12].)

In practice it is difficult to give non-trivial results about the obstruction map, the usual application is some variation of $\mathrm{H}^2 = 0 \Rightarrow H$ *is smooth*. In fact, very few classes of examples of deformation functors have been given for which anything beyond the general Krull dimension estimate is known. By studying modules, one can at least calculate examples as there exists an effective obstruction algorithm. In the present paper we provide a refinement of the obstruction map for modules which has both theoretical and computational consequences. For an application of these ideas, see [8].

Let B be a flat $\mathcal{O}$-algebra which is a quotient of A and let $J = \ker(A \to B)$ and assume M is a $\overline{B} = B \otimes_{\mathcal{O}} k$-module as $\overline{A}$-module, i.e. that $\overline{J} = J \otimes_{\mathcal{O}} k \subseteq \mathrm{Ann}_{\overline{A}}(M)$. Suppose we want to study the deformation functor of B-modules $\mathrm{Def}_{\mathrm{M}}^{\mathrm{B}}$. The $\overline{B}$-cohomology of M may be complicated while A can be chosen as a simpler ring. There is a natural injective map $\mathrm{Def}_{\mathrm{M}}^{\mathrm{B}} \to \mathrm{Def}_{\mathrm{M}}^{\mathrm{A}}$ and the ideal J acts on an A-deformation M_R of M to R through the A_R-action. Let $\mathrm{Def}_{\mathrm{M}}^{(\mathrm{A},\mathrm{J})} \subseteq \mathrm{Def}_{\mathrm{M}}^{\mathrm{A}}$ be the sub-functor of A-deformations annihilated by J.

Lemma 1. *Let A and B be flat $\mathcal{O}$-algebras and M a $\overline{B} = B \otimes_{\mathcal{O}} k$-module. Let J be an ideal in A and assume $B = A/J$. Then*

$$\mathrm{Def}_{\mathrm{M}}^{\mathrm{B}} \cong \mathrm{Def}_{\mathrm{M}}^{(\mathrm{A},\mathrm{J})}.$$

The main idea in this paper emerges from Lemma 1: Lift M as an A-module with trivial J-action and only use $\overline{A}$-cohomology to characterise the tangent space and the obstructions. In Theorem 1 we give a new obstruction class o_{J} which exists (in a small lifting situation) if the obstruction o_{A} for lifting M as A-module is zero, such that $\mathrm{o}_{\mathrm{J}} = 0$ if and only if there exists an A-lifting with trivial J-action. In fact o_{J} will sit in the cokernel of a natural map $\partial_{\overline{\mathrm{J}}} : \mathrm{Ext}^1_{\overline{\mathrm{A}}}(\mathrm{M},\mathrm{M}) \to \mathrm{Hom}_{\overline{\mathrm{A}}}(\overline{\mathrm{J}}, \mathrm{End}_{\overline{\mathrm{A}}}(\mathrm{M}))$. Moreover, the kernel of this map is the tangent space of $\mathrm{Def}_{\mathrm{M}}^{(\mathrm{A},\mathrm{J})}$.

With two natural obstruction classes we can construct two obstruction maps $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$, as stated in Theorem 2, which are compatible with the obstructions, Definition 4. The obstruction maps are defined if the cohomology k-vector spaces are of countable dimension, as in [13]. Remark how these maps depend on each other. In particular, it is not true that o^{A} in the pair $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ is induced by o^{A} for $\mathrm{Def}_{\mathrm{M}}^{\mathrm{A}}$ as we clearly see in Example 5. This example also shows that much of the obstruction space not necessarily is "hit" by obstructions (at least as long as we do not deform over *non-commutative* Artinian algebras). Theorem 3 compares $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ with the traditional o^{B} and is based on the relations of the various obstruction classes which are found by investigating maps in a change of rings spectral sequence, which is undertaken in Section 4 and Section 5. In particular, Theorem 4 ties several of our obstruction classes together by a cup product with the obstruction class for lifting M *non-flat* to $\overline{A}/\overline{J}^2$. Finally, in Section 6, we give three obstruction calculations.

In Example 6 and 7 we find obstructions in mixed characteristic. In Example 8 the obstruction ideal is given as a regular sequence (at least in an infinite set of cases) with two elements while $\dim_k \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{M}) = 4$.

For actual calculation of the obstruction power series, one can lift a free resolution of the module, see [14]. The universal deformation to the relative Zariski tangent space of the deformation functor is given by perturbing the differentials in the resolution with Yoneda-representations for a k-basis of $\mathrm{Ext}^1(\mathrm{M},\mathrm{M})$. The quadratic obstruction is given in terms of cup products and the higher degree obstruction as generalised Massey products which are represented as composition products in the Yoneda algebra. It is therefore not sufficient for our purposes to work in the derived category, and our results describing maps in the change of rings spectral sequence and the comparison of obstruction classes is done by giving explicit representations in the appropriate Yoneda algebra of a free complex. Our result enables the obstruction calculus to be performed entirely within a (truncated) Yoneda complex of an A-free resolution of the B-module M. A formal proof of this (in the case $\mathcal{O} = k$) is given in [7, Thm. 3.3.2], see also Example 6–8.

For explicit non-trivial calculations of obstructions (given by cup products) for the Hilbert functor of space curves, see [22, 6]. A. Siqveland gave the local equations for the compactified Jacobian of the $\mathbf{E_6}$ curve singularity and found the degeneracy diagram of the rank 1 torsion free modules in [19] by calculating the obstruction maps. The Massey product algorithms are given in [20]. Similar ideas have recently been used by I. C. Borge and O. A. Laudal [3] to solve the modular isomorphism problem for p-groups with $\mathbb{F}_p$-coefficients. See also [2].

2. The J-obstruction class

In this section we construct 3 obstruction classes for lifting a module in a relative lifting situation.

Let $A \to B$ be any surjective ring homomorphism and let M and N be A-modules with A-free resolutions F and F' respectively. The corresponding *Yoneda complex* is the differential graded module $\mathrm{Hom}^*_{\mathrm{A}}(\mathrm{F},\mathrm{F}')$ where $\mathrm{Hom}^{\mathrm{n}}_{\mathrm{A}}(\mathrm{F},\mathrm{F}') = \mathrm{Hom}_{\mathrm{A}}(\mathrm{F},\mathrm{F}'[-\mathrm{n}])$ with differential ∂ induced from the ones on F and F'. Our first objective is to define a lifting of a Tor-action to the Yoneda complex, which will enable us to study the J-action on the A-deformations of M. Assume that M and N are B-modules as A-modules and let E be an A-free resolution of the A-module B.

Let $m : E \otimes_A F \to F$ and $m' : E \otimes_A F' \to F'$ lift $B \otimes_A M \cong M$ and $B \otimes_A N \cong N$ respectively and see that for $e \in E$, m, m' give an $m(e) \in \mathrm{End}_{\mathrm{A}}(\mathrm{F})$ respectively $m'(e) \in \mathrm{End}_{\mathrm{A}}(\mathrm{F}')$. Define

$$\partial_{\mathrm{A/B}} : E \longrightarrow \mathrm{End}^*_{\mathrm{A}}(\mathrm{Hom}^*_{\mathrm{A}}(\mathrm{F},\mathrm{F}'))$$

by $\partial_{\mathrm{A/B}}(e)(\phi) = m(e)\phi - (-1)^{|\phi||e|}\phi m'(e)$ where $\phi \in \mathrm{Hom}^*_{\mathrm{A}}(\mathrm{F},\mathrm{F}')$. Clearly $\partial_{\mathrm{A/B}}$ depends on the choices made.

Proposition 1. $\partial_{\mathrm{A/B}}$ *induces a canonical map of graded B-modules*

$$\mathrm{Tor}^{\mathrm{A}}_*(\mathrm{B},\mathrm{B}) \longrightarrow \mathrm{End}^*_{\mathrm{A}}(\mathrm{Ext}^*_{\mathrm{A}}(\mathrm{M},\mathrm{N}))$$

making $\mathrm{Ext}^*_{\mathrm{A}}(\mathrm{M},\mathrm{N})$ *a* $\mathrm{Tor}^{\mathrm{A}}_*(\mathrm{B},\mathrm{B})$*-module. In the case $M = N$,* $\mathrm{Ext}^*_{\mathrm{A}}(\mathrm{M},\mathrm{M})$ *is an algebra-module and* $\mathrm{Tor}^{\mathrm{A}}_{\mathrm{p}}(\mathrm{B},\mathrm{B})$ *acts as degree p-derivations. The map is natural in the sequence $(A \to B, M, N)$.*

Proof. One calculates

$$\partial_{\mathrm{A/B}}(e)(\partial\phi) = \pm\partial\big(\partial_{\mathrm{A/B}}(e)(\phi)\big)$$

hence we get induced a $\partial_{\mathrm{A/B}} : E \longrightarrow \mathrm{End}^*_{\mathrm{A}}(\mathrm{Ext}^*_{\mathrm{A}}(\mathrm{M},\mathrm{N}))$. It factorises via $B\otimes_A E$ and one calculates again:

$$\partial_{\mathrm{A/B}}(d\overline{e})(\phi) = \pm\partial\big(\partial_{\mathrm{A/B}}(\overline{e})(\phi)\big)$$

if $\partial\phi = 0$. We get a map

$$\mathrm{H}(B\otimes_A E) \longrightarrow \mathrm{End}^*_{\mathrm{A}}(\mathrm{Ext}^*_{\mathrm{A}}(\mathrm{M},\mathrm{N}))$$

which is independent of the choices made. □

Let $J = \ker(A \to B)$ and define

$$\partial_{\mathrm{J}} : \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M},\mathrm{N}) \longrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{J},\mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N})) \tag{1}$$

to be the adjoint of $\partial_{\mathrm{A/B}}$ restricted to J through $\mathrm{Tor}^{\mathrm{A}}_1(\mathrm{B},\mathrm{B}) \cong J/J^2$. For convenience we will also use the ∂_{J}-notation in adjoint situations, as in the next theorem.

We are now in the position to formulate necessary and sufficient conditions for the existence of deformations of an A-module with trivial J-action, i.e. a B-module, in a small lifting situation. The standard result here, as given in [14], is to produce a class o_{B} in the B-cohomology of the situation, for modules that would be $\mathrm{Ext}^2_{\mathrm{B}}$, which vanish if and only if there is a lifting. We will instead produce two classes (actually three), the o_{A} in $\mathrm{Ext}^2_{\mathrm{A}}$ which is the old obstruction for lifting A-modules, and if $\mathrm{o}_{\mathrm{A}} = 0$, a new class o_{J}, also given by A-cohomology such that $\mathrm{o}_{\mathrm{J}} = 0$ if and only if there exists an A-lifting with trivial J-action. These two classes will enable us to characterise the hull of $\mathrm{Def}^{\mathrm{B}}_{\mathrm{M}}$ by *two* obstruction maps (see Theorem 2) in the same way as o_{B} gives the tool for constructing the obstruction map in [14].

We formulate the result in the following *relative lifting situation*: Suppose A and B are flat $\mathcal{O}$-algebras where $\mathcal{O}$ is a commutative ring. Let $A \to B$ and $\pi : R \to S$ be surjective maps of $\mathcal{O}$-algebras with kernels the ideals J and I respectively. Let M be a $B_S = B\otimes_{\mathcal{O}} S$-module, (similarly $J_S = J\otimes_{\mathcal{O}} S$ etc.). Assume $I^2 = 0$.

Definition 3. A *lifting* of M to B_R is a B_R-module M_R and a B_R-linear map $\pi : M_R \to M$ with $\pi\otimes S : M_R\otimes_R S \xrightarrow{\simeq} M$, such that $\mathrm{Tor}^{\mathrm{R}}_1(\mathrm{M_R},\mathrm{S}) = 0$.

Theorem 1. *In the above situation we have*:

i) *There exists a class* $\mathrm{o}_{\mathrm{A_S}} = \mathrm{o}_{\mathrm{A_S}}(\pi, M) \in \mathrm{Ext}^2_{\mathrm{A_S}}(\mathrm{M},\mathrm{M}\otimes_{\mathrm{S}}\mathrm{I})$ *such that* $\mathrm{o}_{\mathrm{A_S}} = 0$ *if and only if there exists a lifting of* M *to* A_R.

ii) *Given a lifting* M_R *of* M *to* A_R, *there exists a class*

$$\mathrm{o}(\mathrm{J_S}) \in \mathrm{Hom}_{\mathrm{A_S}}(\mathrm{J_S}\otimes_{\mathrm{A_S}}\mathrm{M},\mathrm{M}\otimes_{\mathrm{S}}\mathrm{I})$$

such that $\mathrm{o}(\mathrm{J_S}) = 0$ *if and only if* M_R *is a* B_R*-module as an* A_R*-module.*

iii) *If* $\mathrm{o}_{\mathrm{A_S}} = 0$ *there exists a class* $\mathrm{o}_{\mathrm{J_S}} = \mathrm{o}_{\mathrm{J_S}}(\pi, M) \in \operatorname{coker}\partial_{\mathrm{J_S}}$ *where*

$$\partial_{\mathrm{J_S}} : \mathrm{Ext}^1_{\mathrm{A_S}}(\mathrm{M},\mathrm{M}\otimes_{\mathrm{S}}\mathrm{I}) \longrightarrow \mathrm{Hom}_{\mathrm{A_S}}(\mathrm{J_S}\otimes_{\mathrm{A_S}}\mathrm{M},\mathrm{M}\otimes_{\mathrm{S}}\mathrm{I})$$

is as given in (1) *such that* $\mathrm{o}_{\mathrm{J_S}} = 0$ *if and only if there is a lifting of* M *to* B_R.

iv) *Assume* $\mathrm{o}_{\mathrm{A_S}} = 0 = \mathrm{o}_{\mathrm{J_S}}$, *then there is a transitive and effective action of* $\ker\partial_{\mathrm{J_S}}$ *on the set of isomorphism classes of liftings of* M *to* B_R *over* M.

Finally, all classes and the action are natural for flat maps of $A \to B$ *and of* $R \to S$ *and the induced modules. If* M *is* S*-flat, the naturality follows for all maps of* $R \to S$.

Remark 2. In the case $\mathcal{O} = k = A = B$ (and $J = 0$) one recovers the standard result and our construction of o_{S} is as in [14].

Remark 3. In fact $\mathrm{o}_{\mathrm{A_S}}$ is in the image of the natural map $\mathrm{Ext}^2_{\mathrm{B_S}} \to \mathrm{Ext}^2_{\mathrm{A_S}}$ as we prove in Lemma 4.

Proof. Since we are mainly interested in the deformation case, we give a proof under the additional assumption of either M or I being S-flat. The complex we use lends itself best to these cases. At the end we comment on the general situation.

i) Let $F = (F, d)$ be an A_S-free resolution of M. By the freeness we can lift the differential to a map $\tilde{d}$ of the graded A_R-free module $\tilde{F}$ which in each degree has the same rank as F, thus $\tilde{d}\otimes_R S = d$. If $\tilde{F} = (\tilde{F}, \tilde{d})$ were a complex, it would be a resolution of a lifting of M to A_R. Tensoring $\tilde{F}$ with the short exact sequence $0 \to I \to R \xrightarrow{\pi} S \to 0$ of R-modules gives an exact sequence $0 \to F\otimes_S I \to \tilde{F} \xrightarrow{\pi} F \to 0$ of graded modules since A is flat as $\mathcal{O}$-module. It follows that $(\tilde{d})^2$ is induced by a map $\rho \in Z^2\mathrm{Hom}_{A_S}(F, F\otimes_S I)$ i.e. a 2-cocycle in the Yoneda complex. Define

$$\mathrm{o}_{A_S} = \mathrm{o}_{A_S}(\pi, M) := [\rho] \in \mathrm{Ext}^2_{A_S}(M, M\otimes_S I).$$

This element is independent of the resolution F and the choice of lifting $(\tilde{F}, \tilde{d})$ and is the obstruction for lifting M along π: If $\mathrm{o}_{A_S} = 0$, there is a $\tau \in \mathrm{Hom}^1_{A_S}(F, F\otimes_S I)$—this is the place where we use the additional hypothesis—with $\partial\tau = \rho$. We perturb $\tilde{d}$ by τ and get a differential $d_R = \tilde{d} - \tau\pi$. Hence $F_R = (\tilde{F}, d_R)$ is a complex which is an extension of resolutions (by the additional hypothesis again), thus itself a resolution of $M_R := H_0(F_R)$. Clearly $M_R\otimes_R S \cong M$ and $\mathrm{Tor}^R_1(M_R, S) = H_1(F) = 0$, in fact M_R is R-flat if M is S-flat. If there is a lifting M_R of M, $\mathrm{o}_{A_S} = 0$ by the independence of the choices we claimed above.

ii) To find the obstruction for M_R to be a B_R-module as A_R-module, we lift the canonical isomorphism $m_0 : A_S\otimes_{A_S} F \to F$ to a map of complexes $m : E_S\otimes_{A_S} F \to F$ where $\ldots \to E_2 \to E_1 \to J$ gives an A-free resolution of J which, together with $J \subset A = E_0$, gives an A-free resolution E of B, and $E_S := E\otimes_{\mathcal{O}} S$ gives an A_S-free resolution of B_S. The lifting m exists since M is a B_S-module as in Proposition 1, i.e. since J_S's action on F is homotopically trivial. Choose an $\tilde{m} : E_R\otimes_{A_R} F_R \to F_R$ with $\tilde{m}\otimes_R S = m$ and with $\tilde{m}_{|(E_R)_0\otimes F_R}$ the canonical isomorphism $A_R\otimes_{A_R} F_R \cong F_R$. Then we can view $\tilde{m}$ as an attempt to kill the action of J_R on F_R. We find $\partial(\tilde{m}) = d(F_R) \circ \tilde{m} - \tilde{m} \circ d(E\otimes F_R)$ to be induced by a $\rho \in Z^0\mathrm{Hom}_{A_S}(E_{\geqslant 1}\otimes_A F, F\otimes_S I)$ where $E_{\geqslant 1} = [\ldots \to E_2 \to E_1][+1]$. Define

$$\mathrm{o}(J_S) := [\rho] \in H^0\mathrm{Hom}_{A_S}(E_{\geqslant 1}\otimes_A F, F\otimes_S I) \cong \mathrm{Hom}_{A_S}(J_S\otimes_{A_S} M, M\otimes_S I). \tag{2}$$

This class only depends on the lifting M_R and is the obstruction for M_R to be a B_R-module as A_R-module. If $\mathrm{o}(J_S) = 0$, there is a $\tau \in \mathrm{Hom}^{-1}_{A_S}(E_{\geqslant 1}\otimes_A F, F\otimes_S I)$ with $\partial(\tau) = \rho$. Perturbing $\tilde{m}$ with τ gives $m_R = \tilde{m} - \tau\pi$ with $\partial(m_R) = 0$.

iii) The $\mathrm{o}(J)$ only checks our specific choice of lifting M_R given by d_R, other choices of M_R could be better. To obtain other A_R-liftings we perturb d_R by $\xi \in Z^1\mathrm{Hom}_{A_S}(F, F\otimes_S I)$ to $d'_R = d_R + \xi\pi$. This gives a new differential ∂' and

$$\begin{aligned}\partial'(\tilde{m}) &= (d_R + \xi\pi)\tilde{m} - \tilde{m}(d_{E\otimes F'_R})\\ &= d_R\tilde{m} - (-1)^{|E|}\tilde{m}(1\otimes d_R) - \tilde{m}(d_E\otimes 1) + \xi\pi\tilde{m} - (-1)^{|E|}\tilde{m}(1\otimes\xi\pi)\\ &= \big(\rho + \partial_{A_S/B_S}(\xi)\big)\pi\end{aligned}$$

where $\partial_{A_S/B_S} : \mathrm{Hom}_{A_S}(F, F\otimes_S I) \longrightarrow \mathrm{Hom}_{A_S}(E\otimes_A F, F\otimes_S I)$ up to adjointness is the one in Proposition 1. Define the class

$$\mathrm{o}_{J_S} = [\mathrm{o}(J_S)] \in \mathrm{coker}\,\partial_{J_S} = \mathrm{coker}\left(\mathrm{Ext}^1_{A_S}(M, M\otimes_S I) \to \mathrm{Hom}_{A_S}(J_S\otimes_{A_S} M, M\otimes_S I)\right),$$

it depends only on M and π and is the obstruction for lifting M to B_R if there exists a lifting of M to A_R. If $\partial'(\tilde{m}) = \partial(\tau)\pi$, with $\tau \in \mathrm{Hom}^0_{A_S}(F, F\otimes_S I)$, we can perturb $\tilde{m}$ to $\tilde{m}' = \tilde{m} - \tau\pi$ and $\partial'(\tilde{m}') = \big(\rho + \partial_{A_S/B_S}(\xi) - \partial\tau\big)\pi = 0$ so $\tilde{m}'$ gives a homotopy to zero for the action of J_R on F'_R, i.e. M'_R is a B_R-module as A_R-module.

iv) It also follows that any $\xi' \in Z^1\mathrm{Hom}_{A_S}(F, F\otimes_S I)$ with $\partial_{A_S/B_S}(\xi') = 0$ gives another lifting to B_R by $d''_R = d'_R + \xi'\pi$ and that the difference $d'_R - d''_R$ of two liftings to B_R gives an element in $\ker\partial_{\mathrm{A_S/B_S}}$. They are isomorphic if and only if this element is zero in $\mathrm{Ext}^1_{\mathrm{A_S}}(\mathrm{M},\mathrm{M}\otimes_{\mathrm{S}}\mathrm{I})$.

For the general case, the main difference is that $F\otimes_S I$ not necessarily is a resolution, and a resolution F_R of M_R will give $\mathrm{H}_*(F_R\otimes_R S)\cong\mathrm{Tor}^{\mathrm{R}}_*(\mathrm{M_R},\mathrm{S})$ and cannot therefore in general be taken as a lifting $\tilde{F}$ of F. But, in fact only the initial part

(3)
$$\begin{array}{ccccc} F_0 & \xleftarrow{d_1} & F_1 & \xleftarrow{d_2} & F_2 \\ & \nearrow_{m} & & & \\ E_1\otimes F_0 & & & & \end{array}$$

where $d_1 m$ equals the map induced by the multiplication by (generators of) J_S on F, is essential to the existence of liftings to A_R and to B_R as will be exploited later on. The $\mathrm{o}_{\mathrm{A_S}}$ is induced by $\tilde{d}_1\tilde{d}_2$ and if $\mathrm{o}_{\mathrm{A_S}}=0$ modify $\tilde{d}_1$ and $\tilde{d}_2$ by τ_1 and τ_2 as before. $M_R=\mathrm{H}_0(\tilde{F})$ has $M_R\otimes_R S=M$, in a resolution F_R for M_R we can choose $(F_R)_i=\tilde{F}_i$ for $i=0,1$ and $(F_R)_2=\tilde{F}_2\oplus K_2$. Then the Tor-condition follows:

$$\mathrm{Tor}^{\mathrm{R}}_1(\mathrm{M_R},\mathrm{S})=\mathrm{H}_1(F_R\otimes_R S)=\frac{\ker(F_1\longrightarrow F_0)}{\mathrm{im}\big(F_2\oplus(K_2\otimes_R S)\longrightarrow F_1\big)}=0$$

so $M_R=\mathrm{H}_0(\tilde{F})$ is certainly a lifting. The $\mathrm{o}(\mathrm{J_S})$ is defined as induced by $\tilde{d}_1\tilde{m}-m_{E_1}$ where $m_{E_1}:E_1\otimes_A\tilde{F}_0\to\tilde{F}_0$ is induced by the multiplication of J_R on $\tilde{F}_0$. The rest follows as above. □

Remark 4. We shall primarily be interested in the deformation situation, Definition 1, and the case of a small lifting situation, Definition 2. If M_S is a deformation of M to S in $\mathsf{Art}_{\mathcal{O}}$ one has natural isomorphisms like $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{A_S}}(\mathrm{M_S},\mathrm{M_S}\otimes_{\mathrm{S}}\mathrm{I})\cong\mathrm{Ext}^{\mathrm{i}}_{\overline{\mathrm{A}}}(\mathrm{M},\mathrm{M})\otimes_k I$ and $\mathrm{Hom}_{\mathrm{A_S}}(\mathrm{J_S}\otimes_{\mathrm{A_S}}\mathrm{M_S},\mathrm{M_S}\otimes_{\mathrm{S}}\mathrm{I})\cong\mathrm{Hom}_{\overline{\mathrm{A}}}(\overline{\mathrm{J}}\otimes_{\overline{\mathrm{A}}}\mathrm{M},\mathrm{M})\otimes_k I$. The existence of such *constant* groups is essential for the existence of an obstruction algorithm. With a fixed k-basis the constant cohomology groups will keep track of the different obstruction "polynomials" in (the varying) I. To simplify the notation in the deformation situation, let $\mathrm{o}_{\mathrm{A_S}}=\mathrm{o}_{\mathrm{A}}$, $\mathrm{o}_{\mathrm{J_S}}=\mathrm{o}_{\mathrm{J}}$ and so on.

Example 3. A *matrix factorisation* (mf) of an element f in a ring A is a pair (ϕ,ψ) of maps of free modules $\phi:F\to G,\ \psi:G\to F$ with $\phi\psi=f\cdot\mathrm{id}_G$ and $\psi\phi=f\cdot\mathrm{id}_F$. Let $B=A/(f)$ then $M=\operatorname{coker}\phi$ is a B-module as A-module since f annihilates M. If f is A-regular then the following 2-periodic complex of free B-modules (necessarily of equal rank if A is Noetherian and $\operatorname{rk}G<\infty$)

(4)
$$\overline{G}\xleftarrow{\overline{\phi}}\overline{F}\xleftarrow{\overline{\psi}}\overline{G}\xleftarrow{\overline{\phi}}\overline{F}\xleftarrow{\overline{\psi}}\dots$$

is a free resolution of M where $\overline{F}=F\otimes_A B$ etc. Maximal Cohen-Macaulay modules over a hypersurface singularity are given by mfs of the hypersurface. Mfs were introduced by D. Eisenbud in [4]. A deformation of M as B-module will be given by a lifting of this resolution, one will therefore have conditions for lifting the equations $\overline{\phi}\,\overline{\psi}=0$ which create the obstruction o_{B} in $\mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{M})$. Instead Theorem 1 offers the possibility of lifting ϕ corresponding to deformations of M as A-module for which the obstruction $\mathrm{o}_{\mathrm{A}}=0$ since $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{A}}(\mathrm{M},\mathrm{M})=0$ for $i>1$, such that there is a lifting of ψ retaining the relation $\phi\psi=f\cdot\mathrm{id}_G$, this gives the non-trivial obstruction o_{J} (with $J=(f)$) in the cokernel of $\partial_{\mathrm{J}}:\mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M},\mathrm{M})\to\mathrm{End}_{\mathrm{A}}(\mathrm{M})$ where $\partial_{\mathrm{J}}=\psi^*$. Even in this most simple example the advantages are clear: The A-cohomology is easier than the B-cohomology and the relation $f=0$ is eliminated from the obstruction calculus. Further simplifications are possible in the $\mathrm{rk}_B(M)=1$-case as we show in [8].

3. The obstruction maps

We define obstruction maps o^{A} and o^{J} for the obstructions o_{A} and o_{J} in Definition 4 and formulate a structure theorem for the hull of $\mathrm{Def}^{\mathrm{B}}_{\mathrm{M}}$ in Theorem 2. A comparison of the A- and J-obstruction maps with the B-obstruction map is given in Theorem 3.

For greater flexibility we will define obstruction maps as continuous maps between local $\mathcal{O}$-algebras which have countably dimensional Zariski tangent spaces. Let V be a countably dimensional vector space over k with a given basis $\{e_i\}_{i=1}^{\infty}$. The set of sub-vector spaces of V which contain almost all the basis elements defines a topology on V such that $V^* = \mathrm{Hom}^{\text{cont.}}_{k\text{-vec.}}(\mathrm{V},\mathrm{k})$ also is countably dimensional, and if we fix the dual basis $\{e_i^*\}_{i=1}^{\infty}$ for V^* then $V^{**} \cong V$ canonically. Let $\widehat{\mathrm{Free}}_{\mathcal{O}}(\mathrm{V}^*)$ be the free $\mathcal{O}$-algebra in variables $\{x_i\}_{i=1}^{\infty}$ completed in the topology given by the basis $\mathcal{I}_{ijl}$ of open ideals around 0 where $\mathcal{I}_{ijl} = \mathfrak{m}^i_{\mathcal{O}} + (x_1, x_2, \dots)^j + (x_l, x_{l+1}, \dots)$. We insist on the continuous identification of the relative cotangent vector space $\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}_{\mathcal{O}})$ of $\widehat{\mathrm{Free}}_{\mathcal{O}}(\mathrm{V}^*)$ with V^* where $\overline{x}_i = e_i^*$ hence also a canonical continuous identification of the relative Zariski tangent space of $\widehat{\mathrm{Free}}_{\mathcal{O}}(\mathrm{V}^*)$ with V. Suppose $\{H_n\}_{n=1}^{\infty}$ is a projective system of surjections in $\mathsf{Art}_{\mathcal{O}}$. Then $H := \varprojlim H_n$ with the induced topology is a continuous quotient of a $\widehat{\mathrm{Free}}_{\mathcal{O}}(\mathrm{V}^*)$ for some V. Conversely every continuous quotient of $\widehat{\mathrm{Free}}_{\mathcal{O}}(\mathrm{V}^*)$ can be given as such a projective limit. Define $\widehat{\mathrm{Def}}^{\mathrm{B}}_{\mathrm{M}}(H) = \varprojlim \mathrm{Def}^{\mathrm{B}}_{\mathrm{M}}(H_n)$.

Recall the map ∂_{J} with $N = M$, see (1). Assume for the rest of this section that the k-vector spaces $\mathrm{H}^2_{\mathrm{A}} = \mathrm{im}\big(\mathrm{Ext}^2_{\overline{\mathrm{B}}}(\mathrm{M},\mathrm{M}) \to \mathrm{Ext}^2_{\overline{\mathrm{A}}}(\mathrm{M},\mathrm{M})\big)$, $\mathrm{H}^2_{\mathrm{J}} = \mathrm{coker}\, \partial_{\overline{\mathrm{J}}}$ and $\mathrm{H}^1 = \ker \partial_{\overline{\mathrm{J}}}$ all are of countable k-dimension and for any choice of k-bases let T^2_A, T^2_J and T^1 be the corresponding complete $\mathcal{O}$-algebras with these vector spaces as relative Zariski tangent spaces.

Definition 4. In the situation described before Lemma 1, *two obstruction maps* for the obstructions o_{A} and o_{J} in Theorem 1 (see Remark 4) are continuous $\mathcal{O}$-algebra homomorphisms $\mathrm{o}^{\mathrm{A}} : T^2_A \to T^1$ and $\mathrm{o}^{\mathrm{J}} : T^2_J \to T^1$ satisfying the following conditions. If $H := \big(T^1 \hat{\otimes}_{T^2_A} \mathcal{O}\big) \hat{\otimes}_{T^2_J} \mathcal{O}$ there is a formal deformation $\widehat{M}$ in $\widehat{\mathrm{Def}}^{\mathrm{B}}_{\mathrm{M}}(H)$ such that for any small lifting situation, Definition 2, there is a continuous $\sigma : H \to S$ with $\sigma_* \widehat{M} = M_S$ and for any such σ we have that the adjoint $\mathrm{o}^{\mathrm{adj}}_{\mathrm{A}}$ of $\mathrm{o}_{\mathrm{A}}(\pi, M_S) \in \mathrm{H}^2_{\mathrm{A}} \otimes_k I$ makes the following diagram commutative

$$
\begin{array}{ccccccc}
\mathfrak{m}_{T^2_X}/(\mathfrak{m}^2_{T^2_X} + \mathfrak{m}_{\mathcal{O}}) & \xrightarrow{=} & \mathrm{H}^2_{\mathrm{X}}{}^* & & \xrightarrow{\mathrm{o}^{\mathrm{adj}}_{\mathrm{X}}} & & I \\
\uparrow & & & & & & \downarrow \\
\mathfrak{m}_{T^2_X} & \hookrightarrow & T^2_X & \xrightarrow{\mathrm{o}^{\mathrm{X}}} & T^1 & \xrightarrow{\theta} & R \\
 & & & & \downarrow & & \downarrow \pi \\
 & & & & H & \xrightarrow{\sigma} & S
\end{array} \tag{5}
$$

where θ is continuous and lifts σ and $X = A$. If $\mathrm{o}_{\mathrm{A}}(\pi, M_S) = 0$ then the adjoint $\mathrm{o}^{\mathrm{adj}}_{\mathrm{J}}$ of $\mathrm{o}_{\mathrm{J}}(\pi, M_S) \in \mathrm{H}^2_{\mathrm{J}} \otimes_k I$ makes the diagram commutative with $X = J$.

Theorem 2 ([7])**.** *Let A and B be flat $\mathcal{O}$-algebras with $B = A/J$ for an ideal $J \subset A$. Let M be a $\overline{B} = \overline{A}/\overline{J}$-module where $\overline{X} = X \otimes_{\mathcal{O}} k$ for $X = A$, B and J. Then $\mathrm{Def}^{\mathrm{B}}_{\mathrm{M}}$ is a functor with two obstructions in $\mathrm{H}^2_{\mathrm{A}}$ and $\mathrm{H}^2_{\mathrm{J}}$ such that if H^1, $\mathrm{H}^2_{\mathrm{A}}$ and $\mathrm{H}^2_{\mathrm{J}}$ have countable k-dimension there are obstruction maps*

$$\mathrm{o}^{\mathrm{A}} : T^2_A \longrightarrow T^1 \qquad \textit{and} \qquad \mathrm{o}^{\mathrm{J}} : T^2_J \longrightarrow T^1$$

for the obstructions $\mathrm{o_A}$ *and* $\mathrm{o_J}$*. In particular the hull of* $\mathrm{Def}_\mathrm{M}^{(\mathrm{A,J})} \cong \mathrm{Def}_\mathrm{M}^\mathrm{B}$ *is given as*

$$H \cong \left(T^1 \hat{\otimes}_{T_A^2} \mathcal{O}\right) \hat{\otimes}_{T_J^2} \mathcal{O}\,.$$

Remark 5. The statement implicitly claims the existence of k-vector bases and hence topologies as described before the Theorem and maps continuous with respect to these topologies.

Example 4. Remark that every $\mathcal{O}$-algebra in the pro-category of $\mathsf{Art}_\mathcal{O}$ is obtained as the hull of the deformation functor of a module. In fact the following argument is valid for the *non-commutative deformation functor* of modules as well as for the commutative one. In the non-commutative case $\mathsf{Art}_\mathcal{O}$ is the category of local not necessarily commutative Artinian $\mathcal{O}$-algebras ($\mathcal{O}$ as in Definition 1) R with k as residue field (i.e. k is the unique simple R-module). A and B may also be non-commutative $\mathcal{O}$-algebras. A deformation of a left $\overline{A}$-module M is defined as in Definition 1 except that M_R is an $A-R$-bimodule which is a left A-module and a right R-module, or equivalent, a left $A\otimes_\mathcal{O} R^\mathrm{o}$-module. Furthermore, $\widehat{\mathrm{Free}}_\mathcal{O}(\mathrm{V}^*)$ is the free non-commutative $\mathcal{O}$-algebra, completed in the topology defined by ideals $\mathcal{I}_{ijl}$ analogous to the ones in the beginning of this section, e.g. where the “power ideal” $(x_1, x_2, \ldots)^j$ is replaced by the (two sided) ideal generated by j-tensors, and so on.

Fix a maximal ideal $\mathfrak{m}$ in an $\mathcal{O}$-algebra B such that $B/\mathfrak{m} \cong k$. Assume $B/(\mathfrak{m}^2 + \mathfrak{m}_\mathcal{O})$ is countably dimensional and let $\hat{B}$ be the completion of B in any topology as given in the beginning of this section (or analogous in the non-commutative case), then

$$\mathrm{Hom}^{\mathrm{cont.}}_{\mathcal{O}-\mathrm{alg./k}}(\hat{\mathrm{B}}, -) \xrightarrow{\simeq} \mathrm{Def}_\mathrm{k}^\mathrm{B}$$

where $\phi \in \mathrm{Hom}^{\mathrm{cont.}}_{\mathcal{O}-\mathrm{alg./k}}(\hat{\mathrm{B}}, \mathrm{R})$ is mapped to the $B\otimes_\mathcal{O} R^\mathrm{o}$-module R with module structure given by (left) multiplication of $B\otimes_\mathcal{O} R^\mathrm{o}$ through the composition $\hat{B}\otimes_\mathcal{O} R^\mathrm{o} \xrightarrow{\phi\otimes\,\mathrm{id}} R\otimes_\mathcal{O} R^\mathrm{o} \xrightarrow{\mathrm{mult.}} R^\mathrm{o}$. It gives a deformation of k to R. For the inverse, any deformation M_R of k to R has $M_R \cong R$ as R-modules since M_R is R-flat, i.e. R-free of rank 1. Hence R has a (left) $B\otimes_\mathcal{O} R^\mathrm{o}$-module structure. Define $\phi : \hat{B} \to R$ by $\phi(b) := (b\otimes 1) \bullet 1_R = r \in R$ for $b \in B$. Then $\phi(b'b) = b'b\otimes 1 \bullet 1_R = (b'\otimes 1)(b\otimes 1) \bullet 1_R = (b'\otimes 1) \bullet r = (b'\otimes 1)(1\otimes r) \bullet 1_R = (1\otimes r)(b'\otimes 1) \bullet 1_R = 1\otimes r \bullet r' = r'r = (b'\otimes 1 \bullet 1_R)(b\otimes 1 \bullet 1_R) = \phi(b')\phi(b)$, $\phi(1) = 1_R$ and ϕ is additive. If $f : \mathcal{O} \to B$ and $g : \mathcal{O} \to R$ define the $\mathcal{O}$-algebra structures, $\phi(f(\lambda)a) = f(\lambda)a\otimes 1 \bullet 1_R = a\otimes g(\lambda) \bullet 1_R = (1\otimes g(\lambda)) \bullet (a\otimes 1 \bullet 1_R) = g(\lambda)(a\otimes 1 \bullet 1_R) = g(\lambda)\phi(a)$, hence ϕ gives a well defined $\mathcal{O}$-algebra homomorphism $\hat{B} \to R$ above k and $\hat{B}$ pro-represents $\mathrm{Def}_\mathrm{k}^\mathrm{B}$. In particular

$$\hat{B} \cong \left(T^1 \hat{\otimes}_{T_A^2} \mathcal{O}\right) \hat{\otimes}_{T_J^2} \mathcal{O}$$

for obstruction maps $\mathrm{o^A}$ and $\mathrm{o^J}$. For instance, if $B = \mathcal{O}$ then $\overline{B} = k$ and $\mathrm{Def}_\mathrm{k}^\mathrm{B}(R)$ is a one element set for all R and $\mathrm{Def}_\mathrm{k}^\mathrm{B}$ is pro-represented by $\mathcal{O}$.

Remark that if B is non-commutative, we can still deform k over commutative Artinian $\mathcal{O}$-algebras, but then the completion $\hat{B}$ will be in the ideals $\mathcal{I}_{ijl} + [B, B]$ and hence in that case be a commutative $\mathcal{O}$-algebra, indeed $\hat{B} = (B/[B, B])\hat{}$.

In [7] we define axiomatically *a functor with* n *obstructions* and corresponding obstruction maps and prove the existence of such maps in the countably dimensional case. Theorem 2 is an instance of this. A single obstruction map defined for the obstruction $\mathrm{o_A}$ has been constructed by O. A. Laudal in [14].

We next state a theorem describing how closely related the obstruction map $\mathrm{o^B}$, defined by $\overline{B}$-cohomology, is to $\mathrm{o^A}$ and $\mathrm{o^J}$, defined by $\overline{A}$-cohomology. Let T_B^2 be a local complete $\mathcal{O}$-algebra with relative Zariski tangent space $\mathrm{Ext}^2_{\overline{\mathrm{B}}}(\mathrm{M}, \mathrm{M})$ for any topology as in Theorem 2. In the next sections (Proposition 2 and Lemma 4) we show that there is a canonical isomorphism $\ker \partial_{\overline{\mathrm{J}}} \cong \mathrm{Ext}^1_{\overline{\mathrm{B}}}(\mathrm{M},\mathrm{M})$ and a natural exact sequence of $\overline{A}$-modules $0 \to \operatorname{coker} \partial_{\overline{\mathrm{J}}} \to \mathrm{Ext}^2_{\overline{\mathrm{B}}}(\mathrm{M},\mathrm{M}) \to \mathrm{Ext}^2_{\overline{\mathrm{A}}}(\mathrm{M},\mathrm{M})$. Hence there is a "short exact sequence" of continuous maps $T^2_A \hookrightarrow T^2_B \twoheadrightarrow T^2_J$. Our main comparison result reads:

Theorem 3. *With assumptions as in Theorem 2, given a pair of obstruction maps* $\mathrm{o}^{\mathrm{A}} : T^2_A \to T^1$ *and* $\mathrm{o}^{\mathrm{J}} : T^2_J \to T^1$ *for the obstructions* o_{A} *and* o_{J}, *defining the hull of* $\mathrm{Def}^{(\mathrm{A},\mathrm{J})}_{\mathrm{M}}$, *there exists an obstruction map* $\mathrm{o}^{\mathrm{B}} : T^2_B \longrightarrow T^1$ *for the obstruction* o_{B}, *defining the hull of* $\mathrm{Def}^{\mathrm{B}}_{\mathrm{M}}$, *such that*

$$\mathrm{o}^{\mathrm{B}}|_{T^2_A} = \mathrm{o}^{\mathrm{A}}\,, \qquad \textit{and} \qquad \mathrm{o}^{\mathrm{B}} \hat{\underset{T^2_A}{\otimes}} \mathcal{O} = \mathrm{o}^{\mathrm{J}} \hat{\underset{T^2_A}{\otimes}} \mathcal{O} \ \ \textit{as maps}\ \ T^2_J \longrightarrow T^1 \hat{\underset{T^2_A}{\otimes}} \mathcal{O}\,. \tag{6}$$

Conversely, given an obstruction map o^{B}, *there exists a pair of obstruction maps* o^{A} *and* o^{J} *such that the following diagram of continuous maps is commutative*:

$$\begin{array}{ccccc} T^2_A & \longrightarrow & T^2_B & \longrightarrow & T^2_J \\ & \searrow^{\mathrm{o}^{\mathrm{A}}} & \downarrow{\scriptstyle \mathrm{o}^{\mathrm{B}}} & \swarrow_{\mathrm{o}^{\mathrm{J}}} & \\ & & T^1 & & \end{array}$$

Remark 6. The o^{B}-map is in general not the "union" of o^{A} and o^{J}, but there is always a pair of obstruction maps $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ such that o^{B} can be taken as the "union" of o^{A} and o^{J}.

We do not necessarily get a trivial $\mathrm{o}^{\mathrm{J}_1+\mathrm{J}_2}$ even if $\mathrm{o}^{\mathrm{J}_i}$ is trivial for $i = 1, 2$. The reason for this is simply that the natural map $\operatorname{coker} \partial_{\mathrm{J}_1+\mathrm{J}_2} \to \oplus \operatorname{coker} \partial_{\mathrm{J}_i}$ does not have to be injective, an explicit example is given in [7, Ex. 4.1.4].

Remark 7. If a choice of o^{A} for $\mathrm{Def}^{\mathrm{A}}_{\mathrm{M}}$ continued to T^1 is trivial, one can choose $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ for $\mathrm{Def}^{(\mathrm{A},\mathrm{J})}_{\mathrm{M}}$ such that o^{A} is trivial. But even if o^{A} in $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ is trivial, o^{A} for $\mathrm{Def}^{\mathrm{A}}_{\mathrm{M}}$ continued to T^1 may be far from trivial as Example 5 shows. There is no way one can find o^{A} "first" and then find o^{J} as this has no meaning. It is not clear to the author whether o^{A} in the pair $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ and the locus it defines has any interesting interpretation.

Proof. Suppose $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ is given, assume $\dim_k \ker \partial_{\mathrm{J}} < \infty$ and let $T^1_n = T^1/\mathfrak{m}^{n-1}\mathfrak{n}$ where $\mathfrak{n} = \mathfrak{m}^2 + \mathfrak{m}_{\mathcal{O}}$. Let $G^A_n = T^1_n/(f^{n-1}_i)\mathfrak{m} + (g^{n-1}_j)\mathfrak{m}$ which maps surjectively to $G^J_n = T^1_n/(f^n_i) + (g^{n-1}_j)\mathfrak{m}$ above $H_{n-1} = T^1_{n-1}/(f^{n-1}_i) + (g^{n-1}_j)$ where $f^n_i = \mathrm{o}^{\mathrm{A}}(y_i)$ and $g^n_j = \mathrm{o}^{\mathrm{J}}(z_j)$ in T^1_n. Observe that G^A_n is "maximal" (with fixed relative Zariski tangent space) such that $\pi^A_n : G^A_n \twoheadrightarrow H_{n-1}$ and M_{n-1}, a versal lifting of M to H_{n-1}, together give a small lifting situation. G^A_n is the test algebra for the o_{A}-obstruction. Similarly G^J_n is maximal such that $\pi^J_n : G^J_n \twoheadrightarrow H_{n-1}$ and M_{n-1} together give a small lifting situation with $\mathrm{o}_{\mathrm{A}}(\pi^J_n, M_{n-1}) = 0$. G^J_n is the test algebra for the o_{J}-obstruction. By Proposition 2 and Lemma 4 we have a "short exact sequence" $T^2_A \hookrightarrow T^2_B \twoheadrightarrow T^2_J$, the last map has a section and we let $\{y_i\} \cup \{z_j\}$ also denote the "generators" in T^2_B. We want to define o^{B}. While we let $\mathrm{o}^{\mathrm{B}}(y_i) := \mathrm{o}^{\mathrm{A}}(y_i)$ which is OK by Lemma 4, we find $\mathrm{o}^{\mathrm{B}}(z_j)$ by induction. Let $I^A_n = \ker \pi^A_n$ and $I^J_n = \ker \pi^J_n$. Then $\mathrm{o}_{\mathrm{B}}(\pi^A_n, M_{n-1}) \in \mathrm{H}^2_{\mathrm{B}} \otimes I^A_n$ maps to $\mathrm{o}_{\mathrm{J}}(\pi^A_n, M_{n-1}) \in \mathrm{H}^2_{\mathrm{J}} \otimes I^J_n$ along $I^A_n \twoheadrightarrow I^J_n$ by Theorem 4. We have chosen representatives g_j (and f_i) in T^1, likewise there is a choice for $\mathrm{o}^{\mathrm{B}}(z_j)$ in T^1. Mapped to I^A_n, $\mathrm{o}^{\mathrm{B}}(z_j)$ and g^n_j may only differ by an element in $K_n := \ker(I^A_n \twoheadrightarrow I^J_n)$. But $K_n = (f^n_i)$, hence we can define o^{B} to "level n" by

$$\mathrm{o}^{\mathrm{B}}(z_j) := g^n_j + \sum_i a^{(n)}_{ji} f^n_i \in T^1_n$$

with $a_{ji}^{(n)} \in \mathcal{O}/\mathfrak{m}_{\mathcal{O}}^n$ and where

$$\mathrm{o}_{\mathrm{B}}(\pi_n^A, M_{n-1}) = \sum_i y_i^* \otimes f_i^n + \sum_j z_j^* \otimes (g_j^n + \sum_i a_{ji}^{(1)} f_i^n) \in \mathrm{H}^2_{\mathrm{B}} \otimes I_n^A \,.$$

By naturality of o_{B} this extends $\mathrm{o}^{\mathrm{B}}(z_j)$ defined on level $n-1$ if we choose $a_{ji}^{(n)} \in \mathcal{O}/\mathfrak{m}^n$ as a lifting of $a_{ji}^{(n-1)} \in \mathcal{O}/\mathfrak{m}^{n-1}$. In the limit we get $\mathrm{o}^{\mathrm{B}}(z_j) = g_j + \sum a_{ji} f_i$ where $a_{ji} = \varprojlim \{a_{ji}^{(n)}\}$ in $\mathcal{O}$ in T^1. This is stronger than our claim. For the general case choose a sequence of finite dimensional k-vector spaces $V_1 \subset V_2 \subset \ldots \subset \mathrm{H}^1$ with $\bigcup V_s = \mathrm{H}^1$. This gives a topology on H^1 and hence on ${\mathrm{H}^1}^* = \varprojlim V_s^*$ and on $T^1 = \varprojlim T_s^1$ where T_s^1 has V_s as relative Zariski tangent space. For each s the argument above may be applied to the composition of o^{A} and of o^{J} with the continuous map $T^1 \twoheadrightarrow T_s^1$. By induction on s (and n) one proves that $\mathrm{o}_{s+1}^{\mathrm{B}}$ may be chosen compatible with $\mathrm{o}_s^{\mathrm{B}}$. Set $\mathrm{o}^{\mathrm{B}} = \varprojlim \{\mathrm{o}_s^{\mathrm{B}}\}$.

Given o^{B}, let $\mathrm{o}^{\mathrm{A}}(y_i) := \mathrm{o}^{\mathrm{B}}(y_i)$ and let $\mathrm{o}^{\mathrm{J}}(z_j) := \mathrm{o}^{\mathrm{B}}(z_j)$ then $(\mathrm{o}^{\mathrm{A}}, \mathrm{o}^{\mathrm{J}})$ are obstruction maps for the two obstructions o_{A} and o_{J}. This follows from Theorem 4 and Lemma 4. □

4. The change of rings spectral sequence

The spectral sequence connects the A- and the B-cohomology and also provides a framework for describing relations between the various obstruction classes. In the following we give detailed descriptions of the maps α, d_2 and γ by representations in the Yoneda complex.

Lemma 2. *Let $A \to B$ be a ring homomorphism and N, M an A- and a B-module respectively. Then there is a first quadrant cohomological spectral sequence*

$$\mathrm{E}_2^{\mathrm{pq}} = \mathrm{Ext}^{\mathrm{p}}_{\mathrm{B}}(\mathrm{M}, \mathrm{Ext}^{\mathrm{q}}_{\mathrm{A}}(\mathrm{B}, \mathrm{N})) \Rightarrow \mathrm{Ext}^*_{\mathrm{A}}(\mathrm{M}, \mathrm{N}) \,.$$

In particular there is a canonical 5-term exact sequence which, in the case $B = A/J$ and N is a B-module as A-module, becomes

$$\begin{aligned} 0 \longrightarrow & \mathrm{Ext}^1_{\mathrm{B}}(\mathrm{M},\mathrm{N}) \longrightarrow \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M},\mathrm{N}) \xrightarrow{\alpha} \mathrm{Hom}_{\mathrm{A}}(\mathrm{J}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N})) \\ \xrightarrow{d_2} & \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{N}) \xrightarrow{\gamma} \mathrm{Ext}^2_{\mathrm{A}}(\mathrm{M},\mathrm{N}) \end{aligned} \tag{7}$$

Proof. Let $G = G. \twoheadrightarrow M$ be a B-projective resolution of M and $N \hookrightarrow I^{\cdot} = I$ an A-injective resolution of N. Then the II-filtration of $\mathrm{Hom}_{\mathrm{B}}(\mathrm{G}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{B}, \mathrm{I}))$ gives a spectral sequence which collapses at stage 2 to the total cohomology. The spectral sequence obtained from the I-filtration gives the E_2-terms. The 5-term exact sequence is the standard one with $\mathrm{E}_2^{01} \cong \mathrm{Hom}_{\mathrm{A}}(\mathrm{J}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N}))$. □

Let $\varepsilon : (F, d) \twoheadrightarrow M$ be an A-free resolution of M, $E \to B$ an A-free resolution of B; $\ldots E_2 \to E_1 \to A \twoheadrightarrow B$. Recall the definition of $m : E \otimes_A F \to F$ before Proposition 1. We change the notation by $s := m_{\geqslant 1} : E_{\geqslant 1} \otimes F \to F$ where $E_{\geqslant 1} = [\ldots \to E_2 \to E_1][+1]$ and let $m_{E_1} : E_1 \otimes F \to F$ be the multiplication with $J = (f_1, \ldots, f_r)$ on F-map pulled back along $E_1 \twoheadrightarrow J$. Then $\partial(s)$ equals m_{E_1} when restricted to $E_1 \otimes F$ and is zero elsewhere. Hence the map ∂_{J} is described simply as induced by the pullback along $s : E_1 \otimes F_0 \to F_1$, (the m in (3)).

Proposition 2. *If M and N are B-modules as A-modules then*

$$\partial_{\mathrm{J}} : \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M},\mathrm{N}) \longrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{J}, \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N}))$$

given in (1) *is the edge map α in the change of rings spectral sequence in Lemma 2. In particular there are canonical isomorphisms*

$$\ker \partial_{\mathrm{J}} \cong \mathrm{Ext}^1_{\mathrm{B}}(\mathrm{M},\mathrm{N}) \quad \textit{and} \quad \operatorname{coker} \partial_{\mathrm{J}} \cong \operatorname{im} d_2 \subseteq \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{N})$$

where d_2 is the spectral sequence differential in Lemma 2.

Proof. Let $\xi \in \mathrm{Hom_A}(\mathrm{F_1}, \mathrm{N})$ be a cocycle representing the class $[\xi] \in \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M}, \mathrm{N})$. Then there is a $\rho \in \mathrm{Hom_A}(\mathrm{F_0}, \mathrm{I^0})$ extending $\iota\xi$ where $\iota : M \hookrightarrow I^0$ is the coaugmentation map. There is also a $\tau \in \mathrm{Hom_A}(\mathrm{M}, \mathrm{I^1})$ extending $d^0\rho$, clearly $[\tau] = [\xi]$. From $\rho d_1 = \iota\xi$ we get $\rho m_{E_1} = \iota\xi s$. The map $\iota\xi s$ represents $\partial_{\mathrm{J}}([\xi])$. If $\overline{\varepsilon} = \varepsilon \otimes_A B$ one is left to prove that the connecting $\mathrm{Hom_B}(\mathrm{M}, \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{B}, \mathrm{M})) \xrightarrow{\simeq} \mathrm{End_B}(\mathrm{M})$ is represented by taking $\tau\overline{\varepsilon}$ to ρm_{E_1}. Applying $\mathrm{Hom_A}(\mathrm{F_0}, \mathrm{Hom_A}(-, \mathrm{I}^{\cdot}))$ to the short exact sequence $0 \to J \to A \to B \to 0$ gives an exact sequence of complexes. Observe $\mathrm{Hom_A}(\mathrm{F_0}, \mathrm{Hom_A}(\mathrm{B}, \mathrm{I}^{\cdot})) \cong \mathrm{Hom_A}(\overline{\mathrm{F}}_0, \mathrm{I}^{\cdot})$ and $\mathrm{Hom_A}(\mathrm{F_0}, \mathrm{Hom_A}(\mathrm{J}, \mathrm{I}^{\cdot})) \cong \mathrm{Hom_A}(\mathrm{J} \otimes \mathrm{F_0}, \mathrm{I}^{\cdot}) \twoheadleftarrow \mathrm{Hom_A}(\mathrm{E_1} \otimes \mathrm{F_0}, \mathrm{I}^{\cdot})$ hence:

$$
\begin{array}{ccccccc}
\mathrm{Hom_A}(\mathrm{E_1}\otimes\mathrm{F_0}, \mathrm{I^1}) & \xleftarrow{m^*_{E_1}} & \mathrm{Hom_A}(\mathrm{F_0}, \mathrm{I^1}) \ni \tau\varepsilon & \longleftarrow & \mathrm{Hom_A}(\overline{\mathrm{F}}_0, \mathrm{I^1}) \ni \tau\overline{\varepsilon} & \leftarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Hom_A}(\mathrm{E_1}\otimes\mathrm{F_0}, \mathrm{I^0}) \ni \rho m_{E_1} & \xleftarrow{m^*_{E_1}} & \mathrm{Hom_A}(\mathrm{F_0}, \mathrm{I^0}) \ni \rho & \longleftarrow & \mathrm{Hom_A}(\overline{\mathrm{F}}_0, \mathrm{I^0}) & \longleftarrow & 0 \\
\Big\uparrow{\scriptstyle \iota_*} & & & & & & \\
\mathrm{Hom_A}(\mathrm{E_1}\otimes\mathrm{F_0}, \mathrm{N}) \ni \xi s & & & & & &
\end{array}
$$

□

Remark 8. If A is an $\mathcal{O}$-algebra for any commutative ring $\mathcal{O}$, there is a restriction of derivations map

$$D \in \mathrm{Der}_{\mathcal{O}}(\mathrm{A}, \mathrm{Hom}_{\mathcal{O}}(\mathrm{M}, \mathrm{N})) \longrightarrow \mathrm{Hom_A}(\mathrm{J}, \mathrm{Hom_A}(\mathrm{M}, \mathrm{N})) \ni D_{|J}$$

which one checks is well defined if M and N are annihilated by J. The inner derivations maps to zero and if A is $\mathcal{O}$-flat we have

$$\mathrm{Der}_{\mathcal{O}}(\mathrm{A}, \mathrm{W})/(\text{inner derivations}) \cong \mathrm{HH}^1(\mathrm{A}; \mathrm{W}) \cong \mathrm{Ext}^1_{\mathrm{A}}(\mathrm{M}, \mathrm{N})$$

where $\mathrm{HH}^*(\mathrm{A}; \mathrm{W})$ is the Hochschild cohomology with values in the A-bimodule $W = \mathrm{Hom}_{\mathcal{O}}(\mathrm{M}, \mathrm{N})$. Via this identification our ∂_{J} equals the restriction of derivations map. This is proved by constructing a specific lifting of the multiplication of J-map on a non-reduced bar complex. Indeed let $s : E_1 \otimes_A A \otimes_{\mathcal{O}} M \to A \otimes_{\mathcal{O}} A \otimes_{\mathcal{O}} M$ be defined by $s(e_i \otimes 1 \otimes m) = 1 \otimes f_i \otimes m$ where $J = (f_1, \dots, f_r)$ and $E_1 = \oplus_{i=1}^{i=r} A e_i$. Then $\partial_{\mathrm{J}} = s^*$. We are indebted to Prof. O. A. Laudal for suggesting this interpretation of ∂_{J} at an early stage, which again led to the above change of rings spectral sequence.

Example 5. Let P be a regular local complete Noetherian k-algebra, and let J_A and J_B be ideals in P with $0 \neq J_A \subseteq \mathfrak{m} J_B$ and $J_B \subseteq \mathfrak{m}^2$. Let $A = P/J_A$, $B = P/J_B$, $\mathcal{O} = k$ and $M = k$ which is assumed to be the residue field of B. Then $\mathrm{Def}^{\mathrm{X}}_{\mathrm{M}}$ is pro-represented by X for $X = P$, A and B, see Example 4. In particular the ideal J_A is given by the image of the maximal ideal under the obstruction map $\mathrm{o^A}$. But if $\overline{J} := J_B \cdot A \subseteq A$ then $\mathrm{o^A}$ in the pair of obstruction maps $(\mathrm{o^A}, \mathrm{o^{\overline{J}}})$ is trivial. The reason for this can be seen from the obstruction calculus. Since $\mathrm{o^{\overline{J}}}$ potentially can generate J_B as obstruction ideal and $\mathrm{o^A}$ only generates $J_A \subseteq \mathfrak{m} J_B$, the $\mathrm{o_A}$-obstruction will be one or more "steps" behind $\mathrm{o_{\overline{J}}}$, the latter thus takes care of all the obstruction. This phenomenon can also be deduced from the 5-term exact sequence. For transparency assume J_A and J_B are generated by regular sequences of length a and b. A- and B-free minimal resolutions of k may be produced from the P-free Koszul resolution of k together with "Eisenbud systems", i.e. systems of homotopies for killing the action of the ideals on the Koszul complex, see [4, Thm. 7.2], [7, Chap. 7.4] and Lemma 3 below. Then the 5-term exact sequence is:

$$0 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2{}^* \xrightarrow{\simeq} \mathfrak{m}_A/\mathfrak{m}_A^2{}^* \xrightarrow{0} (\overline{J} \otimes k)^* \xrightarrow{d_2} k^b \oplus k^{\binom{n}{2}} \longrightarrow k^a \oplus k^{\binom{n}{2}}$$

where $n = \dim_{\mathrm{Krull}} P$. The spectral sequence differential d_2 is injective, and gives the isomorphism $(\overline{J}\otimes k)^* \cong k^b$. This in fact also proves that the o_{B} is confined to k^b and thus maps to zero in $\mathrm{Ext}^2_{\mathrm{A}}(\mathrm{k},\mathrm{k})$. Hence o^{A} in the pair $(\mathrm{o}^{\mathrm{A}},\mathrm{o}^{\mathrm{J}})$ has to be trivial, see Lemma 4. The o_{A} for $\mathrm{Def}^{\mathrm{A}}_{\mathrm{k}}$ is for the same reason confined to k^a, but far from trivial. Remark that this is not in contradiction to Lemma 4. If, by changing the assumptions, some elements in J_A are non-zero in $J_B/\mathfrak{m}J_B$ they will produce identifications between corresponding subspaces of k^b and k^a and hence *some* of o^{A} will be induced from o^{A} in $(\mathrm{o}^{\mathrm{A}},\mathrm{o}^{\mathrm{J}})$. In fact the isomorphism of the "Koszul-part" of the Ext^2s may be explained similarly if we for a moment consider the *non-commutative* deformation functors, i.e. where the local Artinian rings are allowed to be non-commutative. Then $\mathrm{Def}^{\mathrm{B}}_{\mathrm{k}}$ in fact still is pro-represented by the commutative ring B, see Example 4. The $\binom{n}{2}$-part takes care of the commutators, which are given as cup products, and hence appear simultaneously in the obstruction calculus for both deformation functors.

Assume $d_2 : E_2 \to E_1$ is given as the Koszul differential d_K plus a map $d'_E : E'_2 \to E_1 \cong A^r$ and let s' be the restriction of $s : E_2\otimes F_0 \to F_2$ to $E'_2\otimes F_0$.

Lemma 3. *The following maps of A-free modules give a B-free 3-presentation of M after tensoring with B:*

$$F_0 \xleftarrow{\;d\;} F_1 \xleftarrow{(s,d)} \begin{matrix} E_1\otimes F_0 \\ \oplus \\ F_2 \end{matrix} \xleftarrow{\begin{pmatrix} d'_E\otimes 1 & 1\otimes d & 0 \\ s' & s & d \end{pmatrix}} \begin{matrix} E'_2\otimes F_0 \\ \oplus \\ E_1\otimes F_1 \\ \oplus \\ F_3 \end{matrix}$$

Remark 9. The above assumption about $d_2 : E_2 \to E_1$ is no limitation, we can always produce such resolutions of B. The point is that if $J = (f_1,\dots,f_r)$ is a regular sequence, then $\mathrm{H}_1(K(f_1,\dots,f_r)) = 0$ where K is the Koszul complex, $E'_2 = 0$ and our 3-presentation is the beginning of a construction of Eisenbud which gives a B-free resolution from an A-free one, see [4, Theorem 7.1]. In general this resolution is not minimal. Using E_2 instead of E'_2 it is not hard to prove the result by moving elements around.

Proof. Since the 3-presentation (with E_2 instead of E'_2) is the beginning of the mapping cone $K(s)$ of $B\otimes s : B\otimes E_{\geqslant 1}\otimes F \to B\otimes F[+1]$ shifted minus one, and the composition $E_{\geqslant 1}\otimes F \to B\otimes E_{\geqslant 1}\otimes F \to B\otimes F[+1]$ is a quasi-isomorphism inducing the inverse connecting $\mathrm{Tor}^{\mathrm{A}}_*(\mathrm{J},\mathrm{M}) \xrightarrow{\cong} \mathrm{Tor}^{\mathrm{A}}_{*+1}(\mathrm{B},\mathrm{M})$, the map $\mathrm{H}_*(B\otimes s)$ is surjective thus $\mathrm{H_i}K(s) = 0$ for $i = 0,1$ and hence our complex is exact in degree 1 and 2. □

Let $(\mathrm{id}[2],0)$ be the obvious map $F_0\otimes E_1[-1]\oplus F_2 \to F_0\otimes E_1[1]$ tensored down to B. As we will see it gives a 2-cocycle in a B-free Yoneda complex calculating $\mathrm{Ext}^*_{\mathrm{B}}(\mathrm{M},\mathrm{M}\otimes_{\mathrm{B}}\mathrm{J}/\mathrm{J}^2)$ where the resolution of M begins as in Lemma 3 and the resolution of $M\otimes_B J/J^2$ begins with $B\otimes_A(F_0\otimes E_1)$. In fact this cocycle represents $\mathrm{o} := \mathrm{o}_{(\mathrm{A}/\mathrm{J}^2)} \in \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{M}\otimes_{\mathrm{B}}\mathrm{J}/\mathrm{J}^2)$ which is the obstruction for lifting M to A/J^2 as in Definition 3. This element induces via the Yoneda cup product $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}\otimes_{\mathrm{B}}\mathrm{J}/\mathrm{J}^2,\mathrm{N}) \times \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{M}\otimes_{\mathrm{B}}\mathrm{J}/\mathrm{J}^2) \xrightarrow{\cup} \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{N})$ a map

$$-\cup \mathrm{o} : \mathrm{Hom}_{\mathrm{B}}(\mathrm{M}\otimes_{\mathrm{B}}\mathrm{J}/\mathrm{J}^2,\mathrm{N}) \longrightarrow \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{N})\,. \tag{8}$$

Via the natural isomorphism $\mathrm{Hom}_{\mathrm{B}}(\mathrm{M}\otimes_{\mathrm{B}}\mathrm{J}/\mathrm{J}^2,\mathrm{N}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{Ext}^1_{\mathrm{A}}(\mathrm{B},\mathrm{N}))$ we have

Proposition 3. *Assume both M and N are $B = A/J$-modules as A-modules. Then the differential in the change of rings spectral sequence of Lemma 2*

$$d_2 : \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{Ext}^1_{\mathrm{A}}(\mathrm{B},\mathrm{N})) \longrightarrow \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{N})$$

is induced by cupping with the obstruction $o(A/J^2) \in \mathrm{Ext}^2_B(M, M \otimes_B J/J^2)$ *as in* (8) *and* $o(A/J^2)$ *is induced by the cocycle* $(\mathrm{id}[2], 0)$ *in the Yoneda complex of* B*-free resolutions. In particular the obstruction is canonically given as*

$$o(A/J^2) = d_2(\mathrm{id}_{M\otimes J/J^2})\,.$$

Remark 10. This result is only marginally different from L. Illusie's Prop. 3.1.5 and Prop. 3.1.13 combined [9, Chap. IV], in the case of rings (Illusie works with rings over a topos). In his formulation $J^2 = 0$ (but $\mathrm{Ext}^1_A(M, N) \cong \mathrm{Ext}^1_{A/J^2}(M, N)$), and his spectral sequence is $\mathrm{Ext}^p_B(\mathrm{Tor}^{A/J^2}_q(M, B), N) \Rightarrow \mathrm{Ext}^*_{A/J^2}(M, N)$. Illusie's proof depends on the cotangent complex of graded algebras and gives representations in the derived category. We are interested in explicit calculations of the generalised Massey products and the formally versal formal family, and our proof, which gives simple representations of the map and the class in the Yoneda complex, is therefore better suited to our needs.

Remark 11. If J defines a locally complete intersection, i.e. J/J^2 is B-projective of finite rank, it is not hard to extend the result to all the d_2-differentials. We have

$$E_2^{pq} = \mathrm{Ext}^p_B(M, \mathrm{Ext}^q_A(B, N)) \cong \mathrm{Ext}^p_B(M, N) \otimes \bigwedge^q J/J^{2*}$$

and $\psi \in \mathrm{End}_B(J/J^2)$ acts on $f_1 \wedge \ldots \wedge f_q \in \bigwedge J/J^{2*}$ by $\psi \cdot f_1 \wedge \ldots \wedge f_q = \sum f_1 \wedge \ldots \wedge \psi^*(f_i) \wedge \ldots \wedge f_q$ while J/J^2 acts as graded derivations. Combining this with the cup product, any $\xi \in E_2^{pq} = \mathrm{Ext}^p_B(M, \mathrm{Ext}^q_A(B, N))$ defines a natural, vertical map by "multiplication" in the diagram:

$$\begin{array}{ccc} \mathrm{id} \in \mathrm{End}_B(M \otimes_B J/J^2) & \xrightarrow{d_2} & \mathrm{Ext}^2_B(M, M \otimes_B J/J^2) \ni o(A/J^2) \\ \downarrow{\scriptstyle \xi\cdot -} & & \downarrow{\scriptstyle \xi\cdot -} \\ \xi \in \mathrm{Ext}^p_B(M, N) \otimes_B \bigwedge^q J/J^{2*} & \xrightarrow{d_2} & \mathrm{Ext}^{p+2}_B(M, N) \otimes_B \bigwedge^{q-1} J/J^{2*} \ni \xi \cdot o(A/J^2) \end{array}$$

The diagram commutes since d_2 is natural for natural products on the cohomology and hence if $o(A/J^2) = \sum o_i \otimes x_i$ and $\xi = \zeta \otimes f$ then $d_2(\xi) = \sum (\zeta \cup o_i) \otimes f \cdot x_i$.

Proof of Proposition 3. The pullback of endomorphisms by $(\mathrm{id}[2], 0)$ induces a map

$$\mathrm{End}_B(M \otimes J/J^2) \longrightarrow \mathrm{Ext}^2_B(M, M \otimes J/J^2) \tag{9}$$

which possibly depends on the resolution chosen etc. The image of the identity is the obstruction $o(A/J^2)$ since $o(A/J^2)$ is induced from the square of a lifting of the B-differential d^B to A/J^2: By Lemma 3, $d_1^B = d_1 \otimes_A B$, $d_2^B = (s, d_2) \otimes_A B$ hence $\tilde{d_1^B} \circ \tilde{d_2^B} = d_1 \circ (s, d_2) \otimes_A A/J^2 = (m_{E_1}, 0) \otimes_A A/J^2$ which lifts to $(\mathrm{id}[2], 0)$ via $m_{E_1} : F_0 \otimes E_1 \to F_0$. Since we have such a nice representation of $o(A/J^2)$ in the Yoneda complex, the idea of the proof is to take a class $[\xi] \in \mathrm{Hom}_B(M, \mathrm{Ext}^1_A(B, N)) \cong \mathrm{Hom}_B(M \otimes J/J^2, N)$ represented by a cocycle $\xi = (\xi_i)$ in the Yoneda complex: $\xi_2 \in \mathrm{Hom}_B(\overline{F}_0 \otimes \overline{E}_1, \overline{F}'_0)$ where (F', d') is an A-free resolution of N and $\overline{F} = F \otimes_A B$ etc. Move it to a representative for the same class in $E^{01} = \mathrm{Hom}_B(F^B_0, \mathrm{Hom}_A(B, I^1))$, where $N \hookrightarrow I$ is an A-injective resolution of N and calculate d_2 by moving this new representative along the "stairs" in the double complex $\mathrm{Hom}_B(F^B, \mathrm{Hom}_A(B, I))$ to a representative for the image $d_2([\xi])$ in $E^{20} = \mathrm{Hom}_B(F^B_2, \mathrm{Hom}_A(B, I^0))$. Finally we move back to a representative for $d_2([\xi])$ in the Yoneda complex and observe that we may take it to be $(\xi\,, 0) \in \mathrm{Hom}_B(\overline{F}_0 \otimes \overline{E}_1 \oplus \overline{F}_2, \overline{F}_0 \otimes \overline{E}_1)$. Since $\xi \circ (\mathrm{id}[2], 0) = (\xi\,, 0)$, we get $d_2([\xi]) = [\xi] \cup o(A/J^2)$. Hence the map (9) induced by the particular form of the (possibly non-minimal) 3-presentation is indeed canonical and equal to d_2. This is only almost what we do, actually we lift the B-representative ξ from

$\mathrm{Hom_B}(\overline{\mathrm{F}}_0, \mathrm{Hom_B}(\overline{\mathrm{F}}_0, \overline{\mathrm{E}}'_1))$ to an A-representative in $\mathrm{Hom_A}(\mathrm{F}_0, \mathrm{Hom_A}(\mathrm{E}_1, \mathrm{F}'_0))$ and then do the zigzagging with A-representatives. Details are given in [7]. □

5. The formula $\mathrm{o(J)} \cup \mathrm{o(A/J^2)} = \mathrm{o_B}$

Several of our obstruction classes are connected by the d_2-differential in the following theorem.

Theorem 4. *With assumptions as in Theorem 1, assume furthermore that there exists an A_R-module M_R lifting M along $\pi_{A_R} = \mathrm{id}_A \otimes \pi$ in the diagram*

$$\begin{array}{ccc} A_R & \longrightarrow & B_R \\ \downarrow{\scriptstyle \pi_{A_R}} & & \downarrow{\scriptstyle \pi_{B_R}} \\ A_S & \longrightarrow & B_S \end{array}$$

in the sense of Definition 3, in particular $\mathrm{o_{A_S}}(\pi_{A_R}, M) = 0$. If $\mathrm{o_{J_S}} = \mathrm{o_{J_S}}(\pi_{A_R}, M)$ is the obstruction for lifting M to B_R, then, via the natural inclusion $\mathrm{o_{J_S}} \in \operatorname{coker} \partial_{\mathrm{J_S}} \hookrightarrow \mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I})$, it satisfies

$$\mathrm{o_{J_S}} = d_2(\mathrm{o(J_S)}) = \mathrm{o(J_S)} \cup \mathrm{o(A_S/J_S^2)} = \mathrm{o_{B_S}}$$

where $\cup$ is the cup product and $\mathrm{o(J_S)} \in \mathrm{Hom_{B_S}}(\mathrm{M} \otimes_{\mathrm{B_S}} \mathrm{J_S}/\mathrm{J_S^2}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I})$ is the obstruction for M_R to be a B_R-module as A_R-module, $\mathrm{o(A_S/J_S^2)} \in \mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M}, \mathrm{M} \otimes_{\mathrm{B_S}} \mathrm{J_S}/\mathrm{J_S^2})$ is the obstruction for lifting M to A_S/J_S^2, $\mathrm{o_{B_S}} \in \mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I})$ is the obstruction for lifting M to B_R and $d_2 : \mathrm{Hom_{B_S}}(\mathrm{M} \otimes_{\mathrm{B_S}} \mathrm{J_S}/\mathrm{J_S^2}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I}) \to \mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I})$ is the 2^{nd} differential in the change of rings spectral sequence in Lemma 2.

Proof. Let (F, d) be an A_S-free resolution of M, then $\mathrm{o_{A_S}}$ is induced by $\tilde{d}_1 \circ \tilde{d}_2$ where $(\tilde{F}, \tilde{d})$ is a lifting of d to maps of A_R-free modules as explained in the proof of Theorem 1. We choose an A_S-free $E_1 \twoheadrightarrow J_S$ and maps s as in Lemma 3 to produce a B_S-free 3-presentation $F^{B_S} = F^{B_S}(s)$ of M. We observe the bottom row as $B_S \otimes d$ and that the edge map $\mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M}, -) \to \mathrm{Ext}^2_{\mathrm{A_S}}(\mathrm{M}, -)$ in the change of rings spectral sequence is induced by the map $F \to F^{B_S}(s)$ where F maps to $B_S \otimes F$, see Lemma 4. The obstruction $\mathrm{o_{B_S}}$ is induced by $\tilde{d}_1^{B_S} \circ \tilde{d}_2^{B_S}$ where $\tilde{d}^{B_S}$ lifts d^{B_S} to an B_R-free lifting of F^{B_S}, but $d_1^{B_S} = B_S \otimes d_1$ and $d_2^{B_S} = B_S \otimes (s, d_2)$ hence

$$\tilde{d}_1^{B_S} \circ \tilde{d}_2^{B_S} = \widetilde{B_S \otimes d_1} \circ (\widetilde{B_S \otimes s}, \widetilde{B_S \otimes d_2}) = B_R \otimes_{A_R} (\tilde{d}_1 \circ \tilde{s}, \tilde{d}_1 \circ \tilde{d}_2)$$

for liftings $\tilde{d}$, $\tilde{s}$ of d and s to A_R-free modules. Clearly $\tilde{d}_1 \circ \tilde{d}_2$ induces the same element as $B_R \otimes_{A_R} \tilde{d}_1 \circ \tilde{d}_2$ in $\mathrm{Ext}^2_{\mathrm{A_S}}(\mathrm{M}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I})$, hence $\mathrm{o_{B_S}}$ maps to $\mathrm{o_{A_S}}$ by the edge map. We have assumed $\mathrm{o_{A_S}} = 0$, indeed we have chosen an A_R-module M_R lifting M which corresponds to a choice of liftings $\tilde{d}_i$, $i = 1, 2$ with $\tilde{d}_1 \circ \tilde{d}_2 = 0$, hence $\mathrm{o_{B_S}}$ is induced by $\tilde{d}_1^{B_S} \circ \tilde{d}_2^{B_S} = B_R \otimes_{A_R} (\tilde{d}_1 \circ \tilde{s}, 0)$ and $\tilde{d}_1 \circ \tilde{s}$ induces the class $\mathrm{o(J_S)}$ (see the proof of Theorem 1). The only thing we lack is a description of the d_2-differential in terms of our construction. By Proposition 3, d_2 is cupping with the obstruction class $\mathrm{o(A_S/J_S^2)} \in \mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M}, \mathrm{M} \otimes_{\mathrm{B_S}} \mathrm{J_S}/\mathrm{J_S^2})$ which is induced by the composition $\hat{d}_1^{B_S} \circ \hat{d}_2^{B_S}$ of liftings of the B_S-differential $d_i^{B_S}$ to maps of an A_S/J_S^2-free lifting of F^{B_S}. But we already have a lifting to A_S, hence we choose $\hat{d}_1^{B_S} = A_S/J_S^2 \otimes d_1$ and $\hat{d}_2^{B_S} = A_S/J_S^2 \otimes (s_0, d_2)$ and thus $\mathrm{o(A_S/J_S^2)}$ is induced by $\hat{d}_1^{B_S} \circ \hat{d}_2^{B_S} = A_S/J_S^2 \otimes (d_1 \circ s, 0)$, the first coordinate corresponds to $\mathrm{id}_{M \otimes J_S/J_S^2}$. Hence the class $\mathrm{o(J_S)}$ maps to $\mathrm{o_{B_S}}$ under the spectral sequence differential d_2:

$$\mathrm{o(J_S)} \in \mathrm{Hom_{B_S}}(\mathrm{M} \otimes_{\mathrm{B_S}} \mathrm{J_S}/\mathrm{J_S^2}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I}) \xrightarrow{d_2} \mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M}, \mathrm{M} \otimes_{\mathrm{S}} \mathrm{I}) \ni \mathrm{o_{B_S}} = (\mathrm{o(J_S)}, 0)$$

□

For the sake of completeness we note

Lemma 4. *The edge map*

$$\gamma : \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M},\mathrm{N}) \longrightarrow \mathrm{Ext}^2_{\mathrm{A}}(\mathrm{M},\mathrm{N})$$

in the 5*-term exact sequence* (7) *is induced by any comparison map* $F^A \to F^B$ *of an* A*- and a* B*-free resolution of* M*. With notation as in Theorem 4, the obstruction class* $\mathrm{o}_{\mathrm{B_S}}$ *maps to the obstruction class* $\mathrm{o}_{\mathrm{A_S}}$ *under the edge map*

$$\mathrm{Ext}^2_{\mathrm{B_S}}(\mathrm{M},\mathrm{M}{\otimes}_{\mathrm{S}}\mathrm{I}) \longrightarrow \mathrm{Ext}^2_{\mathrm{A_S}}(\mathrm{M},\mathrm{M}{\otimes}_{\mathrm{S}}\mathrm{I})\,.$$

Proof. The edge is clear from inspecting the complex $\mathrm{Hom}_{\mathrm{B}}(\mathrm{F}^{\mathrm{B}},\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{I_A}))$ given in the proof of Lemma 2, since the edge, which is

$$\mathrm{E}_2^{\mathrm{p}0} = \mathrm{H}^{\mathrm{p}}\mathrm{Hom}_{\mathrm{B}}(\mathrm{F}^{\mathrm{B}},\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{N})) \xrightarrow{\ i_*\ } \mathrm{H}^{\mathrm{p}} = \mathrm{H}^{\mathrm{p}}\mathrm{Hom}_{\mathrm{B}}(\mathrm{F}^{\mathrm{B}},\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{I_A}))$$

for $i : N \hookrightarrow I^0$, factorises via

$$\begin{aligned}(10)\quad \mathrm{Hom}_{\mathrm{B}}(\mathrm{F}^{\mathrm{B}},\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{N})) &\longrightarrow \mathrm{Hom}_{\mathrm{A}}(\mathrm{F}^{\mathrm{A}},\mathrm{N}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathrm{A}}(\mathrm{F}^{\mathrm{A}},\mathrm{I}) \\ &\xleftarrow{\ \sim\ } \mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{I}) \cong \mathrm{Hom}_{\mathrm{B}}(\mathrm{M},\mathrm{Hom}_{\mathrm{A}}(\mathrm{B},\mathrm{I}))\,.\end{aligned}$$

That $\mathrm{o}_{\mathrm{B_S}}$ maps to $\mathrm{o}_{\mathrm{A_S}}$ follows as in the proof of Theorem 4. □

In the situation of Theorem 4, the map $\partial_{\mathrm{I}} : \mathrm{Ext}^1_{\mathrm{A_R}}(\mathrm{M},\mathrm{M}{\otimes}_{\mathrm{S}}\mathrm{I}) \to \mathrm{End}_{\mathrm{A_S}}(\mathrm{M}{\otimes}_{\mathrm{S}}\mathrm{I})$ is surjective by Proposition 3. The following result hence gives a characterisation of the class $\mathrm{o}(\mathrm{J_S})$.

Proposition 4. *With assumptions as in Theorem 4,*

$$\mathrm{o}(\mathrm{J_S}) = -\partial_{\mathrm{J_S}}(\xi)$$

for any extension $\xi \in \mathrm{Ext}^1_{\mathrm{A_R}}(\mathrm{M},\mathrm{M}{\otimes}_{\mathrm{S}}\mathrm{I})$ *with* $\partial_{\mathrm{I}}(\xi) = \mathrm{id} \in \mathrm{End}_{\mathrm{A_S}}(\mathrm{M}{\otimes}_{\mathrm{S}}\mathrm{I})$.

A proof in the deformation situation is given in [7], it is easily extended.

6. Explicit examples of obstruction calculations

Finally we give some examples, in the first one we already know the answer by the general Example 4.

Example 6. Let $\mathbb{Z}_2 = \hat{\mathbb{Z}}_{(2)}$ and $B := \mathbb{Z}_2[x]/J$ where $J = (f)$ and $f = 2 + x^2$. Let $M = \overline{B}/(x) \cong \mathbb{F}_2$ as $\overline{B} = B/(2)$-module. We calculate the obstruction polynomial of $\mathrm{Def}^{\mathrm{B}}_{\mathrm{M}} : \mathsf{Art}_{\mathbb{Z}_2} \to \mathsf{Sets}$. Let $A := \mathbb{Z}_2[x]$, $\overline{A} = A/(2)$, then $M \twoheadleftarrow \overline{A} \xleftarrow{x} \overline{A}$ is a length 1 $\overline{A}$-free resolution of M. We have $\mathrm{Ext}^1_{\overline{\mathrm{B}}}(\mathrm{M},\mathrm{M}) \cong \mathrm{Ext}^1_{\overline{\mathrm{A}}}(\mathrm{M},\mathrm{M}) \cong \mathbb{F}_2$. Since $\mathrm{Ext}^2_{\overline{\mathrm{A}}}(\mathrm{M},\mathrm{M}) = 0$, there is no $\mathrm{o_A}$-obstruction. To find the $\mathrm{o_J}$-obstruction we start with a factorisation of the multiplication-by-f-map, given by $f \equiv x{\cdot}x \mod (2)$; $\overline{A} \xleftarrow{x} \overline{A} \xleftarrow{x} \overline{A}$. Let $T^1 = \mathbb{Z}_2[u⟧$ where the image $\overline{u}$ of u in $\mathfrak{m}/(2) + \mathfrak{m}^2$ corresponds to the $\mathbb{F}_2$-dual of $\xi = [-1] = [1] \in \mathrm{Ext}^1_{\overline{\mathrm{B}}}(\mathrm{M},\mathrm{M})$. Then the universal lifting of M to the relative Zariski tangent space $T^1_1 = T^1/\mathfrak{n} = \mathbb{Z}_2[u⟧/(2,u^2) = \mathbb{F}_2[u]/(u^2)$ is given by the factorisation $A{\otimes}_{\mathbb{Z}_2}T^1_1 \xleftarrow{x-\overline{u}} A{\otimes}_{\mathbb{Z}_2}T^1_1 \xleftarrow{x+\overline{u}} A{\otimes}_{\mathbb{Z}_2}T^1_1$ of $f \equiv x^2 \mod (2,u^2)$. The only obstruction appears when we try to lift this factorisation of f to $T^1_2 = T^1/\mathfrak{n}{\cdot}\mathfrak{m} = \mathbb{Z}_2[u⟧/(2^2,2u,u^3)$ and it is represented by $(x-u)(x+u) - f = -(2+u^2)$ in $A{\otimes}_{\mathbb{Z}_2}T^1_2$. In particular is the class $\xi \cup \xi = [-1] = [1] \in \operatorname{coker}\partial_{\overline{\mathrm{J}}} \cong \mathbb{F}_2$ "carrying" the obstruction polynomial $2 + u^2$. There are no more obstructions.

If instead $f = 4 + x^3$, one obtains the factorisation

$$A{\otimes}_{\mathbb{Z}_2}T^1_2 \xleftarrow{x-u} A{\otimes}_{\mathbb{Z}_2}T^1_2 \xleftarrow{x^2+xu+u^2} A{\otimes}_{\mathbb{Z}_2}T^1_2$$

of $f \equiv x^3 \mod (2^2, 2u, u^3)$ which gives a defining system $\mathcal{B}$ for the J-Massey product $\langle \xi, \xi, \xi\,; \mathcal{B}\rangle_J = [-1] \in \operatorname{coker} \partial_{\overline{\mathrm{J}}}$ and the obstruction is $(x-u)(x^2+xu+u^2)-(4+x^3) = -(4+u^3)$ (or more precisely $[-1]\otimes(4+u^3)$). There are no more obstructions.

Example 7. Let C_4 be the cyclic group of order four and $\mathbb{Z}_2 = \hat{\mathbb{Z}}_{(2)}$. Let x be a generator of C_4 such that the group algebra $B := \mathbb{Z}_2 C_4 \cong \mathbb{Z}_2[x]/J$ where $J = (f)$ and $f = x^4 - 1$. Then $\overline{B} = B/(2)$, let $M = \overline{B}/(y^2) \cong \mathbb{F}_2[y]/(y^2)$ where $y = x - 1$. We give obstruction polynomials defining the hull H of $\mathrm{Def}_{\mathrm{M}}^{\mathrm{B}} : \mathsf{Art}_{\mathbb{Z}_2} \to \mathsf{Sets}$ and indicate how to find them. Let $A := \mathbb{Z}_2[x]$, then $\overline{A} = A/(2)$. The 5-term exact sequence of the spectral sequence $\mathrm{Ext}^{\mathrm{p}}_{\overline{\mathrm{B}}}(\mathrm{M}, \mathrm{Ext}^{\mathrm{q}}_{\overline{\mathrm{A}}}(\overline{\mathrm{B}}, \mathrm{M})) \Rightarrow \mathrm{Ext}^{*}_{\overline{\mathrm{A}}}(\mathrm{M}, \mathrm{M})$ is

$$0 \to M \xrightarrow{\simeq} M \xrightarrow{\partial_{\overline{\mathrm{J}}}=0} M \xrightarrow{\simeq} M \to 0\,.$$

Since $\mathrm{pd}_{\overline{A}}(M) = 1$ there is no A-obstruction and o^{A} is trivial. Let $T_J^2 = \mathbb{Z}_2[z_0\,, z_1]\hat{}$ where $\overline{z}_0$ and $\overline{z}_1$, the images in the relative cotangent space, are $\mathbb{F}_2$-duals to the elements -1 and $-y$ of $\operatorname{coker} \partial_{\overline{\mathrm{J}}} \cong M$. Let $T^1 = \mathbb{Z}_2[a, b]\hat{}$ where $\overline{a}$ and $\overline{b}$ likewise are $\mathbb{F}_2$-duals to the elements -1 and $-y$ in $\ker \partial_{\overline{\mathrm{J}}} \cong M$. Then the obstruction map $\mathrm{o}^{\mathrm{J}} : T_J^2 \to T^1$ may be given as

$$\begin{aligned} \mathrm{o}^{\mathrm{J}}(z_0) &= a^2 + 6a + ab^2 + 4ab \\ \mathrm{o}^{\mathrm{J}}(z_1) &= 4 + 6b + b^3 + 4a + 2ab + 4b^2\,. \end{aligned}$$

This is a regular sequence, hence $\dim_{\mathrm{Krull}} H = 1$. The versal family is the cyclic module $A\hat{\otimes}H/(y^2-a-yb)$ where $A\hat{\otimes}H = \varprojlim\{A\otimes H/\operatorname{im}(\mathfrak{n}\cdot\mathfrak{m}_H^i)\}$, $\mathfrak{n} = \mathfrak{m}_H^2 + (2)$ and $\mathfrak{m}_H$ is the maximal ideal in H. To find the obstruction one deforms the pair (y^2, y^2) as a (1×1-matrix) factorisation of f; $x^4 - 1 = y^2\cdot y^2$. In particular $\overline{B} \xleftarrow{\overline{y}^2} \overline{B} \xleftarrow{\overline{y}^2} \overline{B}$ gives a $\overline{B}$-free 2-presentation of M. The versal lifting of M to the tangent space is given by the factorisation $x^4 - 1 = (y^2 - (a + yb))\cdot(y^2 + (a + yb)) \mod \mathfrak{n}$. The obstructions are created as one lifts and expands the factorisation as to be valid over $A\otimes H/\mathfrak{n}\mathfrak{m}_H^i$ successively for all i. Indeed we get $x^4 - 1 = (y^2 - a - yb)\cdot(y^2 + a + yb + 6 + b^2 + 4y + 4b)$ in $A\hat{\otimes}H$.

In the last example we shall see (even clearer) how the change of rings formalism is instrumental both in estimating and calculating the obstruction. The 5-term exact sequence and the A-free Koszul resolution immediately imply that we can have at maximum two obstruction polynomials even though $\dim_k \mathrm{Ext}^2_{\mathrm{B}}(\mathrm{M}, \mathrm{M}) = 4$. Moreover we only have to lift a "generalised matrix factorisation" (see [7]), defined over the regular ring A, to give defining systems for the Massey products which calculate the obstruction for $\mathrm{Def}_{\mathrm{M}}^{\mathrm{B}}$, and hence avoiding the relations in B in the calculus.

Example 8. Let $A = k[x, y]$, $f = x^{m+1} + y^{n+1}$, $B = A/(f)$, $M = B/(y, x^2)$ as B-module and assume $m \geq 3$, $m \equiv 1 \bmod 2$, and $n \geq 1$ (the case $m \equiv 0 \bmod 2$ is similar). We give the obstruction polynomials of $\mathrm{Def}_{\mathrm{M}}^{\mathrm{B}} : \mathsf{Art}_k \to \mathsf{Sets}$ and indicate how to find them. The 5-term exact sequence is

$$0 \to M^{\oplus 2} \xrightarrow{\simeq} M^{\oplus 2} \xrightarrow{0} M \xrightarrow{(\mathrm{id},0)} M \oplus M \xrightarrow{(0,\mathrm{id})} M \to 0$$

and in particular the Zariski tangent space of the hull H of $\mathrm{Def}_{\mathrm{M}}^{\mathrm{B}}$ is 4-dimensional as k-vector space; $\mathrm{Ext}^1_{\mathrm{B}}(\mathrm{M}, \mathrm{M}) \cong M^{\oplus 2} \cong k^{\oplus 4}$. The Koszul complex of (y, x^2) gives an A-free resolution of M and hence there is no A-obstruction and the obstruction map o^{A} is trivial. The d_2-map (7) is injective and the obstruction space $\operatorname{coker} \partial_{\mathrm{J}} \cong M \cong k^{\oplus 2}$ k-linearly where $J = (f)$. Let $T_J^2 = k[[z_0\,, z_1]]$ where the images $\overline{z}_0$ and $\overline{z}_1$ in the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ are the k-dual elements to -1 and $-x$, which give a k-basis for $\operatorname{coker} \partial_{\mathrm{J}}$. Let $T^1 = k[[a, b, c, d]]$ where $\overline{a}$ and $\overline{b}$ in $\mathfrak{m}/\mathfrak{m}^2$ are k-duals to

-1 and $-x$ in M and likewise for $\overline{c}$ and $\overline{d}$. Let $l = \frac{m-1}{2}$. Then the obstruction map $\mathrm{o}^{\mathrm{J}} : T^2_J \to T^1$ may be given as

$$\begin{aligned}
\mathrm{o}^{\mathrm{J}}(z_0) &= \sum_{i=0}^{n} \sum_{j=0}^{\min(i,n-i)} \binom{i}{j}\binom{n}{i+j} a^{n-i-j} b^{i+j} c^{j+1} d^{i-j} \\
&+ \sum_{i=0}^{l} \left(\sum_{j=0}^{\min(i,l-i)} \binom{i}{j}\binom{l}{i+j} \right) c^{l+1-i} d^{2i} \\
\mathrm{o}^{\mathrm{J}}(z_1) &= \sum_{i=0}^{n} \sum_{j=0}^{\min(i+1,n-i)} \binom{i+1}{j}\binom{n}{i+j} a^{n-i-j} b^{i+j} c^{j} d^{i+1-j} \\
&+ \sum_{i=0}^{l} \left(\sum_{j=0}^{\min(i+1,l-i)} \binom{i+1}{j}\binom{l}{i+j} \right) c^{l-i} d^{2i+1} .
\end{aligned}$$

The terms of lowest and highest degrees in the four sums are given by

$$\begin{aligned}
\mathrm{o}^{\mathrm{J}}(z_0) &= a^n c + c^{l+1} + \ldots + b^n c d^n + c d^{2l} \\
\mathrm{o}^{\mathrm{J}}(z_1) &= a^n d + n a^{n-1} bc + (l+1) c^l d + \ldots + b^n d^{n+1} + d^{2l+1} .
\end{aligned}$$

At least in the case $l = n$ we get a regular sequence. But before calculating a single obstruction we have $4 \geq \dim_{\mathrm{Krull}} H \geq 2$ while the standard estimate yields $4 \geq \dim_{\mathrm{Krull}} H \geq 0$. If we localise B (and M) at the maximal ideal $\mathfrak{m} = (x, y)$, then there is a natural isomorphism $\mathrm{Def}^{\mathrm{B}}_{\mathrm{M}} \to \mathrm{Def}^{\mathrm{B}}_{\Omega_{\mathrm{B}}(\mathrm{M})}$ given by mapping a deformation M_R to its B_R-syzygy module. The syzygy induces isomorphisms $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{B}}(\mathrm{M},\mathrm{M}) \xrightarrow{\simeq} \mathrm{Ext}^{\mathrm{i}}_{\mathrm{B}}(\Omega_{\mathrm{B}}(\mathrm{M}), \Omega_{\mathrm{B}}(\mathrm{M}))$ for $i = 1, 2$, see [7]. In [8] we more generally show that the hull of the deformation functor of a rank 1 maximal Cohen Macaulay module N on a hypersurface singularity in particular satisfies the sharpened estimate

$$\dim_k \mathrm{Ext}^1_{\mathrm{B}} \geq \dim_{\mathrm{Krull}} H \geq \dim_k \mathrm{Ext}^1_{\mathrm{B}} - \dim_k \mathrm{Ext}^2_{\mathrm{B}} + \dim_k \mathrm{H}_2(\mathcal{S})$$

where $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{B}} = \mathrm{Ext}^{\mathrm{i}}_{\mathrm{B}}(\mathrm{N},\mathrm{N})$ and $\mathcal{S} = \mathcal{S}(\phi)$ is the "Scandinavian Complex" of the (square) presenting matrix ϕ of N. In our case $\mathrm{H}_2(\mathcal{S}) \cong M \cong k^2$ k-linearly.

The versal family of $\mathrm{Def}^{\mathrm{B}}_{\mathrm{M}}$ is the cyclic module $A \hat{\otimes} H/(y-a-xb\,,\ x^2-c-xd)$ where $A \hat{\otimes} H = \varprojlim \{A \otimes H/\operatorname{im}(\mathfrak{m}^n_H)\}$. To find the obstruction one deforms the "generalised matrix factorisation" (see [7, Def. 6.1.6]) $\big((y, x^2), (y^n, x^{m-1})^{\mathrm{t}}\big)$ of $f = x^{m+1} + y^{n+1}$. The versal lifting to the tangent space is given by

$$x^{m+1} + y^{n+1} = (y - a - xb\,,\ x^2 - c - xd) \cdot \begin{pmatrix} y^n + y^{n-1}(a + xb) \\ x^{m-1} + x^{m-3}(c + xd) \end{pmatrix} \quad \mathrm{mod}\ \mathfrak{m}^2_H .$$

To find the obstruction one has to lift and expand this factorisation as to be valid mod $\mathfrak{m}^n_H$ successively for all $n \geq 2$.

References

[1] Michael Artin. Versal deformations and algebraic stacks. *Invent. Math.*, 27:165–189, 1974.

[2] Inger Christin Borge. A cohomological approach to the modular isomorphism problem. Preprint in Pure Math. No. 15, Dep. of Math., University of Oslo, August 2002. `www.math.uio.no/eprint/pure_math/2002/15-02`, submitted to J. Pure Appl. Algebra.

[3] Inger Christin Borge and Olav Arnfinn Laudal. The modular isomorphism problem. Preprint in Pure Math. No. 19, Dep. of Math., University of Oslo, November 2002. `www.math.uio.no/eprint/pure_math/2002/19-02`, submitted to Invent. Math.

[4] David Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, July 1980.

[5] Barbara Fantechi and Marco Manetti. Obstruction calculus for functors of Artin rings, I. *J. Algebra*, 202:541–576, 1998.

[6] Gunnar Fløystad. Determining obstructions for space curves, with applications to non-reduced components of the Hilbert scheme. *J. Reine Angew. Math.*, 439:11–44, 1993.
[7] Runar Ile. *Obstructions to deforming modules*. PhD thesis, University of Oslo, 2001.
[8] Runar Ile. Deformation theory of rank 1 maximal Cohen-Macaulay modules on hypersurface singularities and the Scandinavian complex, 2002. To appear in Compositio Math.
[9] Luc Illusie. *Complexe cotangent et déformations I*. Number 239 in Lecture Notes in Math. Springer-Verlag, 1971.
[10] Luc Illusie. *Complexe cotangent et déformations II*. Number 283 in Lecture Notes in Math. Springer-Verlag, 1972.
[11] Akira Ishii. Versal deformation of reflexive modules over rational double points. *Math. Ann.*, 317:239–262, 2000.
[12] Yujiro Kawamata. Unobstructed deformations. II. *J. Algebraic Geom.*, 4:277–279, 1995.
[13] Olav Arnfinn Laudal. *Formal Moduli of Algebraic Structures*. Number 754 in Lecture Notes in Math. Springer-Verlag, 1979.
[14] Olav Arnfinn Laudal. Matric Massey products and formal moduli I. In *Algebra, Algebraic Topology and Their Interactions*, number 683 in Lecture Notes in Math., pages 218–240. Springer-Verlag, 1986.
[15] Barry Mazur. Deforming Galois representations. In Y. Ihara, K. Ribet, and J.-P. Serre, editors, *Galois Groups over* $\mathbb{Q}$, number 16 in MSRI Publications, pages 385–437. Springer-Verlag, 1989.
[16] Barry Mazur. Deformation theory of Galois representations. In Gary Cornell, Joseph H. Silvermann, and Glenn Stevens, editors, *Modular Forms and Fermat's Last Theorem*, pages 243–311. Springer-Verlag, 1997.
[17] V. P. Palamodov. Deformations of complex spaces. *Russian Math. Surveys*, 31(3):129–194, 1976.
[18] Michael Schlessinger. Functors of Artin rings. *Trans. Amer. Math. Soc.*, 130:208–222, 1968.
[19] Arvid Siqveland. Global matric Massey products and the compactified Jacobian of the $\mathbf{E_6}$-singularity. *J. Algebra*, 241:259–291, 2001.
[20] Arvid Siqveland. The method of computing formal moduli. *J. Algebra*, 241:292–327, 2001.
[21] Hartwig von Essen. Nonflat deformations of modules and isolated singularities. *Math. Ann.*, 287(3):413–427, 1990.
[22] Charles H. Walter. Some examples of obstructed curves in $\mathbb{P}^3$. In *Complex Projective Geometry* (*Trieste, 1989/Bergen, 1989*), number 179 in London Math. Soc. Lecture Note Ser., pages 324–340. Cambridge Univ. Press, 1992.

Dept. of Mathematics, PO Box 1053 Blindern, NO-0316 Oslo, NORWAY
Current address: Norwegian University of Science and Technology, Dept. of Mathematical Sciences, 7491 Trondheim, NORWAY
E-mail address: ile@math.ntnu.no